On the structure of the 2-category of additive categories

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Contents

Chapter 0

Introduction

0.1 Additive categories and additive functors

An additive category $\mathcal{A}$ is a category with zero object, in which any pair of objects has a direct sum. This yields an associative and commutative addition on the set of morphisms ${}_{\mathcal{A}}(X, X')$ between fixed objects X and X'. We require for each identity an additive inverse to exist, in that we require $\left(\begin{smallmatrix} \mathrm{id}_X & 0 \\ \mathrm{id}_X & \mathrm{id}_X \end{smallmatrix}\right)$ to be an isomorphism for $X \in \mathrm{Ob}(\mathcal{A})$.

Consequently, in an additive category $\mathcal{A}$, the set of morphisms ${}_{\mathcal{A}}(X, X')$ is an abelian group. Composition is distributive with respect to addition.

Given additive categories $\mathcal{A}$ and $\mathcal{B}$, a functor $F : \mathcal{A} \to \mathcal{B}$ is called additive if it maps zero objects to zero objects and if it is compatible with direct sums of each pair of objects in $\mathcal{A}$.

Equivalently, F is additive if it is compatible with addition of morphisms.

0.2 Pure short exact sequences

Suppose given additive categories $\mathcal{A}'$, $\mathcal{A}$ and $\mathcal{A}''$.

Suppose given additive functors $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$.

The full image of F is the full additive subcategory $\mathrm{Im}(F) \subseteq \mathcal{A}$ with

$$\mathrm{Ob}(\mathrm{Im}(F)) \;=\; \{\, X \in \mathrm{Ob}(\mathcal{A}) \,:\, X \simeq X'F \text{ for some } X' \in \mathrm{Ob}(\mathcal{A}') \,\}\,.$$

The kernel of G is the full additive subcategory $\mathrm{Kern}(G) \subseteq \mathcal{A}$ with

$$\mathrm{Ob}(\mathrm{Kern}(G)) \;=\; \{\, X \in \mathrm{Ob}(\mathcal{A}) \,:\, XG \text{ is a zero object in } \mathcal{A}'' \,\}\,.$$

The sequence $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$ is called pure short exact if (P 1–4) hold.

(P 1) The functor F is full and faithful.

(P 2) The functor G is full and dense.

(P 3) We have $\mathrm{Im}(F) = \mathrm{Kern}(G)$.

(P 4) Suppose given a morphism $X \stackrel{u}{\to} \tilde{X}$ in $\mathcal{A}$ such that $uG = 0$. Then there exists $Z' \in \mathrm{Ob}(\mathcal{A}')$ and morphisms $X \stackrel{a}{\to} Z'F \stackrel{\tilde{a}}{\to} \tilde{X}$ such that $a \cdot \tilde{a} = u$.

A functor $\mathcal{A}' \stackrel{F}{\to} \mathcal{A}$ is called a pure monofunctor if there exists a pure short exact sequence $\mathcal{A}' \stackrel{F}{\to} \mathcal{A} \stackrel{G}{\to} \mathcal{A}''$. To indicate that F is a pure monofunctor, we often write $\mathcal{A}' \stackrel{F}{\bullet\!\!\to} \mathcal{A}$.

A functor $\mathcal{A} \stackrel{G}{\to} \mathcal{A}''$ is called a pure epifunctor if there exists a pure short exact sequence $\mathcal{A}' \stackrel{F}{\to} \mathcal{A} \stackrel{G}{\to} \mathcal{A}''$. To indicate that G is a pure epifunctor, we often write $\mathcal{A} \stackrel{G}{\to\!\!\!\!\!+} \mathcal{A}''$.

For instance, suppose given an additive category $\mathcal{A}$ and a full additive subcategory $\mathcal{N} \subseteq \mathcal{A}$ closed under retracts. Then we have the pure short exact sequence

$$\mathcal{N} \stackrel{I}{\to} \mathcal{A} \stackrel{R}{\to} \mathcal{A}/\mathcal{N}\ ,$$

in which I denotes the inclusion functor and R denotes the residue class functor to the factor category $\mathcal{A}/\mathcal{N}$.

0.3 Properties of pure short exact sequences

We collect some properties of pure short exact sequences.

0.3.1 Universal properties

Suppose given a pure short exact sequence $\mathcal{A}' \stackrel{F}{\bullet\!\!\to} \mathcal{A} \stackrel{G}{\to\!\!\!\!\!+} \mathcal{A}''$.

(1) The functor F has the universal property of a kernel of G, up to isomorphy of functors.

(2) The functor G has the universal property of a cokernel of F, up to isomorphy of functors.

0.3.2 Composition properties

Suppose given additive categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Suppose given additive functors

$$\mathcal{A} \stackrel{F}{\to} \mathcal{B} \stackrel{G}{\to} \mathcal{C}\ .$$

(1) Suppose G to be a pure monofunctor.

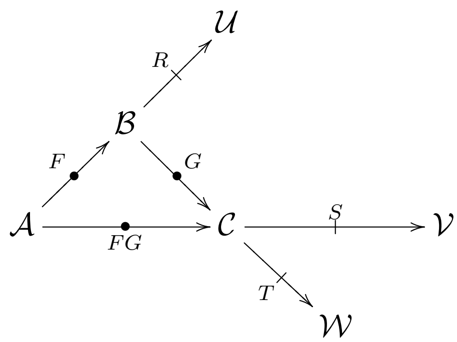
Then F is a pure monofunctor if and only if FG is a pure monofunctor.

(2) Suppose F to be a pure epifunctor.

Then G is a pure epifunctor if and only if FG is a pure epifunctor.

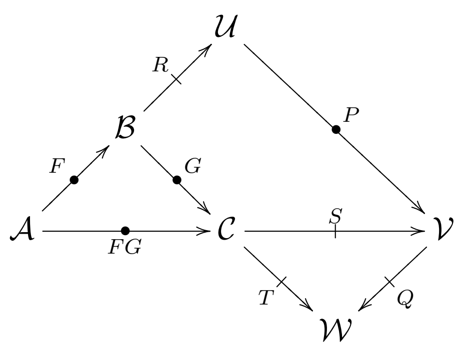
0.3.3 Noetherian properties

(1) Suppose given the following diagram of additive categories and additive functors.

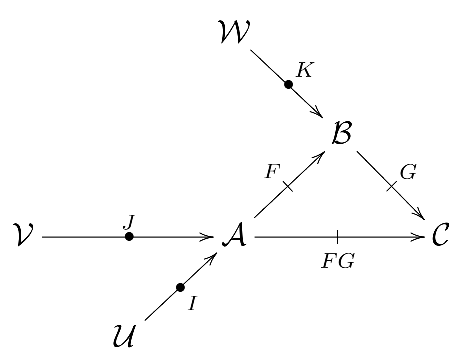


Suppose that $\mathcal{A} \overset{F}{\bullet\!\!\twoheadrightarrow} \mathcal{B} \overset{R}{\twoheadrightarrow} \mathcal{U}$ and $\mathcal{A} \overset{FG}{\bullet\!\!\twoheadrightarrow} \mathcal{C} \overset{S}{\twoheadrightarrow} \mathcal{V}$ and $\mathcal{B} \overset{G}{\bullet\!\!\twoheadrightarrow} \mathcal{C} \overset{T}{\twoheadrightarrow} \mathcal{W}$ are pure short exact.

Then we obtain a pure short exact sequence $\mathcal{U} \overset{P}{\bullet\!\!\twoheadrightarrow} \mathcal{V} \overset{Q}{\twoheadrightarrow} \mathcal{W}$ making the following diagram commutative up to isomorphy.

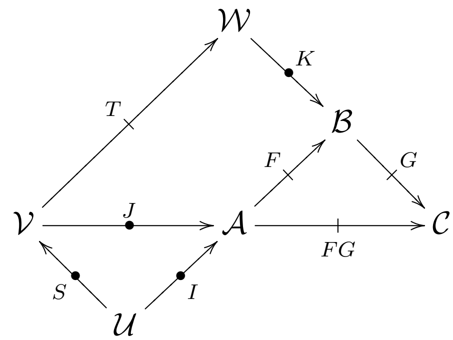


(2) Suppose given the following diagram of additive categories and additive functors.



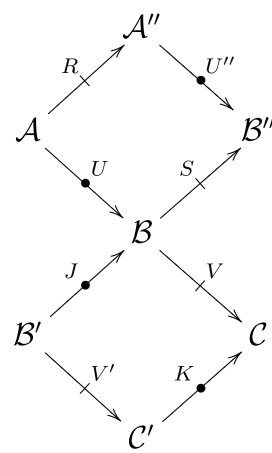
Suppose that $\mathcal{W} \overset{K}{\bullet\!\!\twoheadrightarrow} \mathcal{B} \overset{G}{\twoheadrightarrow} \mathcal{C}$ and $\mathcal{V} \overset{J}{\bullet\!\!\twoheadrightarrow} \mathcal{A} \overset{FG}{\twoheadrightarrow} \mathcal{C}$ and $\mathcal{U} \overset{I}{\bullet\!\!\twoheadrightarrow} \mathcal{A} \overset{F}{\twoheadrightarrow} \mathcal{B}$ are pure short exact.

Then we obtain a pure short exact sequence $\mathcal{U} \overset{S}{\twoheadrightarrow\!\!\!\!\bullet} \mathcal{V} \overset{T}{\longrightarrow\!\!\!\!+} \mathcal{W}$ making the following diagram commutative up to isomorphy.



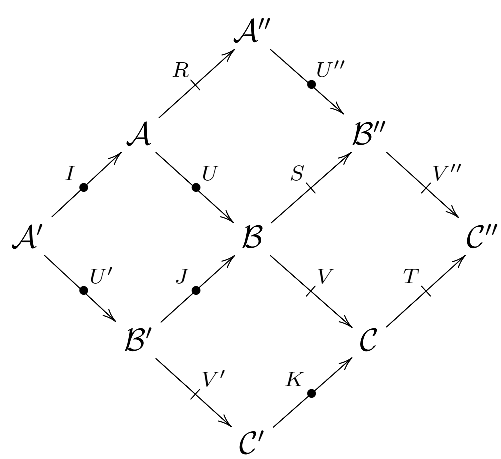
0.3.4 3×3-property

Suppose given the following diagram of additive categories and additive functors, commutative up to isomorphy.



Suppose that $\mathcal{A} \overset{U}{\to} \mathcal{B} \overset{V}{\to} \mathcal{C}$ and $\mathcal{B}' \overset{J}{\to} \mathcal{B} \overset{S}{\to} \mathcal{B}''$ are pure short exact.

Then there exist pure short exact sequences $\mathcal{A}' \overset{U'}{\to} \mathcal{B}' \overset{V'}{\to} \mathcal{C}'$ and $\mathcal{A}' \overset{I}{\to} \mathcal{A} \overset{R}{\to} \mathcal{A}''$ and $\mathcal{C}' \overset{K}{\to} \mathcal{C} \overset{T}{\to} \mathcal{C}''$ and $\mathcal{A}'' \overset{U''}{\to} \mathcal{B}'' \overset{V''}{\to} \mathcal{C}''$ making the following diagram commutative up to isomorphy.



Conventions

(1) Given sets X, Y and a map $X \xrightarrow{f} Y$, we write xf for the image of x under f.

(2) Given sets X, Y, Z and maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, we write their composite as $X \xrightarrow{f \cdot g} Z$. So for $x \in X$, we have $x(f \cdot g) = (xf)g$.

We often write $fg := f \cdot g$.

(3) Categories are understood to be small with respect to a given universe. I.e. for a category $\mathcal{C}$, both $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets in this universe.

(4) The identity morphism on an object X of a category $\mathcal{C}$ is written $\mathrm{id}_X^{\mathcal{C}}$.

We often abbreviate $1 := \mathrm{id}_X := \mathrm{id}_X^{\mathcal{C}}$.

(5) Given a category $\mathcal{C}$ and objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, we denote by ${}_{\mathcal{C}}(X,Y)$ the set of morphisms in $\mathcal{C}$ with source X and target Y.

(6) Given a category $\mathcal{C}$ and an isomorphism f in $\mathcal{C}$, we denote by f^- its inverse.

(7) Suppose given a category $\mathcal{C}$. Suppose given $X, X', Y, Y' \in \mathrm{Ob}(\mathcal{C})$.

The object X is called a retract of the object Y if there exist morphisms $X \xrightarrow{a} Y \xrightarrow{b} X$ such that $a \cdot b = \mathrm{id}_X$.

Note that if $Y \simeq Y'$, then X is a retract of Y if and only if X is a retract of Y'.

Note that if $X \simeq X'$, then X is a retract of Y if and only if X' is a retract of Y.

(8) Suppose given a category $\mathcal{C}$. Suppose given a full subcategory $\mathcal{D} \subseteq \mathcal{C}$.

We say that $\mathcal{D}$ is closed under retracts in $\mathcal{C}$ if the following property (CR) holds.

(CR) Suppose given $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$. Suppose that X is a retract of Y. Then $X \in \mathrm{Ob}(\mathcal{D})$.

(9) Suppose given a category $\mathcal{C}$. Suppose given a full subcategory $\mathcal{D} \subseteq \mathcal{C}$. Then the inclusion functor is denoted by $\mathrm{I}_{\mathcal{D},\mathcal{C}} : \mathcal{D} \to \mathcal{C}$. Often, we abbreviate $\mathrm{I} := \mathrm{I}_{\mathcal{D},\mathcal{C}}$.

(10) Suppose given categories $\mathcal{C}$ and $\mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called dense if for each $Y \in \mathrm{Ob}(\mathcal{D})$ there exists $X \in \mathrm{Ob}(\mathcal{C})$ such that $XF \simeq Y$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called an equivalence if F is full, faithful and dense.

A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{C}}$ and $GF \simeq \mathrm{id}_{\mathcal{D}}$.

(11) Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$. Suppose given $X, X' \in \mathrm{Ob}(\mathcal{C})$. We write

$$\begin{array}{rccl} F_{X,X'} & : & {}_{\mathcal{C}}(X,X') & \to & {}_{\mathcal{D}}(XF, X'F) \\ & & c & \mapsto & cF\,. \end{array}$$

(12) Suppose given categories $\mathcal{C}$ and $\mathcal{D}$. Suppose given functors $F, G : \mathcal{C} \to \mathcal{D}$.

A transformation from F to G is a tuple $a = (Xa : XF \to XG)_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms in $\mathcal{D}$ such that

$$\begin{array}{ccc} XF & \xrightarrow{Xa} & XG \\ {\scriptstyle uF}\downarrow & & \downarrow{\scriptstyle uG} \\ YF & \xrightarrow{Ya} & YG \end{array}$$

commutes for every morphism $X \overset{u}{\to} Y$ in $\mathcal{C}$.

This commutativity for every morphism $X \overset{u}{\to} Y$ in $\mathcal{C}$ is also referred to as the naturality of the tuple a.

A transformation $a = (Xa)_{X \in \mathrm{Ob}(\mathcal{C})}$ is called an isotransformation if Xa is an isomorphism for $X \in \mathrm{Ob}(\mathcal{C})$. In this case, $a^- = (X(a^-))_{X \in \mathrm{Ob}(\mathcal{C})} = ((Xa)^-)_{X \in \mathrm{Ob}(\mathcal{C})} = (Xa^-)_{X \in \mathrm{Ob}(\mathcal{C})}$ is an isotransformation, too.

See also §1.3.

Chapter 1

2-categories

1.1 Definition of a 2-category

Definition 1. A 2-*category* $\mathfrak{K}$ consists of the following data.

- A set of 0-*morphisms* $\operatorname{Mor}_0(\mathfrak{K})$, also called *objects*. We also write $\operatorname{Ob}(\mathfrak{K}) := \operatorname{Mor}_0(\mathfrak{K})$.
- A set of 1-*morphisms* $\operatorname{Mor}_1(\mathfrak{K})$.
- A set of 2-*morphisms* $\operatorname{Mor}_2(\mathfrak{K})$.
- Maps

$$\begin{array}{llll} \operatorname{Mor}_1(\mathfrak{K}) & \xrightarrow{\mathrm{s}_0^{\mathfrak{K}}} & \operatorname{Ob}(\mathfrak{K})\,, & \text{mapping a 1-morphism to its } \textit{source}, \\ \operatorname{Mor}_1(\mathfrak{K}) & \xleftarrow{\mathrm{i}_0^{\mathfrak{K}}} & \operatorname{Ob}(\mathfrak{K})\,, & \text{mapping an object to its } \textit{identity}, \\ \operatorname{Mor}_1(\mathfrak{K}) & \xrightarrow{\mathrm{t}_0^{\mathfrak{K}}} & \operatorname{Ob}(\mathfrak{K})\,, & \text{mapping a 1-morphism to its } \textit{target}. \end{array}$$

- Maps

$$\begin{array}{llll} \operatorname{Mor}_2(\mathfrak{K}) & \xrightarrow{\mathrm{s}_1^{\mathfrak{K}}} & \operatorname{Mor}_1(\mathfrak{K})\,, & \text{mapping a 2-morphism to its } \textit{source}, \\ \operatorname{Mor}_2(\mathfrak{K}) & \xleftarrow{\mathrm{i}_1^{\mathfrak{K}}} & \operatorname{Mor}_1(\mathfrak{K})\,, & \text{mapping an 1-morphism to its } \textit{identity}, \\ \operatorname{Mor}_2(\mathfrak{K}) & \xrightarrow{\mathrm{t}_1^{\mathfrak{K}}} & \operatorname{Mor}_1(\mathfrak{K})\,, & \text{mapping a 2-morphism to its } \textit{target}. \end{array}$$

- A map

$$\begin{array}{rcl} \{\, (F,G) \in \operatorname{Mor}_1(\mathfrak{K}) \times \operatorname{Mor}_1(\mathfrak{K}) \;:\; F\mathrm{t}_0^{\mathfrak{K}} = G\mathrm{s}_0^{\mathfrak{K}} \,\} & \xrightarrow{(\,\overset{\mathfrak{K}}{*}\,)} & \operatorname{Mor}_1(\mathfrak{K}) \\ (F,G) & \mapsto & F \overset{\mathfrak{K}}{*} G\,, \end{array}$$

called *composition* of 1-morphisms.

- A map

$$\begin{array}{rcl} \{\, (a,a') \in \operatorname{Mor}_2(\mathfrak{K}) \times \operatorname{Mor}_2(\mathfrak{K}) \;:\; a\mathrm{t}_1^{\mathfrak{K}} = a'\mathrm{s}_1^{\mathfrak{K}} \,\} & \xrightarrow{(\,\underset{\mathfrak{K}}{\cdot}\,)} & \operatorname{Mor}_2(\mathfrak{K}) \\ (a,a') & \mapsto & a \underset{\mathfrak{K}}{\cdot} a'\,, \end{array}$$

called *vertical composition* of 2-morphisms.

- A map

$$\{ (a,b) \in \mathrm{Mor}_2(\mathfrak{K}) \times \mathrm{Mor}_2(\mathfrak{K}) \,:\, a\mathrm{s}_1^{\mathfrak{K}}\mathrm{t}_0^{\mathfrak{K}} = b\mathrm{s}_1^{\mathfrak{K}}\mathrm{s}_0^{\mathfrak{K}} \} \xrightarrow{(\underset{\mathfrak{K}}{*})} \mathrm{Mor}_2(\mathfrak{K})$$
$$(a,b) \mapsto a \underset{\mathfrak{K}}{*} b\,,$$

called *horizontal composition* of 2-morphisms.

Given $F \in \mathrm{Mor}_1(\mathfrak{K})$ and $\mathcal{C}$, $\mathcal{D} \in \mathrm{Ob}(\mathfrak{K})$, to express that $F\mathrm{s}_0^{\mathfrak{K}} = \mathcal{C}$ and that $F\mathrm{t}_0^{\mathfrak{K}} = \mathcal{D}$, we write

$$\mathcal{C} \xrightarrow{F} \mathcal{D}\,.$$

Given $a \in \mathrm{Mor}_2(\mathfrak{K})$ and F, $F' \in \mathrm{Mor}_1(\mathfrak{K})$, to express that $a\mathrm{s}_1^{\mathfrak{K}} = F$ and that $a\mathrm{t}_1^{\mathfrak{K}} = F'$, we write

$$F \xrightarrow{a} F'\,.$$

Given $\mathcal{C} \in \mathrm{Ob}(\mathfrak{K})$, we write $\mathrm{id}_{\mathcal{C}} := \mathrm{id}_{\mathcal{C}}^{\mathfrak{K}} := \mathcal{C}\mathrm{i}_0^{\mathfrak{K}}$.
Given $F \in \mathrm{Mor}_1(\mathfrak{K})$, we write $\mathrm{id}_F := \mathrm{id}_F^{\mathfrak{K}} := F\mathrm{i}_1^{\mathfrak{K}}$.

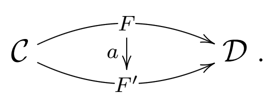
The following properties (1–13) are required to hold in $\mathfrak{K}$.

(1) We have $\mathrm{i}_0^{\mathfrak{K}} \cdot \mathrm{s}_0^{\mathfrak{K}} = \mathrm{id}_{\mathrm{Ob}(\mathfrak{K})}$ and $\mathrm{i}_0^{\mathfrak{K}} \cdot \mathrm{t}_0^{\mathfrak{K}} = \mathrm{id}_{\mathrm{Ob}(\mathfrak{K})}$. I.e. for $\mathcal{C} \in \mathrm{Ob}(\mathfrak{K})$, we have $\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C}$.

(2) We have $\mathrm{i}_1^{\mathfrak{K}} \cdot \mathrm{s}_1^{\mathfrak{K}} = \mathrm{id}_{\mathrm{Mor}_1(\mathfrak{K})}$ and $\mathrm{i}_1^{\mathfrak{K}} \cdot \mathrm{t}_1^{\mathfrak{K}} = \mathrm{id}_{\mathrm{Mor}_1(\mathfrak{K})}$. I.e. for $F \in \mathrm{Mor}_1(\mathfrak{K})$, we have $F \xrightarrow{\mathrm{id}_F} F$.

(3) We have $\mathrm{s}_1^{\mathfrak{K}} \cdot \mathrm{s}_0^{\mathfrak{K}} = \mathrm{t}_1^{\mathfrak{K}} \cdot \mathrm{s}_0^{\mathfrak{K}}$ and $\mathrm{s}_1^{\mathfrak{K}} \cdot \mathrm{t}_0^{\mathfrak{K}} = \mathrm{t}_1^{\mathfrak{K}} \cdot \mathrm{t}_0^{\mathfrak{K}}$.

So given $a \in \mathrm{Mor}_2(\mathfrak{K})$, writing $a\mathrm{s}_1^{\mathfrak{K}} =: F$, $a\mathrm{t}_1^{\mathfrak{K}} =: F'$, $F\mathrm{s}_0^{\mathfrak{K}} = F'\mathrm{s}_0^{\mathfrak{K}} =: \mathcal{C}$ and $F\mathrm{t}_0^{\mathfrak{K}} = F'\mathrm{t}_0^{\mathfrak{K}} =: \mathcal{D}$, we have



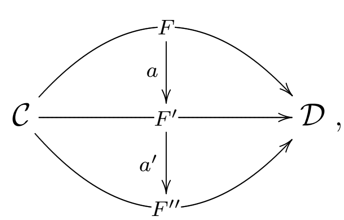
(4) Given 1-morphisms

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}\,,$$

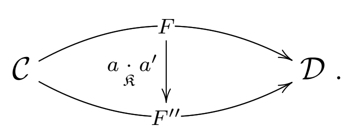
we have

$$\mathcal{C} \xrightarrow{F \overset{\mathfrak{K}}{*} G} \mathcal{E}\,.$$

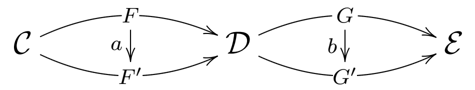
(5) Given 2-morphisms



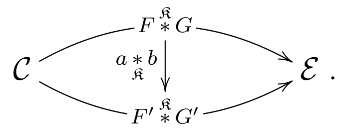
we have



(6) Given 2-morphisms



we have



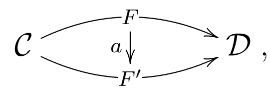
(7) Given a 1-morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$, we have $\mathrm{id}_{\mathcal{C}} \overset{\mathfrak{K}}{*} F = F$ and $F \overset{\mathfrak{K}}{*} \mathrm{id}_{\mathcal{D}} = F$.

(8) Given 1-morphisms

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \xrightarrow{H} \mathcal{B}\,,$$

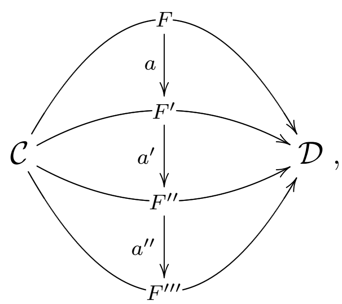
we have $(F \overset{\mathfrak{K}}{*} G) \overset{\mathfrak{K}}{*} H = F \overset{\mathfrak{K}}{*} (G \overset{\mathfrak{K}}{*} H) =: F \overset{\mathfrak{K}}{*} G \overset{\mathfrak{K}}{*} H$.

(9) Given a 2-morphism



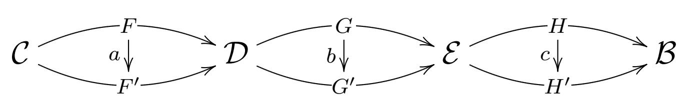
we have $\mathrm{id}_F \underset{\mathfrak{K}}{\cdot} a = a$ and $a \underset{\mathfrak{K}}{\cdot} \mathrm{id}_{F'} = a$ and $\mathrm{id}_{\mathrm{id}_{\mathcal{C}}} \underset{\mathfrak{K}}{*} a = a$ and $a \underset{\mathfrak{K}}{*} \mathrm{id}_{\mathrm{id}_{\mathcal{D}}} = a$.

(10) Given 2-morphisms



we have $(a \underset{\mathfrak{K}}{\cdot} a') \underset{\mathfrak{K}}{\cdot} a'' = a \underset{\mathfrak{K}}{\cdot} (a' \underset{\mathfrak{K}}{\cdot} a'') =: a \underset{\mathfrak{K}}{\cdot} a' \underset{\mathfrak{K}}{\cdot} a''$.

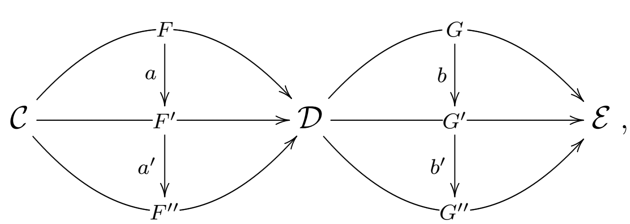
(11) Given 2-morphisms



we have $(a \underset{\mathfrak{K}}{*} b) \underset{\mathfrak{K}}{*} c = a \underset{\mathfrak{K}}{*} (b \underset{\mathfrak{K}}{*} c) =: a \underset{\mathfrak{K}}{*} b \underset{\mathfrak{K}}{*} c$.

(12) Given 1-morphisms $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$, we have $\mathrm{id}_F \underset{\mathfrak{K}}{*} \mathrm{id}_G = \mathrm{id}_{F \overset{\mathfrak{K}}{*} G}$.

(13) Given 2-morphisms



we have $(a \underset{\mathfrak{K}}{\cdot} a') \underset{\mathfrak{K}}{*} (b \underset{\mathfrak{K}}{\cdot} b') = (a \underset{\mathfrak{K}}{*} b) \underset{\mathfrak{K}}{\cdot} (a' \underset{\mathfrak{K}}{*} b')$.

1.2 1-morphism category

Suppose given a 2-category $\mathfrak{K}$. Suppose given $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$, $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}'' \in \mathrm{Ob}(\mathfrak{K})$.

Definition 2. Let

$$ {}_{\mathfrak{K}}(\mathcal{C},\mathcal{D}) $$

be the category having

$$ \mathrm{Ob}({}_{\mathfrak{K}}(\mathcal{C},\mathcal{D})) := \{ F \in \mathrm{Mor}_1(\mathfrak{K}) : \mathcal{C} \xrightarrow{F} \mathcal{D} \} $$

$$ \mathrm{Mor}({}_{\mathfrak{K}}(\mathcal{C},\mathcal{D})) := \{ a \in \mathrm{Mor}_2(\mathfrak{K}) : \mathcal{C} \underset{F'}{\overset{F}{\rightrightarrows}} \mathcal{D} \text{ with } a : F \Rightarrow F' \text{ for some } F,\, F' \in \mathrm{Ob}({}_{\mathfrak{K}}(\mathcal{C},\mathcal{D})) \} , $$

with, abbreviating $(\mathcal{C},\mathcal{D}) := {}_{\mathfrak{K}}(\mathcal{C},\mathcal{D})$,

$$ \mathrm{Mor}((\mathcal{C},\mathcal{D})) \xrightarrow{\mathrm{s}^{(\mathcal{C},\mathcal{D})}} \mathrm{Ob}((\mathcal{C},\mathcal{D})) : \left(\mathcal{C} \underset{F'}{\overset{F}{\rightrightarrows}} \mathcal{D},\ a{\downarrow} \right) \mapsto (\mathcal{C} \xrightarrow{F} \mathcal{D}) $$

$$ \mathrm{Mor}((\mathcal{C},\mathcal{D})) \xleftarrow{\mathrm{i}^{(\mathcal{C},\mathcal{D})}} \mathrm{Ob}((\mathcal{C},\mathcal{D})) : \left(\mathcal{C} \underset{F}{\overset{F}{\rightrightarrows}} \mathcal{D},\ \mathrm{id}_F{\downarrow} \right) \leftarrow\!\shortmid (\mathcal{C} \xrightarrow{F} \mathcal{D}) $$

$$ \mathrm{Mor}((\mathcal{C},\mathcal{D})) \xrightarrow{\mathrm{t}^{(\mathcal{C},\mathcal{D})}} \mathrm{Ob}((\mathcal{C},\mathcal{D})) : \left(\mathcal{C} \underset{F'}{\overset{F}{\rightrightarrows}} \mathcal{D},\ a{\downarrow} \right) \mapsto (\mathcal{C} \xrightarrow{F'} \mathcal{D}) $$

and with

$$ a \cdot a' := a \underset{\mathfrak{K}}{\cdot} a' $$

for

$$ \mathcal{C} \xrightarrow{F,\ F',\ F''} \mathcal{D} , \quad F \overset{a}{\Rightarrow} F' \overset{a'}{\Rightarrow} F'' . $$

By Definition 1.(2, 3, 5, 9, 10), this in fact defines the category ${}_{\mathfrak{K}}(\mathcal{C},\mathcal{D})$.

We often abbreviate $(\mathcal{C},\mathcal{D}) := {}_{\mathfrak{K}}(\mathcal{C},\mathcal{D})$.

Definition 3. Suppose given a 1-morphism $\mathcal{D} \xrightarrow{H} \mathcal{D}'$.

We obtain the functor

$$ {}_{\mathfrak{K}}(\mathcal{C},\mathcal{D}) \xrightarrow{{}_{\mathfrak{K}}(\mathcal{C},H)} {}_{\mathfrak{K}}(\mathcal{C},\mathcal{D}') $$

$$ \left(\mathcal{C} \underset{F'}{\overset{F}{\rightrightarrows}} \mathcal{D},\ a{\downarrow} \right) \mapsto \left(\mathcal{C} \underset{F' \overset{\mathfrak{K}}{*} H}{\overset{F \overset{\mathfrak{K}}{*} H}{\rightrightarrows}} \mathcal{D}',\ a \underset{\mathfrak{K}}{*} \mathrm{id}_H{\downarrow} \right) $$

by Definition 1.(4, 6, 9, 12, 13).

Definition 4. Suppose given a 1-morphism $\mathcal{C}' \xrightarrow{G} \mathcal{C}$.

$$ {}_{\mathfrak{K}}(\mathcal{C},\mathcal{D}) \xrightarrow{{}_{\mathfrak{K}}(G,\mathcal{D})} {}_{\mathfrak{K}}(\mathcal{C}',\mathcal{D}) $$

$$ \left(\mathcal{C} \underset{F'}{\overset{F}{\rightrightarrows}} \mathcal{D},\ a{\downarrow} \right) \mapsto \left(\mathcal{C}' \underset{G \overset{\mathfrak{K}}{*} F'}{\overset{G \overset{\mathfrak{K}}{*} F}{\rightrightarrows}} \mathcal{D},\ \mathrm{id}_G \underset{\mathfrak{K}}{*} a{\downarrow} \right) $$

by Definition 1.(4, 6, 9, 12, 13).

Remark 5.

(1) Suppose given 1-morphisms $\mathcal{D} \xrightarrow{H} \mathcal{D}' \xrightarrow{H'} \mathcal{D}''$. Then

$$_{\mathfrak{K}}(\mathcal{C}, H) \cdot {}_{\mathfrak{K}}(\mathcal{C}, H') \;=\; {}_{\mathfrak{K}}(\mathcal{C}, H \overset{\mathfrak{K}}{*} H') \;.$$

This follows by Definition 1.$(8, 11, 12)$.

(2) Suppose given 1-morphisms $\mathcal{C}'' \xrightarrow{G'} \mathcal{C}' \xrightarrow{G} \mathcal{C}$. Then

$$_{\mathfrak{K}}(G, \mathcal{D}) \cdot {}_{\mathfrak{K}}(G', \mathcal{D}) \;=\; {}_{\mathfrak{K}}(G' \overset{\mathfrak{K}}{*} G, \mathcal{D}) \;.$$

This follows by Definition 1.$(8, 11, 12)$.

(3) Suppose given 1-morphisms $\mathcal{C}' \xrightarrow{G} \mathcal{C}$ and $\mathcal{D} \xrightarrow{H} \mathcal{D}'$. Then

$$_{\mathfrak{K}}(G, \mathcal{D}) \cdot {}_{\mathfrak{K}}(\mathcal{C}, H) \;=\; {}_{\mathfrak{K}}(\mathcal{C}, H) \cdot {}_{\mathfrak{K}}(G, \mathcal{D}) \;=:\; {}_{\mathfrak{K}}(G, H) \;.$$

This follows by Definition 1.$(8, 11)$.

(4) We have $_{\mathfrak{K}}(\mathcal{C}, \mathrm{id}_{\mathcal{D}}) = \mathrm{id}_{\,_{\mathfrak{K}}(\mathcal{C},\mathcal{D})}$. This follows by Definition 1.$(1, 7, 9)$.

(5) We have $_{\mathfrak{K}}(\mathrm{id}_{\mathcal{C}}, \mathcal{D}) = \mathrm{id}_{\,_{\mathfrak{K}}(\mathcal{C},\mathcal{D})}$. This follows by Definition 1.$(1, 7, 9)$.

Definition 6. Suppose given a 1-morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$.

Then F is called a 1-isomorphism if there exists a 1-morphism $\mathcal{C} \xleftarrow{G} \mathcal{D}$ such that $F \overset{\mathfrak{K}}{*} G \simeq \mathrm{id}_{\mathcal{C}}$ in $_{\mathfrak{K}}(\mathcal{C}, \mathcal{C})$ and such that $G \overset{\mathfrak{K}}{*} F \simeq \mathrm{id}_{\mathcal{D}}$ in $_{\mathfrak{K}}(\mathcal{D}, \mathcal{D})$.

1.3 The 2-category of categories, called Cat

We recall the notions of functors and transformations in order to fix notation.

Reminder 7. Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{B}$ be categories.

(1) We have maps

$$\begin{array}{rcl} \mathrm{Mor}(\mathcal{C}) & \xrightarrow{\mathrm{s}^{\mathcal{C}}} & \mathrm{Ob}(\mathcal{C}) \\ (X \xrightarrow{f} Y) & \mapsto & X \\ \mathrm{Mor}(\mathcal{C}) & \xleftarrow{\mathrm{i}^{\mathcal{C}}} & \mathrm{Ob}(\mathcal{C}) \\ (X \xrightarrow{\mathrm{id}_X} X) & \leftarrow\!\shortmid & X \\ \mathrm{Mor}(\mathcal{C}) & \xrightarrow{\mathrm{t}^{\mathcal{C}}} & \mathrm{Ob}(\mathcal{C}) \\ (X \xrightarrow{f} Y) & \mapsto & Y \end{array}$$

(2) We have the composition map

$$\begin{array}{rcl} \{\, (u,v) \in \operatorname{Mor}(\mathcal{C}) \times \operatorname{Mor}(\mathcal{C}) \;:\; u\mathrm{t}^{\mathcal{C}} = v\mathrm{s}^{\mathcal{C}} \,\} & \xrightarrow{(\,\underset{\mathcal{C}}{\cdot}\,)} & \operatorname{Mor}(\mathcal{C}) \\ (u,v) & \mapsto & u \underset{\mathcal{C}}{\cdot} v \;. \end{array}$$

We often write $uv := u \cdot v := u \underset{\mathcal{C}}{\cdot} v$.

So given $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$, composition gives the composite morphism $X \xrightarrow{uv} Z$.

Given $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} W$ in $\mathcal{C}$, we require $\mathrm{id}_X\, u = u = u\, \mathrm{id}_Y$ and $(uv)w = u(vw) =: uvw$.

(3) A functor F from $\mathcal{C}$ to $\mathcal{D}$, often written $\mathcal{C} \xrightarrow{F} \mathcal{D}$, consists of maps

$$\begin{array}{rcrcl} \operatorname{Ob}(F) & : & \operatorname{Ob}(\mathcal{C}) & \to & \operatorname{Ob}(\mathcal{D}) \\ \operatorname{Mor}(F) & : & \operatorname{Mor}(\mathcal{C}) & \to & \operatorname{Mor}(\mathcal{D}) \end{array}$$

such that the following conditions (i, ii) hold.

(i) We have $\operatorname{Mor}(F) \cdot \mathrm{s}^{\mathcal{D}} = \mathrm{s}^{\mathcal{C}} \cdot \operatorname{Ob}(F)$.
We have $\operatorname{Mor}(F) \cdot \mathrm{t}^{\mathcal{D}} = \mathrm{t}^{\mathcal{C}} \cdot \operatorname{Ob}(F)$.
We have $\operatorname{Ob}(F) \cdot \mathrm{i}^{\mathcal{D}} = \mathrm{i}^{\mathcal{C}} \cdot \operatorname{Mor}(F)$.

(ii) Given $u,\, v \in \operatorname{Mor}(\mathcal{C})$ such that $u\mathrm{t}^{\mathcal{C}} = v\mathrm{s}^{\mathcal{C}}$, we have

$$(u \underset{\mathcal{C}}{\cdot} v)\operatorname{Mor}(F) = u\operatorname{Mor}(F) \underset{\mathcal{D}}{\cdot} v\operatorname{Mor}(F) \;.$$

Often, we write $XF := X\operatorname{Ob}(F)$ for $X \in \operatorname{Ob}(\mathcal{C})$ and $uF := u\operatorname{Mor}(F)$ for $u \in \operatorname{Mor}(\mathcal{C})$.

So (i, ii) can be expressed as follows.

First, we have $\mathrm{id}_{XF} = \mathrm{id}_X F$ for $X \in \operatorname{Ob}(\mathcal{C})$.

Second, given $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$, we have $XF \xrightarrow{uF} YF \xrightarrow{vF} ZF$ in $\mathcal{D}$ and the composite uv is sent to $(uv)F = (uF)(vF)$.

For instance, we have the functor $\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C}$ with $\operatorname{Ob}(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{\operatorname{Ob}(\mathcal{C})}$ and $\operatorname{Mor}(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{\operatorname{Mor}(\mathcal{C})}$.

(4) Given functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \xrightarrow{H} \mathcal{B}$, the composite functor $\mathcal{C} \xrightarrow{F*G} \mathcal{E}$ is defined by $\operatorname{Ob}(F*G) := \operatorname{Ob}(F) \cdot \operatorname{Ob}(G)$ and by $\operatorname{Mor}(F*G) := \operatorname{Mor}(F) \cdot \operatorname{Mor}(G)$. We often write $FG := F*G$. We have $(FG)H = F(GH) = FGH$.

(5) Let F and F' be functors from $\mathcal{C}$ to $\mathcal{D}$. A transformation a from F to F' is a tuple of morphisms

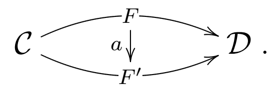
$$a \;=\; (XF \xrightarrow{Xa} XF')_{X \in \operatorname{Ob}(\mathcal{C})}$$

such that for every morphism $X \xrightarrow{u} Y$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} XF & \xrightarrow{Xa} & XF' \\ {\scriptstyle uF}\big\downarrow & & \big\downarrow{\scriptstyle uF'} \\ YF & \xrightarrow{Ya} & YF' \end{array}$$

is commutative, i.e. $(uF)(Ya) = (Xa)(uF')$.

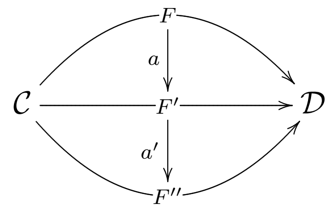
Graphically, the transformation a can be displayed as $F \xrightarrow{a} F'$ or as



For instance, we have the transformation $\mathrm{id}_F := (XF \xrightarrow{\mathrm{id}_{XF}} XF)_{X \in \mathrm{Ob}(\mathcal{C})}$, so $F \xrightarrow{\mathrm{id}_F} F$.

We have $X\mathrm{id}_F = \mathrm{id}_{XF}$ for $X \in \mathrm{Ob}(\mathcal{C})$.

(6) Let F, F' and F'' be functors from $\mathcal{C}$ to $\mathcal{D}$. Let a be a transformation from F to F'. Let a' be a transformation from F' to F''.



So $a = (XF \xrightarrow{Xa} XF')_{X \in \mathrm{Ob}(\mathcal{C})}$ and $a' = (XF' \xrightarrow{Xa'} XF'')_{X \in \mathrm{Ob}(\mathcal{C})}$.

Then the vertical composite of a and a' is given by $a \cdot a' = (XF \xrightarrow{Xa \cdot Xa'} XF'')_{X \in \mathrm{Ob}(\mathcal{C})}$.

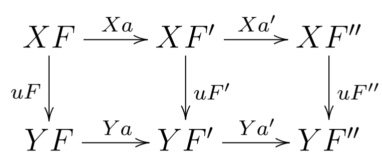
So $X(a \cdot a') = Xa \cdot Xa'$ for $X \in \mathrm{Ob}(\mathcal{C})$.

We often write $aa' := a \cdot a'$.

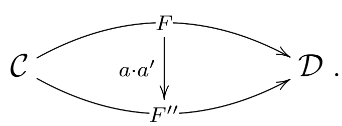
Then aa' is a transformation from F to F'', since

$$(uF)(Y(aa')) \;=\; (uF)(Ya)(Ya') \;=\; (Xa)(uF')(Ya') \;=\; (Xa)(Xa')(uF'') \;=\; (X(aa'))(uF'')$$

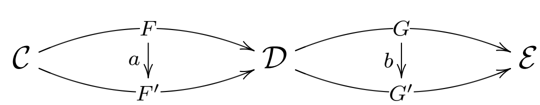
for $X \xrightarrow{u} Y$ in $\mathrm{Mor}(\mathcal{C})$.



So



(7) Let F and F' be functors from $\mathcal{C}$ to $\mathcal{D}$. Let G and G' be functors from $\mathcal{D}$ to $\mathcal{E}$. Let a be a transformation from F to F'. Let b a transformation from G to G'.



Let $aG := (XFG \xrightarrow{(Xa)G} XF'G)_{X \in \mathrm{Ob}(\mathcal{C})}$.

So we have $X(aG) = (Xa)G =: XaG$ for $X \in \mathrm{Ob}(\mathcal{C})$.

Note that the following diagram commutes.

$$\begin{array}{ccc} XF & \xrightarrow{Xa} & XF' \\ {\scriptstyle uF}\downarrow & & \downarrow{\scriptstyle uF'} \\ YF & \xrightarrow{Ya} & YF' \end{array}$$

Then aG is a transformation from FG to $F'G$, since

$$(uFG)(YaG) \;=\; ((uF)(Ya))G \;=\; ((Xa)(uF'))G \;=\; (XaG)(uF'G)$$

for $X \xrightarrow{u} Y$ in $\mathrm{Mor}(\mathcal{C})$.

$$\begin{array}{ccc} XFG & \xrightarrow{XaG} & XF'G \\ {\scriptstyle uFG}\downarrow & & \downarrow{\scriptstyle uF'G} \\ YFG & \xrightarrow{YaG} & YF'G \end{array}$$

Let $Fb := (XFG \xrightarrow{(XF)b} XFG')_{X \in \mathrm{Ob}(\mathcal{C})}$.

So we have $X(Fb) = (XF)b =: XFb$ for $X \in \mathrm{Ob}(\mathcal{C})$.

Then Fb is a transformation from FG to FG', since

$$(uFG)(YFb) \;=\; ((uF)G)((YF)b) \;=\; ((XF)b)((uF)G') \;=\; (XFb)(uFG')$$

for $X \xrightarrow{u} Y$ in $\mathrm{Mor}(\mathcal{C})$.

$$\begin{array}{ccc} XFG & \xrightarrow{XFb} & XFG' \\ {\scriptstyle uFG}\downarrow & & \downarrow{\scriptstyle uFG'} \\ YFG & \xrightarrow{YFb} & YFG' \end{array}$$

Finally, the horizontal composite of a and b is given by

$$a * b \;:=\; (XFG \xrightarrow{(XFb)(XaG')} XF'G')_{X \in \mathrm{Ob}(\mathcal{C})} \;=\; (XFG \xrightarrow{(XaG)(XF'b)} XF'G')_{X \in \mathrm{Ob}(\mathcal{C})} \,.$$

Equality holds since the following diagram commutes.

$$\begin{array}{ccc} XFG & \xrightarrow{XFb} & XFG' \\ {\scriptstyle XaG}\downarrow & & \downarrow{\scriptstyle XaG'} \\ XF'G & \xrightarrow{XF'b} & XF'G' \end{array}$$

So $a * b = Fb \cdot aG' = aG \cdot F'b$ is a transformation from $FG = F * G$ to $F'G' = F' * G'$ as a vertical composite of two transformations. I.e.

$$\mathcal{C} \; \underset{F' * G'}{\overset{F * G}{\rightrightarrows}} \; \mathcal{E} \quad (a * b \,\Downarrow) \,.$$

In particular, $a * \mathrm{id}_G = aG$ and $\mathrm{id}_F * b = Fb$.

Remark 8. A transformation $a = (XF \xrightarrow{Xa} XG)_{X \in \mathrm{Ob}(\mathcal{C})}$ as in Reminder 7.(5) is classically often written as $(F(X) \xrightarrow{a_X} G(X))_{X \in \mathrm{Ob}(\mathcal{C})}$.

The condition in loc. cit. then classically reads $a_Y \circ F(u) = G(u) \circ a_X$.

Proposition 9. *Suppose given a universe $\mathfrak{U}$. Suppose given a universe $\mathfrak{V}$ such that $\mathfrak{U} \in \mathfrak{V}$.*

We have the following 2*-category* Cat *in* $\mathfrak{V}$*, called the* 2-category of categories.

- *It has* $\mathrm{Mor}_0(\mathrm{Cat}) = \mathrm{Ob}(\mathrm{Cat}) := \{\, \mathcal{C} : \mathcal{C} \text{ is a category in } \mathfrak{U} \,\}$.
- *It has* $\mathrm{Mor}_1(\mathrm{Cat}) := \{\, F : F \text{ is a functor between categories in } \mathfrak{U} \,\}$.
- *It has* $\mathrm{Mor}_2(\mathrm{Cat}) := \{\, a : a \text{ is a transformation between functors between categories in } \mathfrak{U} \,\}$.
- *We have maps*

$$\begin{array}{rcl} \mathrm{Mor}_1(\mathrm{Cat}) & \xrightarrow{\mathrm{s}_0^{\mathrm{Cat}}} & \mathrm{Ob}(\mathrm{Cat}) \\ (\mathcal{C} \xrightarrow{F} \mathcal{D}) & \mapsto & \mathcal{C} \\ \mathrm{Mor}_1(\mathrm{Cat}) & \xleftarrow{\mathrm{i}_0^{\mathrm{Cat}}} & \mathrm{Ob}(\mathrm{Cat}) \\ (\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C}) & \leftarrow\!\shortmid & \mathcal{C} \\ \mathrm{Mor}_1(\mathrm{Cat}) & \xrightarrow{\mathrm{t}_0^{\mathrm{Cat}}} & \mathrm{Ob}(\mathrm{Cat}) \\ (\mathcal{C} \xrightarrow{F} \mathcal{D}) & \mapsto & \mathcal{D} \,. \end{array}$$

- *Maps*

$$\begin{array}{rcl} \mathrm{Mor}_2(\mathrm{Cat}) & \xrightarrow{\mathrm{s}_1^{\mathrm{Cat}}} & \mathrm{Mor}_1(\mathrm{Cat}) \\ (F \xrightarrow{a} F') & \mapsto & F \\ \mathrm{Mor}_2(\mathrm{Cat}) & \xleftarrow{\mathrm{i}_1^{\mathrm{Cat}}} & \mathrm{Mor}_1(\mathrm{Cat}) \\ (F \xrightarrow{\mathrm{id}_F} F) & \leftarrow\!\shortmid & F \\ \mathrm{Mor}_2(\mathrm{Cat}) & \xrightarrow{\mathrm{t}_1^{\mathrm{Cat}}} & \mathrm{Mor}_1(\mathrm{Cat}) \\ (F \xrightarrow{a} F') & \mapsto & F' \,. \end{array}$$

- *Composition of* 1*-morphisms is given by*

$$\begin{array}{rcl} \{\, (F,G) \in \mathrm{Mor}_1(\mathrm{Cat}) \times \mathrm{Mor}_1(\mathrm{Cat}) \,:\, F\mathrm{t}_0^{\mathrm{Cat}} = G\mathrm{s}_0^{\mathrm{Cat}} \,\} & \xrightarrow{\left(\overset{\mathrm{Cat}}{*}\right)} & \mathrm{Mor}_1(\mathrm{Cat}) \\ (F,G) & \mapsto & F \overset{\mathrm{Cat}}{*} G \;:=\; F * G \;=\; FG \,. \end{array}$$

- *Vertical composition of* 2*-morphisms is given by*

$$\begin{array}{rcl} \{\, (a,a') \in \mathrm{Mor}_2(\mathrm{Cat}) \times \mathrm{Mor}_2(\mathrm{Cat}) \,:\, a\mathrm{t}_1^{\mathrm{Cat}} = a'\mathrm{s}_1^{\mathrm{Cat}} \,\} & \xrightarrow{\left(\underset{\mathrm{Cat}}{\cdot}\right)} & \mathrm{Mor}_2(\mathrm{Cat}) \\ (a,a') & \mapsto & a \underset{\mathrm{Cat}}{\cdot} a' \;:=\; a \cdot a' \;=\; aa' \,. \end{array}$$

- *Horizontal composition of* 2*-morphisms is given by*

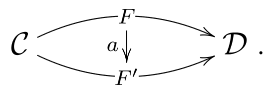
$$\begin{array}{rcl} \{\, (a,b) \in \mathrm{Mor}_2(\mathrm{Cat}) \times \mathrm{Mor}_2(\mathrm{Cat}) \,:\, a\mathrm{s}_1^{\mathrm{Cat}}\mathrm{t}_0^{\mathrm{Cat}} = b\mathrm{s}_1^{\mathrm{Cat}}\mathrm{s}_0^{\mathrm{Cat}} \,\} & \xrightarrow{\left(\underset{\mathrm{Cat}}{*}\right)} & \mathrm{Mor}_2(\mathrm{Cat}) \\ (a,b) & \mapsto & a \underset{\mathrm{Cat}}{*} b \;:=\; a * b \,. \end{array}$$

Proof. We verify properties (1–13) from Definition 1.

Ad (1). Given $\mathcal{C} \in \mathrm{Ob}(\mathrm{Cat})$, we have $\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C}$.

Ad (2). Given $F \in \mathrm{Mor}_1(\mathrm{Cat})$, we have $F \xrightarrow{\mathrm{id}_F} F$.

Ad (3). Given $a \in \mathrm{Mor}_2(\mathrm{Cat})$, we have



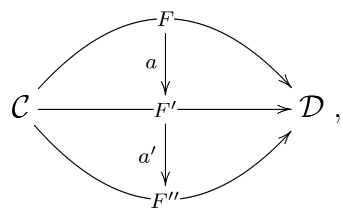
Ad (4). Given 1-morphisms

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}\ ,$$

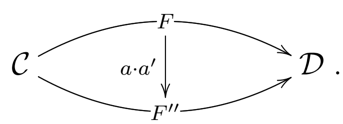
we have

$$\mathcal{C} \xrightarrow{F*G} \mathcal{E}\ .$$

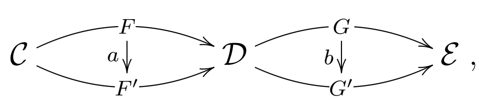
Ad (5). Given 2-morphisms



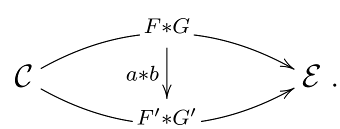
we have



Ad (6). Given 2-morphisms



we have



Ad (7). Suppose given a 1-morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$.

We have

$$\begin{aligned}
\mathrm{Ob}(\mathrm{id}_{\mathcal{C}} * F) &= \mathrm{Ob}(\mathrm{id}_{\mathcal{C}}) \cdot \mathrm{Ob}(F) = \mathrm{id}_{\mathrm{Ob}(\mathcal{C})} \cdot \mathrm{Ob}(F) = \mathrm{Ob}(F) \\
\mathrm{Mor}(\mathrm{id}_{\mathcal{C}} * F) &= \mathrm{Mor}(\mathrm{id}_{\mathcal{C}}) \cdot \mathrm{Mor}(F) = \mathrm{id}_{\mathrm{Mor}(\mathcal{C})} \cdot \mathrm{Mor}(F) = \mathrm{Mor}(F)\ .
\end{aligned}$$

So $\mathrm{id}_{\mathcal{C}} * F = F$.

We have

$$\begin{aligned}
\mathrm{Ob}(F * \mathrm{id}_{\mathcal{D}}) &= \mathrm{Ob}(F) \cdot \mathrm{Ob}(\mathrm{id}_{\mathcal{D}}) = \mathrm{Ob}(F) \cdot \mathrm{id}_{\mathrm{Ob}(\mathcal{D})} = \mathrm{Ob}(F) \\
\mathrm{Mor}(F * \mathrm{id}_{\mathcal{D}}) &= \mathrm{Mor}(F) \cdot \mathrm{Mor}(\mathrm{id}_{\mathcal{D}}) = \mathrm{Mor}(F) \cdot \mathrm{id}_{\mathrm{Mor}(\mathcal{D})} = \mathrm{Mor}(F)
\end{aligned}$$

So $F * \mathrm{id}_{\mathcal{D}} = F$.

Ad (8). Given 1-morphisms

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \xrightarrow{H} \mathcal{B} ,$$

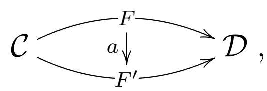
we have

$$\begin{aligned}\mathrm{Ob}((F * G) * H) = \mathrm{Ob}(F * G) \cdot \mathrm{Ob}(H) = \mathrm{Ob}(F) \cdot \mathrm{Ob}(G) \cdot \mathrm{Ob}(H) \\ = \mathrm{Ob}(F) \cdot \mathrm{Ob}(G * H) = \mathrm{Ob}(F * (G * H))\end{aligned}$$

$$\begin{aligned}\mathrm{Mor}((F * G) * H) = \mathrm{Mor}(F * G) \cdot \mathrm{Mor}(H) = \mathrm{Mor}(F) \cdot \mathrm{Mor}(G) \cdot \mathrm{Mor}(H) \\ = \mathrm{Mor}(F) \cdot \mathrm{Mor}(G * H) = \mathrm{Mor}(F * (G * H)) .\end{aligned}$$

Hence $(F * G) * H = F * (G * H)$.

Ad (9). Given a 2-morphism



we have

$$\begin{aligned}\mathrm{id}_F &= (XF \xrightarrow{\mathrm{id}_{XF}} XF)_{X \in \mathrm{Ob}(\mathcal{C})} \\ \mathrm{id}_{F'} &= (XF' \xrightarrow{\mathrm{id}_{XF'}} XF')_{X \in \mathrm{Ob}(\mathcal{C})}\end{aligned}$$

hence

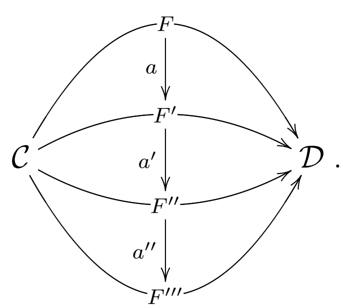
$$\mathrm{id}_F \cdot a = (XF \xrightarrow{\mathrm{id}_{XF} \cdot Xa} XF')_{X \in \mathrm{Ob}(\mathcal{C})} = (XF \xrightarrow{Xa} XF')_{X \in \mathrm{Ob}(\mathcal{C})} = a .$$

and

$$a \cdot \mathrm{id}_{F'} = (XF \xrightarrow{Xa \cdot \mathrm{id}_{XF'}} XF')_{X \in \mathrm{Ob}(\mathcal{C})} = (XF \xrightarrow{Xa} XF')_{X \in \mathrm{Ob}(\mathcal{C})} = a .$$

Moreover, we have $\mathrm{id}_{\mathrm{id}_{\mathcal{C}}} * a = \mathrm{id}_{\mathcal{C}}\, a = a$ and $a * \mathrm{id}_{\mathrm{id}_{\mathcal{D}}} = a\, \mathrm{id}_{\mathcal{D}} = a$.

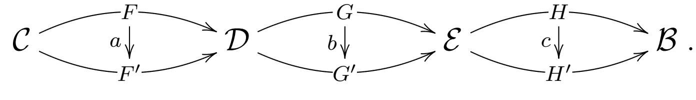
Ad (10). Suppose given 2-morphisms



We have

$$\begin{aligned}(a \cdot a') \cdot a'' &= (XF \xrightarrow{(Xa \cdot Xa') \cdot Xa''} XF''')_{X \in \mathrm{Ob}(\mathcal{C})} \\ &= (XF \xrightarrow{Xa \cdot (Xa' \cdot Xa'')} XF''')_{X \in \mathrm{Ob}(\mathcal{C})} \\ &= a \cdot (a' \cdot a'') .\end{aligned}$$

Ad (11). Suppose given 2-morphisms



For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned}X((a * b) * c) &= (X(FG))c \cdot (X(a * b))H' \\ &= (X(FG))c \cdot ((XF)b \cdot (Xa)G')H' \\ &= (XFG)c \cdot ((XF)b)H' \cdot (Xa)G'H'\end{aligned}$$

and

$$\begin{array}{rcl} X(a*(b*c)) & = & (XF)(b*c)\cdot(Xa)(G'H') \\ & = & ((XF)G)c\cdot((XF)b)H'\cdot(Xa)(G'H') \\ & = & (XFG)c\cdot((XF)b)H'\cdot(Xa)G'H'\ . \end{array}$$

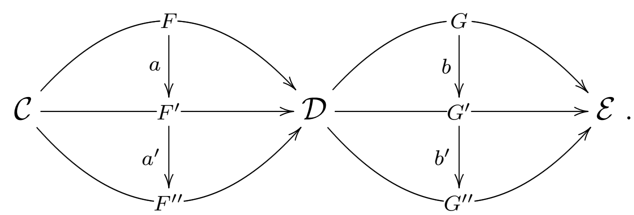
Hence $(a*b)*c = a*(b*c)$.

Ad (12). Given 1-morphisms $\mathcal{C}\xrightarrow{F}\mathcal{D}\xrightarrow{G}\mathcal{E}$ and $X\in\mathrm{Ob}(\mathcal{C})$, we have

$$\begin{array}{rcl} (XFG\xrightarrow{X(\mathrm{id}_F*\mathrm{id}_G)}XFG) & = & (XFG\xrightarrow{XF\mathrm{id}_G\cdot X\mathrm{id}_F G}XFG) \\ & = & (XFG\xrightarrow{\mathrm{id}_{XFG}\cdot\mathrm{id}_{XFG}}XFG) \\ & = & (XFG\xrightarrow{X(\mathrm{id}_{FG})}XFG)\ . \end{array}$$

Hence $\mathrm{id}_F*\mathrm{id}_G = \mathrm{id}_{FG}$.

Ad (13). Suppose given 2-morphisms



For $X\in\mathrm{Ob}(\mathcal{C})$, we have

$$\begin{array}{rcl} X((a\cdot a')*(b\cdot b')) & = & XF(b\cdot b')\cdot X(a\cdot a')G'' \\ & = & XFb\cdot XFb'\cdot XaG''\cdot Xa'G'' \\ & = & XFb\cdot(XFb'\cdot XaG'')\cdot Xa'G'' \\ & = & XFb\cdot X(a*b')\cdot Xa'G'' \end{array}$$

and

$$\begin{array}{rcl} X((a*b)\cdot(a'*b')) & = & X(a*b)\cdot X(a'*b') \\ & = & XFb\cdot XaG'\cdot XF'b'\cdot Xa'G'' \\ & = & XFb\cdot(XaG'\cdot XF'b')\cdot Xa'G'' \\ & = & XFb\cdot X(a*b')\cdot Xa'G''\ . \end{array}$$

Hence $(a\cdot a')*(b\cdot b') = (a*b)\cdot(a'*b')$. □

1.4 2-subcategories

Suppose given a 2-category $\mathfrak{K}$; cf. Definition 1.1.

Definition 10. A 2-*subcategory* $\check{\mathfrak{K}}$ of $\mathfrak{K}$ consists of subsets

$$\begin{array}{rcl} \mathrm{Ob}(\check{\mathfrak{K}}) & \subseteq & \mathrm{Ob}(\mathfrak{K}) \\ \mathrm{Mor}_1(\check{\mathfrak{K}}) & \subseteq & \mathrm{Mor}_1(\mathfrak{K}) \\ \mathrm{Mor}_2(\check{\mathfrak{K}}) & \subseteq & \mathrm{Mor}_2(\mathfrak{K}) \end{array}$$

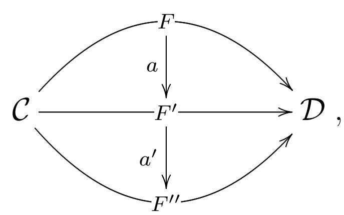
such that the following properties (1–5) hold.

(1) The map $\mathrm{s}_0^{\mathfrak{K}}$ restricts to a map $\mathrm{s}_0^{\check{\mathfrak{K}}}$ from $\mathrm{Mor}_1(\check{\mathfrak{K}})$ to $\mathrm{Ob}(\check{\mathfrak{K}})$.
The map $\mathrm{i}_0^{\mathfrak{K}}$ restricts to a map $\mathrm{i}_0^{\check{\mathfrak{K}}}$ from $\mathrm{Ob}(\check{\mathfrak{K}})$ to $\mathrm{Mor}_1(\check{\mathfrak{K}})$.
The map $\mathrm{t}_0^{\mathfrak{K}}$ restricts to a map $\mathrm{t}_0^{\check{\mathfrak{K}}}$ from $\mathrm{Mor}_1(\check{\mathfrak{K}})$ to $\mathrm{Ob}(\check{\mathfrak{K}})$.
I.e. given a 1-morphism in $\mathrm{Mor}_1(\check{\mathfrak{K}})$, its source and its target are in $\mathrm{Ob}(\check{\mathfrak{K}})$.
Conversely, given an object in $\mathrm{Ob}(\check{\mathfrak{K}})$, its identity is in $\mathrm{Mor}_1(\check{\mathfrak{K}})$.

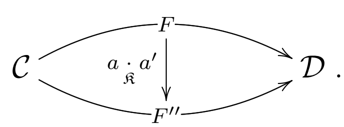
(2) The map $\mathrm{s}_1^{\mathfrak{K}}$ restricts to a map $\mathrm{s}_1^{\check{\mathfrak{K}}}$ from $\mathrm{Mor}_2(\check{\mathfrak{K}})$ to $\mathrm{Mor}_1(\check{\mathfrak{K}})$.
The map $\mathrm{i}_1^{\mathfrak{K}}$ restricts to a map $\mathrm{i}_1^{\check{\mathfrak{K}}}$ from $\mathrm{Mor}_1(\check{\mathfrak{K}})$ to $\mathrm{Mor}_2(\check{\mathfrak{K}})$.
The map $\mathrm{t}_1^{\mathfrak{K}}$ restricts to a map $\mathrm{t}_1^{\check{\mathfrak{K}}}$ from $\mathrm{Mor}_2(\check{\mathfrak{K}})$ to $\mathrm{Mor}_1(\check{\mathfrak{K}})$.
I.e. given a 2-morphism in $\mathrm{Mor}_2(\check{\mathfrak{K}})$, its source and its target are in $\mathrm{Mor}_1(\check{\mathfrak{K}})$.
Conversely, given a 1-morphism in $\mathrm{Mor}_1(\check{\mathfrak{K}})$, its identity is in $\mathrm{Mor}_2(\check{\mathfrak{K}})$.

(3) The map $(\overset{\mathfrak{K}}{*})$ restricts to a map $(\overset{\check{\mathfrak{K}}}{*})$ from $\{\,(F,G)\in \mathrm{Mor}_1(\check{\mathfrak{K}})\times\mathrm{Mor}_1(\check{\mathfrak{K}}) \,:\, F\mathrm{t}_0^{\check{\mathfrak{K}}} = G\mathrm{s}_0^{\check{\mathfrak{K}}}\,\}$ to $\mathrm{Mor}_1(\check{\mathfrak{K}})$.
I.e. given $\mathcal{C}\xrightarrow{F}\mathcal{D}\xrightarrow{G}\mathcal{E}$ in $\mathfrak{K}$ with $F,G\in\mathrm{Mor}_1(\check{\mathfrak{K}})$, their composite $\mathcal{C}\xrightarrow{F\overset{\mathfrak{K}}{*}G}\mathcal{E}$ is in $\mathrm{Mor}_1(\check{\mathfrak{K}})$ and is written $F\overset{\check{\mathfrak{K}}}{*}G := F\overset{\mathfrak{K}}{*}G$.

(4) The map $(\underset{\mathfrak{K}}{\cdot})$ restricts to a map $(\underset{\check{\mathfrak{K}}}{\cdot})$ from $\{\,(a,a')\in \mathrm{Mor}_2(\check{\mathfrak{K}})\times\mathrm{Mor}_2(\check{\mathfrak{K}}) \,:\, a\mathrm{t}_1^{\check{\mathfrak{K}}} = a'\mathrm{s}_1^{\check{\mathfrak{K}}}\,\}$ to $\mathrm{Mor}_2(\check{\mathfrak{K}})$.
I.e. given

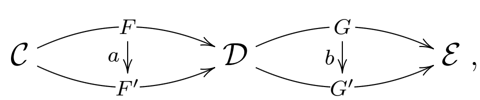


in $\mathfrak{K}$ with $a,\ a'\in\mathrm{Mor}_2(\check{\mathfrak{K}})$, their vertical composite

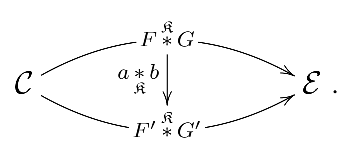


is in $\mathrm{Mor}_2(\check{\mathfrak{K}})$ and is written $a\underset{\check{\mathfrak{K}}}{\cdot}a' := a\underset{\mathfrak{K}}{\cdot}a'$.

(5) The map $(\underset{\mathfrak{K}}{*})$ restricts to a map $(\underset{\check{\mathfrak{K}}}{*})$ from $\{\,(a,b)\in \mathrm{Mor}_2(\check{\mathfrak{K}})\times\mathrm{Mor}_2(\check{\mathfrak{K}}) \,:\, a\mathrm{s}_1^{\check{\mathfrak{K}}}\mathrm{t}_0^{\check{\mathfrak{K}}} = b\mathrm{s}_1^{\check{\mathfrak{K}}}\mathrm{s}_0^{\check{\mathfrak{K}}}\,\}$ to $\mathrm{Mor}_2(\check{\mathfrak{K}})$.
I.e. given



in $\mathfrak{K}$ with $a,\ b\in\mathrm{Mor}_2(\check{\mathfrak{K}})$, their horizontal composite



is in $\mathrm{Mor}_2(\check{\mathfrak{K}})$ and is written $a\underset{\check{\mathfrak{K}}}{*}b := a\underset{\mathfrak{K}}{*}b$.

Then $\mathrm{Ob}(\breve{\mathfrak{K}})$, $\mathrm{Mor}_1(\breve{\mathfrak{K}})$, $\mathrm{Mor}_2(\breve{\mathfrak{K}})$, together with $\mathrm{s}_0^{\breve{\mathfrak{K}}}$, $\mathrm{i}_0^{\breve{\mathfrak{K}}}$, $\mathrm{t}_0^{\breve{\mathfrak{K}}}$, $\mathrm{s}_1^{\breve{\mathfrak{K}}}$, $\mathrm{i}_1^{\breve{\mathfrak{K}}}$, $\mathrm{t}_1^{\breve{\mathfrak{K}}}$, $(\overset{\breve{\mathfrak{K}}}{*})$, $(\underset{\breve{\mathfrak{K}}}{\cdot})$, $(\underset{\breve{\mathfrak{K}}}{*})$ form a 2-category $\breve{\mathfrak{K}}$.

Proof. We verify (1–13) from Definition 1.

Ad (1). For $\mathcal{C} \in \mathrm{Ob}(\breve{\mathfrak{K}})$, we have $\mathcal{C} \xrightarrow{\mathrm{id}_\mathcal{C}} \mathcal{C}$ in $\mathrm{Mor}_1(\breve{\mathfrak{K}})$ by property (1).

Ad (2). For $F \in \mathrm{Mor}_1(\breve{\mathfrak{K}})$, we have $F \xrightarrow{\mathrm{id}_F} F$ in $\mathrm{Mor}_2(\breve{\mathfrak{K}})$ by property (2).

Ad (3). Given $a \in \mathrm{Mor}_2(\breve{\mathfrak{K}})$, we have

$$a\mathrm{s}_1^{\breve{\mathfrak{K}}}\mathrm{s}_0^{\breve{\mathfrak{K}}} = a\mathrm{s}_1^{\mathfrak{K}}\mathrm{s}_0^{\mathfrak{K}} = a\mathrm{t}_1^{\mathfrak{K}}\mathrm{s}_0^{\mathfrak{K}} = a\mathrm{t}_1^{\breve{\mathfrak{K}}}\mathrm{s}_0^{\breve{\mathfrak{K}}}$$

and

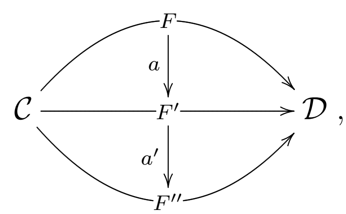
$$a\mathrm{s}_1^{\breve{\mathfrak{K}}}\mathrm{t}_0^{\breve{\mathfrak{K}}} = a\mathrm{s}_1^{\mathfrak{K}}\mathrm{t}_0^{\mathfrak{K}} = a\mathrm{t}_1^{\mathfrak{K}}\mathrm{t}_0^{\mathfrak{K}} = a\mathrm{t}_1^{\breve{\mathfrak{K}}}\mathrm{t}_0^{\breve{\mathfrak{K}}}\ .$$

Ad (4). Given 1-morphisms

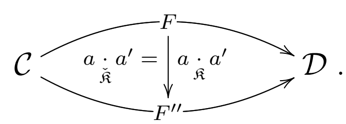
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

in $\mathrm{Mor}_1(\breve{\mathfrak{K}})$, we have $(\mathcal{C} \xrightarrow{F \overset{\breve{\mathfrak{K}}}{*} G} \mathcal{E}) = (\mathcal{C} \xrightarrow{F \overset{\mathfrak{K}}{*} G} \mathcal{E})$ in $\mathrm{Mor}_1(\breve{\mathfrak{K}})$; cf. property (3).

Ad (5). Given 2-morphisms

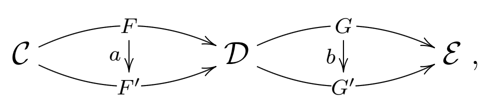


in $\mathrm{Mor}_2(\breve{\mathfrak{K}})$, we have

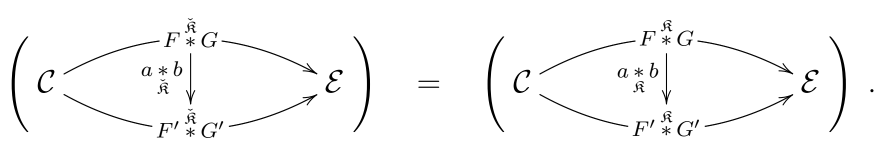


in $\mathrm{Mor}_2(\breve{\mathfrak{K}})$; cf. property (4).

Ad (6). Given



in $\mathfrak{K}$ with a, $b \in \mathrm{Mor}_2(\breve{\mathfrak{K}})$, we have $F \overset{\breve{\mathfrak{K}}}{*} G = F \overset{\mathfrak{K}}{*} G$, $F' \overset{\breve{\mathfrak{K}}}{*} G' = F' \overset{\mathfrak{K}}{*} G'$ and $a \underset{\breve{\mathfrak{K}}}{*} b = a \underset{\mathfrak{K}}{*} b$ and thus



in $\mathrm{Mor}_2(\breve{\mathfrak{K}})$; cf. properties $(3, 5)$.

Ad (7). Given a 1-morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathrm{Mor}_1(\breve{\mathfrak{K}})$, we have

$$\mathrm{id}_\mathcal{C} \overset{\breve{\mathfrak{K}}}{*} F \;=\; \mathrm{id}_\mathcal{C} \overset{\mathfrak{K}}{*} F \;=\; F$$

and

$$F \overset{\check{\mathfrak{K}}}{*} \mathrm{id}_{\mathcal{D}} \;=\; F \overset{\mathfrak{K}}{*} \mathrm{id}_{\mathcal{D}} \;=\; F\;;$$

cf. property (3).

Ad (8). Given 1-morphisms

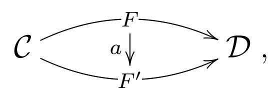
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \xrightarrow{H} \mathcal{B}$$

in $\mathrm{Mor}_1(\check{\mathfrak{K}})$, we have

$$(F \overset{\check{\mathfrak{K}}}{*} G) \overset{\check{\mathfrak{K}}}{*} H \;=\; (F \overset{\mathfrak{K}}{*} G) \overset{\mathfrak{K}}{*} H \;=\; F \overset{\mathfrak{K}}{*} (G \overset{\mathfrak{K}}{*} H) \;=\; F \overset{\check{\mathfrak{K}}}{*} (G \overset{\check{\mathfrak{K}}}{*} H)\;;$$

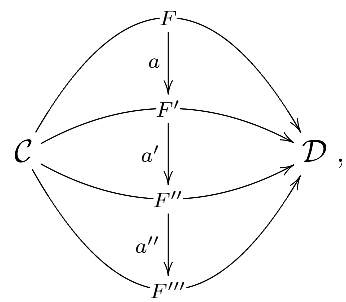
cf. property (3).

Ad (9). Given a 2-morphism in $\mathrm{Mor}_2(\check{\mathfrak{K}})$



we have $\mathrm{id}_F \underset{\check{\mathfrak{K}}}{\cdot} a = \mathrm{id}_F \underset{\mathfrak{K}}{\cdot} a = a$ and $a \underset{\check{\mathfrak{K}}}{\cdot} \mathrm{id}_{F'} = a \underset{\mathfrak{K}}{\cdot} \mathrm{id}_{F'} = a$ and $\mathrm{id}_{\mathrm{id}_{\mathcal{C}}} \underset{\check{\mathfrak{K}}}{*} a = \mathrm{id}_{\mathrm{id}_{\mathcal{C}}} \underset{\mathfrak{K}}{*} a = a$ and $a \underset{\check{\mathfrak{K}}}{*} \mathrm{id}_{\mathrm{id}_{\mathcal{D}}} = a \underset{\mathfrak{K}}{*} \mathrm{id}_{\mathrm{id}_{\mathcal{D}}} = a$; cf. properties $(4,5)$.

Ad (10). Given 2-morphisms in $\mathrm{Mor}_2(\check{\mathfrak{K}})$

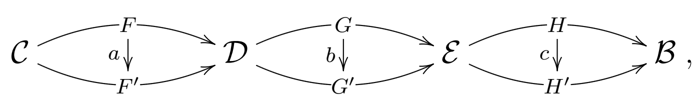


we have

$$(a \underset{\check{\mathfrak{K}}}{\cdot} a') \underset{\check{\mathfrak{K}}}{\cdot} a'' \;=\; (a \underset{\mathfrak{K}}{\cdot} a') \underset{\mathfrak{K}}{\cdot} a'' \;=\; a \underset{\mathfrak{K}}{\cdot} (a' \underset{\mathfrak{K}}{\cdot} a'') \;=\; a \underset{\check{\mathfrak{K}}}{\cdot} (a' \underset{\check{\mathfrak{K}}}{\cdot} a'')\;;$$

cf. property (4).

Ad (11). Given 2-morphisms in $\mathrm{Mor}_2(\check{\mathfrak{K}})$



we have

$$(a \underset{\check{\mathfrak{K}}}{*} b) \underset{\check{\mathfrak{K}}}{*} c \;=\; (a \underset{\mathfrak{K}}{*} b) \underset{\mathfrak{K}}{*} c \;=\; a \underset{\mathfrak{K}}{*} (b \underset{\mathfrak{K}}{*} c) \;=\; a \underset{\check{\mathfrak{K}}}{*} (b \underset{\check{\mathfrak{K}}}{*} c)\;;$$

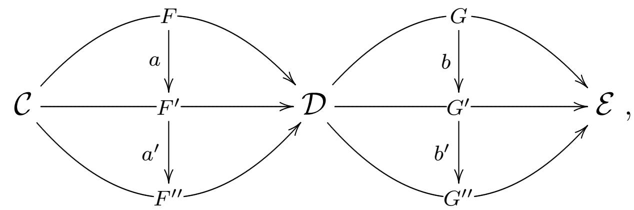
cf. property (5).

Ad (12). Given 1-morphisms $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ in $\mathrm{Mor}_1(\check{\mathfrak{K}})$, we have

$$\mathrm{id}_F \underset{\check{\mathfrak{K}}}{*} \mathrm{id}_G \;=\; \mathrm{id}_F \underset{\mathfrak{K}}{*} \mathrm{id}_G \;=\; \mathrm{id}_{F \overset{\mathfrak{K}}{*} G} \;=\; \mathrm{id}_{F \overset{\check{\mathfrak{K}}}{*} G}\;;$$

cf. properties $(3,5)$.

Ad (13). Given 2-morphisms



in $\mathrm{Mor}_2(\breve{\mathfrak{K}})$, we have

$$(a \underset{\breve{\mathfrak{K}}}{\cdot} a') \underset{\breve{\mathfrak{K}}}{*} (b \underset{\breve{\mathfrak{K}}}{\cdot} b') \;=\; (a \underset{\mathfrak{K}}{\cdot} a') \underset{\mathfrak{K}}{*} (b \underset{\mathfrak{K}}{\cdot} b') \;=\; (a \underset{\mathfrak{K}}{*} b) \underset{\mathfrak{K}}{\cdot} (a' \underset{\mathfrak{K}}{*} b') \;=\; (a \underset{\breve{\mathfrak{K}}}{*} b) \underset{\breve{\mathfrak{K}}}{\cdot} (a' \underset{\breve{\mathfrak{K}}}{*} b') \;;$$

cf. properties $(4, 5)$. □

1.5 Zero objects in 2-categories

Suppose given a 2-category $\mathfrak{K}$.

Definition 11. An object $\mathcal{Z} \in \mathrm{Ob}(\mathfrak{K})$ is called a zero object if ${}_{\mathfrak{K}}(\mathcal{C}, \mathcal{Z})$ contains one isoclass and ${}_{\mathfrak{K}}(\mathcal{Z}, \mathcal{D})$ contains one isoclass for each $\mathcal{C}$, $\mathcal{D} \in \mathrm{Ob}(\mathfrak{K})$. Cf. Definition 2.

Definition 12. Suppose given $\mathcal{C}$, $\mathcal{D} \in \mathrm{Ob}(\mathfrak{K})$.

A 1-morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is called a *zero* 1*-morphism,* if there exists a zero object $\mathcal{Z}$ and 1-morphisms $\mathcal{C} \xrightarrow{U} \mathcal{Z} \xrightarrow{V} \mathcal{D}$ such that in ${}_{\mathfrak{K}}(\mathcal{C}, \mathcal{D})$. Cf. Definitions 11 and 2.

Remark 13. *Suppose given zero objects $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ in $\mathfrak{K}$.*

Suppose given a 1*-morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathfrak{K}$.*

Then the following assertions $(1, 2)$ *are equivalent.*

(1) *There exist* 1*-morphisms $\mathcal{C} \xrightarrow{U} \mathcal{Z} \xrightarrow{V} \mathcal{D}$ such that $F \simeq U \overset{\mathfrak{K}}{*} V$.*

(2) *There exist* 1*-morphisms $\mathcal{C} \xrightarrow{\tilde{U}} \tilde{\mathcal{Z}} \xrightarrow{\tilde{V}} \mathcal{D}$ such that $F \simeq \tilde{U} \overset{\mathfrak{K}}{*} \tilde{V}$.*

Proof. It suffices to show that (1) implies (2).

So suppose that (1) holds and that we have $F \simeq U \overset{\mathfrak{K}}{*} V$ as stated there.

Choose 1-morphisms $G : \mathcal{Z} \to \tilde{\mathcal{Z}}$ and $H : \tilde{\mathcal{Z}} \to \mathcal{Z}$, which is possible since $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are zero objects in $\mathfrak{K}$. Note that $G \overset{\mathfrak{K}}{*} H \simeq \mathrm{id}_{\mathcal{Z}}$ since $\mathcal{Z}$ is a zero object in $\mathfrak{K}$.

Let $\tilde{U} := U \overset{\mathfrak{K}}{*} G : \mathcal{C} \to \tilde{\mathcal{Z}}$ and $\tilde{V} := H \overset{\mathfrak{K}}{*} V : \tilde{\mathcal{Z}} \to \mathcal{D}$. We obtain, using $(9, 13)$ of Definition 1,

$$\tilde{U} \overset{\mathfrak{K}}{*} \tilde{V} \;=\; U \overset{\mathfrak{K}}{*} G \overset{\mathfrak{K}}{*} H \overset{\mathfrak{K}}{*} V \;\simeq\; U \overset{\mathfrak{K}}{*} \mathrm{id}_{\mathcal{Z}} \overset{\mathfrak{K}}{*} V \;=\; U \overset{\mathfrak{K}}{*} V \;\simeq\; F \,.$$

□

Chapter 2

The 2-category of additive categories

2.1 Pointed categories

Definition 14. Suppose given a category $\mathcal{C}$.

(1) An object $Z \in \mathrm{Ob}(\mathcal{C})$ is called a zero object if $|_{\mathcal{C}}(X, Z)| = 1$ and $|_{\mathcal{C}}(Z, Y)| = 1$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$. I.e. Z is a zero object if it is initial and terminal.

(2) The category $\mathcal{C}$ is called *pointed* if there exists a zero object in $\mathcal{C}$.

Remark 15. *Suppose given a category $\mathcal{C}$ and zero objects Z, $Z' \in \mathrm{Ob}(\mathcal{C})$.*

Then the unique morphism from Z to Z' is an isomorphism.

In particular, we have $Z \simeq Z'$.

Proof. There exists a unique morphism $f : Z \to Z'$ and a unique morphism $g : Z' \to Z$.

We have $f \cdot g = \mathrm{id}_Z$, since $f \cdot g \in {}_{\mathcal{C}}(Z, Z) = \{\mathrm{id}_Z\}$.

We have $g \cdot f = \mathrm{id}_{Z'}$, since $g \cdot f \in {}_{\mathcal{C}}(Z', Z') = \{\mathrm{id}_{Z'}\}$. □

Remark 16. *Suppose given a category $\mathcal{C}$ and a zero object $Z \in \mathrm{Ob}(\mathcal{C})$.*

Suppose given $X \in \mathrm{Ob}(\mathcal{C})$. Suppose that X is a retract of Z. Then X is a zero object in $\mathcal{C}$.

In particular, if $X \simeq Z$, then X is a zero object in $\mathcal{C}$.

Proof. We may choose morphisms $X \xrightarrow{a} Z \xrightarrow{b} X$ such that $a \cdot b = \mathrm{id}_X$.

Suppose given $T \in \mathrm{Ob}(\mathcal{T})$.

We have to show that there exists a unique morphism from T to X.

Existence. There exists a morphism from T to Z, since Z is a zero object. Composing this morphism with $Z \xrightarrow{b} X$, we obtain a morphism $T \to X$.

Uniqueness. Suppose given morphisms u, $v : T \to X$. Then $u \cdot a$, $v \cdot a : T \to Z$. Since Z is a zero object, we have $u \cdot a = v \cdot a$. Thus

$$u \;=\; u \cdot a \cdot b \;=\; v \cdot a \cdot b \;=\; v\ .$$

We have to show that there exists a unique morphism from X to T.

Existence. There exists a morphism from Z to T, since Z is a zero object. Composing this morphism with $X \overset{a}{\to} Z$, we obtain a morphism $X \to T$.

Uniqueness. Suppose given morphisms $u,\, v\, :\, X \to T$. Then $b \cdot u,\, b \cdot v\, :\, Z \to T$. Since Z is a zero object, we have $b \cdot u = b \cdot v$. Thus

$$u \;=\; a \cdot b \cdot u = a \cdot b \cdot v \;=\; v\;.$$

□

Definition 17. *Suppose given a category $\mathcal{C}$. Suppose given $X, Y \in \mathrm{Ob}(\mathcal{C})$.*

A morphism $X \overset{f}{\to} Y$ is called a zero morphism, if there exists a zero object Z and morphisms $X \overset{u}{\to} Z \overset{v}{\to} Y$ such that $f = u \cdot v$.

Lemma 18. *Suppose given a pointed category $\mathcal{C}$ and $X, Y \in \mathrm{Ob}(\mathcal{C})$. The assertions $(1, 2)$ hold.*

(1) *There exists an unique zero morphism with source X and target Y. That zero morphism is written as $X \xrightarrow{0_{X,Y}} Y$ or simply as $0 := 0_{X,Y}$.*

(2) *Given morphisms $u : X' \to X$ and $v : Y \to Y'$, we have $u \cdot 0_{X,Y} \cdot v = 0_{X',Y'}$.*

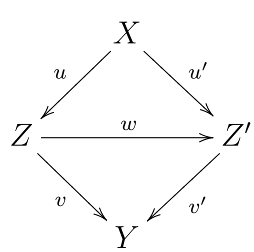
Proof. *Ad* (1).

Existence. There exists a zero object $Z \in \mathrm{Ob}(\mathcal{C})$ since $\mathcal{C}$ is pointed. So we have morphisms $u \in {}_{\mathcal{C}}(X, Z)$ and $v \in {}_{\mathcal{C}}(Z, Y)$. Therefore $u \cdot v$ is a zero morphism with source X and target Y.

Uniqueness. Suppose given zero morphisms $f = u \cdot v$ and $f' = u' \cdot v'$ in ${}_{\mathcal{C}}(X, Y)$ with $X \overset{u}{\to} Z \overset{v}{\to} Y$ and $X \overset{u'}{\to} Z' \overset{v'}{\to} Y$, where Z and Z' are zero objects in $\mathcal{C}$. We have a unique morphism $Z \overset{w}{\to} Z'$ since Z and Z' are zero objects. Note that $u \cdot w$ is a morphism with source X and target Z'. Since Z' is terminal, we have $u \cdot w = u'$. Analogously, $w \cdot v' = v$ since Z is initial. Thus we have

$$f' \;=\; u' \cdot v' \;=\; u \cdot w \cdot v' \;=\; u \cdot v \;=\; f\;.$$

So there exists at most one zero morphism from X to Y.



Ad (2). Since $0_{X,Y}$ factors over a zero object, so does $u \cdot 0_{X,Y} \cdot v$. □

Remark 19. *Suppose given a pointed category $\mathcal{C}$.*

Suppose given $X \in \mathrm{Ob}(\mathcal{C})$.

(1) *We have $\mathrm{id}_X = 0_{X,X}$ if and only if X is a zero object in $\mathcal{C}$.*

(2) *We have* $|{}_{\mathcal{C}}(X,X)| = 1$ *if and only if* X *is a zero object in* $\mathcal{C}$.

Proof. *Ad* (1). If X is a zero object in $\mathcal{C}$, the morphism id_X factors over X, hence $\mathrm{id}_X = 0_{X,X}$.

Conversely, if $\mathrm{id}_X = 0_{X,X}$, then id_X factors over a zero object Z of $\mathcal{C}$. Thus X is a retract of Z, which is a zero object. Therefore X is a zero object; cf. Remark 16.

Ad (2). If X is a zero object, then $|{}_{\mathcal{C}}(X,X)| = 1$.

Conversely, if $|{}_{\mathcal{C}}(X,X)| = 1$, then $\mathrm{id}_X = 0_{X,X}$. So X is a zero object by (1). □

2.2 Additive categories

2.2.1 Direct sums

Suppose given a pointed category $\mathcal{A}$.

Definition 20. Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$.

A *direct sum* of the tuple $(X_1, \ldots, X_m)$ is an object $S \in \mathrm{Ob}(\mathcal{A})$ together with *inclusion morphisms*

$$X_j \xrightarrow{\iota_j^S} S$$

and *projection morphisms*

$$S \xrightarrow{\pi_j^S} X_j$$

for $j \in [1,m]$, if the following axioms (Sum 1–3) hold.

(Sum 1) For $U \in \mathrm{Ob}(\mathcal{A})$ and each tuple of morphisms $(U \xrightarrow{u_i} X_i)_{i\in[1,m]}$ in $\mathcal{A}$, there exists a unique morphism $U \xrightarrow{a} S$ such that $a \cdot \pi_i^S = u_i$ for $i \in [1,m]$.
We write $a =: (\, u_1 \cdots u_m \,)^S$.

(Sum 2) For $V \in \mathrm{Ob}(\mathcal{A})$ and each tuple of morphisms $(X_i \xrightarrow{v_i} V)_{i\in[1,m]}$ in $\mathcal{A}$, there exists a unique morphism $S \xrightarrow{b} V$ such that $\iota_i^S \cdot b = v_i$ for $i \in [1,m]$.
We write $b =: {}^{S}\!\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$.

(Sum 3) We have $\iota_i^S \cdot \pi_i^S = \mathrm{id}_{X_i}$ for $i \in [1,m]$.
We have $\iota_i^S \cdot \pi_j^S = 0_{X_i,X_j}$ for $i, j \in [1,m]$ with $i \neq j$.

By abuse of notation, we often write $\pi_j := \pi_j^S$ and $\iota_j := \iota_j^S$.

Example 21. Suppose $m = 1$. Suppose given $X_1 \in \mathrm{Ob}(\mathcal{A})$.

Then $S := X_1$ is a direct sum of the single-entry-tuple of objects (X_1), with inclusion morphism $\iota_1 := \mathrm{id}_{X_1} : X_1 \to X_1$ and projection morphism $\pi_1 := \mathrm{id}_{X_1} : X_1 \to X_1$.

We verify (Sum 1–3).

In the situation of (Sum 1), we have to take $a = u_1$.

In the situation of (Sum 2), we have to take $b = v_1$.

For (Sum 3), we obtain $\iota_1 \cdot \pi_1 = \mathrm{id}_{X_1} \cdot \mathrm{id}_{X_1} = \mathrm{id}_{X_1}$.

Example 22. Suppose $m = 0$.

Let S be a zero object in $\mathcal{A}$. Then S is a direct sum of the empty tuple of objects $(\;)$, with zero inclusion morphisms and zero projection morphisms.

We verify (Sum 1–3).

Ad (Sum 1). Suppose given $U \in \mathrm{Ob}(\mathcal{A})$, together with an empty tuple of morphisms. Since S is terminal, there exists a unique morphism $a : U \to S$ satisfying the empty condition.

Ad (Sum 2). Suppose given $V \in \mathrm{Ob}(\mathcal{A})$, together with an empty tuple of morphisms. Since S is initial, there exists a unique morphism $b : S \to V$ satisfying the empty condition.

Ad (Sum 3). The condition is empty.

Conversely, suppose $\tilde{S}$ to be a direct sum of the empty tuple of objects $(\;)$. Then (Sum 1) shows that given $U \in \mathrm{Ob}(\mathcal{A})$, there exists a unique morphism $U \xrightarrow{a} \tilde{S}$. Moreover, (Sum 2) shows that given $V \in \mathrm{Ob}(\mathcal{A})$, there exists a unique morphism $\tilde{S} \xrightarrow{b} V$. Thus $\tilde{S}$ is a zero object in $\mathcal{A}$; cf. Definition 14.(1).

Example 23. Suppose $m = 2$.

Let Z, Z' and Z'' be zero objects in $\mathcal{A}$.

We *claim* that Z'', together with inclusion morphisms $\iota_1 := 0_{Z,Z''}$ and $\iota_2 := 0_{Z',Z''}$ and projection morphisms $\pi_1 := 0_{Z'',Z}$ and $\pi_2 := 0_{Z'',Z'}$, is a direct sum of the pair (Z, Z').

Ad (Sum 1). Suppose given an object U in $\mathcal{A}$ and morphisms $U \xrightarrow{u_1} Z$ and $U \xrightarrow{u_2} Z'$. Then $u_1 = 0_{U,Z}$ and $u_2 = 0_{U,Z'}$. Therefore there exists a unique morphism $U \xrightarrow{a} Z''$ such that $v \cdot \pi_1 = u_1$ and $v \cdot \pi_2 = u_2$, namely $a := 0_{U,Z''}$.

Ad (Sum 2). Suppose given an object V in $\mathcal{A}$ and morphisms $Z \xrightarrow{v_1} V$ and $Z' \xrightarrow{v_2} V$. Then $v_1 = 0_{Z,V}$ and $v_2 = 0_{Z',V}$. Therefore there exists a unique morphism $Z'' \xrightarrow{b} V$ such that $\iota_1 \cdot b = v_1$ and $\iota_2 \cdot b = v_2$, namely $b := 0_{Z'',V}$.

Ad (Sum 3). We have $\iota_1 \cdot \pi_1 = \mathrm{id}_Z$, $\iota_1 \cdot \pi_2 = 0_{Z,Z'}$, $\iota_2 \cdot \pi_1 = 0_{Z',Z}$ and $\iota_2 \cdot \pi_2 = \mathrm{id}_{Z'}$.

Remark 24. *Suppose given $m \geqslant 0$ and $X_1, X_2, \dots, X_m \in \mathrm{Ob}(\mathcal{A})$.*

Suppose given a direct sum S of the tuple $(X_1, \dots, X_m)$ as in Definition 20.

Suppose given an isomorphism $X_i \xrightarrow[\sim]{f_i} X_i'$ in $\mathcal{A}$ for each $i \in [1, m]$.

Suppose given an isomorphism $S \xrightarrow[\sim]{g} S'$ in $\mathcal{A}$.

Then S' is a direct sum of $(X_1', \dots, X_m')$, with inclusion morphisms $f_i^- \cdot \iota_i \cdot g : X_i' \to S'$ and projection morphisms $g^- \cdot \pi_i \cdot f_i : S' \to X_i'$ for $i \in [1, m]$.

Proof.

Ad (Sum 1). Suppose given $U \in \mathrm{Ob}(\mathcal{A})$ and a tuple of morphisms $(U \xrightarrow{u_i'} X_i')_{i \in [1,m]}$. We have to show that there exists a unique morphism $U \xrightarrow{a'} S'$ such that $a' \cdot g^- \cdot \pi_i \cdot f_i = u_i'$ for $i \in [1, m]$.

Existence. By (Sum 1) for the given direct sum, there exists a morphism $U \xrightarrow{a} S$ with $a \cdot \pi_i = u_i' \cdot f_i^- : U \to X_i$ for $i \in [1, m]$. Let $a' := a \cdot g : U \to S'$. Then $a' \cdot g^- \cdot \pi_i \cdot f_i = a \cdot \pi_i \cdot f_i = u_i' : U \to X_i'$ for $i \in [1, m]$.

Uniqueness. Suppose given a', $\tilde{a}' : U \to S'$ with $a' \cdot g^- \cdot \pi_i \cdot f_i = u'_i$ and $\tilde{a}' \cdot g^- \cdot \pi_i \cdot f_i = u'_i$ for $i \in [1, m]$. We have to show that $a' \overset{!}{=} \tilde{a}'$.

We obtain $a' \cdot g^- \cdot \pi_i = u'_i \cdot f_i^-$ and $\tilde{a}' \cdot g^- \cdot \pi_i = u'_i \cdot f_i^-$ for $i \in [1, m]$. By (Sum 1) for the given direct sum, we conclude that $a' \cdot g^- = \tilde{a}' \cdot g^-$. Hence $a' = \tilde{a}'$.

Ad (Sum 2). Suppose given $V \in \mathrm{Ob}(\mathcal{A})$ and a tuple of morphisms $(X'_i \xrightarrow{v'_i} V)_{i \in [1,m]}$ in $\mathcal{A}$. We have to show that there exists an unique morphism $S' \xrightarrow{b'} V$ such that $f_i^- \cdot \iota_i \cdot g \cdot b' = v'_i$ for $i \in [1, m]$.

Existence. By (Sum 2) for the given direct sum, there exists a morphism $S \xrightarrow{b} V$ with $\iota_i \cdot b = f_i \cdot v'_i : X_i \to V$ for $i \in [1, m]$. Let $b' := g^- \cdot b : S' \to V$. Then $f_i^- \cdot \iota_i \cdot g \cdot b' = f_i^- \cdot \iota_i \cdot b = v'_i : X'_i \to V$ for $i \in [1, m]$.

Uniqueness. Suppose given b', $\tilde{b}' : S' \to V$ such that $f_i^- \cdot \iota_i \cdot g \cdot b' = v'_i$ and $f_i^- \cdot \iota_i \cdot g \cdot \tilde{b}' = v'_i$ for $i \in [1, m]$. We have to show that $b' \overset{!}{=} \tilde{b}'$.

We obtain $\iota_i \cdot g \cdot b' = f_i \cdot v'_i$ and $\iota_i \cdot g \cdot \tilde{b}' = f_i \cdot v'_i$ for $i \in [1, m]$. By (Sum 2) for the given direct sum, we conclude that $g \cdot b' = g \cdot \tilde{b}'$. Hence $b' = \tilde{b}'$.

Ad (Sum 3).

We have $(f_i^- \cdot \iota_i \cdot g) \cdot (g^- \cdot \pi_i \cdot f_i) = f_i^- \cdot \iota_i \cdot \pi_i \cdot f_i = f_i^- \cdot \mathrm{id}_{X_i} \cdot f_i = \mathrm{id}_{X'_i}$ for $i \in [1, m]$, using (Sum 3) for the given direct sum.

We have $(f_i^- \cdot \iota_i \cdot g) \cdot (g^- \cdot \pi_j \cdot f_j) = f_i^- \cdot \iota_i \cdot \pi_j \cdot f_j = f_i^- \cdot 0_{X_i, X_j} \cdot f_j = 0_{X'_i, X'_j}$ for $i, j \in [1, m]$ with $i \neq j$, using (Sum 3) for the given direct sum; cf. Lemma 18.(2). □

Remark 25. *Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$.*

Suppose given a direct sum $S \in \mathrm{Ob}(\mathcal{A})$ of $(X_1, \ldots, X_m)$ as in Definition 20.

Suppose that S is a zero object. Then X_i is a zero object for $i \in [1, m]$.

Proof. Suppose given $i \in [1, m]$. Since $\iota_i \cdot \pi_i = \mathrm{id}_{X_i}$, we conclude from S being a zero object that X_i is a zero object; cf. Remark 16. □

Remark 26. Suppose that we are in the situation of Definition 20.

Suppose given $j \in [1, m]$.

Suppose given $Y \xrightarrow{f} X_j \xrightarrow{g} Z$ in $\mathcal{A}$.

By (Sum 3), we have $f \cdot \iota_j \cdot \pi_j = f \cdot 1 = f$ and $f \cdot \iota_j \cdot \pi_i = f \cdot 0 = 0$ for $i \in [1, m] \smallsetminus \{j\}$. Therefore,

$$f \cdot \iota_j \;=\; \underset{\substack{\uparrow \\ \text{position } j}}{\begin{pmatrix} 0 \ldots 0 \; f \; 0 \ldots 0 \end{pmatrix}^S} \;.$$

By (Sum 3), we have $\iota_j \cdot \pi_j \cdot g = 1 \cdot g = g$ and $\iota_i \cdot \pi_j \cdot g = 0 \cdot g = 0$ for $i \in [1, m] \smallsetminus \{j\}$. Therefore,

$$\pi_j \cdot g \;=\; {}^S\!\begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{position } j \;.$$

Lemma 27 (and Definition)**.**

Suppose given $m \geqslant 0$ *and* $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$. *Suppose given a direct sum* S *of* $(X_1, \ldots, X_m)$, *with inclusion morphisms* $X_i \xrightarrow{\iota_i} S$ *and projection morphisms* $S \xrightarrow{\pi_i} X_i$ *for* $i \in [1, m]$.

Suppose given $n \geqslant 0$ *and* $Y_1, Y_2, \ldots, Y_n \in \mathrm{Ob}(\mathcal{A})$. *Suppose given a direct sum* T *of* $(Y_1, \ldots, Y_n)$, *with inclusion morphisms* $Y_j \xrightarrow{\iota_j} T$ *and projection morphisms* $T \xrightarrow{\pi_j} Y_j$ *for* $j \in [1, n]$.

Suppose given morphisms $X_i \xrightarrow{u_{i,j}} Y_j$ *in* $\mathcal{A}$ *for* $i \in [1, m]$ *and* $j \in [1, n]$.

Then there exists a unique morphism $S \xrightarrow{w} T$ *with* $\iota_i \cdot w \cdot \pi_j = u_{i,j}$ *for* $i \in [1, m]$ *and* $j \in [1, n]$.

We write w *as a matrix, i.e.*

$$ {}^{S}\!\begin{pmatrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & & \vdots \\ u_{m,1} & \cdots & u_{m,n} \end{pmatrix}^{T} \;=\; {}^{S}(u_{i,j})_{i,j}^{T} \;:=\; w \;:S \;\to\; T\,. $$

In particular, given a morphism $S \xrightarrow{v} T$, *we have* $v = {}^{S}(\iota_i \cdot v \cdot \pi_j)_{i,j}^{T}$.

Proof. Existence. For each $j \in [1, n]$, (Sum 2) yields a morphism $S \xrightarrow{v_j} Y_j$ such that $\iota_i \cdot v_j = u_{i,j}$ for $i \in [1, m]$. Then, (Sum 1) yields a morphism $S \xrightarrow{w} T$ such that $w \cdot \pi_j = v_j$ for $j \in [1, n]$. Altogether,

$$ \iota_i \cdot w \cdot \pi_j \;=\; \iota_i \cdot v_j \;=\; u_{i,j} $$

for $i \in [1, m]$ and $j \in [1, n]$.

Uniqueness. Suppose given morphisms $S \xrightarrow{w} T$ and $S \xrightarrow{\tilde{w}} T$ with $\iota_i \cdot w \cdot \pi_j = u_{i,j} = \iota_i \cdot \tilde{w} \cdot \pi_j$ for $i \in [1, m]$ and $j \in [1, n]$. We have to show that $w \overset{!}{=} \tilde{w}$.

By (Sum 2), we get $w \cdot \pi_j = \tilde{w} \cdot \pi_j$ for $j \in [1, n]$. By (Sum 1), we get $w = \tilde{w}$ for $j \in [1, n]$. □

Remark 28. *Suppose that we are in the situation of Definition* 20.

Suppose given $U, V \in \mathrm{Ob}(\mathcal{A})$.

Recall that U *is a direct sum of* U, *with inclusion morphism* id_U *and projection morphism* id_U *; cf. Example* 21.

Recall that V *is a direct sum of* V, *with inclusion morphism* id_V *and projection morphism* id_V *; cf. Example* 21.

(1) *For* $m \geqslant 0$ *and each tuple of morphisms* $(U \xrightarrow{u_i} X_i)_{i \in [1,m]}$, *we have*

$$ \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}^{S} \;=\; {}^{U}\!\begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}^{S}. $$

(2) *For* $m \geqslant 0$ *and each tuple of morphisms* $(X_i \xrightarrow{v_i} V)_{i \in [1,m]}$, *we have*

$$ {}^{S}\!\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \;=\; {}^{S}\!\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}^{V} $$

(3) *Given a morphism* $f : U \to V$, *we have* ${}^{U}(f)^{V} = f$.

Proof.

Ad (1). Write a for the left hand side and b for the right hand side of the equation. Then a is characterized by $a \cdot \pi_i = u_i$ for $i \in [1,m]$; cf. Definition 20. Moreover, b is characterized by $\iota_1 \cdot b \cdot \pi_i = u_i$ for $i \in [1,m]$; cf. Lemma 27. Since $\iota_1 = \mathrm{id}_U$, we conclude that $a = b$.

Ad (2). Write a for the left hand side and b for the right hand side of the equation. Then a is characterized by $\iota_i \cdot a = v_i$ for $i \in [1,m]$; cf. Definition 20. Moreover, b is characterized by $\iota_i \cdot b \cdot \pi_1 = v_i$ for $i \in [1,m]$; cf. Lemma 27. Since $\pi_1 = \mathrm{id}_V$, we conclude that $a = b$.

Ad (3). We have $f = \iota_1 \cdot {}^U(f)^V \cdot \pi_1 = \mathrm{id}_U \cdot {}^U(f)^V \cdot \mathrm{id}_V = {}^U(f)^V$. □

Remark 29. *Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$. Suppose given a direct sum S of $(X_1, \ldots, X_m)$, with inclusion morphisms $X_i \xrightarrow{\iota_i} S$ and projection morphisms $S \xrightarrow{\pi_i} X_i$ for $i \in [1,m]$.*

For $i,\, j \in [1,m]$, we write

$$\delta_{i,j} := \begin{cases} \mathrm{id}_{X_i} & \text{if } i = j \\ 0_{X_i,X_j} & \text{if } i \neq j\,. \end{cases}$$

Then

$$\mathrm{id}_S = {}^S(\delta_{i,j})^S_{i,j}\,.$$

Proof. We have to show that $\iota_i \cdot \mathrm{id}_S \cdot \pi_j \overset{!}{=} \delta_{i,j}$ for $i,\, j \in [1,m]$; cf. Lemma 27.

In fact, if $i \neq j$, then $\iota_i \cdot \mathrm{id}_S \cdot \pi_j = \iota_i \cdot \pi_j = 0_{X_i,X_j} = \delta_{i,j}$ by (Sum 3). If $i = j$, then $\iota_i \cdot \mathrm{id}_S \cdot \pi_j = \iota_i \cdot \pi_i = \mathrm{id}_{X_i} = \delta_{i,j}$ by (Sum 3). □

Remark 30.

Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$.

Suppose given direct sums S and S' of $(X_1, \ldots, X_m)$.

Suppose given $n \geqslant 0$ and $Y_1, Y_2, \ldots, Y_n \in \mathrm{Ob}(\mathcal{A})$.

Suppose given direct sums T and T' of $(Y_1, \ldots, Y_n)$.

Suppose given morphisms $X_i \xrightarrow{u_{i,j}} Y_j$ in $\mathcal{A}$ for $i \in [1,m]$ and $j \in [1,n]$.

Write

$$\delta_{i,i'} := \begin{cases} \mathrm{id}_{X_i} & \text{if } i = i' \\ 0_{X_i,X_{i'}} & \text{if } i \neq i' \end{cases}$$

for $i,\, i' \in [1,m]$ and

$$\tilde{\delta}_{j,j'} := \begin{cases} \mathrm{id}_{Y_j} & \text{if } j = j' \\ 0_{Y_j,Y_{j'}} & \text{if } j \neq j' \end{cases}$$

for $j,\, j' \in [1,n]$.

The following assertions $(1,2)$ hold.

(1) *We have*

$${}^S(\delta_{i,i'})^{S'}_{i,i'} \cdot {}^{S'}(u_{i',j'})^{T'}_{i',j'} \cdot {}^{T'}(\tilde{\delta}_{j',j})^T_{j',j} = {}^S(u_{i,j})^T_{i,j}\,.$$

(2) *We have* ${}^{S}(\delta_{i,i'})_{i,i'}^{S'} \cdot {}^{S'}(\delta_{i',i})_{i',i}^{S} = \mathrm{id}_S$ *and* ${}^{S'}(\delta_{i',i})_{i',i}^{S} \cdot {}^{S}(\delta_{i,i'})_{i,i'}^{S'} = \mathrm{id}_{S'}$.
In particular, ${}^{S}(\delta_{i,i'})_{i,i'}^{S'} : S \to S'$ *is an isomorphism.*

(3) *Suppose given* $\ell \in [1,n]$. *We have* ${}^{S}(u_{i,j})_{i,j}^{T} \cdot \pi_\ell = {}^{S}\!\begin{pmatrix} u_{1,\ell} \\ \vdots \\ u_{m,\ell} \end{pmatrix}$.

(4) *Suppose given* $k \in [1,m]$. *We have* $\iota_k \cdot {}^{S}(u_{i,j})_{i,j}^{T} = (u_{k,1} \,\cdots\, u_{k,n})^{T}$.

(5) *We have* ${}^{S}(u_{i,j})_{i,j}^{T} = {}^{S}\!\begin{pmatrix} (u_{1,1} \,\cdots\, u_{1,n})^{T} \\ \vdots \\ (u_{m,1} \,\cdots\, u_{m,n})^{T} \end{pmatrix}$.

(6) *We have* ${}^{S}(u_{i,j})_{i,j}^{T} = \left({}^{S}\!\begin{pmatrix} u_{1,1} \\ \vdots \\ u_{m,1} \end{pmatrix} \cdots {}^{S}\!\begin{pmatrix} u_{1,n} \\ \vdots \\ u_{m,n} \end{pmatrix} \right)^{T}$.

Proof.

Ad (1). We have to show that

$$\iota_k \cdot \left({}^{S}(\delta_{i,i'})_{i,i'}^{S'} \cdot {}^{S'}(u_{i',j'})_{i',j'}^{T'} \cdot {}^{T'}(\tilde{\delta}_{j',j})_{j',j}^{T} \right) \cdot \pi_\ell \overset{!}{=} u_{k,\ell}$$

for $k \in [1,m]$ and $\ell \in [1,n]$; cf. Lemma 27.

We claim that $\iota_k \cdot {}^{S}(\delta_{i,i'})_{i,i'}^{S'} \overset{!}{=} \iota_k$. By (Sum 1), it suffices to show that $\iota_k \cdot {}^{S}(\delta_{i,i'})_{i,i'}^{S'} \cdot \pi_p \overset{!}{=} \iota_k \cdot \pi_p$ for $p \in [1,m]$. In fact, $\iota_k \cdot {}^{S}(\delta_{i,i'})_{i,i'}^{S'} \cdot \pi_p = \delta_{k,p} = \iota_k \cdot \pi_p$ by (Sum 3).

We claim that ${}^{T'}(\tilde{\delta}_{j',j})_{j',j}^{T} \cdot \pi_\ell \overset{!}{=} \pi_\ell$. By (Sum 2), it suffices to show that $\iota_q \cdot {}^{T'}(\tilde{\delta}_{j',j})_{j',j}^{T} \cdot \pi_\ell \overset{!}{=} \iota_q \cdot \pi_\ell$ for $q \in [1,n]$. In fact, $\iota_q \cdot {}^{T'}(\tilde{\delta}_{j',j})_{j',j}^{T} \cdot \pi_\ell = \tilde{\delta}_{q,\ell} = \iota_q \cdot \pi_\ell$ by (Sum 3).

So we get

$$\iota_k \cdot \left({}^{S}(\delta_{i,i'})_{i,i'}^{S'} \cdot {}^{S'}(u_{i',j'})_{i',j'}^{T'} \cdot {}^{T'}(\tilde{\delta}_{j',j})_{j',j}^{T} \right) \cdot \pi_\ell \;=\; \iota_k \cdot {}^{S'}(u_{i',j'})_{i',j'}^{T'} \cdot \pi_\ell \;=\; u_{k,\ell} \,.$$

Ad (2). Using Remark 29 and (1), we obtain

$$\begin{aligned} {}^{S}(\delta_{i,i'})_{i,i'}^{S'} \cdot {}^{S'}(\delta_{i',i})_{i',i}^{S} &= {}^{S}(\delta_{i,i'})_{i,i'}^{S'} \cdot {}^{S'}(\delta_{i,i'})_{i,i'}^{S} \cdot {}^{S}(\delta_{p,q})_{p,q}^{S} \\ &= {}^{S}(\delta_{i,i'})_{i,i'}^{S} \\ &= \mathrm{id}_S \,. \end{aligned}$$

Likewise, we obtain ${}^{S'}(\delta_{i',i})_{i',i}^{S} \cdot {}^{S}(\delta_{i,i'})_{i,i'}^{S'} = \mathrm{id}_{S'}$.

Ad (3). For $k \in [1,m]$, we have $\iota_k \cdot {}^{S}(u_{i,j})_{i,j}^{T} \cdot \pi_\ell = u_{k,\ell}$. This proves the required equality by (Sum 2).

Ad (4). For $\ell \in [1,n]$, we have $\iota_k \cdot {}^{S}(u_{i,j})_{i,j}^{T} \cdot \pi_\ell = u_{k,\ell}$. This proves the required equality by (Sum 1).

Ad (5). For $k \in [1,m]$ and $\ell \in [1,n]$, we have $\iota_k \cdot {}^{S}\!\begin{pmatrix} (u_{1,1} \,\cdots\, u_{1,n})^{T} \\ \vdots \\ (u_{m,1} \,\cdots\, u_{m,n})^{T} \end{pmatrix} \cdot \pi_\ell = (u_{k,1} \,\cdots\, u_{k,n})^{T} \cdot \pi_\ell = u_{k,\ell}$; cf. Lemma 27.

Ad (6). For $k \in [1,m]$ and $\ell \in [1,n]$, we have $\iota_k \cdot \left({}^{S}\!\begin{pmatrix} u_{1,1} \\ \vdots \\ u_{m,1} \end{pmatrix} \cdots {}^{S}\!\begin{pmatrix} u_{1,n} \\ \vdots \\ u_{m,n} \end{pmatrix} \right)^{T} \cdot \pi_\ell = \iota_k \cdot {}^{S}\!\begin{pmatrix} u_{1,\ell} \\ \vdots \\ u_{m,\ell} \end{pmatrix} = u_{k,\ell}$; cf. Lemma 27. □

2.2.2 Definition of additive categories

Suppose given a pointed category $\mathcal{A}$.

Definition 31. The pointed category $\mathcal{A}$ is called *additive* if the following axioms (Add 1–2) hold.

(Add 1) For each $(X, Y) \in \operatorname{Ob}(\mathcal{A}) \times \operatorname{Ob}(\mathcal{A})$, there exists a direct sum of (X, Y); cf. Definition 20.

(Add 2) For each $X \in \operatorname{Ob}(\mathcal{A})$ and each direct sum S of (X, X), the morphism ${}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{S} : S \to S$ is an isomorphism; cf. Lemma 27.

We reformulate (Add 2) of Definition 31.

Remark 32. *Suppose that the pointed category $\mathcal{A}$ satisfies* (Add 1)*.*

Consider the following condition.

(Add 2′) For each $X \in \operatorname{Ob}(\mathcal{A})$ and some direct sum S of (X, X), the morphism ${}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{S} : S \to S$ is an isomorphism.

Then (Add 2) *holds if and only if* (Add 2′) *holds.*

Proof.

Suppose that (Add 2) *holds.* By (Add 1), there exists a direct sum S of (X, X). By (Add 2), ${}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{S}$ is an isomorphism. So (Add 2′) holds.

Suppose that (Add 2′) *holds.* So we may choose a direct sum S of (X, X) such that ${}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{S}$ is an isomorphism.

We want to show (Add 2). Suppose given a direct sum $\tilde{S}$ of (X, X). By Remark 30.(1), we obtain

$$
{}^{\tilde{S}}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{\tilde{S}} = {}^{\tilde{S}}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ 0_{X,X} & \mathrm{id}_X \end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ 0_{X,X} & \mathrm{id}_X \end{pmatrix}^{\tilde{S}} ,
$$

of which all three factors are isomorphisms; cf. Remark 30.(2). So ${}^{\tilde{S}}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{\tilde{S}}$ is an isomorphism. This shows that (Add 2) holds.

2.2.3 Elementary properties of additive categories

Suppose given an additive category $\mathcal{A}$; cf. Definition 31.

Definition 33 (and Remark)**.** Suppose given $X, Y \in \operatorname{Ob}(\mathcal{A})$.

We choose a direct sum S of (X, X) and a direct sum T of (Y, Y).

Suppose given morphisms $a, b : X \to Y$ in $\mathcal{A}$.

Let

$$a+b \ :=\ \begin{pmatrix}\mathrm{id}_X & \mathrm{id}_X\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix} a & 0_{X,Y} \\ 0_{X,Y} & b \end{pmatrix}^{T} \cdot {}^{T}\!\begin{pmatrix}\mathrm{id}_Y \\ \mathrm{id}_Y\end{pmatrix} .$$

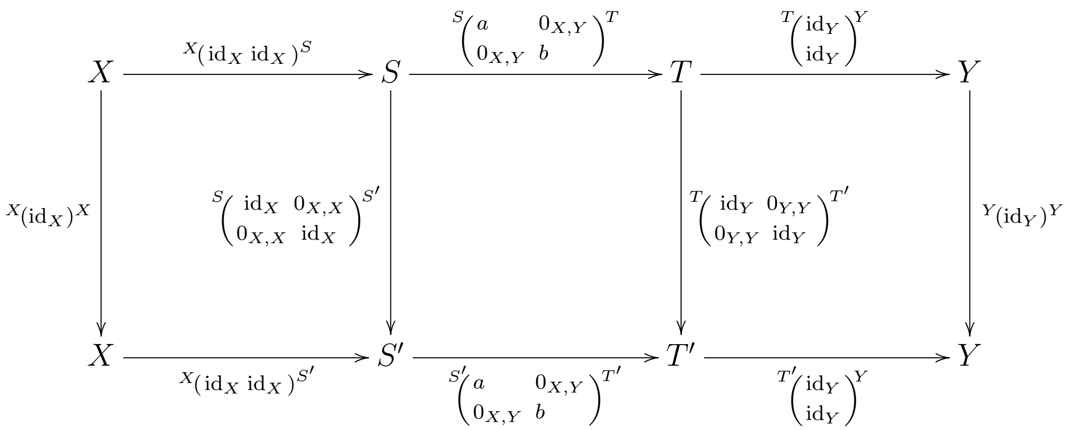
This is independent of the choice of the direct sum S of (X, X) and of the direct sum T of (Y, Y).

Proof. We shall prove the claimed independence of choice. So suppose given a direct sum S' of (X, X) and a direct sum T' of (Y, Y).

Note that $\begin{pmatrix}\mathrm{id}_X & \mathrm{id}_X\end{pmatrix}^{S} = {}^{X}\!\begin{pmatrix}\mathrm{id}_X & \mathrm{id}_X\end{pmatrix}^{S}$ and ${}^{T}\!\begin{pmatrix}\mathrm{id}_Y \\ \mathrm{id}_Y\end{pmatrix} = {}^{T}\!\begin{pmatrix}\mathrm{id}_Y \\ \mathrm{id}_Y\end{pmatrix}^{Y}$; cf. Remark 28.(1, 2).

Note that ${}^{X}(\mathrm{id}_X)^{X} = \mathrm{id}_X$ and ${}^{Y}(\mathrm{id}_Y)^{Y} = \mathrm{id}_Y$; cf. Remark 28.(3).

The following diagram commutes by Remarks 30.(1) and 29.



Hence the choice of S and T yields the same result as the choice of S' and T'. □

Notation 34. Given $X \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z$ and $X \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z$ in $\mathcal{A}$, we write

$$f_1 \cdot g_1 + f_2 \cdot g_2 \ :=\ (f_1 \cdot g_1) + (f_2 \cdot g_2) .$$

Remark 35. *Suppose that we are in the situation of Definition* 33. *Recall that* $a,\, b : X \to Y$, *that* S *is a direct sum of* (X, X) *and that* T *is a direct sum of* (Y, Y).

Then

$$a+b \ =\ \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{T} \cdot {}^{T}\!\begin{pmatrix}1 \\ 1\end{pmatrix} \ =\ \begin{pmatrix}a & b\end{pmatrix}^{T} \cdot {}^{T}\!\begin{pmatrix}1 \\ 1\end{pmatrix} \ =\ \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a \\ b\end{pmatrix} .$$

Proof. To show the equality of the second and the third term, we want to show

$$u \ :=\ \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{T} \ \overset{!}{=}\ \begin{pmatrix}a & b\end{pmatrix}^{T} .$$

We have to show that $u \cdot \pi_1 \overset{!}{=} a$ and that $u \cdot \pi_2 \overset{!}{=} b$; cf. (Sum 1).

Writing $v := {}^{S}\!\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{T}$, we obtain $v \cdot \pi_1 = {}^{S}\!\begin{pmatrix}a \\ 0\end{pmatrix}$, since $\iota_1 \cdot v \cdot \pi_1 = a$ and $\iota_2 \cdot v \cdot \pi_1 = 0$; cf. (Sum 2).

Moreover, ${}^{S}\!\begin{pmatrix}a \\ 0\end{pmatrix} = \pi_1 \cdot a$; cf. Remark 26. So we get

$$u \cdot \pi_1 \ =\ \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot v \cdot \pi_1 \ =\ \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot \pi_1 \cdot a \ =\ 1 \cdot a \ =\ a .$$

Analogously, we obtain $u \cdot \pi_2 = b$.

To show the equality of the second and the fourth term, we want to show

$$w \ := \ {}^{S}\!\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{T} \cdot {}^{T}\!\begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \overset{!}{=} \ {}^{S}\!\begin{pmatrix} a \\ b \end{pmatrix} \ .$$

We have to show that $\iota_1 \cdot w \overset{!}{=} a$ and that $\iota_2 \cdot w \overset{!}{=} b$; cf. (Sum 2).

Writing $v = {}^{S}\!\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{T}$, we obtain $\iota_1 \cdot v = \begin{pmatrix} a & 0 \end{pmatrix}^{T}$, since $\iota_1 \cdot v \cdot \pi_1 = a$ and $\iota_1 \cdot v \cdot \pi_2 = 0$; cf. (Sum 1).

Moreover, $\begin{pmatrix} a & 0 \end{pmatrix}^{T} = a \cdot \iota_1$; cf. Remark 26. So we get

$$\iota_1 \cdot w \ = \ \iota_1 \cdot v \cdot {}^{T}\!\begin{pmatrix} 1 \\ 1 \end{pmatrix} \ = \ a \cdot \iota_1 \cdot {}^{T}\!\begin{pmatrix} 1 \\ 1 \end{pmatrix} \ = \ a \cdot 1 \ = \ a \ .$$

Analogously, we obtain $\iota_2 \cdot w = b$. □

Lemma 36. *Suppose given $m \geqslant 3$ and a tuple of objects $(X_1, \ldots, X_m)$ in $\mathcal{A}$.*

Suppose given a direct sum S' of $(X_1, \ldots, X_{m-1})$. Write its inclusion morphisms ι'_i and its projection morphisms π'_i for $i \in [1, m-1]$.

Suppose given a direct sum S of (S', X_m). Write its inclusion morphisms ι''_i and its projection morphisms π''_i for $i \in [1, 2]$.

Then S is a direct sum of $(X_1, \ldots, X_{m-1}, X_m)$ with inclusion morphisms

$$\iota_i \ := \ \begin{cases} \iota'_i \cdot \iota''_1 & \text{if } i \in [1, m-1] \\ \iota''_2 & \text{if } i = m \end{cases}$$

and projection morphisms

$$\pi_i \ := \ \begin{cases} \pi''_1 \cdot \pi'_i & \text{if } i \in [1, m-1] \\ \pi''_2 & \text{if } i = m \end{cases}$$

for $i \in [1, m]$.

Proof.

Ad (Sum 1). Suppose given $U \in \mathrm{Ob}(\mathcal{A})$ and a tuple of morphisms $(U \xrightarrow{u_i} X_i)_{i \in [1,m]}$. We have to show the existence and the uniqueness of a morphism $U \xrightarrow{v} S$ such that $v \cdot \pi_i = u_i$ for $i \in [1, m]$.

Existence. Let $v := \big((u_1 \, \ldots \, u_{m-1})^{S'} \; u_m \big)^{S}$.

For $i \in [1, m-1]$, we have $v \cdot \pi_i = v \cdot \pi''_1 \cdot \pi'_i = (u_1 \, \ldots \, u_{m-1})^{S'} \cdot \pi'_i = u_i$.

For $i = m$, we have $v \cdot \pi_m = v \cdot \pi''_2 = u_m$.

Uniqueness. Suppose given $U \xrightarrow{\tilde{v}} S$ such that $\tilde{v} \cdot \pi_i = u_i$ for $i \in [1, m]$. So we have $\tilde{v} \cdot \pi''_1 \cdot \pi'_i = u_i$ for $i \in [1, m-1]$ and $\tilde{v} \cdot \pi''_2 = u_m$.

We have to show that $\tilde{v} \overset{!}{=} \big((u_1 \, \ldots \, u_{m-1})^{S'} \; u_m \big)^{S}$. So we have to show that $\tilde{v} \cdot \pi''_1 \overset{!}{=} (u_1 \, \ldots \, u_{m-1})^{S'}$ and $\tilde{v} \cdot \pi''_2 \overset{!}{=} u_m$. The second equality holds.

To show the first equality, we have to show that $\tilde{v} \cdot \pi''_1 \cdot \pi'_i \overset{!}{=} u_i$ for $i \in [1, m-1]$. This equality holds.

Ad (Sum 2). Suppose given $U \in \mathrm{Ob}(\mathcal{A})$ and a tuple of morphisms $(X_i \xrightarrow{u_i} U)_{i\in[1,m]}$. We have to show the existence and the uniqueness of a morphism $S \xrightarrow{v} U$ such that $\iota_i \cdot v = u_i$ for $i \in [1, m]$.

Existence. Let $v := {}^{S}\!\begin{pmatrix} {}^{S'}\!\begin{pmatrix} u_1 \\ \vdots \\ u_{m-1} \end{pmatrix} \\ u_m \end{pmatrix}$.

For $i \in [1, m-1]$, we have $\iota_i \cdot v = \iota'_i \cdot \iota''_1 \cdot v = \iota'_i \cdot {}^{S'}\!\begin{pmatrix} u_1 \\ \vdots \\ u_{m-1} \end{pmatrix} = u_i$.

For $i = m$, we have $\iota_m \cdot v = \iota''_2 \cdot v = u_m$.

Uniqueness. Suppose given $U \xrightarrow{\tilde{v}} S$ such that $\iota_i \cdot \tilde{v} = u_i$ for $i \in [1, m]$. So we have $\iota'_i \cdot \iota''_1 \cdot \tilde{v} = u_i$ for $i \in [1, m-1]$ and $\iota''_2 \cdot \tilde{v} = u_m$.

We have to show that $\tilde{v} \overset{!}{=} {}^{S}\!\begin{pmatrix} {}^{S'}\!\begin{pmatrix} u_1 \\ \vdots \\ u_{m-1} \end{pmatrix} \\ u_m \end{pmatrix}$. So we have to show that $\iota''_1 \cdot \tilde{v} \overset{!}{=} {}^{S'}\!\begin{pmatrix} u_1 \\ \vdots \\ u_{m-1} \end{pmatrix}$ and $\iota''_2 \cdot \tilde{v} \overset{!}{=} u_m$. The second equality holds.

To show the first equality, we have to show that $\iota'_i \cdot \iota''_1 \cdot \tilde{v} \overset{!}{=} u_i$ for $i \in [1, m-1]$. This equality holds.

Ad (Sum 3).

Suppose given $i \in [1, m]$. We have to show that $\iota_i \cdot \pi_i \overset{!}{=} \mathrm{id}_{X_i}$.

If $i \in [1, m-1]$, then $\iota_i \cdot \pi_i = \iota'_i \cdot \iota''_1 \cdot \pi''_1 \cdot \pi'_i = \iota'_i \cdot \mathrm{id}_{S'} \cdot \pi'_i = \mathrm{id}_{X_i}$.

If $i = m$, then $\iota_m \cdot \pi_m = \iota''_2 \cdot \pi''_2 = \mathrm{id}_{X_m}$.

Suppose given $i,\ j \in [1, m]$ such that $i \neq j$. We have to show that $\iota_i \cdot \pi_j \overset{!}{=} 0_{X_i, X_j}$.

If $i \in [1, m-1]$ and $j \in [1, m-1]$, then $\iota_i \cdot \pi_j = \iota'_i \cdot \iota''_1 \cdot \pi''_1 \cdot \pi'_j = \iota'_i \cdot \mathrm{id}_{S'} \cdot \pi'_j = 0_{X_i, X_j}$.

If $i \in [1, m-1]$ and $j = m$, then $\iota_i \cdot \pi_m = \iota'_i \cdot \iota''_1 \cdot \pi''_2 = \iota'_i \cdot 0_{S', X_m} = 0_{X_i, X_m}$.

If $i = m$ and $j \in [1, m-1]$, then $\iota_m \cdot \pi_j = \iota''_2 \cdot \pi''_1 \cdot \pi'_j = 0_{X_m, S'} \cdot \pi'_j = 0_{X_m, X_j}$. □

Lemma 37. *Suppose given $m \geqslant 0$ and a tuple of objects $(X_1, \ldots, X_m)$ in $\mathcal{A}$.*

Then $(X_1, \ldots, X_m)$ has a direct sum; cf. Definition 20.

Proof. We proceed by induction on $m \geqslant 0$.

In the case $m = 0$, the assertion holds by Example 22.

In the case $m = 1$, the assertion holds by Example 21.

In the case $m = 2$, the assertion holds by (Add 1).

Suppose given $m \geqslant 3$.

Choose a direct sum S' of $(X_1, \ldots, X_{m-1})$, which is possible by induction.

Choose a direct sum S of (S', X_m), which is possible by (Add 1).

Then S is a direct sum of $(X_1, \ldots, X_m)$ by Lemma 36. □

Lemma 38. *Suppose given $a, b : X \to Y$ in $\mathcal{A}$.*

(1) *We have $a + b = b + a$.*

(2) *We have $a + 0 = a$.*

(3) *Suppose given $u : X' \to X$ and $v : Y \to Y'$ in $\mathcal{A}$. Then $u \cdot (a + b) \cdot v = u \cdot a \cdot v + u \cdot b \cdot v$.*

Proof. We choose a direct sum S of (X, X).

Ad (1). We have $\begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{S} = \begin{pmatrix}1 & 1\end{pmatrix}^{S}$ since $\begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{S} \cdot \pi_1 = \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot \pi_2 = 1$ and $\begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{S} \cdot \pi_2 = \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot \pi_1 = 1$; cf. Remark 30.(3), Remark 26, (Sum 1).

We have ${}^{S}\!\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} = {}^{S}\!\begin{pmatrix}b\\ a\end{pmatrix}$, since $\iota_1 \cdot {}^{S}\!\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} = \iota_2 \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} = b$ and $\iota_2 \cdot {}^{S}\!\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} = \iota_1 \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} = a$; cf. Remarks 30.(4) and 26, (Sum 2).

So

$$a + b \;=\; \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} \;=\; \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} \;=\; \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}b\\ a\end{pmatrix} \;=\; b + a\,;$$

cf. Remark 35.

Ad (2). We have

$$a + 0 \;=\; \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ 0\end{pmatrix} \;=\; \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot \pi_1 \cdot a \;=\; 1 \cdot a = a\,;$$

cf. Remarks 35 and 26, (Sum 1).

Ad (3). We have ${}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} \cdot v = {}^{S}\!\begin{pmatrix}a\cdot v\\ b\cdot v\end{pmatrix}$ since $\iota_1 \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} \cdot v = a \cdot v$ and $\iota_2 \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} \cdot v = b \cdot v$; cf. (Sum 2). Thus

$$(a + b) \cdot v \;=\; \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\\ b\end{pmatrix} \cdot v \;=\; \begin{pmatrix}1 & 1\end{pmatrix}^{S} \cdot {}^{S}\!\begin{pmatrix}a\cdot v\\ b\cdot v\end{pmatrix} \;=\; a \cdot v + b \cdot v\,,$$

cf. Remark 35.

We choose a direct sum T' of (Y', Y').

We have $u \cdot \begin{pmatrix}a\cdot v & b\cdot v\end{pmatrix}^{T'} = \begin{pmatrix}u\cdot a\cdot v & u\cdot b\cdot v\end{pmatrix}^{T'}$ since $u \cdot \begin{pmatrix}a\cdot v & b\cdot v\end{pmatrix}^{T'} \cdot \pi_1 = u \cdot a \cdot v$ and $u \cdot \begin{pmatrix}a\cdot v & b\cdot v\end{pmatrix}^{T'} \cdot \pi_2 = u \cdot b \cdot v$; cf. (Sum 1). Thus

$$u \cdot (a + b) \cdot v \;=\; u \cdot (a \cdot v + b \cdot v) \;=\; u \cdot \begin{pmatrix}a\cdot v & b\cdot v\end{pmatrix}^{T'} \cdot {}^{T'}\!\begin{pmatrix}1\\ 1\end{pmatrix} \;=\; \begin{pmatrix}u\cdot a\cdot v & u\cdot b\cdot v\end{pmatrix}^{T'} \cdot {}^{T'}\!\begin{pmatrix}1\\ 1\end{pmatrix} \;=\; u \cdot a \cdot v + u \cdot b \cdot v\,,$$

cf. Remark 35. □

Lemma 39.

Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum S of $(X_1, \ldots, X_m)$.

Suppose given $n \geqslant 0$ and $Y_1, Y_2, \ldots, Y_n \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum T of $(Y_1, \ldots, Y_n)$.

Suppose given morphisms $X_i \xrightarrow{u_{i,j}} Y_j$ and $X_i \xrightarrow{v_{i,j}} Y_j$ in $\mathcal{A}$ for $i \in [1, m]$ and $j \in [1, n]$.

Then

$${}^{S}(u_{i,j})^{T}_{i,j} + {}^{S}(v_{i,j})^{T}_{i,j} \;=\; {}^{S}(u_{i,j} + v_{i,j})^{T}_{i,j}\,.$$

Proof. For $k \in [1, m]$ and $\ell \in [1, n]$, we have

$$\begin{aligned} \iota_k \cdot ({}^S(u_{i,j})_{i,j}^T + {}^S(v_{i,j})_{i,j}^T) \cdot \pi_\ell &\overset{\text{L. 38.(3)}}{=} \iota_k \cdot {}^S(u_{i,j})_{i,j}^T \cdot \pi_\ell + \iota_k \cdot {}^S(v_{i,j})_{i,j}^T \cdot \pi_\ell \\ &\overset{\text{L. 27}}{=} u_{k,\ell} + v_{k,\ell} \\ &\overset{\text{L. 27}}{=} \iota_k \cdot {}^S(u_{i,j} + v_{i,j})_{i,j}^T \cdot \pi_\ell \ . \end{aligned}$$

Thus ${}^S(u_{i,j})_{i,j}^T + {}^S(v_{i,j})_{i,j}^T = {}^S(u_{i,j} + v_{i,j})_{i,j}^T$; cf. Lemma 27. □

Lemma 40. *Suppose given $X, Y \in \mathrm{Ob}(\mathcal{A})$. Suppose given $a, b, c : X \to Y$.*

Then $a + (b + c) = (a + b) + c$.

We shall write $a + b + c := a + (b + c) = (a + b) + c$, etc.

Proof. Let S be a direct sum of (X, X). Let T be a direct sum of (Y, Y, Y).

We have

$$a + (b + c) \overset{\text{R. 35}}{=} \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)^S \cdot {}^S\!\left(\begin{smallmatrix} a \\ b+c \end{smallmatrix}\right) .$$

We *claim* that ${}^S\!\left(\begin{smallmatrix} a \\ b+c \end{smallmatrix}\right) \overset{!}{=} {}^S\!\left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & c \end{smallmatrix}\right)^T \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$.

It suffices to show equality after composition with ι_1 and ι_2 from the left; cf. (Sum 2).

We have $\iota_1 \cdot {}^S\!\left(\begin{smallmatrix} a \\ b+c \end{smallmatrix}\right) = a$.

We have $\iota_1 \cdot {}^S\!\left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & c \end{smallmatrix}\right)^T \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) \overset{\text{R. 30.(4)}}{=} \left(\begin{smallmatrix} a & 0 & 0 \end{smallmatrix}\right)^T \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) \overset{\text{R. 26}}{=} a \cdot \iota_1 \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) = a \cdot 1 = a$.

We have $\iota_2 \cdot {}^S\!\left(\begin{smallmatrix} a \\ b+c \end{smallmatrix}\right) = b + c$.

We have

$$\begin{aligned} &\iota_2 \cdot {}^S\!\left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & c \end{smallmatrix}\right)^T \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) \overset{\text{R. 30.(4)}}{=} \left(\begin{smallmatrix} 0 & b & c \end{smallmatrix}\right)^T \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) \overset{\text{L. 39}}{=} \left(\left(\begin{smallmatrix} 0 & b & 0 \end{smallmatrix}\right)^T + \left(\begin{smallmatrix} 0 & 0 & c \end{smallmatrix}\right)^T\right) \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) \\ &\overset{\text{L. 38.(3)}}{=} \left(\begin{smallmatrix} 0 & b & 0 \end{smallmatrix}\right)^T \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 & c \end{smallmatrix}\right)^T \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) \overset{\text{R. 26}}{=} b \cdot \iota_2 \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) + c \cdot \iota_3 \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) = b \cdot 1 + c \cdot 1 = b + c \ . \end{aligned}$$

This proves the *claim*.

So we can continue to calculate

$$\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)^S \cdot {}^S\!\left(\begin{smallmatrix} a \\ b+c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)^S \cdot {}^S\!\left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & c \end{smallmatrix}\right)^T \cdot {}^T\!\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) .$$

We *claim* that $\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)^S \cdot {}^S\!\left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & c \end{smallmatrix}\right)^T \overset{!}{=} \left(\begin{smallmatrix} a & b & c \end{smallmatrix}\right)^T$. It suffices to show equality after composition with π_1, π_2 and π_3 from the right; cf. (Sum 1).

We have $\left(\begin{smallmatrix} a & b & c \end{smallmatrix}\right)^T \cdot \pi_1 = a$.

We have $\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)^S \cdot {}^S\!\left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & c \end{smallmatrix}\right)^T \cdot \pi_1 \overset{\text{R. 30.(3)}}{=} \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)^S \cdot {}^S\!\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) \overset{\text{R. 35}}{=} a + 0 \overset{\text{L. 38}}{=} a$.

We have $\left(\begin{smallmatrix} a & b & c \end{smallmatrix}\right)^T \cdot \pi_2 = b$.

We have $\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)^S \cdot {}^S\!\left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & c \end{smallmatrix}\right)^T \cdot \pi_2 \overset{\text{R. 30.(3)}}{=} \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)^S \cdot {}^S\!\left(\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right) \overset{\text{R. 35}}{=} 0 + b \overset{\text{L. 38}}{=} b$.

We have $\left(\begin{smallmatrix} a & b & c \end{smallmatrix}\right)^T \cdot \pi_3 = c$.

We have $\begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \end{pmatrix}^T \cdot \pi_3 \overset{\text{R. 30.(3)}}{=} \begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} 0 \\ c \end{pmatrix} \overset{\text{R. 35}}{=} 0 + c \overset{\text{L. 38}}{=} c.$

This proves the *claim*.

So we can continue to calculate

$$\begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} a & b & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;.$$

So far, we have

$$a + (b + c) \;=\; \begin{pmatrix} a & b & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;.$$

We have

$$(a + b) + c \overset{\text{R. 35}}{=} \begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a+b \\ c \end{pmatrix} .$$

We *claim* that ${}^S\!\begin{pmatrix} a+b \\ c \end{pmatrix} \overset{!}{=} {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

It suffices to show equality after composition with ι_1 and ι_2 from the left; cf. (Sum 2).

We have $\iota_2 \cdot {}^S\!\begin{pmatrix} a+b \\ c \end{pmatrix} = c$.

We have $\iota_2 \cdot {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \overset{\text{R. 30.(4)}}{=} \begin{pmatrix} 0 & 0 & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \overset{\text{R. 26}}{=} c \cdot \iota_3 \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = c \cdot 1 = c.$

We have $\iota_1 \cdot {}^S\!\begin{pmatrix} a+b \\ c \end{pmatrix} = a + b$.

We have

$$\begin{aligned} & \iota_1 \cdot {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \overset{\text{R. 30.(4)}}{=} \begin{pmatrix} a & b & 0 \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \overset{\text{L. 39}}{=} \left(\begin{pmatrix} a & 0 & 0 \end{pmatrix}^T + \begin{pmatrix} 0 & b & 0 \end{pmatrix}^T\right) \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ & \overset{\text{L. 38.(3)}}{=} \begin{pmatrix} a & 0 & 0 \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & b & 0 \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \overset{\text{R. 26}}{=} a \cdot \iota_1 \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \cdot \iota_2 \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = a \cdot 1 + b \cdot 1 = a + b \;. \end{aligned}$$

This proves the *claim*.

So we can continue to calculate

$$\begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a+b \\ c \end{pmatrix} \;=\; \begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;.$$

We *claim* that $\begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \overset{!}{=} \begin{pmatrix} a & b & c \end{pmatrix}^T$. It suffices to show equality after composition with π_1, π_2 and π_3 from the right; cf. (Sum 1).

We have $\begin{pmatrix} a & b & c \end{pmatrix}^T \cdot \pi_1 = a$.

We have $\begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \cdot \pi_1 \overset{\text{R. 30.(3)}}{=} \begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a \\ 0 \end{pmatrix} \overset{\text{R. 35}}{=} a + 0 \overset{\text{L. 38}}{=} a.$

We have $\begin{pmatrix} a & b & c \end{pmatrix}^T \cdot \pi_2 = b$.

We have $\begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \cdot \pi_2 \overset{\text{R. 30.(3)}}{=} \begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} b \\ 0 \end{pmatrix} \overset{\text{R. 35}}{=} b + 0 \overset{\text{L. 38}}{=} b.$

We have $\begin{pmatrix} a & b & c \end{pmatrix}^T \cdot \pi_3 = c$.

We have $\begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \cdot \pi_3 \overset{\text{R. 30.(3)}}{=} \begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} 0 \\ c \end{pmatrix} \overset{\text{R. 35}}{=} 0 + c \overset{\text{L. 38}}{=} c.$

This proves the *claim*.

So we can continue to calculate

$$\begin{pmatrix} 1 & 1 \end{pmatrix}^S \cdot {}^S\!\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} a & b & c \end{pmatrix}^T \cdot {}^T\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;.$$

So now, we also have

$$(a+b)+c \;=\; \begin{pmatrix} a & b & c \end{pmatrix}^{T} \cdot {}^{T}\!\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .$$

Altogether, we have $a+(b+c) = (a+b)+c$. □

Remark 41. *Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$.*

Suppose given a direct sum S of $(X_1, \ldots, X_m)$, with inclusion morphisms ι_i and projection morphisms π_i , where $i \in [1,m]$.

Then

$$\mathrm{id}_S \;=\; \sum_{i \in [1,m]} \pi_i \cdot \iota_i \,.$$

Proof. Suppose given $j \in [1,m]$. It suffices to show that

$$\iota_j \cdot \mathrm{id}_S \;\overset{!}{=}\; \iota_j \cdot \Big(\sum_{i \in [1,m]} \pi_i \cdot \iota_i \Big) .$$

In fact,

$$\iota_j \cdot \Big(\sum_{i \in [1,n]} \pi_i \cdot \iota_i \Big) \;=\; \sum_{i \in [1,n]} \iota_j \cdot \pi_i \cdot \iota_i \;=\; \iota_j$$

by (Sum 3) and Lemma 38.(2). □

Lemma 42.

Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum S of $(X_1, \ldots, X_m)$, with inclusion morphisms ι_i and projection morphisms π_i , where $i \in [1,m]$.

Suppose given $n \geqslant 0$ and $Y_1, Y_2, \ldots, Y_n \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum T of $(Y_1, \ldots, Y_n)$, with inclusion morphisms ι'_j and projection morphisms π'_j , where $j \in [1,n]$.

Suppose given $p \geqslant 0$ and $Z_1, Z_2, \ldots, Z_p \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum W of $(Z_1, \ldots, Z_p)$, with inclusion morphisms ι''_k and projection morphisms π''_k , where $k \in [1,p]$.

Suppose given morphisms $X_i \xrightarrow{u_{i,j}} Y_j$ in $\mathcal{A}$ for $i \in [1,m]$ and $j \in [1,n]$.

Suppose given morphisms $Y_j \xrightarrow{v_{j,k}} Z_k$ in $\mathcal{A}$ for $j \in [1,n]$ and $k \in [1,p]$.

Then

$${}^{S}(u_{i,j})^{T}_{i,j} \cdot {}^{T}(v_{j,k})^{W}_{j,k} \;=\; {}^{S}\Big(\sum_{j \in [1,n]} u_{i,j} \cdot v_{j,k} \Big)^{W}_{i,k} .$$

Proof. Write $u := {}^{S}(u_{i,j})^{T}_{i,j}$ and $v := {}^{T}(v_{j,k})^{W}_{j,k}$.

Suppose given $a \in [1,m]$ and $c \in [1,p]$. We have to show that

$$\iota_a \cdot u \cdot v \cdot \pi''_c \;\overset{!}{=}\; \iota_a \cdot {}^{S}\Big(\sum_{j \in [1,n]} u_{i,j} \cdot v_{j,k} \Big)^{W}_{i,k} \cdot \pi''_c \,.$$

In fact

$$\begin{aligned}
\iota_a \cdot {}^{S}\!\Big(\sum_{j\in[1,n]} u_{i,j}\cdot v_{j,k}\Big)^{W}_{i,k}\cdot \pi''_c &= \sum_{j\in[1,n]} u_{a,j}\cdot v_{j,c} \\
&= \sum_{j\in[1,n]} \iota_a\cdot u\cdot \pi'_j\cdot \iota'_j\cdot v\cdot \pi''_c \\
&= \iota_a\cdot u\cdot \Big(\sum_{j\in[1,n]} \pi'_j\cdot \iota'_j\Big)\cdot v\cdot \pi''_c \\
&\overset{\text{R. 41}}{=} \iota_a\cdot u\cdot \mathrm{id}_T\cdot v\cdot \pi''_c \\
&= \iota_a\cdot u\cdot v\cdot \pi''_c\ .
\end{aligned}$$

□

Lemma 43. *Suppose given $X, Y \in \mathrm{Ob}(\mathcal{A})$.*

Suppose given $a \in {}_{\mathcal{A}}(X,Y)$.

Then there exists a unique $b \in {}_{\mathcal{A}}(X,Y)$ such that $a+b=0$.

We write $-a := b$.

Proof.

Existence. Suppose we find a morphism $m \in {}_{\mathcal{A}}(X,X)$ such that $\mathrm{id}_X + m = 0$. Then $a + m\cdot a = \mathrm{id}_X\cdot a + m\cdot a = (\mathrm{id}_X + m)\cdot a = 0\cdot a = 0$ by Lemma 38.(3).

So it suffices to find m as described.

We choose a direct sum S of (X,X).

By (Add 2), ${}^{S}\!\left(\begin{smallmatrix}\mathrm{id}_X & 0\\ \mathrm{id}_X & \mathrm{id}_X\end{smallmatrix}\right)^{S}$ is an isomorphism. Write ${}^{S}\!\left(\begin{smallmatrix}s & t\\ u & v\end{smallmatrix}\right)^{S} := \left({}^{S}\!\left(\begin{smallmatrix}\mathrm{id}_X & 0\\ \mathrm{id}_X & \mathrm{id}_X\end{smallmatrix}\right)^{S}\right)^{-}$; cf. Lemma 27. Then

$${}^{S}\!\begin{pmatrix}\mathrm{id}_X & 0\\ 0 & \mathrm{id}_X\end{pmatrix}^{S} = {}^{S}\!\begin{pmatrix}\mathrm{id}_X & 0\\ \mathrm{id}_X & \mathrm{id}_X\end{pmatrix}^{S}\cdot {}^{S}\!\begin{pmatrix}s & t\\ u & v\end{pmatrix}^{S} \overset{\text{L. 42, L. 38.(2)}}{=} {}^{S}\!\begin{pmatrix}s & t\\ s+u & t+v\end{pmatrix}^{S}\ .$$

So $s = \mathrm{id}_X$. Hence $0 = s+u = \mathrm{id}_X + u$. Choose $m := u$.

Uniqueness. Suppose given $b, \tilde{b} \in {}_{\mathcal{A}}(X,Y)$ such that $a+b=0$ and $a+\tilde{b}=0$. Then

$$b \overset{\text{L. 38.(2)}}{=} b + a + \tilde{b} \overset{\text{L. 38.(1,2)}}{=} \tilde{b}\ .$$

□

Corollary 44. *Suppose given $X, Y \in \mathrm{Ob}(\mathcal{A})$.*

The set ${}_{\mathcal{A}}(X,Y)$, together with the addition

$${}_{\mathcal{A}}(X,Y)\times {}_{\mathcal{A}}(X,Y) \to {}_{\mathcal{A}}(X,Y) : (a,b)\mapsto a+b$$

introduced in Definition 33*, is an abelian group.*

Proof. This follows from Lemmas 38.(1,2), 40 and 43. □

Corollary 45. *Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$.*

We have

$$-(f\cdot g) = (-f)\cdot g = f\cdot(-g)\ .$$

Proof. We have to show that $(-f)\cdot g + (f\cdot g) \overset{!}{=} 0$. But $(-f)\cdot g + f\cdot g = ((-f)+f)\cdot g = 0_{X,Y}\cdot g = 0_{X,Z}$; cf. Lemmas 38 and 18.(2).

We have to show that $f\cdot(-g)+(f\cdot g) \overset{!}{=} 0$. But $f\cdot(-g)+f\cdot g = f\cdot((-g)+g) = f\cdot 0_{Y,Z} = 0_{X,Z}$; cf. Lemmas 38 and 18.(2). □

2.3 Additive functors

Suppose given additive categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

Definition 46. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *additive* if it satisfies the properties (AddFun 1, 2) below.

(AddFun 1) Suppose given a zero object $Z \in \mathrm{Ob}(\mathcal{A})$.

Then ZF is a zero object in $\mathcal{B}$.

(AddFun 2) Suppose given $X,\ X' \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum S of (X, X') with inclusion morphisms ι_1 and ι_2 and projection morphisms π_1 and π_2 .

Then SF is a direct sum of $(XF, X'F)$ with inclusion morphisms $\iota_1 F$ and $\iota_2 F$ and projection morphisms $\pi_1 F$ and $\pi_2 F$.

Remark 47. *Suppose given an additive functor $F : \mathcal{A} \to \mathcal{B}$.*

Suppose given a functor $\tilde{F} : \mathcal{A} \to \mathcal{B}$ such that $F \simeq \tilde{F}$.

Then $\tilde{F}$ is an additive functor.

Proof. We choose an isotransformation $F \overset{a}{\underset{\sim}{\to}} \tilde{F}$.

Ad (AddFun 1). Suppose given a zero object $Z \in \mathrm{Ob}(\mathcal{A})$. We have to show that $Z\tilde{F}$ is a zero object in $\mathcal{B}$.

Since F is additive, ZF is a zero object in $\mathcal{B}$. We have the isomorphism $ZF \overset{Za}{\underset{\sim}{\to}} Z\tilde{F}$ in $\mathcal{B}$. Hence $Z\tilde{F}$ is a zero object in $\mathcal{B}$; cf. Remark 16.

Ad (AddFun 2). Suppose given $X,\ X' \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum S of (X, X') with inclusion morphisms ι_1 and ι_2 and projection morphisms π_1 and π_2 .

Since F is additive, SF is a direct sum of $(XF, X'F)$ with inclusion morphisms $\iota_1 F$ and $\iota_2 F$ and projection morphisms $\pi_1 F$ and $\pi_2 F$.

We have to show that $S\tilde{F}$ is a direct sum of $(X\tilde{F}, X'\tilde{F})$ with inclusion morphisms $\iota_1\tilde{F}$ and $\iota_2\tilde{F}$ and projection morphisms $\pi_1\tilde{F}$ and $\pi_2\tilde{F}$.

By Remark 24, $S\tilde{F}$ is a direct sum of $(X\tilde{F}, X'\tilde{F})$ with inclusion morphisms $Xa^- \cdot \iota_1 F \cdot Sa$ and $X'a^- \cdot \iota_2 F \cdot Sa$ and projection morphisms $Sa^- \cdot \pi_1 F \cdot Xa$ and $Sa^- \cdot \pi_2 F \cdot X'a$.

Now $Xa^{-} \cdot \iota_1 F \cdot Sa = \iota_1 \tilde{F}$ and $X'a^{-} \cdot \iota_2 F \cdot Sa = \iota_2 \tilde{F}$ by naturality of a.

$$\begin{array}{ccc} XF & \xrightarrow[\sim]{Xa} & X\tilde{F} \\ {\scriptstyle \iota_1 F}\big\downarrow & & \big\downarrow{\scriptstyle \iota_1 \tilde{F}} \\ SF & \xrightarrow[\sim]{Sa} & S\tilde{F} \end{array} \qquad\qquad \begin{array}{ccc} X'F & \xrightarrow[\sim]{X'a} & X'\tilde{F} \\ {\scriptstyle \iota_2 F}\big\downarrow & & \big\downarrow{\scriptstyle \iota_2 \tilde{F}} \\ SF & \xrightarrow[\sim]{Sa} & S\tilde{F} \end{array}$$

Moreover, $Sa^{-} \cdot \pi_1 F \cdot Xa = \pi_1 \tilde{F}$ and $Sa^{-} \cdot \pi_2 F \cdot X'a = \pi_2 \tilde{F}$ by naturality of a.

$$\begin{array}{ccc} XF & \xrightarrow[\sim]{Xa} & X\tilde{F} \\ {\scriptstyle \pi_1 F}\big\uparrow & & \big\uparrow{\scriptstyle \pi_1 \tilde{F}} \\ SF & \xrightarrow[\sim]{Sa} & S\tilde{F} \end{array} \qquad\qquad \begin{array}{ccc} X'F & \xrightarrow[\sim]{X'a} & X'\tilde{F} \\ {\scriptstyle \pi_2 F}\big\uparrow & & \big\uparrow{\scriptstyle \pi_2 \tilde{F}} \\ SF & \xrightarrow[\sim]{Sa} & S\tilde{F} \end{array}$$

Hence $S\tilde{F}$ is a direct sum of $(X\tilde{F}, X'\tilde{F})$ with inclusion morphisms $\iota_1 \tilde{F}$ and $\iota_2 \tilde{F}$ and projection morphisms $\pi_1 \tilde{F}$ and $\pi_2 \tilde{F}$, as was to be shown. □

Remark 48. *Suppose given an additive functor* $F : \mathcal{A} \to \mathcal{B}$.

Suppose given X, $X' \in \mathrm{Ob}(\mathcal{A})$. *Then* $0_{X,X'} F = 0_{XF,X'F}$.

Proof. Choose a zero object Z. We have morphisms $u : X \to Z$ and $v : Z \to X'$. Then $0_{X,X'} = u \cdot v$; cf. Definition 17. Hence $0_{X,X'} F = uF \cdot vF$ factors over the zero object ZF. So $0_{X,X'} F = 0_{XF,X'F}$. □

Remark 49.

(1) *The identity functor* $\mathrm{id}_{\mathcal{A}}$ *is additive.*

(2) *Suppose given additive functors* $F : \mathcal{A} \to \mathcal{B}$ *and* $G : \mathcal{B} \to \mathcal{C}$.

Then their composite $FG : \mathcal{A} \to \mathcal{C}$ *is additive.*

Proof. *Ad* (2).

Ad (AddFun 1). Suppose given a zero object $Z \in \mathrm{Ob}(\mathcal{A})$. Since F is additive, ZF is a zero object in $\mathcal{B}$. Since G is additive, ZFG is a zero object in $\mathcal{C}$.

Ad (AddFun 2). Suppose given X, $X' \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum S of (X, X') with inclusion morphisms ι_1 and ι_2 and projection morphisms π_1 and π_2.

Since F is additive, SF is a direct sum of $(XF, X'F)$ with inclusion morphisms $\iota_1 F$ and $\iota_2 F$ and projection morphisms $\pi_1 F$ and $\pi_2 F$.

Since G is additive, SFG is a direct sum of $(XFG, X'FG)$ with inclusion morphisms $\iota_1 FG$ and $\iota_2 FG$ and projection morphisms $\pi_1 FG$ and $\pi_2 FG$. □

Lemma 50. *Suppose given a functor $F : \mathcal{A} \to \mathcal{B}$.*

The following assertions $(1, 2, 3)$ *are equivalent.*

(1) *The functor F is additive.*

(2) *Given $X, Y \in \mathrm{Ob}(\mathcal{A})$ and morphisms $a, a' : X \to Y$, we have $(a + a')F = aF + a'F$.*

(3) *Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$.*

Suppose given a direct sum $S \in \mathrm{Ob}(\mathcal{A})$ of $(X_1, \ldots, X_m)$, with inclusion morphisms $X_i \xrightarrow{\iota_i} S$ and projection morphisms $S \xrightarrow{\pi_i} X_i$ for $i \in [1, m]$.

Then $SF \in \mathrm{Ob}(\mathcal{B})$ is a direct sum of $(X_1F, \ldots, X_mF)$, with inclusion morphisms $X_iF \xrightarrow{\iota_iF} SF$ and projection morphisms $SF \xrightarrow{\pi_iF} X_iF$ for $i \in [1, m]$.

Moreover, if F is additive and we are given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$ and a direct sum $S \in \mathrm{Ob}(\mathcal{A})$ of $(X_1, \ldots, X_m)$, and $n \geqslant 0$ and $Y_1, Y_2, \ldots, Y_n \in \mathrm{Ob}(\mathcal{A})$ and a direct sum $T \in \mathrm{Ob}(\mathcal{A})$ of $(Y_1, \ldots, Y_n)$, and morphisms $X_i \xrightarrow{u_{i,j}} Y_j$ in $\mathcal{A}$ for $i \in [1, m]$ and $j \in [1, n]$, then

$$(*) \qquad \left({}^{S}(u_{i,j})_{i,j}^{T} \right) F \;=\; {}^{SF}(u_{i,j}F)_{i,j}^{TF} \,.$$

Proof.

Claim 1. If (3) holds, then $(*)$ holds. Since ι_kF are the inclusion morphisms for SF for $k \in [1, m]$ and $\pi_\ell F$ are the projection morphisms for TF for $\ell \in [1, n]$, it suffices to show that $\iota_kF \cdot \left({}^{S}(u_{i,j})_{i,j}^{T} \right) F \cdot \pi_\ell F \overset{!}{=} u_{k,\ell}F$ for $k \in [1, m]$ and $\ell \in [1, n]$. In fact,

$$\iota_kF \cdot \left({}^{S}(u_{i,j})_{i,j}^{T} \right) F \cdot \pi_\ell F \;=\; \left(\iota_k \cdot {}^{S}(u_{i,j})_{i,j}^{T} \cdot \pi_\ell \right) F \overset{\text{L. 27}}{=} u_{k,\ell}F$$

This proves *Claim* 1.

Claim 2. Suppose that (1) holds, i.e. that F is additive. Suppose given $X, Y \in \mathrm{Ob}(\mathcal{A})$ and morphisms $a, a' : X \to Y$ in $\mathcal{A}$. Suppose given a direct sum S of (X, X).

Then $\left((\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{S} \right) F \overset{!}{=} (\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{SF}$ and $\left({}^{S}\!\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \right) F \overset{!}{=} {}^{SF}\!\left(\begin{smallmatrix} aF \\ a'F \end{smallmatrix}\right)$.

In fact, we have $\left((\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{S} \right) F \cdot \pi_1F = \left((\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{S} \cdot \pi_1 \right) F = 1F = 1$ and $\left((\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{S} \right) F \cdot \pi_2F = \left((\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{S} \cdot \pi_2 \right) F = 1F = 1$.

Moreover, we get $\iota_1F \cdot \left({}^{S}\!\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \right) F = \left(\iota_1 \cdot {}^{S}\!\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \right) F = aF$ and $\iota_2F \cdot \left({}^{S}\!\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \right) F = \left(\iota_2 \cdot {}^{S}\!\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \right) F = a'F$.

This proves the *Claim* 2.

Ad (3) $\Rightarrow$ (1). We have to show that F satisfies (AddFun 1). Suppose given a zero object $Z \in \mathrm{Ob}(\mathcal{A})$. Then Z is a direct sum of the empty tuple of objects in $\mathcal{A}$; cf. Example 22. Using (3) in the case $m = 0$, we obtain that ZF is a direct sum of the empty tuple of objects in $\mathcal{B}$. Thus ZF is a zero object in $\mathcal{B}$; cf. Example 22.

Ad (1) $\Rightarrow$ (2). By Remark 35 and Claim 2, we have

$$(a + a')F \;=\; \left((\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{S} \cdot {}^{S}\!\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \right) F \;=\; \left((\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{S} \right) F \cdot \left({}^{S}\!\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \right) F \;=\; (\begin{smallmatrix} 1 & 1 \end{smallmatrix})^{SF} \cdot {}^{SF}\!\left(\begin{smallmatrix} aF \\ a'F \end{smallmatrix}\right) \;=\; aF + a'F \,.$$

Ad (2) $\Rightarrow$ (3). Suppose given $m \geqslant 0$ and $X_1, X_2, \ldots, X_m \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum $S \in \mathrm{Ob}(\mathcal{A})$ of $(X_1, \ldots, X_m)$, with inclusion morphisms $X_i \xrightarrow{\iota_i} S$ and projection morphisms $S \xrightarrow{\pi_i} X_i$ for $i \in [1, m]$.

We have to show that $SF \in \mathrm{Ob}(\mathcal{B})$ is a direct sum of $(X_1F, \ldots, X_mF)$, with inclusion morphisms $X_iF \xrightarrow{\iota_i F} SF$ and projection morphisms $SF \xrightarrow{\pi_i F} X_iF$ for $i \in [1, m]$.

We choose a direct sum T of $(X_1F, \ldots, X_mF)$ in $\mathcal{B}$, together with inclusion morphisms ι_i' and projection morphisms π_i' for $i \in [1, m]$; cf. Lemma 37.

Recall that SF is a direct sum of (SF); cf. Example 21, Remark 28.

Let $\varphi := {}^{T}\!\begin{pmatrix} \iota_1 F \\ \vdots \\ \iota_m F \end{pmatrix}^{SF} : T \to SF$.

Let $\psi := {}^{SF}\!\begin{pmatrix} \pi_1 F & \ldots & \pi_m F \end{pmatrix}^{T} : SF \to T$.

Write

$$\delta_{i,j} := \begin{cases} \mathrm{id}_{X_i} & \text{if } i = j \\ 0_{X_i, X_j} & \text{if } i \neq j \end{cases}$$

and

$$\delta'_{i,j} := \begin{cases} \mathrm{id}_{X_iF} & \text{if } i = j \\ 0_{X_iF, X_jF} & \text{if } i \neq j\ . \end{cases}$$

for $i,\, j \in [1, m]$.

We have

$$\begin{aligned} \varphi \cdot \psi &= {}^{T}\!\begin{pmatrix} \iota_1 F \\ \vdots \\ \iota_m F \end{pmatrix}^{SF} \cdot {}^{SF}\!\begin{pmatrix} \pi_1 F & \ldots & \pi_m F \end{pmatrix}^{T} \\ &\overset{\text{L. 42}}{=} {}^{T}(\iota_i F \cdot \pi_j F)^{T}_{i,j} \\ &\overset{\text{(Sum 3)}}{=} {}^{T}(\delta_{i,j} F)^{T}_{i,j} \\ &\overset{\text{R. 48}}{=} {}^{T}(\delta'_{i,j})^{T}_{i,j} \\ &\overset{\text{R. 29}}{=} \mathrm{id}_T\ . \end{aligned}$$

We have

$$\begin{aligned} \psi \cdot \varphi &= {}^{SF}\!\begin{pmatrix} \pi_1 F & \ldots & \pi_m F \end{pmatrix}^{T} \cdot {}^{T}\!\begin{pmatrix} \iota_1 F \\ \vdots \\ \iota_m F \end{pmatrix}^{SF} \\ &\overset{\text{L. 42}}{=} {}^{SF}\Big(\sum_{i \in [1,m]} \pi_i F \cdot \iota_i F\Big)^{SF} \\ &\overset{\text{R. 28.(3)}}{=} \sum_{i \in [1,m]} \pi_i F \cdot \iota_i F \\ &= \sum_{i \in [1,m]} (\pi_i \cdot \iota_i) F \\ &\overset{(2)}{=} \Big(\sum_{i \in [1,m]} \pi_i \cdot \iota_i\Big) F \\ &\overset{\text{R. 41}}{=} \mathrm{id}_S F \\ &= \mathrm{id}_{SF}\ . \end{aligned}$$

So φ and ψ are mutually inverse isomorphisms.

By Remark 24, SF is a direct sum of $(X_1F, \ldots, X_mF)$, with inclusion morphisms $\iota_i' \cdot \varphi = \iota_i F$ and projection morphisms $\psi \cdot \pi_i' = \pi_i F$ for $i \in [1, m]$. $\square$

Remark 51. *Suppose given an additive functor $F : \mathcal{A} \to \mathcal{B}$.*

Suppose given $X,\ Y \in \mathrm{Ob}(\mathcal{A})$.

Then the map

$$\begin{array}{rcl} {}_{\mathcal{A}}(X,Y) & \xrightarrow{F_{X,Y}} & {}_{\mathcal{B}}(XF,YF) \\ a & \mapsto & aF \end{array}$$

is a group morphism; cf. Corollary 44.

Proof. This follows from Lemma 50, implication (1) $\Rightarrow$ (2). □

Remark 52. *Suppose given an equivalence $F : \mathcal{A} \to \mathcal{B}$.*

Then F is additive.

Proof. We have to show the properties (AddFun 1, 2) for F from Definition 46.

Ad (AddFun 1). Suppose given a zero object Z in $\mathcal{A}$.

We have to show that ZF is a zero object in $\mathcal{B}$.

We show that ZF is terminal. Suppose given $Y \in \mathrm{Ob}(\mathcal{B})$. We have to show that $|\,{}_{\mathcal{B}}(Y,ZF)| \overset{!}{=} 1$.

The functor F being dense, we may choose $X \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $b : XF \to Y$. The object Z being terminal, we have $|\,{}_{\mathcal{A}}(X,Z)| = 1$. The functor F being full and faithful, we have $1 = |\,{}_{\mathcal{A}}(X,Z)| = |\,{}_{\mathcal{B}}(XF,ZF)|$. The map ${}_{\mathcal{B}}(b,ZF) : {}_{\mathcal{B}}(Y,ZF) \to {}_{\mathcal{B}}(XF,ZF)$ being bijective, we conclude that $|\,{}_{\mathcal{B}}(Y,ZF)| = 1$.

We show that ZF is initial. Suppose given $Y \in \mathrm{Ob}(\mathcal{B})$. We have to show that $|\,{}_{\mathcal{B}}(ZF,Y)| \overset{!}{=} 1$.

The functor F being dense, we may choose $X \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $b : XF \to Y$. The object Z being initial, we have $|\,{}_{\mathcal{A}}(Z,X)| = 1$. The functor F being full and faithful, we have $1 = |\,{}_{\mathcal{A}}(Z,X)| = |\,{}_{\mathcal{B}}(ZF,XF)|$. The map ${}_{\mathcal{B}}(ZF,b) : {}_{\mathcal{B}}(ZF,XF) \to {}_{\mathcal{B}}(ZF,Y)$ being bijective, we conclude that $|\,{}_{\mathcal{B}}(ZF,Y)| = 1$.

In particular, given $X,\ X' \in \mathrm{Ob}(\mathcal{A})$, we have $0_{X,X'}F = 0_{XF,X'F}$. In fact, we have a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{0_{X,X'}} & X' \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle b} & \\ & Z & \end{array}$$

with Z being a zero object, and thus a commutative triangle

$$\begin{array}{ccc} XF & \xrightarrow{0_{X,X'}F} & X'F \\ & {\scriptstyle aF}\searrow \quad \nearrow{\scriptstyle bF} & \\ & ZF & \end{array}$$

with ZF being a zero object.

Ad (AddFun 2).

Suppose given $X,\ X' \in \mathrm{Ob}(\mathcal{A})$.

Suppose given a direct sum S of (X, X') with inclusion morphisms ι_1 and ι_2 and projection morphisms π_1 and π_2 .

We have to show that SF is a direct sum of $(XF, X'F)$ with inclusion morphisms $\iota_1 F$ and $\iota_2 F$ and projection morphisms $\pi_1 F$ and $\pi_2 F$.

Ad (Sum 1). We remark that property (Sum 1) for the direct sum S means that for $U \in \mathrm{Ob}(\mathcal{A})$, we have the bijection

$$\begin{array}{ccccc} {}_{\mathcal{A}}(U,S) & \xrightarrow[\sim]{\varphi} & {}_{\mathcal{A}}(U,X) & \times & {}_{\mathcal{A}}(U,X') \\ w & \mapsto & (w \cdot \pi_1 & , & w \cdot \pi_2)\ . \end{array}$$

So given $V \in \mathrm{Ob}(\mathcal{B})$, we have to show that the map

$$\begin{array}{ccccc} {}_{\mathcal{B}}(V,SF) & \xrightarrow{\psi} & {}_{\mathcal{B}}(V,XF) & \times & {}_{\mathcal{B}}(V,X'F) \\ \tilde{w} & \mapsto & (\tilde{w} \cdot \pi_1 F & , & \tilde{w} \cdot \pi_2 F) \end{array}$$

is a bijection.

The functor F being dense, we may choose $U \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $UF \xrightarrow[\sim]{b} V$.

It suffices to show that the following diagram commutes.

$$\begin{array}{ccc} {}_{\mathcal{B}}(V,SF) & \xrightarrow{\psi} & {}_{\mathcal{B}}(V,XF) \times {}_{\mathcal{B}}(V,X'F) \\ {\scriptstyle {}_{\mathcal{B}}(b^-,SF)}\uparrow\wr & & \wr\uparrow{\scriptstyle {}_{\mathcal{B}}(b^-,XF)\times {}_{\mathcal{B}}(b^-,X'F)} \\ {}_{\mathcal{B}}(UF,SF) & & {}_{\mathcal{B}}(UF,XF) \times {}_{\mathcal{B}}(UF,X'F) \\ {\scriptstyle F_{U,S}}\uparrow\wr & & \wr\uparrow{\scriptstyle F_{U,X}\times F_{U,X'}} \\ {}_{\mathcal{A}}(U,S) & \xrightarrow[\sim]{\varphi} & {}_{\mathcal{A}}(U,X) \times {}_{\mathcal{A}}(U,X') \end{array}$$

In fact, a morphism $w \in {}_{\mathcal{A}}(U,S)$ is mapped in both ways to $(b^- \cdot wF \cdot \pi_1 F,\, b^- \cdot wF \cdot \pi_2 F)$ in ${}_{\mathcal{B}}(V,XF) \times {}_{\mathcal{B}}(V,X'F)$.

Ad (Sum 2).

We remark that property (Sum 2) for the direct sum S means that for $U \in \mathrm{Ob}(\mathcal{A})$, we have the bijection

$$\begin{array}{ccccc} {}_{\mathcal{A}}(S,U) & \xrightarrow[\sim]{\varphi'} & {}_{\mathcal{A}}(X,U) & \times & {}_{\mathcal{A}}(X',U) \\ w & \mapsto & (\iota_1 \cdot w & , & \iota_2 \cdot w)\ . \end{array}$$

So given $V \in \mathrm{Ob}(\mathcal{B})$, we have to show that the map

$$\begin{array}{ccccc} {}_{\mathcal{B}}(SF,V) & \xrightarrow{\psi'} & {}_{\mathcal{B}}(XF,V) & \times & {}_{\mathcal{B}}(X'F,V) \\ \tilde{w} & \mapsto & (\iota_1 F \cdot \tilde{w} & , & \iota_2 F \cdot \tilde{w}) \end{array}$$

is a bijection.

The functor F being dense, we may choose $U \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $UF \xrightarrow[\sim]{b} V$.

It suffices to show that the following diagram commutes.

$$
\begin{array}{ccc}
{}_{\mathcal{B}}(SF,V) & \xrightarrow{\psi'} & {}_{\mathcal{B}}(XF,V) \times {}_{\mathcal{B}}(X'F,V) \\
{\scriptstyle {}_{\mathcal{B}}(SF,b)}\uparrow\wr & & \wr\uparrow{\scriptstyle {}_{\mathcal{B}}(XF,b)\times {}_{\mathcal{B}}(X'F,b)} \\
{}_{\mathcal{B}}(SF,UF) & & {}_{\mathcal{B}}(XF,UF) \times {}_{\mathcal{B}}(X'F,UF) \\
{\scriptstyle F_{S,U}}\uparrow\wr & & \wr\uparrow{\scriptstyle F_{X,U}\times F_{X',U}} \\
{}_{\mathcal{A}}(S,U) & \xrightarrow[\sim]{\varphi'} & {}_{\mathcal{A}}(X,U) \times {}_{\mathcal{A}}(X',U)
\end{array}
$$

In fact, a morphism $w \in {}_{\mathcal{A}}(S,U)$ is mapped in both ways to $(\iota_1 F \cdot wF \cdot b,\, \iota_2 F \cdot wF \cdot b)$ in ${}_{\mathcal{B}}(XF,V) \times {}_{\mathcal{B}}(X'F,V)$.

Ad (Sum 3). We have $\iota_1 F \cdot \pi_1 F = (\iota_1 \cdot \pi_1)F = \mathrm{id}_X F = \mathrm{id}_{XF}$ and $\iota_2 F \cdot \pi_2 F = (\iota_2 \cdot \pi_2)F = \mathrm{id}_{X'} F = \mathrm{id}_{X'F}$. We have $\iota_1 F \cdot \pi_2 F = (\iota_1 \cdot \pi_2)F = 0_{X,X'} F = 0_{XF,X'F}$ and $\iota_2 F \cdot \pi_1 F = (\iota_2 \cdot \pi_1)F = 0_{X',X} F = 0_{X'F,XF}$. □

2.4 Full additive subcategories

Let $\mathcal{A}$ be an additive category.

Definition 53. A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called a *full additive subcategory* of $\mathcal{A}$ if the following properties (1–3) hold.

(1) Given $X \in \mathrm{Ob}(\mathcal{A})$ and $Y \in \mathrm{Ob}(\mathcal{B})$ such that $X \simeq Y$ in $\mathcal{A}$, then $X \in \mathrm{Ob}(\mathcal{B})$.

(2) There exists a zero object Z in $\mathcal{A}$ such that $Z \in \mathrm{Ob}(\mathcal{B})$.

(3) Given $Y, Y' \in \mathrm{Ob}(\mathcal{B})$, there exists a direct sum S of (Y, Y') in $\mathcal{A}$ such that $S \in \mathrm{Ob}(\mathcal{B})$; cf. Definition 20.

A full additive subcategory $\mathcal{B}$ of $\mathcal{A}$ is called *closed under summands* if given $X, Y, S \in \mathrm{Ob}(\mathcal{A})$ such that S is a direct sum of (X, Y) in $\mathcal{A}$ and such that $S \in \mathrm{Ob}(\mathcal{B})$, we have $X, Y \in \mathrm{Ob}(\mathcal{B})$.

If $\mathcal{B}$ is full additive subcategory of $\mathcal{A}$ closed under retracts, it is in particular closed under summands. In fact, in the situation above, X and Y are retract of S.

Remark 54. *Suppose given a full additive subcategory $\mathcal{B}$ of $\mathcal{A}$.*

(1) *Let Z be a zero object of $\mathcal{A}$. Then $Z \in \mathrm{Ob}(\mathcal{B})$.*

Given $Y, Y' \in \mathrm{Ob}(\mathcal{B})$, the zero morphism $0_{Y,Y'}$ formed in $\mathcal{A}$ is also the zero morphism formed in $\mathcal{B}$.

(2) *Given $Y, Y' \in \mathrm{Ob}(\mathcal{B})$ and a direct sum S of (Y, Y') in $\mathcal{A}$, then $S \in \mathrm{Ob}(\mathcal{B})$.*

Proof. *Ad* (1). By Definition 53.(2), there exists a zero object Z' in $\mathcal{A}$ such that $Z' \in \mathrm{Ob}(\mathcal{B})$. By Remark 15, we have $Z \simeq Z'$. So $Z \in \mathrm{Ob}(\mathcal{B})$ by Definition 53.(1).

Now the zero morphism $0_{Y,Y'}$, formed in $\mathcal{A}$, factors over Z, which is a zero object in $\mathcal{A}$ and in $\mathcal{B}$. So $0_{Y,Y'}$ is a zero morphism in $\mathcal{B}$ as well.

Ad (2). By Definition 53.(2), there exists a direct sum S' of (Y, Y') in $\mathcal{A}$ such that $S' \in \mathrm{Ob}(\mathcal{B})$. By Remark 30.(2), we have $S \simeq S'$. So $S \in \mathrm{Ob}(\mathcal{B})$ by Definition 53.(1). □

Remark 55. *Suppose given a full additive subcategory $\mathcal{B}$ of $\mathcal{A}$.*

Suppose given $m \geqslant 0$ and $Y_1, Y_2, \ldots, Y_m \in \mathrm{Ob}(\mathcal{B})$.

Suppose given a direct sum S of $(Y_1, \ldots, Y_m)$ in $\mathcal{A}$, with inclusion morphisms ι_i and projection morphisms π_i for $i \in [1, m]$.

Suppose given $n \geqslant 0$ and $Y'_1, Y'_2, \ldots, Y'_n \in \mathrm{Ob}(\mathcal{B})$.

Suppose given a direct sum T of $(Y'_1, \ldots, Y'_n)$ in $\mathcal{A}$, with inclusion morphisms ι'_j and projection morphisms π'_j for $i \in [1, n]$.

Suppose given morphisms $Y_i \xrightarrow{u_{i,j}} Y'_j$ in $\mathcal{B}$ for $i \in [1, m]$ and $j \in [1, n]$.

(1) *We have $S \in \mathrm{Ob}(\mathcal{B})$.*

Moreover, S, with inclusion morphisms ι_i and projection morphisms π_i for $i \in [1, m]$, is a direct sum of $(Y_1, \ldots, Y_m)$ also in $\mathcal{B}$.

(2) *We may form the morphism ${}^{S}(u_{i,j})^{T}_{i,j}$ with respect to $\mathcal{A}$.*

We may form the morphism ${}^{S}(u_{i,j})^{T}_{i,j}$ with respect to $\mathcal{B}$.

These morphisms coincide.

(3) *Suppose given morphisms g, $g' : Y \to Y'$ in $\mathcal{B}$.*

We may form the morphism $g + g' : Y \to Y'$ with respect to $\mathcal{A}$.

We may form the morphism $g + g' : Y \to Y'$ with respect to $\mathcal{B}$.

These morphisms coincide.

Proof. Ad (1).

We show that $S \overset{!}{\in} \mathrm{Ob}(\mathcal{B})$. We proceed by induction on $m \geqslant 0$.

In the case $m = 0$, the assertion holds by Example 22, Remark 30.(2), Remark 54.(1).

In the case $m = 1$, the assertion holds by Example 21, Remark 30.(2), Definition 53.(1).

In the case $m = 2$, the assertion holds by Remark 54.(2).

Suppose given $m \geqslant 3$.

Choose a direct sum S' of $(Y_1, \ldots, Y_{m-1})$ in $\mathcal{A}$. Then $S' \in \mathrm{Ob}(\mathcal{B})$ by induction.

Choose a direct sum S'' of (S', Y_m) in $\mathcal{A}$. Then $S'' \in \mathrm{Ob}(\mathcal{B})$ by Remark 54.(2).

Then S'' is a direct sum of $(Y_1, \ldots, Y_m)$ by Lemma 36. So $S \simeq S''$ by Remark 30.(2). Hence $S \in \mathrm{Ob}(\mathcal{B})$ by Definition 53.(1).

We show that S, with inclusion morphisms ι_i and projection morphisms π_i for $i \in [1, m]$, is a direct sum of $(Y_1, \ldots, Y_m)$ also in $\mathcal{B}$.

Consider Definition 20, in the notation used there.

In (Sum 1), the morphism a is contained in $\mathcal{B}$ since $\mathcal{B}$ is a full subcategory of $\mathcal{A}$.

In (Sum 2), the morphism b is contained in $\mathcal{B}$ since $\mathcal{B}$ is a full subcategory of $\mathcal{A}$.

In (Sum 3), the identity morphisms and the zero morphisms are valid in $\mathcal{B}$; cf. Remark 54.(1).

Ad (2).

We have the direct sum S of $(Y_1, \ldots, Y_m)$ in $\mathcal{B}$, with inclusion morphisms ι_i for $i \in [1, m]$; cf. (1).

We have the direct sum T of $(Y'_1, \ldots, Y'_n)$ in $\mathcal{B}$ with projection morphisms π'_j for $j \in [1, n]$; cf. (1).

Moreover, $\iota_k \cdot {}^S(u_{i,j})^T_{i,j} \cdot \pi'_\ell = u_{k,\ell}$ for $k \in [1, m]$ and $\ell \in [1, n]$; cf. Lemma 27.

Ad (3). Let S' be a direct sum of (Y', Y') in $\mathcal{A}$.

In $\mathcal{A}$, we may write

$$g + g' \;=\; (\,g\;g'\,)^{S'} \cdot {}^{S'}\!\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \;;$$

cf. Remark 35.

Now S' is also a direct sum of (Y', Y') in $\mathcal{B}$, and the factors on the right hand side are the same when formed in $\mathcal{B}$ as when formed in $\mathcal{A}$; cf. (1, 2).

So by Remark 35, the left hand side equals $g + g'$, formed in $\mathcal{B}$. □

Lemma 56. *Suppose given a full additive subcategory $\mathcal{B}$ of $\mathcal{A}$.*

Then $\mathcal{B}$ is an additive category.

Proof.

Ad (Add 1). For each $(Y, Y') \in \operatorname{Ob}(\mathcal{B}) \times \operatorname{Ob}(\mathcal{B})$, there exists a direct sum of (Y, Y'); cf. Definition 53.(3), Remark 55.(1).

Ad (Add 2). Suppose given $Y \in \operatorname{Ob}(\mathcal{B})$ and a direct sum S of (Y, Y). We have to show that ${}^S\!\left(\begin{smallmatrix}\mathrm{id}_Y & 0_{Y,Y}\\ \mathrm{id}_Y & \mathrm{id}_Y\end{smallmatrix}\right)^S$, formed in $\mathcal{B}$, is an isomorphism.

But ${}^S\!\left(\begin{smallmatrix}\mathrm{id}_Y & 0_{Y,Y}\\ \mathrm{id}_Y & \mathrm{id}_Y\end{smallmatrix}\right)^S$, formed in $\mathcal{A}$, is an isomorphism, since $\mathcal{A}$ is an additive category. So the result follows from Remark 55.(2). □

Remark 57. *Suppose given a full additive subcategory $\mathcal{B}$ of $\mathcal{A}$.*

The inclusion functor

$$\begin{array}{rcl} \mathrm{I} = \mathrm{I}_{\mathcal{B},\mathcal{A}} : \quad \mathcal{B} & \to & \mathcal{A} \\ (Y \xrightarrow{g} Y') & \mapsto & (Y \xrightarrow{g} Y') \end{array}$$

is additive.

Proof. By Lemma 50.(2), given $g,\, g' : Y \to Y'$ in $\mathcal{B}$, it suffices to show that $(g+g')\mathrm{I} \stackrel{!}{=} g\mathrm{I}+g'\mathrm{I}$. But this follows from Remark 55.(3). □

Remark 58. *Suppose given a full additive subcategory $\mathcal{B}$ of $\mathcal{A}$.*

(1) *Suppose given an additive category $\mathcal{T}$. Suppose given an additive functor $F : \mathcal{T} \to \mathcal{A}$ such that $UF \in \operatorname{Ob}(\mathcal{B})$ for $U \in \operatorname{Ob}(\mathcal{T})$. Then there exists a unique functor $\check{F} : \mathcal{T} \to \mathcal{B}$ such that $\check{F}\mathrm{I} = F$.*

Then $\check{F}$ is additive.

We often write $F|^{\mathcal{B}} := \check{F}$.

(2) *Suppose given an additive category $\mathcal{T}$. Suppose given additive functors $F,\, F' : \mathcal{T} \to \mathcal{A}$ such that $UF,\, UF' \in \mathrm{Ob}(\mathcal{B})$ for $U \in \mathrm{Ob}(\mathcal{T})$. Suppose given a transformation $a : F \to F'$.*

Then there exists a unique transformation $\check{a} : \check{F} \to \check{F}'$ such that $\check{a}\mathrm{I} = a$.

We often write $a|^{\mathcal{B}} := \check{a}$.

Proof. Ad (1).

Uniqueness. Suppose given a functor $G : \mathcal{T} \to \mathcal{B}$ such that $G\mathrm{I} = F$. For $h : U \to U'$ in $\mathcal{T}$, we obtain $hG = hG\mathrm{I} = hF$.

Existence. For $h : U \to U'$ in $\mathcal{T}$, we let $(U \xrightarrow{h} U')\check{F} := (UF \xrightarrow{hF} U'F)$. This is a well-defined functor by assumption on F. We have $\check{F}\mathrm{I} = F$ by construction.

We show that $\check{F}$ is additive. We use Lemma 50. Suppose given $h,\, \tilde{h} : U \to U'$ in $\mathcal{T}$. Note that the sum of two morphisms from $\mathcal{B}$ does not depend on whether we form it in $\mathcal{B}$ or in $\mathcal{A}$; cf. Remark 55.(3). We obtain

$$(h + \tilde{h})\check{F} \;=\; (h + \tilde{h})F \;=\; hF + \tilde{h}F \;=\; h\check{F} + \tilde{h}\check{F}\ .$$

Ad (2).

Uniqueness. Suppose given a transformation $b : \check{F} \to \check{F}'$ such that $b\mathrm{I} = a$. For U in $\mathrm{Ob}(\mathcal{T})$, we obtain $Ub = Ub\mathrm{I} = Ua$.

Existence. For U in $\mathrm{Ob}(\mathcal{T})$, we let $U\check{a} := Ua$, going from $U\check{F} = UF$ to $U\check{F}' = UF'$. This defines a transformation from $\check{F}$ to $\check{F}'$, since given $h : U \to U'$ in $\mathcal{T}$, we obtain

$$h\check{F} \cdot U'\check{a} \;=\; hF \cdot U'a \;=\; Ua \cdot hF' \;=\; U\check{a} \cdot h\check{F}'\ .$$

We have $\check{a}\mathrm{I} = a$ since $U\check{a}\mathrm{I} = U\check{a} = Ua$ for $U \in \mathrm{Ob}(\mathcal{T})$. □

2.5 The zero category and zero functors

Definition 59. A category $\mathcal{Z}$ is called a *zero category* if $\mathrm{Ob}(\mathcal{Z})$ is nonempty and if each object of $\mathcal{Z}$ is a zero object.

Remark 60. *Suppose given a zero category $\mathcal{Z}$.*

Then $\mathcal{Z}$ is additive.

Proof. Since there exists a zero object in $\mathcal{Z}$, the category $\mathcal{Z}$ is pointed.

Ad (Add 1). Suppose given $Z,\, Z' \in \mathrm{Ob}(\mathcal{Z})$. Choose $Z'' \in \mathrm{Ob}(\mathcal{Z})$. By Example 23, the object Z'', together with inclusion morphisms $\iota_1 := 0_{Z,Z''}$ and $\iota_2 := 0_{Z',Z''}$ and projection morphisms $\pi_1 := 0_{Z'',Z}$ and $\pi_2 := 0_{Z'',Z'}$, is a direct sum of (Z, Z').

Ad (Add 2). Suppose given $Z \in \mathrm{Ob}(\mathcal{Z})$. Suppose given a direct sum Z'' of (Z, Z).

We have to show that ${}^{Z''}\!\begin{pmatrix} \mathrm{id}_Z & 0_{Z,Z} \\ \mathrm{id}_Z & \mathrm{id}_Z \end{pmatrix}^{Z''} : Z'' \to Z''$ is an isomorphism.

But ${}^{Z''}\!\begin{pmatrix} \mathrm{id}_Z & 0_{Z,Z} \\ \mathrm{id}_Z & \mathrm{id}_Z \end{pmatrix}^{Z''} = \mathrm{id}_{Z''}$, which is an isomorphism. □

Definition 61. Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$.

A functor $F : \mathcal{A} \to \mathcal{B}$ is called a *zero functor* if XF is a zero object for $X \in \mathrm{Ob}(\mathcal{A})$.

Remark 62. *Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$.*

Suppose given a zero functor $F : \mathcal{A} \to \mathcal{B}$.

Then F is additive.

Proof. We make use of Lemma 50.

Suppose given $X,\, X' \in \mathrm{Ob}(\mathcal{A})$. Suppose given $a,\, \tilde{a} : X \to X'$ in $\mathcal{A}$. We have to show that $(a + \tilde{a})F \overset{!}{=} aF + \tilde{a}F$.

Since XF and $X'F$ are zero objects, we have $|\,{}_{\mathcal{B}}(XF, X'F)| = 1$.

Since $(a + \tilde{a})F,\, aF + \tilde{a}F \in {}_{\mathcal{B}}(XF, X'F)$, we conclude that $(a + \tilde{a})F = aF + \tilde{a}F$. □

Remark 63. *Suppose given an additive category $\mathcal{A}$.*

Let $\mathcal{Z}_{\mathcal{A}} \subseteq \mathcal{A}$ be the full subcategory with

$$\mathrm{Ob}(\mathcal{Z}_{\mathcal{A}}) := \{\, Z \in \mathrm{Ob}(\mathcal{A}) : Z \textit{ is a zero object in } \mathcal{A}\}$$

Then $\mathcal{Z}_{\mathcal{A}}$ is a zero category.

Moreover, $\mathcal{Z}_{\mathcal{A}}$ is a full additive subcategory of $\mathcal{A}$.

Proof. The category $\mathcal{Z}_{\mathcal{A}}$ is a zero category, since each object of $\mathcal{Z}_{\mathcal{A}}$ is a zero object in $\mathcal{A}$, hence a zero object in $\mathcal{Z}_{\mathcal{A}}$.

We show that $\mathcal{Z}_{\mathcal{A}}$ is a full additive subcategory of $\mathcal{A}$; cf. Definition 53.

Ad (1). Suppose given an object X in $\mathcal{A}$ that is isomorphic to an object $Z \in \mathrm{Ob}(\mathcal{Z}_{\mathcal{A}})$. Then X is a zero object in $\mathcal{A}$, hence $X \in \mathrm{Ob}(\mathcal{Z}_{\mathcal{A}})$; cf. Remark 16.

Ad (2). There exists a zero object Z in $\mathcal{A}$, so $Z \in \mathrm{Ob}(\mathcal{Z}_{\mathcal{A}})$.

Ad (3). Suppose given $Z,\, Z' \in \mathrm{Ob}(\mathcal{Z}_{\mathcal{A}})$. We have to show that there exists a direct sum of Z and Z' in $\mathcal{A}$ that is contained in $\mathrm{Ob}(\mathcal{Z}_{\mathcal{A}})$.

Choose $Z'' \in \mathrm{Ob}(\mathcal{Z}_{\mathcal{A}})$. By Example 23, Z'', together with inclusion morphisms $\iota_1 := 0_{Z,Z''}$ and $\iota_2 := 0_{Z',Z''}$ and projection morphisms $\pi_1 := 0_{Z'',Z}$ and $\pi_2 := 0_{Z'',Z'}$, is a direct sum of (Z, Z'). □

Remark 64. Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$.

Suppose given a zero functor $F : \mathcal{A} \to \mathcal{B}$.

Then F factors over $\mathcal{Z}_{\mathcal{B}}$; cf. Remark 63.

Remark 65. *Suppose given additive categories $\mathcal{A}'$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}'$.*

Suppose given additive functors $\mathcal{A}' \xrightarrow{U} \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{V} \mathcal{B}'$.

If F is a zero functor, then UFV is a zero functor.

Proof. We have to show that $X'UFV$ is a zero object for each $X' \in \mathrm{Ob}(\mathcal{A}')$.

But since F is a zero functor, the object $X'UF$ is a zero object. Since V is additive, it follows that $X'UFV$ is a zero object by (AddFun 1). □

Remark 66. *Suppose given additive categories $\mathcal{B}$ and $\mathcal{C}$.*

Suppose given an additive functor $\mathcal{B} \xrightarrow{F} \mathcal{C}$. Suppose that F is full and faithful.

(1) *Suppose given $Y \in \mathrm{Ob}(\mathcal{B})$ such that YF is a zero object. Then Y is a zero object.*

(2) *Suppose given an additive category $\mathcal{A}$ and an additive functor $\mathcal{A} \xrightarrow{U} \mathcal{B}$ such that UF is a zero functor. Then U is a zero functor.*

Proof.

Ad (1). Choose a zero object $Z \in \mathrm{Ob}(\mathcal{B})$. Then ZF is a zero object in $\mathcal{C}$; cf. Definition 46. Hence $ZF \simeq YF$; cf. Remark 15. Since F is full and faithful, we conclude that $Z \simeq Y$. Hence Y is a zero object in $\mathcal{B}$; cf. Remark 16.

Ad (2). Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that XU is a zero object in $\mathcal{B}$.

Since UF is a zero functor, XUF is a zero object in $\mathcal{C}$. Since F is full and faithful, we may apply (1) to conclude that XU is a zero object in $\mathcal{B}$. □

2.6 Standard notation by choice of direct sums

Let $\mathcal{A}$ be an additive category.

2.6.1 Standard direct sums

By Lemma 37, for each $m \geqslant 0$ and each tuple $(X_1, \dots, X_m)$ of objects of $\mathcal{A}$, we may choose a direct sum

$$\bigoplus_{i \in [1,m]} X_i \;=\; X_1 \oplus \ldots \oplus X_m \;,$$

called *standard direct sum.*

In particular, we choose an object

$$0 \;=\; 0_{\mathcal{A}} \;\in\; \mathrm{Ob}(\mathcal{A})$$

as direct sum of the empty tuple of objects.

Given $X_1 \in \mathrm{Ob}(\mathcal{A})$, we stipulate that we choose X_1 as standard direct sum of (X_1), with $\iota_1 = \mathrm{id}_{X_1}$ and $\pi_1 = \mathrm{id}_{X_1}$; cf. Example 21.

For short, in $\mathcal{A}$ we may *choose finite standard direct sums.*

2.6.2 Standard matrices

Suppose that we have chosen standard direct sums in $\mathcal{A}$.

Suppose given $m \geqslant 0$ and $X_1, X_2, \dots, X_m \in \mathrm{Ob}(\mathcal{A})$.

Suppose given $n \geqslant 0$ and $Y_1, Y_2, \dots, Y_n \in \mathrm{Ob}(\mathcal{A})$.

Suppose given morphisms $X_i \xrightarrow{u_{i,j}} Y_j$ in $\mathcal{A}$ for $i \in [1,m]$ and $j \in [1,n]$.

We write the *standard matrix*

$$\begin{pmatrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & & \vdots \\ u_{m,1} & \cdots & u_{m,n} \end{pmatrix} = (u_{i,j})_{i,j}$$

$$:= {}^{\bigoplus_{i\in[1,m]} X_i}\!\begin{pmatrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & & \vdots \\ u_{m,1} & \cdots & u_{m,n} \end{pmatrix}^{\bigoplus_{j\in[1,n]} Y_j} = {}^{\bigoplus_{i\in[1,m]} X_i}(u_{i,j})_{i,j}^{\bigoplus_{j\in[1,n]} Y_j} .$$

as morphism between the standard direct sums. So

$$(u_{i,j})_{i,j} \;:\; \bigoplus_{i\in[1,m]} X_i \;\to\; \bigoplus_{j\in[1,n]} Y_j \;.$$

Since we stipulated X_1 to be the standard direct sum of (X_1) for $X_1 \in \mathrm{Ob}(\mathcal{A})$, with id_{X_1} as inclusion and projection morphism, this is in accordance with the notation introduced in Definition 20; cf. Remark 28.

2.6.3 Characterisation of additive functors using standard direct sums

Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$. We choose finite standard direct sums in $\mathcal{A}$ and $\mathcal{B}$.

Lemma 67. *Suppose given a functor $F : \mathcal{A} \to \mathcal{B}$.*

The following assertions $(1, 2)$ are equivalent.

(1) *The functor F is additive.*

(2) *We have $0_{\mathcal{A}}F \simeq 0_{\mathcal{B}}$.*
Given X_1 , $X_2 \in \mathrm{Ob}(\mathcal{A})$, we have mutually inverse isomorphisms

$$\begin{pmatrix} \iota_1 F \\ \iota_2 F \end{pmatrix} \;:\; X_1F \oplus X_2F \;\to\; (X_1 \oplus X_2)F$$

and

$$\begin{pmatrix} \pi_1 F & \pi_2 F \end{pmatrix} \;:\; (X_1 \oplus X_2)F \;\to\; X_1F \oplus X_2F \;.$$

Proof. *Ad* $(1) \Rightarrow (2)$.

Since F is additive, the object $0_{\mathcal{A}}F$ is a zero object in $\mathcal{B}$; cf. (AddFun 1) in Definition 46. Hence $0_{\mathcal{A}}F \simeq 0_{\mathcal{B}}$; cf. Remark 15.

Suppose given U, $V \in \mathrm{Ob}(\mathcal{A})$. We have

$$0_{U,V}F \;=\; 0_{UF,VF} \;;$$

cf. Remark 48.

So we obtain

$$\begin{aligned}
\begin{pmatrix} \iota_1 F \\ \iota_2 F \end{pmatrix} \cdot \begin{pmatrix} \pi_1 F & \pi_2 F \end{pmatrix} &= \begin{pmatrix} \iota_1F\cdot\pi_1F & \iota_1F\cdot\pi_2F \\ \iota_2F\cdot\pi_1F & \iota_2F\cdot\pi_2F \end{pmatrix} = \begin{pmatrix} (\iota_1\cdot\pi_1)F & (\iota_1\cdot\pi_2)F \\ (\iota_2\cdot\pi_1)F & (\iota_2\cdot\pi_2)F \end{pmatrix} \\
&\overset{\text{(Sum 3)}}{=} \begin{pmatrix} (\mathrm{id}_{X_1})F & (0_{X_1,\,X_2})F \\ (0_{X_2,\,X_1})F & (\mathrm{id}_{X_2})F \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{X_1F} & 0_{X_1F,\,X_2F} \\ 0_{X_2F,\,X_1F} & \mathrm{id}_{X_2F} \end{pmatrix} \overset{\text{R. 29}}{=} \mathrm{id}_{X_1F\oplus X_2F} \;.
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \left(\begin{smallmatrix} \pi_1 F & \pi_2 F \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} \iota_1 F \\ \iota_2 F \end{smallmatrix}\right) \;&=\; \pi_1 F \cdot \iota_1 F + \pi_2 F \cdot \iota_2 F \;=\; (\pi_1 \cdot \iota_1)F + (\pi_2 \cdot \iota_2)F \\ &\overset{\text{L. 50}}{=} (\pi_1 \cdot \iota_1 + \pi_2 \cdot \iota_2)F \overset{\text{R. 41}}{=} \mathrm{id}_{X_1 \oplus X_2} F \;=\; \mathrm{id}_{(X_1 \oplus X_2)F}\,. \end{aligned}$$

Ad (2) $\Rightarrow$ (1).

Suppose given $a,\, a' \,:\, X \to Y$ in $\mathcal{A}$. We have to show that $aF + a'F \overset{!}{=} (a+a')F$; cf. Lemma 50. We obtain

$$\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) F \cdot \left(\begin{smallmatrix} \pi_1 F & \pi_2 F \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} (1\,1)F \cdot \pi_1 F & (1\,1)F \cdot \pi_2 F \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} ((1\,1)\cdot \pi_1)F & ((1\,1)\cdot \pi_2)F \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} 1F & 1F \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)$$

and

$$\left(\begin{smallmatrix} \iota_1 F \\ \iota_2 F \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) F \;=\; \begin{pmatrix} \iota_1 F \cdot \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) F \\ \iota_2 F \cdot \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) F \end{pmatrix} \;=\; \begin{pmatrix} (\iota_1 \cdot \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right))F \\ (\iota_2 \cdot \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right))F \end{pmatrix} \;=\; \left(\begin{smallmatrix} aF \\ a'F \end{smallmatrix}\right)\,.$$

So we obtain

$$\begin{aligned} (a+a')F \;&=\; ((\begin{smallmatrix} 1 & 1 \end{smallmatrix}) \cdot \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right))F \\ &=\; \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) F \cdot \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) F \\ &=\; \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) F \cdot \left(\begin{smallmatrix} \pi_1 F & \pi_2 F \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} \iota_1 F \\ \iota_2 F \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) F \\ &=\; \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} aF \\ a'F \end{smallmatrix}\right) \\ &=\; aF + a'F\,. \end{aligned}$$

□

2.7 The 2-category of additive categories, called AddCat

Definition 68. We consider the 2-category of categories Cat ; cf. Proposition 9.

We define the 2-subcategory AddCat in Cat by letting

$$\begin{aligned} \mathrm{Ob}(\mathrm{AddCat}) \;&:=\; \{\, \mathcal{C} \in \mathrm{Ob}(\mathrm{Cat}) \,:\, \mathcal{C} \text{ is an additive category} \,\} \\ &\subseteq\; \mathrm{Ob}(\mathrm{Cat}) \\ \mathrm{Mor}_1(\mathrm{AddCat}) \;&:=\; \{\, F \in \mathrm{Mor}_1(\mathrm{Cat}) \,:\, F \text{ is an additive functor between additive categories} \,\} \\ &\subseteq\; \mathrm{Mor}_1(\mathrm{Cat}) \\ \mathrm{Mor}_2(\mathrm{AddCat}) \;&:=\; \left\{ a \in \mathrm{Mor}_2(\mathrm{Cat}) \,:\, \begin{array}{l} a \text{ is a transformation between additive functors} \\ \text{between additive categories} \end{array} \right\} \\ &\subseteq\; \mathrm{Mor}_2(\mathrm{Cat})\,. \end{aligned}$$

The 2-category AddCat is called the 2-*category of additive categories*. Cf. Definition 10.

Proof. We verify the properties (1–5) from Definition 10.

Ad (1). Given a 1-morphism $F \in \mathrm{Mor}_1(\mathrm{AddCat})$, its source and target are in Ob(AddCat) by construction.

Given an object $\mathcal{A} \in \mathrm{Ob}(\mathrm{AddCat})$, its identity $\mathrm{id}_{\mathcal{A}}$ is an additive functor by Remark 49.(1), i.e. it is in $\mathrm{Mor}_1(\mathrm{AddCat})$.

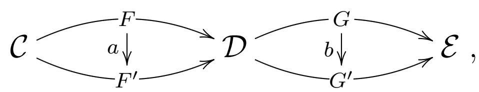
Ad (2). Given a 2-morphism in $\mathrm{Mor}_2(\mathrm{AddCat})$, its source and target are additive functors, i.e. they are in $\mathrm{Mor}_1(\mathrm{AddCat})$.

Conversely, given a 1-morphism in $\mathrm{Mor}_1(\mathrm{AddCat})$, its identity is a transformation between additive functors, i.e. it is in $\mathrm{Mor}_2(\mathrm{AddCat})$.

Ad (3). Given $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ in Cat with $F, G \in \mathrm{Mor}_1(\mathrm{AddCat})$, the categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are additive by construction. The composite FG of F and G is additive; cf. Remark 49.(2). So we have $(\mathcal{C} \xrightarrow{F \overset{\mathrm{Cat}}{*} G = FG} \mathcal{E}) \in \mathrm{Mor}_1(\mathrm{AddCat})$.

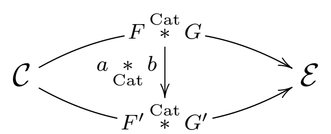
Ad (4). The vertical composite of two transformations between additive functors is still a transformation between additive functors, so it is in $\mathrm{Mor}_2(\mathrm{AddCat})$.

Ad (5). Given



in Cat with a, $b \in \mathrm{Mor}_2(\mathrm{AddCat})$, the functors F, F', G and G' are additive by construction. So $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are additive by construction.

Hence the horizontal composite



is a transformation between additive functors $FG = F \overset{\mathrm{Cat}}{*} G$ and $F'G' = F' \overset{\mathrm{Cat}}{*} G'$; cf. Remark 49.(2). So $a * b = a \underset{\mathrm{Cat}}{*} b \in \mathrm{Mor}_2(\mathrm{AddCat})$. □

Definition 69. Let $\mathcal{O}$ be the category defined as follows.

Let $\mathrm{Ob}(\mathcal{O})$ contain a single object, called 0. So $\mathrm{Ob}(\mathcal{O}) = \{0\}$.

Let $\mathrm{Mor}(\mathcal{O}) := \{\mathrm{id}_0\}$. Composition is defined by $\mathrm{id}_0 \cdot \mathrm{id}_0 := \mathrm{id}_0$.

Then $\mathcal{O}$ is a category.

Now 0 is a zero object in $\mathcal{O}$. So by Remark 60, $\mathcal{O}$ is an additive category.

Remark 70. *The category $\mathcal{O}$ is a zero object in* AddCat *; cf. Definitions* 69 *and* 11.

Proof. Suppose given $\mathcal{A} \in \mathrm{Ob}(\mathrm{AddCat})$. We have to show that the category ${}_{\mathrm{AddCat}}(\mathcal{A}, \mathcal{O})$ contains one isoclass.

There exists a unique functor $F : \mathcal{A} \to \mathcal{O}$, mapping each morphism $(X \xrightarrow{f} \tilde{X})$ in $\mathcal{A}$ to $(0 \xrightarrow{\mathrm{id}_0} 0)$ in $\mathcal{O}$. By Remark 62, the functor F is additive.

Suppose given $\mathcal{B} \in \mathrm{Ob}(\mathrm{AddCat})$. We have to show that the category ${}_{\mathrm{AddCat}}(\mathcal{O}, \mathcal{B})$ contains one isoclass.

For each object $Y \in \mathrm{Ob}(\mathcal{B})$, we have the functor G_Y mapping $(0 \xrightarrow{\mathrm{id}_0} 0)$ in $\mathcal{O}$ to $(Y \xrightarrow{\mathrm{id}_Y} Y)$ in $\mathcal{B}$. Conversely, each functor from $\mathcal{O}$ to $\mathcal{B}$ is of that form.

For $Y \in \mathrm{Ob}(\mathcal{B})$, the functor G_Y is additive if and only if Y is a zero object in $\mathcal{B}$; cf. Remark 62, (AddFun 1).

Since $\mathcal{B}$ has a zero object Z, there exists the corresponding object G_Z in ${}_{\mathrm{AddCat}}(\mathcal{O}, \mathcal{B})$.

Suppose given zero objects Z and $\tilde{Z}$ in $\mathcal{B}$. We have to show that G_Z and $G_{\tilde{Z}}$ are isomorphic.

We have an isomorphism $Z \overset{f}{\underset{\sim}{\to}} \tilde{Z}$; cf. Remark 15. Letting $0\varphi := f : 0G_Z \to 0G_{\tilde{Z}}$, we obtain an isotransformation $\varphi : G_Z \to G_{\tilde{Z}}$. In fact, for the unique morphism $0 \xrightarrow{\mathrm{id}_0} 0$, we obtain the following commutative quadrangle.

$$\begin{array}{ccc} 0G_Z & \overset{0\varphi}{\underset{\sim}{\longrightarrow}} & 0G_{\tilde{Z}} \\ {\scriptstyle \mathrm{id}_0 G_Z}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_0 G_{\tilde{Z}}} \\ 0G_Z & \overset{0\varphi}{\underset{\sim}{\longrightarrow}} & 0G_{\tilde{Z}} \end{array}$$

□

Remark 71. *Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$ and an additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$.*

The following statements $(1, 2, 3)$ are equivalent.

(1) *The functor F is a zero functor, i.e. the object XF is zero for $X \in \mathrm{Ob}(\mathcal{A})$.*

(2) *There exist additive functors $\mathcal{A} \xrightarrow{U} \mathcal{O} \xrightarrow{V} \mathcal{B}$ such that $F \simeq UV$.*

(3) *The functor F is a zero 1-morphism in* AddCat.

Cf. Definitions 61 and 12.

Proof.

Ad $(1) \Rightarrow (3)$. This follows by Remarks 64, 63.

Ad $(3) \Rightarrow (2)$. This follows by Remark 13.

Ad $(2) \Rightarrow (1)$. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have $XF \simeq XUV$. So it suffices to show that XUV is a zero object in $\mathcal{B}$; cf. Remark 16.

Now $XU = 0$ in $\mathcal{O}$, which is a zero object in $\mathcal{O}$. Since V is additive, we conclude that XUV is a zero object in $\mathcal{B}$; cf. (AddFun 1). □

Remark 72. *Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$.*

(1) *The category ${}_{\mathrm{AddCat}}(\mathcal{A}, \mathcal{B})$ is additive.*

The zero objects therein are the zero functors from $\mathcal{A}$ to $\mathcal{B}$.

(2) *Suppose given additive functors F, $\tilde{F} : \mathcal{A} \to \mathcal{B}$.*

Suppose given transformations a, $a' : F \to \tilde{F}$.

We have $a + a' = (Xa + Xa')_{X \in \mathrm{Ob}(\mathcal{A})}$.

Proof.

Ad (1). We choose finite standard direct sums in $\mathcal{B}$; cf. §2.6.1.

We verify that ${}_{\mathrm{AddCat}}(\mathcal{A}, \mathcal{B})$ is pointed and satisfies (Add 1, 2′); cf. Definition 31, Remark 32.

Ad (pointed). We define the functor $0 : \mathcal{A} \to \mathcal{B} : (X \xrightarrow{a} X') \mapsto (0 \xrightarrow{0} 0)$.

This is in fact a functor. On the one hand, for $X \in \mathrm{Ob}(\mathcal{A})$, we have $\mathrm{id}_X 0 = 0 = \mathrm{id}_0 = \mathrm{id}_{X0}$. On the other hand, for $X \xrightarrow{a} X' \xrightarrow{a'} X''$ in $\mathcal{A}$, we have $(aa')0 = 0 = 0 \cdot 0 = a0 \cdot a'0$.

The functor $0 : \mathcal{A} \to \mathcal{B}$ is a zero functor, whence it is additive by Remark 62.

Suppose given an additive functor $U : \mathcal{A} \to \mathcal{B}$.

We show that there exists a unique morphism from 0 to U. Since such a morphism is necessarily equal to $(X0 \to XU)_{X \in \mathrm{Ob}(\mathcal{A})} = (0 \xrightarrow{0} XU)_{X \in \mathrm{Ob}(\mathcal{A})}$, uniqueness follows. For the existence, it suffices to show naturality of this tuple. In fact, for $X \xrightarrow{a} X'$ in $\mathcal{A}$, the quadrangle

$$\begin{array}{ccc} X0 & \xrightarrow{0} & XU \\ {\scriptstyle a0}\downarrow & & \downarrow{\scriptstyle aU} \\ X'0 & \xrightarrow{0} & X'U \end{array}$$

commutes because of $X0 = 0$.

We show that there exists a unique morphism from U to 0. Since such a morphism is necessarily equal to $(XU \to X0)_{X \in \mathrm{Ob}(\mathcal{A})} = (XU \xrightarrow{0} 0)_{X \in \mathrm{Ob}(\mathcal{A})}$, uniqueness follows. For the existence, it suffices to show naturality of this tuple. In fact, for $X \xrightarrow{a} X'$ in $\mathcal{A}$, the quadrangle

$$\begin{array}{ccc} XU & \xrightarrow{0} & X0 \\ {\scriptstyle aU}\downarrow & & \downarrow{\scriptstyle a0} \\ X'U & \xrightarrow{0} & X'0 \end{array}$$

commutes because of $X'0 = 0$.

Moreover, any zero functor from $\mathcal{A}$ to $\mathcal{B}$ is isomorphic to the functor $0 : \mathcal{A} \to \mathcal{B}$, the isotransformation consisting of the unique morphisms between the respective zero objects in $\mathcal{B}$.

Conversely, an additive functor from $\mathcal{A}$ to $\mathcal{B}$ isomorphic to the functor $0 : \mathcal{A} \to \mathcal{B}$ is a zero functor.

Ad (Add 1). Suppose given additive functors $F_1,\, F_2 : \mathcal{A} \to \mathcal{B}$.

Note that $0_{F_1, F_2} = 0_{F_1, 0} \cdot 0_{0, F_2} = (0_{XF_1, X0})_{X \in \mathrm{Ob}(\mathcal{A})} \cdot (0_{X0, XF_2})_{X \in \mathrm{Ob}(\mathcal{A})} = (0_{XF_1, XF_2})_{X \in \mathrm{Ob}(\mathcal{A})}$.

We have to show that there exists a direct sum of F_1 and F_2 in ${}_{\mathrm{AddCat}}(\mathcal{A}, \mathcal{B})$ in the sense of Definition 20.

Let

$$F_1 \oplus F_2 \;:\; \mathcal{A} \to \mathcal{B} \;:\; (X \xrightarrow{a} X') \mapsto (X \xrightarrow{a} X')(F_1 \oplus F_2) \;:=\; (XF_1 \oplus XF_2 \xrightarrow{\left(\begin{smallmatrix} aF_1 & 0 \\ 0 & aF_2 \end{smallmatrix}\right)} X'F_1 \oplus X'F_2)\,.$$

We show that $F_1 \oplus F_2$ is a functor. On the one hand, for $X \in \mathrm{Ob}(\mathcal{A})$, we have

$$\mathrm{id}_X(F_1 \oplus F_2) \;=\; \begin{pmatrix} \mathrm{id}_X F_1 & 0 \\ 0 & \mathrm{id}_X F_2 \end{pmatrix} \;=\; \begin{pmatrix} \mathrm{id}_{XF_1} & 0 \\ 0 & \mathrm{id}_{XF_2} \end{pmatrix} \;\overset{\text{R. 29}}{=}\; \mathrm{id}_{XF_1 \oplus XF_2} \;=\; \mathrm{id}_{X(F_1 \oplus F_2)}\,.$$

On the other hand, for $X \xrightarrow{a} X' \xrightarrow{a'} X''$, we have

$$a(F_1 \oplus F_2) \cdot a'(F_1 \oplus F_2) \;=\; \begin{pmatrix} aF_1 & 0 \\ 0 & aF_2 \end{pmatrix} \cdot \begin{pmatrix} a'F_1 & 0 \\ 0 & a'F_2 \end{pmatrix} \;\overset{\text{L. 42}}{=}\; \begin{pmatrix} aF_1 \cdot a'F_1 & 0 \\ 0 & aF_2 \cdot a'F_2 \end{pmatrix} \;=\; \begin{pmatrix} (a \cdot a')F_1 & 0 \\ 0 & (a \cdot a')F_2 \end{pmatrix} \;=\; (a \cdot a')(F_1 \oplus F_2)\,.$$

We show that the functor $F_1 \oplus F_2$ is additive. In fact, by Lemma 50.($2 \Rightarrow 1$), it suffices to show that for $X \underset{\tilde{a}}{\overset{a}{\rightrightarrows}} X'$ in $\mathcal{A}$, we have $(a + \tilde{a})(F_1 \oplus F_2) \overset{!}{=} a(F_1 \oplus F_2) + \tilde{a}(F_1 \oplus F_2)$. We obtain

$$a(F_1 \oplus F_2) + \tilde{a}(F_1 \oplus F_2) = \begin{pmatrix} aF_1 & 0 \\ 0 & aF_2 \end{pmatrix} + \begin{pmatrix} \tilde{a}F_1 & 0 \\ 0 & \tilde{a}F_2 \end{pmatrix} \overset{\text{L. 39}}{=} \begin{pmatrix} aF_1 + \tilde{a}F_1 & 0 \\ 0 & aF_2 + \tilde{a}F_2 \end{pmatrix} \overset{\text{L. 50}}{=} \begin{pmatrix} (a+\tilde{a})F_1 & 0 \\ 0 & (a+\tilde{a})F_2 \end{pmatrix} = (a+\tilde{a})(F_1 \oplus F_2) .$$

We define the transformation

$$\iota_1 := \left(XF_1 \xrightarrow{(1\ 0)} XF_1 \oplus XF_2 \right)_{X \in \mathrm{Ob}(\mathcal{A})} : F_1 \to F_1 \oplus F_2 .$$

We have to show naturality of the tuple ι_1. Suppose given $X \xrightarrow{a} X'$ in $\mathcal{A}$. The quadrangle

$$\begin{array}{ccc} XF_1 & \xrightarrow{(1\ 0)} & XF_1 \oplus XF_2 \\ {\scriptstyle aF_1}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} aF_1 & 0 \\ 0 & aF_2 \end{pmatrix}} \\ X'F_1 & \xrightarrow{(1\ 0)} & X'F_1 \oplus X'F_2 \end{array}$$

commutes because of $(1\ 0) \cdot \begin{pmatrix} aF_1 & 0 \\ 0 & aF_2 \end{pmatrix} = (aF_1\ 0) = aF_1 \cdot (1\ 0)$.

We define the transformation

$$\iota_2 := \left(XF_2 \xrightarrow{(0\ 1)} XF_1 \oplus XF_2 \right)_{X \in \mathrm{Ob}(\mathcal{A})} : F_2 \to F_1 \oplus F_2 .$$

We have to show naturality of the tuple ι_2. Suppose given $X \xrightarrow{a} X'$ in $\mathcal{A}$. The quadrangle

$$\begin{array}{ccc} XF_2 & \xrightarrow{(0\ 1)} & XF_1 \oplus XF_2 \\ {\scriptstyle aF_2}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} aF_1 & 0 \\ 0 & aF_2 \end{pmatrix}} \\ X'F_2 & \xrightarrow{(0\ 1)} & X'F_1 \oplus X'F_2 \end{array}$$

commutes because of $(0\ 1) \cdot \begin{pmatrix} aF_1 & 0 \\ 0 & aF_2 \end{pmatrix} = (0\ aF_2) = aF_2 \cdot (0\ 1)$.

We define the transformation

$$\pi_1 := \left(XF_1 \oplus XF_2 \xrightarrow{\binom{1}{0}} XF_1 \right)_{X \in \mathrm{Ob}(\mathcal{A})} : F_1 \oplus F_2 \to F_1 .$$

We have to show naturality of the tuple π_1. Suppose given $X \xrightarrow{a} X'$ in $\mathcal{A}$. The quadrangle

$$\begin{array}{ccc} XF_1 \oplus XF_2 & \xrightarrow{\binom{1}{0}} & XF_1 \\ {\scriptstyle \begin{pmatrix} aF_1 & 0 \\ 0 & aF_2 \end{pmatrix}}\downarrow & & \downarrow{\scriptstyle aF_1} \\ X'F_1 \oplus X'F_2 & \xrightarrow{\binom{1}{0}} & X'F_1 \end{array}$$

commutes because of $\begin{pmatrix} aF_1 & 0 \\ 0 & aF_2 \end{pmatrix} \cdot \binom{1}{0} = \binom{aF_1}{0} = \binom{1}{0} \cdot aF_1$.

We define the transformation

$$\pi_2 := \left(XF_1 \oplus XF_2 \xrightarrow{\binom{0}{1}} XF_2 \right)_{X \in \mathrm{Ob}(\mathcal{A})} : F_1 \oplus F_2 \to F_2 .$$

We have to show naturality of the tuple π_2. Suppose given $X \overset{a}{\to} X'$ in $\mathcal{A}$. The quadrangle

$$\begin{array}{ccc} XF_1 \oplus XF_2 & \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} & XF_2 \\ {\scriptstyle \left(\begin{smallmatrix} aF_1 & 0 \\ 0 & aF_2 \end{smallmatrix}\right)} \downarrow & & \downarrow {\scriptstyle aF_2} \\ X'F_1 \oplus X'F_2 & \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} & X'F_2 \end{array}$$

commutes because of $\left(\begin{smallmatrix} aF_1 & 0 \\ 0 & aF_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ aF_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \cdot aF_2$.

We show that $F_1 \oplus F_2$, together with the inclusion morphisms ι_1 and ι_2 and the projection morphisms π_1 and π_2, is a direct sum of (F_1, F_2). We have to verify the properties (Sum 1–3) from Definition 20.

Ad (Sum 1). Suppose given an additive functor $U : \mathcal{A} \to \mathcal{B}$ and transformations $u_1 : U \to F_1$ and $u_2 : U \to F_2$. We have to show that there exists a unique transformation $u : U \to F_1 \oplus F_2$ such that $u \cdot \pi_1 = u_1$ and $u \cdot \pi_2 = u_2$.

Existence. We define the transformation

$$u \;:=\; \left(XU \xrightarrow{(Xu_1\ Xu_2)} XF_1 \oplus XF_2 \right)_{X \in \mathrm{Ob}(\mathcal{A})} \;:\; U \;\to\; F_1 \oplus F_2 \,.$$

We have to show naturality of the tuple u. Suppose given $X \overset{a}{\to} X'$ in $\mathcal{A}$. The quadrangle

$$\begin{array}{ccc} XU & \xrightarrow{(Xu_1\ Xu_2)} & XF_1 \oplus XF_2 \\ {\scriptstyle aU} \downarrow & & \downarrow {\scriptstyle \left(\begin{smallmatrix} aF_1 & 0 \\ 0 & aF_2 \end{smallmatrix}\right)} \\ X'U & \xrightarrow{(X'u_1\ X'u_2)} & X'F_1 \oplus X'F_2 \end{array}$$

commutes because of

$$\left(Xu_1\ Xu_2 \right) \cdot \left(\begin{smallmatrix} aF_1 & 0 \\ 0 & aF_2 \end{smallmatrix}\right) \;=\; \left(Xu_1 \cdot aF_1 \quad Xu_2 \cdot aF_2 \right) \;=\; \left(aU \cdot X'u_1 \quad aU \cdot X'u_2 \right) \;=\; aU \cdot \left(X'u_1\ X'u_2 \right) \,.$$

We show that $u \cdot \pi_1 \overset{!}{=} u_1$. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that $X(u \cdot \pi_1) \overset{!}{=} Xu_1$. In fact,

$$X(u \cdot \pi_1) \;=\; Xu \cdot X\pi_1 \;=\; \left(Xu_1\ Xu_2 \right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \;=\; Xu_1 \,.$$

We show that $u \cdot \pi_2 \overset{!}{=} u_2$. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that $X(u \cdot \pi_2) \overset{!}{=} Xu_2$. In fact,

$$X(u \cdot \pi_2) \;=\; Xu \cdot X\pi_2 \;=\; \left(Xu_1\ Xu_2 \right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \;=\; Xu_2 \,.$$

Uniqueness. Suppose given a transformation $\tilde{u} : U \to F_1 \oplus F_2$ such that $\tilde{u} \cdot \pi_1 = u_1$ and $\tilde{u} \cdot \pi_2 = u_2$. We have to show that $\tilde{u} \overset{!}{=} u$. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that $X\tilde{u} \overset{!}{=} Xu : XU \to XF_1 \oplus XF_2$. Writing $X\tilde{u} =: \left(s_1\ s_2 \right)$ with $s_1 : XU \to XF_1$ and $s_2 : XU \to XF_2$ and recalling that $Xu = \left(Xu_1\ Xu_2 \right)$, we have to show that $s_1 \overset{!}{=} Xu_1$ and $s_2 \overset{!}{=} Xu_2$. In fact,

$$s_1 \;=\; \left(s_1\ s_2 \right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \;=\; X\tilde{u} \cdot X\pi_1 \;=\; X(\tilde{u} \cdot \pi_1) \;=\; Xu_1$$

and

$$s_2 \;=\; \left(s_1\ s_2 \right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \;=\; X\tilde{u} \cdot X\pi_2 \;=\; X(\tilde{u} \cdot \pi_2) \;=\; Xu_2 \,.$$

Ad (Sum 2). Suppose given an additive functor $V : \mathcal{A} \to \mathcal{B}$ and transformations $v_1 : F_1 \to V$ and $v_2 : F_2 \to V$. We have to show that there exists a unique transformation $v : F_1 \oplus F_2 \to V$ such that $\iota_1 \cdot v = v_1$ and $\iota_2 \cdot v = v_2$.

Existence. We define the transformation

$$v \;:=\; \left(XF_1 \oplus XF_2 \xrightarrow{\left(\begin{smallmatrix} Xv_1 \\ Xv_2 \end{smallmatrix}\right)} XV \right)_{X \in \mathrm{Ob}(\mathcal{A})} \;:\; F_1 \oplus F_2 \;\to\; V \,.$$

We have to show naturality of the tuple v. Suppose given $X \xrightarrow{a} X'$ in $\mathcal{A}$. The quadrangle

$$\begin{array}{ccc} XF_1 \oplus XF_2 & \xrightarrow{\left(\begin{smallmatrix} Xv_1 \\ Xv_2 \end{smallmatrix}\right)} & XV \\ {\scriptstyle \left(\begin{smallmatrix} aF_1 & 0 \\ 0 & aF_2 \end{smallmatrix}\right)} \downarrow & & \downarrow {\scriptstyle aV} \\ X'F_1 \oplus X'F_2 & \xrightarrow{\left(\begin{smallmatrix} X'v_1 \\ X'v_2 \end{smallmatrix}\right)} & X'V \end{array}$$

commutes because of

$$\left(\begin{smallmatrix} aF_1 & 0 \\ 0 & aF_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} X'v_1 \\ X'v_2 \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} aF_1 \cdot X'v_1 \\ aF_2 \cdot X'v_2 \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} Xv_1 \cdot aV \\ Xv_2 \cdot aV \end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} Xv_1 \\ Xv_2 \end{smallmatrix}\right) \cdot aV \,.$$

We show that $\iota_1 \cdot v \overset{!}{=} v_1$. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that $X(\iota_1 \cdot v) \overset{!}{=} Xv_1$. In fact,

$$X(\iota_1 \cdot v) \;=\; X\iota_1 \cdot Xv \;=\; (\begin{smallmatrix} 1 & 0 \end{smallmatrix}) \cdot \left(\begin{smallmatrix} Xv_1 \\ Xv_2 \end{smallmatrix}\right) \;=\; Xv_1 \,.$$

We show that $\iota_2 \cdot v \overset{!}{=} v_2$. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that $X(\iota_2 \cdot v) \overset{!}{=} Xv_2$. In fact,

$$X(\iota_2 \cdot v) \;=\; X\iota_2 \cdot Xv \;=\; (\begin{smallmatrix} 0 & 1 \end{smallmatrix}) \cdot \left(\begin{smallmatrix} Xv_1 \\ Xv_2 \end{smallmatrix}\right) \;=\; Xv_2 \,.$$

Uniqueness. Suppose given a transformation $\tilde{v} : F_1 \oplus F_2 \to V$ such that $\iota_1 \cdot \tilde{v} = v_1$ and $\iota_2 \cdot \tilde{v} = v_2$. We have to show that $\tilde{v} \overset{!}{=} v$. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that $X\tilde{v} \overset{!}{=} Xv : XF_1 \oplus XF_2 \to XV$. Writing $X\tilde{v} =: \left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right)$ with $t_1 : XF_1 \to XV$ and $t_2 : XF_2 \to XV$ and recalling that $Xv = \left(\begin{smallmatrix} Xv_1 \\ Xv_2 \end{smallmatrix}\right)$, we have to show that $t_1 \overset{!}{=} Xv_1$ and $t_2 \overset{!}{=} Xv_2$. In fact,

$$t_1 \;=\; (\begin{smallmatrix} 1 & 0 \end{smallmatrix}) \cdot \left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right) \;=\; X\iota_1 \cdot X\tilde{v} \;=\; X(\iota_1 \cdot \tilde{v}) \;=\; Xv_1$$

and

$$t_2 \;=\; (\begin{smallmatrix} 0 & 1 \end{smallmatrix}) \cdot \left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right) \;=\; X\iota_2 \cdot X\tilde{v} \;=\; X(\iota_2 \cdot \tilde{v}) \;=\; Xv_2 \,.$$

Ad (Sum 3). We have to show that

$$\iota_1 \cdot \pi_1 \overset{!}{=} \mathrm{id}_{F_1} \,, \quad \iota_1 \cdot \pi_2 \overset{!}{=} 0_{F_1,\,F_2} \,, \quad \iota_2 \cdot \pi_1 \overset{!}{=} 0_{F_2,\,F_1} \,, \quad \iota_2 \cdot \pi_2 \overset{!}{=} \mathrm{id}_{F_2} \,.$$

Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that

$$X\iota_1 \cdot X\pi_1 \overset{!}{=} \mathrm{id}_{XF_1} \,, \quad X\iota_1 \cdot X\pi_2 \overset{!}{=} 0_{XF_1,\,XF_2} \,, \quad X\iota_2 \cdot X\pi_1 \overset{!}{=} 0_{XF_2,\,XF_1} \,, \quad X\iota_2 \cdot X\pi_2 \overset{!}{=} \mathrm{id}_{XF_2} \,.$$

We obtain

$$\begin{array}{rclclcl} X\iota_1 \cdot X\pi_1 &=& \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) &=& 1 &=& \mathrm{id}_{XF_1} \\ X\iota_1 \cdot X\pi_2 &=& \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) &=& 0 &=& 0_{XF_1,\, XF_2} \\ X\iota_2 \cdot X\pi_1 &=& \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) &=& 0 &=& 0_{XF_2,\, XF_1} \\ X\iota_2 \cdot X\pi_2 &=& \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) &=& 1 &=& \mathrm{id}_{XF_2}\ . \end{array}$$

Ad (Add 2′). Suppose given an additive functor $F : \mathcal{A} \to \mathcal{B}$. We use the construction of $F \oplus F$ from the proof of (Add 1). We have to show that the transformation

$$F \oplus F \xrightarrow{\;{}^{F\oplus F}\left(\begin{smallmatrix} \mathrm{id}_F & 0_{F,\,F} \\ \mathrm{id}_F & \mathrm{id}_F \end{smallmatrix}\right)^{F\oplus F}\;} F \oplus F$$

is an isotransformation.

We write $\varphi := {}^{F\oplus F}\left(\begin{smallmatrix} \mathrm{id}_F & 0_{F,\,F} \\ \mathrm{id}_F & \mathrm{id}_F \end{smallmatrix}\right)^{F\oplus F}$.

Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that $X\varphi$ is an isomorphism in $\mathcal{B}$.

By definition, φ is the unique morphism from $F \oplus F$ to $F \oplus F$ satisfying

$$\begin{array}{rclcrcl} \iota_1 \cdot \varphi \cdot \pi_1 &=& \mathrm{id}_F & \qquad & \iota_1 \cdot \varphi \cdot \pi_2 &=& 0_{F,\,F} \\ \iota_2 \cdot \varphi \cdot \pi_1 &=& \mathrm{id}_F & & \iota_2 \cdot \varphi \cdot \pi_2 &=& \mathrm{id}_F\ . \end{array}$$

Hence

$$\begin{array}{rclcrcl} X\iota_1 \cdot X\varphi \cdot X\pi_1 &=& \mathrm{id}_{XF} & \qquad & X\iota_1 \cdot X\varphi \cdot X\pi_2 &=& 0_{XF,\,XF} \\ X\iota_2 \cdot X\varphi \cdot X\pi_1 &=& \mathrm{id}_{XF} & & X\iota_2 \cdot X\varphi \cdot X\pi_2 &=& \mathrm{id}_{XF}\ . \end{array}$$

Writing $X\varphi = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) : XF \oplus XF \to XF \oplus XF$, this amounts to

$$\begin{array}{rclclcrclcl} s &=& \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) &=& 1 & \qquad & t &=& \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) &=& 0 \\ u &=& \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) &=& 1 & & v &=& \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) &=& 1\ . \end{array}$$

Hence $X\varphi = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, which is an isomorphism in $\mathcal{B}$.

Ad (2). We shall use the inclusion morphisms and projection morphisms as calculated for (1).

Let $\psi := \left(\begin{smallmatrix} \mathrm{id}_F & \mathrm{id}_F \end{smallmatrix}\right)^{F\oplus F}$. Then ψ is the unique morphism having $\psi \cdot \pi_1 = \mathrm{id}_F$ and $\psi \cdot \pi_2 = \mathrm{id}_F$. So for $X \in \mathrm{Ob}(\mathcal{A})$, we obtain $X\psi \cdot X\pi_1 = X\mathrm{id}_F = \mathrm{id}_{XF}$ and $X\psi \cdot X\pi_2 = X\mathrm{id}_F = \mathrm{id}_{XF}$. Thus $X\psi = \left(\begin{smallmatrix} \mathrm{id}_{XF} & \mathrm{id}_{XF} \end{smallmatrix}\right)$. Hence

$$\left(\begin{smallmatrix} \mathrm{id}_F & \mathrm{id}_F \end{smallmatrix}\right)^{F\oplus F} \;=\; \psi \;=\; \left(\left(\begin{smallmatrix} \mathrm{id}_{XF} & \mathrm{id}_{XF} \end{smallmatrix}\right)\right)_{X\in\mathrm{Ob}(\mathcal{A})}\ .$$

Let $\alpha := {}^{F\oplus F}\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right)$. Then α is the unique morphism having $\iota_1 \cdot \alpha = a$ and $\iota_2 \cdot \alpha = a'$. So for $X \in \mathrm{Ob}(\mathcal{A})$, we obtain $X\iota_1 \cdot X\alpha = Xa$ and $X\iota_2 \cdot X\alpha = Xa'$. Thus $X\alpha = \left(\begin{smallmatrix} Xa \\ Xa' \end{smallmatrix}\right)$. Hence

$${}^{F\oplus F}\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \;=\; \alpha \;=\; \left(\left(\begin{smallmatrix} Xa \\ Xa' \end{smallmatrix}\right)\right)_{X\in\mathrm{Ob}(\mathcal{A})}\ .$$

We obtain

$$\begin{aligned} a+a' &= \left(\mathrm{id}_F\ \mathrm{id}_F\right)^{F\oplus F} \cdot {}^{F\oplus F}\!\left(\begin{smallmatrix} a \\ a' \end{smallmatrix}\right) \\ &= \left(\left(\mathrm{id}_{XF}\ \mathrm{id}_{XF}\right)\right)_{X\in\mathrm{Ob}(\mathcal{A})} \cdot \left(\left(\begin{smallmatrix} Xa \\ Xa' \end{smallmatrix}\right)\right)_{X\in\mathrm{Ob}(\mathcal{A})} \\ &= \left(\left(\mathrm{id}_{XF}\ \mathrm{id}_{XF}\right) \cdot \left(\begin{smallmatrix} Xa \\ Xa' \end{smallmatrix}\right)\right)_{X\in\mathrm{Ob}(\mathcal{A})} \\ &= (Xa+Xa')_{X\in\mathrm{Ob}(\mathcal{A})} \;. \end{aligned}$$

□

Remark 73. *Suppose given additive categories $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, $\mathcal{B}'$.*

Suppose given additive functors $\mathcal{A}' \xrightarrow{G} \mathcal{A}$ and $\mathcal{B} \xrightarrow{H} \mathcal{B}'$.

(1) *The functor ${}_{\mathrm{AddCat}}(\mathcal{A}, H) : {}_{\mathrm{AddCat}}(\mathcal{A}, \mathcal{B}) \to {}_{\mathrm{AddCat}}(\mathcal{A}, \mathcal{B}')$ is additive; cf. Definition* 3.

(2) *The functor ${}_{\mathrm{AddCat}}(G, \mathcal{B}) : {}_{\mathrm{AddCat}}(\mathcal{A}, \mathcal{B}) \to {}_{\mathrm{AddCat}}(\mathcal{A}', \mathcal{B})$ is additive; cf. Definition* 4.

Proof.

Ad (1). Suppose given additive functors $F, \tilde{F} : \mathcal{A} \to \mathcal{B}$. Suppose given transformations $a, \tilde{a} : F \to \tilde{F}$. We have to show that

$$(a+\tilde{a})\,{}_{\mathrm{AddCat}}(\mathcal{A}, H) \overset{!}{=} (a)\,{}_{\mathrm{AddCat}}(\mathcal{A}, H) + (\tilde{a})\,{}_{\mathrm{AddCat}}(\mathcal{A}, H)\;;$$

cf. Lemma 50.

I.e. we have to show that $(a+\tilde{a})H \overset{!}{=} aH + \tilde{a}H$.

Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show that $X(a+\tilde{a})H \overset{!}{=} X(aH+\tilde{a}H)$. In fact, we obtain

$$X(a+\tilde{a})H \overset{\text{R. 72.(2)}}{=} (Xa+X\tilde{a})H \overset{\text{L. 50.(2)}}{=} XaH + X\tilde{a}H \overset{\text{R. 72.(2)}}{=} X(aH+\tilde{a}H)\;.$$

Ad (2). Suppose given additive functors $F, \tilde{F} : \mathcal{A} \to \mathcal{B}$. Suppose given transformations $a, \tilde{a} : F \to \tilde{F}$. We have to show that

$$(a+\tilde{a})\,{}_{\mathrm{AddCat}}(G, \mathcal{B}) \overset{!}{=} (a)\,{}_{\mathrm{AddCat}}(G, \mathcal{B}) + (\tilde{a})\,{}_{\mathrm{AddCat}}(G, \mathcal{B})\;;$$

cf. Lemma 50.

I.e. we have to show that $G(a+\tilde{a}) \overset{!}{=} Ga + G\tilde{a}$.

Suppose given $X' \in \mathrm{Ob}(\mathcal{A}')$. We have to show that $X'G(a+\tilde{a}) \overset{!}{=} X'(Ga+G\tilde{a})$. In fact, we obtain

$$X'G(a+\tilde{a}) \overset{\text{R. 72.(2)}}{=} X'Ga + X'G\tilde{a} \overset{\text{R. 72.(2)}}{=} X'(Ga+G\tilde{a})\;.$$

□

2.8 Factor categories

2.8.1 Construction

Let $\mathcal{A}$ be an additive category. We choose finite standard direct sums in $\mathcal{A}$.

Suppose given a full additive subcategory $\mathcal{N}$ of $\mathcal{A}$ such that $\mathcal{N}$ is closed under retracts in $\mathcal{A}$.

Definition 74. For $X, Y \in \mathrm{Ob}(\mathcal{A})$, we define

$$\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y) := \left\{ f \in {}_{\mathcal{A}}(X,Y) : \begin{array}{l} \text{There exists a factorization } f = f' \cdot f'' \\ \text{such that } X \xrightarrow{f'} N \xrightarrow{f''} Y \text{ for some } N \in \mathrm{Ob}(\mathcal{N}) \end{array} \right\}$$

Remark 75. The set $\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$ is a subgroup of ${}_{\mathcal{A}}(X,Y)$; cf. Corollary 44.

Proof. The zero object 0 of $\mathcal{A}$ is contained in $\mathcal{N}$; cf. Remark 54.(1). We have

$$0_{X,Y} \;=\; 0_{X,0} \cdot 0_{0,Y} \;\in\; \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)\,.$$

Suppose given $f_1, f_2 \in \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$. There exist factorizations

$$f_1 \;=\; f_1' \cdot f_1'' \;\text{ such that } X \xrightarrow{f_1'} N_1 \xrightarrow{f_1''} Y \text{ with } N_1 \in \mathrm{Ob}(\mathcal{N})$$

and

$$f_2 \;=\; f_2' \cdot f_2'' \;\text{ such that } X \xrightarrow{f_2'} N_2 \xrightarrow{f_2''} Y \text{ with } N_2 \in \mathrm{Ob}(\mathcal{N})\,.$$

We get a factorization

$$-f_1 \;=\; (-f_1') \cdot f_1''\,,$$

where $X \xrightarrow{-f_1'} N_1 \xrightarrow{f_1''} Y$, so that

$$-f_1 \;\in\; \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)\,;$$

cf. Corollary 45.

The direct sum $N_1 \oplus N_2$ is contained in $\mathrm{Ob}(\mathcal{N})$; cf. Remark 54.(2). We get a factorization

$$f_1 + f_2 \;=\; (f_1'\; f_2') \cdot \begin{pmatrix} f_1'' \\ f_2'' \end{pmatrix}\,,$$

where

$$X \xrightarrow{(f_1'\; f_2')} N_1 \oplus N_2 \xrightarrow{\begin{pmatrix} f_1'' \\ f_2'' \end{pmatrix}} Y\,,$$

so that $f_1 + f_2 \in \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$. □

Definition 76 (and Remark)**.** We may define the *factor category* $\mathcal{A}/\mathcal{N}$ as follows.

(1) Let $\mathrm{Ob}(\mathcal{A}/\mathcal{N}) := \mathrm{Ob}(\mathcal{A})$.

(2) For $X, Y \in \mathrm{Ob}(\mathcal{A}/\mathcal{N})$, the set of morphisms between X and Y is defined as the factor group

$${}_{\mathcal{A}/\mathcal{N}}(X,Y) := {}_{\mathcal{A}}(X,Y)/\,\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)\,;$$

cf. Remark 75. Given $f \in {}_{\mathcal{A}}(X,Y)$, we write

$$[f] \;:=\; f + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y) \;\in\; {}_{\mathcal{A}/\mathcal{N}}(X,Y)\,.$$

for its residue class.

(3) For X, Y, $Z \in \operatorname{Ob}(\mathcal{A}/\mathcal{N})$, composition is defined by

$$\begin{array}{ccccc} {}_{\mathcal{A}/\mathcal{N}}(X,Y) & \times & {}_{\mathcal{A}/\mathcal{N}}(Y,Z) & \xrightarrow{(\cdot)} & {}_{\mathcal{A}/\mathcal{N}}(X,Z) \\ ([f] & , & [g]) & \mapsto & [f]\cdot[g] \;:=\; [f\cdot g]\;. \end{array}$$

This is independent of the choice of the representative f of $[f]$ and of the representative g of $[g]$.

We have $\mathrm{id}_X^{\mathcal{A}/\mathcal{N}} = [\mathrm{id}_X^{\mathcal{A}}]$ for $X \in \operatorname{Ob}(\mathcal{A}) = \operatorname{Ob}(\mathcal{A}/\mathcal{N})$.

Then $\mathcal{A}/\mathcal{N}$ is in fact a category.

Moreover, we have the full and dense *residue class functor*

$$\begin{array}{rcl} \mathrm{R} \;=\; \mathrm{R}_{\mathcal{A},\mathcal{N}} \;:\; \mathcal{A} & \to & \mathcal{A}/\mathcal{N} \\ (X \xrightarrow{f} Y) & \mapsto & (X \xrightarrow{[f]} Y) \end{array}$$

Proof. We show independence in the construction of (3).

Suppose given $f,\, f' \in {}_{\mathcal{A}}(X,Y)$ and $g,\, g' \in {}_{\mathcal{A}}(Y,Z)$ such that $f - f' \in \operatorname{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$ and $g - g' \in \operatorname{Null}_{\mathcal{A},\mathcal{N}}(Y,Z)$. Then $f\cdot g - f'\cdot g' = f\cdot(g-g') + (f-f')\cdot g'$.

Since $g - g' \in \operatorname{Null}_{\mathcal{A},\mathcal{N}}(Y,Z)$, the morphism $g - g'$ factors over an object N in $\mathcal{N}$. Therefore $f\cdot(g-g')$ is a morphism between X and Z that factors over N, so we have $f\cdot(g-g') \in \operatorname{Null}_{\mathcal{A},\mathcal{N}}(X,Z)$.

Since $f - f' \in \operatorname{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$, the morphism $f - f'$ factors over an object N' in $\mathcal{N}$. Therefore $(f-f')\cdot g'$ is a morphism between X and Z that factors over N', so we have $(f-f')\cdot g' \in \operatorname{Null}_{\mathcal{A},\mathcal{N}}(X,Z)$.

Since $\operatorname{Null}_{\mathcal{A},\mathcal{N}}(X,Z)$ is a subgroup of ${}_{\mathcal{A}}(X,Z)$ by Remark 75, we have

$$f\cdot g - f'\cdot g' \;=\; f\cdot(g-g') + (f-f')\cdot g' \in \operatorname{Null}_{\mathcal{A},\mathcal{N}}(X,Z)\;.$$

That means $[f\cdot g] = [f'\cdot g']$, as we had to show.

We show that $\mathcal{A}/\mathcal{N}$ is in fact a category. The composition of morphisms in $\operatorname{Mor}(\mathcal{A}/\mathcal{N})$ is defined by the composites of representatives in $\operatorname{Mor}(\mathcal{A})$. So it is associative. Moreover, given $X \xrightarrow{f} Y$ in $\mathcal{A}$, we have $[\mathrm{id}_X^{\mathcal{A}}]\cdot[f] = [\mathrm{id}_X^{\mathcal{A}}\cdot f] = [f]$ and $[f]\cdot[\mathrm{id}_Y^{\mathcal{A}}] = [f\cdot\mathrm{id}_Y^{\mathcal{A}}] = [f]$. So $\mathrm{id}_X^{\mathcal{A}/\mathcal{N}} = [\mathrm{id}_X^{\mathcal{A}}]$ for $X \in \operatorname{Ob}(\mathcal{A}) = \operatorname{Ob}(\mathcal{A}/\mathcal{N})$.

We show that R is in fact a functor.

We have $(\mathrm{id}_X^{\mathcal{A}})\mathrm{R} = [\mathrm{id}_X^{\mathcal{A}}] = \mathrm{id}_X^{\mathcal{A}/\mathcal{N}}$ for $X \in \operatorname{Ob}(\mathcal{A})$.

Suppose given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \in \operatorname{Mor}(\mathcal{A})$. We obtain

$$(f\cdot g)\mathrm{R} = [f\cdot g] = [f]\cdot[g] = (f)\mathrm{R}\cdot(g)\mathrm{R}$$

So R is indeed a functor. □

Lemma 77. *The following assertions* $(1,2)$ *hold.*

(1) *The category* $\mathcal{A}/\mathcal{N}$ *is additive.*

More precisely, we have the following.

First, for $X \in \mathrm{Ob}(\mathcal{A})$*, we have that* X *is a zero object in* $\mathcal{A}/\mathcal{N}$ *if and only if* $X \in \mathrm{Ob}(\mathcal{N})$*. In particular, for* $f : X \to Y$ *in* $\mathcal{A}$*, the morphism* $[f]$ *is a zero morphism in* $\mathcal{A}/\mathcal{N}$ *if and only if* $f \in \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$*.*

Second, suppose given $m \geqslant 0$ *and* $X_i \in \mathrm{Ob}(\mathcal{A}/\mathcal{N}) = \mathrm{Ob}(\mathcal{A})$ *for* $i \in [1,m]$*.*

Suppose given a direct sum S *of* $(X_1, \ldots, X_m)$ *in* $\mathcal{A}$ *with inclusion morphisms* ι_1 *, ...,* ι_m *and projection morphisms* π_1 *, ...,* π_m *.*

Then S *is also a direct sum of* $(X_1, \ldots, X_m)$ *in* $\mathcal{A}/\mathcal{N}$*, with inclusion morphisms* $[\iota_1]$*, ...,* $[\iota_m]$ *and projection morphisms* $[\pi_1]$*, ...,* $[\pi_m]$*.*

Third, given $f,\, f' : X \to Y$ *in* $\mathcal{A}$*, the sum of* $[f]$ *and* $[f']$ *in the additive category* $\mathcal{A}/\mathcal{N}$ *equals the sum of* $[f]$ *and* $[f']$ *formed in the abelian factor group* ${}_{\mathcal{A}/\mathcal{N}}(X,Y) = {}_{\mathcal{A}}(X,Y)/\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$*, i.e.*

$$[f] + [f'] \;=\; [f + f'] \,.$$

(2) *The residue class functor*

$$\begin{array}{rcl} \mathrm{R} \;=\; \mathrm{R}_{\mathcal{A},\mathcal{N}} \;:\; & \mathcal{A} & \to \; \mathcal{A}/\mathcal{N} \\ & (X \xrightarrow{f} Y) & \mapsto \; (X \xrightarrow{[f]} Y) \end{array}$$

is additive.

Proof.

Ad (1).

First, suppose given $X \in \mathrm{Ob}(\mathcal{A})$.

If $X \in \mathrm{Ob}(\mathcal{N})$, then given $U \in \mathrm{Ob}(\mathcal{A}/\mathcal{N}) = \mathrm{Ob}(\mathcal{A})$, we have $\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,U) = {}_{\mathcal{A}}(X,U)$ and thus $|\,{}_{\mathcal{A}/\mathcal{N}}(X,U)| = |\,{}_{\mathcal{A}}(X,U)/\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,U)| = 1$, and, likewise, $\mathrm{Null}_{\mathcal{A},\mathcal{N}}(U,X) = {}_{\mathcal{A}}(U,X)$ and thus $|\,{}_{\mathcal{A}/\mathcal{N}}(U,X)| = |\,{}_{\mathcal{A}}(U,X)/\mathrm{Null}_{\mathcal{A},\mathcal{N}}(U,X)| = 1$. Hence X is a zero object in $\mathcal{A}/\mathcal{N}$.

Conversely, if X is a zero object in $\mathcal{A}/\mathcal{N}$, then $1 = |\,{}_{\mathcal{A}/\mathcal{N}}(X,X)| = |\,{}_{\mathcal{A}}(X,X)/\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,X)|$, i.e. ${}_{\mathcal{A}}(X,X) = \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,X)$. So $\mathrm{id}_X \in \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,X)$. Hence there exist an object $N \in \mathrm{Ob}(\mathcal{N})$ and morphisms $X \xrightarrow{a} N$ and $N \xrightarrow{b} X$ such that $\mathrm{id}_X = a \cdot b$. Therefore, X is a retract of $N \in \mathrm{Ob}(\mathcal{N})$. Thus $X \in \mathrm{Ob}(\mathcal{N})$.

Second, suppose given $m \geqslant 0$ and $X_i \in \mathrm{Ob}(\mathcal{A}/\mathcal{N}) = \mathrm{Ob}(\mathcal{A})$ for $i \in [1,m]$.

Suppose given a direct sum S of $(X_1, \ldots, X_m)$ in $\mathcal{A}$ with inclusion morphisms ι_1 , ..., ι_m and projection morphisms π_1 , ..., π_m .

Such a direct sum exists; cf. Lemma 37. Note that $S \in \mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A}/\mathcal{N})$.

We *claim* that S is also a direct sum in $\mathcal{A}/\mathcal{N}$, with inclusion morphisms $[\iota_1]$, ..., $[\iota_m]$ and projection morphisms $[\pi_1]$, ..., $[\pi_m]$.

Ad (Sum 1). Suppose given $U \in \mathrm{Ob}(\mathcal{A}/\mathcal{N})$ and a tuple of morphisms $(U \xrightarrow{u_i} X_i)_{i\in[1,m]}$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$. For each $i \in [1,m]$, we choose a representative $u'_i \in \mathrm{Mor}(\mathcal{A})$ of u_i , i.e. $u_i = [u'_i]$.

We consider the object U with the tuple of morphisms $(U \xrightarrow{u'_i} X_i)_{i\in[1,m]}$ in $\mathcal{A}$. Since S is a direct sum in $\mathcal{A}$, there is a unique morphism $U \xrightarrow{v} S$ such that $v \cdot \pi_i = u'_i$ for $i \in [1,m]$.

We have $[v] \cdot [\pi_i] = [v \cdot \pi_i] = [u'_i] = u_i$ for $i \in [1,m]$. So there is a morphism $U \xrightarrow{[v]} S$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$ such that $[v] \cdot [\pi_i] = u_i$ for $i \in [1,m]$.

Now we need to show the uniqueness. Suppose given a morphism $U \xrightarrow{w} S$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$ such that $w \cdot [\pi_i] = u_i$ for $i \in [1,m]$. So

$$[v \cdot \pi_i] \;=\; [v] \cdot [\pi_i] \;=\; u_i \;=\; w \cdot [\pi_i]$$

for $i \in [1,m]$.

We have to show that $w \overset{!}{=} [v]$.

We choose a representative $w' \in \mathrm{Mor}(\mathcal{A})$ of w, i.e. $w = [w']$.

For $i \in [1,m]$, we get

$$[v \cdot \pi_i] \;=\; w \cdot [\pi_i] \;=\; [w'] \cdot [\pi_i] \;=\; [w' \cdot \pi_i] \;,$$

hence

$$(v - w') \cdot \pi_i \;=\; v \cdot \pi_i - w' \cdot \pi_i \;\in\; \mathrm{Null}_{\mathcal{A},\mathcal{N}}(U, X_i)$$

and therefore

$$(v - w') \cdot \pi_i \cdot \iota_i \;\in\; \mathrm{Null}_{\mathcal{A},\mathcal{N}}(U, S) \;.$$

Hence

$$v - w' \;=\; (v - w') \cdot \Big(\sum_{i\in[1,m]} \pi_i \cdot \iota_i \Big) \;=\; \sum_{i\in[1,m]} (v - w') \cdot \pi_i \cdot \iota_i \;\in\; \mathrm{Null}_{\mathcal{A},\mathcal{N}}(U, S) \;.$$

Cf. Remark 41. Hence $w = [w'] = [v]$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$.

Ad (Sum 2). Suppose given $U \in \mathrm{Ob}(\mathcal{A}/\mathcal{N})$ and a tuple of morphisms $(X_i \xrightarrow{u_i} U)_{i\in[1,m]}$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$. For each $i \in [1,m]$, we choose a representative $u'_i \in \mathrm{Mor}(\mathcal{A})$ of u_i, i.e. $u_i = [u'_i]$. We consider the object U with the tuple of morphisms $(X_i \xrightarrow{u'_i} U)_{i\in[1,m]}$ in $\mathcal{A}$. Since S is a direct sum in $\mathcal{A}$, there is a unique morphism $S \xrightarrow{v} U$ such that $\iota_i \cdot v = u'_i$ for $i \in [1,m]$.

We have $[\iota_i] \cdot [v] = [\iota_i \cdot v] = [u'_i] = u_i$ for $i \in [1,m]$. So there is a morphism $S \xrightarrow{[v]} U$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$ such that $[\iota_i] \cdot [v] = u_i$ for $i \in [1,m]$.

Now we need to show the uniqueness. Suppose given a morphism $S \xrightarrow{w} U$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$ such that $[\iota_i] \cdot w = u_i$ for $i \in [1,m]$. So

$$[\iota_i \cdot v] \;=\; [\iota_i] \cdot [v] \;=\; u_i \;=\; [\iota_i] \cdot w$$

for $i \in [1,m]$.

We have to show that $w \overset{!}{=} [v]$.

We choose a representative $w' \in \mathrm{Mor}(\mathcal{A})$ of w, i.e. $w = [w']$.

For $i \in [1,m]$, we get

$$[\iota_i \cdot v] \;=\; [\iota_i] \cdot w \;=\; [\iota_i] \cdot [w'] \;=\; [\iota_i \cdot w'] \;,$$

hence

$$\iota_i \cdot (v - w') \;=\; \iota_i \cdot v - \iota_i \cdot w' \;\in\; \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X_i, U)$$

and therefore

$$\pi_i \cdot \iota_i \cdot (v - w') \;\in\; \mathrm{Null}_{\mathcal{A},\mathcal{N}}(S,U)\;.$$

Hence

$$v - w' \;=\; \Big(\sum_{i\in[1,m]} \pi_i \cdot \iota_i\Big) \cdot (v - w') \;=\; \sum_{i\in[1,m]} \pi_i \cdot \iota_i \cdot (v - w') \;\in\; \mathrm{Null}_{\mathcal{A},\mathcal{N}}(S,U)\;.$$

Cf. Remark 41. Hence $w = [w'] = [v]$ in $\mathrm{Mor}(\mathcal{A}/\mathcal{N})$.

Ad (Sum 3). We have $[\iota_i]\cdot[\pi_i] = [\iota_i \cdot \pi_i] = [\mathrm{id}_{X_i}] = \mathrm{id}_{X_i}$ for $i \in [1,m]$.

We have $[\iota_i]\cdot[\pi_j] = [\iota_i \cdot \pi_j] = [0_{X_i\,,\,X_j}] = 0_{X_i\,,\,X_j}$ for $i,j \in [1,m]$ with $i \neq j$.

This proves the *claim*.

Now we aim to show that $\mathcal{A}/\mathcal{N}$ is additive.

Note that $\mathcal{A}/\mathcal{N}$ is a pointed category, since there exists a zero object of $\mathcal{A}$ that is contained in $\mathrm{Ob}(\mathcal{N})$, which is thus a zero object in $\mathcal{A}/\mathcal{N}$. Alternatively, we can choose $m = 0$ in the argument above.

Ad (Add 1). This follows by letting $m = 2$ in the argument above.

Ad (Add 2). Suppose given $X \in \mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A}/\mathcal{N})$. Let S be a direct sum of (X,X) in $\mathcal{A}$, with inclusion morphisms ι_1, ι_2 and projection morphisms π_1, π_2. Then S is a direct sum of (X,X) in $\mathcal{A}/\mathcal{N}$, with inclusion morphisms $[\iota_1]$, $[\iota_2]$ and projection morphisms $[\pi_1]$, $[\pi_2]$.

We *claim* that the morphism ${}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{\!S} : S \to S$ formed in $\mathcal{A}/\mathcal{N}$ equals the residue class of the morphism $\varphi := {}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{\!S}$ formed in $\mathcal{A}$.

In fact, we have

$$\begin{array}{lllllll} [\iota_1]\cdot[\varphi]\cdot[\pi_1] & = & [\iota_1\cdot\varphi\cdot\pi_1] & = & [\mathrm{id}_X] & = & \mathrm{id}_X \\ [\iota_1]\cdot[\varphi]\cdot[\pi_2] & = & [\iota_1\cdot\varphi\cdot\pi_2] & = & [0_{X,X}] & = & 0_{X,X} \\ [\iota_2]\cdot[\varphi]\cdot[\pi_1] & = & [\iota_2\cdot\varphi\cdot\pi_1] & = & [\mathrm{id}_X] & = & \mathrm{id}_X \\ [\iota_2]\cdot[\varphi]\cdot[\pi_2] & = & [\iota_2\cdot\varphi\cdot\pi_2] & = & [\mathrm{id}_X] & = & \mathrm{id}_X\;. \end{array}$$

The morphism φ is an isomorphism, since $\mathcal{A}$ is additive.

Hence the morphism ${}^{S}\!\begin{pmatrix} \mathrm{id}_X & 0_{X,X} \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}^{\!S} : S \to S$ formed in $\mathcal{A}/\mathcal{N}$, which equals $[\varphi] = (\varphi)\mathrm{R}$, is an isomorphism as well since R is a functor.

Third. Suppose given $f,\, f' : X \to Y$ in $\mathcal{A}$. We want to calculate the sum of $[f]$ and $[f']$ in the additive category $\mathcal{A}/\mathcal{N}$ using Remark 35.

Suppose given $X \in \mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A}/\mathcal{N})$. Let S be a direct sum of (X,X) in $\mathcal{A}$, with inclusion morphisms ι_1, ι_2 and projection morphisms π_1, π_2. Then S is a direct sum of (X,X) in $\mathcal{A}/\mathcal{N}$, with inclusion morphisms $[\iota_1]$, $[\iota_2]$ and projection morphisms $[\pi_1]$, $[\pi_2]$.

We *claim* that the morphism ${}^{S}\!\begin{pmatrix} [f] \\ [f'] \end{pmatrix} : S \to Y$ formed in $\mathcal{A}/\mathcal{N}$ equals the residue class of the morphism $\varphi := {}^{S}\!\begin{pmatrix} f \\ f' \end{pmatrix}$. In fact, we have $[\iota_1]\cdot[\varphi] = [\iota_1\cdot\varphi] = [f]$ and $[\iota_2]\cdot[\varphi] = [\iota_2\cdot\varphi] = [f']$. This proves the *claim*.

We *claim* that the morphism $\left(\mathrm{id}_X\ \mathrm{id}_X\right)^S$, formed in $\mathcal{A}/\mathcal{N}$, equals the residue class of the morphism $\psi := \left(\mathrm{id}_X\ \mathrm{id}_X\right)^S$, formed in $\mathcal{A}$. In fact, we have $[\psi]\cdot[\pi_1] = [\psi\cdot\pi_1] = [\mathrm{id}_X] = \mathrm{id}_X$ and $[\psi]\cdot[\pi_2] = [\psi\cdot\pi_2] = [\mathrm{id}_X] = \mathrm{id}_X$. This proves the *claim*.

Now the sum of $[f]$ and $[f']$, formed in the additive category $\mathcal{A}/\mathcal{N}$, is given by

$$\left(\mathrm{id}_X\ \mathrm{id}_X\right)^S \cdot {}^S\!\begin{pmatrix}[f]\\ [f']\end{pmatrix} \;=\; \left[\left(\mathrm{id}_X\ \mathrm{id}_X\right)^S\right]\cdot\left[{}^S\!\begin{pmatrix}f\\ f'\end{pmatrix}\right] \;=\; \left[\left(\mathrm{id}_X\ \mathrm{id}_X\right)^S \cdot {}^S\!\begin{pmatrix}f\\ f'\end{pmatrix}\right] \;=\; [f+f']$$

But this is the sum of $[f]$ and $[f']$, formed in the abelian factor group ${}_{\mathcal{A}/\mathcal{N}}(X,Y) = {}_{\mathcal{A}}(X,Y)/\operatorname{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$.

Ad (2). We want to use Lemma 50.

Suppose given $f,\ f' : X \to Y$ in $\mathcal{A}$. We obtain

$$(f+f')R \;=\; [f+f'] \overset{(1)}{=} [f]+[f'] \;=\; (f)\mathrm{R} + (f')\mathrm{R}\ .$$

$\square$

2.8.2 Universal property of the factor category

Let $\mathcal{A}$ be an additive category.

Let $\mathcal{N} \subseteq \mathcal{A}$ be a full additive subcategory; cf. Definition 53.

Write $\mathrm{I} = \mathrm{I}_{\mathcal{N},\mathcal{A}} : \mathcal{N} \to \mathcal{A}$ for the inclusion functor.

Write $\mathrm{R} = \mathrm{R}_{\mathcal{A},\mathcal{N}} : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ for the residue class functor; cf. Definition 76.

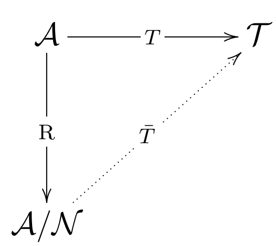
Note that $\mathrm{I}\,\mathrm{R}$ is a zero functor; cf. Lemma 77.(1).

Lemma 78. *The following assertions* $(1,2)$ *hold.*

(1) *Suppose given an additive category* $\mathcal{T}$ *and an additive functor* $\mathcal{A} \xrightarrow{T} \mathcal{T}$ *such that* $\mathrm{I}T$ *is a zero functor.*

Then there exists a unique functor $\mathcal{A}/\mathcal{N} \xrightarrow{\bar{T}} \mathcal{T}$ *such that* $\mathrm{R}\bar{T} = T$.

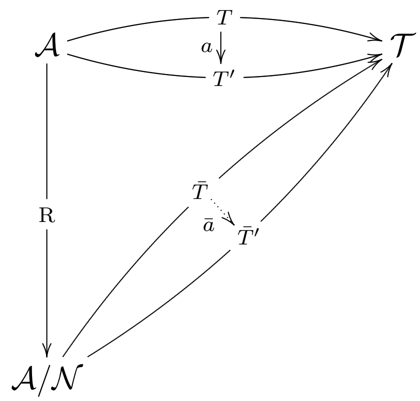
Moreover, $\bar{T}$ *is additive.*



(2) *Suppose given an additive category* $\mathcal{T}$ *and an additive functors* $\mathcal{A} \underset{T'}{\overset{T}{\rightrightarrows}} \mathcal{T}$ *such that* $\mathrm{I}T$ *and* $\mathrm{I}T'$ *are zero functors.*

Suppose given a transformation $a : T \to T'$.

Then there exists a unique transformation $\bar{a} : \bar{T} \to \bar{T}'$ *such that* $\mathrm{R}\bar{a} = a$ *; cf.* (1).



Proof. Ad (1). *Uniqueness* of $\bar{T}$ follows from the surjectivity of R on morphisms.

Existence. On objects, we have $\mathrm{Ob}(\mathcal{A}/\mathcal{N}) = \mathrm{Ob}(\mathcal{A})$. Therefore, we may define

$$X\bar{T} \ := \ XT$$

for $X \in \mathrm{Ob}(\mathcal{A}/\mathcal{N})$.

On morphisms, given $X \xrightarrow{f} Y$ in $\mathcal{A}$, we let

$$(X \xrightarrow{[f]} Y)\bar{T} \ := \ (XT \xrightarrow{fT} YT)\ .$$

We have to show independence of the choice of the representative f. Suppose given $X \xrightarrow{f} Y$ and $X \xrightarrow{f'} Y$ such that $[f] = [f']$. We have to show that $fT \overset{!}{=} f'T$.

We choose morphisms $X \xrightarrow{a} N \xrightarrow{b} Y$, where $N \in \mathrm{Ob}(\mathcal{N})$, such that $f - f' = a \cdot b$. Note that NT is a zero object in $\mathcal{T}$. So

$$fT - f'T \overset{\mathrm{R.}\,51}{=} (f - f')T \ = \ (a \cdot b)T \ = \ aT \cdot bT \ = \ 0\ .$$

We show that $\bar{T}$ is a functor.

Suppose given $X \in \mathrm{Ob}(\mathcal{A}/\mathcal{N})$. Then $(\mathrm{id}_X^{\mathcal{A}/\mathcal{N}})\bar{T} = [\mathrm{id}_X^{\mathcal{A}}]\bar{T} = \mathrm{id}_X^{\mathcal{A}}T = \mathrm{id}_{XT}^{\mathcal{T}} = \mathrm{id}_{X\bar{T}}^{\mathcal{T}}$.

Suppose given $X \xrightarrow{[f]} Y \xrightarrow{[g]} Z$ in $\mathcal{A}/\mathcal{N}$. Then $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$. We obtain

$$([f] \cdot [g])\bar{T} \ = \ [f \cdot g]\bar{T} \ = \ (f \cdot g)T \ = \ fT \cdot gT \ = \ [f]\bar{T} \cdot [g]\bar{T}\ .$$

By construction, $\mathrm{R}\bar{T} = T$.

We show that $\bar{T}$ is additive using Lemma 50.

Suppose given $[f], [f'] : X \to Y$ in $\mathcal{A}/\mathcal{N}$. We obtain

$$([f] + [f'])\bar{T} \ = \ [f + f']\bar{T} \ = \ (f + f')T \ = \ fT + f'T \ = \ [f]\bar{T} + [f']\bar{T}\ .$$

Ad (2). To show *uniqueness,* we show that a transformation $b : \bar{T} \to \bar{T}'$ satisfying $Rb = a$ is uniquely determined by a. Given $X \in \mathrm{Ob}(\mathcal{A}/\mathcal{N}) = \mathrm{Ob}(\mathcal{A})$, we have $Xb = X\mathrm{R}b = Xa$. So b is uniquely determined by a.

We show *existence* of the transformation $\bar{a}$ such that $R\bar{a} = a$. For $X \in \mathrm{Ob}(\mathcal{A}/\mathcal{N}) = \mathrm{Ob}(\mathcal{A})$, we let $X\bar{a} := Xa$, which is a morphism from $XT = X\bar{T}$ to $XT' = X\bar{T}'$. We show that $\bar{a}$ is a transformation from $\bar{T}$ to $\bar{T}'$. Suppose given $X \xrightarrow{[f]} Y$ in $\mathcal{A}/\mathcal{N}$. Then

$$\begin{array}{ccc} X\bar{T} & \xrightarrow{X\bar{a}} & X\bar{T}' \\ {\scriptstyle [f]\bar{T}}\downarrow & & \downarrow{\scriptstyle [f]\bar{T}'} \\ Y\bar{T} & \xrightarrow{Y\bar{a}} & Y\bar{T}' \end{array}$$

commutes, since we obtain, using that $a : T \to T'$ is a transformation,

$$[f]\bar{T} \cdot Y\bar{a} \;=\; fT \cdot Ya \;=\; Xa \cdot fT' \;=\; X\bar{a} \cdot [f]\bar{T}' \,.$$

By construction, $\mathrm{R}\bar{a} = a$. □

2.9 The kernel of an additive functor

Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$. Suppose given an additive functor $F : \mathcal{A} \to \mathcal{B}$.

Definition 79. Let $\mathrm{Kern}(F)$ be the full subcategory of $\mathcal{A}$ defined by

$$\mathrm{Ob}(\mathrm{Kern}(F)) \;:=\; \{\, X \in \mathrm{Ob}(\mathcal{A}) \;:\; XF \text{ is a zero object in } \mathcal{B} \,\} \,,$$

called the *kernel* of F.

Remark 80. *The kernel* $\mathrm{Kern}(F)$ *is a full additive subcategory of* $\mathcal{A}$ *that is closed under retracts.*

Proof. We show (1–3) from Definition 53.

Ad (1). Suppose given $X \in \mathrm{Ob}(\mathcal{A})$ and $Y \in \mathrm{Ob}(\mathrm{Kern}(F))$ such that $X \simeq Y$. We have to show that $X \overset{!}{\in} \mathrm{Ob}(\mathrm{Kern}(F))$.

In fact, $XF \simeq YF$. Since YF is a zero object, so is XF; cf. Remark 16. Hence $X \in \mathrm{Ob}(\mathrm{Kern}(F))$.

Ad (2). We have to show that there exists a zero object of $\mathcal{A}$ that lies in $\mathrm{Kern}(F)$.

Choose a zero object Z in $\mathcal{A}$. Then ZF is a zero object; cf. Definition 46. So $Z \in \mathrm{Ob}(\mathrm{Kern}(F))$.

Ad (3). Given $Y, Y' \in \mathrm{Ob}(\mathrm{Kern}(F))$, we have to show that there exists a direct sum S of (Y, Y') in $\mathcal{A}$ such that $S \overset{!}{\in} \mathrm{Ob}(\mathrm{Kern}(F))$.

Choose a direct sum S of (Y, Y') in $\mathcal{A}$. Then SF is a direct sum of $(YF, Y'F)$; cf. Definition 46. Since YF and $Y'F$ are zero objects, so is SF; cf. Remarks 63 and 54. Hence $S \in \mathrm{Ob}(\mathrm{Kern}(F))$.

We show that $\mathrm{Kern}(F)$ is closed under retracts in $\mathcal{A}$. Suppose given $Y \in \mathrm{Ob}(\mathrm{Kern}(F))$, i.e. $Y \in \mathrm{Ob}(\mathcal{A})$ such that YF is a zero object. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$ such that X is a retract of Y. We have to show that $X \overset{!}{\in} \mathrm{Ob}(\mathrm{Kern}(F))$, i.e. that XF is a zero object.

Since X is a retract of Y, there exist morphisms $X \xrightarrow{a} Y \xrightarrow{b} X$ such that $a \cdot b = \mathrm{id}_X$. Thus $aF \cdot bF = \mathrm{id}_{XF}$. Hence XF is a retract of YF. Since YF is a zero object, we conclude that XF is a zero object; cf. Remark 16. □

Remark 81. *Write* $\mathrm{I} : \mathrm{Kern}(F) \to \mathcal{A}$ *for the inclusion functor. Note that* $\mathrm{I}F$ *is a zero functor.*

(1) *Suppose given an additive category* $\mathcal{T}$. *Suppose given an additive functor* $T : \mathcal{T} \to \mathcal{A}$ *such that* TF *is a zero functor. Then there exists a unique additive functor* $\check{T} : \mathcal{T} \to \mathrm{Kern}(F)$ *such that* $\check{T}\mathrm{I} = T$.

(2) *Suppose given an additive category* $\mathcal{T}$. *Suppose given additive functors* $T, T' : \mathcal{T} \to \mathcal{A}$ *such that* TF *and* $T'F$ *are zero functors. Suppose given a transformation* $a : T \to T'$.

Then there exists a unique transformation $\check{a} : \check{T} \to \check{T}'$ *such that* $\check{a}\mathrm{I} = a$.

Proof. Note that for an additive functor $T : \mathcal{T} \to \mathcal{A}$, the composite TF is a zero functor if and only if T maps each object of $\mathcal{T}$ to $\mathrm{Ob}(\mathrm{Kern}(F))$.

Thus the assertion follow from Remark 58. □

2.10 The full image of an additive functor

Suppose given additive categories $\mathcal{A}$ and $\mathcal{B}$. Suppose given an additive functor $F : \mathcal{A} \to \mathcal{B}$.

Definition 82. Let $\mathrm{Im}(F)$ be the full subcategory of $\mathcal{B}$ given by

$$\mathrm{Ob}(\mathrm{Im}(F)) := \{\, Y \in \mathrm{Ob}(\mathcal{B}) \,:\, \text{there exists } X \in \mathrm{Ob}(\mathcal{A}) \text{ with } Y \simeq XF \,\}$$

The category $\mathrm{Im}(F)$ is called the *full image* of $\mathcal{A}$ under F.

We often write $\mathcal{A}F := \mathrm{Im}(F)$.

Remark 83. *We have that* $\mathcal{A}F$ *is a full additive subcategory of* $\mathcal{B}$.

Proof. We have to show the properties (1–3) of Definition 53.

Ad (1). An object of $\mathcal{B}$ isomorphic to an object of $\mathcal{A}F$ is in $\mathcal{A}F$ by construction.

Ad (2). The functor F maps a zero object of $\mathcal{A}$ to a zero object of $\mathcal{B}$; cf. Definition 46. Thus $\mathcal{A}F$ contains a zero object of $\mathcal{B}$.

Ad (3). Suppose given $Y, \tilde{Y} \in \mathrm{Ob}(\mathcal{A}F)$. Then there exist $X, \tilde{X} \in \mathrm{Ob}(\mathcal{A})$ such that $Y \simeq XF$ and $\tilde{Y} \simeq \tilde{X}F$. Let $S \in \mathrm{Ob}(\mathcal{A})$ be a direct sum of $(X, \tilde{X})$. Then SF is a direct sum of $(XF, \tilde{X}F)$; cf. Definition 46. Thus SF is also a direct sum of $(Y, \tilde{Y})$; cf. Remark 24. □

2.11 Pure short exact sequences in AddCat

In each additive category appearing in §2.11, we choose finite standard direct sums.

2.11.1 Definition and first properties

Definition 84. Suppose given additive categories $\mathcal{A}'$, $\mathcal{A}$ and $\mathcal{A}''$.

Suppose given additive functors $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$.

The sequence $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$ is called a *pure short exact sequence* in AddCat if the following conditions (P 1–4) hold.

(P 1) The functor F is full and faithful.

(P 2) The functor G is full and dense.

(P 3) We have $\mathrm{Im}(F) = \mathrm{Kern}(G)$ as full additive subcategories of $\mathcal{A}$.

(P 4) Suppose given a morphism $X \xrightarrow{u} \tilde{X}$ in $\mathcal{A}$ such that $uG = 0$. Then there exists $Z' \in \mathrm{Ob}(\mathcal{A}')$ and morphisms $X \xrightarrow{a} Z'F \xrightarrow{\tilde{a}} \tilde{X}$ in $\mathcal{A}$ such that $a \cdot \tilde{a} = u$.

An additive functor $\mathcal{A}' \xrightarrow{F} \mathcal{A}$ in AddCat for which there exists a pure short exact sequence $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$ in AddCat is called a *pure monofunctor*.

To indicate that $\mathcal{A}' \xrightarrow{F} \mathcal{A}$ is a pure monofunctor, we write $\mathcal{A}' \overset{F}{\bullet\!\!\twoheadrightarrow} \mathcal{A}$.

An additive functor $\mathcal{A} \xrightarrow{G} \mathcal{A}''$ in AddCat for which there exists a pure short exact sequence $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$ in AddCat is called a *pure epifunctor*.

To indicate that $\mathcal{A} \xrightarrow{G} \mathcal{A}''$ is a pure epifunctor, we write $\mathcal{A} \overset{G}{+\!\!\twoheadrightarrow} \mathcal{A}''$.

Remark 85. *Suppose given a pure short exact sequence* $\mathcal{A}' \overset{F}{\bullet\!\!\twoheadrightarrow} \mathcal{A} \overset{G}{+\!\!\twoheadrightarrow} \mathcal{A}''$.

(1) *The full additive subcategory* $\mathcal{A}'F \subseteq \mathcal{A}$ *is closed under retracts.*

(2) *The functor* $FG : \mathcal{A}' \to \mathcal{A}''$ *is a zero functor.*

Proof. *Ad* (1). The full additive subcategory $\mathcal{A}'F = \mathrm{Kern}(G) \subseteq \mathcal{A}$ is closed under retracts; cf. (P 3), Remark 80.

Ad (2). Suppose given $X' \in \mathrm{Ob}(\mathcal{A}')$. Then $XF \in \mathrm{Ob}(\mathcal{A}'F) = \mathrm{Kern}(G)$. Hence XFG is a zero object in $\mathcal{A}''$. Thus FG is a zero functor; cf. Definition 61. □

Remark 86. *Suppose given a sequence* $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$ *in* AddCat.

It is a pure short exact sequence if and only if (P 1), (P 2), (P 3a), (P 3b) *and* (P 4) *hold.*

(P 1) *The functor* F *is full and faithful.*

(P 2) *The functor* G *is full and dense.*

(P 3a) *The composite* $FG : \mathcal{A}' \to \mathcal{A}''$ *is a zero functor.*

(P 3b) *The full additive subcategory* $\mathrm{Im}(F) \subseteq \mathcal{A}$ *is closed under retracts.*

(P 4) *Suppose given a morphism* $X \xrightarrow{u} \tilde{X}$ *in* $\mathcal{A}$ *such that* $uG = 0$. *Then there exists* $Z' \in \mathrm{Ob}(\mathcal{A}')$ *and morphisms* $X \xrightarrow{a} Z'F \xrightarrow{\tilde{a}} \tilde{X}$ *in* $\mathcal{A}$ *such that* $a \cdot \tilde{a} = u$.

Proof. If (F, G) satisfies (P 1–4), it satisfies (P1, 2, 3a, 3b, 4) by Remark 85.

Conversely, if (F, G) satisfies (P1, 2, 3a, 3b, 4), we have to show (P 3). By (P 3a), it suffices to show that $\mathcal{A}'F \overset{!}{\supseteq} \mathrm{Kern}(G)$.

Suppose given $X \in \mathrm{Ob}(\mathrm{Kern}(G)) \subseteq \mathrm{Ob}(\mathcal{A})$. Then XG is a zero object, whence $\mathrm{id}_X G = 0$. By (P 4), we conclude that there exists $Z' \in \mathrm{Ob}(\mathcal{A}')$ and a commutative triangle in $\mathcal{A}$ as follows.

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ & \searrow \quad \nearrow & \\ & Z'F & \end{array}$$

Since $\mathcal{A}'F$ is closed under retracts in $\mathcal{A}$ by (P 3b), we conclude that $X \in \mathrm{Ob}(\mathcal{A}'F)$. □

Remark 87. *Suppose given an additive category $\mathcal{A}$.*

Suppose given a full additive subcategory $\mathcal{N} \subseteq \mathcal{A}$ that is closed under retracts.

Denote by $\mathcal{N} \xrightarrow{\mathrm{I}} \mathcal{A}$ the inclusion functor and by $\mathcal{A} \xrightarrow{\mathrm{R}} \mathcal{A}/\mathcal{N}$ the residue class functor.

Then $\mathcal{N} \xrightarrow{\mathrm{I}} \mathcal{A} \xrightarrow{\mathrm{R}} \mathcal{A}/\mathcal{N}$ is a pure short exact sequence.

Proof. *Ad* (P 1, 2, 4). This follows by construction; cf. Definition 76.

Ad (P 3). This follows from Lemma 77.(1). □

2.11.2 Stability under equivalences

Lemma 88. *Suppose given a diagram in* AddCat *as follows.*

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{F} & \mathcal{A} \\ {\scriptstyle U'}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{B}' & \xrightarrow{G} & \mathcal{B} \end{array}$$

Suppose that U' and U are equivalences.

Suppose that $U'G \simeq FU$.

Then the following statements (1, 2, 3) *hold.*

(1) *If F is full, then G is full.*

(2) *If F is faithful, then G is faithful.*

(3) *If F is dense, then G is dense.*

(4) *If $\mathcal{A}'F \subseteq \mathcal{A}$ is closed under retracts, then $\mathcal{B}'G \subseteq \mathcal{B}$ is closed under retracts.*

Proof. We choose an isotransformation $m : FU \xrightarrow{\sim} U'G$.

Ad (1, 2). Suppose given $Y', \tilde{Y}' \in \mathrm{Ob}(\mathcal{B}')$.

We choose $X' \in \mathrm{Ob}(\mathcal{A}')$ and an isomorphism $f' : X'U' \stackrel{\sim}{\to} Y'$.

We choose $\tilde{X}' \in \mathrm{Ob}(\mathcal{A}')$ and an isomorphism $\tilde{f}' : \tilde{X}'U' \stackrel{\sim}{\to} \tilde{Y}'$.

We have the bijective map

$$\begin{array}{rcl} {}_{\mathcal{B}'}(X'U', \tilde{X}'U') & \xrightarrow[\sim]{\varphi'} & {}_{\mathcal{B}'}(Y', \tilde{Y}') \\ g & \stackrel{\varphi'}{\mapsto} & f'^{-} \cdot g \cdot \tilde{f}' \\ f' \cdot h \cdot \tilde{f}'^{-} & \stackrel{\varphi'^{-}}{\leftarrow\!\shortmid} & h \end{array}$$

$$\begin{array}{ccc} X'U' & \xrightarrow{g} & \tilde{X}'U' \\ {\scriptstyle f'}\downarrow\wr & & \wr\downarrow{\scriptstyle \tilde{f}'} \\ Y' & \xrightarrow{h} & \tilde{Y}' \end{array}$$

We have the bijective map

$$\begin{array}{rcl} {}_{\mathcal{B}}(X'U'G, \tilde{X}'U'G) & \xrightarrow[\sim]{\psi'} & {}_{\mathcal{B}}(Y'G, \tilde{Y}'G) \\ g & \stackrel{\psi'}{\mapsto} & f'^{-}G \cdot g \cdot \tilde{f}'G \\ f'G \cdot h \cdot \tilde{f}'^{-}G & \stackrel{\psi'^{-}}{\leftarrow\!\shortmid} & h \end{array}$$

$$\begin{array}{ccc} X'U'G & \xrightarrow{g} & \tilde{X}'U'G \\ {\scriptstyle f'G}\downarrow\wr & & \wr\downarrow{\scriptstyle \tilde{f}'G} \\ Y'G & \xrightarrow{h} & \tilde{Y}'G \end{array}$$

We have the bijective map

$$\begin{array}{rcl} {}_{\mathcal{B}}(X'FU, \tilde{X}'FU) & \xrightarrow[\sim]{\vartheta'} & {}_{\mathcal{B}}(X'U'G, \tilde{X}'U'G) \\ g & \stackrel{\vartheta'}{\mapsto} & X'm^{-} \cdot g \cdot \tilde{X}'m \\ X'm \cdot h \cdot \tilde{X}'m^{-} & \stackrel{\vartheta'^{-}}{\leftarrow\!\shortmid} & h \end{array}$$

$$\begin{array}{ccc} X'FU & \xrightarrow{g} & \tilde{X}'FU \\ {\scriptstyle X'm}\downarrow\wr & & \wr\downarrow{\scriptstyle \tilde{X}'m} \\ X'U'G & \xrightarrow{h} & \tilde{X}'U'G \end{array}$$

We have the following commutative diagram of sets and maps.

$$\begin{array}{ccc} {}_{\mathcal{A}'}(X', \tilde{X}') & \xrightarrow{F_{X', \tilde{X}'}} & {}_{\mathcal{A}}(X'F, \tilde{X}'F) \\ & & \wr\downarrow{\scriptstyle U_{X'F, \tilde{X}'F}} \\ {\scriptstyle U'_{X', \tilde{X}'}}\downarrow\wr & & {}_{\mathcal{B}}(X'FU, \tilde{X}'FU) \\ & & \wr\downarrow{\scriptstyle \vartheta'} \\ {}_{\mathcal{B}'}(X'U', \tilde{X}'U') & \xrightarrow{G_{X'U', \tilde{X}'U'}} & {}_{\mathcal{B}}(X'U'G, \tilde{X}'U'G) \\ {\scriptstyle \varphi'}\downarrow\wr & & \wr\downarrow{\scriptstyle \psi'} \\ {}_{\mathcal{B}'}(Y', \tilde{Y}') & \xrightarrow{G_{Y', \tilde{Y}'}} & {}_{\mathcal{B}}(Y'G, \tilde{Y}'G) \end{array}$$

In fact, for $a' \in {}_{\mathcal{A}'}(X', \tilde{X}')$, we obtain, by naturality of m,

$$(a')(F_{X',\tilde{X}'} \cdot U_{X'F,\tilde{X}'F} \cdot \vartheta') \;=\; (a'FU)\vartheta' \;=\; X'm^- \cdot a'FU \cdot \tilde{X}'m \;=\; a'U'G \;=\; (a')(U'_{X',\tilde{X}'} \cdot G_{X'U',\tilde{X}'U'}) \;.$$

Moreover, for $b' \in {}_{\mathcal{B}'}(X'U', \tilde{X}'U')$, we obtain

$$(b')(\varphi' \cdot G_{Y',\tilde{Y}'}) \;=\; (f'^- \cdot b' \cdot \tilde{f}')G \;=\; f'^-G \cdot b'G \cdot \tilde{f}'G \;=\; (b'G)\psi' \;=\; (b')(G_{X'U',\tilde{X}'U'} \cdot \psi')$$

If $F_{X',\tilde{X}'}$ is surjective, then $G_{Y',\tilde{Y}'}$ is surjective. Hence, if F is full, then G is full.

If $F_{X',\tilde{X}'}$ is injective, then $G_{Y',\tilde{Y}'}$ is injective. Hence, if F is faithful, then G is faithful.

Ad (3). Suppose that F is dense. We have to show that G is dense.

Suppose given $Y \in \mathrm{Ob}(\mathcal{B})$. We have to find an object $Y' \in \mathrm{Ob}(\mathcal{B}')$ such that $Y'G \simeq Y$.

Since U is dense, we may choose $X \in \mathrm{Ob}(\mathcal{A})$ such that $XU \simeq Y$.

Since F is dense, we may choose $X' \in \mathrm{Ob}(\mathcal{A}')$ such that $X'F \simeq X$.

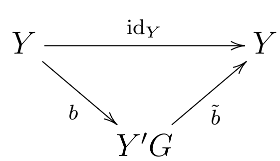
Letting $Y' := X'U'$, we obtain

$$Y'G \;=\; X'U'G \;\simeq\; X'FU \;\simeq\; XU \;\simeq\; Y \;.$$

Ad (4). Suppose that $\mathcal{A}'F \subseteq \mathcal{A}$ is closed under retracts.

We have to show that $\mathcal{B}'G \subseteq \mathcal{B}$ is closed under retracts.

Suppose given $Y' \in \mathrm{Ob}(\mathcal{B}')$, $Y \in \mathrm{Ob}(\mathcal{B})$ and a commutative triangle in $\mathcal{B}$ as follows.



We have to show that $Y \overset{!}{\in} \mathrm{Ob}(\mathcal{B}'G)$.

We choose $X \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $t : XU \overset{\sim}{\to} Y$.

We choose $X' \in \mathrm{Ob}(\mathcal{A}')$ and an isomorphism $t' : X'U' \overset{\sim}{\to} Y'$.

We have

$$XU \;\underset{\sim}{\overset{t}{\to}}\; Y \;\overset{b}{\to}\; Y'G \;\underset{\sim}{\xrightarrow{t'^-G}}\; X'U'G \;\underset{\sim}{\xrightarrow{X'm^-}}\; X'FU \;.$$

Since U is full and faithful, there exists a unique morphism $a : X \to X'F$ such that

$$aU \;=\; t \cdot b \cdot t'^-G \cdot X'm^- \;.$$

We have

$$X'FU \;\underset{\sim}{\xrightarrow{X'm}}\; X'U'G \;\underset{\sim}{\xrightarrow{t'G}}\; Y'G \;\overset{\tilde{b}}{\to}\; Y \;\underset{\sim}{\xrightarrow{t^-}}\; XU \;.$$

Since U is full and faithful, there exists a unique morphism $\tilde{a} : X'F \to X$ such that

$$\tilde{a}U \;=\; X'm \cdot t'G \cdot \tilde{b} \cdot t^- \;.$$

We have the following commutative triangle in $\mathcal{A}$.

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle \tilde{a}} & \\ & X'F & \end{array}$$

To see this, it suffices to show that $(a \cdot \tilde{a})U \overset{!}{=} \mathrm{id}_X U$. In fact,

$$(a \cdot \tilde{a})U \;=\; aU \cdot \tilde{a}U \;=\; t \cdot b \cdot t'^{-} G \cdot X' m^{-} \cdot X' m \cdot t' G \cdot \tilde{b} \cdot t^{-} \;=\; t \cdot b \cdot \tilde{b} \cdot t^{-} \;=\; t \cdot t^{-} \;=\; \mathrm{id}_{XU} \;=\; \mathrm{id}_X U \;.$$

Since $\mathcal{A}'F \subseteq \mathcal{A}$ is closed under retracts, there exists $\tilde{X}' \in \mathrm{Ob}(\mathcal{A}')$ with $\tilde{X}'F \simeq X$. Hence

$$(\tilde{X}'U')G \;\simeq\; \tilde{X}'FU \;\simeq\; XU \;\simeq\; Y \;.$$

So $Y \in \mathrm{Ob}(\mathcal{B}'G)$. □

Lemma 89. *Suppose given a diagram in* AddCat *as follows.*

$$\begin{array}{ccccc} \mathcal{A}' & \xrightarrow{F'} & \mathcal{A} & \xrightarrow{F''} & \mathcal{A}'' \\ {\scriptstyle U'}\downarrow & & {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U''} \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{B} & \xrightarrow{G''} & \mathcal{B}'' \end{array}$$

Suppose that U', U and U'' are equivalences.

Suppose that $U'G' \simeq F'U$ and $UG'' \simeq F''U''$.

Then the sequence $\mathcal{A}' \xrightarrow{F'} \mathcal{A} \xrightarrow{F''} \mathcal{A}''$ is pure short exact if and only if the sequence $\mathcal{B}' \xrightarrow{G'} \mathcal{B} \xrightarrow{G''} \mathcal{B}''$ is pure short exact.

In particular, this applies if $U' = \mathrm{id}_{\mathcal{A}'}$ and $U = \mathrm{id}_{\mathcal{A}}$ and $U'' = \mathrm{id}_{\mathcal{A}''}$.

Proof.

We choose an isotransformation $m' : F'U \overset{\sim}{\to} U'G'$.

We choose an isotransformation $m'' : F''U'' \overset{\sim}{\to} UG''$.

Suppose that $\mathcal{A}' \xrightarrow{F'} \mathcal{A} \xrightarrow{F''} \mathcal{A}''$ is pure short exact. We have to show that $\mathcal{B}' \xrightarrow{G'} \mathcal{B} \xrightarrow{G''} \mathcal{B}''$ is pure short exact. We may verify conditions (P 1, 2, 3a, 3b, 4) from Remark 86.

Ad (P 1). Since F' is full and faithful, we conclude by Lemma 88.$(1, 2)$ that G' is full and faithful.

Ad (P 2). Since F'' is full and dense, we conclude by Lemma 88.$(1, 3)$ that G'' is full and dense.

Ad (P 3a). Suppose given $Y' \in \mathrm{Ob}(\mathcal{B}')$. We have to show that $Y'G'G''$ is zero.

Since U' is dense, we may choose $X' \in \mathrm{Ob}(\mathcal{A}')$ such that $X'U' \simeq Y'$. Then

$$Y'G'G'' \;\simeq\; X'U'G'G'' \;\simeq\; X'F'UG'' \;\simeq\; X'F'F''U'' \;,$$

which is zero, since $X'F'F''$ is zero and U'' is additive.

Ad (P 3b). Since $\mathcal{A}'F' \subseteq \mathcal{A}$ is closed under retracts, we conclude by Lemma 88.(4) that $\mathcal{B}'G' \subseteq \mathcal{B}$ is closed under retracts.

Ad (P 4). Suppose given a morphism $Y \xrightarrow{b} \tilde{Y}$ in $\mathcal{B}$ such that $bG'' = 0$. We have to show that there exists $Y' \in \mathrm{Ob}(\mathcal{B}')$ and morphisms $Y \xrightarrow{b_1} Y'G' \xrightarrow{b_2} \tilde{Y}$ such that $b_1 \cdot b_2 = b$.

Using that U is dense, we choose $X, \tilde{X} \in \mathrm{Ob}(\mathcal{A})$ and isomorphisms $XU \overset{s}{\underset{\sim}{\to}} Y$ and $\tilde{X}U \overset{\tilde{s}}{\underset{\sim}{\to}} \tilde{Y}$. Then $s \cdot b \cdot \tilde{s}^- : XU \to \tilde{X}U$. Since U is full and faithful, there exists a unique $X \xrightarrow{a} \tilde{X}$ with $aU = s \cdot b \cdot \tilde{s}^-$.

We have

$$aUG'' \;=\; (s \cdot b \cdot \tilde{s}^-)G'' \;=\; sG'' \cdot bG'' \cdot \tilde{s}^-G'' \;=\; 0\,.$$

We have a commutative quadrangle by naturality of m'' as follows.

$$\begin{array}{ccc} XF''U'' & \xrightarrow[\sim]{Xm''} & XUG'' \\ {\scriptstyle aF''U''}\downarrow & & \downarrow{\scriptstyle aUG''} \\ \tilde{X}F''U'' & \xrightarrow[\sim]{\tilde{X}m''} & \tilde{X}UG'' \end{array}$$

We conclude that $aF''U'' = 0$.

Since U'' is full and faithful, Remarks 52 and 51 give the following isomorphism of abelian groups.

$${}_{\mathcal{A}''}(XF'', \tilde{X}F'') \xrightarrow[\sim]{U''_{XF'',\tilde{X}F''}} {}_{\mathcal{B}''}(XF''U'', \tilde{X}F''U'')$$

Hence $aF'' = 0$.

By (P 4) for $\mathcal{A}' \overset{F'}{\bullet\!\!\to} \mathcal{A} \overset{F''}{\to\!\!\!\!+\!\!\!\!\to} \mathcal{A}''$, we may choose $X' \in \mathrm{Ob}(\mathcal{A}')$ and a commutative triangle in $\mathcal{A}$ as follows.

$$\begin{array}{ccccc} X & & \xrightarrow{\quad a \quad} & & \tilde{X} \\ & {\scriptstyle a_1}\searrow & & \nearrow{\scriptstyle a_2} & \\ & & X'F' & & \end{array}$$

Application of U yields the following diagram in $\mathcal{B}$.

$$\begin{array}{ccccc} Y & & \xrightarrow{\quad b \quad} & & \tilde{Y} \\ {\scriptstyle s}\uparrow\wr & & & & \wr\uparrow{\scriptstyle \tilde{s}} \\ XU & & \xrightarrow{\quad aU \quad} & & \tilde{X}U \\ & {\scriptstyle a_1U}\searrow & & \nearrow{\scriptstyle a_2U} & \\ & & X'F'U & & \\ & & \wr\downarrow{\scriptstyle X'm'} & & \\ & & X'U'G' & & \end{array}$$

Hence

$$\begin{aligned} b &= s^- \cdot (sb\tilde{s}^-) \cdot \tilde{s} \\ &= s^- \cdot aU \cdot \tilde{s} \\ &= s^- \cdot a_1U \cdot a_2U \cdot \tilde{s} \\ &= (s^- \cdot a_1U \cdot X'm') \cdot (X'm'^- \cdot a_2U \cdot \tilde{s}) \end{aligned}$$

So letting $Y' := X'U' \in \mathrm{Ob}(\mathcal{B}')$, $b_1 := s^- \cdot a_1 U \cdot X'm'$ and $b_2 := X'm'^- \cdot a_2 U \cdot \tilde{s}$, we get the factorisation $b = b_1 \cdot b_2$ over $Y'G'$.

Hence $\mathcal{B}' \overset{G'}{\bullet\!\!\rightarrow} \mathcal{B} \overset{G''}{\twoheadrightarrow} \mathcal{B}''$ is pure short exact.

Conversely, suppose that $\mathcal{B}' \overset{G'}{\bullet\!\!\rightarrow} \mathcal{B} \overset{G''}{\twoheadrightarrow} \mathcal{B}''$ is pure short exact. We have to show that $\mathcal{A}' \xrightarrow{F'} \mathcal{A} \xrightarrow{F''} \mathcal{A}''$ is pure short exact.

We choose $V' : \mathcal{B}' \to \mathcal{A}'$ such that $U'V' \simeq \mathrm{id}_{\mathcal{A}'}$ and $V'U' \simeq \mathrm{id}_{\mathcal{B}'}$.

We choose $V : \mathcal{B} \to \mathcal{A}$ such that $UV \simeq \mathrm{id}_{\mathcal{A}}$ and $VU \simeq \mathrm{id}_{\mathcal{B}}$.

We choose $V'' : \mathcal{B}'' \to \mathcal{A}''$ such that $U''V'' \simeq \mathrm{id}_{\mathcal{A}''}$ and $V''U'' \simeq \mathrm{id}_{\mathcal{B}''}$.

Then $V'F'U \simeq V'U'G' \simeq G' \simeq G'VU$, whence $V'F' \simeq G'V$ since U is full and faithful.

And $VF''U'' \simeq VUG'' \simeq G'' \simeq G''V''U''$, whence $VF'' \simeq G''V''$ since U'' is full and faithful.

Now the argument above applies, so that we may conclude that $\mathcal{A}' \overset{F'}{\bullet\!\!\rightarrow} \mathcal{A} \overset{F''}{\twoheadrightarrow} \mathcal{A}''$ is pure short exact.

$$\begin{array}{ccccc} \mathcal{A}' & \xrightarrow{F'} & \mathcal{A} & \xrightarrow{F''} & \mathcal{A}'' \\ {\scriptstyle V'}\uparrow & & {\scriptstyle V}\uparrow & & \uparrow{\scriptstyle V''} \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{B} & \xrightarrow{G''} & \mathcal{B}'' \end{array}$$

□

2.11.3 Universal properties

We give variants of the universal properties from Lemma 78 and Remark 81 for pure short exact sequences in AddCat.

Lemma 90. *Suppose given a pure short exact sequence*

$$\mathcal{A}' \overset{F}{\bullet\!\!\rightarrow} \mathcal{A} \overset{G}{\twoheadrightarrow} \mathcal{A}''$$

in AddCat*.*

Suppose given an additive category $\mathcal{T}$*.*

We have induced functors

$$(\mathcal{A}', \mathcal{T}) \xleftarrow{(F,\mathcal{T})} (\mathcal{A}, \mathcal{T}) \xleftarrow{(G,\mathcal{T})} (\mathcal{A}'', \mathcal{T})$$

We have a full additive subcategory $\mathrm{Kern}(F, \mathcal{T}) \subseteq (\mathcal{A}, \mathcal{T})$*.*

We obtain a functor

$$(\mathcal{A}'', \mathcal{T}) \xrightarrow{(G,\mathcal{T})|^{\mathrm{Kern}(F,\mathcal{T})}} \mathrm{Kern}(F, \mathcal{T}) .$$

This functor is an equivalence.

Proof. Preliminary remark. By (P 2), the functor $G : \mathcal{A} \to \mathcal{A}''$ is dense. So given $X'' \in \mathrm{Ob}(\mathcal{A}'')$, we may choose an object $U_{X''} \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $u''_{X''} : U_{X''}G \xrightarrow{\sim} X''$ in $\mathcal{A}''$.

Now, we have to show that $(G,\mathcal{T})$ maps each object T'' of $(\mathcal{A}'',\mathcal{T})$ to an object of $\mathrm{Kern}(F,\mathcal{T})$. In fact, T'' is mapped to GT'', and since $(GT'')(F,\mathcal{T}) = FGT''$ is a zero functor by Remarks 85.(2) and 65, the functor GT'' is an object of $\mathrm{Kern}(F,\mathcal{T})$.

So the additive functor $G^* := (G,\mathcal{T})|^{\mathrm{Kern}(F,\mathcal{T})} : (\mathcal{A}'',\mathcal{T}) \to \mathrm{Kern}(F,\mathcal{T})$ exists; cf. Remark 58.(1). We have to show that it is faithful, full and dense.

Ad G^* faithful. Since G^* is additive, it suffices to show that a morphism that is sent to zero by G^* is zero.

So suppose given T'', $\tilde{T}'' \in \mathrm{Ob}(\mathcal{A}'',\mathcal{T})$, i.e. suppose given additive functors T'', $\tilde{T}'' : \mathcal{A}'' \to \mathcal{T}$. Suppose given a transformation $a'' : T'' \to \tilde{T}''$ such that $0 = a''G^* = a''(G,\mathcal{T}) = Ga''$. So $XGa'' = 0$ for $X \in \mathrm{Ob}(\mathcal{A})$.

We have to show that $a'' \stackrel{!}{=} 0$. I.e. we have to show that $X''a'' = 0$ for $X'' \in \mathrm{Ob}(\mathcal{A}'')$.

Since a'' is a transformation, we have the following commutative quadrangle.

$$\begin{array}{ccc} U_{X''}GT'' & \xrightarrow{U_{X''}Ga''} & U_{X''}G\tilde{T}'' \\ {\scriptstyle u''_{X''}T''}\downarrow\wr & & \wr\downarrow{\scriptstyle u''_{X''}\tilde{T}''} \\ X''T'' & \xrightarrow{X''a''} & X''\tilde{T}'' \end{array}$$

Now $U_{X''}Ga'' = 0$. So $X''a'' = (u''_{X''}T'')^- \cdot U_{X''}Ga'' \cdot u''_{X''}\tilde{T}'' = 0$; cf. Lemma 18.(2).

Ad G^* full. Suppose given T'', $\tilde{T}'' \in \mathrm{Ob}(\mathcal{A}'',\mathcal{T})$. We have to show that the map

$$\begin{array}{ccccc} {}_{(\mathcal{A}'',\mathcal{T})}(T'',\, \tilde{T}'') & \to & {}_{\mathrm{Kern}(F,\mathcal{T})}(T''G^*,\, \tilde{T}''G^*) & = & {}_{(\mathcal{A},\mathcal{T})}(GT'',\, G\tilde{T}'') \\ a'' & \mapsto & a''G^* & = & Ga'' \end{array}$$

is surjective. So suppose given a transformation $a : GT'' \to G\tilde{T}''$. We have to show that there exists a transformation $a'' : T'' \to \tilde{T}''$ such that $Ga'' = a$.

Let $X''a'' := (u''_{X''}T'')^- \cdot U_{X''}a \cdot u''_{X''}\tilde{T}''$. Then we get the following commutative quadrangle.

$$\begin{array}{ccc} U_{X''}GT'' & \xrightarrow{U_{X''}a} & U_{X''}G\tilde{T}'' \\ {\scriptstyle u''_{X''}T''}\downarrow\wr & & \wr\downarrow{\scriptstyle u''_{X''}\tilde{T}''} \\ X''T'' & \xrightarrow{X''a''} & X''\tilde{T}'' \end{array}$$

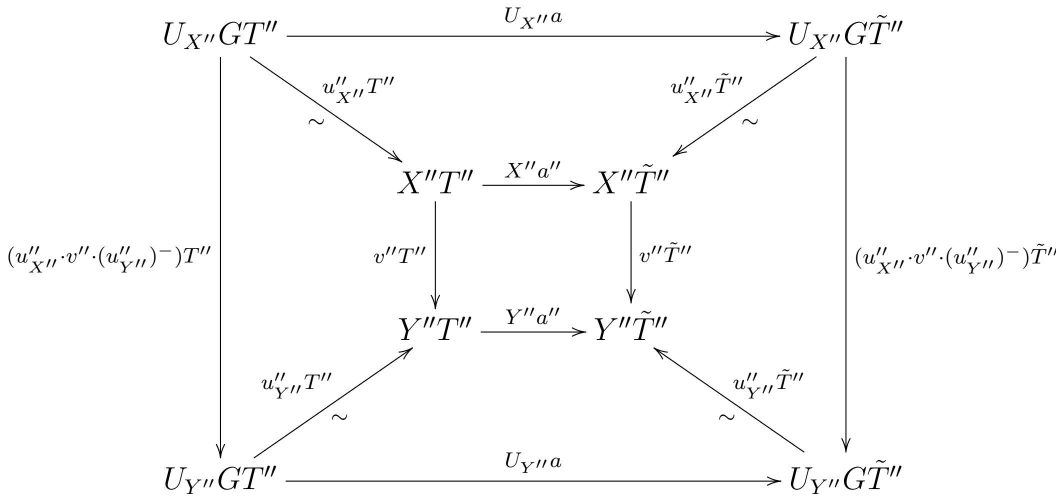
We *claim* that $a'' := (X''a'')_{X''\in\mathrm{Ob}(\mathcal{A}'')}$ is a transformation from T'' to $\tilde{T}''$ satisfying $Ga'' \stackrel{!}{=} a$.

We show that a'' is a transformation.

Suppose given a morphism $X'' \xrightarrow{v''} Y''$ in $\mathcal{A}''$. We have to show that the following quadrangle commutes.

$$\begin{array}{ccc} X''T'' & \xrightarrow{X''a''} & X''\tilde{T}'' \\ {\scriptstyle v''T''}\downarrow & & \downarrow{\scriptstyle v''\tilde{T}''} \\ Y''T'' & \xrightarrow{Y''a''} & Y''\tilde{T}'' \end{array}$$

To this end, we form the following diagram.



The left and the right quadrangle commute by construction.

The upper and the lower quadrangle commute by definition of a''.

Since G is full by (P 2), we may choose $v : U_{X''} \to U_{Y''}$ in $\mathcal{A}$ with $vG = u''_{X''} \cdot v'' \cdot (u''_{Y''})^-$.

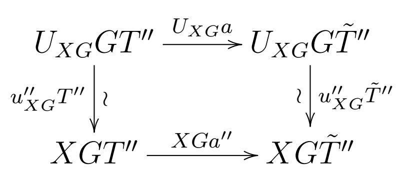
Since $a : GT'' \to G\tilde{T}''$ is a transformation, we conclude that the outer quadrangle $(U_{X''}GT'', U_{X''}G\tilde{T}'', U_{Y''}GT'', U_{Y''}G\tilde{T}'')$ commutes.

Hence the inner quadrangle $(X''T'', X''\tilde{T}'', Y''T'', Y''\tilde{T}'')$ commutes:

$$\begin{array}{rcl}
X''a'' \cdot v''\tilde{T}'' & = & (u''_{X''}T'')^- \cdot U_{X''}a \cdot u''_{X''}\tilde{T}'' \cdot v''\tilde{T}'' \\
& = & (u''_{X''}T'')^- \cdot U_{X''}a \cdot (u''_{X''} \cdot v'' \cdot (u''_{Y''})^-)\tilde{T}'' \cdot u''_{Y''}\tilde{T}'' \\
& = & (u''_{X''}T'')^- \cdot (u''_{X''} \cdot v'' \cdot (u''_{Y''})^-)T'' \cdot U_{Y''}a \cdot u''_{Y''}\tilde{T}'' \\
& = & (u''_{X''}T'')^- \cdot (u''_{X''} \cdot v'' \cdot (u''_{Y''})^-)T'' \cdot u''_{Y''}T'' \cdot Y''a'' \\
& = & (u''_{X''}T'')^- \cdot u''_{X''}T'' \cdot v''T'' \cdot Y''a'' \\
& = & v''T'' \cdot Y''a'' \,.
\end{array}$$

We show that $Ga'' \stackrel{!}{=} a$. Suppose given $X \in \mathrm{Ob}(\mathcal{A})$. We have to show $XGa'' \stackrel{!}{=} Xa$.

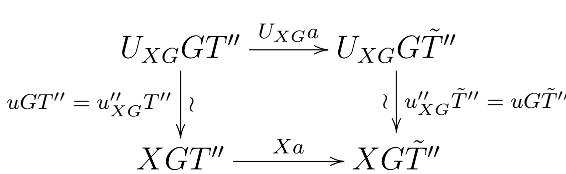
By construction of a'', we have the following commutative quadrangle.



Since G is full by (P 2), we may choose a morphism $U_{XG} \xrightarrow{u} X$ in $\mathcal{A}$ with

$$(U_{XG}G \xrightarrow{uG} XG) \;=\; (U_{XG}G \xrightarrow{u''_{XG}} XG)\,.$$

Since $a : GT'' \to G\tilde{T}''$ is a transformation, the following quadrangle commutes.



This proves the required equation:

$$\begin{array}{rcl} XGa'' & = & (u''_{XG}T'')^- \cdot U_{XG}a \cdot u''_{XG}\tilde{T}'' \\ & = & (uGT'')^- \cdot U_{XG}a \cdot uG\tilde{T}'' \\ & = & Xa\,. \end{array}$$

This proves the *claim*.

Ad G^* dense. Suppose given $T \in \mathrm{Ob}(\mathrm{Kern}(F, \mathcal{T}))$. I.e. T is an additive functor $\mathcal{A} \xrightarrow{T} \mathcal{T}$ such that FT is a zero functor.

First, suppose given a morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ such that $fG = 0$. By (P 4), we may choose a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle b}\searrow \quad \nearrow{\scriptstyle \tilde{b}} & \\ & Z'F & \end{array}$$

in $\mathcal{A}$, for some $Z' \in \mathrm{Ob}(\mathcal{A}')$. Application of T yields the commutative triangle

$$\begin{array}{ccc} XT & \xrightarrow{fT} & YT \\ & {\scriptstyle bT}\searrow \quad \nearrow{\scriptstyle \tilde{b}T} & \\ & Z'FT & \end{array}$$

in $\mathcal{T}$. Since FT maps every object to a zero object, this shows that $fT = 0$.

We have to show that there exists $T'' \in \mathrm{Ob}(\mathcal{A}'', \mathcal{T})$ such that $T''G^* \overset{!}{\simeq} T$. I.e. we have to show that there exists an additive functor $\mathcal{A}'' \xrightarrow{T''} \mathcal{T}$ such that $GT'' \overset{!}{\simeq} T$.

Suppose given $X'' \xrightarrow{f''} Y''$ in $\mathcal{A}''$. We set out to construct the image of this morphism under T'' in $\mathcal{T}$.

We have the following commutative quadrangle.

$$\begin{array}{ccc} U_{X''}G & \xrightarrow{u''_{X''} \cdot f'' \cdot (u''_{Y''})^-} & U_{Y''}G \\ {\scriptstyle u''_{X''}}\downarrow\wr & & \wr\downarrow{\scriptstyle u''_{Y''}} \\ X'' & \xrightarrow{f''} & Y'' \end{array}$$

Since G is full by (P 2), we may choose $U_{X''} \xrightarrow{v_{f''}} U_{Y''}$ in $\mathcal{A}$ such that

$$v_{f''}G \;=\; u''_{X''} \cdot f'' \cdot (u''_{Y''})^-\,.$$

We let

$$(X'' \xrightarrow{f''} Y'')T'' \;:=\; (U_{X''}T \xrightarrow{v_{f''}T} U_{Y''}T)\,.$$

We have to show the following assertions.

(i) T'' is a functor from $\mathcal{A}''$ to $\mathcal{T}$.

(ii) We have $GT'' \simeq T$.

(iii) T'' is additive.

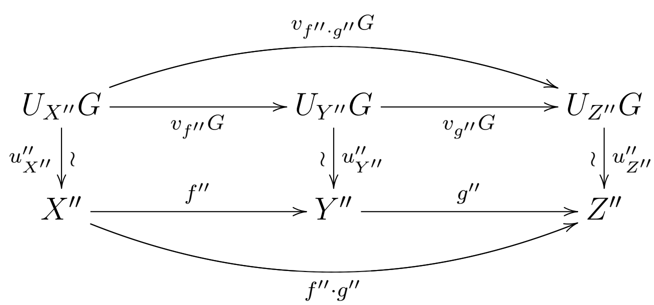
Ad (i). Suppose given $X'' \in \mathrm{Ob}(\mathcal{A}'')$. We have to show that $\mathrm{id}_{X''}T'' \overset{!}{=} \mathrm{id}_{X''T''}$.

We have $\mathrm{id}_{X''}T'' = v_{\mathrm{id}_{X''}}T$ by construction. Moreover, we have $v_{\mathrm{id}_{X''}}G = u''_{X''} \cdot \mathrm{id}_{X''} \cdot (u''_{X''})^- = \mathrm{id}_{U_{X''}G} = \mathrm{id}_{U_{X''}}G$. So $(v_{\mathrm{id}_{X''}} - \mathrm{id}_{U_{X''}})G = 0$. Hence, by what was remarked above, we obtain $(v_{\mathrm{id}_{X''}} - \mathrm{id}_{U_{X''}})T = 0$. So

$$\mathrm{id}_{X''}T'' \;=\; v_{\mathrm{id}_{X''}}T \;=\; \mathrm{id}_{U_{X''}}T \;=\; \mathrm{id}_{U_{X''}T} \;=\; \mathrm{id}_{X''T''}\ .$$

Suppose given $X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z''$ in $\mathcal{A}''$. We have to show that $f''T'' \cdot g''T'' \overset{!}{=} (f'' \cdot g'')T''$.

We have the following diagram.



Since its lower triangle and its quadrangles commute, we conclude that its upper triangle commutes. So we obtain

$$v_{f''\cdot g''}G \;=\; v_{f''}G \cdot v_{g''}G \;=\; (v_{f''} \cdot v_{g''})G\ ,$$

i.e. $(v_{f''\cdot g''} - v_{f''} \cdot v_{g''})G = 0$. Hence, by what was remarked above, $(v_{f''\cdot g''} - v_{f''} \cdot v_{g''})T = 0$. So

$$f''T'' \cdot g''T'' \;=\; v_{f''}T \cdot v_{g''}T \;=\; (v_{f''} \cdot v_{g''})T \;=\; v_{f''\cdot g''}T \;=\; (f'' \cdot g'')T''\ .$$

Ad (ii). Suppose given $X \in \mathrm{Ob}(\mathcal{A})$.

Consider the isomorphism $u''_{XG} : U_{XG}G \xrightarrow{\sim} XG$ in $\mathcal{A}''$. Since G is full by (P 2), we may choose a morphism $w_X : U_{XG} \to X$ such that $w_X G = u''_{XG}$. Again since G is full, we may choose a morphism $\tilde{w}_X : X \to U_{XG}$ such that $\tilde{w}_X G = {u''_{XG}}^-$.

Recall that $XGT'' = U_{XG}T$.

Let

$$X\varphi \;:=\; w_X T \;:\; U_{XG}T \;=\; XGT'' \;\to\; XT\ .$$

Let

$$X\tilde{\varphi} \;:=\; \tilde{w}_X T \;:\; XT \;\to\; U_{XG}T \;=\; XGT''\ .$$

Then

$$\begin{array}{rclcl}(w_X \cdot \tilde{w}_X - \mathrm{id}_{U_{XG}})G & = & u''_{XG} \cdot u''^{-}_{XG} - \mathrm{id}_{U_{XG}G} & = & 0 \\ (\tilde{w}_X \cdot w_X - \mathrm{id}_X)G & = & u''^{-}_{XG} \cdot u''_{XG} - \mathrm{id}_{XG} & = & 0\,.\end{array}$$

Hence, as remarked above, we also have

$$\begin{array}{rclcl}0 & = & (w_X \cdot \tilde{w}_X - \mathrm{id}_{U_{XG}})T & = & X\varphi \cdot X\tilde{\varphi} - \mathrm{id}_{XGT''} \\ 0 & = & (\tilde{w}_X \cdot w_X - \mathrm{id}_X)T & = & X\tilde{\varphi} \cdot X\varphi - \mathrm{id}_{XT}\,.\end{array}$$

So $X\varphi$ is an isomorphism.

It remains to show that $\varphi := (X\varphi)_{X\in\mathrm{Ob}(\mathcal{A})}$ is a transformation from GT'' to T. Suppose given a morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$. We have to show that $X\varphi \cdot fT \overset{!}{=} fGT'' \cdot Y\varphi$. I.e. we have to show that

$$(w_X \cdot f - v_{fG} \cdot w_Y)T \overset{!}{=} 0\,.$$

As remarked above, it suffices to show that

$$(w_X \cdot f - v_{fG} \cdot w_Y)G \overset{!}{=} 0\,.$$

In fact, we obtain

$$(w_X \cdot f)G = w_X G \cdot fG = u''_{XG} \cdot fG$$

and

$$(v_{fG} \cdot w_Y)G = v_{fG}G \cdot w_Y G = u''_{XG} \cdot fG \cdot (u''_{YG})^{-} \cdot u''_{YG} = u''_{XG} \cdot fG\,,$$

which is the same.

Ad (iii). We want to show that T'' is additive. Suppose given f'', $g'' : X'' \to Y''$ in $\mathcal{A}''$. By Lemma 50, it suffices to show that

$$(f'' + g'')T'' \overset{!}{=} f''T'' + g''T''\,.$$

I.e. we have to show that

$$v_{f''+g''}T \overset{!}{=} v_{f''}T + v_{g''}T\,.$$

Since T is additive, this amounts to showing that

$$(v_{f''+g''} - (v_{f''} + v_{g''}))T \overset{!}{=} 0\,.$$

As remarked above, it suffices to show that

$$(v_{f''+g''} - (v_{f''} + v_{g''}))G \overset{!}{=} 0\,.$$

Since G is additive, this amounts to showing that

$$v_{f''+g''}G \overset{!}{=} v_{f''}G + v_{g''}G\,.$$

In fact,

$$v_{f''+g''}G = u''_{X''} \cdot (f'' + g'') \cdot (u''_{Y''})^{-} = u''_{X''} \cdot f'' \cdot (u''_{Y''})^{-} + u''_{X''} \cdot g'' \cdot (u''_{Y''})^{-} = v_{f''}G + v_{g''}G\,.$$

□

Lemma 91. *Suppose given a pure short exact sequence*

$$\mathcal{A}' \overset{F}{\longrightarrow\!\!\!\!\!\bullet\!\!\!\!\rightarrow} \mathcal{A} \overset{G}{\longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow} \mathcal{A}''$$

in AddCat.

Suppose given an additive category $\mathcal{T}$.

We have induced functors

$$(\mathcal{T},\mathcal{A}') \xrightarrow{(\mathcal{T},F)} (\mathcal{T},\mathcal{A}) \xrightarrow{(\mathcal{T},G)} (\mathcal{T},\mathcal{A}'')$$

We have a full additive subcategory $\mathrm{Kern}(\mathcal{T},G) \subseteq (\mathcal{T},\mathcal{A})$.

We obtain a functor

$$(\mathcal{T},\mathcal{A}') \xrightarrow{(\mathcal{T},F)|^{\mathrm{Kern}(\mathcal{T},G)}} \mathrm{Kern}(\mathcal{T},G)\ .$$

This functor is an equivalence.

Proof. First, we have to show that $(\mathcal{T},F)$ maps each object T' of $(\mathcal{T},\mathcal{A}')$ to an object of $\mathrm{Kern}(\mathcal{T},G)$. In fact, T' is mapped to $T'F$, and since $(T'F)(\mathcal{T},G) = T'FG$ is a zero functor by Remarks 85.(2) and 65, the functor $T'F$ is an object of $\mathrm{Kern}(\mathcal{T},G)$.

So the additive functor $F_* := (\mathcal{T},F)|^{\mathrm{Kern}(\mathcal{T},G)} : (\mathcal{T},\mathcal{A}') \to \mathrm{Kern}(\mathcal{T},G)$ exists; cf. Remark 58.(1). We have to show that it is faithful, full and dense.

Ad F_* faithful. Suppose given T', $\tilde{T}' \in \mathrm{Ob}(\mathcal{T},\mathcal{A}')$ and a morphism $T' \xrightarrow{a'} \tilde{T}'$. So T' and $\tilde{T}'$ are additive functors from $\mathcal{T}$ to $\mathcal{A}'$, and a' is a transformation from T' to $\tilde{T}'$.

Suppose that $a'F_* = 0$. We have to show that $a' \overset{!}{=} 0$. I.e. given $U \in \mathrm{Ob}(\mathcal{T})$, we have to show that $Ua' \overset{!}{=} 0$.

We have $0 = a'F_* = a'F$. Hence $Ua'F = 0 : UT'F \to U\tilde{T}'F$. By (P 1), F is faithful. Hence $Ua' = 0$.

Ad F_* full. Suppose given T', $\tilde{T}' \in \mathrm{Ob}(\mathcal{T},\mathcal{A}')$ and a morphism $T'F_* \xrightarrow{a} \tilde{T}'F_*$. So T' and $\tilde{T}'$ are additive functors from $\mathcal{T}$ to $\mathcal{A}'$, and a is a transformation from $T'F$ to $\tilde{T}'F$.

We need to find a transformation $T' \xrightarrow{a'} \tilde{T}'$ with $a'F_* = a'F \overset{!}{=} a$.

Suppose given $U \in \mathrm{Ob}(\mathcal{T})$. We have $Ua : UT'F \to U\tilde{T}'F$. By (P 1), F is full and faithful. Hence there exists a unique morphism $Ua' : UT' \to U\tilde{T}'$ such that $Ua'F = Ua$.

It suffices to show that $(Ua')_{U\in\mathrm{Ob}(\mathcal{T})}$ is a transformation from T' to $\tilde{T}'$.

Suppose given a morphism $U \xrightarrow{g} V$ in $\mathcal{T}$. Since a is a transformation, we have the following commutative quadrangle.

$$\begin{array}{ccc} UT'F & \xrightarrow{Ua\,=\,Ua'F} & U\tilde{T}'F \\ \downarrow{\scriptstyle gT'F} & & \downarrow{\scriptstyle g\tilde{T}'F} \\ VT'F & \xrightarrow{Va\,=\,Va'F} & V\tilde{T}'F \end{array}$$

Since F is faithful by (P 1), we obtain the following commutative quadrangle.

$$\begin{array}{ccc} UT' & \xrightarrow{Ua'} & U\tilde{T}' \\ {\scriptstyle gT'}\downarrow & & \downarrow{\scriptstyle g\tilde{T}'} \\ VT' & \xrightarrow{Va'} & V\tilde{T}' \end{array}$$

Ad F_* *dense.* Suppose given $T \in \mathrm{Ob}(\mathrm{Kern}(\mathcal{T}, G))$. I.e. T is an additive functor from $\mathcal{T}$ to $\mathcal{A}$ such that TG is a zero functor.

We have to construct an additive functor $T' : \mathcal{T} \to \mathcal{A}'$ such that $T'F_* = T'F \overset{!}{\simeq} T$.

Given $U \in \mathrm{Ob}(\mathcal{T})$, the object $(UT)G$ is zero. So by (P 3) we may choose $UT' \in \mathrm{Ob}(\mathcal{A}')$ and an isomorphism $UT'F \xrightarrow[\sim]{Ub} UT$; cf. Definition 82.

Suppose given $U \xrightarrow{g} V$ in $\mathcal{T}$. We have the composite $Ub \cdot gT \cdot Vb^- : UT'F \to VT'F$. Since F is full and faithful by (P 1), there exists a unique morphism $gT' : UT' \to VT'$ such that $gT'F = Ub \cdot gT \cdot Vb^-$. So we have the following commutative quadrangle.

$$\begin{array}{ccc} UT'F & \xrightarrow{gT'F} & VT'F \\ {\scriptstyle Ub}\downarrow\wr & & \wr\downarrow{\scriptstyle Vb} \\ UT & \xrightarrow{gT} & VT \end{array}$$

We have to show the following assertions.

(i) T' is a functor.

(ii) We have $T'F \simeq T$.

(iii) T' is additive.

Ad (i). Suppose given $U \in \mathrm{Ob}(\mathcal{T})$. We have the following commutative quadrangle.

$$\begin{array}{ccc} UT'F & \xrightarrow{\mathrm{id}_U T'F} & UT'F \\ {\scriptstyle Ub}\downarrow\wr & & \wr\downarrow{\scriptstyle Ub} \\ UT & \xrightarrow{\mathrm{id}_U T = \mathrm{id}_{UT}} & UT \end{array}$$

Hence $\mathrm{id}_U T'F = Ub \cdot \mathrm{id}_{UT} \cdot Ub^- = \mathrm{id}_{UT'F} = \mathrm{id}_{UT'}F$. Since, by (P 1), F is faithful, we obtain $\mathrm{id}_U T' = \mathrm{id}_{UT'}$.

Suppose given $U \xrightarrow{g} V \xrightarrow{h} W$ in $\mathcal{T}$. We have the following commutative diagram.

$$\begin{array}{ccccc} UT'F & \xrightarrow{gT'F} & VT'F & \xrightarrow{hT'F} & WT'F \\ {\scriptstyle Ub}\downarrow\wr & & \wr\downarrow{\scriptstyle Vb} & & \wr\downarrow{\scriptstyle Wb} \\ UT & \xrightarrow{gT} & VT & \xrightarrow{hT} & WT \end{array}$$

Hence $(gT' \cdot hT')F = gT'F \cdot hT'F = Ub \cdot gT \cdot Vb^- \cdot Vb \cdot hT \cdot Wb^- = Ub \cdot gT \cdot hT \cdot Wb^- = Ub \cdot (g \cdot h)T \cdot Wb^- = (g \cdot h)T'F$. Since, by (P 1), F is faithful, we obtain $gT' \cdot hT' = (g \cdot h)T'$.

Ad (ii). By the commutative quadrangle above, $b := (Ub)_{U\in\mathrm{Ob}(\mathcal{T})}$ is an isotransformation from $T'F$ to T. Hence $T'F \simeq T$.

Ad (iii). We want to show that T' is additive. Suppose given $g,\, \tilde{g} : U \to V$ in $\mathcal{T}$. By Lemma 50, it suffices to show that

$$(g+\tilde{g})T' \overset{!}{=} gT' + \tilde{g}T' \, .$$

Since F is faithful by (P 1), it suffices to show that

$$(g+\tilde{g})T'F \overset{!}{=} (gT' + \tilde{g}T')F \, .$$

In fact,

$$\begin{aligned}(g+\tilde{g})T'F &= Ub\cdot(g+\tilde{g})T\cdot Vb^- \\ &= Ub\cdot(gT+\tilde{g}T)\cdot Vb^- \\ &= Ub\cdot gT\cdot Vb^- + Ub\cdot\tilde{g}T\cdot Vb^- \\ &= gT'F + \tilde{g}T'F \\ &= (gT'+\tilde{g}T')F \, .\end{aligned}$$

□

2.11.4 Pure monofunctors

2.11.4.1 Characterisation of pure monofunctors

Suppose given $\mathcal{A}' \xrightarrow{F} \mathcal{A}$ in AddCat.

Remark 92. *The functor F is a pure monofunctor if and only if the following properties* (PM 1, 2) *hold.*

(PM 1) *The functor F is full and faithful.*

(PM 2) *The full additive subcategory $\mathcal{A}'F$ of $\mathcal{A}$ is closed under retracts.*

Proof. Suppose that F satisfies properties (PM 1, 2). By (PM 2), we may form the sequence

$$\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{\mathrm{R}} \mathcal{A}/\mathcal{A}'F \, ;$$

cf. Definition 76, Lemma 77. We have to show that it is pure short exact.

Now (P 1) follows by (PM 1). Moreover, (P 2) follows by Definition 76; cf. Lemma 77.(2). Finally, (P 3, 4) follow by Lemma 77.(1); cf. Definition 82.

Conversely, suppose that F is a pure monofunctor. So we have a pure short exact sequence

$$\mathcal{A}' \overset{F}{\bullet\!\!\rightarrow} \mathcal{A} \overset{G}{+\!\!\rightarrow} \mathcal{A}'' \, .$$

We have to show properties (PM 1, 2) for F.

Property (PM 1) follows by (P 1).

Property (PM 2) holds by Remark 85.(1); cf. Definition 82. □

Lemma 93. *Suppose given a diagram in* AddCat *as follows.*

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{F'} & \mathcal{A} \\ {\scriptstyle U'}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{B} \end{array}$$

Suppose that U' and U are equivalences.

Suppose that $U'G' \simeq F'U$.

Then $\mathcal{A}' \xrightarrow{F'} \mathcal{A}$ is a pure monofunctor if and only if $\mathcal{B}' \xrightarrow{G'} \mathcal{B}$ is a pure monofunctor.

In particular, this applies if $U' = \mathrm{id}_{\mathcal{A}'}$ and $U = \mathrm{id}_{\mathcal{A}}$.

Proof.

Suppose that $\mathcal{A}' \overset{F'}{-\!\!\bullet\!\!\!\rightarrow} \mathcal{A}$. We have to show that $\mathcal{B}' \xrightarrow{G'} \mathcal{B}$ is a pure monofunctor. We may verify conditions (PM 1, 2) from Remark 92.

Ad (PM 1). Since F' is full and faithful, we conclude by Lemma 88.(1, 2) that G' is full and faithful.

Ad (PM 2). Since $\mathcal{A}'F' \subseteq \mathcal{A}$ is closed under retracts, we conclude by Lemma 88.(4) that $\mathcal{B}'G' \subseteq \mathcal{B}$ is closed under retracts.

This shows that $\mathcal{B}' \overset{G'}{-\!\!\bullet\!\!\!\rightarrow} \mathcal{B}$.

Conversely, suppose that $\mathcal{B}' \overset{G'}{-\!\!\bullet\!\!\!\rightarrow} \mathcal{B}$. We have to show that $\mathcal{A}' \xrightarrow{F'} \mathcal{A}$ is a pure monofunctor.

We choose $V' : \mathcal{B}' \to \mathcal{A}'$ such that $U'V' \simeq \mathrm{id}_{\mathcal{A}'}$ and $V'U' \simeq \mathrm{id}_{\mathcal{B}'}$.

We choose $V : \mathcal{B} \to \mathcal{A}$ such that $UV \simeq \mathrm{id}_{\mathcal{A}}$ and $VU \simeq \mathrm{id}_{\mathcal{B}}$.

Then $V'F'U \simeq V'U'G' \simeq G' \simeq G'VU$, whence $V'F' \simeq G'V$ since U is full and faithful.

Now the argument above applies, so that we may conclude that $\mathcal{A}' \overset{F'}{-\!\!\bullet\!\!\!\rightarrow} \mathcal{A}$.

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{F'} & \mathcal{A} \\ {\scriptstyle V'}\uparrow & & \uparrow{\scriptstyle V} \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{B} \end{array}$$

□

2.11.4.2 Properties of pure monofunctors

Lemma 94. *Suppose given composable pure monofunctors $\mathcal{A} \overset{F}{-\!\!\bullet\!\!\!\rightarrow} \mathcal{B}$ and $\mathcal{B} \overset{G}{-\!\!\bullet\!\!\!\rightarrow} \mathcal{C}$ in* AddCat.

Then we have the pure monofunctor $\mathcal{A} \overset{FG}{-\!\!\bullet\!\!\!\rightarrow} \mathcal{C}$.

Proof. We have to show (PM 1, 2) for FG; cf. Remark 92

Ad (PM 1). Since F and G are full and faithful, so is FG.

Ad (PM 2). We have to show that the full image $\mathcal{A}(FG) \subseteq \mathcal{C}$ is closed under retracts.

Suppose given $Z \in \mathrm{Ob}(\mathcal{A}(FG))$. Suppose given a retract $W \in \mathrm{Ob}(\mathcal{C})$ of Z. We have to show that $W \overset{!}{\in} \mathrm{Ob}(\mathcal{A}(FG))$.

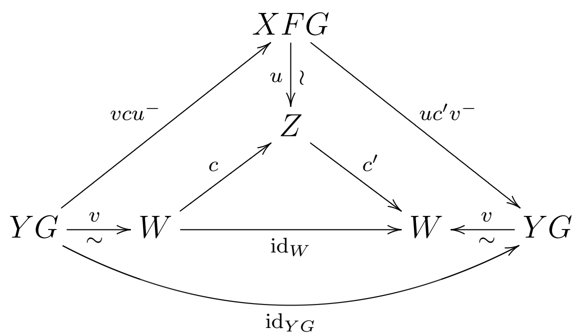
Since $Z \in \mathrm{Ob}(\mathcal{A}(FG))$, we may choose an object $X \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $u : XFG \overset{\sim}{\to} Z$ in $\mathcal{C}$.

We may choose morphisms $W \overset{c}{\to} Z \overset{c'}{\to} W$ in $\mathcal{C}$ such that $c \cdot c' = \mathrm{id}_W$.

Since $Z \in \mathrm{Ob}(\mathcal{B}G)$, we may conclude that $W \in \mathrm{Ob}(\mathcal{B}G)$ since G satisfies (PM 2).

So we may choose an object $Y \in \mathrm{Ob}(\mathcal{B})$ and an isomorphism $v : YG \overset{\sim}{\to} W$ in $\mathcal{C}$.

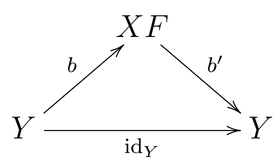
Altogether, we have the following commutative diagram.



In fact, we have $(vcu^-) \cdot (uc'v^-) = vcc'v^- = vv^- = \mathrm{id}_{YG}$.

Since G is full and faithful by (PM 1), there exist unique morphisms $b : Y \to XF$ and $b' : XF \to Y$ in $\mathcal{B}$ such that $bG = vcu^-$ and $b'G = uc'v^-$.

Then $(b \cdot b')G = bG \cdot b'G = \mathrm{id}_{YG} = \mathrm{id}_Y G$. Since G is faithful by (PM 1), we obtain $b \cdot b' = \mathrm{id}_Y$.



Hence Y is a retract of XF. We may conclude that $Y \in \mathrm{Ob}(\mathcal{A}F)$ since F satisfies (PM 2). So we may choose an object $\tilde{X} \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $r : \tilde{X}F \overset{\sim}{\to} Y$ in $\mathcal{B}$.

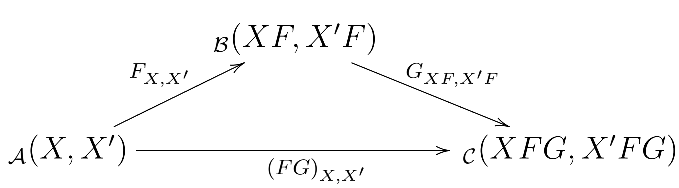
Altogether, we have the isomorphism $rG \cdot v : \tilde{X}FG \overset{\sim}{\to} W$. Therefore, $W \in \mathrm{Ob}(\mathcal{A}(FG))$. □

Lemma 95. *Suppose given a diagram* $\mathcal{A} \overset{F}{\to} \mathcal{B} \overset{G}{\to} \mathcal{C}$ *in* AddCat.

Suppose that FG *and* G *are pure monofunctors.*

Then F *is a pure monofunctor.*

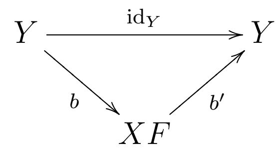
Proof. Ad (PM 1). Suppose given $X, X' \in \mathrm{Ob}(\mathcal{A})$. We have the following commutative triangle of maps.



Since G and FG are pure monofunctors, they are full and faithful. Hence the maps $G_{XF,X'F}$ and $(FG)_{X,X'}$ are bijective. Hence, the map $F_{X,X'}$ is bijective.

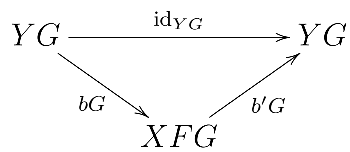
Therefore, F is full and faithful.

Ad (PM 2). Suppose given a commutative triangle in $\mathcal{B}$ as follows, where $X \in \mathrm{Ob}(\mathcal{A})$.



We have to show that $Y \overset{!}{\in} \mathrm{Ob}(\mathcal{A}F)$.

Application of G yields the following commutative triangle in $\mathcal{C}$.



Since FG is a pure monofunctor, $\mathcal{A}FG$ is closed under retracts in $\mathcal{C}$. We conclude that $YG \in \mathrm{Ob}(\mathcal{A}FG)$. So we may choose $\tilde{X} \in \mathrm{Ob}(\mathcal{A})$ such that $YG \simeq \tilde{X}FG$.

Since G is a pure monofunctor, it is full and faithful. We conclude that $Y \simeq \tilde{X}F$. Therefore, $Y \in \mathrm{Ob}(\mathcal{A}F)$. □

2.11.5 Pure epifunctors

2.11.5.1 Characterisation of pure epifunctors

Remark 96. *Suppose given* $\mathcal{A} \xrightarrow{G} \mathcal{A}''$ *in* AddCat.

Then G is a pure epifunctor if and only if the following properties (PE 1, 2) *hold.*

(PE 1) *The functor G is full and dense.*

(PE 2) *Suppose given a morphism $X \xrightarrow{u} \tilde{X}$ in $\mathcal{A}$ such that $uG = 0$. Then there exist $Z \in \mathrm{Ob}(\mathrm{Kern}(G))$ and morphisms $X \xrightarrow{a} Z \xrightarrow{\tilde{a}} \tilde{X}$ in $\mathcal{A}$ such that $a \cdot \tilde{a} = u$.*

Proof. Suppose that G is a pure epifunctor. Then (PE 1) holds by Definition 84, property (P 2). Moreover, (PE 2) holds by Definition 84, properties (P 4, 3).

Now suppose that G satisfies (PE 1, 2). We have to show that G is a pure epifunctor. Consider the sequence

$$\mathrm{Kern}(G) \xrightarrow{\mathrm{I}} \mathcal{A} \xrightarrow{G} \mathcal{A}'' ;$$

cf. Definition 79.

It suffices to show that this sequence is a pure short exact sequence in AddCat. We have to show properties (P 1–4) from Definition 84.

Ad (P 1). The inclusion functor I of the full subcategory $\mathrm{Kern}(G)$ is full and faithful.

Ad (P 2). The functor G is full and dense by (PE 1).

Ad (P 3). We have $\mathrm{Kern}(G)\mathrm{I} = \mathrm{Kern}(G)$, using that $\mathrm{Kern}(G)$ is closed under retracts and thus under isomorphy; cf. Definition 82, Remark 80.

Ad (P 4). Suppose given a morphism $X \xrightarrow{u} Y$ in $\mathcal{A}$ such that $uG = 0$. Then there exists $Z \in \mathrm{Ob}(\mathrm{Kern}(G))$ and morphisms $X \xrightarrow{a} Z \xrightarrow{b} Y$ such that $a \cdot b = u$ by (PE 2). Moreover, $Z = Z\mathrm{I}$. So (P 4) holds. □

Lemma 97. *Suppose given a diagram in* AddCat *as follows.*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F''} & \mathcal{A}'' \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U''} \\ \mathcal{B} & \xrightarrow{G''} & \mathcal{B}'' \end{array}$$

Suppose that U and U'' are equivalences.

Suppose that $UG'' \simeq F''U''$.

Then $\mathcal{A} \xrightarrow{F''} \mathcal{A}''$ is a pure epifunctor if and only if $\mathcal{B} \xrightarrow{G''} \mathcal{B}''$ is a pure epifunctor.

In particular, this applies if $U = \mathrm{id}_{\mathcal{A}}$ and $U'' = \mathrm{id}_{\mathcal{A}''}$.

Proof. We choose an isotransformation $m'' : F''U'' \overset{\sim}{\to} UG''$.

Suppose that $\mathcal{A} \overset{F''}{\twoheadrightarrow} \mathcal{A}''$. We have to show that $\mathcal{B} \xrightarrow{G''} \mathcal{B}''$ is a pure epifunctor. We may verify conditions (PE 1, 2) from Remark 96.

Ad (PE 1). Since F'' is full and dense, we conclude by Lemma 88.(1, 3) that G'' is full and dense.

Ad (PE 2). Suppose given a morphism $Y \xrightarrow{b} \tilde{Y}$ in $\mathcal{B}$ such that $bG'' = 0$. We have to show that there exists $Y' \in \mathrm{Ob}(\mathrm{Kern}(G''))$ and morphisms $Y \xrightarrow{b_1} Y' \xrightarrow{b_2} \tilde{Y}$ such that $b_1 \cdot b_2 = b$.

Using that U is dense, we choose $X, \tilde{X} \in \mathrm{Ob}(\mathcal{A})$ and isomorphisms $XU \overset{s}{\underset{\sim}{\to}} Y$ and $\tilde{X}U \overset{\tilde{s}}{\underset{\sim}{\to}} \tilde{Y}$.

Then $s \cdot b \cdot \tilde{s}^- : XU \to \tilde{X}U$. Since U is full and faithful, there exists a unique $X \xrightarrow{a} \tilde{X}$ with $aU = s \cdot b \cdot \tilde{s}^-$.

We have

$$aUG'' \;=\; (s \cdot b \cdot \tilde{s}^-)G'' \;=\; sG'' \cdot bG'' \cdot \tilde{s}^-G'' \;=\; 0\,.$$

We have a commutative quadrangle by naturality of m'' as follows.

$$\begin{array}{ccc} XF''U'' & \overset{Xm''}{\underset{\sim}{\longrightarrow}} & XUG'' \\ {\scriptstyle aF''U''}\downarrow & & \downarrow{\scriptstyle aUG''} \\ \tilde{X}F''U'' & \overset{\tilde{X}m''}{\underset{\sim}{\longrightarrow}} & \tilde{X}UG'' \end{array}$$

We conclude that $aF''U'' = 0$.

Since U'' is full and faithful, Remarks 52 and 51 give the following isomorphism of abelian groups.

$$_{\mathcal{A}''}(XF'', \tilde{X}F'') \xrightarrow[\sim]{U''_{XF'',\tilde{X}F''}} {}_{\mathcal{B}''}(XF''U'', \tilde{X}F''U'')$$

Hence $aF'' = 0$.

By (PE 2) for $\mathcal{A} \overset{F''}{\twoheadrightarrow} \mathcal{A}''$, we may choose $X' \in \mathrm{Ob}(\mathrm{Kern}(F''))$ and a commutative triangle in $\mathcal{A}$ as follows.

$$\begin{array}{ccc} X & \xrightarrow{\quad a \quad} & \tilde{X} \\ & {\scriptstyle a_1}\searrow \qquad \nearrow{\scriptstyle a_2} & \\ & X' & \end{array}$$

Application of U yields the following diagram in $\mathcal{B}$.

$$\begin{array}{ccc} Y & \xrightarrow{\qquad b \qquad} & \tilde{Y} \\ {\scriptstyle s}\uparrow\wr & & \wr\uparrow{\scriptstyle \tilde{s}} \\ XU & \xrightarrow{\qquad aU \qquad} & \tilde{X}U \\ & {\scriptstyle a_1U}\searrow \qquad \nearrow{\scriptstyle a_2U} & \\ & X'U & \end{array}$$

Hence

$$\begin{aligned} b &= s^- \cdot (s \cdot b \cdot \tilde{s}^-) \cdot \tilde{s} \\ &= s^- \cdot aU \cdot \tilde{s} \\ &= (s^- \cdot a_1U) \cdot (a_2U \cdot \tilde{s})\,. \end{aligned}$$

So letting $Y' := X'U \in \mathrm{Ob}(\mathcal{B})$, $b_1 := s^- \cdot a_1U$ and $b_2 := a_2U \cdot \tilde{s}$, we get the factorisation $b = b_1 \cdot b_2$ over Y'. Moreover, $Y'G'' = X'UG'' \simeq X'F''U''$, which is zero since $X'F''$ is.

Hence $\mathcal{B} \overset{G''}{\twoheadrightarrow} \mathcal{B}''$.

Conversely, suppose that $\mathcal{B} \overset{G''}{\twoheadrightarrow} \mathcal{B}''$. We have to show that $\mathcal{A} \xrightarrow{F''} \mathcal{A}''$ is a pure epifunctor.

We choose $V : \mathcal{B} \to \mathcal{A}$ such that $UV \simeq \mathrm{id}_{\mathcal{A}}$ and $VU \simeq \mathrm{id}_{\mathcal{B}}$.

We choose $V'' : \mathcal{B}'' \to \mathcal{A}''$ such that $U''V'' \simeq \mathrm{id}_{\mathcal{A}''}$ and $V''U'' \simeq \mathrm{id}_{\mathcal{B}''}$.

Then $VF''U'' \simeq VUG'' \simeq G'' \simeq G''V''U''$, whence $VF'' \simeq G''V''$ since U'' is full and faithful.

Now the argument above applies, so that we may conclude that $\mathcal{A} \overset{F''}{\twoheadrightarrow} \mathcal{A}''$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F''} & \mathcal{A}'' \\ {\scriptstyle V}\uparrow & & \uparrow{\scriptstyle V''} \\ \mathcal{B} & \xrightarrow{G''} & \mathcal{B}'' \end{array}$$

□

2.11.5.2 Properties of pure epifunctors

Lemma 98. *Suppose given composable pure epifunctors* $\mathcal{A} \overset{F}{\twoheadrightarrow} \mathcal{B}$ *and* $\mathcal{B} \overset{G}{\twoheadrightarrow} \mathcal{C}$ *in* AddCat.

Then we have the pure epifunctor $\mathcal{A} \overset{FG}{\twoheadrightarrow} \mathcal{C}$.

Proof. We use Remark 96.

Ad (PE 1). Since F and G are full and dense, so is FG.

Ad (PE 2). Suppose given $X \xrightarrow{u} X'$ in $\mathcal{A}$ such that $uFG = 0$.

We have to show that there exists a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & X' \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle a'} & \\ & X'' & \end{array}$$

in $\mathcal{A}$ such that $X''FG$ is a zero object in $\mathcal{C}$.

Since G satisfies (PE 2) and since $(uF)G = 0$, we may choose a commutative triangle

$$\begin{array}{ccc} XF & \xrightarrow{\quad uF \quad} & X'F \\ & {\scriptstyle b}\searrow \quad \nearrow{\scriptstyle b'} & \\ & Y'' & \end{array}$$

in $\mathcal{B}$ such that $Y''G$ is a zero object in $\mathcal{C}$.

Since F is dense, we may choose $X''_1 \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $\tilde{b} : X''_1F \xrightarrow{\sim} Y''$. So we obtain the following commutative triangle.

$$\begin{array}{ccc} XF & \xrightarrow{\quad uF \quad} & X'F \; . \\ & {\scriptstyle b\cdot\tilde{b}^-}\searrow \quad \nearrow{\scriptstyle \tilde{b}\cdot b'} & \\ & X''_1F & \end{array}$$

Since F is full, we may choose $X \xrightarrow{a_1} X''_1 \xrightarrow{a'_1} X'$ in $\mathcal{A}$ such that $a_1F = b \cdot \tilde{b}^-$ and $a'_1F = \tilde{b} \cdot b'$.

So we have

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & X' \; , \\ & {\scriptstyle a_1}\searrow \quad \nearrow{\scriptstyle a'_1} & \\ & X''_1 & \end{array}$$

where X''_1FG is isomorphic to $Y''G$, hence a zero object, and where $uF = (b \cdot \tilde{b}^-) \cdot (\tilde{b} \cdot b') = a_1F \cdot a'_1F = (a_1 \cdot a'_1)F$.

Since F satisfies (PE 2) and $(u - a_1 \cdot a'_1)F \overset{\mathrm{R.}\,51}{=} uF - (a_1 \cdot a'_1)F = 0$, we may choose an object $X''_2 \in \mathrm{Ob}(\mathcal{A})$ such that X''_2F is a zero object in $\mathcal{B}$ and a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad u - a_1\cdot a'_1 \quad} & X' \; . \\ & {\scriptstyle a_2}\searrow \quad \nearrow{\scriptstyle a'_2} & \\ & X''_2 & \end{array}$$

So $u = a_1 \cdot a'_1 + a_2 \cdot a'_2$. Hence we have a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & X' \\ & {\scriptstyle (a_1\; a_2)}\searrow \quad \nearrow{\scriptstyle \binom{a'_1}{a'_2}} & \\ & X''_1 \oplus X''_2 & \end{array}$$

in $\mathcal{A}$; cf. Lemma 42.

Now $X_2''F$ is a zero object. Thus $X_2''FG$ is a zero object; cf. Definition 46.

So $(X_1'' \oplus X_2'')FG \overset{\text{L.67}}{\simeq} X_1''FG \oplus X_2''FG$ is a zero object; cf. Example 23, Lemma 30.(2).

Letting $a := (a_1\ a_2)$, $X'' := X_1'' \oplus X_2''$ and $a' := \begin{pmatrix} a_1' \\ a_2' \end{pmatrix}$, we get a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle a'} & \\ & X'' & \end{array}$$

in $\mathcal{A}$ such that $X''FG$ is a zero object in $\mathcal{C}$, as required. $\square$

Lemma 99. *Suppose given a diagram* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ *in* AddCat.

Suppose that F and FG are pure epifunctors.

Then G is a pure epifunctor.

Proof. *Ad* (PE 1). We have to show that G is dense. Suppose given $Z \in \mathrm{Ob}(\mathcal{C})$. Since FG is dense, we may choose $X \in \mathrm{Ob}(\mathcal{A})$ with $XFG \simeq Z$. Hence $Z \in \mathrm{Ob}(\mathcal{B}G)$. So G is dense.

We have to show that G is full. Suppose given $Y, Y' \in \mathrm{Ob}(\mathcal{B})$. We have to show that the map

$$_{\mathcal{B}}(Y, Y') \xrightarrow{G_{Y,Y'}} {}_{\mathcal{C}}(YG, Y'G)$$

is surjective.

Since F is a pure epifunctor, it is dense. So we may choose $X, X' \in \mathrm{Ob}(\mathcal{A})$ and isomorphisms $u : XF \overset{\sim}{\to} Y$ and $u' : X'F \overset{\sim}{\to} Y'$. We have the following commutative triangle of maps.

$$\begin{array}{ccc} & _{\mathcal{B}}(XF, X'F) & \\ {\scriptstyle F_{X,X'}}\nearrow & & \searrow{\scriptstyle G_{XF,X'F}} \\ _{\mathcal{A}}(X, X') & \xrightarrow[(FG)_{X,X'}]{} & _{\mathcal{C}}(XFG, X'FG) \end{array}$$

Since FG is a pure epifunctor, it is full. Hence the map $(FG)_{X,X'}$ is surjective. Thus the map $G_{XF,X'F}$ is surjective.

We have the following bijection.

$$\begin{array}{rcl} _{\mathcal{B}}(XF, X'F) & \overset{\alpha}{\to} & _{\mathcal{B}}(Y, Y') \\ f & \mapsto & u^- \cdot f \cdot u' \\ u \cdot g \cdot u'^- & \leftarrow\!\shortmid & g \end{array}$$

In fact,

$$f \;\mapsto\; u^- \cdot f \cdot u' \;\mapsto\; u \cdot u^- \cdot f \cdot u' \cdot u'^- \;=\; f$$

and

$$g \;\mapsto\; u \cdot g \cdot u'^- \;\mapsto\; u^- \cdot u \cdot g \cdot u'^- \cdot u' \;=\; g\,.$$

$$\begin{array}{ccc} XF & \xrightarrow{f} & X'F \\ {\scriptstyle u}\downarrow\wr & & \wr\downarrow{\scriptstyle u'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

We have the following bijection.

$$\begin{array}{rcl} {}_{\mathcal{C}}(XFG, X'FG) & \overset{\beta}{\to} & {}_{\mathcal{C}}(YG, Y'G) \\ f & \mapsto & u^-G \cdot f \cdot u'G \\ uG \cdot g \cdot u'^-G & \leftarrow\!\shortmid & g \end{array}$$

In fact,

$$f \;\mapsto\; u^-G \cdot f \cdot u'G \;\mapsto\; uG \cdot u^-G \cdot f \cdot u'G \cdot u'^-G \;=\; f$$

and

$$g \;\mapsto\; uG \cdot g \cdot u'^-G \;\mapsto\; u^-G \cdot uG \cdot g \cdot u'^-G \cdot u'G \;=\; g\,.$$

$$\begin{array}{ccc} XFG & \xrightarrow{f} & X'FG \\ {\scriptstyle uG}\downarrow\wr & & \wr\downarrow{\scriptstyle u'G} \\ YG & \xrightarrow[g]{} & Y'G \end{array}$$

We have the following commutative quadrangle of maps.

$$\begin{array}{ccc} {}_{\mathcal{B}}(XF, X'F) & \xrightarrow[\sim]{\alpha} & {}_{\mathcal{B}}(Y, Y') \\ {\scriptstyle G_{XF,X'F}}\downarrow & & \downarrow{\scriptstyle G_{Y,Y'}} \\ {}_{\mathcal{C}}(XFG, X'FG) & \xrightarrow[\sim]{\beta} & {}_{\mathcal{C}}(YG, Y'G) \end{array}$$

In fact, given $f \in {}_{\mathcal{B}}(XF, X'F)$, we obtain $((f)\alpha)G_{Y,Y'} = (u^- \cdot f \cdot u')G = u^-G \cdot fG \cdot u'G$ and $((f)G_{XF,X'F})\beta = (fG)\beta = u^-G \cdot fG \cdot u'G$, which is the same.

Since $G_{XF,X'F}$ is surjective and since α and β are bijective, we conclude that $G_{Y,Y'}$ is surjective.

Therefore, G is full.

Ad (PE 2). Suppose given $Y \overset{b}{\to} Y'$ in $\mathcal{B}$ such that $bG = 0$.

We have to show that there exists $\tilde{Y} \in \operatorname{Ob}(\mathcal{B})$ such that $\tilde{Y}G$ is a zero object and such that there exists a commutative triangle in $\mathcal{B}$ as follows.

$$\begin{array}{ccccc} Y & & \xrightarrow{b} & & Y' \\ & {\scriptstyle b_1}\searrow & & \nearrow{\scriptstyle b_2} & \\ & & \tilde{Y} & & \end{array}$$

Since F is a pure epifunctor, it is dense. So we may choose X, $X' \in \operatorname{Ob}(\mathcal{A})$ and isomorphisms $u : XF \overset{\sim}{\to} Y$ and $u' : X'F \overset{\sim}{\to} Y'$.

Since F is full, we may choose $a : X \to X'$ such that $aF = u \cdot b \cdot u'^-$. This yields the following commutative quadrangle.

$$\begin{array}{ccc} XF & \xrightarrow{aF \,=\, u \cdot b \cdot u'^-} & X'F \\ {\scriptstyle u}\downarrow\wr & & \wr\downarrow{\scriptstyle u'} \\ Y & \xrightarrow[b]{} & Y' \end{array}$$

Then $aFG = (u \cdot b \cdot u'^{-})G = uG \cdot bG \cdot u'^{-}G = 0$. Since FG is a pure epifunctor, we may choose $\tilde{X} \in \mathrm{Ob}(\mathcal{A})$ such that $\tilde{X}FG$ is a zero object and a commutative triangle in $\mathcal{A}$ as follows.

$$\begin{array}{ccc} X & \xrightarrow{\quad a \quad} & X' \\ & {\scriptstyle a_1}\searrow \quad \nearrow{\scriptstyle a_2} & \\ & \tilde{X} & \end{array}$$

Applying F, we obtain a commutative diagram in $\mathcal{B}$ as follows.

$$\begin{array}{ccc} Y & \xrightarrow{\quad b \quad} & Y' \\ {\scriptstyle u}\uparrow\wr & & \wr\uparrow{\scriptstyle u'} \\ XF & \xrightarrow{\quad aF \quad} & X'F \\ & {\scriptstyle a_1F}\searrow \quad \nearrow{\scriptstyle a_2F} & \\ & \tilde{X}F & \end{array}$$

Letting $\tilde{Y} := \tilde{X}F$, the object $\tilde{Y}G = \tilde{X}FG$ is zero. Moreover, letting

$$b_1 \; := \; u^{-} \cdot a_1F \; : \; Y \to \tilde{X}F = \tilde{Y}$$

and

$$b_2 \; := \; a_2F \cdot u' \; : \; \tilde{Y} = \tilde{X}F \to Y' \; ,$$

we obtain $b_1 \cdot b_2 \; = \; u^{-} \cdot a_1F \cdot a_2F \cdot u' \; = \; u^{-} \cdot aF \cdot u' \; = \; b$. I.e. we have obtained the required commutative triangle as follows.

$$\begin{array}{ccc} Y & \xrightarrow{\quad b \quad} & Y' \\ & {\scriptstyle b_1}\searrow \quad \nearrow{\scriptstyle b_2} & \\ & \tilde{Y} & \end{array}$$

□

2.11.6 Stability under isomorphisms

Corollary 100.

(1) *Suppose given a pure short exact sequence* $\mathcal{A}' \overset{F'}{-\!\!\bullet\!\!\!\rightarrowtail} \mathcal{A} \overset{F''}{-\!\!\!+\!\!\!\twoheadrightarrow} \mathcal{A}''$ *in* AddCat.

Suppose given $\mathcal{A}' \xrightarrow{\tilde{F}'} \mathcal{A} \xrightarrow{\tilde{F}''} \mathcal{A}''$ *in* AddCat.

Suppose that $\tilde{F}' \simeq F'$ *and* $\tilde{F}'' \simeq F''$.

Then we also have the pure short exact sequence $\mathcal{A}' \overset{\tilde{F}'}{-\!\!\bullet\!\!\!\rightarrowtail} \mathcal{A} \overset{\tilde{F}''}{-\!\!\!+\!\!\!\twoheadrightarrow} \mathcal{A}''$.

(2) *Suppose given a pure monofunctor* $\mathcal{A}' \overset{F'}{-\!\!\bullet\!\!\!\rightarrowtail} \mathcal{A}$ *in* AddCat.

Suppose given $\mathcal{A}' \xrightarrow{\tilde{F}'} \mathcal{A}$ *in* AddCat.

Suppose that $\tilde{F}' \simeq F'$.

Then we also have the pure monofunctor $\mathcal{A}' \overset{\tilde{F}'}{-\!\!\bullet\!\!\!\rightarrowtail} \mathcal{A}$.

(3) *Suppose given a pure epifunctor* $\mathcal{A} \xrightarrow{F''} \mathcal{A}''$ *in* AddCat.

Suppose given $\mathcal{A} \xrightarrow{\tilde{F}''} \mathcal{A}''$ *in* AddCat.

Suppose that $\tilde{F}'' \simeq F''$.

Then we also have the pure epifunctor $\mathcal{A} \xrightarrow{\tilde{F}''} \mathcal{A}''$.

Proof.

Ad (1). This follows from Lemma 89, letting $U' = \mathrm{id}_{\mathcal{A}'}$, $U = \mathrm{id}_{\mathcal{A}}$ and $U'' = \mathrm{id}_{\mathcal{A}''}$.

Ad (2). This follows from Lemma 93, letting $U' = \mathrm{id}_{\mathcal{A}'}$ and $U = \mathrm{id}_{\mathcal{A}}$.

Ad (3). This follows from Lemma 97, letting $U = \mathrm{id}_{\mathcal{A}}$ and $U'' = \mathrm{id}_{\mathcal{A}''}$. □

2.11.7 Two composable pure monofunctors

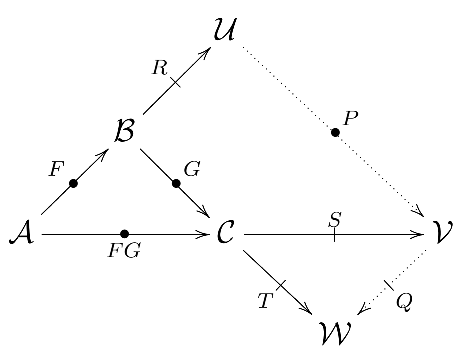
Lemma 101. *Suppose given additive categories* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$.

Suppose given pure short exact sequences $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{R} \mathcal{U}$ *and* $\mathcal{A} \xrightarrow{FG} \mathcal{C} \xrightarrow{S} \mathcal{V}$ *and* $\mathcal{B} \xrightarrow{G} \mathcal{C} \xrightarrow{T} \mathcal{W}$.

We may choose an additive functor $\mathcal{U} \xrightarrow{P} \mathcal{V}$ *such that* $RP \simeq GS$. *This condition determines* P *up to isomorphy.*

We may choose an additive functor $\mathcal{V} \xrightarrow{Q} \mathcal{W}$ *such that* $SQ \simeq T$. *This condition determines* Q *up to isomorphy.*

Then we have the pure short exact sequence $\mathcal{U} \xrightarrow{P} \mathcal{V} \xrightarrow{Q} \mathcal{W}$.



So we have a pure monofunctor F and a pure monofunctor G, which compose to a pure monofunctor FG; cf. Lemma 94.

Proof. Since $\mathcal{A} \xrightarrow{FG} \mathcal{C} \xrightarrow{S} \mathcal{V}$ is a pure short exact sequence, FGS is a zero functor. Hence we may choose an additive functor $\mathcal{U} \xrightarrow{P} \mathcal{V}$ such that $RP \simeq GS$. This condition determines P up to isomorphy. Cf. Lemma 90.

We choose an isotransformation $RP \xrightarrow[\sim]{m} GS$.

Since $\mathcal{B} \xrightarrow{G} \mathcal{C} \xrightarrow{T} \mathcal{W}$ is a pure short exact sequence, GT is a zero functor. Hence also FGT maps each object to a zero object, i.e. FGT is a zero functor. Thus we may choose an additive functor $\mathcal{V} \xrightarrow{Q} \mathcal{W}$ such that $SQ \simeq T$. This condition determines Q up to isomorphy. Cf. Lemma 90.

We choose an isotransformation $SQ \overset{n}{\underset{\sim}{\to}} T$.

We have to show that $\mathcal{U} \xrightarrow{P} \mathcal{V} \xrightarrow{Q} \mathcal{W}$ is a pure short exact sequence; cf. Definition 84. We may verify conditions (P 1, 2, 3a, 3b, 4) from Remark 86.

Ad (P 1). We show that P is full. Suppose given $X,\, X' \in \operatorname{Ob}(\mathcal{U})$. Suppose given a morphism $v : XP \to X'P$. We have to show that v is in the image of $P_{X,X'}$.

Since R is a pure epifunctor, we may choose objects $\hat{X},\, \hat{X}' \in \operatorname{Ob}(\mathcal{B})$ and isomorphisms $\hat{X}R \overset{f}{\underset{\sim}{\to}} X$ and $\hat{X}'R \overset{f'}{\underset{\sim}{\to}} X'$. So $fP \cdot v \cdot (f'P)^- : \hat{X}RP \to \hat{X}'RP$. So

$$(\hat{X}m)^- \cdot fP \cdot v \cdot (f'P)^- \cdot \hat{X}'m \;:\; \hat{X}GS \;\to\; \hat{X}'GS\,.$$

Since S is a pure epifunctor, S is full. Since G is a pure monofunctor, G is full. So GS is full. Hence we may choose $b : \hat{X} \to \hat{X}'$ such that

$$bGS \;=\; (\hat{X}m)^- \cdot fP \cdot v \cdot (f'P)^- \cdot \hat{X}'m\,.$$

We have the following commutative quadrangle.

$$\begin{array}{ccc} \hat{X}RP & \overset{\hat{X}m}{\underset{\sim}{\longrightarrow}} & \hat{X}GS \\ {\scriptstyle bRP}\downarrow & & \downarrow{\scriptstyle bGS} \\ \hat{X}'RP & \overset{\hat{X}'m}{\underset{\sim}{\longrightarrow}} & \hat{X}'GS \end{array}$$

So we get

$$\begin{aligned} v &= (fP)^- \cdot \hat{X}m \cdot bGS \cdot (\hat{X}'m)^- \cdot f'P \\ &= (fP)^- \cdot bRP \cdot \hat{X}'m \cdot (\hat{X}'m)^- \cdot f'P \\ &= (fP)^- \cdot bRP \cdot f'P \\ &= (f^- \cdot bR \cdot f')P\,. \end{aligned}$$

We show that P is faithful. Suppose given $X,\, X' \in \operatorname{Ob}(\mathcal{U})$. We have the group morphism

$$\begin{array}{ccc} {}_{\mathcal{U}}(X,X') & \xrightarrow{P_{X,X'}} & {}_{\mathcal{V}}(XP,X'P) \\ u & \mapsto & uP\,; \end{array}$$

cf. Remark 51. We have to show that it is injective. So we have to show that its kernel is zero. Suppose given $u : X \to X'$ in $\mathcal{U}$ such that $uP = 0$. We have to show that $u \overset{!}{=} 0$.

Since R is a pure epifunctor, we may choose objects $\hat{X},\, \hat{X}' \in \operatorname{Ob}(\mathcal{B})$ and isomorphisms $\hat{X}R \overset{f}{\underset{\sim}{\to}} X$ and $\hat{X}'R \overset{f'}{\underset{\sim}{\to}} X'$. So $f \cdot u \cdot f'^- : \hat{X}R \to \hat{X}'R$. Since R is full, we may choose $\hat{u} : \hat{X} \to \hat{X}'$ such that $\hat{u}R = f \cdot u \cdot f'^-$.

We have the following commutative quadrangle.

$$\begin{array}{ccc} \hat{X}RP & \overset{\hat{X}m}{\underset{\sim}{\longrightarrow}} & \hat{X}GS \\ {\scriptstyle \hat{u}RP}\downarrow & & \downarrow{\scriptstyle \hat{u}GS} \\ \hat{X}'RP & \overset{\hat{X}'m}{\underset{\sim}{\longrightarrow}} & \hat{X}'GS \end{array}$$

We obtain

$$\begin{array}{rcl} (\hat{u}G)S & = & (\hat{X}m)^{-} \cdot \hat{u}RP \cdot \hat{X}'m \\ & = & (\hat{X}m)^{-} \cdot fP \cdot uP \cdot (f'P)^{-} \cdot \hat{X}'m \\ & = & 0\,. \end{array}$$

Since $\mathcal{A} \xrightarrow{FG} \mathcal{C} \xrightarrow{S} \mathcal{V}$ is a pure short exact sequence, by (P 4) there exists a factorisation of $\hat{u}G : \hat{X}G \to \hat{X}'G$ over an object ZFG for some $Z \in \mathrm{Ob}(\mathcal{A})$. I.e. we may choose morphisms $\hat{X}G \xrightarrow{c} ZFG \xrightarrow{c'} \hat{X}'G$ in $\mathcal{C}$ such that $c \cdot c' = \hat{u}G$.

Since G is a pure monofunctor, it is full and faithful. Since G is full, we may choose morphisms $\hat{X} \xrightarrow{b} ZF \xrightarrow{b'} \hat{X}'$ in $\mathcal{B}$ such that $bG = c$ and $b'G = c'$. So $(b \cdot b')G = bG \cdot b'G = c \cdot c' = \hat{u}G$. Since G is faithful, we obtain $b \cdot b' = \hat{u}$.

Hence $bR \cdot b'R = (b \cdot b')R = \hat{u}R$. Moreover, $bR \cdot b'R$ factors over ZFR, which is a zero object since $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{R} \mathcal{U}$ is a pure short exact sequence. So $\hat{u}R = bR \cdot b'R = 0$. We conclude that

$$u \;=\; f^{-} \cdot \hat{u}R \cdot f' \;=\; 0\,.$$

Ad (P 2). We show that Q is full. Suppose given $Y,\, Y' \in \mathrm{Ob}(\mathcal{V})$ and $w : YQ \to Y'Q$. We have to show that w is in the image of $Q_{Y,Y'}$.

Since S is a pure epifunctor, we may choose objects $\hat{Y},\, \hat{Y}' \in \mathrm{Ob}(\mathcal{C})$ and isomorphisms $\hat{Y}S \xrightarrow[\sim]{f} Y$ and $\hat{Y}'S \xrightarrow[\sim]{f'} Y'$.

We obtain the following commutative diagram.

$$\begin{array}{ccc} YQ & \xrightarrow{\quad w \quad} & Y'Q \\ {\scriptstyle fQ}\uparrow\wr & & \wr\uparrow{\scriptstyle f'Q} \\ \hat{Y}SQ & & \hat{Y}'SQ \\ {\scriptstyle \hat{Y}n}\downarrow\wr & & \wr\downarrow{\scriptstyle \hat{Y}'n} \\ \hat{Y}T & \xrightarrow[(\hat{Y}n)^{-} \cdot fQ \cdot w \cdot (f'Q)^{-} \cdot \hat{Y}'n]{} & \hat{Y}'T \end{array}$$

Since T is full, we may choose $c : \hat{Y} \to \hat{Y}'$ such that $cT = (\hat{Y}n)^{-} \cdot fQ \cdot w \cdot (f'Q)^{-} \cdot \hat{Y}'n$. So we obtain the following commutative diagram.

$$\begin{array}{ccc} YQ & \xrightarrow{\quad w \quad} & Y'Q \\ {\scriptstyle fQ}\uparrow\wr & & \wr\uparrow{\scriptstyle f'Q} \\ \hat{Y}SQ & \xrightarrow{\quad cSQ \quad} & \hat{Y}'SQ \\ {\scriptstyle \hat{Y}n}\downarrow\wr & & \wr\downarrow{\scriptstyle \hat{Y}'n} \\ \hat{Y}T & \xrightarrow[\quad cT \quad]{} & \hat{Y}'T \end{array}$$

Hence $w = (fQ)^{-} \cdot cSQ \cdot f'Q = (f^{-} \cdot cS \cdot f')Q$.

We show that Q is dense. Suppose given $Z \in \mathrm{Ob}(\mathcal{W})$. Since T is dense, we may choose an object $\hat{Z} \in \mathrm{Ob}(\mathcal{C})$ such that $\hat{Z}T \simeq Z$. Then $(\hat{Z}S)Q \simeq \hat{Z}T \simeq Z$.

Ad (P 3a). We show that PQ is a zero functor. Suppose given $X \in \mathrm{Ob}(\mathcal{U})$. We have to show that XPQ is a zero object.

Since R is a pure epifunctor, there we may choose an object $\hat{X} \in \mathrm{Ob}(\mathcal{B})$ such that $\hat{X}R \simeq X$. Now $XPQ \simeq \hat{X}RPQ \simeq \hat{X}GSQ \simeq \hat{X}GT \simeq 0$ since $\mathcal{B} \overset{G}{\bullet\!\!\rightarrowtail} \mathcal{C} \overset{T}{\twoheadrightarrow} \mathcal{W}$ is a pure short exact sequence.

Ad (P 3b). We show that $\mathcal{U}P$ is closed under retracts in $\mathcal{V}$. Suppose given $X \in \mathrm{Ob}(\mathcal{U})$. Suppose given $Y \in \mathrm{Ob}(\mathcal{V})$ and morphisms $Y \xrightarrow{v} XP \xrightarrow{v'} Y$ in $\mathcal{V}$ such that $v \cdot v' = \mathrm{id}_Y$. We have to show that Y is in $\mathrm{Ob}(\mathcal{U}P)$, i.e. that there exists an object $X' \in \mathrm{Ob}(\mathcal{U})$ such that $X'P \simeq Y$.

Since S is a pure epifunctor, we may choose an object $\hat{Y} \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $\hat{Y}S \overset{g}{\underset{\sim}{\rightarrow}} Y$.

Since R is a pure epifunctor, we may choose an object $\hat{X} \in \mathrm{Ob}(\mathcal{B})$ and an isomorphism $\hat{X}R \overset{f}{\underset{\sim}{\rightarrow}} X$.

So

$$\hat{Y}S \xrightarrow{g \cdot v \cdot (fP)^-} \hat{X}RP \xrightarrow{fP \cdot v' \cdot g^-} \hat{Y}S\,.$$

So

$$\hat{Y}S \xrightarrow{g \cdot v \cdot (fP)^- \cdot \hat{X}m} \hat{X}GS \xrightarrow{(\hat{X}m)^- \cdot fP \cdot v' \cdot g^-} \hat{Y}S\,.$$

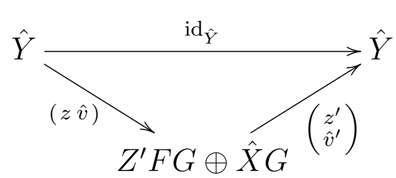
Since S is a pure epifunctor, S is full. So we may choose morphisms $\hat{Y} \xrightarrow{\hat{v}} \hat{X}G \xrightarrow{\hat{v}'} \hat{Y}$ in $\mathcal{C}$ such that $\hat{v}S = g \cdot v \cdot (fP)^- \cdot \hat{X}m$ and $\hat{v}'S = (\hat{X}m)^- \cdot fP \cdot v' \cdot g^-$.

We obtain

$$\begin{aligned}
(\mathrm{id}_{\hat{Y}} - \hat{v} \cdot \hat{v}')S &= \mathrm{id}_{\hat{Y}}S - (\hat{v} \cdot \hat{v}')S \\
&= \mathrm{id}_{\hat{Y}S} - \hat{v}S \cdot \hat{v}'S \\
&= \mathrm{id}_{\hat{Y}S} - (g \cdot v \cdot (fP)^- \cdot \hat{X}m) \cdot ((\hat{X}m)^- \cdot fP \cdot v' \cdot g^-) \\
&= \mathrm{id}_{\hat{Y}S} - (g \cdot v \cdot v' \cdot g^-) \\
&= \mathrm{id}_{\hat{Y}S} - (g \cdot \mathrm{id}_Y \cdot g^-) \\
&= \mathrm{id}_{\hat{Y}S} - \mathrm{id}_{\hat{Y}S} \\
&= 0\,;
\end{aligned}$$

cf. Remark 51.

Since $\mathcal{A} \overset{FG}{\bullet\!\!\rightarrowtail} \mathcal{C} \overset{S}{\twoheadrightarrow} \mathcal{V}$ is a pure short exact sequence, by (P 4) we obtain a factorisation of $\mathrm{id}_{\hat{Y}} - \hat{v} \cdot \hat{v}'$ over $Z'FG$ for some object $Z' \in \mathrm{Ob}(\mathcal{A})$. I.e. we may choose morphisms $\hat{Y} \xrightarrow{z} Z'FG \xrightarrow{z'} \hat{Y}$ such that $z \cdot z' = \mathrm{id}_{\hat{Y}} - \hat{v} \cdot \hat{v}'$. So $\mathrm{id}_{\hat{Y}} = z \cdot z' + \hat{v} \cdot \hat{v}'$. I.e. we have the following commutative triangle.



Since $Z'FG \oplus \hat{X}G$ is isomorphic to $(Z'F \oplus \hat{X})G$ by Lemma 67.(2), $\hat{Y}$ is a retract of an object in $\mathcal{B}G$.

Since $\mathcal{B} \overset{G}{\bullet\!\!\rightarrowtail} \mathcal{C} \overset{T}{\twoheadrightarrow} \mathcal{W}$ is a pure short exact sequence, we conclude that $\hat{Y} \simeq \tilde{X}G$ for some $\tilde{X} \in \mathrm{Ob}(\mathcal{B})$. So $Y \simeq \hat{Y}S \simeq \tilde{X}GS \simeq (\tilde{X}R)P$. I.e. we have used $X' := \tilde{X}R \in \mathrm{Ob}(\mathcal{U})$.

Ad (P 4). Suppose given $Y \xrightarrow{v} Y'$ in $\mathcal{V}$ such that $vQ = 0$. We have to show that v factors over an object in $\mathcal{U}P$.

Since S is a pure epifunctor, we may choose objects $\hat{Y}$, $\hat{Y}' \in \mathrm{Ob}(\mathcal{C})$ and isomorphisms $\hat{Y}S \underset{\sim}{\xrightarrow{f}} Y$ and $\hat{Y}'S \underset{\sim}{\xrightarrow{f'}} Y'$.

So $\hat{Y}S \xrightarrow{f \cdot v \cdot f'^{-}} \hat{Y}'S$. Since S is a pure epifunctor, S is full. Therefore, we may choose $\hat{Y} \xrightarrow{\hat{v}} \hat{Y}'$ in $\mathcal{C}$ such that $\hat{v}S = f \cdot v \cdot f'^{-}$.

We have the following commutative quadrangle.

$$\begin{array}{ccc} \hat{Y}SQ & \xrightarrow{\hat{v}SQ} & \hat{Y}'SQ \\ {\scriptstyle \hat{Y}n}\downarrow\wr & & \wr\downarrow{\scriptstyle \hat{Y}'n} \\ \hat{Y}T & \xrightarrow[\hat{v}T]{} & \hat{Y}'T \end{array}$$

We obtain

$$\begin{aligned} \hat{v}T &= (\hat{Y}n)^{-} \cdot \hat{v}SQ \cdot \hat{Y}'n \\ &= (\hat{Y}n)^{-} \cdot (f \cdot v \cdot f'^{-})Q \cdot \hat{Y}'n \\ &= (\hat{Y}n)^{-} \cdot fQ \cdot vQ \cdot (f'Q)^{-} \cdot \hat{Y}'n \\ &= 0\,. \end{aligned}$$

Since $\mathcal{B} \overset{G}{\rightarrowtail} \mathcal{C} \overset{T}{\twoheadrightarrow} \mathcal{W}$ is a pure short exact sequence, $\hat{v}$ factors over $\hat{X}G$ for some $\hat{X} \in \mathrm{Ob}(\mathcal{B})$. I.e. we may choose morphisms $\hat{Y} \xrightarrow{\tilde{z}} \hat{X}G \xrightarrow{\tilde{z}'} \hat{Y}'$ such that $\hat{v} = \tilde{z} \cdot \tilde{z}'$.

So

$$\begin{aligned} v &= f^{-} \cdot \hat{v}S \cdot f' \\ &= f^{-} \cdot (\tilde{z} \cdot \tilde{z}')S \cdot f' \\ &= f^{-} \cdot \tilde{z}S \cdot \tilde{z}'S \cdot f' \\ &= f^{-} \cdot \tilde{z}S \cdot (\hat{X}m)^{-} \cdot \hat{X}m \cdot \tilde{z}'S \cdot f', \end{aligned}$$

which factors over $(\hat{X}R)P$. □

2.11.8 Two composable pure epifunctors

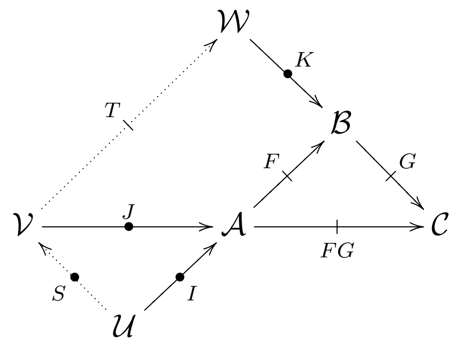
Lemma 102. *Suppose given additive categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$.*

Suppose given pure short exact sequences $\mathcal{U} \overset{I}{\rightarrowtail} \mathcal{A} \overset{F}{\twoheadrightarrow} \mathcal{B}$ and $\mathcal{V} \overset{J}{\rightarrowtail} \mathcal{A} \overset{FG}{\twoheadrightarrow} \mathcal{C}$ and $\mathcal{W} \overset{K}{\rightarrowtail} \mathcal{B} \overset{G}{\twoheadrightarrow} \mathcal{C}$.

We may choose an additive functor $\mathcal{U} \xrightarrow{S} \mathcal{V}$ such that $SJ \simeq I$. This condition determines S up to isomorphy.

We may choose an additive functor $\mathcal{V} \xrightarrow{T} \mathcal{W}$ such that $TK \simeq JF$. This condition determines T up to isomorphy.

Then we have the pure short exact sequence $\mathcal{U} \xrightarrow{S} \mathcal{V} \xrightarrow{T} \mathcal{W}$.



So we have a pure epifunctor F and a pure epifunctor G, which compose to a pure epifunctor FG; cf. Lemma 98.

Proof. Since $\mathcal{U} \xrightarrow{I} \mathcal{A} \xrightarrow{F} \mathcal{B}$ is a pure short exact sequence, IF is a zero functor, i.e. it maps each object to a zero object. Since G is additive, also $I(FG)$ is a zero functor; cf. Definition 46.

Since $\mathcal{V} \xrightarrow{J} \mathcal{A} \xrightarrow{FG} \mathcal{C}$ is a pure short exact sequence, we may choose an additive functor $\mathcal{U} \xrightarrow{S} \mathcal{V}$ such that $SJ \simeq I$. This condition determines S up to isomorphy. Cf. Lemma 91.

We choose an isotransformation $SJ \overset{n}{\underset{\sim}{\to}} I$.

Since $\mathcal{V} \xrightarrow{J} \mathcal{A} \xrightarrow{FG} \mathcal{C}$ is a pure short exact sequence, $(JF)G$ is a zero functor. Since $\mathcal{W} \xrightarrow{K} \mathcal{B} \xrightarrow{G} \mathcal{C}$ is a pure short exact sequence, we may choose an additive functor $\mathcal{V} \xrightarrow{T} \mathcal{W}$ such that $TK = JF$. This condition determines T up to isomorphy. Cf. Lemma 91.

We choose an isotransformation $TK \overset{m}{\underset{\sim}{\to}} JF$.

We have to show that $\mathcal{U} \xrightarrow{S} \mathcal{V} \xrightarrow{T} \mathcal{W}$ is a pure short exact sequence; cf. Definition 84. We may verify conditions (P 1, 2, 3a, 3b, 4) from Remark 86.

Ad (P 1). We have to show that S is full and faithful.

Suppose given $X,\, X' \in \operatorname{Ob}(\mathcal{U})$. We have to show that the group morphism

$$\begin{array}{ccc} {}_{\mathcal{U}}(X, X') & \xrightarrow{S_{X,X'}} & {}_{\mathcal{V}}(XS, X'S) \\ f & \mapsto & fS \end{array}$$

is an isomorphism of abelian groups; cf. Remark 51.

Since I is full and faithful, we have the following group isomorphism.

$$\begin{array}{ccc} {}_{\mathcal{U}}(X, X') & \xrightarrow[\sim]{I_{X,X'}} & {}_{\mathcal{A}}(XI, X'I) \\ f & \mapsto & fI \end{array}$$

Since J is full and faithful, we have the following group isomorphism.

$$\begin{array}{ccc} {}_{\mathcal{V}}(XS, X'S) & \xrightarrow[\sim]{J_{XS,X'S}} & {}_{\mathcal{A}}(XSJ, X'SJ) \\ g & \mapsto & gJ \end{array}$$

We have the group isomorphism

$$\begin{array}{rcl} {}_{\mathcal{A}}(XI, X'I) & \overset{c}{\underset{\sim}{\to}} & {}_{\mathcal{A}}(XSJ, X'SJ) \\ h & \mapsto & Xn \cdot h \cdot (X'n)^- \ , \end{array}$$

with inverse

$$\begin{array}{rcl} {}_{\mathcal{A}}(XI, X'I) & \overset{c^-}{\underset{\sim}{\leftarrow}} & {}_{\mathcal{A}}(XSJ, X'SJ) \\ (Xn)^- \cdot k \cdot X'n & \leftarrow\!\shortmid & k \ . \end{array}$$

Cf. Lemma 38.(3).

It remains to show that the following quadrangle of abelian groups commutes.

$$\begin{array}{ccc} {}_{\mathcal{U}}(X, X') & \xrightarrow{S_{X,X'}} & {}_{\mathcal{V}}(XS, X'S) \\ {\scriptstyle I_{X,X'}} \downarrow \wr & & \wr \downarrow {\scriptstyle J_{XS,X'S}} \\ {}_{\mathcal{A}}(XI, X'I) & \underset{\sim}{\xrightarrow{c}} & {}_{\mathcal{A}}(XSJ, X'SJ) \end{array}$$

In fact, for $f \in {}_{\mathcal{U}}(X, X')$, we obtain

$$\begin{array}{rcl} fI_{X,X'}c & = & (fI)c \\ & = & Xn \cdot fI \cdot (X'n)^- \\ & = & fSJ \cdot X'n \cdot (X'n)^- \\ & = & fSJ \\ & = & fS_{X,X'}J_{XS,X'S} \end{array}$$

because of the following commutative quadrangle.

$$\begin{array}{ccc} XSJ & \underset{\sim}{\xrightarrow{Xn}} & XI \\ {\scriptstyle fSJ} \downarrow & & \downarrow {\scriptstyle fI} \\ X'SJ & \underset{\sim}{\xrightarrow{X'n}} & X'I \end{array}$$

We conclude that $S_{X,X'}$ is a group isomorphism, as was to be shown.

Ad (P 2). We have to show that T is full. Suppose given $Y, Y' \in \mathrm{Ob}(\mathcal{V})$. Suppose given a morphism $YT \xrightarrow{w} Y'T$ in $\mathcal{W}$. We have to find a morphism from Y to Y' that maps to w under T.

We have $YTK \xrightarrow{wK} Y'TK$ in $\mathcal{B}$. So we have $YJF \xrightarrow{(Ym)^- \cdot wK \cdot Y'm} Y'JF$ in $\mathcal{B}$.

Since J and F are full, so is JF. Hence we may choose a morphism $Y \xrightarrow{v} Y'$ such that

$$vJF = (Ym)^- \cdot wK \cdot Y'm \ .$$

We have the following commutative quadrangle.

$$\begin{array}{ccc} YTK & \underset{\sim}{\xrightarrow{Ym}} & YJF \\ {\scriptstyle vTK} \downarrow & & \downarrow {\scriptstyle vJF} \\ Y'TK & \underset{\sim}{\xrightarrow{Y'm}} & Y'JF \end{array}$$

Hence

$$vTK \;=\; Ym \cdot vJF \cdot (Y'm)^- \;=\; Ym \cdot ((Ym)^- \cdot wK \cdot Y'm) \cdot (Y'm)^- \;=\; wK\,.$$

Since K is faithful, we obtain $vT = w : YT \to Y'T$.

We have to show that T is dense. Suppose given $Z \in \mathrm{Ob}(\mathcal{W})$. We have to show that there exists an object $Y \in \mathrm{Ob}(\mathcal{V})$ such that $YT \overset{!}{\simeq} Z$. Since K is full and faithful, it suffices to show that $YTK \overset{!}{\simeq} ZK$.

Since F is dense, we may choose $X \in \mathrm{Ob}(\mathcal{A})$ such that $XF \simeq ZK$. Then $XFG \simeq ZKG$, which is a zero object since $\mathcal{W} \xrightarrow{K} \mathcal{B} \xrightarrow{G} \mathcal{C}$ is pure short exact. So $X \in \mathrm{Ob}(\mathrm{Kern}(FG))$. By (P 3), we conclude that $X \in \mathrm{Ob}(\mathcal{V}J)$. Hence there exists $Y \in \mathrm{Ob}(\mathcal{V})$ such that $YJ \simeq X$. Altogether, we obtain $YTK \simeq YJF \simeq XF \simeq ZK$.

Ad (P 3a). We have to show that ST is a zero functor. Since K is full and faithful, it suffices to show that STK is a zero functor; cf. Remark 66.(2).

In fact, $STK \simeq SJF \simeq IF$, which is a zero functor since $\mathcal{U} \xrightarrow{I} \mathcal{A} \xrightarrow{F} \mathcal{B}$ is a pure short exact sequence.

Ad (P 3b). We have to show that $\mathcal{U}S$ is closed under retracts in $\mathcal{V}$.

Suppose given $X \in \mathrm{Ob}(\mathcal{U})$, $Y \in \mathrm{Ob}(\mathcal{V})$ and a commutative triangle

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ & {\scriptstyle v}\searrow \quad \nearrow{\scriptstyle v'} & \\ & XS & \end{array}$$

in $\mathcal{V}$. We have to show that $Y \overset{!}{\in} \mathrm{Ob}(\mathcal{U}S)$.

We have the commutative triangle

$$\begin{array}{ccc} YJ & \xrightarrow{\mathrm{id}_{YJ}} & YJ \\ & {\scriptstyle vJ}\searrow \quad \nearrow{\scriptstyle v'J} & \\ & XSJ & \end{array}$$

in $\mathcal{A}$. Since $XSJ \simeq XI$, the object YJ is a retract of XI.

Since $\mathcal{U} \xrightarrow{I} \mathcal{A} \xrightarrow{F} \mathcal{B}$ is a pure short exact sequence, $\mathcal{U}I$ is closed under retracts in $\mathcal{A}$. Hence $YJ \in \mathrm{Ob}(\mathcal{U}I)$. We choose $\tilde{X} \in \mathrm{Ob}(\mathcal{U})$ with $\tilde{X}I \simeq YJ$.

Now $YJ \simeq \tilde{X}I \simeq \tilde{X}SJ$. Since J is full and faithful, we obtain that $Y \simeq \tilde{X}S$, whence $Y \in \mathrm{Ob}(\mathcal{U}S)$.

Ad (P 4). Suppose given $Y \xrightarrow{v} Y'$ in $\mathcal{V}$ such that $vT = 0$. We have to show that v factors over an object in $\mathcal{U}S$.

We have the following commutative quadrangle.

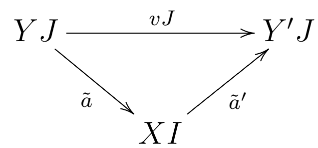
$$\begin{array}{ccc} YTK & \xrightarrow[\sim]{Ym} & YJF \\ {\scriptstyle vTK}\downarrow & & \downarrow{\scriptstyle vJF} \\ Y'TK & \xrightarrow[\sim]{Y'm} & Y'JF \end{array}$$

We obtain

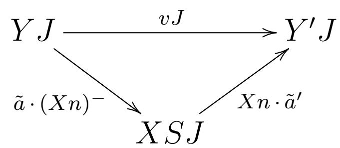
$$(vJ)F \;=\; (Ym)^{-} \cdot vTK \cdot Y'm \;=\; 0\,;$$

cf. Remark 51.

Since $\mathcal{U} \overset{I}{\longrightarrow} \mathcal{A} \overset{F}{\longrightarrow} \mathcal{B}$ is a pure short exact sequence, we may choose a commutative triangle

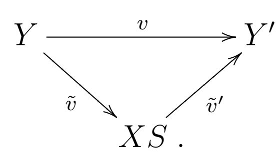


in $\mathcal{A}$ with $X \in \mathrm{Ob}(\mathcal{U})$. So we obtain the commutative triangle



in $\mathcal{A}$. Since J is full, we find morphisms $Y \overset{\tilde v}{\to} XS$ such that $\tilde v J = \tilde a \cdot (Xn)^{-}$ and $XS \overset{\tilde v'}{\to} Y'$ such that $\tilde v' J = Xn \cdot \tilde a'$.

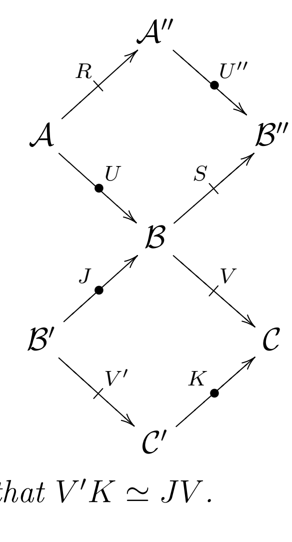
Now $(\tilde v \cdot \tilde v')J = \tilde v J \cdot \tilde v' J = \tilde a \cdot (Xn)^{-} \cdot Xn \cdot \tilde a' = \tilde a \cdot \tilde a' = vJ$. Since J is faithful, we obtain $\tilde v \cdot \tilde v' = v$, i.e. the commutative triangle



□

2.11.9 Completion to a 3×3-diagram

Lemma 103. *Suppose given the following diagram in* AddCat*.*



Suppose that $US \simeq RU''$. Suppose that $V'K \simeq JV$.

Suppose that $\mathcal{B}' \overset{J}{\longrightarrow} \mathcal{B} \overset{S}{\longrightarrow} \mathcal{B}''$ is a pure short exact sequence.

Suppose that $\mathcal{A} \xrightarrow{U} \mathcal{B} \xrightarrow{V} \mathcal{C}$ *is a pure short exact sequence.*

(1) *Suppose given a pure short exact sequence* $\mathcal{A}' \xrightarrow{I} \mathcal{A} \xrightarrow{R} \mathcal{A}''$ *with* R *as in the diagram.*

There exists an additive functor $\mathcal{A}' \xrightarrow{U'} \mathcal{B}'$ *such that* $U'J \simeq IU$. *This condition determines* U' *up to isomorphy.*

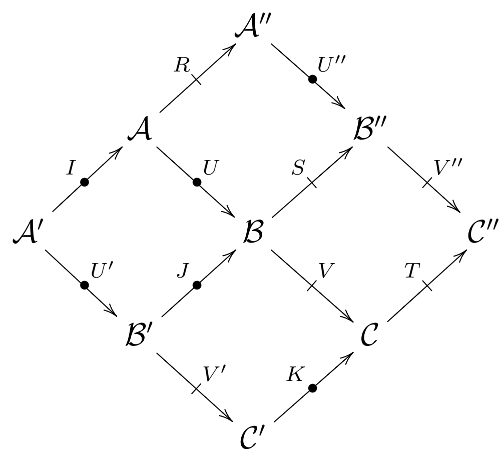
Then we have the pure short exact sequence $\mathcal{A}' \xrightarrow{U'} \mathcal{B}' \xrightarrow{V'} \mathcal{C}'$.

(2) *Suppose given a pure short exact sequence* $\mathcal{A}'' \xrightarrow{U''} \mathcal{B}'' \xrightarrow{V''} \mathcal{C}''$ *with* U'' *as in the diagram.*

There exists an additive functor $\mathcal{C} \xrightarrow{T} \mathcal{C}''$ *such that* $VT \simeq SV''$. *This condition determines* T *up to isomorphy.*

Then we have the pure short exact sequence $\mathcal{C}' \xrightarrow{K} \mathcal{C} \xrightarrow{T} \mathcal{C}''$.

(3) *There exists a diagram in* AddCat *as follows, in which* $IU \simeq U'J$ *and* $VT \simeq SV''$.



Moreover, $\mathcal{A}' \xrightarrow{U'} \mathcal{B}' \xrightarrow{V'} \mathcal{C}'$, $\mathcal{A}' \xrightarrow{I} \mathcal{A} \xrightarrow{R} \mathcal{A}''$, $\mathcal{C}' \xrightarrow{K} \mathcal{C} \xrightarrow{T} \mathcal{C}''$, $\mathcal{A}'' \xrightarrow{U''} \mathcal{B}'' \xrightarrow{V''} \mathcal{C}''$ *are pure short exact sequences.*

Proof.

Ad (1). To show that we may choose an additive functor U' such that

$$U'J \ \simeq \ IU \ ,$$

we have to show that $(IU)S$ is a zero functor; cf. Lemma 91. However, since $IUS \simeq IRU''$, this follows from (I, R) being a pure short exact sequence; cf. (P 3) in Definition 84.

Again by Lemma 91, the condition that $U'J \simeq IU$ determines U' up to isomorphy.

Since I and U are pure monofunctors, so is IU; cf. Lemma 94. Since $IU \simeq U'J$, so is $U'J$; cf. Corollary 100.(2). Since $U'J$ and J are pure monofunctors, we conclude by Lemma 95 that U' is a pure monofunctor.

To show that $\mathcal{A}' \overset{U'}{\longrightarrow} \mathcal{B}' \overset{V'}{\longrightarrow} \mathcal{B}'$ is a pure short exact sequence, we may verify conditions (P 1, 2, 3a, 3b, 4) from Remark 86.

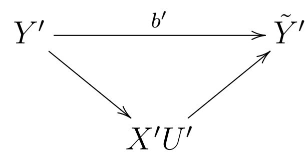
Ad (P 1). The functor U' is full and faithful by (PM 1).

Ad (P 2). The functor V' is full and dense by (PE 1).

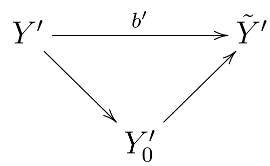
Ad (P 3b). The full additive subcategory $\mathcal{A}'U'$ of $\mathcal{B}'$ is closed under retracts by (PM 1) for U'.

Ad (P 3a). We have $U'V'K \simeq U'JV \simeq IUV$. Since UV is a zero functor, so is IUV and thus $U'V'K$. Since K is full and faithful by (PM 1), it follows that $U'V'$ is a zero functor; cf. Remark 66.(2).

Ad (P 4). Suppose given $Y' \xrightarrow{b'} \tilde{Y}'$ in $\mathcal{B}'$ such that $b'V' = 0$. We have to show the existence of a commutative triangle as follows, for some $X' \in \mathrm{Ob}(\mathcal{A}')$.



By (PE 2) for V', we may choose $Y'_0 \in \mathrm{Ob}(\mathrm{Kern}(V'))$ and a commutative triangle as follows.



It suffices to show that $Y'_0 \overset{!}{\in} \mathrm{Ob}(\mathcal{A}'U')$.

Since Y'_0V' is zero, so is $Y'_0V'K \simeq (Y'_0J)V$.

Since (U, V) is pure short exact, we may choose $X \in \mathrm{Ob}(\mathcal{A})$ such that $XU \simeq Y'_0J$; cf. (P 3).

So $XRU'' \simeq XUS \simeq Y'_0JS$, which is zero since (J, S) is pure short exact; cf. (P 3).

Since U'' is full and faithful by (PM 1), we conclude that XR is zero; cf. Remark 66.(1).

Since (I, R) is pure short exact, we conclude by (P 3) that we may choose $X' \in \mathrm{Ob}(\mathcal{A}')$ such that $X'I \simeq X$.

So $Y'_0J \simeq XU \simeq X'IU \simeq X'U'J$. Since J is full and faithful by (PM 1), we conclude that $Y'_0 \simeq X'U'$. Hence $Y'_0 \in \mathrm{Ob}(\mathcal{A}'U')$.

Ad (2). To show that we may choose an additive functor T such that

$$VT \simeq SV'' ,$$

we have to show that $U(SV'')$ is a zero functor; cf. Lemma 90. However, since $USV'' \simeq RU''V''$, this follows from (U'', V'') being a pure short exact sequence; cf. (P 3) in Definition 84.

Again by Lemma 90, the condition that $VT \simeq SV''$ determines T up to isomorphy.

Since S and V'' are pure epifunctors, so is SV'' ; cf. Lemma 98. Since $VT \simeq SV''$, so is VT ; cf. Corollary 100.(3). Since VT and V are pure epifunctors, we conclude by Lemma 99 that T is a pure epifunctor.

We choose an isotransformation $m : VT \overset{\sim}{\Rightarrow} SV''$.

We choose an isotransformation $n : RU'' \overset{\sim}{\Rightarrow} US$.

We choose an isotransformation $p : JV \xrightarrow{\sim} V'K$.

To show that $\mathcal{C}' \xrightarrow{K} \mathcal{C} \xrightarrow{T} \mathcal{C}''$ is a pure short exact sequence, we may verify conditions (P 1, 2, 3a, 3b, 4) from Remark 86.

Ad (P 1). The functor K is full and faithful by (PM 1).

Ad (P 2). The functor T is full and dense by (PE 1).

Ad (P 3a). We have to show that KT is a zero functor. Suppose given $Z' \in \mathrm{Ob}(\mathcal{C}')$. We have to show that $Z'KT$ is a zero object.

Since V' is dense by (PE 1), we may choose $Y' \in \mathrm{Ob}(\mathcal{B}')$ with $Y'V' \simeq Z'$. Then

$$Z'KT \;\simeq\; Y'V'KT \;\simeq\; Y'JVT \;\simeq\; Y'JSV'' ,$$

which is a zero object since JS is a zero functor.

Ad (P 3b). The full additive subcategory $\mathcal{C}'K \subseteq \mathcal{C}$ is closed under retracts by (PM 2) for K.

Ad (P 4). Suppose given $Z \xrightarrow{c} \tilde{Z}$ in $\mathcal{C}$ such that $cT = 0$. We have to find a commutative triangle

$$\begin{array}{ccc} Z & \xrightarrow{c} & \tilde{Z} \\ & \searrow \quad \nearrow & \\ & Z'K & \end{array}$$

with $Z' \in \mathrm{Ob}(\mathcal{C}')$.

Since V is dense by (P 2), we may choose $Y, \tilde{Y} \in \mathrm{Ob}(\mathcal{B})$ and isomorphisms $YV \overset{f}{\underset{\sim}{\to}} Z$ and $\tilde{Y}V \overset{\tilde{f}}{\underset{\sim}{\to}} \tilde{Z}$ in $\mathcal{C}$.

Since V is full, we may choose $Y \xrightarrow{b} \tilde{Y}$ with $bV = f \cdot c \cdot \tilde{f}^{-}$. So the following quadrangle commutes.

$$\begin{array}{ccc} YV & \xrightarrow{bV} & \tilde{Y}V \\ {\scriptstyle f}\downarrow\wr & & \wr\downarrow{\scriptstyle \tilde{f}} \\ Z & \xrightarrow{c} & \tilde{Z} \end{array}$$

We apply T to obtain the following commutative quadrangle.

$$\begin{array}{ccc} YVT & \xrightarrow{bVT} & \tilde{Y}VT \\ {\scriptstyle fT}\downarrow\wr & & \wr\downarrow{\scriptstyle \tilde{f}T} \\ ZT & \xrightarrow{cT} & \tilde{Z}T \end{array}$$

Since $cT = 0$, we conclude that $bVT = 0$. We have a commutative quadrangle as follows.

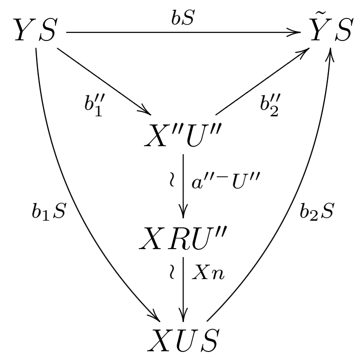
$$\begin{array}{ccc} YVT & \xrightarrow{bVT} & \tilde{Y}VT \\ {\scriptstyle Ym}\downarrow\wr & & \wr\downarrow{\scriptstyle \tilde{Y}m} \\ YSV'' & \xrightarrow{bSV''} & \tilde{Y}SV'' \end{array}$$

Hence $(bS)V'' = 0$. Since (U'', V'') is pure short exact, (P 4) gives a commutative triangle in $\mathcal{B}''$ as follows, where $X'' \in \mathrm{Ob}(\mathcal{A}'')$.

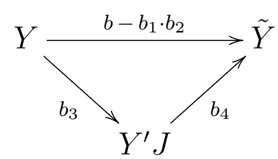
$$\begin{array}{ccc} YS & \xrightarrow{bS} & \tilde{Y}S \\ & {\scriptstyle b''_1}\searrow \quad \nearrow{\scriptstyle b''_2} & \\ & X''U'' & \end{array}$$

By (PE 1), R is dense. Hence we may choose $X \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $XR \overset{a''}{\underset{\sim}{\longrightarrow}} X''$ in $\mathcal{A}''$.

Since S is full, we may choose $Y \xrightarrow{b_1} XU$ and $XU \xrightarrow{b_2} \tilde{Y}$ such that the following diagram commutes.

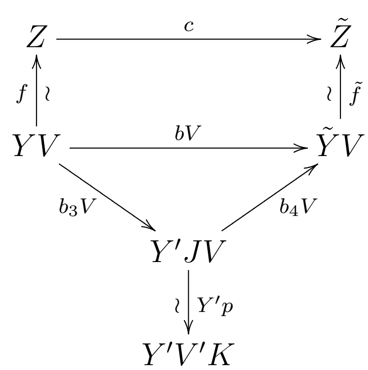


Hence $(b - b_1 \cdot b_2)S = bS - b_1S \cdot b_2S = 0$. Since (J, S) is pure short exact, (P 4) gives a commutative triangle in $\mathcal{B}$ as follows, where $Y' \in \mathrm{Ob}(\mathcal{B}')$.



So $b = b_1 \cdot b_2 + b_3 \cdot b_4$. Hence $bV = b_1V \cdot b_2V + b_3V \cdot b_4V$; cf. Lemma 50.(2).

Now $YV \xrightarrow{b_1V} XUV \xrightarrow{b_2V} \tilde{Y}V$. Since XUV is a zero object, we conclude that $b_1V \cdot b_2V = 0$. Hence $bV = b_3V \cdot b_4V$. We obtain the following commutative diagram.

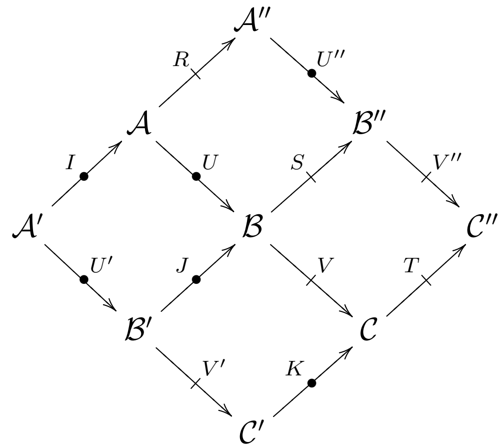


So we may let $Z' := Y'V'$ and obtain the factorisation $c = (f^{-} \cdot b_3V \cdot Y'p) \cdot (Y'p^{-} \cdot b_4V \cdot \tilde{f})$ over $Z'K$.

Ad (3). Since $\mathcal{A} \xrightarrow{R} \mathcal{A}''$ is a pure epifunctor, we may choose a pure short exact sequence $\mathcal{A}' \xrightarrow{I} \mathcal{A} \xrightarrow{R} \mathcal{A}''$; cf. Definition 84.

Since $\mathcal{A}'' \xrightarrow{U''} \mathcal{B}''$ is a pure monofunctor, we may choose a pure short exact sequence $\mathcal{A}'' \xrightarrow{U''} \mathcal{B}'' \xrightarrow{V''} \mathcal{C}''$; cf. Definition 84.

The assertion now follows from (1) and (2).



□

Chapter 3

Provisional notion of an exact 2-category

We subsume the properties of AddCat found in §2.11 under a notion of a provisional exact 2-category, in the style of the notion of an exact category; cf. [**3,** §2, first def.]; cf. also [**1,** Ex. 3.11].

Definition 104. Suppose given a 2-category $\mathfrak{K}$.

Suppose given a set $\mathfrak{P}$ of diagrams of the form $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$ in $\mathfrak{K}$, consisting of objects and 1-morphisms.

Its elements are called *pure short exact sequences.*

A 1-morphism $\mathcal{A}' \xrightarrow{F} \mathcal{A}$ in $\mathfrak{K}$ is called a *pure* 1-*monomorphism* if there exists a pure short exact sequence of the form $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$.

To state that $\mathcal{A}' \xrightarrow{F} \mathcal{A}$ is a pure 1-monomorphism, we write $\mathcal{A}' \overset{F}{\bullet\!\!\rightarrow} \mathcal{A}$.

A 1-morphism $\mathcal{A} \xrightarrow{G} \mathcal{A}''$ in $\mathfrak{K}$ is called a *pure* 1-*epimorphism* if there exists a pure short exact sequence of the form $\mathcal{A}' \xrightarrow{F} \mathcal{A} \xrightarrow{G} \mathcal{A}''$.

To state that $\mathcal{A} \xrightarrow{G} \mathcal{A}''$ is a pure 1-epimorphism, we write $\mathcal{A} \overset{G}{-\!\!\!+\!\!\!\rightarrow} \mathcal{A}''$.

The 2-category $\mathfrak{K}$, together with the set of diagrams $\mathfrak{P}$, is called a *provisional exact* 2-*category*, provided the following properties (PEx 1–7) hold.

(PEx 1) (1) Given $\mathcal{A}$, $\mathcal{B} \in \mathrm{Ob}(\mathfrak{K})$, the category ${}_{\mathfrak{K}}(\mathcal{A}, \mathcal{B})$ is additive.

(2) Given 1-morphisms $\mathcal{A}' \xrightarrow{F} \mathcal{A}$ and $\mathcal{B} \xrightarrow{G} \mathcal{B}'$, the functor ${}_{\mathfrak{K}}(\mathcal{A}, \mathcal{B}) \xrightarrow{{}_{\mathfrak{K}}(F,G)} {}_{\mathfrak{K}}(\mathcal{A}', \mathcal{B}')$ is additive.

(3) There exists a zero object in $\mathfrak{K}$; cf. Definition 11.

(PEx 2) Suppose given a diagram in $\mathfrak{K}$ as follows.

$$\begin{array}{ccccc} \mathcal{A}' & \xrightarrow{F'} & \mathcal{A} & \xrightarrow{F''} & \mathcal{A}'' \\ {\scriptstyle U'}\downarrow & & {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U''} \\ \mathcal{B}' & \xrightarrow{G'} & \mathcal{B} & \xrightarrow{G''} & \mathcal{B}'' \end{array}$$

Suppose that U', U and U'' are 1-isomorphisms; cf. Definition 6.

Suppose that $U' \overset{\mathfrak{K}}{*} G' \simeq F' \overset{\mathfrak{K}}{*} U$ and $U \overset{\mathfrak{K}}{*} G'' \simeq F'' \overset{\mathfrak{K}}{*} U''$.

Then the sequence $\mathcal{A}' \xrightarrow{F'} \mathcal{A} \xrightarrow{F''} \mathcal{A}''$ is pure short exact if and only if the sequence $\mathcal{B}' \xrightarrow{G'} \mathcal{B} \xrightarrow{G''} \mathcal{B}''$ is pure short exact.

(PEx 3) Suppose given a pure short exact sequence in $\mathfrak{K}$ as follows.

$$\mathcal{A}' \overset{F}{\bullet\!\!\!\rightarrowtail} \mathcal{A} \overset{G}{\twoheadrightarrow\!\!\!\!\!+} \mathcal{A}''$$

(1) Then $F \overset{\mathfrak{K}}{*} G$ is a zero 1-morphism; cf. Definition 12.

(2) Suppose given $\mathcal{T} \in \mathrm{Ob}(\mathfrak{K})$.

We have induced functors

$$_{\mathfrak{K}}(\mathcal{T}, \mathcal{A}') \xrightarrow{_{\mathfrak{K}}(\mathcal{T},F)} {_{\mathfrak{K}}(\mathcal{T}, \mathcal{A})} \xrightarrow{_{\mathfrak{K}}(\mathcal{T},G)} {_{\mathfrak{K}}(\mathcal{T}, \mathcal{A}'')}$$

We have a full additive subcategory $\mathrm{Kern}({_{\mathfrak{K}}(\mathcal{T}, G)}) \subseteq {_{\mathfrak{K}}(\mathcal{T}, \mathcal{A})}$.
We obtain a functor

$$_{\mathfrak{K}}(\mathcal{T}, \mathcal{A}') \xrightarrow{_{\mathfrak{K}}(\mathcal{T},F)|^{\mathrm{Kern}(_{\mathfrak{K}}(\mathcal{T},G))}} \mathrm{Kern}({_{\mathfrak{K}}(\mathcal{T}, G)})\ .$$

This functor is an equivalence.

(3) Suppose given $\mathcal{T} \in \mathrm{Ob}(\mathfrak{K})$.

We have induced functors

$$_{\mathfrak{K}}(\mathcal{A}', \mathcal{T}) \xleftarrow{_{\mathfrak{K}}(F,\mathcal{T})} {_{\mathfrak{K}}(\mathcal{A}, \mathcal{T})} \xleftarrow{_{\mathfrak{K}}(G,\mathcal{T})} {_{\mathfrak{K}}(\mathcal{A}'', \mathcal{T})}$$

We have a full additive subcategory $\mathrm{Kern}({_{\mathfrak{K}}(F, \mathcal{T})}) \subseteq {_{\mathfrak{K}}(\mathcal{A}, \mathcal{T})}$.

We obtain a functor

$$_{\mathfrak{K}}(\mathcal{A}'', \mathcal{T}) \xrightarrow{_{\mathfrak{K}}(G,\mathcal{T})|^{\mathrm{Kern}(_{\mathfrak{K}}(F,\mathcal{T}))}} \mathrm{Kern}({_{\mathfrak{K}}(F, \mathcal{T})})\ .$$

This functor is an equivalence.

(PEx 4) (1) Suppose given $\mathcal{C} \overset{F}{\bullet\!\!\!\rightarrowtail} \mathcal{D}$ and $\mathcal{D} \overset{G}{\bullet\!\!\!\rightarrowtail} \mathcal{E}$ in $\mathfrak{K}$.

Then $\mathcal{C} \overset{F \overset{\mathfrak{K}}{*} G}{\bullet\!\!\!\rightarrowtail} \mathcal{E}$.

(2) Suppose given $\mathcal{C} \overset{F}{\twoheadrightarrow} \mathcal{D}$ and $\mathcal{D} \overset{G}{\twoheadrightarrow} \mathcal{E}$ in $\mathfrak{K}$.

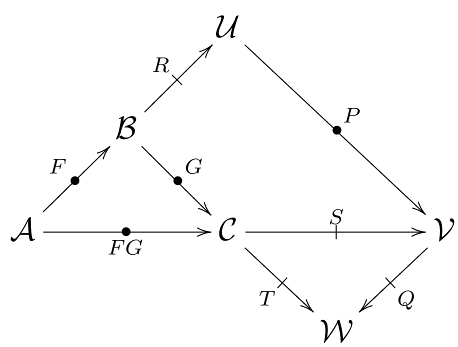
Then $\mathcal{C} \overset{F \overset{\mathfrak{K}}{*} G}{\twoheadrightarrow} \mathcal{E}$.

(PEx 5) (1) Suppose given pure short exact sequences $\mathcal{A} \overset{F}{\rightarrowtail} \mathcal{B} \overset{R}{\twoheadrightarrow} \mathcal{U}$ and $\mathcal{A} \overset{FG}{\rightarrowtail} \mathcal{C} \overset{S}{\twoheadrightarrow} \mathcal{V}$ and $\mathcal{B} \overset{G}{\rightarrowtail} \mathcal{C} \overset{T}{\twoheadrightarrow} \mathcal{W}$ in $\mathfrak{K}$.

By (PEx 3.3), we may choose a 1-morphism $\mathcal{U} \xrightarrow{P} \mathcal{V}$ such that $RP \simeq GS$. This condition determines P up to isomorphy.

By (PEx 3.3), we may choose a 1-morphism $\mathcal{V} \xrightarrow{Q} \mathcal{W}$ such that $SQ \simeq T$. This condition determines Q up to isomorphy.

Then we have the pure short exact sequence $\mathcal{U} \overset{P}{\rightarrowtail} \mathcal{V} \overset{Q}{\twoheadrightarrow} \mathcal{W}$.

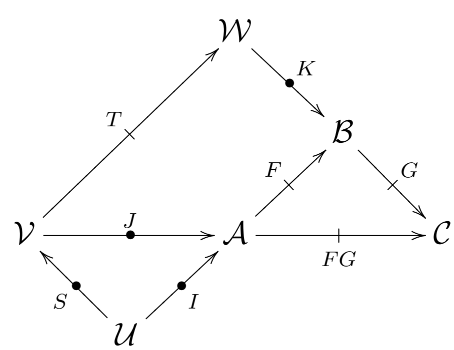


(2) Suppose given pure short exact sequences $\mathcal{U} \overset{I}{\rightarrowtail} \mathcal{A} \overset{F}{\twoheadrightarrow} \mathcal{B}$ and $\mathcal{V} \overset{J}{\rightarrowtail} \mathcal{A} \overset{FG}{\twoheadrightarrow} \mathcal{C}$ and $\mathcal{W} \overset{K}{\rightarrowtail} \mathcal{B} \overset{G}{\twoheadrightarrow} \mathcal{C}$.

By (PEx 3.2), we may choose a 1-morphism $\mathcal{U} \xrightarrow{S} \mathcal{V}$ such that $SJ \simeq I$. This condition determines S up to isomorphy.

By (PEx 3.2), we may choose a 1-morphism $\mathcal{V} \xrightarrow{T} \mathcal{W}$ such that $TK \simeq JF$. This condition determines T up to isomorphy.

Then we have the pure short exact sequence $\mathcal{U} \overset{S}{\rightarrowtail} \mathcal{V} \overset{T}{\twoheadrightarrow} \mathcal{W}$.



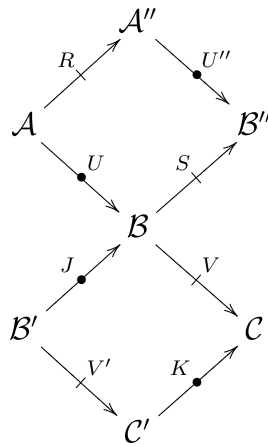
(PEx 6) (1) Suppose given 1-morphisms $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$.

Suppose that G and $F \overset{\mathfrak{K}}{*} G$ are pure 1-monomorphisms.

Then F is a pure 1-monomorphism.

(2) Suppose given 1-morphisms $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$.
Suppose that F and $F \overset{\mathfrak{K}}{*} G$ are pure 1-epimorphisms.
Then G is a pure 1-epimorphism.

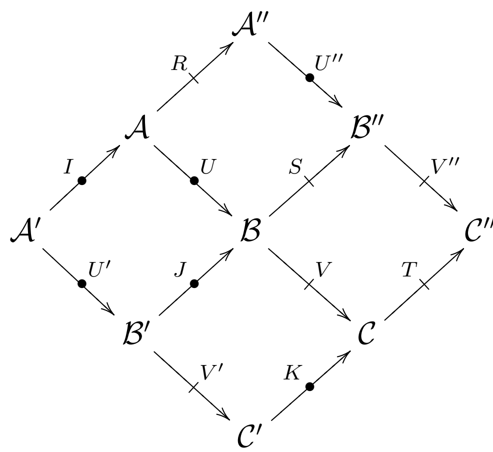
(PEx 7) Suppose given the following diagram in AddCat.



Suppose that $US \simeq RU''$. Suppose that $V'K \simeq JV$.

Suppose that $\mathcal{B}' \xrightarrow{J} \mathcal{B} \xrightarrow{S} \mathcal{B}''$ is a pure short exact sequence.

Suppose that $\mathcal{A} \xrightarrow{U} \mathcal{B} \xrightarrow{V} \mathcal{C}$ is a pure short exact sequence.
Then there exists a diagram in $\mathfrak{K}$ as follows, in which we require $IU \simeq U'J$ and $VT \simeq SV''$.



Moreover, $\mathcal{A}' \xrightarrow{U'} \mathcal{B}' \xrightarrow{V'} \mathcal{C}'$, $\mathcal{A}' \xrightarrow{I} \mathcal{A} \xrightarrow{R} \mathcal{A}''$, $\mathcal{C}' \xrightarrow{K} \mathcal{C} \xrightarrow{T} \mathcal{C}''$, $\mathcal{A}'' \xrightarrow{U''} \mathcal{B}'' \xrightarrow{V''} \mathcal{C}''$ are required to be pure short exact sequences.

Remark 105.

(1) As to (PEx 1) in Defintion 104, one might additionally require direct sums of pairs of objects in $\mathfrak{K}$ in a suitable sense.

(2) Also of interest would be a notion of exact 2-functors and of exact 2-transformations. What about modifications?

(3) Do abelian categories, with localisation sequences being pure short exact, form an exact 2-category? Same question for triangulated categories with localisation sequences and with recollement sequences.

(4) Putting $\mathcal{C}'$ to be zero in (PEx 7), we obtain a diagram as in (PEx 5). There might be further connections.

Remark 106. *The* 2*-category*

$$\text{AddCat}\ ,$$

together with pure short exact sequences, is a provisional exact 2*-category.*

Cf. Definitions 68*,* 84*,* 104*.*

Proof.

Ad (PEx 1).

Ad (PEx 1.1). See Remark 72.(1).

Ad (PEx 1.2). See Remark 73.

Ad (PEx 1.3). See Remark 70.

Ad (PEx 2). See Lemma 89.

Ad (PEx 3).

Ad (PEx 3.1). See Remark 71 and Remark 86, (P 3a).

Ad (PEx 3.2). See Lemma 91.

Ad (PEx 3.3). See Lemma 90.

Ad (PEx 4).

Ad (PEx 4.1). See Lemma 94.

Ad (PEx 4.2). See Lemma 98.

Ad (PEx 5).

Ad (PEx 5.1). See Lemma 101.

Ad (PEx 5.2). See Lemma 102.

Ad (PEx 6).

Ad (PEx 6.1). See Lemma 95.

Ad (PEx 6.2). See Lemma 99.

Ad (PEx 7). See Lemma 103.(3). □

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[1] BÜHLER, T., *Exact categories,* Exp. Math. 28.(1), p. 1–69, 2010.

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Zusammenfassung

Additive Kategorien und additive Funktoren

Eine additive Kategorie $\mathcal{A}$ ist eine Kategorie mit Nullobjekt, in welcher jedes Paar von Objekten eine direkte Summe besitzt. Damit erhält man eine assoziative und kommutative Addition auf der Menge der Morphismen ${}_{\mathcal{A}}(X, X')$ zwischen festen Objekten X und X'. Wir verlangen für jede Identität die Existenz eines additiv Inversen, indem wir den Isomorphismus $\begin{pmatrix} \mathrm{id}_X & 0 \\ \mathrm{id}_X & \mathrm{id}_X \end{pmatrix}$ fordern für $X \in \mathrm{Ob}(\mathcal{A})$.

Infolgedessen ist in einer additiven Kategorie $\mathcal{A}$ die Menge der Morphismen ${}_{\mathcal{A}}(X, X')$ eine abelsche Gruppe. Komposition ist distributiv bezüglich der Addition.

Für additive Kategorien $\mathcal{A}$ und $\mathcal{B}$ heißt ein Funktor $F : \mathcal{A} \to \mathcal{B}$ additiv, falls er Nullobjekte auf Nullobjekte abbildet und kompatibel ist mit direkten Summen von Paaren von Objekten von $\mathcal{A}$.

Äquivalent hierzu ist ein Funktor additiv, falls er Addition von Morphismen respektiert.

Rein kurz exakte Sequenzen

Seien $\mathcal{A}'$, $\mathcal{A}$ und $\mathcal{A}''$ additive Kategorien.

Seien $\mathcal{A}' \overset{F}{\to} \mathcal{A} \overset{G}{\to} \mathcal{A}''$ additive Funktoren.

Das volle Bild von F ist die volle additive Teilkategorie $\mathrm{Im}(F) \subseteq \mathcal{A}$ mit

$$\mathrm{Ob}(\mathrm{Im}(F)) \;=\; \{\, X \in \mathrm{Ob}(\mathcal{A}) \,:\, X \simeq X'F \text{ für ein } X' \in \mathrm{Ob}(\mathcal{A}') \,\} \,.$$

Der Kern von G ist die volle additive Teilkategorie $\mathrm{Kern}(G) \subseteq \mathcal{A}$ mit

$$\mathrm{Ob}(\mathrm{Kern}(G)) \;=\; \{\, X \in \mathrm{Ob}(\mathcal{A}) \,:\, XG \text{ ist ein Nullobjekt in } \mathcal{A}'' \,\} \,.$$

Die Sequenz $\mathcal{A}' \overset{F}{\to} \mathcal{A} \overset{G}{\to} \mathcal{A}''$ heißt rein kurz exakt, falls (P 1–4) gelten.

(P 1) Der Funktor F ist voll und treu.

(P 2) Der Funktor G ist voll und dicht.

(P 3) Es ist $\mathrm{Im}(F) = \mathrm{Kern}(G)$.

(P 4) Sei $X \overset{u}{\to} \tilde{X}$ ein Morphismus in $\mathcal{A}$ mit $uG = 0$. Dann gibt es ein $Z' \in \mathrm{Ob}(\mathcal{A}')$ und Morphismen $X \overset{a}{\to} Z'F \overset{\tilde{a}}{\to} \tilde{X}$ mit $a \cdot \tilde{a} = u$.

Ein Funktor $\mathcal{A}' \overset{F}{\to} \mathcal{A}$ heißt reiner Monofunktor, falls es eine rein kurz exakte Sequenz $\mathcal{A}' \overset{F}{\to} \mathcal{A} \overset{G}{\to} \mathcal{A}''$ gibt. Um F als reinen Monofunktor zu kennzeichnen, schreiben wir oft $\mathcal{A}' \overset{F}{\bullet\!\!\rightarrowtail} \mathcal{A}$.

Ein Funktor $\mathcal{A} \overset{G}{\to} \mathcal{A}''$ heißt reiner Epifunktor, falls es eine rein kurz exakte Sequenz $\mathcal{A}' \overset{F}{\to} \mathcal{A} \overset{G}{\to} \mathcal{A}''$ gibt. Um G als reinen Epifunktor zu kennzeichnen, schreiben wir oft $\mathcal{A} \overset{G}{\twoheadrightarrow} \mathcal{A}''$.

Zum Beispiel haben wir für eine additive Kategorie $\mathcal{A}$ und eine volle additive Teilkategorie $\mathcal{N} \subseteq \mathcal{A}$, die unter Retrakten abgeschlossen ist, die rein kurz exakte Sequenz

$$\mathcal{N} \stackrel{I}{\to} \mathcal{A} \stackrel{R}{\to} \mathcal{A}/\mathcal{N} \,,$$

in welcher I den Inklusionsfunktor und R den Restklassenfunktor bezeichnet.

Eigenschaften rein kurz exakter Sequenzen

Wir stellen Eigenschaften kurz exakter Sequenzen zusammen.

Universelle Eigenschaften.

Sei $\mathcal{A}' \stackrel{F}{\to} \mathcal{A} \stackrel{G}{\to} \mathcal{A}''$ eine rein kurz exakte Sequenz.

(1) Der Funktor F hat die universelle Eigenschaft des Kerns von G, bis auf Isomorphie von Funktoren.

(2) Der Funktor G hat die universelle Eigenschaft des Cokerns von F, bis auf Isomorphie von Funktoren.

Kompositionseigenschaften.

Seien $\mathcal{A}$, $\mathcal{B}$ und $\mathcal{C}$ additive Kategorien. Seien

$$\mathcal{A} \stackrel{F}{\to} \mathcal{B} \stackrel{G}{\to} \mathcal{C}$$

additive Funktoren.

(1) Sei G ein reiner Monofunktor.

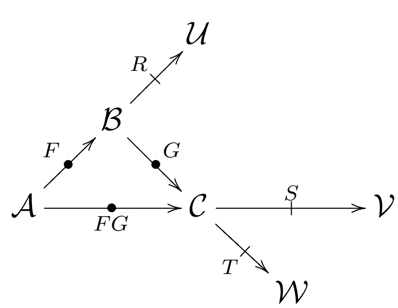
Dann ist F genau dann ein reiner Monofunktor, wenn FG ein reiner Monofunktor ist.

(2) Sei F ein reiner Epifunktor.

Dann ist G genau dann ein reiner Epifunktor, wenn FG ein reiner Epifunktor ist.

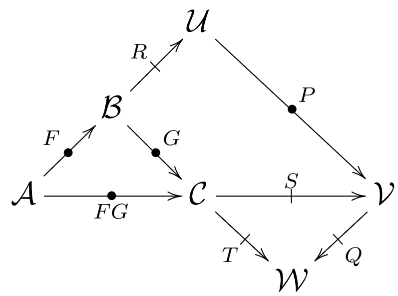
Noethersche Eigenschaften.

(1) Sei folgendes Diagramm additiver Kategorien und additiver Funktoren gegeben.

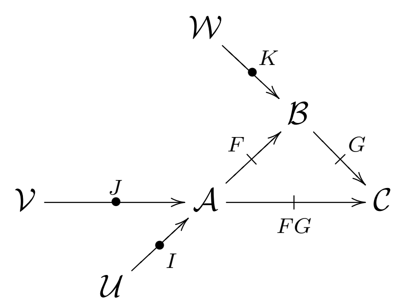


Seien darin $\mathcal{A} \overset{F}{\rightarrowtail} \mathcal{B} \overset{R}{\twoheadrightarrow} \mathcal{U}$ und $\mathcal{A} \overset{FG}{\rightarrowtail} \mathcal{C} \overset{S}{\twoheadrightarrow} \mathcal{V}$ und $\mathcal{B} \overset{G}{\rightarrowtail} \mathcal{C} \overset{T}{\twoheadrightarrow} \mathcal{W}$ rein kurz exakt.

Dann erhalten wir eine rein kurz exakte Sequenz $\mathcal{U} \overset{P}{\rightarrowtail} \mathcal{V} \overset{Q}{\twoheadrightarrow} \mathcal{W}$, die folgendes Diagram bis auf Isomorphie kommutativ macht.

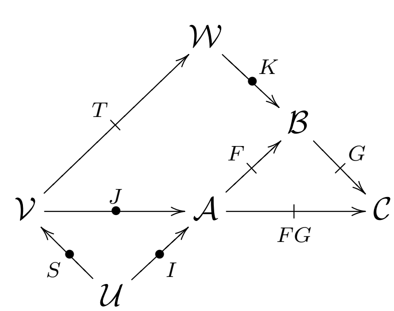


(2) Sei folgendes Diagramm additiver Kategorien und additiver Funktoren gegeben.



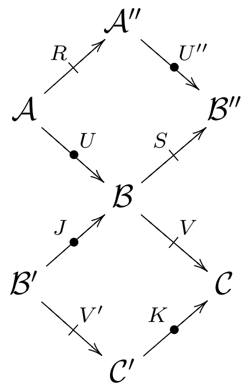
Seien darin $\mathcal{W} \overset{K}{\rightarrowtail} \mathcal{B} \overset{G}{\twoheadrightarrow} \mathcal{C}$ und $\mathcal{V} \overset{J}{\rightarrowtail} \mathcal{A} \overset{FG}{\twoheadrightarrow} \mathcal{C}$ und $\mathcal{U} \overset{I}{\rightarrowtail} \mathcal{A} \overset{F}{\twoheadrightarrow} \mathcal{B}$ rein kurz exakt.

Dann erhalten wir eine rein kurz exakte Sequenz $\mathcal{U} \overset{S}{\rightarrowtail} \mathcal{V} \overset{T}{\twoheadrightarrow} \mathcal{W}$, die folgendes Diagram bis auf Isomorphie kommutativ macht.



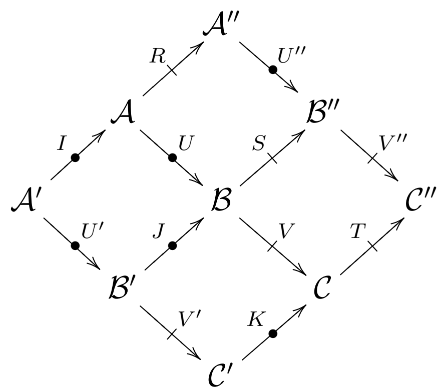
3×3-*Eigenschaft.*

Sei folgendes Diagramm additiver Kategorien und additiver Funktoren gegeben, welches bis auf Isomorphie kommutativ sei.



Seien darin $\mathcal{A} \xrightarrow{U} \mathcal{B} \xrightarrow{V} \mathcal{C}$ und $\mathcal{B}' \xrightarrow{J} \mathcal{B} \xrightarrow{S} \mathcal{B}''$ rein kurz exakt.

Dann gibt es rein kurz exakte Sequenzen $\mathcal{A}' \xrightarrow{U'} \mathcal{B}' \xrightarrow{V'} \mathcal{C}'$ und $\mathcal{A}' \xrightarrow{I} \mathcal{A} \xrightarrow{R} \mathcal{A}''$ und $\mathcal{C}' \xrightarrow{K} \mathcal{C} \xrightarrow{T} \mathcal{C}''$ und $\mathcal{A}'' \xrightarrow{U''} \mathcal{B}'' \xrightarrow{V''} \mathcal{C}''$, die folgendes Diagramm bis auf Isomorphie kommutativ machen.



Versicherung

Hiermit versichere ich,

1. dass ich meine Arbeit selbstständig verfasst habe,
2. dass ich keine anderen als die angegeben Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,
3. dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und
4. dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, 22.04.2022

Chen Zhang