Does “Quillen A with an extra direction” hold?

1. Some notation

A bicategory $X$ is a bisimplicial set such that the simplicial sets $X_{i,*}$ and $X_{*,j}$ are (nerves of) categories for all $i, j \geq 0$. A bifunctor between bicategories is simply a bisimplicial map.

We write maps on the right.

The face operators in a bisimplicial set are denoted by $d^{(1)}_i$ in the first, and by $d^{(2)}_j$ in the second direction. Analogously in a trisimplicial set.

Write $d^{(2)}_{[m+n+1,m+1]} := d^{(2)}_{m+n+1} \cdots d^{(2)}_{m+1}$. Etc.

If $X$ is a bisimplicial set, denote by $X_{\text{const}_2}$ the trisimplicial set that has $(X_{\text{const}_2})_{i,j,k} = X_{i,k}$. Etc.

A bisimplicial map is called a weak homotopy equivalence if its diagonalisation is a weak homotopy equivalence. Likewise for a trisimplicial map.

2. The question on "Quillen A with an extra direction"

Let $X$ and $Y$ be bicategories, and let $X \xrightarrow{f} Y$ be a bifunctor.

Form the trisimplicial set $T_f$ that has

$$(T_f)_{s,m,n} = \{(x, y) \in X_{s,m} \times Y_{s,m+n+1} : xf = yd^{(2)}_{[m+n+1,m+1]}\}$$

for $s, m, n \geq 0$. To sketch a schematic picture,

The trisimplicial operation in the first (aka $s$-) direction on $(x, y)$ is the operation on $x$ and on $y$ in the first direction.

The trisimplicial operation in the second (aka $m$-) direction on $(x, y)$ is the operation on $x$ in the second direction and the operation on $y$ in the "front part" in the second direction.

So e.g. if $(x, y) \in (T_f)_{s,m,n}$, then $(x, y)d^{(2)}_i = (xd^{(2)}_i, yd^{(2)}_i)$.

The trisimplicial operation in the third (aka $n$-) direction on $(x, y)$ is the identical operation on $x$ and the operation on $y$ in the "back part" in the second direction. So e.g. if $(x, y) \in (T_f)_{s,m,n}$, then $(x, y)d^{(3)}_i = (x, yd^{(2)}_{m+1+i})$. 

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We have trisimplicial “projection” maps

\[
\begin{align*}
T_f & \xrightarrow{p_1.f} X \text{ const}_3 \\
(x, y) & \mapsto x \\
T_f & \xrightarrow{p_2.f} Y \text{ const}_2 \\
(x, y) & \mapsto yd^{(2)}_{[m, 0]} 
\end{align*}
\]

For a schematic picture of \( p_{2, f} \), cf. §3 (upper row of picture).

**Remark.** Imitating Quillen’s proof of Theorem A, it is not difficult to show that \( f \) is a weak homotopy equivalence if and only if \( p_{2, f} \) is a weak homotopy equivalence.

Consider the bisimplicial map \( p_{2, f}|_{n=0} \) that is given at \((s, m)\) by \((T_f)_{s, m, 0} \xrightarrow{(p_{2, f})_{s, m, 0}} Y_{s, 0} \). For a schematic picture of \( p_{2, f}|_{n=0} \), cf. §3 (lower row of picture).

**Question.** If \( p_{2, f}|_{n=0} \) is a weak homotopy equivalence, is then \( p_{2, f} \) (and thus \( f \)) a weak homotopy equivalence?

**Remark.** If \( X \) and \( Y \) are constant in the first (aka \( s \)-) direction (that is, if this simplicial direction is “not there”), then the answer is affirmative by Quillen A. In fact, \((T_f)_{0, s, 0} \) is the disjoint union of the over-categories \( f_{0, s}/y \), indexed by \( y \in Y_{0, 0} \), and \( p_{2, f}|_{n=0} \) maps an element in that disjoint union just to its indexing element.

**Remark.** If the simplicial subset \( p_{2, f}^{-1}(\tilde{y}) \) of \((T_f)_{s, s, 0} \) is weakly contractible for all \( s \geq 0 \) and all \( \tilde{y} \in Y_{s, 0} \), then \( p_{2, f} \) and \( p_{2, f}|_{n=0} \) are both weak homotopy equivalences. In fact, in this case it follows that \( p_{2, f}^{-1}(\tilde{y}) \) is weakly contractible for all \( s, n \geq 0 \) and all \( \tilde{y} \in Y_{s, n} \), for \( p_{2, f}^{-1}(\tilde{y}) \simeq p_{2, f}^{-1}(\tilde{y}d^{(2)}_{[n, 1]}) \) (isomorphism of simplicial sets).

### 3. A question for a homotopy pullback

Fix \( n \geq 0 \). Consider the following commutative quadrangle of bisimplicial sets.

\[
\begin{array}{ccc}
(T_f)_{s, \tilde{s}, n} & \xrightarrow{(p_{2, f})_{s, \tilde{s}, n}} & Y_{s, n} \\
\downarrow d^{(3)}_{[n, 1]} & & \downarrow d^{(2)}_{[n, 1]} \\
(T_f)_{s, \tilde{s}, 0} & \xrightarrow{(p_{2, f})_{s, \tilde{s}, 0}} & Y_{s, 0}
\end{array}
\]

\((\ast)\)
To sketch a schematic picture,

\[ \begin{array}{c}
\begin{array}{c}
\text{m} \\
m\text{x}
\end{array}
\end{array},
\begin{array}{c}
\text{m} \\
m\text{xf}
\end{array}
\end{array} \xrightarrow{y} \begin{array}{c}
\text{n}
\end{array}
\]

\[ (p_{2,f})_{s,m,n} \]

\[ \begin{array}{c}
\text{d}^{(2)}_{[m,0]} \xrightarrow{y}
\end{array} \]

\[ \begin{array}{c}
\text{d}^{(2)}_{[n,1]}
\end{array} \]

\[ \begin{array}{c}
\text{d}^{(3)}_{[n,1]}
\end{array} \]

Question. Is \((\ast)\) a homotopy pullback?

Remark. If the answer to this question is affirmative, so is the answer to the question in §2.

Remark. If \(X\) and \(Y\) are constant in the first direction, then, as far as I can see, this is true.

Speculation. Is there a categorical model for the homotopy pullback (of categories, to begin with; then of bicategories)? With objects like in the comma category, only with an eventually bothsided constant zigzag instead of simply a morphism? Such a model could then be used to compare – one would need to get rid of the zigzag again somehow.