# A counterexample on nilpotent endomorphisms in triangulated categories

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**Abstract**

We give a counterexample to an assertion on nilpotent endomorphisms of degree 3 in triangulated categories. Roughly speaking, it is in general not possible to lower the nilpotency degree by passing to a certain cone.

## 1 Nilpotency degree 2 in triangulated categories

Let $\mathcal{C}$ be a Verdier triangulated category.

Suppose given $X \in \operatorname{Ob}\mathcal{C}$ and $d \in \operatorname{End}_{\mathcal{C}} X$ such that $d^2 = 0$. Let $X \xrightarrow{d} X \xrightarrow{f} C \xrightarrow{g} X^{+1}$ be a (distinguished) triangle. M. Breuning made use of the following endomorphism of triangles.

$$\begin{array}{ccccccc} X & \xrightarrow{d} & X & \xrightarrow{f} & C & \xrightarrow{g} & X^{+1} \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ X & \xrightarrow{d} & X & \xrightarrow{f} & C & \xrightarrow{g} & X^{+1} \end{array}$$

So the pair $(0, d)$ can be extended to an endomorphism of triangles $(0, d, e)$ such that $e = e^1 = 0$.

## 2 Nilpotency degree 3 in abelian categories

Let $\mathcal{A}$ be an abelian category.

Suppose given $X \in \operatorname{Ob}\mathcal{A}$ and $d \in \operatorname{End}_{\mathcal{A}} X$ such that $d^3 = 0$. Let $f$ be a cokernel of $d$. We obtain an endomorphism of sequences.

$$\begin{array}{ccccc} X & \xrightarrow{d^2} & X & \xrightarrow{f} & C \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ X & \xrightarrow{d^2} & X & \xrightarrow{f} & C \end{array}$$

Now $fe^2 = d^2 f = 0$, whence $e^2 = 0$. So the pair $(0, d)$ can be (uniquely) extended to an endomorphism of sequences $(0, d, e)$ such that $e^2 = 0$.

# 3 Nilpotency degree 3 in triangulated categories

Let $\mathcal{C}$ be a Verdier triangulated category.

Suppose given $X \in \operatorname{Ob}\mathcal{C}$ and $d \in \operatorname{End}_{\mathcal{C}} X$ such that $d^3 = 0$. Let $X \xrightarrow{d} X \xrightarrow{f} C \xrightarrow{g} X^{+1}$ be a triangle. There exists $C \xrightarrow{e} C$ fitting into an endomorphism of triangles as follows.

$$\begin{array}{ccccccc} X & \xrightarrow{d^2} & X & \xrightarrow{f} & C & \xrightarrow{g} & X^{+1} \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle 0} \\ X & \xrightarrow{d^2} & X & \xrightarrow{f} & C & \xrightarrow{g} & X^{+1} \end{array}$$

Now, $eg = 0$, whence $e = bf$ for some $C \xrightarrow{b} X$. Hence $e^3 = bfe^2 = bd^2f = 0$.

Motivated by §1 and §2, we consider the following

**Assertion.** *There exists an endomorphism $C \xrightarrow{\tilde{e}} C$ with $\tilde{e}^2 = 0$ such that $(0, d, \tilde{e})$ is an endomorphism of triangles.*

*Counterexample.* Let $\mathcal{C} = \underline{\mathbf{Z}/64\text{-mod}}$; cf. [**1**]. Let $(X \xrightarrow{d} X) := (\mathbf{Z}/8 \xrightarrow{2} \mathbf{Z}/8)$. Suppose given the following endomorphism

$$\begin{array}{ccccccc} \mathbf{Z}/8 & \xrightarrow{4} & \mathbf{Z}/8 & \xrightarrow{(1\,2)} & \mathbf{Z}/4 \oplus \mathbf{Z}/16 & \xrightarrow{\binom{-2}{1}} & \mathbf{Z}/8 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle \tilde{e} = \binom{a\ 4b}{c\ d}} & & \downarrow{\scriptstyle 0} \\ \mathbf{Z}/8 & \xrightarrow{4} & \mathbf{Z}/8 & \xrightarrow{(1\,2)} & \mathbf{Z}/4 \oplus \mathbf{Z}/16 & \xrightarrow{\binom{-2}{1}} & \mathbf{Z}/8 \end{array}$$

of triangles, in which $a$, $b$, $c$, $d \in \mathbf{Z}/4$.

The middle quadrangle yields $\binom{a+2c\ \ 4b+2d} = \binom{2\,4}$, as morphisms from $\mathbf{Z}/8$ to $\mathbf{Z}/4 \oplus \mathbf{Z}/16$. So $a + 2c \equiv_4 2$ and $4b + 2d \equiv_8 4$. Hence $a = 2 - 2c$ and $d = 2 - 2b$. Thus $\tilde{e} = \binom{2-2c\ \ 4b}{c\ \ 2-2b}$.

Now the right hand side quadrangle yields $\binom{-4+4b+4c}{2-2b-2c} = \binom{0}{0}$ as morphisms from $\mathbf{Z}/4 \oplus \mathbf{Z}/16$ to $\mathbf{Z}/8$. So $b + c \equiv_2 1$.

We calculate $\tilde{e}^2 = \begin{pmatrix} 0 & 8b(b+c) \\ 2c(b+c) & 0 \end{pmatrix} = \binom{0\ \ 8b}{2c\ \ 0}$, which is nonzero because of $b + c \equiv_2 1$. ▫

## References

[1] Künzer, M., *Nonisomorphic Verdier octahedra on the same base,* J. Homotopy and Related Structures 4(1), p. 7–38, 2009.

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