Plancherel

L : field

G : finite group acting on L

$K := \operatorname{Fix}_G L$

N : kernel of action

$H := G/N$

$$\begin{array}{c} L \\ \Big| \, H \\ K \end{array}$$

Remark : $L \wr G$ semisimple $\iff$ $|N|$ is invertible in K.

Assume from now on $|N|$ to be invertible in K.

X_i : simple $L \wr G$-modules, $i \in [1, k]$

$K_i := \operatorname{End}_{L \wr G} X_i$

$x_i := \dim_{K_i} X_i$, $d_i := [K_i : \mathrm{Z}(K_i)]^{1/2}$, $r_i := [K_i : K]$

Remark : $x_i d_i / |H| \in \mathbf{Z}$

Wedderburn isomorphism

$$\begin{array}{ccc} L \wr G & \overset{\omega}{\underset{\sim}{\longrightarrow}} & \prod_{i \in [1,k]} \operatorname{End}_{K_i} X_i \\ \xi & \longmapsto & (\xi \omega_i)_i \end{array}$$

u : chosen element of L such that $\operatorname{Tr}_{L|K}(u) = 1$

Bilinear form on $L \wr G$:

$$(gx, g'x') \;:=\; \partial_{g, g'^{-1}} \operatorname{Tr}_{L|K}(x^{g'} x') \;=\; \frac{1}{|N|} \operatorname{Tr}_K \left(u(-) g x g' x' \right) \; ,$$

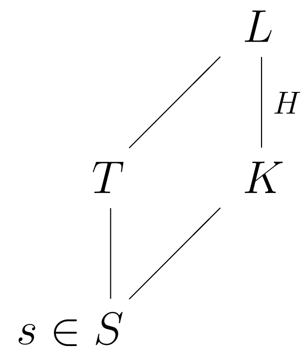
where g, $g' \in G$, x, $x' \in L$.

Plancherel formula : For ξ, $\eta \in L \wr G$, we have

$$(\xi, \eta) \;=\; \sum_{i \in [1,k]} \frac{x_i d_i}{|G|} \underbrace{\operatorname{tr}_{K_i|K}}_{\text{red. tr.}} \underbrace{\operatorname{Tr}_{K_i} \big((\xi\eta) \omega_i \big)}_{\text{“matrix trace”}}$$

$\rightsquigarrow$ explicit inverse to ω

Colength formula



$S_i \subseteq K_i$: maximal S-order

$V_i \subseteq X_i$: suitable $T \wr G$-lattice

ℓ : Jordan-Hölder length of an S-module

Discriminant valuations :

$S_i^+ \; := \; \{y \in K_i \; : \; \mathrm{tr}_K(yS_i) \subseteq S\}\;,\;\; \delta_{S_i|S} := \ell(S_i^+/S_i)$

$T^+ \; := \; \{y \in L \; : \; \mathrm{Tr}_{L|K}(yT) \subseteq S\}\;,\;\; \delta_{T|S} := \ell(T^+/T)$

Now Plancherel $\rightsquigarrow$

Remark : The colength of the Wedderburn embedding

$$\begin{array}{rcl} T \wr G & \stackrel{\omega}{\hookrightarrow} & \prod_{i\in[1,k]} \mathrm{End}_{S_i} V_i \\ \xi & \longmapsto & (\xi\omega_i)_i \end{array}$$

is given by

$$\ell(\mathrm{Cokern}\,\omega) \;=\; \frac{1}{2}\left(|G|\left(\delta_{T|S} + \mathrm{v}_s(|N|)|H|\right) - \sum_{i\in[1,k]} x_i^2 \left(\delta_{S_i|S} + \mathrm{v}_s\left(\frac{x_i d_i}{|H|}\right) r_i\right)\right).$$

Open questions :

- Length of cokernel of $T \wr G \xrightarrow{\omega_i} \mathrm{End}_{S_i} V_i$?
  (quasiblock colength, where quasiblock := $\mathrm{Im}(\omega_i)$)
- Criterion for a quasiblock of an untwisted group ring to be a twisted group ring ? (examples in thesis of H. Weber)

Example

$S := \mathbf{Z}_{(3)}$, $s := 3$, $t := (\zeta_9 - 1)(\zeta_9^{-1} - 1)$, $T := S[t]$, $a = 2$

$\rho :$ induced by $\zeta_9 \longmapsto \zeta_9^4$, restricted from $\mathbf{Q}(\zeta_9)$ to T.

$G := \langle \rho \rangle \simeq \mathcal{C}_3$

Then $\dot{\rho} = \begin{pmatrix} 1 & 0 & 0 \\ 6 & -5 & 1 \\ 24 & -21 & 4 \end{pmatrix}$ (difficult to control), $\dot{t} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -9 & 6 \end{pmatrix}$, $\ddot{t} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix}$.

So an isomorphic copy of $T \wr G$ is given by its Wedderburn image

$$\left\{ \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ 3a_{1,2} & a_{1,0} & a_{1,1} \\ 3a_{2,1} & 3a_{2,2} & a_{2,0} \end{pmatrix} \in \mathbf{Z}_{(3)}^{3\times 3} \; : \; \underbrace{a_{0,1} \equiv_3 a_{1,1} \equiv_3 a_{2,1}}_{\text{resulting from } (t^\rho - t)^1}, \; \underbrace{a_{0,2} + a_{1,2} + a_{2,2} \equiv_3 0}_{\text{resulting from } (t^\rho - t)^2} \right\} .$$

This can be used to calculate

$$\mathrm{H}^*(G, T; S) \simeq \mathbf{Z}_{(3)}[h^{(1)}, h^{(2)}]/(3h^{(1)}, 3h^{(2)}, {h^{(1)}}^2) ,$$

where $\deg h^{(i)} = i$.