On adjoint functors of the Heller operator

Matthias Künzer

December 16, 2011

Abstract

Given an abelian category $\mathcal{A}$ with enough projectives, we can form its stable category $\underline{\mathcal{A}} := \mathcal{A}/\mathrm{Proj}(\mathcal{A})$. The Heller operator $\Omega : \underline{\mathcal{A}} \longrightarrow \underline{\mathcal{A}}$ is characterised on an object X by a choice of a short exact sequence $\Omega X \rightarrowtail P \twoheadrightarrow X$ in $\mathcal{A}$ with P projective. If $\mathcal{A}$ is Frobenius, then Ω is an equivalence, hence has a left and a right adjoint. If $\mathcal{A}$ is hereditary, then Ω is zero, hence has a left and a right adjoint. In general, Ω is neither an equivalence nor zero. In the examples we have calculated via Magma, it has a left adjoint, but in general not a right adjoint. If $\mathcal{A}$ has projective covers, then Ω preserves monomorphisms; this would also follow from Ω having a left adjoint. I do not know an example where Ω does not have a left adjoint.

Contents

MSC 2010: 18E30.

0 Introduction

0.1 The question

Let $\mathcal{E}$ be an exact category. Let $\operatorname{Proj}(\mathcal{E})$ be its full additive subcategory of relative projectives, i.e. for $P \in \operatorname{Ob}\mathcal{E}$ we have $P \in \operatorname{Ob}\operatorname{Proj}(\mathcal{E})$ if and only if ${}_{\mathcal{E}}(P,-)$ maps pure short exact sequences of $\mathcal{E}$ to short exact sequences of abelian groups. Suppose that $\mathcal{E}$ has enough relative projectives, i.e. suppose that for any $X \in \operatorname{Ob}\mathcal{E}$, there exists a pure epimorphism $P \twoheadrightarrow X$ in $\mathcal{E}$ with $P \in \operatorname{Ob}\operatorname{Proj}(\mathcal{E})$.

Write $\underline{\mathcal{E}} := \mathcal{E}/\operatorname{Proj}(\mathcal{E})$. The Heller operator $\Omega : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ is characterised on a given $X \in \operatorname{Ob}\mathcal{E}$ by a choice of a pure short exact sequence

$$\Omega X \rightarrowtail P \twoheadrightarrow X$$

in $\mathcal{E}$ with P relatively projective. This then is extended to morphisms.

We ask whether Ω has a left adjoint; cf. Question 1. I do not know a counterexample.

If $\mathcal{E}$ is a Frobenius category, then Ω is an equivalence, thus has both a left and a right adjoint.

If $\mathcal{E}$ is hereditary, i.e. if $\Omega \simeq 0$, then Ω has both a left and a right adjoint, viz. 0.

0.2 Monomorphisms

If a functor has a left adjoint, then it preserves monomorphisms. So first of all, we ask whether the Heller operator $\Omega : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ preserves monomorphisms.

It turns out that if $\mathcal{E}$ is weakly idempotent complete and has relative projective covers in the sense of §1.3, then Ω maps monomorphisms even to coretractions; cf. Proposition 4.

0.3 Construction of a left adjoint to the Heller operator Ω

Let $p \in [2, 997]$ be a prime. Let $R := \mathbf{F}_p[X]$ and $\pi := X$.

Let $A := (R/\pi^3)(e \xrightarrow{a} f) \simeq \begin{pmatrix} R/\pi^3 & R/\pi^3 \\ 0 & R/\pi^3 \end{pmatrix}$. Let $\mathcal{E} := \text{mod-}A$.

Using MAGMA [**1**], we construct a left adjoint $S : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ to Ω. We do so likewise for certain factor rings of A. Cf. Propositions 6, 9 and 11.

Let now k be a field, $R := k[X]$ and $\pi := X$. Let $n \geqslant 1$. An $(R/\pi^n)(e \xrightarrow{a} f)$-module is given by a morphism $X \xrightarrow{f} Y$ in mod-(R/π^n). The full subcategory of mod-$(R/\pi^n)(e \xrightarrow{a} f)$ consisting of injective morphisms $X \xrightarrow{f} Y$ as modules has been intensely studied; it is of finite type if $n \leqslant 5$, tame if $n = 6$, wild if $n \geqslant 7$; cf. [**7,** (0.1), (0.6)].

0.4 Two counterexamples

The functor $\Omega : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ does not have a right adjoint in general; cf. Remark 13.

Provided $S \dashv \Omega$ exists, the composite $\Omega \circ S : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ is not idempotent in general; cf. Remark 12.

0.5 Acknowledgements

Sebastian Thomas asked whether there exists a category with set of weak equivalences $(\mathcal{C}, W)$ that carries the structure of a Brown fibration category, but whose Ω on $\mathcal{C}[W^{-1}]$ ([1]) does not have a left adjoint. In our exact category context, this is Question 1.

I thank Steffen König for help with monomorphisms, cf. §1.2. I thank Sebastian Thomas and Markus Kirschmer for help with Magma. I thank Markus Schmidmeier for help with mod-$(R/\pi^n)(e \xrightarrow{a} f)$.

0.6 Notations and conventions

- Given $a, b \in \mathbf{Z}$, we write $[a, b] := \{ z \in \mathbf{Z} : a \leqslant z \leqslant b \}$.
- Composition of morphisms is written naturally, $(\xrightarrow{a}\xrightarrow{b}) = \xrightarrow{ab} = \xrightarrow{a\cdot b}$.

  Composition of functors is written traditionally, $(\xrightarrow{F}\xrightarrow{G}) = \xrightarrow{G\circ F}$.
- In a category $\mathcal{C}$, given $X, Y \in \operatorname{Ob}\mathcal{C}$, we write ${}_{\mathcal{C}}(X, Y)$ for the set of morphisms from X to Y.
- Given an isomorphism f, we write f^- for its inverse.
- In an additive category, a morphism of the form $X \xrightarrow{(1\ 0)} X \oplus Y$, or isomorphic to such a morphism, is called split monomorphic; a morphism of the form $X \oplus Y \xrightarrow{\binom{1}{0}} X$, or isomorphic to such a morphism, is called split epimorphic.
- In exact categories, pure monomorphisms are denoted by $\rightarrowtail$, pure epimorphisms by $\twoheadrightarrow$ and pure squares, i.e. bicartesian squares with pure short exact diagonal sequence, by a box $\Box$ in the diagram.
- Given a ring A, an A-*module* is a finitely generated right A-module.
- Given a commutative ring A and $a \in A$, we often write $A/a := A/(a) = A/aA$.
- Given a noetherian ring A, we write $\underline{\text{mod}}\text{-}A := \underline{\text{mod-}A}$ for the factor category of mod-A modulo the full additive subcategory of projectives.

  So in the language of §1.1 below, we consider the abelian category mod-A as an exact category with all short exact sequences declared to be pure and write $\underline{\text{mod}}$-A for its classical stable category.

1 The Heller operator Ω

1.1 Notation

Let $\mathcal{E}$ be an exact category in the sense of Quillen [**5,** p. 99] with enough relative projectives. We will use the notation of [**4,** §A.2] concerning pure short exact sequences, pure monomorphisms and pure epimorphisms.

Let $\operatorname{Proj}(\mathcal{E}) \subseteq \mathcal{E}$ denote the full subcategory of relative projectives. Let

$$\underline{\mathcal{E}} := \mathcal{E} / \operatorname{Proj}(\mathcal{E})$$

denote the classical stable category of $\mathcal{E}$. The residue class functor shall be denoted by

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \underline{\mathcal{E}} \\ (X \xrightarrow{f} Y) & \longmapsto & (X \xrightarrow{[f]} Y) \,. \end{array}$$

[1] Cf. also [**6,** p. 210].

For each $X \in \operatorname{Ob}\mathcal{E}$, we choose a pure short exact sequence

$$(*) \qquad \Omega X \xrightarrow{i_X} \mathrm{P}X \xrightarrow{p_X} X$$

with $\mathrm{P}X$ relatively projective. Let the *Heller operator* **[3]**

$$\Omega \,:\, \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$$

be defined on the objects by the choice just made. Suppose given a morphism $X \xrightarrow{[f]} Y$ in $\underline{\mathcal{E}}$. Choose a morphism

$$\begin{array}{ccccc} \Omega X & \xrightarrow{i_X} & \mathrm{P}X & \xrightarrow{p_X} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle f} \\ \Omega Y & \xrightarrow{i_Y} & \mathrm{P}Y & \xrightarrow{p_Y} & Y \end{array}$$

of pure short exact sequences. Let

$$\Omega[f] \;:=\; [f'] \,.$$

Different choices of pure short exact sequences $(*)$ yield mutually isomorphic Heller operators.

Question 1 *Does Ω have a left adjoint?*

I do not know a counterexample.

1.2 Preservation of monomorphisms

If $\Omega : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ has a left adjoint, then it preserves monomorphisms. So if, for some $\mathcal{E}$, the functor Ω did not preserve monomorphisms, then Ω could not have a left adjoint. Under certain finiteness assumptions, however, we will show that Ω maps monomorphisms to coretractions, so in particular to monomorphisms. This is to be compared to the case of $\mathcal{E}$ being Frobenius, where in the triangulated category $\underline{\mathcal{E}}$ all monomorphisms are split.

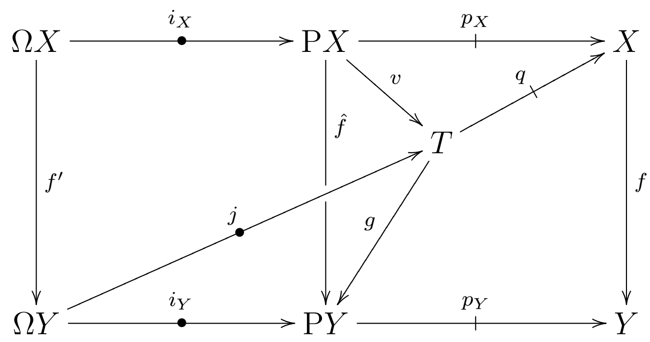
Lemma 2 *Suppose that for $X \in \operatorname{Ob}\mathcal{E}$ and for $s \in {}_{\mathcal{E}}(\mathrm{P}X, \mathrm{P}X)$ such that $sp_X = p_X$, the endomorphism s is an isomorphism.*

Suppose given $X \xrightarrow{f} Y$ in $\mathcal{E}$.

If $[f]$ is a monomorphism, then $\Omega[f]$ is a coretraction.

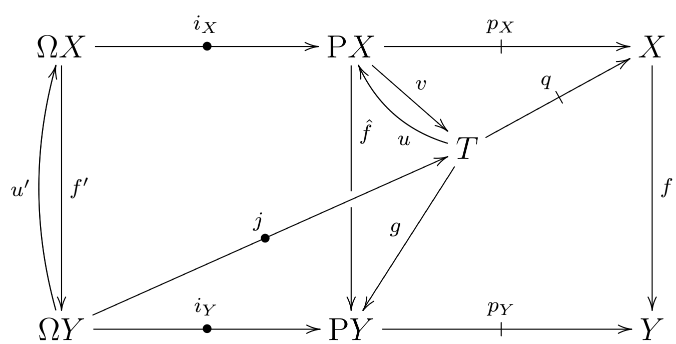
In particular, Ω preserves monomorphisms.

Proof. Choose a morphism of pure short exact sequences as shown below. Insert a pullback $(T, X, \mathrm{P}Y, Y)$ and the induced morphism $\mathrm{P}X \xrightarrow{v} T$, having $vg = \hat{f}$ and $vq = p_X$. Insert a kernel j of q with $jg = i_Y$.



We have $f'j = i_X v$, since $f'jq = 0 = i_X p_X = i_X vq$ and $f'jg = f'i_Y = i_X \hat{f} = i_X vg$.

We have $[q][f] = [gp_Y] = 0$. Since $[f]$ is monomorphic, we infer that $[q] = 0$. Hence there exists $T \xrightarrow{u} \mathrm{P}X$ such that $up_X = q$. On the kernels, we obtain $\Omega Y \xrightarrow{u'} \Omega X$ such that $u'i_X = ju$.



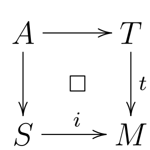
We have $vup_X = vq = p_X$. Hence vu is an isomorphism by assumption.

We obtain $f'u'i_X = f'ju = i_X vu$. So $(f'u', vu, \mathrm{id}_X)$ is a morphism of pure short exact sequences. Hence $f'u'$ is an isomorphism. Thus f' is a coretraction. We conclude that $\Omega[f] = [f']$ is a coretraction. □

1.3 Relative projective covers

Suppose $\mathcal{E}$ to be weakly idempotent complete; cf. [**2,** Def. 7.2].

A morphism $S \xrightarrow{i} M$ in $\mathcal{E}$ is called *small* if in each pure square



in $\mathcal{E}$, the morphism $T \xrightarrow{t} M$ is purely epimorphic; cf. [**8,** Def. 2.8.30]. In other words, i is small iff $\binom{i}{t}$ being purely epimorphic entails t being purely epimorphic for each morphism t with same target as i. E.g. $i = 0$ is small, for $\binom{0}{t} = \binom{0}{1} t$ is purely epimorphic only if t is; cf. [**2,** Prop. 2.16].

If $S \overset{i}{\rightarrowtail} M$ is small and split monomorphic, then there exists

$$\begin{array}{ccc} 0 & \longrightarrow & S' \\ \downarrow & \Box & \downarrow{\scriptstyle i'} \\ S & \overset{i}{\rightarrowtail} & M \end{array},$$

forcing i' to be an isomorphism and thus S to be isomorphic to 0.

Given $\tilde{S} \xrightarrow{j} S \xrightarrow{i} M$ with $S \xrightarrow{i} M$ small, then $\tilde{S} \xrightarrow{ji} M$ is small. In fact, given t such that $\left(\begin{smallmatrix} ji \\ t \end{smallmatrix}\right)$ is a pure epimorphism, the factorisation $\left(\begin{smallmatrix} ji \\ t \end{smallmatrix}\right) = \left(\begin{smallmatrix} j & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} i \\ t \end{smallmatrix}\right)$ shows that $\left(\begin{smallmatrix} i \\ t \end{smallmatrix}\right)$ is a pure epimorphism; cf. [**2,** Cor. 7.7]. Thus t is purely epimorphic by smallness of i.

A *relative projective cover* of $X \in \operatorname{Ob}\mathcal{E}$ is a pure epimorphism $P \overset{p}{\twoheadrightarrow} X$ in $\mathcal{E}$ such that P is relatively projective and such that $\operatorname{Kern} p \rightarrowtail P$ is small; cf. [**8,** 2.8.31].

We say that $\mathcal{E}$ *has relative projective covers* if for each $X \in \operatorname{Ob}\mathcal{E}$, there exists a relative projective cover $P \overset{p}{\twoheadrightarrow} X$.

Lemma 3 *Suppose given a relative projective cover $P \overset{p}{\twoheadrightarrow} X$ in $\mathcal{E}$. Suppose given $P \xrightarrow{s} P$ such that $sp = p$. Then s is an isomorphism.*

Proof. We complete to a pure short exact sequence $K \overset{k}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} X$. We obtain a morphism

$$\begin{array}{ccccc} K & \overset{k}{\rightarrowtail} & P & \overset{p}{\twoheadrightarrow} & X \\ {\scriptstyle g}\downarrow & \Box & {\scriptstyle s}\downarrow & & \| \\ K & \overset{k}{\rightarrowtail} & P & \overset{p}{\twoheadrightarrow} & X \end{array}.$$

of pure short exact sequences. Since the left hand side quadrangle is a pure square, we conclude that s is purely epimorphic by smallness of $K \overset{k}{\rightarrowtail} P$. Hence s is split epimorphic by relative projectivity of P; cf. [**2,** Rem. 7.4]. Let $L \overset{\ell}{\rightarrowtail} P$ be a kernel of s. Since ℓ factors over the small morphism k, it is small as well. Since ℓ is split monomorphic, we have $L \simeq 0$. Thus s is an isomorphism. ▫

Proposition 4 *Suppose that the exact category $\mathcal{E}$ is weakly idempotent complete and has relative projective covers.*

Then $\Omega : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}}$ maps each monomorphism to a coretraction. In particular, Ω preserves monomorphisms.

Proof. We may use relative projective covers to construct Ω in $(*)$. Then Lemma 3 allows us to apply Lemma 2. ▫

2 Examples for adjoints of the Heller operator Ω

Let R be a principal ideal domain, with a maximal ideal generated by an element $\pi \in R$.

Let

$$A \;:=\; (R/\pi^3)(e \xrightarrow{a} f)\;.$$

I.e. A is the path algebra of $e \xrightarrow{a} f$ over the ground ring R/π^3. It has primitive idempotents e and f, and $a \in eAf$.

An object in mod-A is given by a morphism $X \longrightarrow Y$ in mod-(R/π^3). A morphism in mod-A is given by a commutative quadrangle in mod-(R/π^3).

2.1 Example of a left adjoint

2.1.1 A list of indecomposables

Define the following objects in mod-A.

$$
\begin{array}{lcllcl}
P_1 & := & (R/\pi^3 \xrightarrow{1} R/\pi^3) & P_2 & := & (0 \longrightarrow R/\pi^3) \\[2ex]
X_1 & := & (R/\pi \xrightarrow{1} R/\pi) & X_{14} & := & (R/\pi^3 \xrightarrow{\pi^2} R/\pi^3) \\
X_2 & := & (R/\pi^2 \xrightarrow{1} R/\pi^2) & X_{15} & := & (R/\pi^3 \longrightarrow 0) \\
X_3 & := & (R/\pi^2 \xrightarrow{1} R/\pi) & X_{16} & := & (R/\pi^2 \oplus R/\pi^3 \xrightarrow{\left(\begin{smallmatrix}1 & \pi\\ 1 & 0\end{smallmatrix}\right)} R/\pi \oplus R/\pi^3) \\
X_4 & := & (R/\pi^3 \xrightarrow{1} R/\pi^2) & X_{17} & := & (R/\pi \oplus R/\pi^3 \xrightarrow{\left(\begin{smallmatrix}\pi & \pi^2\\ 1 & 0\end{smallmatrix}\right)} R/\pi^2 \oplus R/\pi^3) \\
X_5 & := & (R/\pi^2 \xrightarrow{\pi} R/\pi^3) & X_{18} & := & (R/\pi \oplus R/\pi^3 \xrightarrow{\left(\begin{smallmatrix}0 & \pi^2\\ 1 & \pi\end{smallmatrix}\right)} R/\pi \oplus R/\pi^3) \\
X_6 & := & (R/\pi \xrightarrow{\pi} R/\pi^2) & X_{19} & := & (R/\pi \xrightarrow{\pi^2} R/\pi^3) \\
X_7 & := & (R/\pi^2 \xrightarrow{(1\ \pi)} R/\pi \oplus R/\pi^3) & X_{20} & := & (R/\pi^3 \xrightarrow{1} R/\pi) \\
X_8 & := & (R/\pi \oplus R/\pi^3 \xrightarrow{\left(\begin{smallmatrix}\pi\\ 1\end{smallmatrix}\right)} R/\pi^2) & X_{21} & := & (R/\pi \longrightarrow 0) \\
X_9 & := & (R/\pi^3 \xrightarrow{\pi} R/\pi^3) & X_{22} & := & (0 \longrightarrow R/\pi) \\
X_{10} & := & (R/\pi^2 \longrightarrow 0) & X_{23} & := & (R/\pi^3 \xrightarrow{\pi} R/\pi^2) \\
X_{11} & := & (0 \longrightarrow R/\pi^2) & X_{24} & := & (R/\pi^2 \xrightarrow{\pi^2} R/\pi^3) \\
X_{12} & := & (R/\pi \oplus R/\pi^3 \xrightarrow{\left(\begin{smallmatrix}\pi^2\\ \pi\end{smallmatrix}\right)} R/\pi^3) & X_{25} & := & (R/\pi^2 \xrightarrow{\pi} R/\pi^2) \\
X_{13} & := & (R/\pi^3 \xrightarrow{(1\ \pi)} R/\pi \oplus R/\pi^3) & & &
\end{array}
$$

A matrix inspection yields the

Lemma 5

(1) *For each projective indecomposable A-module P, there exists a unique $i \in [1, 2]$ such that $P \simeq P_i$.*

(2) *For each nonprojective indecomposable A-module X, there exists a unique $i \in [1, 25]$ such that $X \simeq X_i$.*

2.1.2 Construction of a left adjoint

Our aim in this section is to computationally verify the

Proposition 6 *Suppose given a prime $p \in [2, 997]$. Suppose that $R = \mathbf{F}_p[X]$ and $\pi = X$. Then the Heller operator $\Omega : \underline{\mathrm{mod}}\text{-}A \longrightarrow \underline{\mathrm{mod}}\text{-}A$ has a left adjoint.*

For ease of Magma input, we have used that

$$A \;\simeq\; \mathbf{F}_p\Big({}^{u}\circlearrowleft e \xrightarrow{a} f \circlearrowright^{v} \Big)/(u^3,\ v^3,\ ua - av)$$

as $\mathbf{F}_p$-algebras.

To reduce the calculation of this adjoint functor to the proof of the representability of certain functors, we use

Lemma 7 ([9, 16.4.5, 4.5.1]) *Suppose given categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$. Suppose that*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{{}_{\mathcal{D}}(Y,F(-))} & \text{Sets} \\ (X \xrightarrow{x} X') & \longmapsto & \big({}_{\mathcal{D}}(Y,FX) \xrightarrow{{}_{\mathcal{D}}(Y,Fx)} {}_{\mathcal{D}}(Y,FX')\big) \end{array}$$

is representable for each $Y \in \mathrm{Ob}\,\mathcal{D}$.

Then F has a left adjoint.

More precisely, given a map $\mathrm{Ob}\,\mathcal{C} \xleftarrow{\gamma} \mathrm{Ob}\,\mathcal{D}$ and an isomorphism

$${}_{\mathcal{D}}(Y, F(-)) \xrightarrow[\sim]{\varphi_Y} {}_{\mathcal{C}}(Y\gamma, -)$$

for $Y \in \mathrm{Ob}\,\mathcal{D}$, there exists a left adjoint $\mathcal{C} \xleftarrow{G} \mathcal{D}$ to F, i.e. $G \dashv F$, such that, writing

$$\varepsilon Y \;:=\; (1_{Y\gamma})(\varphi_Y(Y\gamma))^{-} \;:\; Y \longrightarrow F(Y\gamma)$$

for $Y \in \mathrm{Ob}\,\mathcal{D}$, we have

$$G(Y \xrightarrow{y} Y') \;=\; (Y\gamma \xrightarrow{(y\cdot\varepsilon Y')(\varphi_Y(Y'\gamma))} Y'\gamma)$$

for $Y \xrightarrow{y} Y$ in $\mathcal{D}$.

Thus in order to construct the left adjoint to Ω on $\underline{\mathrm{mod}}$-A, it suffices to show that the functor ${}_{\underline{\mathrm{mod}}\text{-}A}(X_i\,, \Omega(-))$ is representable $i \in [1, 25]$. We shall do so by an actual construction of an isotransformation from a Hom-functor.

Suppose given $i \in [1, 25]$. Such an isotransformation is necessarily of the form

$$\begin{array}{ccc} {}_{\underline{\mathrm{mod}}\text{-}A}(SX_i\,, Y) & \longrightarrow & {}_{\underline{\mathrm{mod}}\text{-}A}(X_i\,, \Omega Y) \\ [f] & \longmapsto & [\epsilon_i] \cdot \Omega[f] \end{array}$$

for some $SX_i \in \mathrm{Ob}\,\underline{\mathrm{mod}}\text{-}A$ and some A-linear map $\epsilon_i : X_i \longrightarrow \Omega SX_i$, where $Y \in \mathrm{Ob}\,\underline{\mathrm{mod}}\text{-}A$.

So it suffices to find an A-module SX_i and an A-linear map $\epsilon_i : X_i \longrightarrow \Omega SX_i$ such that the induced map

$$(**) \qquad \begin{array}{rcl} {}_{\underline{\mathrm{mod}}\text{-}A}(SX_i\,, X_j) & \longrightarrow & {}_{\underline{\mathrm{mod}}\text{-}A}(X_i\,, \Omega X_j) \\ [f] & \longmapsto & [\epsilon_i]\cdot\Omega[f] \end{array}$$

is an isomorphism for $j \in [1, 25]$.

In particular, given an automorphism α of X_i in $\underline{\mathrm{mod}}\text{-}A$, an automorphism β of SX_i in $\underline{\mathrm{mod}}\text{-}A$ and a valid such morphism $[\epsilon_i]$, then $\alpha\cdot[\epsilon_i]\cdot\Omega\beta$ is another valid such morphism, sometimes of a simpler shape.

To show that a guess for SX_i is in fact the sought-for representing object, we make use of the fact that ${}_{\underline{\mathrm{mod}}\text{-}A}(X_i\,, \Omega SX_i)$ is finite, so that we have only a finite set of candidates for ϵ_i . Then to check whether the candidate-induced maps $(**)$ are isomorphisms, is also feasible via Magma, using in particular its commands `ProjectiveCover`, `AHom` and `PHom`; cf. [**1**].

We obtain

$$\begin{array}{lcl lcl lcl lcl}
SX_1 &=& X_2 & \Omega SX_1 &=& X_1 & SX_{14} &=& X_5 & \Omega SX_{14} &=& X_{19} \\
SX_2 &=& X_1 & \Omega SX_2 &=& X_2 & SX_{15} &=& 0 & \Omega SX_{15} &=& 0 \\
SX_3 &=& X_2 & \Omega SX_3 &=& X_1 & SX_{16} &=& X_2 \oplus X_{19} & \Omega SX_{16} &=& X_1 \oplus X_5 \\
SX_4 &=& X_1 & \Omega SX_4 &=& X_2 & SX_{17} &=& X_1 \oplus X_5 & \Omega SX_{17} &=& X_2 \oplus X_{19} \\
SX_5 &=& X_{19} & \Omega SX_5 &=& X_5 & SX_{18} &=& X_6 & \Omega SX_{18} &=& X_7 \\
SX_6 &=& X_7 & \Omega SX_6 &=& X_6 & SX_{19} &=& X_5 & \Omega SX_{19} &=& X_{19} \\
SX_7 &=& X_6 & \Omega SX_7 &=& X_7 & SX_{20} &=& X_2 & \Omega SX_{20} &=& X_1 \\
SX_8 &=& X_1 & \Omega SX_8 &=& X_2 & SX_{21} &=& 0 & \Omega SX_{21} &=& 0 \\
SX_9 &=& X_{19} & \Omega SX_9 &=& X_5 & SX_{22} &=& X_{11} & \Omega SX_{22} &=& X_{22} \\
SX_{10} &=& 0 & \Omega SX_{10} &=& 0 & SX_{23} &=& X_7 & \Omega SX_{23} &=& X_6 \\
SX_{11} &=& X_{22} & \Omega SX_{11} &=& X_{11} & SX_{24} &=& X_5 & \Omega SX_{24} &=& X_{19} \\
SX_{12} &=& X_{19} & \Omega SX_{12} &=& X_5 & SX_{25} &=& X_7 & \Omega SX_{25} &=& X_6 \\
SX_{13} &=& X_6 & \Omega SX_{13} &=& X_7 & & & & & &
\end{array}$$

and

$$\begin{pmatrix} X_1 \\ \epsilon_1\downarrow \\ \Omega SX_1 \end{pmatrix} = \begin{pmatrix} R/\pi & \xrightarrow{1} & R/\pi \\ 1\downarrow & & \downarrow 1 \\ R/\pi & \xrightarrow{1} & R/\pi \end{pmatrix} \qquad \begin{pmatrix} X_{14} \\ \epsilon_{14}\downarrow \\ \Omega SX_{14} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{\pi^2} & R/\pi^3 \\ 1\downarrow & & \downarrow 1 \\ R/\pi & \xrightarrow{\pi^2} & R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_2 \\ \epsilon_2\downarrow \\ \Omega SX_2 \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{1} & R/\pi^2 \\ 1\downarrow & & \downarrow 1 \\ R/\pi^2 & \xrightarrow{1} & R/\pi^2 \end{pmatrix} \qquad \begin{pmatrix} X_{15} \\ \epsilon_{15}\downarrow \\ \Omega SX_{15} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_3 \\ \epsilon_3\downarrow \\ \Omega SX_3 \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{1} & R/\pi \\ 1\downarrow & & \downarrow 1 \\ R/\pi & \xrightarrow{1} & R/\pi \end{pmatrix} \begin{pmatrix} X_{16} \\ \epsilon_{16}\downarrow \\ \Omega SX_{16} \end{pmatrix} = \begin{pmatrix} R/\pi^2 \oplus R/\pi^3 & \xrightarrow{\left(\begin{smallmatrix} 1 & \pi \\ 1 & 0 \end{smallmatrix}\right)} & R/\pi \oplus R/\pi^3 \\ \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ R/\pi \oplus R/\pi^2 & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi \end{smallmatrix}\right)} & R/\pi \oplus R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_4 \\ {\scriptstyle \epsilon_4}\downarrow \\ \Omega S X_4 \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{1} & R/\pi^2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{1} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_5 \\ {\scriptstyle \epsilon_5}\downarrow \\ \Omega S X_5 \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{\pi} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{\pi} & R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_6 \\ {\scriptstyle \epsilon_6}\downarrow \\ \Omega S X_6 \end{pmatrix} = \begin{pmatrix} R/\pi & \xrightarrow{\pi} & R/\pi^2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{\pi} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_7 \\ {\scriptstyle \epsilon_7}\downarrow \\ \Omega S X_7 \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{(1\ \pi)} & R/\pi \oplus R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \\ R/\pi^2 & \xrightarrow{(1\ \pi)} & R/\pi \oplus R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_8 \\ {\scriptstyle \epsilon_8}\downarrow \\ \Omega S X_8 \end{pmatrix} = \begin{pmatrix} R/\pi \oplus R/\pi^3 & \xrightarrow{\left(\begin{smallmatrix}\pi\\1\end{smallmatrix}\right)} & R/\pi^2 \\ {\scriptstyle \left(\begin{smallmatrix}\pi\\1\end{smallmatrix}\right)}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{1} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_9 \\ {\scriptstyle \epsilon_9}\downarrow \\ \Omega S X_9 \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{\pi} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{\pi} & R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_{10} \\ {\scriptstyle \epsilon_{10}}\downarrow \\ \Omega S X_{10} \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_{11} \\ {\scriptstyle \epsilon_{11}}\downarrow \\ \Omega S X_{11} \end{pmatrix} = \begin{pmatrix} 0 & \longrightarrow & R/\pi^2 \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{12} \\ {\scriptstyle \epsilon_{12}}\downarrow \\ \Omega S X_{12} \end{pmatrix} = \begin{pmatrix} R/\pi \oplus R/\pi^3 & \xrightarrow{\left(\begin{smallmatrix}\pi^2\\ \pi\end{smallmatrix}\right)} & R/\pi^3 \\ {\scriptstyle \left(\begin{smallmatrix}\pi\\1\end{smallmatrix}\right)}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{\pi} & R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_{13} \\ {\scriptstyle \epsilon_{13}}\downarrow \\ \Omega S X_{13} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{(1\ \pi)} & R/\pi \oplus R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \\ R/\pi^2 & \xrightarrow{(1\ \pi)} & R/\pi \oplus R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_{17} \\ {\scriptstyle \epsilon_{17}}\downarrow \\ \Omega S X_{17} \end{pmatrix} = \begin{pmatrix} R/\pi \oplus R/\pi^3 & \xrightarrow{\left(\begin{smallmatrix}\pi&\pi^2\\1&0\end{smallmatrix}\right)} & R/\pi^2 \oplus R/\pi^3 \\ {\scriptstyle \left(\begin{smallmatrix}\pi&1\\1&0\end{smallmatrix}\right)}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \\ R/\pi^2 \oplus R/\pi & \xrightarrow{\left(\begin{smallmatrix}1&0\\0&\pi^2\end{smallmatrix}\right)} & R/\pi^2 \oplus R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_{18} \\ {\scriptstyle \epsilon_{18}}\downarrow \\ \Omega S X_{18} \end{pmatrix} = \begin{pmatrix} R/\pi \oplus R/\pi^3 & \xrightarrow{\left(\begin{smallmatrix}0&\pi^2\\1&\pi\end{smallmatrix}\right)} & R/\pi \oplus R/\pi^3 \\ {\scriptstyle \left(\begin{smallmatrix}\pi\\1\end{smallmatrix}\right)}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \\ R/\pi^2 & \xrightarrow{(1\ \pi)} & R/\pi \oplus R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_{19} \\ {\scriptstyle \epsilon_{19}}\downarrow \\ \Omega S X_{19} \end{pmatrix} = \begin{pmatrix} R/\pi & \xrightarrow{\pi^2} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{\pi^2} & R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_{20} \\ {\scriptstyle \epsilon_{20}}\downarrow \\ \Omega S X_{20} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{1} & R/\pi \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{1} & R/\pi \end{pmatrix}$$

$$\begin{pmatrix} X_{21} \\ {\scriptstyle \epsilon_{21}}\downarrow \\ \Omega S X_{21} \end{pmatrix} = \begin{pmatrix} R/\pi & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_{22} \\ {\scriptstyle \epsilon_{22}}\downarrow \\ \Omega S X_{22} \end{pmatrix} = \begin{pmatrix} 0 & \longrightarrow & R/\pi \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & R/\pi \end{pmatrix}$$

$$\begin{pmatrix} X_{23} \\ {\scriptstyle \epsilon_{23}}\downarrow \\ \Omega S X_{23} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{\pi} & R/\pi^2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{\pi} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{24} \\ {\scriptstyle \epsilon_{24}}\downarrow \\ \Omega S X_{24} \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{\pi^2} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{\pi^2} & R/\pi^3 \end{pmatrix}$$

$$\begin{pmatrix} X_{25} \\ {\scriptstyle \epsilon_{25}}\downarrow \\ \Omega S X_{25} \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{\pi} & R/\pi^2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{\pi} & R/\pi^2 \end{pmatrix}.$$

Remark 8 Keep the assumptions of Proposition 6.

We have $(\Omega \circ S)^2 Y \simeq (\Omega \circ S) Y$ for $Y \in \mathrm{Ob}\, \underline{\mathrm{mod}}\text{-}A$.

The unit of the adjunction $S \dashv \Omega$ at an A-module $X \xrightarrow{f} Y$ is represented by a factorisation

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \bar f}\downarrow & & \| \\ \mathrm{I}_f & \xrightarrow{\dot f} & Y \end{array}$$

over an image I_f of the module-defining morphism f.

I do not know why.

2.2 Another example of a left adjoint

Recall that R is a principal ideal domain, with a maximal ideal generated by an element $\pi \in R$.

2.2.1 A list of indecomposables

Let

$$B \;:=\; A/(\pi^2 a) \;=\; (R/\pi^3)(e \xrightarrow{a} f)/(\pi^2 a)\;.$$

Indecomposable nonprojective B-modules become indecomposable nonprojective A-modules via restriction along the residue class map $A \longrightarrow B$.

We list the 24 representatives of isoclasses of indecomposable nonprojective B-modules in the numbering used in §2.1.1 as follows.

$X_1\,,\ X_2\,,\ X_3\,,\ X_5\,,\ X_6\,,\ X_7\,,\ X_8\,,\ X_9\,,\ X_{10}\,,\ X_{11}\,,\ X_{12}\,,\ X_{13}\,,\ X_{14}\,,\ X_{15}\,,\ X_{16}\,,$
$X_{17}\,,\ X_{18}\,,\ X_{19}\,,\ X_{20}\,,\ X_{21}\,,\ X_{22}\,,\ X_{23}\,,\ X_{24}\,,\ X_{25}$

2.2.2 Construction of a left adjoint

Our aim in this section is to computationally verify the

Proposition 9 *Suppose given a prime $p \in [2, 997]$. Suppose that $R = \mathbf{F}_p[X]$ and $\pi = X$. Then the Heller operator $\Omega : \underline{\mathrm{mod}}\text{-}B \longrightarrow \underline{\mathrm{mod}}\text{-}B$ has a left adjoint.*

We proceed analogously to §2.1.

We obtain

$$
\begin{array}{lll|lll|lll|lll}
SX_1 &=& X_2 & \Omega SX_1 &=& X_{21} & SX_{14} &=& X_2\oplus X_9 & \Omega SX_{14} &=& X_{21}\oplus X_{11}\\
SX_2 &=& X_1 & \Omega SX_2 &=& X_3 & SX_{15} &=& X_2 & \Omega SX_{15} &=& X_{21}\\
SX_3 &=& X_1 & \Omega SX_3 &=& X_3 & SX_{16} &=& X_1\oplus X_{17} & \Omega SX_{16} &=& X_3\oplus X_{25}\\
SX_5 &=& X_{17} & \Omega SX_5 &=& X_{25} & SX_{17} &=& X_1\oplus X_2\oplus X_9 & \Omega SX_{17} &=& X_{21}\oplus X_3\oplus X_{11}\\
SX_6 &=& X_2\oplus X_{13} & \Omega SX_6 &=& X_{21}\oplus X_{22} & SX_{18} &=& X_1\oplus X_2\oplus X_9 & \Omega SX_{18} &=& X_{21}\oplus X_3\oplus X_{11}\\
SX_7 &=& X_1\oplus X_9 & \Omega SX_7 &=& X_3\oplus X_{11} & SX_{19} &=& X_2\oplus X_9 & \Omega SX_{19} &=& X_{21}\oplus X_{11}\\
SX_8 &=& X_1\oplus X_2 & \Omega SX_8 &=& X_{21}\oplus X_3 & SX_{20} &=& X_1 & \Omega SX_{20} &=& X_3\\
SX_9 &=& X_{17} & \Omega SX_9 &=& X_{25} & SX_{21} &=& X_2 & \Omega SX_{21} &=& X_{21}\\
SX_{10} &=& X_2 & \Omega SX_{10} &=& X_{21} & SX_{22} &=& X_{13} & \Omega SX_{22} &=& X_{22}\\
SX_{11} &=& X_9 & \Omega SX_{11} &=& X_{11} & SX_{23} &=& X_{17} & \Omega SX_{23} &=& X_{25}\\
SX_{12} &=& X_2\oplus X_{17} & \Omega SX_{12} &=& X_{21}\oplus X_{25} & SX_{24} &=& X_2\oplus X_9 & \Omega SX_{24} &=& X_{21}\oplus X_{11}\\
SX_{13} &=& X_1\oplus X_9 & \Omega SX_{13} &=& X_3\oplus X_{11} & SX_{25} &=& X_{17} & \Omega SX_{25} &=& X_{25}
\end{array}
$$

and

$$\begin{pmatrix} X_1 \\ {\scriptstyle\epsilon_1}\downarrow \\ \Omega SX_1 \end{pmatrix} = \begin{pmatrix} R/\pi & \xrightarrow{1} & R/\pi \\ {\scriptstyle 1}\downarrow & & \downarrow \\ R/\pi & \longrightarrow & 0 \end{pmatrix} \qquad \begin{pmatrix} X_{14} \\ {\scriptstyle\epsilon_{14}}\downarrow \\ \Omega SX_{14} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{\pi^2} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{0} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_2 \\ {\scriptstyle\epsilon_2}\downarrow \\ \Omega SX_2 \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{1} & R/\pi^2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{1} & R/\pi \end{pmatrix} \qquad \begin{pmatrix} X_{15} \\ {\scriptstyle\epsilon_{15}}\downarrow \\ \Omega SX_{15} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \longrightarrow & 0 \\ {\scriptstyle 1}\downarrow & & \downarrow \\ R/\pi & \longrightarrow & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_3 \\ {\scriptstyle\epsilon_3}\downarrow \\ \Omega SX_3 \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{1} & R/\pi \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{1} & R/\pi \end{pmatrix} \qquad \begin{pmatrix} X_{16} \\ {\scriptstyle\epsilon_{16}}\downarrow \\ \Omega SX_{16} \end{pmatrix} = \begin{pmatrix} R/\pi^2\oplus R/\pi^3 & \xrightarrow{\left(\begin{smallmatrix}1&\pi\\1&0\end{smallmatrix}\right)} & R/\pi\oplus R/\pi^3 \\ \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \\ R/\pi^2\oplus R/\pi^2 & \xrightarrow{\left(\begin{smallmatrix}1&0\\0&\pi\end{smallmatrix}\right)} & R/\pi\oplus R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{17} \\ {\scriptstyle\epsilon_{17}}\downarrow \\ \Omega SX_{17} \end{pmatrix} = \begin{pmatrix} R/\pi\oplus R/\pi^3 & \xrightarrow{\left(\begin{smallmatrix}\pi&\pi^2\\1&0\end{smallmatrix}\right)} & R/\pi^2\oplus R/\pi^3 \\ \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \\ R/\pi\oplus R/\pi^2 & \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} & R/\pi\oplus R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_5 \\ {\scriptstyle\epsilon_5}\downarrow \\ \Omega SX_5 \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{\pi} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{\pi} & R/\pi^2 \end{pmatrix} \qquad \begin{pmatrix} X_{18} \\ {\scriptstyle\epsilon_{18}}\downarrow \\ \Omega SX_{18} \end{pmatrix} = \begin{pmatrix} R/\pi\oplus R/\pi^3 & \xrightarrow{\left(\begin{smallmatrix}0&\pi^2\\1&\pi\end{smallmatrix}\right)} & R/\pi\oplus R/\pi^3 \\ \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\downarrow & & \downarrow\left(\begin{smallmatrix}1&-\pi\\0&1\end{smallmatrix}\right) \\ R/\pi\oplus R/\pi^2 & \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} & R/\pi\oplus R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_6 \\ {\scriptstyle\epsilon_6}\downarrow \\ \Omega SX_6 \end{pmatrix} = \begin{pmatrix} R/\pi & \xrightarrow{\pi} & R/\pi^2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{0} & R/\pi \end{pmatrix} \qquad \begin{pmatrix} X_{19} \\ {\scriptstyle\epsilon_{19}}\downarrow \\ \Omega SX_{19} \end{pmatrix} = \begin{pmatrix} R/\pi & \xrightarrow{\pi^2} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{0} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_7 \\ {\scriptstyle\epsilon_7}\downarrow \\ \Omega SX_7 \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{(1\ \pi)} & R/\pi\oplus R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow\left(\begin{smallmatrix}1&-\pi\\0&1\end{smallmatrix}\right) \\ R/\pi^2 & \xrightarrow{(1\ 0)} & R/\pi\oplus R/\pi^2 \end{pmatrix} \qquad \begin{pmatrix} X_{20} \\ {\scriptstyle\epsilon_{20}}\downarrow \\ \Omega SX_{20} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{1} & R/\pi \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{1} & R/\pi \end{pmatrix}$$

$$\begin{pmatrix} X_8 \\ {\scriptstyle \epsilon_8}\downarrow \\ \Omega S X_8 \end{pmatrix} = \begin{pmatrix} R/\pi \oplus R/\pi^3 & \xrightarrow{\binom{\pi}{1}} & R/\pi^2 \\ \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi \oplus R/\pi^2 & \xrightarrow{\binom{0}{1}} & R/\pi \end{pmatrix}$$

$$\begin{pmatrix} X_9 \\ {\scriptstyle \epsilon_9}\downarrow \\ \Omega S X_9 \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{\pi} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{\pi} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{10} \\ {\scriptstyle \epsilon_{10}}\downarrow \\ \Omega S X_{10} \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \longrightarrow & 0 \\ {\scriptstyle 1}\downarrow & & \downarrow \\ R/\pi & \longrightarrow & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_{11} \\ {\scriptstyle \epsilon_{11}}\downarrow \\ \Omega S X_{11} \end{pmatrix} = \begin{pmatrix} 0 & \longrightarrow & R/\pi^2 \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{12} \\ {\scriptstyle \epsilon_{12}}\downarrow \\ \Omega S X_{12} \end{pmatrix} = \begin{pmatrix} R/\pi \oplus R/\pi^3 & \xrightarrow{\binom{\pi^2}{\pi}} & R/\pi^3 \\ \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi \oplus R/\pi^2 & \xrightarrow{\binom{0}{\pi}} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{13} \\ {\scriptstyle \epsilon_{13}}\downarrow \\ \Omega S X_{13} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{(1\ \pi)} & R/\pi \oplus R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow\left(\begin{smallmatrix}1&-\pi\\0&1\end{smallmatrix}\right) \\ R/\pi^2 & \xrightarrow{(1\ 0)} & R/\pi \oplus R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{21} \\ {\scriptstyle \epsilon_{21}}\downarrow \\ \Omega S X_{21} \end{pmatrix} = \begin{pmatrix} R/\pi & \longrightarrow & 0 \\ {\scriptstyle 1}\downarrow & & \downarrow \\ R/\pi & \longrightarrow & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_{22} \\ {\scriptstyle \epsilon_{22}}\downarrow \\ \Omega S X_{22} \end{pmatrix} = \begin{pmatrix} 0 & \longrightarrow & R/\pi \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & R/\pi \end{pmatrix}$$

$$\begin{pmatrix} X_{23} \\ {\scriptstyle \epsilon_{23}}\downarrow \\ \Omega S X_{23} \end{pmatrix} = \begin{pmatrix} R/\pi^3 & \xrightarrow{\pi} & R/\pi^2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{\pi} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{24} \\ {\scriptstyle \epsilon_{24}}\downarrow \\ \Omega S X_{24} \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{\pi^2} & R/\pi^3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi & \xrightarrow{0} & R/\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} X_{25} \\ {\scriptstyle \epsilon_{25}}\downarrow \\ \Omega S X_{25} \end{pmatrix} = \begin{pmatrix} R/\pi^2 & \xrightarrow{\pi} & R/\pi^2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ R/\pi^2 & \xrightarrow{\pi} & R/\pi^2 \end{pmatrix} .$$

Cf. §2.1.

Remark 10 Keep the assumptions of Proposition 9.

We have $(\Omega \circ S)^2 Y \simeq (\Omega \circ S) Y$ for $Y \in \operatorname{Ob} \underline{\text{mod}}\text{-}B$.

Given an R/π^3-module X, we write $\bar{X} := X/\pi^2 X$ and $\operatorname{Ann}_\pi \bar{X} := \{\, \bar{x} \in \bar{X} \,:\, \pi\bar{x} = 0 \,\}$.

The unit of the adjunction $S \dashv \Omega$ at a B-module $X \xrightarrow{f} Y$ is represented by the composite

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & & \downarrow \\ \bar{X} & \xrightarrow{\quad \bar{f} \quad} & \bar{Y} \\ \downarrow & & \downarrow \\ \bar{X}/\pi \operatorname{Kern} \bar{f} & \longrightarrow & \bar{Y}/(\operatorname{Ann}_\pi \bar{X})\bar{f} \end{array} ,$$

where the vertical maps are the respective residue class maps, and the middle and lower horizontal maps are the induced maps.

I do not know why.

2.3 Further examples of left adjoints

Let

$$\begin{array}{lcl} C_1 &:=& A/(\pi^2 f) &=& (R/\pi^3)(e \xrightarrow{a} f)/(\pi^2 f) \\ C_2 &:=& A/(\pi f) &=& (R/\pi^3)(e \xrightarrow{a} f)/(\pi f) \\ C_3 &:=& A/(\pi^2 e, \pi^2 f) &=& (R/\pi^2)(e \xrightarrow{a} f) \\ C_4 &:=& A/(\pi a) &=& (R/\pi^3)(e \xrightarrow{a} f)/(\pi a) \\ C_5 &:=& A/(\pi a, \pi^2 f) &=& (R/\pi^3)(e \xrightarrow{a} f)/(\pi a, \pi^2 f) \\ C_6 &:=& A/(\pi^2 e) &=& (R/\pi^3)(e \xrightarrow{a} f)/(\pi^2 e) \\ C_7 &:=& A/(\pi e) &=& (R/\pi^3)(e \xrightarrow{a} f)/(\pi e) \\ C_8 &:=& A/(\pi^2 e, \pi a) &=& (R/\pi^3)(e \xrightarrow{a} f)/(\pi^2 e, \pi a) \end{array}$$

Proposition 11 *Suppose given a prime $p \in [2, 997]$. Suppose that $R = \mathbf{F}_p[X]$ and $\pi = X$.*

Then the Heller operator $\Omega : \underline{\mathrm{mod}}\text{-}C_j \longrightarrow \underline{\mathrm{mod}}\text{-}C_j$ has a left adjoint for $j \in [1, 8]$.

Remark 12 Keep the assumptions of Proposition 11.

We have $(\Omega \circ S)^2 Y \simeq (\Omega \circ S)Y$ for $Y \in \mathrm{Ob}\, \underline{\mathrm{mod}}\text{-}C_j$ for $j \in [1, 8] \smallsetminus \{5\}$.

For $j = 5$, we have

$$\begin{array}{lcl} (\Omega \circ S)X_{10} &=& X_{10} \oplus X_{21} \\ (\Omega \circ S)X_{21} &=& X_{21} \end{array}$$

in the notation of §2.1.1, i.e.

$$\begin{array}{lcl} (\Omega \circ S)(R/\pi^2 \longrightarrow 0) &=& (R/\pi^2 \longrightarrow 0) \oplus (R/\pi \longrightarrow 0) \\ (\Omega \circ S)(R/\pi \longrightarrow 0) &=& (R/\pi \longrightarrow 0)\ . \end{array}$$

2.4 Counterexample: no right adjoint

Recall from §2.3 that $C_3 = (R/\pi^2)(e \xrightarrow{a} f)$. As representatives of isoclasses of nonprojective C_3-modules we obtain, in the notation of §2.1.1,

$$\begin{array}{lclcl} Y_1 &:=& X_1 &=& (R/\pi \xrightarrow{1} R/\pi) \\ Y_2 &:=& X_3 &=& (R/\pi^2 \xrightarrow{1} R/\pi) \\ Y_3 &:=& X_6 &=& (R/\pi \xrightarrow{\pi} R/\pi^2) \\ Y_4 &:=& X_{10} &=& (R/\pi \longrightarrow 0) \end{array} \qquad \begin{array}{lclcl} Y_5 &:=& X_{21} &=& (R/\pi \longrightarrow 0) \\ Y_6 &:=& X_{22} &=& (0 \longrightarrow R/\pi) \\ Y_7 &:=& X_{25} &=& (R/\pi^2 \xrightarrow{\pi} R/\pi^2)\ . \end{array}$$

Remark 13 *Suppose that $R = \mathbf{F}_3[X]$ and $\pi = X$.*

The functor $\Omega : \underline{\mathrm{mod}}\text{-}C_3 \longrightarrow \underline{\mathrm{mod}}\text{-}C_3$ does not have a right adjoint.

Proof. Magma yields

$$H \;:=\; \big(\dim_{\mathbf{F}_3}(\,{}_{\underline{\mathrm{mod}}\text{-}C_3}(Y_i\,, Y_j))\big)_{i,j} \;=\; \begin{pmatrix} 1&0&1&0&1&0&0 \\ 1&1&1&1&1&0&1 \\ 0&1&1&0&1&1&0 \\ 0&1&0&2&1&0&1 \\ 0&1&0&1&1&0&1 \\ 1&1&0&0&0&1&0 \\ 0&1&1&1&1&1&1 \end{pmatrix} \;\in\; (\mathbf{Z}_{\geqslant 0})^{7\times 7}$$

and

$$H' \;:=\; \left(\dim_{\mathbf{F}_3}(\,\underline{\mathrm{mod}}\text{-}C_3(\Omega Y_i\,,Y_j))\right)_{i,j} \;=\; \begin{pmatrix} 1&0&1&0&1&0&0\\ 1&1&0&0&0&1&0\\ 0&1&1&0&1&1&0\\ 0&0&0&0&0&0&0\\ 0&1&1&0&1&1&0\\ 1&1&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \;\in\; (\mathbf{Z}_{\geqslant 0})^{7\times 7}\,.$$

Assume that Ω has right adjoint $T : \underline{\mathrm{mod}}\text{-}C_3 \longrightarrow \underline{\mathrm{mod}}\text{-}C_3$.

Write $TY_j \simeq \bigoplus_{k\in[1,7]} Y_k^{\oplus u_{k,j}}$ for $j \in [1,7]$, where $U := (u_{k,j})_{k,j} \in (\mathbf{Z}_{\geqslant 0})^{7\times 7}$. We obtain

$$\begin{array}{rcl} H' & = & \left(\dim_{\mathbf{F}_3}(\,\underline{\mathrm{mod}}\text{-}C_3(\Omega Y_i\,,Y_j))\right)_{i,j} \\ & = & \left(\dim_{\mathbf{F}_3}(\,\underline{\mathrm{mod}}\text{-}C_3(Y_i\,,TY_j))\right)_{i,j} \\ & = & \left(\dim_{\mathbf{F}_3}(\,\underline{\mathrm{mod}}\text{-}C_3(Y_i\,,\bigoplus_{k\in[1,7]} Y_k^{\oplus u_{k,j}}))\right)_{i,j} \\ & = & \left(\sum_{k\in[1,7]} \dim_{\mathbf{F}_3}(\,\underline{\mathrm{mod}}\text{-}C_3(Y_i\,,Y_k))\cdot u_{k,j}\right)_{i,j} \\ & = & H\cdot U\,. \end{array}$$

So every column of H' is a linear combination of columns in H with coefficients in $\mathbf{Z}_{\geqslant 0}$. However, the third column of H' would afford a coefficient $\in \mathbf{Z}_{>0}$ at the first, third or fifth column of H because its first entry equals 1. But then its second entry would also be in $\mathbf{Z}_{>0}$, because these columns of H all have second entry equal to 1. But this second entry equals 0. We have arrived at a *contradiction.* ▫

References

[1] Bosma, W.; Cannon, J.; Playoust, C., *The Magma algebra system. I. The user language,* J. Symbolic Comput. 24 (3-4), p. 235–265, 1997 (cf. magma.maths.usyd.edu.au).

[2] Bühler, T., *Exact Categories,* Exp. Math. 28 (1), p. 1–69, 2010.

[3] Heller, A., *The loop-space functor in Homological Algebra,* Trans. Am. Math. Soc. 96, p. 382–394, 1960.

[4] Künzer, M., *Heller triangulated categories,* Homology, Homotopy and Applications 9 (2), p. 233–320, 2007.

[5] Quillen, D., *Higher algebraic K-theory: I,* SLN 341, p. 85–147, 1973.

[6] Raussen, M; Skau, C., *Interview with Jean-Pierre Serre,* Notices AMS 51 (2), p. 210-214, 2004.

[7] Ringel, C. M., Schmidmeier, M., *Invariant subspaces of nilpotent linear operators, I,* J. reine angew. Math. 614, p. 1–52, 2008.

[8] Rowen, L. H., *Ring Theory,* Student Edition, Acad. Press, 1991.

[9] Schubert, H., *Categories,* Springer, 1972.

Matthias Künzer
Lehrstuhl D für Mathematik
RWTH Aachen
Templergraben 64
D-52062 Aachen
kuenzer@math.rwth-aachen.de
www.math.rwth-aachen.de/∼kuenzer