A counterexample on nilpotent endomorphisms in triangulated categories

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Abstract

We give a counterexample to an assertion on nilpotent endomorphisms of degree 3 in triangulated categories. Roughly speaking, it is in general not possible to lower the nilpotency degree by passing to a certain cone.

1 Nilpotency degree 2 in triangulated categories

Let $C$ be a Verdier triangulated category.

Suppose given $X \in \text{Ob} C$ and $d \in \text{End}_C X$ such that $d^2 = 0$. Let $X \xrightarrow{d} X \xrightarrow{f} C \xrightarrow{g} X^{+1}$ be a (distinguished) triangle. M. Breuning made use of the following endomorphism of triangles.

\[
\begin{array}{ccc}
X & \xrightarrow{d} & X \\
\downarrow 0 & & \downarrow d \\
X & \xrightarrow{d} & X \\
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{f} & C \\
\downarrow 0 & & \downarrow 0 \\
X & \xrightarrow{f} & C \\
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{g} & X^{+1} \\
\downarrow 0 & & \downarrow 0 \\
X & \xrightarrow{g} & X^{+1} \\
\end{array}
\]

So the pair $(0, d)$ can be extended to an endomorphism of triangles $(0, d, e)$ such that $e = e^1 = 0$.

2 Nilpotency degree 3 in abelian categories

Let $A$ be an abelian category.

Suppose given $X \in \text{Ob} A$ and $d \in \text{End}_A X$ such that $d^3 = 0$. Let $f$ be a cokernel of $d$. We obtain an endomorphism of sequences.

\[
\begin{array}{ccc}
X & \xrightarrow{d^2} & X \\
\downarrow 0 & & \downarrow d \\
X & \xrightarrow{d^2} & X \\
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{f} & C \\
\downarrow e & & \downarrow e \\
X & \xrightarrow{f} & C \\
\end{array}
\]

Now $fe^2 = d^2f = 0$, whence $e^2 = 0$. So the pair $(0, d)$ can be (uniquely) extended to an endomorphism of sequences $(0, d, e)$ such that $e^2 = 0$. 
3 Nilpotency degree 3 in triangulated categories

Let \( C \) be a Verdier triangulated category.

Suppose given \( X \in \text{Ob} \, C \) and \( d \in \text{End}_C \, X \) such that \( d^3 = 0 \). Let \( X \xrightarrow{d} X \xrightarrow{f} C \xrightarrow{g} X^{+1} \) be a triangle. There exists \( C \xrightarrow{e} C \) fitting into an endomorphism of triangles as follows.

\[
\begin{array}{c}
X \xrightarrow{d^2} X \xrightarrow{f} C \xrightarrow{g} X^{+1} \\
\downarrow 0 \quad \downarrow d \quad \downarrow e \quad \downarrow 0 \\
X \xrightarrow{d^2} X \xrightarrow{f} C \xrightarrow{g} X^{+1}
\end{array}
\]

Now, \( eg = 0 \), whence \( e = bf \) for some \( C \xrightarrow{b} X \). Hence \( e^3 = bfe^2 = bd^2 f = 0 \).

Motivated by §1 and §2, we consider the following

**Assertion.** There exists an endomorphism \( C \xrightarrow{\tilde{e}} C \) with \( \tilde{e}^2 = 0 \) such that \( (0, d, \tilde{e}) \) is an endomorphism of triangles.

**Counterexample.** Let \( C = \mathbb{Z}/64\text{-mod} \); cf. [1]. Let \( (X \xrightarrow{d} X) := (\mathbb{Z}/8 \xrightarrow{2} \mathbb{Z}/8) \). Suppose given the following endomorphism

\[
\begin{array}{c}
\mathbb{Z}/8 \xrightarrow{4} \mathbb{Z}/8 \xrightarrow{(12)} \mathbb{Z}/4 \oplus \mathbb{Z}/16 \xrightarrow{(-2)} \mathbb{Z}/8 \\
\downarrow 0 \quad \downarrow 2 \quad \downarrow \tilde{e} = \begin{pmatrix} a & 4b \\ c & d \end{pmatrix} \quad \downarrow 0 \\
\mathbb{Z}/8 \xrightarrow{4} \mathbb{Z}/8 \xrightarrow{(12)} \mathbb{Z}/4 \oplus \mathbb{Z}/16 \xrightarrow{(-2)} \mathbb{Z}/8
\end{array}
\]

of triangles, in which \( a, b, c, d \in \mathbb{Z}/4 \).

The middle quadrangle yields \( (a+2c, 4b+2d) = (2, 4) \), as morphisms from \( \mathbb{Z}/8 \) to \( \mathbb{Z}/4 \oplus \mathbb{Z}/16 \). So \( a + 2c \equiv 2 \) and \( 4b + 2d \equiv 8 \). Hence \( a = 2 - 2c \) and \( d = 2 - 2b \). Thus \( \tilde{e} = \begin{pmatrix} 2-2c & 4b \\ -2 & 2-2b \end{pmatrix} \).

Now the right hand side quadrangle yields \( \begin{pmatrix} -4+4b+4c \\ 2-2b-2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) as morphisms from \( \mathbb{Z}/4 \oplus \mathbb{Z}/16 \) to \( \mathbb{Z}/8 \). So \( b + c \equiv 2 \).

We calculate \( \tilde{e}^2 = \begin{pmatrix} 0 & 8b(b+c) \\ 2c(b+c) & 0 \end{pmatrix} \), which is nonzero because of \( b + c \equiv 2 \).

**References**