A relative Yoneda Lemma (manuscript)

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Abstract

We construct set-valued right Kan extensions via a relative Yoneda Lemma.

A remark of the referee

As the referee pointed out, (2.1) 'can essentially be found in much greater generality' in

[Ke 82] G. M. Kelly, *Basic concepts of enriched category theory,* LMS Lecture Notes 64, Cambridge University Press, 1982.

He continued to explain that to this end, one reformulates the formula [Ke 82, 4.6 (ii), p. 113], given in terms of weighted limits, by means of [Ke 82, 3.10, p. 99] and [Ke 82, 2.2, p. 58].

Therefore, we withdraw this note as a preprint. Since (2.1, 3.1) might be of some use for the working mathematician, we leave it accessible as a manuscript. When using (2.1), the reader is asked to cite [Ke 82], when using (3.1), he is asked to cite [K 58].

0 Introduction

0.1 A relative Yoneda Lemma

The category of set-valued presheaves on a category C shall be denoted by $C^\wedge$. The Yoneda embedding, sending x to ${}_C(-,x)$, shall be denoted by $C \xrightarrow{y_C} C^\wedge$. The presheaf category construction being contravariantly functorial, we obtain the

Proposition (2.1, the relative Yoneda Lemma). *Given a functor $C \xrightarrow{f} D$, we have*

$$f^\wedge \dashv y_D^\wedge \circ f^{\wedge\wedge} \circ y_{C^\wedge}.$$

There is a set theoretical caveat. In particular, this formula is correct only after some additional comments, see (2.1).

The right hand side is also known as the set-valued right Kan extension functor along $C^o \xrightarrow{f^o} D^o$. There are several formulas for right Kan extensions in the literature, for instance using ends [ML 71, X.4], all of them necessarily yielding the same result up to natural equivalence, by uniqueness of the adjoint. In particular, (2.1) is merely still another such a formula.

Letting $f = 1_C$, we recover the (absolute) Yoneda Lemma, thus giving a solution to the exercise [ML 71, X.7, ex. 2]. Concerning its origin, Mac Lane recalled the following incident, taking place in around 1954/55 [ML 98].

> Mac Lane, then visiting Paris, was anxious to learn from Yoneda, and commenced an interview with Yoneda in a café at the Gare du Nord. The interview was continued on Yoneda's train until its departure. In its course, Mac Lane learned about the lemma and subsequently baptized it.

0.2 A relative co-Yoneda Lemma

Suppose given a functor $C \xrightarrow{k} Z$. Kan constructed in [K 58, Th. 14.1] the left adjoint to the functor from Z to $C^\wedge$ that sends $z \in Z$ to ${}_Z(k(-), z) \in C^\wedge$. Now given a functor $C \xrightarrow{f} D$, we can specialize to $Z = D^\wedge$ and to a functor k that sends $c \in C$ to ${}_D(-, fc) \in D^\wedge$, thus obtaining a left adjoint to $C^\wedge \xleftarrow{f^\wedge} D^\wedge$. (Kan's notation is as follows. Identify $\mathcal{Z} = Z$, $\mathcal{V} = C$, $\mathfrak{M}^\mathcal{V} = C^\wedge$. The functor $H^\mathcal{V}(\mathcal{Z}_\mathcal{V}, \mathcal{Z})$ maps from $\mathcal{Z}_\mathcal{V}, \mathcal{Z}$ to $\mathfrak{M}^\mathcal{V}$, i.e. from (the category of covariant functors from $\mathcal{V}$ to $\mathcal{Z}$) $\times\, \mathcal{Z}$ to the category of presheaves on $\mathcal{V}$. Given $\mathcal{V} \xrightarrow{k} \mathcal{Z}$ and an object $z \in \mathcal{Z}$, the pair (k, z) is mapped to ${}_\mathcal{Z}(-, z) \circ k$.) We shall rephrase this special case of Kan's formula as follows, in order to be able to compare the left and the right adjoint of $f^\wedge$.

Let $C^\vee := (C^o)^\wedge$. The co-Yoneda embedding, sending x to ${}_C(x, -)$, shall be denoted by $C \xrightarrow{y'_C} C^\vee$. There is a tensor product $C^\wedge \times C^\vee \longrightarrow (\mathrm{Set})$, sending $v \times v'$ to $v \otimes_C v'$. The according univalent functor that sends v to $v \otimes_C -$ shall be denoted by $C^\wedge \xrightarrow{z_C} C^{\vee\vee}$.

Proposition ([K 58], cf. 3.1, the relative co-Yoneda Lemma). *Given a functor $C \xrightarrow{f} D$, we have*

$$y_D'^{\vee} \circ f^{\vee\vee} \circ z_C \dashv f^\wedge.$$

Again, there is a set-theoretical caveat.

0.3 Acknowledgements

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0.4 Notation

(i) We write composition of morphisms on the right, $\xrightarrow{a}\xrightarrow{b} = \xrightarrow{ab}$. However, we write composition of functors on the left, $\xrightarrow{f}\xrightarrow{g} = \xrightarrow{g\circ f}$.

(ii) Given a functor $C \xrightarrow{f} D$, the opposite functor between the opposite categories is denoted by $C^o \xrightarrow{f^o} D^o$.

(iii) Given functors $C \underset{f'}{\overset{f}{\rightrightarrows}} D \underset{g'}{\overset{g}{\rightrightarrows}} E$ and natural transformations $f \xrightarrow{\alpha} f'$ and $g \xrightarrow{\beta} g'$, we denote by $g \circ f \xrightarrow{\beta\circ f} g' \circ f$ the natural transformation defined by $(\beta \circ f)c = \beta(fc)$ for $c \in C$ and by $g \circ f \xrightarrow{g\circ\alpha} g \circ f'$ the natural transformation defined by $(g \circ \alpha)c = g(\alpha c)$ for $c \in C$. More generally,

$$(g \circ f \xrightarrow{\beta\circ\alpha} g' \circ f') := (g \circ f \xrightarrow{\beta\circ f} g' \circ f \xrightarrow{g'\circ\alpha} g' \circ f') = (g \circ f \xrightarrow{g\circ\alpha} g \circ f' \xrightarrow{\beta\circ f'} g' \circ f').$$

1 Universes

Since we will iterate the construction 'forming the presheaf category over a category' once, we shall work in the setting of universes, which enables us to do such 'large' constructions when keeping track of the universe needed. Therefore, we start with a preliminary section to recall this well-known technique from [SGA 4 I, App.] and to fix some notation.

Definition 1.1 (N. Bourbaki, **[SGA 4 I, App., 1, Déf. 1]**)
A universe *is a set $\mathfrak{U}$ that satisfies the conditions (U1-4).*

(U1) $x \in y \in \mathfrak{U}$ *implies* $x \in \mathfrak{U}$.

(U2) $x, y \in \mathfrak{U}$ *implies* $\{x, y\} \in \mathfrak{U}$.

(U3) $x \in \mathfrak{U}$ *implies* $\mathfrak{P}(x) \in \mathfrak{U}$.

(U4) Given $I \in \mathfrak{U}$ *and a map* $I \longrightarrow \mathfrak{U}$, $i \longmapsto x_i$, *the union* $\bigcup_{i\in I} x_i$ *is in* $\mathfrak{U}$.

Here $\mathfrak{P}(x)$ *denotes the power set of* x.

Remark 1.2

(i) $x \in \mathfrak{U}$ implies $\{x\} = \{x, x\} \in \mathfrak{U}$.

(ii) $x \subseteq y \in \mathfrak{U}$ implies $x \in \mathfrak{U}$. In particular, if y surjects onto some set z, then z is in bijection to an element of $\mathfrak{U}$.

(iii) Let $(x, y) := \{x, \{x, y\}\}$. The element x is the unique element of

$$\{a \in (x, y) \mid \text{for all } \; b \in (x, y) \backslash \{a\} \text{ we have } a \in b\},$$

since $\{x, y\} \in x$ would contradict VON NEUMANN's axiom, asserting that any nonempty set S contains an element that has an empty intersection with S. Now $\{x, y\}$ is the unique element of $(x, y)\backslash\{x\}$, and y is the unique element of $\{x, y\}\backslash\{x\}$.

(iv) $X \in \mathfrak{U}$ and $Y \in \mathfrak{U}$ implies $X \times Y := \bigcup_{x \in X} \bigcup_{y \in Y} \{(x, y)\} \in \mathfrak{U}$.

(v) Given $I \in \mathfrak{U}$ and a map $I \longrightarrow \mathfrak{U}$, $i \longmapsto x_i$, the disjoint union $\coprod_{i \in I} x_i := \bigcup_{i \in I} X_i \times \{i\}$ is in $\mathfrak{U}$.

(vi) Given $I \in \mathfrak{U}$ and a map $I \longrightarrow \mathfrak{U}$, $i \longmapsto x_i$, the product $\prod_{i \in I} x_i \subseteq \mathfrak{P}(\coprod_{i \in I} x_i)$ is in $\mathfrak{U}$.

Assume given universes $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{W}$ such that

$$\emptyset \neq \mathfrak{U} \in \mathfrak{V} \in \mathfrak{W}.$$

This may be achieved by means of BOURBAKI's axiom (A.6) in [SGA 4 I, App., 4], which says that any set is element of some universe. Note that $\mathfrak{U} \subseteq \mathfrak{V} \subseteq \mathfrak{W}$.

For set theoretical purposes the following definition is convenient.

Definition 1.3 (cf. [M 65, I.2]) *A* small category *C is a tuple $(M, K \subseteq M \times M \times M)$, M being a set, subject to the conditions (i-iii). We write $M =: \operatorname{Mor} C$, the set of* morphisms *of C. K is the* composition law *of C.*

(i) For every $(a, b) \in M \times M$ there exists at most one $c \in M$ such that $(a, b, c) \in K$. In this case, we write $c = ab$ and say that '(the composite*) ab exists'.*

Let

$$\begin{array}{rll} \operatorname{Ob} C := \{ \; a \in M & \mid & \textit{for all } b \textit{ for which } ba \textit{ exists, we have } ba = b, \textit{ and} \\ & & \textit{for all } c \textit{ for which } ac \textit{ exists, we have } ac = c.\} \; \subseteq \; \operatorname{Mor} C \end{array}$$

be the set of objects *of C. If $a \in \operatorname{Ob} C$, we also write $a =: 1_a$.*

(ii) For every $a \in M$ there exists a unique source *element $s(a) \in \operatorname{Ob} C$ such that $s(a)a$ exists, and a unique* target *element $t(a) \in \operatorname{Ob} C$ such that $at(a)$ exists.*

(iii) For $a, b \in M$ such that the composite ab exists, the composites $t(a)b$ and $as(b)$ exist.

(iv) Given $a, b, c \in M$ such that ab and bc exist. Then $(ab)c$ and $a(bc)$ exist and equal each other.

The set of morphisms with start x and target y shall be denoted by ${}_C(x,y)$.

Suppose given $a, b \in M$. Note that if ab exists, we have $s(ab) = s(a)$ and $t(ab) = t(b)$. Note that $s(s(a)) = s(a)$, $t(t(a)) = t(a)$. Note that ab exists iff $t(a) = s(b)$.

Definition 1.4 *A set X is said to be* $\mathfrak{U}$-small *if there exists a bijection from X to an element of $\mathfrak{U}$. The category of $\mathfrak{U}$-small sets is denoted by $(\mathrm{Set}_{\mathfrak{U}})$. Note that by a skeleton argument, $(\mathrm{Set}_{\mathfrak{U}})$ is equivalent to the category of sets contained as elements in $\mathfrak{U}$, denoted by $(\mathrm{Set}^0_{\mathfrak{U}})$.*

A small category C is said to be $\mathfrak{U}$-small *if* $\mathrm{Mor}\, C$ *is $\mathfrak{U}$-small. A category C is said to be* essentially $\mathfrak{U}$-small *if it is equivalent to a $\mathfrak{U}$-small category, or, equivalently, if it has a $\mathfrak{U}$-small skeleton.*

Remark 1.5 The category $(\mathrm{Set}^0_{\mathfrak{U}})$ is $\mathfrak{V}$-small since $\coprod\limits_{(x,y)\in\mathfrak{U}\times\mathfrak{U}} {}_{(\mathrm{Set}^0_{\mathfrak{U}})}(x,y)$ is an element of $\mathfrak{V}$. Hence the category $(\mathrm{Set}_{\mathfrak{U}})$ is essentially $\mathfrak{V}$-small.

Given categories C and D, we denote the category of functors mapping from C to D by $[\![C, D]\!]$.

Lemma 1.6 *If C and D are $\mathfrak{U}$-small, then $[\![C, D]\!]$ is $\mathfrak{U}$-small. If C and D are essentially $\mathfrak{U}$-small, then $[\![C, D]\!]$ is essentially $\mathfrak{U}$-small.*

Using induced equivalences, it suffices to prove the first assertion. But then

$$\mathrm{Ob}\,[\![C, D]\!] \subseteq \mathfrak{P}(\mathrm{Mor}\, C \times \mathrm{Mor}\, D)$$

is $\mathfrak{U}$-small. Moreover, given $f, g \in [\![C, D]\!]$, the set

$${}_{[\![C,D]\!]}(f,g) \subseteq \coprod_{x\in \mathrm{Ob}\, C} {}_D(fx, gx)$$

is $\mathfrak{U}$-small. Hence $\mathrm{Mor}\,[\![C, D]\!]$ is $\mathfrak{U}$-small.

Lemma 1.7 *Given an essentially $\mathfrak{U}$-small category C, the category*

$$C^{\wedge_{\mathfrak{U}}} := [\![C^o, (\mathrm{Set}_{\mathfrak{U}})]\!]$$

of $(\mathrm{Set}_{\mathfrak{U}})$-valued presheaves *over C is essentially $\mathfrak{V}$-small. Sometimes we write $C^\wedge = C^{\wedge_{\mathfrak{U}}}$ if the universe is unambiguous. Likewise, the category*

$$C^{\vee_{\mathfrak{U}}} := [\![C, (\mathrm{Set}_{\mathfrak{U}})]\!]$$

of $(\mathrm{Set}_{\mathfrak{U}})$-valued copresheaves *over C is essentially $\mathfrak{V}$-small. Sometimes we write $C^\vee = C^{\vee_{\mathfrak{U}}}$.*

This follows from (1.5, 1.6).

Definition 1.8 *Given an essentially* $\mathfrak{U}$*-small category* C*, we have the* Yoneda embedding

$$\begin{array}{ccc} C & \xrightarrow{y_C} & C^{\wedge\mathfrak{U}} \\ x & \longrightarrow & {}_C(-,x) \end{array}$$

and the co-Yoneda embedding

$$\begin{array}{ccc} C & \xrightarrow{y'_C} & C^{\vee\mathfrak{U}} \\ x & \longrightarrow & {}_C(x,-). \end{array}$$

For a functor $C \xrightarrow{f} D$*, we denote*

$$\begin{array}{ccc} C^{\wedge\mathfrak{U}} & \xleftarrow{f^{\wedge}} & D^{\wedge\mathfrak{U}} \\ (u \circ f^{o} \xrightarrow{\beta \circ f^{o}} u' \circ f^{o}) & \longleftarrow & (u \xrightarrow{\beta} u'). \end{array}$$

Given functors $C \underset{g}{\overset{f}{\rightrightarrows}} D$ *and a natural transformation* $f \xrightarrow{\alpha} g$*, we denote by* $f^{\wedge} \xleftarrow{\alpha^{\wedge}} g^{\wedge}$ *the natural transformation that is given at* $u \in D^{\wedge\mathfrak{U}}$ *by the morphism* $u \circ f^{o} \xleftarrow{u \circ \alpha^{o}} u \circ g^{o}$ *in* $C^{\wedge\mathfrak{U}}$*, that evaluated at* $c \in C^{o}$ *in turn yields* $ufc \xleftarrow{u\alpha c} ugc$*.*
Analogously for $C^{\vee\mathfrak{U}} \xleftarrow{f^{\vee}} D^{\vee\mathfrak{U}}$ *and* $f^{\vee} \xrightarrow{\alpha^{\vee}} g^{\vee}$*.*

2 The right Kan extension

Proposition 2.1 (the relative Yoneda Lemma) *Given essentially* $\mathfrak{U}$*-small categories* C *and* D*, and a functor* $C \xrightarrow{f} D$*. Then the right adjoint* $C^{\wedge\mathfrak{U}} \xrightarrow{\varphi} D^{\wedge\mathfrak{U}}$ *of* $C^{\wedge\mathfrak{U}} \xleftarrow{f^{\wedge}} D^{\wedge\mathfrak{U}}$ *is given by*

$$\begin{tikzcd} C^{\wedge\mathfrak{U}\wedge\mathfrak{V}} \arrow[r, "f^{\wedge\wedge}"] & D^{\wedge\mathfrak{U}\wedge\mathfrak{V}} \arrow[d, "y_D^{\wedge}"] \\ & D^{\wedge\mathfrak{V}} \\ C^{\wedge\mathfrak{U}} \arrow[uu, "y_{C^{\wedge}}"] \arrow[r, "\varphi"] & D^{\wedge\mathfrak{U}} \arrow[u, hook] \end{tikzcd}$$

Keeping the name of the functor after restricting the image to $D^{\wedge\mathfrak{U}}$*, we write shorthand*

$$\boxed{f^{\wedge} \dashv y_D^{\wedge} \circ f^{\wedge\wedge} \circ y_{C^{\wedge}}.}$$

The unit of this adjunction

$$\boxed{1_{D^{\wedge}} \xrightarrow{\varepsilon} \varphi \circ f^{\wedge}}$$

at $u \in D^{\wedge_{\mathfrak{U}}}$*, i.e.*

$$u \xrightarrow{\varepsilon u} {}_{C^\wedge}(f^{\wedge,o} \circ y_D^o(-), u \circ f^o),$$

applied to $d \in D$*, is given by*

$$\begin{array}{rcl} ud & \xrightarrow{\varepsilon ud} & {}_{C^\wedge}({}_D(f^o(-), d), u \circ f^o) \\ x & \longrightarrow & (x)\varepsilon ud, \end{array}$$

where the natural transformation $(x)\varepsilon ud$ *sends at* $c \in C$

$$\begin{array}{rcl} {}_D(fc, d) & \xrightarrow{(x)\varepsilon ud} & ufc \\ a & \longrightarrow & (a)\Big[\Big((x)\varepsilon ud\Big)c\Big] = (x)(ua^o). \end{array}$$

The counit of this adjunction

$$\boxed{f^\wedge \circ \varphi \xrightarrow{\eta} 1_{C^\wedge}}$$

at $v \in C^\wedge$*, i.e.*

$${}_{C^\wedge}(f^{\wedge,o} \circ y_D^o \circ f^o(-), v) \xrightarrow{\eta v} v,$$

applied to $c \in C$*, is given by*

$$\begin{array}{rcl} {}_{C^\wedge}({}_D(f^o(-), fc), v) & \xrightarrow{\eta vc} & vc \\ \xi & \longrightarrow & (\xi)\eta vc = (1_{fc})\xi c \end{array}$$

Since the set

$${}_{C^\wedge}({}_D(f(-), d), v) \subseteq \prod_{c \in \operatorname{Ob} C} {}_{(\mathrm{Set}_{\mathfrak{U}})}({}_D(fc, d), vc)$$

is $\mathfrak{U}$-small, φ exists as the factorization of $y_D^\wedge \circ f^{\wedge\wedge} \circ y_{C^\wedge}$ over the inclusion $D^{\wedge_{\mathfrak{U}}} \subseteq D^{\wedge_{\mathfrak{V}}}$. Various compatibilities need to be verified to ensure the well-definedness of ε and η ([1]).

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[1] Given $u \in D^{\wedge_{\mathfrak{U}}}$, we need to see that $u \xrightarrow{\varepsilon u} \varphi \circ f^\wedge(u)$ is a natural transformation. Suppose given $d' \xrightarrow{a} d$ in D. We have to show that for any $x \in ud$

$$(x)\Big(\varepsilon ud\Big)\Big((\varphi \circ f^\wedge)ua^o\Big) = (x)\Big(ua^o\Big)\Big(\varepsilon ud'\Big)$$

is an equality of natural transformations from ${}_D(f^o(-), d')$ to $u \circ f^o$. At $c \in C$, the element $fc \xrightarrow{a'} d'$ is mapped by the left hand side to $(x)(u(a'a)^o)$, and by the right hand side to $(x(ua^o))(ua'^o)$.

We need to see that $1_{D^{\wedge_{\mathfrak{U}}}} \xrightarrow{\varepsilon} \varphi \circ f^\wedge$ is a natural transformation. Suppose given $u \xrightarrow{s} u'$ in $D^{\wedge_{\mathfrak{U}}}$. We have to show that for any $d \in D$ and any $x \in ud$

$$(x)\Big(sd\Big)\Big(\varepsilon u'd\Big) = (x)\Big(\varepsilon ud\Big)\Big((\varphi \circ f^\wedge(s))d\Big)$$

is an equality of natural transformations from ${}_D(f^o(-), d)$ to $u' \circ f$. At $c \in C$, the element $fc \xrightarrow{a} d$ is mapped by the left hand side to $(x)(sd)(u'a^o)$, and by the right hand side to $(x)(ua^o)(s(fc))$.

We have to show that $(f^\wedge \circ \varepsilon)(\eta \circ f^\wedge) = 1_{f^\wedge}$. Suppose given $u \in D^\wedge$, $c \in C$ and $x \in (f^\wedge u)c = u(fc)$. We obtain

$$\begin{array}{rcl} (x)\Big((f^\wedge \circ \varepsilon)uc\Big)\Big((\eta \circ f^\wedge)uc\Big) & = & (x)\Big(\varepsilon u(fc)\Big)\Big(\eta(u \circ f^o)c\Big) \\ & = & (1_{fc})\,\Big[\Big((x)(\varepsilon u(fc))\Big)c\Big] \\ & = & (x)(u1^o_{fc}) \\ & = & x. \end{array}$$

We have to show that $(\varepsilon \circ \varphi)(\varphi \circ \eta) = 1_\varphi$. Suppose given $v \in C^{\wedge\mathfrak{U}}$ and $d \in D$. Note that $\varphi v = {}_{C^\wedge}(f^{\wedge,o} \circ y^o_D(-), v)$ and therefore $\varphi v d = {}_{C^\wedge}({}_D(f^o(-), d), v)$. The application $\Big((\varepsilon \circ \varphi)vd\Big)\Big((\varphi \circ \eta)vd\Big)$ writes

$${}_C({}_D(f^o(-), d), v) \xrightarrow{\varepsilon(\varphi v)d} {}_{C^\wedge}\Big({}_D(f^o(-), d),\ {}_{C^\wedge}(f^{\wedge,o} \circ y^o_D \circ f^o(-), v)\Big) \xrightarrow{(-)\eta v} {}_{C^\wedge}({}_D(f^o(-), d), v)$$

An element $\xi \in \varphi v d$, i.e. ${}_D(f^o(-), d) \xrightarrow{\xi} v$, is thus mapped to the composite

$${}_D(f^o(-), d) \xrightarrow{(\xi)\varepsilon(\varphi v)d} {}_{C^\wedge}(f^{\wedge,o} \circ y^o_D \circ f^o(-), v) \xrightarrow{\eta v} v.$$

Now suppose given $c \in C$ and $fc \xrightarrow{a} d$. We have

$$\begin{array}{ccccc} {}_D(fc, d) & \xrightarrow{((\xi)\varepsilon(\varphi v)d)c} & {}_{C^\wedge}\Big({}_D(f^o(-), fc), v\Big) & \xrightarrow{\eta v c} & vc \\ a & \longrightarrow & (\xi)((\varphi v)a^o) & & \\ & & = (\xi)\,{}_{C^\wedge}\Big({}_D(f^o(-), a), v\Big) & & \\ & & = {}_D(f^o(-), a)\xi & \longrightarrow & (1_{fc})\,{}_D(fc, a)(\xi c) \\ & & & & = (a)(\xi c), \end{array}$$

whence $\xi\Big((\varepsilon \circ \varphi)vd\Big)\Big((\varphi \circ \eta)vd\Big) = \xi$.

Remark 2.2 (the absolute Yoneda Lemma) *In case $f = 1_C$, we obtain $1 \dashv y^\wedge_C \circ y_{C^\wedge}$, which by uniqueness of the right adjoint yields the comparison isomorphism*

$$\begin{array}{ccc} {}_{C^\wedge}({}_C(-, c), v) & \xrightarrow[\sim]{\eta v c} & vc \\ \xi & \longrightarrow & (1_c)\xi c, \end{array}$$

Given $v \in C^{\wedge\mathfrak{U}}$, we need to see that $f^\wedge \circ \varphi(v) \xrightarrow{\eta v} v$ is a natural transformation. Suppose given $c' \xrightarrow{b} c$ in C. We have to show that for any $\xi \in (f^\wedge \circ \varphi(v))c = {}_{C^\wedge}({}_D(f^o(-), fc), v)$ we have

$$(x)\Big(\eta v c\Big)\Big(v b^o\Big) = (x)\Big((f^\wedge \circ \varphi(v))b^o\Big)\Big(\eta v c'\Big).$$

The left hand side yields $(1_{fc})(\xi c)(vb^o)$, the right hand side yields $(fb)\xi c'$.

We need to see that $f^\wedge \circ \varphi \xrightarrow{\eta} 1_{C^{\wedge\mathfrak{U}}}$ is a natural transformation. Suppose given $v \xrightarrow{t} v'$ in $C^{\wedge\mathfrak{U}}$. We have to show that for any $c \in C$ and any $\xi \in (f^\wedge \circ \varphi(v))c = {}_{C^\wedge}({}_D(f^o(-), fc), v)$ we have

$$(\xi)\Big(\eta v c\Big)\Big(tc\Big) = (\xi)\Big((f^\wedge \circ \varphi(t))c\Big)\Big(\eta v' c\Big).$$

The left hand side yields $(1_{fc})(\xi c)(tc)$. The right hand side yields $(1_{fc})((\xi t)c)$.

at $v \in C^{\wedge_{\mathfrak{U}}}$ and $c \in C$, with inverse given by εvc.

Corollary 2.3 *If f is full, then the counit η of the adjunction $f^{\wedge} \dashv y_D^{\wedge} \circ f^{\wedge\wedge} \circ y_{C^{\wedge}}$ is a monomorphism. If f is full and faithful, then η is an isomorphism.*

3 The left Kan extension

For sake of comparison to (2.1), we rephrase the pertinent case of Kan's formula in our setting.

Let C be a $\mathfrak{U}$-small category. Let $v \in C^{\wedge_{\mathfrak{U}}}$, let $w \in C^{\vee_{\mathfrak{U}}}$. We define the set $v \otimes_C w$ as the quotient of the disjoint union

$$v \times_C w := \coprod_{c \in C} vc \times wc$$

modulo the equivalence relation generated by the following relation $\sim_C$. The equivalence class of $(p,q) \in vc \times wc$, $c \in C$, shall be denoted by $p \otimes q$.

Given $(p,q) \in vc \times wc$, $(p',q') \in vc' \times wc'$, we say that $(p,q) \sim_C (p',q')$ if there exists a morphism $c \xrightarrow{a} c'$ such that

$$\begin{aligned} (p')va^o &= p \\ (q)wa &= q'. \end{aligned}$$

Thus the quotient map $v \times_C w \xrightarrow{\nu} v \otimes_C w$ has the following universal property. Given a map $v \times_C w \xrightarrow{\nu'} X$ such that for any morphism $c \xrightarrow{a} c'$, any $p' \in vc'$ and any $q \in wc$ we have

$$((p')va^o, q)\nu' = (p', (q)wa)\nu',$$

there exists a unique map $v \otimes_C w \xrightarrow{\tilde{\nu}'} X$ such that $\nu' = \nu\tilde{\nu}'$.

In particular, given morphisms $v \xrightarrow{m} v'$ and $w \xrightarrow{n} w'$, we obtain a map $m \otimes_C n$ that maps an element represented by $(p,q) \in vc \times wc$, $c \in C$, as follows.

$$\begin{array}{ccccccc} v & \otimes_C & w & \xrightarrow{m \otimes_C n} & v' & \otimes_C & w' \\ p & \otimes & q & \longrightarrow & (p)mc & \otimes & (q)nc \end{array}$$

Thus the tensor product defines a functor $C^{\wedge_{\mathfrak{U}}} \times C^{\vee_{\mathfrak{U}}} \xrightarrow{= \otimes_C -} (\mathrm{Set}_{\mathfrak{U}})$. We denote the univalent tensor product functor by

$$\begin{array}{ccc} C^{\wedge_{\mathfrak{U}}} & \xrightarrow{z_C} & C^{\vee_{\mathfrak{U}} \vee_{\mathfrak{U}}} \\ v & \longrightarrow & v \otimes_C -. \end{array}$$

Proposition 3.1 (the relative co-Yoneda Lemma, Kan [K 58, Th. 14.1]) *Given $\mathfrak{U}$-small categories C and D, and a functor $C \xrightarrow{f} D$. The left adjoint $C^{\wedge_{\mathfrak{U}}} \xrightarrow{\psi} D^{\wedge_{\mathfrak{U}}}$ of $C^{\wedge_{\mathfrak{U}}} \xleftarrow{f^{\wedge}} D^{\wedge_{\mathfrak{U}}}$ is given by*

$$\begin{tikzcd}
C^{\vee\mathfrak{U}\vee\mathfrak{U}} \arrow[r, "f^{\vee\vee}"] & D^{\vee\mathfrak{U}\vee\mathfrak{U}} \arrow[d, "y_D'^{\wedge}"] \\
C^{\wedge\mathfrak{U}} \arrow[u, "z_C"] \arrow[r, "\psi"] & (D^o)^{\vee\mathfrak{U}} = D^{\wedge\mathfrak{U}}
\end{tikzcd}$$

For short,

$$\boxed{y_D'^{\vee} \circ f^{\vee\vee} \circ z_C \dashv f^{\wedge}.}$$

The unit of this adjunction

$$\boxed{1_{C^\wedge} \xrightarrow{\varepsilon} f^\wedge \circ \psi}$$

at $v \in C^\wedge$, *i.e.*

$$v \xrightarrow{\varepsilon v} v \otimes_C f^\vee \circ y_D' \circ f^o(-),$$

applied to $c \in C$, *is given by*

$$\begin{array}{ccc}
vc & \xrightarrow{\varepsilon vc} & v \otimes_C {}_D(fc, f(-)) \\
x & \longrightarrow & x \otimes 1_{fc}\ .
\end{array}$$

The counit of this adjunction

$$\boxed{\psi \circ f^\wedge \xrightarrow{\eta} 1_{D^\wedge}}$$

at $u \in D^\wedge$, *i.e.*

$$u \circ f^o \otimes_C f^\vee \circ y_D'(-) \xrightarrow{\eta u} u,$$

applied to $d \in D$, *is given by*

$$\begin{array}{cccc}
u \circ f^o & \otimes_C & {}_D(d, f(-)) & \xrightarrow{\eta ud} & ud \\
p & \otimes & q & \longrightarrow & (p)uq^o,
\end{array}$$

where $p \otimes q$ *is represented by* $(p, q) \in ufc \times {}_D(d, fc)$ *for some* $c \in C$.

Various compatibilities have to be verified to ensure the well-definedness of ε and η ([2]).

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[2] Given $v \in C^\wedge$, we need to see that $v \xrightarrow{\varepsilon v} f^\wedge \circ \psi(v)$ is a natural transformation. Suppose given $c' \xrightarrow{b} c$. We have to show that for any $x \in vc$

$$(x)\Big(\varepsilon vc\Big)\Big((f^\vee \circ \psi(v))b^o\Big) = (x)\Big(vb^o\Big)\Big(\varepsilon vc'\Big).$$

The left hand side yields $x \otimes fb$. The right hand side yields $x(vb^o) \otimes 1_{fc'}$.

We need to see that $1_{C^\wedge} \xrightarrow{\varepsilon} f^\wedge \circ \psi$ is a natural transformation. Suppose given $v \xrightarrow{t} v'$ in $C^\wedge$. For any $c \in C$ and any $x \in vc$ we have

$$(x)\Big(tc\Big)\Big(\varepsilon v'c\Big) = (x)\Big(\varepsilon vc\Big)\Big(tc \otimes {}_D(fc, f(-))\Big) = (x)tc \otimes 1_{fc}.$$

Given $u \in D^\wedge$ and $d \in D$, we need to see that ηud is a well-defined map. Suppose given $c, c' \in C$, We have to show that $(\varepsilon \circ f^\wedge)(f^\wedge \circ \eta) = 1_{f^\wedge}$. Suppose given $u \in D^\wedge$, $c \in C$ and $x \in (f^\wedge u)c$. We obtain

$$\begin{aligned}
(x)\Big((\varepsilon \circ f^\wedge)uc\Big)\Big((f^\wedge \circ \eta)uc\Big) &= (x)\Big(\varepsilon(u \circ f^o)c\Big)\Big(\eta u(fc)\Big) \\
&= (x \otimes 1_{fc})\Big(\eta u(fc)\Big) \\
&= (x)u1^o_{fc}. \\
&= x.
\end{aligned}$$

We have to show that $(\psi \circ \varepsilon)(\eta \circ \psi) = 1_\psi$. Suppose given $v \in C^\wedge$, $d \in D$, $c \in C$, $s \in vc$, $t \in {}_D(d, fc)$, so that $s \otimes t \in \psi v d = v \otimes_C {}_D(d, f(-))$. We obtain

$$\begin{aligned}
(s \otimes t)\Big((\psi \circ \varepsilon)vd\Big)\Big((\eta \circ \psi)vd\Big) &= (s \otimes t)\Big(\varepsilon v \otimes {}_D(d, f(-))\Big)\Big(\eta(v \otimes_C f^\vee \circ y'_D)d\Big) \\
&= \Big((s \otimes 1_{fc}) \otimes t\Big)\Big(\eta(v \otimes_C f^\vee \circ y'_D)d\Big) \\
&= (s \otimes 1_{fc})\Big(v \otimes_C {}_D(t, f(-))\Big) \\
&= (s \otimes t).
\end{aligned}$$

Remark 3.2 (the absolute co-Yoneda Lemma) *In case $f = 1_C$, we obtain $y'^\vee_D \circ z_C \dashv 1_{C^\wedge}$, which by uniqueness of the left adjoint yields the comparison isomorphism*

$$\begin{array}{ccccc}
v & \otimes_C & {}_C(c, -) & \xrightarrow[\sim]{\eta vc} & vc \\
s & \otimes & t & \longrightarrow & (s)vt^o
\end{array}$$

at $v \in C^{\wedge\mathfrak{U}}$ and $c \in C$, with inverse given by εvc.

Corollary 3.3 *If f is full, then the unit ε of the adjunction $y'^\vee_D \circ f^{\vee\vee} \circ z_C \dashv f^\wedge$ is an epimorphism. If f is full and faithful, then ε is an isomorphism.*

$c \xrightarrow{b} c'$ in C and $p' \in ufc'$, $q \in {}_D(d, fc)$. Since

$$\Big((p')uf^o b^o\Big)uq^o = \Big(p'\Big)u(q(fb))^o,$$

the universal property applies.

Given $u \in D^\wedge$, we need to see that $\psi \circ f^\wedge(u) \xrightarrow{\eta u} u$ is a natural transformation. Suppose given $d' \xrightarrow{a} d$. For all $c \in C$, $p \in ufc$ and $q \in {}_D(d, fc)$ we obtain

$$(p \otimes q)\Big(\eta ud\Big)\Big(ua^o\Big) = (p \otimes q)\Big((\psi \circ f^\wedge(u))a^o\Big)\Big(\eta ud'\Big) = (p)u(aq)^o.$$

We need to see that $\psi \circ f^\wedge \xrightarrow{\eta} 1_{D^\wedge}$ is a natural transformation. Suppose given $u' \xrightarrow{s} u$ in $D^\wedge$. We have to show that for all $d \in D$, $c \in C$, $p \in ufc$ and $q \in {}_D(d, fc)$, we have

$$(p \otimes q)\Big(\eta ud\Big)\Big(sd\Big) = (p \otimes q)\Big((\psi \circ f^\wedge(s))d\Big)\Big(\eta u'd\Big).$$

The left hand side yields $(p)(uq^o)(sd)$. The right hand side yields $(p)(s(fc))(u'q^o)$.

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