A relative Yoneda Lemma (manuscript)

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Abstract

We construct set-valued right Kan extensions via a relative Yoneda Lemma.

A remark of the referee

As the referee pointed out, (2.1) ‘can essentially be found in much greater generality’ in [Ke 82] G. M. Kelly, Basic concepts of enriched category theory, LMS Lecture Notes 64, Cambridge University Press, 1982.

He continued to explain that to this end, one reformulates the formula [Ke 82, 4.6 (ii), p. 113], given in terms of weighted limits, by means of [Ke 82, 3.10, p. 99] and [Ke 82, 2.2, p. 58].

Therefore, we withdraw this note as a preprint. Since (2.1, 3.1) might be of some use for the working mathematician, we leave it accessible as a manuscript. When using (2.1), the reader is asked to cite [Ke 82], when using (3.1), he is asked to cite [K 58].

0 Introduction

0.1 A relative Yoneda Lemma

The category of set-valued presheaves on a category $C$ shall be denoted by $C^\wedge$. The Yoneda embedding, sending $x$ to $C(-, x)$, shall be denoted by $C \rightarrowtail_{C^\wedge}$. The presheaf category construction being contravariantly functorial, we obtain the
Proposition (2.1, the relative Yoneda Lemma). Given a functor $C \xrightarrow{f} D$, we have
\[ f^\vee \vdash y_D^\vee \circ f^\vee \circ y_C^\vee. \]

There is a set theoretical caveat. In particular, this formula is correct only after some additional comments, see (2.1).

The right hand side is also known as the set-valued right Kan extension functor along $C^\circ \xrightarrow{f^\circ} D^\circ$. There are several formulas for right Kan extensions in the literature, for instance using ends [ML 71, X.4], all of them necessarily yielding the same result up to natural equivalence, by uniqueness of the adjoint. In particular, (2.1) is merely still another such a formula.

Letting $f = 1_C$, we recover the (absolute) Yoneda Lemma, thus giving a solution to the exercise [ML 71, X.7, ex. 2]. Concerning its origin, Mac Lane recalled the following incident, taking place in around 1954/55 [ML 98].

**Mac Lane,** then visiting Paris, was anxious to learn from Yoneda, and commenced an interview with Yoneda in a café at the Gare du Nord. The interview was continued on Yoneda’s train until its departure. In its course, Mac Lane learned about the lemma and subsequently baptized it.

0.2 A relative co-Yoneda Lemma

Suppose given a functor $C \xrightarrow{k} Z$. Kan constructed in [K 58, Th. 14.1] the left adjoint to the functor from $Z$ to $C^\vee$ that sends $z \in Z$ to $\mathcal{Z}(k(-), z) \in C^\vee$. Now given a functor $C \xrightarrow{f} D$, we can specialize to $Z = D^\vee$ and to a functor $k$ that sends $c \in C$ to $\mathcal{Z}(-, fc) \in D^\vee$, thus obtaining a left adjoint to $C^\vee \xrightarrow{f^\vee} D^\vee$. (Kan’s notation is as follows. Identify $\mathcal{Z} = Z$, $\mathcal{V} = C$, $\mathcal{M}^{\mathcal{V}} = C^\vee$. The functor $H^\mathcal{V}(\mathcal{Z}, \mathcal{Z})$ maps from $\mathcal{Z}$, $\mathcal{Z}$ to $\mathcal{M}^{\mathcal{V}}$, i.e. from the category of covariant functors from $\mathcal{V}$ to $\mathcal{Z}$ to the category of presheaves on $\mathcal{V}$. Given $\mathcal{V} \xrightarrow{k} \mathcal{Z}$ and an object $z \in \mathcal{Z}$, the pair $(k, z)$ is mapped to $\mathcal{Z}(z, z) \circ k$.) We shall rephrase this special case of Kan’s formula as follows, in order to be able to compare the left and the right adjoint of $f^\vee$.

Let $C^\vee := (C^\circ)^\vee$. The co-Yoneda embedding, sending $x$ to $\mathcal{C}(x, -)$, shall be denoted by $C \xrightarrow{y_C^\vee} C^\vee$. There is a tensor product $C^\vee \times C'^\vee \longrightarrow \text{Set}$, sending $v \times v'$ to $v \otimes_C v'$. The according univalent functor that sends $v$ to $v \otimes_C -$ shall be denoted by $C^\vee \xrightarrow{z_C} C'^\vee$.

**Proposition** ([K 58], cf. 3.1, the relative co-Yoneda Lemma). Given a functor $C \xrightarrow{f^\vee} D$, we have
\[ y_D'^\vee \circ f'^\vee \circ z_C \vdash f^\vee. \]

Again, there is a set-theoretical caveat.
0.3 Acknowledgements

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0.4 Notation

(i) We write composition of morphisms on the right, \( \xrightarrow{a} \xrightarrow{b} = \xrightarrow{ab} \). However, we write composition of functors on the left, \( \xrightarrow{f} \xrightarrow{g} = \xrightarrow{gf} \).

(ii) Given a functor \( C \xrightarrow{f} D \), the opposite functor between the opposite categories is denoted by \( C^o \xrightarrow{f^o} D^o \).

(iii) Given functors \( C \xrightarrow{f} f' \) \( D \xrightarrow{g} g' \) \( E \) and natural transformations \( \xrightarrow{\alpha} \) \( \xrightarrow{\beta} \) \( f' \) and \( \xrightarrow{\alpha'} \) \( \xrightarrow{\beta'} \) \( g' \), we denote by \( \xrightarrow{g \circ f \beta \circ f' \alpha' \circ f'} \) \( g' \circ f' \) the natural transformation defined by \( (\beta \circ f)c = \beta(fc) \) for \( c \in C \) and by \( \xrightarrow{g \circ f \alpha \circ f'} \) \( g \circ f' \) the natural transformation defined by \( (g \circ \alpha)c = g(\alpha c) \) for \( c \in C \). More generally,

\[
(g \circ f \beta \circ f' \alpha' \circ f') := (g \circ f \beta \circ f' \circ f' \circ f' \circ f') = (g \circ f \beta \circ f' \circ f') = (g \circ f \beta \circ f' \circ f').
\]

1 Universes

Since we will iterate the construction ‘forming the presheaf category over a category’ once, we shall work in the setting of universes, which enables us to do such ‘large’ constructions when keeping track of the universe needed. Therefore, we start with a preliminary section to recall this well-known technique from [SGA 4 I, App.] and to fix some notation.

**Definition 1.1** (N. Bourbaki, [SGA 4 I, App., 1, Déf. 1])

A universe is a set \( \mathcal{U} \) that satisfies the conditions \( (U1-4) \).

- \( (U1) \ x \in y \in \mathcal{U} \) implies \( x \in \mathcal{U} \).
- \( (U2) \ x, y \in \mathcal{U} \) implies \( \{x, y\} \in \mathcal{U} \).
- \( (U3) \ x \in \mathcal{U} \) implies \( \mathcal{P}(x) \in \mathcal{U} \).
- \( (U4) \ Given \ I \in \mathcal{U} \ and \ a \ map \ I \xrightarrow{i} \mathcal{U}, \ i \mapsto x_i, \ the \ union \ \bigcup_{i \in I} x_i \ is \ in \ \mathcal{U}. \)

Here \( \mathcal{P}(x) \) denotes the power set of \( x \).

**Remark 1.2**

- \( (i) \ x \in \mathcal{U} \) implies \( \{x\} = \{x, x\} \in \mathcal{U} \).
(ii) $x \subseteq y \in \mathcal{U}$ implies $x \in \mathcal{U}$. In particular, if $y$ surjects onto some set $z$, then $z$ is in bijection to an element of $\mathcal{U}$.

(iii) Let $(x, y) := \{x, \{x, y\}\}$. The element $x$ is the unique element of

$$\{ a \in (x, y) \mid \text{for all } b \in (x, y) \backslash \{a\} \text{ we have } a \in b \},$$

since $\{x, y\} \in x$ would contradict von Neumann’s axiom, asserting that any nonempty set $S$ contains an element that has an empty intersection with $S$. Now $\{x, y\}$ is the unique element of $(x, y) \backslash \{x\}$, and $y$ is the unique element of $(x, y) \backslash \{x\}$.

(iv) $X \in \mathcal{U}$ and $Y \in \mathcal{U}$ implies $X \times Y := \bigcup_{x \in X} \bigcup_{y \in Y} \{(x, y)\} \in \mathcal{U}$.

(v) Given $I \in \mathcal{U}$ and a map $I \rightarrow \mathcal{U}$, $i \mapsto x_i$, the disjoint union $\biguplus_{i \in I} x_i := \bigcup_{i \in I} X_i \times \{i\}$ is in $\mathcal{U}$.

(vi) Given $I \in \mathcal{U}$ and a map $I \rightarrow \mathcal{U}$, $i \mapsto x_i$, the product $\prod_{i \in I} x_i \subseteq \mathcal{P}(\biguplus_{i \in I} x_i)$ is in $\mathcal{U}$.

Assume given universes $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ such that

$$\emptyset \neq \mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}.$$ 

This may be achieved by means of Bourbaki’s axiom (A.6) in [SGA 4 I, App., 4], which says that any set is element of some universe. Note that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}$.

For set theoretical purposes the following definition is convenient.

**Definition 1.3 (cf. [M 65, I.2])** A small category $C$ is a tuple $(M, K \subseteq M \times M \times M)$, $M$ being a set, subject to the conditions (i-iii). We write $M =: \text{Mor} C$, the set of morphisms of $C$. $K$ is the composition law of $C$.

(i) For every $(a, b) \in M \times M$ there exists at most one $c \in M$ such that $(a, b, c) \in K$. In this case, we write $c = ab$ and say that ‘(the composite) $ab$ exists’.

Let

$$\text{Ob} C := \{ a \in M \mid \text{for all } b \text{ for which } ba \text{ exists, we have } ba = b, \text{ and}$$

$$\text{for all } c \text{ for which } ac \text{ exists, we have } ac = c. \} \subseteq \text{Mor} C$$ 

be the set of objects of $C$. If $a \in \text{Ob} C$, we also write $a =: 1_a$.

(ii) For every $a \in M$ there exists a unique source element $s(a) \in \text{Ob} C$ such that $s(a)a$ exists, and a unique target element $t(a) \in \text{Ob} C$ such that $at(a)$ exists.

(iii) For $a, b \in M$ such that the composite $ab$ exists, the composites $t(a)b$ and $a(s(b)$ exist.

(iv) Given $a, b, c \in M$ such that $ab$ and $bc$ exist. Then $(ab)c$ and $a(bc)$ exist and equal each other.
The set of morphisms with start $x$ and target $y$ shall be denoted by $c(x, y)$.

Suppose given $a, b \in M$. Note that if $ab$ exists, we have $s(ab) = s(a)$ and $t(ab) = t(b)$. Note that $s(s(a)) = s(a)$, $t(t(a)) = t(a)$. Note that $ab$ exists iff $t(a) = s(b)$.

**Definition 1.4** A set $X$ is said to be $\mathfrak{U}$-small if there exists a bijection from $X$ to an element of $\mathfrak{U}$. The category of $\mathfrak{U}$-small sets is denoted by $(\text{Set}_\mathfrak{U})$. Note that by a skeleton argument, $(\text{Set}_\mathfrak{U})$ is equivalent to the category of sets contained as elements in $\mathfrak{U}$, denoted by $(\text{Set}_0^\mathfrak{U})$.

A small category $C$ is said to be $\mathfrak{U}$-small if $\text{Mor} C$ is $\mathfrak{U}$-small. A category $C$ is said to be essentially $\mathfrak{U}$-small if it is equivalent to a $\mathfrak{U}$-small category, or, equivalently, if it has a $\mathfrak{U}$-small skeleton.

**Remark 1.5** The category $(\text{Set}_0^\mathfrak{U})$ is $\mathfrak{V}$-small since $\coprod_{(x,y)\in \mathfrak{U}\times \mathfrak{U}} (\text{Set}_0^\mathfrak{U})(x,y)$ is an element of $\mathfrak{V}$. Hence the category $(\text{Set}_\mathfrak{U})$ is essentially $\mathfrak{V}$-small.

Given categories $C$ and $D$, we denote the category of functors mapping from $C$ to $D$ by $\mathfrak{F}[C, D]$.

**Lemma 1.6** If $C$ and $D$ are $\mathfrak{U}$-small, then $\mathfrak{F}[C, D]$ is $\mathfrak{U}$-small. If $C$ and $D$ are essentially $\mathfrak{U}$-small, then $\mathfrak{F}[C, D]$ is essentially $\mathfrak{U}$-small.

Using induced equivalences, it suffices to prove the first assertion. But then

$$\text{Ob } \mathfrak{F}[C, D] \subseteq \mathfrak{P}(\text{Mor } C \times \text{Mor } D)$$

is $\mathfrak{U}$-small. Moreover, given $f, g \in \mathfrak{F}[C, D]$, the set

$$\mathfrak{F}[C, D](f, g) \subseteq \coprod_{x \in \text{Ob } C} \mathfrak{P}(fx, gx)$$

is $\mathfrak{U}$-small. Hence Mor $\mathfrak{F}[C, D]$ is $\mathfrak{U}$-small.

**Lemma 1.7** Given an essentially $\mathfrak{U}$-small category $C$, the category

$$C^{\wedge \mathfrak{U}} := \mathfrak{F}[C^\circ, (\text{Set}_\mathfrak{U})]$$

of $(\text{Set}_\mathfrak{U})$-valued presheaves over $C$ is essentially $\mathfrak{V}$-small. Sometimes we write $C^{\wedge} = C^{\wedge \mathfrak{U}}$ if the universe is unambiguous. Likewise, the category

$$C^{\vee \mathfrak{U}} := \mathfrak{F}[C, (\text{Set}_\mathfrak{U})]$$

of $(\text{Set}_\mathfrak{U})$-valued copresheaves over $C$ is essentially $\mathfrak{V}$-small. Sometimes we write $C^{\vee} = C^{\vee \mathfrak{U}}$.

This follows from (1.5, 1.6).
Definition 1.8 Given an essentially $\mathcal{U}$-small category $C$, we have the Yoneda embedding

$$C \overset{y_C}{\rightarrow} C^{\land_\mathcal{U}}$$

$$x \rightarrow c(-, x)$$

and the co-Yoneda embedding

$$C \overset{y'_C}{\rightarrow} C^{\lor_\mathcal{U}}$$

$$x \rightarrow c(x, -).$$

For a functor $C \overset{f}{\rightarrow} D$, we denote

$$C^{\land_\mathcal{U}} \overset{f^\land}{\leftarrow} D^{\land_\mathcal{U}}$$

$$(u \circ f^o \overset{\beta}{\rightarrow} u' \circ f^o) \leftarrow (u \overset{\beta}{\rightarrow} u').$$

Given functors $C \overset{f}{\rightarrow} D$ and a natural transformation $f \overset{\alpha}{\rightarrow} g$, we denote by $f^\land \overset{\alpha^\land}{\leftarrow} g^\land$ the natural transformation that is given at $u \in D^{\land_\mathcal{U}}$ by the morphism $u \circ f^o \overset{u \circ \alpha \circ u}{\rightarrow} u \circ g^o$ in $C^{\land_\mathcal{U}}$, that evaluated at $c \in C^o$ in turn yields $ufc \overset{u \circ \alpha \circ u}{\rightarrow} ugc$.

Analogously for $C^{\lor_\mathcal{U}} \overset{f^\lor}{\leftarrow} D^{\lor_\mathcal{U}}$ and $f^\lor \overset{\alpha^\lor}{\rightarrow} g^\lor$.

2 The right Kan extension

Proposition 2.1 (the relative Yoneda Lemma) Given essentially $\mathcal{U}$-small categories $C$ and $D$, and a functor $C \overset{f}{\rightarrow} D$. Then the right adjoint $C^{\land_\mathcal{U}} \overset{\varphi}{\rightarrow} D^{\land_\mathcal{U}}$ of $C^{\land_\mathcal{U}} \overset{f^\land}{\leftarrow} D^{\land_\mathcal{U}}$ is given by

$$C^{\land_\mathcal{U}} \overset{f^\land}{\rightarrow} D^{\land_\mathcal{U}}$$

Keeping the name of the functor after restricting the image to $D^{\land_\mathcal{U}}$, we write shorthand

$$f^\land \overset{y_D \circ f^\land \circ y_C^\land}{\rightarrow}.$$

The unit of this adjunction

$$1_{D^{\land_\mathcal{U}}} \overset{\varepsilon}{\rightarrow} \varphi \circ f^\land$$
at \( u \in D^{\wedge u} \), i.e.

\[
u \xrightarrow{\varepsilon u} C^\wedge(f^{\wedge, o} \circ y_D^o(-), u \circ f^o),\]

applied to \( d \in D \), is given by

\[
u d \xrightarrow{\varepsilon ud} C^\wedge(D(f^o(-), d), u \circ f^o)
\]

\[
x \xrightarrow{(x)\varepsilon ud} (x)\varepsilon ud,
\]

where the natural transformation \((x)\varepsilon ud\) sends at \( c \in C\)

\[
d(f c, d) \xrightarrow{(x)\varepsilon ud} u f c
\]

\[
a \xrightarrow{(a)\left[\left((x)\varepsilon ud\right)c\right]} (x)(u a^o).
\]

The counit of this adjunction

\[
\begin{array}{c}
f^\wedge \circ \varphi \\
\eta \xrightarrow{} 1_C^\wedge
\end{array}
\]

at \( v \in C^\wedge \), i.e.

\[
c^\wedge(f^{\wedge, o} \circ y_D^o \circ f^o(-), v) \xrightarrow{\eta v} v,
\]

applied to \( c \in C \), is given by

\[
c^\wedge(D(f(-), f c), v) \xrightarrow{\eta vc} vc
\]

\[
\xi \xrightarrow{} (\xi)\eta vc = (1_{f c})\xi c
\]

Since the set

\[
c^\wedge(D(f(-), d), v) \subseteq \prod_{c \in \text{Ob} C} (D(f c, d), vc)
\]

is \( \mathfrak{U} \)-small, \( \varphi \) exists as the factorization of \( y_D^o \circ f^{\wedge} \circ y_{C^\wedge} \) over the inclusion \( D^{\wedge u} \subseteq D^{\wedge u} \).

Various compatibilities need to be verified to ensure the well-definedness of \( \varepsilon \) and \( \eta \).

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\[1\] Given \( u \in D^{\wedge u} \), we need to see that \( u \xrightarrow{\varepsilon u} \varphi \circ f^{\wedge}(u) \) is a natural transformation. Suppose given \( d' \xrightarrow{a'} d \) in \( D \). We have to show that for any \( x \in ud \)

\[
(x)(\varepsilon ud)(\varepsilon f^{\wedge}) = (x)(u a^o)(\varepsilon ud')
\]

is an equality of natural transformations from \( D(f^o(-), d') \) to \( u \circ f^o \). At \( c \in C \), the element \( f c \xrightarrow{a'} d' \) is mapped by the left hand side to \( (x)(u a^o)(\varepsilon ud') \), and by the right hand side to \( (x)(u a^o)(\varepsilon ud') \).

We need to see that \( 1_{D^{\wedge u}} \xrightarrow{\varepsilon} \varphi \circ f^{\wedge} \) is a natural transformation. Suppose given \( u \xrightarrow{s} u' \) in \( D^{\wedge u} \). We have to show that for any \( d \in D \) and any \( x \in ud \)

\[
(x)(s d)(\varepsilon u' d) = (x)(\varepsilon ud)(\varepsilon f^{\wedge}(s))d
\]

is an equality of natural transformations from \( D(f^o(-), d) \) to \( u' \circ f \). At \( c \in C \), the element \( f c \xrightarrow{a} d \) is mapped by the left hand side to \( (x)(s d)(u' a^o) \), and by the right hand side to \( (x)(u a^o)(s f c) \).
We have to show that \((f^\wedge \circ \varepsilon)(\eta \circ f^\wedge) = 1_{f^\wedge}\). Suppose given \(u \in D^\wedge\), \(c \in C\) and \(x \in (f^\wedge uc) = u(fc)\). We obtain
\[
(x)((f^\wedge \circ \varepsilon)uc)((\eta \circ f^\wedge)uc) = (x)(\varepsilon u(fc))(\eta(u \circ f^\circ)c) = (1_{fc})[((x)(\varepsilon u(fc)))c] = (x)(u1^\circ_{fc}) = x.
\]

We have to show that \((\varepsilon \circ \varphi)(\varphi \circ \eta) = 1_{\varphi}\). Suppose given \(v \in C^\wedge u\) and \(d \in D\). Note that \(\varphi v = C^\wedge(f^\wedge \circ y^\circ_D (-), v)\) and therefore \(\varphi vd = C^\wedge(d(f^\circ(-), d), v)\). The application \((\varepsilon \circ \varphi)vd\) writes
\[
C(d(f^\circ(-), d), v) \xrightarrow{\varepsilon \circ \varphi vd} C^\wedge(d(f^\circ(-), d), C^\wedge(f^\wedge \circ y^\circ_D \circ f^\circ(-), v))(\eta c) = 1\]
An element \(\xi \in \varphi vd\), i.e. \(d(f^\circ(-), d) \xrightarrow{\xi} v\), is thus mapped to the composite
\[
d(f^\circ(-), d) \xrightarrow{(\xi \circ \varphi)vd} C^\wedge(f^\wedge \circ y^\circ_D \circ f^\circ(-), v) \xrightarrow{\eta c} v.
\]
Now suppose given \(c \in C\) and \(fc \xrightarrow{a} d\). We have
\[
d(fc, d) \xrightarrow{(\xi \circ \varphi)vd} C^\wedge(d(f^\circ(-), fc), v) \xrightarrow{\eta c} vc
\]
whence \(\xi((\varepsilon \circ \varphi)vd)((\varphi \circ \eta)vd) = \xi\).

**Remark 2.2 (the absolute Yoneda Lemma)** In case \(f = 1_C\), we obtain \(1 \Rightarrow y^\circ_C \circ y_C^\wedge\), which by uniqueness of the right adjoint yields the comparison isomorphism
\[
C^\wedge(-, c), v) \xrightarrow{\eta c} vc
\]
\[
\xi \xrightarrow{(1_c)\xi c},
\]
Given \(v \in C^\wedge u\), we need to see that \(f^\circ \circ \varphi(v) \xrightarrow{\eta c} v\) is a natural transformation. Suppose given \(c' \xrightarrow{b} c\) in \(C\). We have to show that for any \(\xi \in (f^\wedge \circ \varphi(v))c = C^\wedge(d(f^\circ(-), fc), v)\) we have
\[
(x)((\varepsilon \circ \varphi)(\eta c)) = (x)((f^\wedge \circ \varphi(v))b)(\eta c').
\]
The left hand side yields \((1fc)(\xi)(v b)\), the right hand side yields \((fb)\xi c'\).

We need to see that \(f^\wedge \circ \varphi \xrightarrow{\eta c} 1_{C^\wedge u}\) is a natural transformation. Suppose given \(v \xrightarrow{b} v'\) in \(C^\wedge u\). We have to show that for any \(c \in C\) and any \(\xi \in (f^\wedge \circ \varphi(v))c = C^\wedge(d(f^\circ(-), fc), v)\) we have
\[
(\xi)(\eta c))(tc) = (\xi)((f^\wedge \circ \varphi(t))c)(\eta c').
\]
The left hand side yields \((1fc)(\xi)(tc)\). The right hand side yields \((1fc)((\xi t)c)\).
at \( v \in C^{\wedge U} \) and \( c \in C \), with inverse given by \( \varepsilon vc \).

**Corollary 2.3** If \( f \) is full, then the counit \( \eta \) of the adjunction \( f^{\wedge} \dashv y_D^{\wedge} \circ f^{\wedge} \circ y_C^{\wedge} \) is a monomorphism. If \( f \) is full and faithful, then \( \eta \) is an isomorphism.

### 3 The left Kan extension

For sake of comparison to (2.1), we rephrase the pertinent case of Kan’s formula in our setting.

Let \( C \) be a \( \U \)-small category. Let \( v \in C^{\wedge U} \), let \( w \in C^{\vee U} \). We define the set \( v \otimes_C w \) as the quotient of the disjoint union

\[
v \times_C w := \bigsqcup_{c \in C} vc \times wc
\]

modulo the equivalence relation generated by the following relation \( \sim_C \). The equivalence class of \( (p, q) \in vc \times wc \), \( c \in C \), shall be denoted by \( p \otimes q \).

Given \( (p, q) \in vc \times wc \), \( (p', q') \in vc' \times wc' \), we say that \( (p, q) \sim_C (p', q') \) if there exists a morphism \( c \xrightarrow{a} c' \) such that

\[
(p')va^o = p
\]
\[
(q)wa = q'.
\]

Thus the quotient map \( v \times_C w \xrightarrow{\nu} v \otimes_C w \) has the following universal property. Given a map \( v \times_C w \xrightarrow{\nu'} X \) such that for any morphism \( c \xrightarrow{a} c' \), any \( p' \in vc' \) and any \( q \in wc \) we have

\[
((p')va^o, q)\nu' = (p', (q)wa)\nu',
\]

there exists a unique map \( v \otimes_C w \xrightarrow{\nu''} X \) such that \( \nu' = \nu \nu'' \).

In particular, given morphisms \( v \xrightarrow{m} v' \) and \( w \xrightarrow{n} w' \), we obtain a map \( m \otimes_C n \) that maps an element represented by \( (p, q) \in vc \times wc \), \( c \in C \), as follows.

\[
\begin{array}{ccc}
v \otimes_C w & m \otimes_C n & v' \otimes_C w' \\
p \otimes q & (p)mc & (q)nc
\end{array}
\]

Thus the tensor product defines a functor \( C^{\wedge U} \times C^{\vee U} \xrightarrow{=} \otimes_C (\text{Set}_U) \). We denote the univalent tensor product functor by

\[
C^{\wedge U} \xrightarrow{=} \otimes_C (\text{Set}_U).
\]

**Proposition 3.1** (the relative co-Yoneda Lemma, Kan [K 58, Th. 14.1]) Given \( \U \)-small categories \( C \) and \( D \), and a functor \( C \xrightarrow{f} D \). The left adjoint \( C^{\wedge U} \xrightarrow{\psi} D^{\wedge U} \) of \( C^{\wedge U} \xrightarrow{f^*} D^{\wedge U} \) is given by
\[ C^\wedge u \xrightarrow{f^\wedge} D^\wedge u \]

\[
\begin{array}{c}
\uparrow z_C \\
\downarrow y' D^\wedge
\end{array}
\]

\[ C^\wedge u \xrightarrow{\psi} (D^o)^\wedge u = D^\wedge u \]

For short,
\[ y'_D \circ f^\wedge \circ z_C \vdash f^\wedge. \]

The unit of this adjunction
\[
\begin{array}{c}
1_{C^\wedge} \xrightarrow{\varepsilon} f^\wedge \circ \psi
\end{array}
\]

at \( v \in C^\wedge \), i.e.
\[ v \xrightarrow{\varepsilon v} v \otimes_C f^\wedge \circ y'_D \circ f^o(-), \]

applied to \( c \in C \), is given by
\[ vc \xrightarrow{\varepsilon vc} v \otimes_C D(f c, f(-)) \]

\[ x \xrightarrow{} x \otimes 1_{f c}. \]

The counit of this adjunction
\[
\begin{array}{c}
\psi \circ f^\wedge \xrightarrow{\eta} 1_{D^\wedge}
\end{array}
\]

at \( u \in D^\wedge \), i.e.
\[ u \circ f^o \otimes_C f^\wedge \circ y'_D(-) \xrightarrow{\eta u} u, \]

applied to \( d \in D \), is given by
\[ u \circ f^o \otimes_C D(d, f(-)) \xrightarrow{\eta ud} ud \]

\[ p \otimes q \xrightarrow{} (p)u q^o, \]

where \( p \otimes q \) is represented by \((p, q) \in u f c \times D(d, f c)\) for some \( c \in C \).

Various compatibilities have to be verified to ensure the well-definedness of \( \varepsilon \) and \( \eta \). \(^2\)

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\(^2\) Given \( v \in C^\wedge \), we need to see that \( v \xrightarrow{\varepsilon v} f^\wedge \circ \psi(v) \) is a natural transformation. Suppose given \( c' \xrightarrow{b} c \). We have to show that for any \( x \in vc \)

\[ (x) (\varepsilon vc) \left((f^\wedge \circ \psi(v)) b^o\right) = (x) (v b^o) \left(\varepsilon vc'\right). \]

The left hand side yields \( x \otimes f b \). The right hand side yields \( x(v b^o) \otimes 1_{f c'}. \)

We need to see that \( 1_{C^\wedge} \xrightarrow{\varepsilon} f^\wedge \circ \psi \) is a natural transformation. Suppose given \( v \xrightarrow{\varepsilon} v' \) in \( C^\wedge \). For any \( c \in C \) and any \( x \in vc \) we have

\[ (x) \left(tc\right) \left(\varepsilon vc'\right) = (x) \left(\varepsilon vc\right) \left(tc \otimes D(f c, f(-))\right) = (x)tc \otimes 1_{f c}. \]

Given \( u \in D^\wedge \) and \( d \in D \), we need to see that \( \eta ud \) is a well-defined map. Suppose given \( c, c' \in C \),
We have to show that \((\varepsilon \circ f^\wedge)(f^\wedge \circ \eta) = 1_{f^\wedge}\). Suppose given \(u \in D^\wedge, c \in C\) and \(x \in (f^\wedge u)c\). We obtain
\[
(x)((\varepsilon \circ f^\wedge)uc)((f^\wedge \circ \eta)uc) = (x)(\varepsilon(u \circ f^\wedge)c)(\eta(u(fc)) = (x)(1_{f^\wedge}c)(\eta(u(fc)) = (x)(u\eta(u(fc))) = x.
\]
We have to show that \((\psi \circ \varepsilon)(\eta \circ \psi) = 1_\psi\). Suppose given \(v \in C^\wedge, d \in D, c \in C, s \in vc, t \in \rho(d, fc)\), so that \(s \otimes t \in \psi vd = v \otimes C \, \rho(d, f(-))\). We obtain
\[
(s \otimes t)((\psi \circ \varepsilon)vd)((\eta \circ \psi)vd) = (s \otimes t)(\varepsilon v \otimes \rho(d, f(-)))(\eta(v \otimes C \, f^\wedge \circ \eta_D)d) = ((s \otimes 1_{f^\wedge}) \otimes t)(\eta(v \otimes C \, f^\wedge \circ \eta_D)d) = (s \otimes 1_{f^\wedge})(v \otimes C \, \rho(t, f(-))) = (s \otimes t).
\]

**Remark 3.2 (the absolute co-Yoneda Lemma)** In case \(f = 1_C\), we obtain \(y_D^{\wedge} \circ z_C \dashv 1_{C^\wedge}\), which by uniqueness of the left adjoint yields the comparison isomorphism
\[
v \otimes C \, C(c, -) \xrightarrow{\eta_{vc}} vc
\]
at \(v \in C^\wedge \) and \(c \in C\), with inverse given by \(\varepsilon vc\).

**Corollary 3.3** If \(f\) is full, then the unit \(\varepsilon\) of the adjunction \(y_D^{\wedge} \circ f^{\wedge} \circ z_C \dashv f^\wedge\) is an epimorphism. If \(f\) is full and faithful, then \(\varepsilon\) is an isomorphism.
4 References

Bourbaki, N.

Kan, D. M.

Mac Lane, S.

Mitchell, B.