A counterexample on nilpotent endomorphisms in triangulated categories

Matthias Künzer

September 16, 2009

Abstract

We give a counterexample to an assertion on nilpotent endomorphisms of degree 3 in triangulated categories. Roughly speaking, it is in general not possible to lower the nilpotency degree by passing to a certain cone.

1 Nilpotency degree 2 in triangulated categories

Let $\mathcal{C}$ be a Verdier triangulated category.

Suppose given $X \in \operatorname{Ob}\mathcal{C}$ and $d \in \operatorname{End}_{\mathcal{C}} X$ such that $d^2 = 0$. Let $X \xrightarrow{d} X \xrightarrow{f} C \xrightarrow{g} X^{+1}$ be a (distinguished) triangle. M. Breuning made use of the following endomorphism of triangles.

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[d, "0"] & X \arrow[r, "f"] \arrow[d, "d"] & C \arrow[r, "g"] \arrow[d, "0"] & X^{+1} \arrow[d, "0"] \\
X \arrow[r, "d"] & X \arrow[r, "f"] & C \arrow[r, "g"] & X^{+1}
\end{tikzcd}$$

So the pair $(0, d)$ can be extended to an endomorphism of triangles $(0, d, e)$ such that $e = e^1 = 0$.

2 Nilpotency degree 3 in abelian categories

Let $\mathcal{A}$ be an abelian category.

Suppose given $X \in \operatorname{Ob}\mathcal{A}$ and $d \in \operatorname{End}_{\mathcal{A}} X$ such that $d^3 = 0$. Let f be a cokernel of d. We obtain an endomorphism of sequences.

$$\begin{tikzcd}
X \arrow[r, "d^2"] \arrow[d, "0"] & X \arrow[r, "f"] \arrow[d, "d"] & C \arrow[d, "e"] \\
X \arrow[r, "d^2"] & X \arrow[r, "f"] & C
\end{tikzcd}$$

Now $fe^2 = d^2 f = 0$, whence $e^2 = 0$. So the pair $(0, d)$ can be (uniquely) extended to an endomorphism of sequences $(0, d, e)$ such that $e^2 = 0$.

3 Nilpotency degree 3 in triangulated categories

Let $\mathcal{C}$ be a Verdier triangulated category.

Suppose given $X \in \operatorname{Ob}\mathcal{C}$ and $d \in \operatorname{End}_{\mathcal{C}} X$ such that $d^3 = 0$. Let $X \xrightarrow{d} X \xrightarrow{f} C \xrightarrow{g} X^{+1}$ be a triangle. There exists $C \xrightarrow{e} C$ fitting into an endomorphism of triangles as follows.

$$\begin{tikzcd} X \arrow[r, "d^2"] \arrow[d, "0"] & X \arrow[r, "f"] \arrow[d, "d"] & C \arrow[r, "g"] \arrow[d, "e"] & X^{+1} \arrow[d, "0"] \\ X \arrow[r, "d^2"] & X \arrow[r, "f"] & C \arrow[r, "g"] & X^{+1} \end{tikzcd}$$

Now, $eg = 0$, whence $e = bf$ for some $C \xrightarrow{b} X$. Hence $e^3 = bfe^2 = bd^2f = 0$.

Motivated by §1 and §2, we consider the following

Assertion. *There exists an endomorphism $C \xrightarrow{\tilde{e}} C$ with $\tilde{e}^2 = 0$ such that $(0, d, \tilde{e})$ is an endomorphism of triangles.*

Counterexample. Let $\mathcal{C} = \underline{\mathbf{Z}/64\text{-mod}}$; cf. [**1**]. Let $(X \xrightarrow{d} X) := (\mathbf{Z}/8 \xrightarrow{2} \mathbf{Z}/8)$. Suppose given the following endomorphism

$$\begin{tikzcd} \mathbf{Z}/8 \arrow[r, "4"] \arrow[d, "0"] & \mathbf{Z}/8 \arrow[r, "(1\ 2)"] \arrow[d, "2"] & \mathbf{Z}/4 \oplus \mathbf{Z}/16 \arrow[r, "\left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right)"] \arrow[d, "\tilde{e} = \left(\begin{smallmatrix} a & 4b \\ c & d \end{smallmatrix}\right)"'] & \mathbf{Z}/8 \arrow[d, "0"] \\ \mathbf{Z}/8 \arrow[r, "4"] & \mathbf{Z}/8 \arrow[r, "(1\ 2)"] & \mathbf{Z}/4 \oplus \mathbf{Z}/16 \arrow[r, "\left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right)"] & \mathbf{Z}/8 \end{tikzcd}$$

of triangles, in which a, b, c, $d \in \mathbf{Z}/4$.

The middle quadrangle yields $\left(\begin{smallmatrix} a+2c & 4b+2d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 4 \end{smallmatrix}\right)$, as morphisms from $\mathbf{Z}/8$ to $\mathbf{Z}/4 \oplus \mathbf{Z}/16$. So $a + 2c \equiv_4 2$ and $4b + 2d \equiv_8 4$. Hence $a = 2 - 2c$ and $d = 2 - 2b$. Thus $\tilde{e} = \left(\begin{smallmatrix} 2-2c & 4b \\ c & 2-2b \end{smallmatrix}\right)$.

Now the right hand side quadrangle yields $\left(\begin{smallmatrix} -4+4b+4c \\ 2-2b-2c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ as morphisms from $\mathbf{Z}/4 \oplus \mathbf{Z}/16$ to $\mathbf{Z}/8$. So $b + c \equiv_2 1$.

We calculate $\tilde{e}^2 = \begin{pmatrix} 0 & 8b(b+c) \\ 2c(b+c) & 0 \end{pmatrix} = \left(\begin{smallmatrix} 0 & 8b \\ 2c & 0 \end{smallmatrix}\right)$, which is nonzero because of $b + c \equiv_2 1$. $\square$

References

[1] Künzer, M., *Nonisomorphic Verdier octahedra on the same base,* J. Homotopy and Related Structures 4(1), p. 7–38, 2009.

MSC2000: 18E30.