A ONE-BOX-SHIFT MORPHISM BETWEEN SPECHT MODULES

MATTHIAS KÜNZER

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Abstract. We give a formula for a morphism between Specht modules over \((\mathbb{Z}/m)S_n\), where \(n \geq 1\), and where the partition indexing the target Specht module arises from that indexing the source Specht module by a downwards shift of one box, \(m\) being the box shift length. Our morphism can be reinterpreted integrally as an extension of order \(m\) of the corresponding Specht lattices.

0. Notation

We write composition of maps on the right, \(\alpha \circ \beta = \alpha \beta\). Intervals are to be read as subsets of \(\mathbb{Z}\). Let \(n \geq 1\), let \(S_n = \text{Aut}_{\text{Sets}}[1,n]\) denote the symmetric group on \(n\) letters and let \(\varepsilon_\sigma\) denote the sign of a permutation \(\sigma \in S_n\). Let

\[
N \xrightarrow{\lambda} N_0
\]

be a partition of \(n\), i.e. assume \(\sum_i \lambda_i = n\) and \(\lambda_i \geq \lambda_i + 1\) for \(i \in \mathbb{N}\). Let \([\lambda] := \{i \times j \in \mathbb{N} \times \mathbb{N} \mid j \leq \lambda_i\}\) denote the diagram of \(\lambda\). We say that \(i \times j \in [\lambda]\) lies in row \(i\) and in column \(j\). A \(\lambda\)-tableau is a bijection

\[
[\lambda] \xrightarrow{[a]} [1,n] \quad i \times j \xrightarrow{a_{i,j}}
\]

The element \(\sigma \in S_n\) acts on the set \(T^\lambda\) of \(\lambda\)-tableaux via composition \(\sigma \circ [a] = [\sigma][a]\). Let \(F^\lambda\) be the free \(\mathbb{Z}\)-module on \(T^\lambda\) with the induced operation of \(S_n\). Let

\[
\lambda_i \xrightarrow{\rho} N \quad i \xrightarrow{\kappa} N
\]

denote the projections. We denote by \(\{a\} := [a]^{-1}\rho\) the \(\lambda\)-tableloid associated to the \(\lambda\)-tableaux \([a]\). The free \(\mathbb{Z}\)-module on the set of tabloids, equipped with the inherited \(S_n\)-operation, is denoted by \(M^\lambda\). Let

\[
C_{\{a\}} := \{\sigma \in S_n \mid [a]^{-1}\kappa = ([a]\sigma)^{-1}\kappa\}
\]

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be the column stabilizer of \([a]\). Let the Specht lattice \(S^\lambda\) be the \(\mathbb{Z}S_n\)-sublattice of \(M^\lambda\) generated over \(\mathbb{Z}\) by the \(\lambda\)-polytabloids
\[
\langle a \rangle := \sum_{\sigma \in C[a]} \{a\} \sigma \in \mathbb{Z}.
\]
Let \(\lambda'\) denote the transposed partition of \(\lambda\), i.e. \(j \leq \lambda_i \iff i \times j \in [\lambda] \iff i \leq \lambda'_j\).

1. Carter-Payne

Let \(d \in [1, n]\) be the number of shifted boxes. Let \(1 \leq s < t \leq n\), \(s\) being the row of \([\lambda]\) from which the boxes are shifted, and \(t\) being the row into which the boxes are shifted. Suppose
\[
\mu_i := \begin{cases} 
\lambda_i - d & \text{for } i = s, \\
\lambda_i + d & \text{for } i = t, \\
\lambda_i & \text{else}
\end{cases}
\]
defines a partition of \(n\). Let the box shift length be denoted by
\[
m := (\lambda_s - s) - (\lambda_t - t) - d.
\]
Let \(m[p] := p^{v_p(m)}\) be the \(p\)-part of \(m\). Using [1], Carter and Payne proved the following

**Theorem 1.1** ([2]). Let \(K\) be an infinite field of characteristic \(p\). Suppose \(d < m[p]\). Then
\[
\text{Hom}_{K \otimes \mathbb{Z}}(K \otimes \mathbb{Z} S^\lambda, K \otimes \mathbb{Z} S^\mu) \neq 0.
\]

2. Integral reinterpretation

Assume \(d = 1\), i.e. \([\mu]\) arises from \([\lambda]\) by a one-box-shift. The condition \(d < m[p]\) translates into \(p|m\).

As we will see below, this particular case of the result of Carter and Payne already holds over \(K = \mathbb{F}_p\). So we obtain a nonzero element in
\[
\text{Hom}_{\mathbb{Z}S_n}(S^\lambda/pS^\lambda, S^\mu/pS^\mu) \twoheadrightarrow \text{Hom}_{\mathbb{Z}S_n}(S^\lambda, S^\mu/pS^\mu).
\]
We consider a part of the long exact \(\text{Ext}^*_{\mathbb{Z}S_n}(S^\lambda, -)\)-sequence on
\[
0 \rightarrow S^\mu \xrightarrow{p} S^\mu \rightarrow S^\mu/pS^\mu \rightarrow 0,
\]
viz.
\[
0 \rightarrow \text{Hom}_{\mathbb{Z}S_n}(S^\lambda, S^\mu) \xrightarrow{p} \text{Hom}_{\mathbb{Z}S_n}(S^\lambda, S^\mu) \rightarrow \text{Hom}_{\mathbb{Z}S_n}(S^\lambda, S^\mu/pS^\mu) = 0
\]
\[
\rightarrow \text{Ext}^1_{\mathbb{Z}S_n}(S^\lambda, S^\mu) \xrightarrow{p} \text{Ext}^1_{\mathbb{Z}S_n}(S^\lambda, S^\mu) = 0.
\]
Mapping our morphism into \(\text{Ext}^1\), we obtain a nonzero element of \(\text{Ext}^1_{\mathbb{Z}S_n}(S^\lambda, S^\mu)\) which is annihilated by \(p\). Conversely, the \(p\)-torsion elements of \(\text{Ext}^1\) are given by morphisms modulo \(p\).

Since \(n!\) annihilates \(\text{Ext}^1_{\mathbb{Z}S_n}(S^\lambda, S^\mu)\), replacement of \(p\) by \(n!\) shows that any element in \(\text{Ext}^1\) is given by a modular morphism modulo \(n!\),
\[
\text{Hom}_{\mathbb{Z}S_n}(S^\lambda, S^\mu/n!S^\mu) \twoheadrightarrow \text{Ext}^1_{\mathbb{Z}S_n}(S^\lambda, S^\mu).
\]
Therefore, in order to get hold of the whole \(\text{Ext}^1\), we need to calculate modulo prime powers in general.
3. One-box-shift formula

We keep the assumption $d = 1$. Let $s' := \lambda_s$ and let $t' := \lambda_t + 1$. A path of length $l \in [1, s' - t']$ is a map

$$[0, l]_k \quad \gamma \quad [\lambda] \cup [\mu]_{\alpha_k \times \beta_k}$$

such that $k < k'$ implies $\beta_k < \beta_{k'}$, and such that $\alpha_0 \times \beta_0 = t \times t'$ and $\beta_l = s'$. For a $\lambda$-tableau $[a]$, we define the $\mu$-tableau $[\gamma a]$ by

$$a_{i,j}^\gamma := a_{i,j} \quad \text{for } i \times j \in [\mu] \setminus (\gamma([1, l]) \cup N \times \{s'\}),$$
$$a_{\alpha k, \beta k}^\gamma := a_{\alpha k+1, \beta k+1} \quad \text{for } k \in [0, l - 1],$$
$$a_{i, s'}^\gamma := a_{i, s'} \quad \text{for } i < \alpha_l,$$
$$a_{i, s'}^\gamma := a_{i+1, s'} \quad \text{for } i \geq \alpha_l.$$

For $i \in [t' + 1, s' - 1]$, we denote

$$X_i := (s' - \lambda_{s'}') - (i - \lambda_i').$$

Let

$$x_\gamma := (-1)^{\alpha_1} \prod_{i \in [t' + 1, s' - 1]} X_i \prod_{k \in [1, l - 1]} X_{\beta_k}.$$

Let $\Gamma$ be the set of paths of some length $l \in [1, s' - t']$.

**Theorem 3.1** ([4], 4.3.31, cf. 0.7.1). The abelian group $\text{Hom}_{\mathbb{Z}[S_n]}(S_\lambda, S_\mu/mS_\mu)$ contains an element $f$ of order $m = (\lambda_n - s) - (\lambda_t - t) - 1$ which is given by the commutative diagram of $\mathbb{Z}[S_n]$-linear maps

$$\begin{array}{ccc}
[a] & \longrightarrow & \sum_{\gamma \in \Gamma} x_\gamma \otimes (a^\gamma) \\
[a] & \longmapsto & F^\lambda \\
\downarrow & & \downarrow \\
\langle a \rangle & \longrightarrow & S^\lambda \\
\downarrow & & \downarrow \\
\langle b \rangle & \longrightarrow & S^\mu/mS^\mu \\
\downarrow & & \downarrow \\
1 \otimes \langle b \rangle & \longrightarrow & S^\mu \\
\downarrow & & \downarrow \\
\langle b \rangle & \longrightarrow & b + mS^\mu.
\end{array}$$

Reducing modulo a prime dividing $m$, this recovers the case $d = 1$ of the result of CARTER and PAYNE. By the long exact sequence as above, but with $p$ replaced by $m$, we obtain a nonzero element in $\text{Ext}_{\mathbb{Z}[S_n]}^1(S_\lambda, S_\mu)$ of order $m$.  

The proof of this theorem proceeds by showing that a sufficient set of Garnir relations in $F^\lambda$ is annihilated by $F^\lambda \longrightarrow S^\mu/mS^\mu$.

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1I do not know the structure of $\text{Ext}_{\mathbb{Z}[S_n]}^1(S_\lambda, S_\mu)$ as an abelian group. At least in case $n \leq 7$, direct computation yields that the projection of our element to its $2'$-part generates this $2'$-part. We have, however, for example $\text{Ext}_{\mathbb{Z}[S_n]}^1(S^{(4,1^2)}, S^{(3,1^3)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. 


4. Example

Let \( n = 9, \lambda = (4, 3, 2), \mu = (3, 3, 2, 1), t' = 1 \) and \( s' = 4 \), whence \( m = 6, X_2 = 4, X_3 = 2 \). We obtain a morphism of order 6 that maps

\[
S^{(4,3,2)} \xrightarrow{f} S^{(3,3,2,1)} / 6 S^{(3,3,2,1)}
\]

\[
\begin{pmatrix}
1 & 4 & 7 & 9 \\
2 & 5 & 8 \\
3 & 6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 7 & 9 \\
2 & 5 & 8 \\
3 & 6
\end{pmatrix}
+ \begin{pmatrix}
1 & 4 & 9 \\
2 & 7 & 8 \\
3 & 6
\end{pmatrix}
+ \begin{pmatrix}
1 & 4 & 9 \\
2 & 5 & 8 \\
3 & 7
\end{pmatrix}
+ \begin{pmatrix}
1 & 8 & 7 \\
2 & 5 & 9 \\
3 & 6
\end{pmatrix}
+ \begin{pmatrix}
1 & 4 & 7 \\
2 & 8 & 9 \\
3 & 6
\end{pmatrix}
+ \begin{pmatrix}
1 & 4 & 7 \\
2 & 5 & 9 \\
3 & 8
\end{pmatrix}
\]

\[
+ 4^{020} \left( \begin{pmatrix}
1 & 4 & 9 \\
2 & 5 & 8 \\
3 & 6
\end{pmatrix} + \begin{pmatrix}
2 & 5 & 9 \\
3 & 6 \\
8
\end{pmatrix} + \begin{pmatrix}
1 & 4 & 7 \\
2 & 5 & 9 \\
3 & 8
\end{pmatrix} \right)
\]

\[
+ 4^{120} \left( \begin{pmatrix}
1 & 4 & 9 \\
2 & 5 & 8 \\
3 & 6
\end{pmatrix} + \begin{pmatrix}
2 & 5 & 9 \\
3 & 6 \\
8
\end{pmatrix} + \begin{pmatrix}
1 & 4 & 7 \\
2 & 9 & 8 \\
3 & 6
\end{pmatrix}
\right)
\]

\[
+ 4^{021} \left( \begin{pmatrix}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6
\end{pmatrix} + \begin{pmatrix}
2 & 9 & 8 \\
3 & 6 \\
9
\end{pmatrix} + \begin{pmatrix}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 9
\end{pmatrix}
\right)
\]

\[
+ 4^{121} \left( \begin{pmatrix}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6
\end{pmatrix} \right)
\]

The \([0, l - 1]\)-part of the respective path is highlighted.

5. Motivation

We consider the rational Wedderburn isomorphism

\[
Q S_n \xrightarrow{\sigma} \prod_\lambda (Q)_{n_\lambda \times n_\lambda}
\]

where \( \lambda \) runs over the partitions of \( n \) and where \( \rho^\lambda_\sigma \) denotes the matrix describing the operation of \( \sigma \in S_n \) on \( S^\lambda \) with respect to a chosen tuple of integral bases. The restriction

\[
Z S_n \xrightarrow{e^\lambda} \prod_\lambda (Z)_{n_\lambda \times n_\lambda}
\]

of this isomorphism, viewed as an embedding of abelian groups, has index \(^2\)

\[
\Pi_\lambda \left( \frac{n!}{n_\lambda} \right)^{n_\lambda/2}.
\]

\(^2\)Question. Given a central primitive idempotent \( e^\lambda \) of \( \Gamma := \prod_\lambda (Z)_{n_\lambda \times n_\lambda} \), what is the index of \( e^\lambda Z S_n \) in \( e^\lambda \Gamma \)? Cf. ([4], Section 1.1.3).
In particular, for \( n \geq 2 \) it is no longer an isomorphism.

Suppose, for partitions \( \lambda \) and \( \mu \) of \( n \) and for some modulus \( m \geq 2 \), we are given a \( \mathbb{Z}S_n \)-linear map

\[
S^\lambda \to S^\mu/mS^\mu.
\]

Let \( G \) be the matrix, with respect to the chosen integral bases of \( S^\lambda \) and \( S^\mu \), of a lifting of \( g \) to a \( \mathbb{Z} \)-linear map \( S^\lambda \to S^\mu \). The \( \mathbb{Z}S_n \)-linearity of \( g \) reads

\[
G\rho^\mu_\sigma - \rho^\lambda_\sigma G \in m(\mathbb{Z})_{n_\lambda \times n_\mu} \quad \text{for all } \sigma \in S_n.
\]

Thus such a morphism yields a necessary condition for a tuple of matrices to lie in the image of the Wedderburn embedding.

For example, the evaluations of our one-box-shift morphism at hook partitions, i.e. at \( \lambda = (k, 1^{n-k}) \) and \( \mu = (k-1, 1^{n-k+1}) \), \( k \in [2, n] \), furnish a long exact sequence. In the (simple) case of \( n = p \) prime, and localized at \( (p) \), the set of necessary conditions imposed by these morphisms already turns out to be sufficient for a tuple of matrices over \( \mathbb{Z}(p) \) to lie in the image of the localized Wedderburn embedding ([4], Section 4.2.1). Therefore, it is advisable to choose a tuple of locally integral bases adapted to this long exact sequence. For instance, we obtain

\[
\mathbb{Z}(3)S_3 \to \left\{ a \times \begin{bmatrix} b & c \\ d & e \end{bmatrix} \times f \mid a \equiv_3 b, \ d \equiv_3 0, \ e \equiv_3 f \right\} \subseteq \mathbb{Z}(3) \times \left[ \begin{bmatrix} \mathbb{Z}(3) \\ \mathbb{Z}(3) \\ \mathbb{Z}(3) \end{bmatrix} \right] \times \mathbb{Z}(3),
\]

the embedding not being written in the combinatorial standard polytabloid bases.

For an approach to the general case, see ([4], Chapters 3 and 5). Further examples may be found in ([4], Chapter 2).

6. Acknowledgments

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References