The minimal projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$

Master's Thesis

University of Stuttgart
Germany

Sebastian Nitsche

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Contents

Chapter 0

Introduction

0.1 Minimal projective resolutions

Let G be a finite group. Let R be a commutative ring.

We consider projective resolutions of the trivial module R over RG.

$$\cdots \longrightarrow P_3 \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} R \longrightarrow 0 \longrightarrow \cdots$$

Suppose R is a discrete valuation ring such that $|G|$ is nonzero in R. Then R has a not necessarily minimal projective resolution which can be written as a tensor product of periodic complexes. This is shown by Benson and Carlson in [**4,** Theorems 3.4 and 4.4] and for a more general case in [**3,** Theorem 5.14.5].

Suppose G has a dihedral Sylow 2-subgroup and R is a splitting field of characteristic 2 of G. Then Alperin states in [**1,** Theorem 1] that there exists a double complex of projective indecomposable modules such that the total complex is a minimal projective resolution of R over RG.

Suppose R is a field of prime characteristic p. In [**2,** p. 4] Alperin considers minimal projective resolutions of an RG-module M for a finite group G. If such a minimal projective resolution can be written as a total complex of an n-fold complex with projective entries of bounded dimension, then he calls M a module of bounded complex type.

He gives some examples of regularly behaving double complexes with projective indecomposable modules as entries such that the total complex is a minimal projective resolution of the trivial module [**2,** p. 6-7].

In [**5,** Diagrams 12.15 and 13.4] and [**4,** Figure 1] Benson and Carlson give further examples of such double complexes with regular behavior. In [**3,** Remark on p. 200] Benson notes that in all examples calculated the minimal projective resolution of R can be written as the total complex of an n-fold complex, where all rows, columns, etc. eventually become periodic.

In [**10,** Theorem 30] Hofmann constructs the minimal projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}D_8$, alternatively to the Wall-Hamada resolution constructed in [**9**] and [**14**]. Both show regular behavior.

The minimal projective resolution of $\mathbb{Z}_{(2)}[\zeta_3]$ over $\mathbb{Z}_{(2)}[\zeta_3]\mathrm{A}_4$ has been constructed in [**13,** Theorem 28]. This resolution also shows a regular behavior. Beforehand, Carlson has constructed the first few terms of a minimal projective resolution of $\mathbb{F}_4$ over $\mathbb{F}_4\mathrm{A}_4$ in [**8,** Example on pages 50-51].

We consider the case $G = \mathrm{S}_4$ and aim to construct a minimal projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$ with regular behavior.

0.2 Wedderburn image

Let $\tilde{\omega}$ be the Wedderburn isomorphism $\mathbb{Q}\mathrm{S}_4 \xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{2\times 2}$. We restrict $\tilde{\omega}$ to $\mathbb{Z}\mathrm{S}_4$ to obtain the Wedderburn embedding $\omega'_+ : \mathbb{Z}\mathrm{S}_4 \longrightarrow \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{2\times 2}$.

Restricting ω'_+ to its image Λ', we obtain an isomorphism ω' of $\mathbb{Z}$-algebras; cf. Lemma 13.

$$\begin{array}{ccc} \mathbb{Q}\mathrm{S}_4 & \xrightarrow[\sim]{\tilde{\omega}} & \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{2\times 2} \\ \uparrow & & \uparrow \\ \mathbb{Z}\mathrm{S}_4 & \xrightarrow{\omega'_+} & \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{2\times 2} \\ & {\scriptstyle \omega',\,\sim} \searrow \quad \Lambda' \quad \nearrow & \end{array}$$

By localization at 2 we obtain an isomorphism of $\mathbb{Z}_{(2)}$-algebras

$$\mathbb{Z}_{(2)}\mathrm{S}_4 \xrightarrow[\sim]{\omega} \Lambda := \Lambda'_{(2)} \subseteq \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}^{3\times 3} \times \mathbb{Z}_{(2)}^{3\times 3} \times \mathbb{Z}_{(2)}^{2\times 2}\,;$$

cf. Corollary 15.

We describe the Wedderburn image Λ via congruences of matrix entries. Thus we can read off an orthogonal decomposition of 1_Λ into primitive idempotents $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ to obtain the projective indecomposable modules $P := \Lambda\mathcal{E}_0$ and $Q := \Lambda\mathcal{E}_1 \simeq \Lambda\mathcal{E}_2$ of $\mathbb{Z}_{(2)}\mathrm{S}_4$; cf. Definition 18 in §2.3.

Furthermore, we define certain $\mathbb{Z}_{(2)}\mathrm{S}_4$-linear maps between the projective indecomposable modules as multiplication with suitable elements of Λ; cf. Lemma 20 in §2.3.

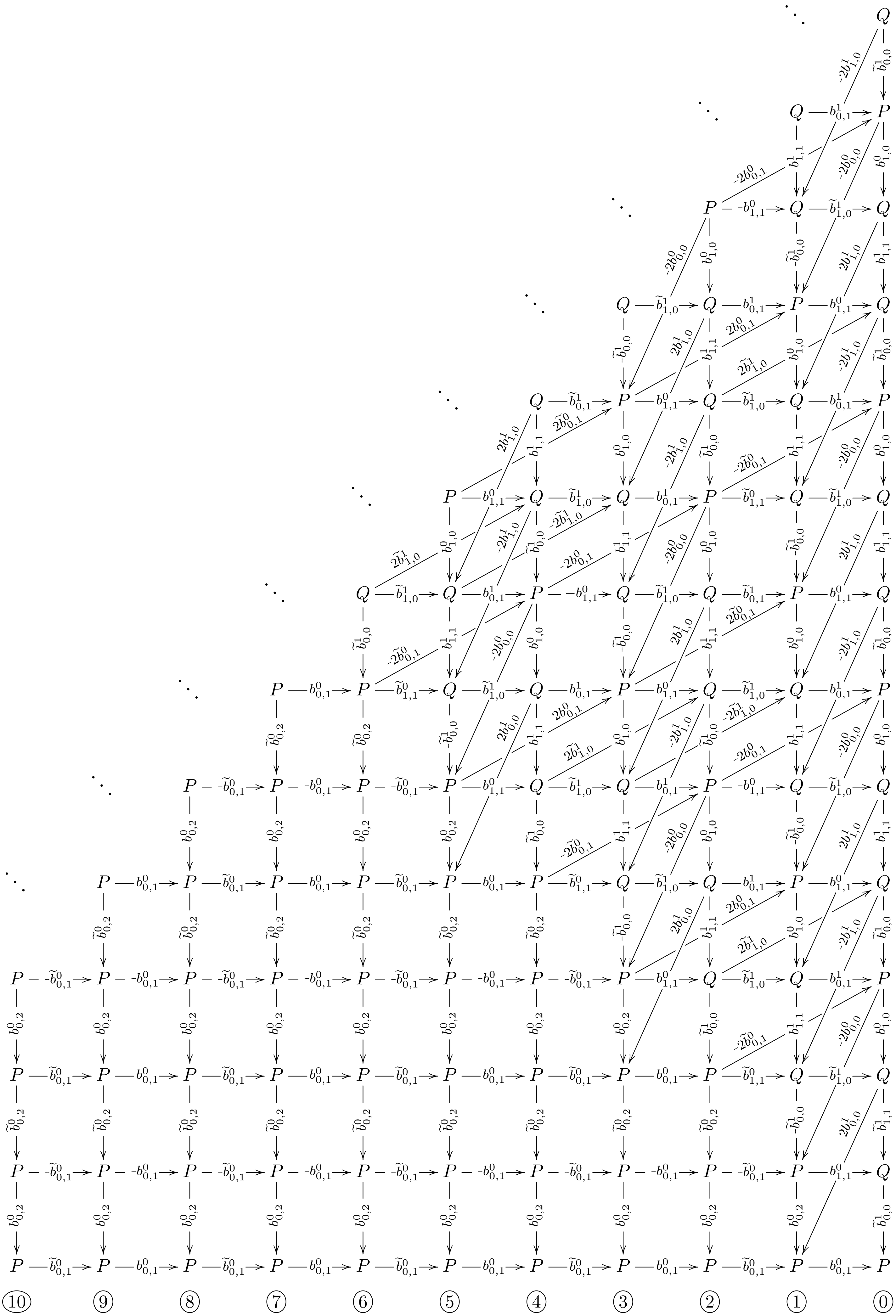
We shall make use of these preparations to construct the minimal projective resolutions of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$.

0.3 Projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$

In §2 we construct the minimal projective resolution X of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$; cf. Theorem 38.

$$X = \Big(\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \longrightarrow 0 \longrightarrow \cdots \Big)$$

The projective resolution can be visualized in a diagram, which is similar to a double complex but with extra maps. The lower right corner of this diagram can be depicted as follows.



The diagram consists of the projective indecomposable modules P and Q and maps $\Lambda\mathcal{E}_i \xrightarrow{b^i_{j,k}} \Lambda\mathcal{E}_j$ between them as prepared in §2.3.

Then X is a total complex of this diagram, but with the extra maps included. I.e. X_k is the direct sum over the modules in the kth diagonal. Moreover, the differential d_k is a matrix consisting of the maps between the $(k+1)$st and kth diagonal.

Note that the terms $X_0, \ldots, X_{10}$ are fully visible in the depicted part of the diagram as sums over the respective diagonals.

This diagram shows an eventually periodic behavior. For instance, the maps in column number 0 eventually repeat with period length 3:

$$\begin{array}{llllll} \widetilde{b}^1_{0,0}, & \widetilde{b}^1_{1,1}, & b^0_{1,0}, & \widetilde{b}^1_{0,0}, & b^1_{1,1}, & \\ & & b^0_{1,0}, & \widetilde{b}^1_{0,0}, & b^1_{1,1}, & \ldots \end{array}$$

Similarly the diagonal maps from column number 2 to column number 0 eventually repeat with period length 6:

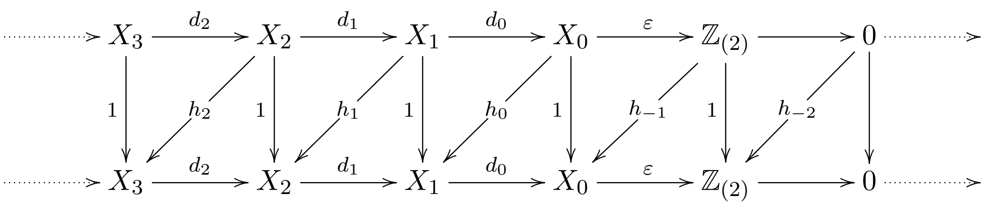
$$\begin{array}{lllllllll} 0, & 0, & -2\widetilde{b}^0_{0,1}, & 2\widetilde{b}^1_{1,0}, & 0, & -2b^0_{0,1}, & 0, & 0, & \\ & & -2\widetilde{b}^0_{0,1}, & 2\widetilde{b}^1_{1,0}, & 0, & -2b^0_{0,1}, & \ldots & & \end{array}$$

Note that minimal projective resolutions are unique up to isomorphism in the sense of Lemma 11. We give a representative in this isomorphy class with projective terms and differentials which show a regular behavior.

So we obtain a closed formula for the projective terms X_k and the differentials d_k. Each formula is divided into four cases, depending on k modulo 4; cf. Definition 24 and Definition 28.

By reduction modulo 2 we obtain a double complex since all diagonal maps vanish. This is in accordance with Alperin's Theorem [**2,** Theorem 1]; cf. §0.1.

To verify that X is in fact a projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$ we construct $\mathbb{Z}_{(2)}$-linear homotopy maps h_k. These homotopy maps also show a regular behavior; cf. Definition 33.



0.4 Cohomology of S_4 at 2

In §3 we calculate the 2-part of the cohomology of S_4. To do so, we calculate the cohomology of S_4 over $\mathbb{Z}_{(2)}$ using the minimal projective resolution X of §2.

For $l \in \mathbb{Z}_{\geq 0}$ we write $\underline{\underline{l}}$ for the unique element in $\mathbb{Z}$ with $l = 3\underline{\underline{l}} + \overline{\overline{l}}$, where $\overline{\overline{l}} \in [0, 2]$.

We obtain the following cohomology groups; cf. Theorem 54.

$$
\begin{aligned}
\mathrm{H}^{0}(\mathrm{S}_4)_{(2)} &\simeq \mathbb{Z}_{(2)} \\
\mathrm{H}^{4l}(\mathrm{S}_4)_{(2)} &\simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus l+\underline{\underline{l}}} \ \oplus\ (\mathbb{Z}/4\mathbb{Z}) \text{ if } l \geq 1 \\
\mathrm{H}^{4l+1}(\mathrm{S}_4)_{(2)} &\simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus l+1+\underline{\underline{l-2}}} \\
\mathrm{H}^{4l+2}(\mathrm{S}_4)_{(2)} &\simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus l+2+\underline{\underline{l-1}}} \\
\mathrm{H}^{4l+3}(\mathrm{S}_4)_{(2)} &\simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus l+1+\underline{\underline{l}}}
\end{aligned}
$$

In [**15,** Theorem 4] Thomas gives a description of the cohomology ring of S_4 as a graded commutative polynomial ring modulo an ideal generated by certain elements.

In order to compare the results, we use the computer algebra system Magma to derive from Thomas's results the cohomology groups $\mathrm{H}^n(\mathrm{S}_4)$ for $n \in [0, 115]$.

In this range, we could confirm that both ways to calculate $\mathrm{H}^n(\mathrm{S}_4)_{(2)}$ yield the same result; cf. Remark 55.

0.5 Acknowledgments

I would like to thank my advisor Matthias Künzer for the very helpful and constructive discussions and for the time he spend on reviewing this thesis.

0.6 Conventions

Let X, Y and Z be sets. Let A be a commutative ring.

- Let $a, b \in \mathbb{Z}$. We write $[a, b] := \{i \in \mathbb{Z} : a \leq i \leq b\}$.
- Given $x, y \in X$, let $\delta_{x,y} = 1$ for $x = y$ and $\delta_{x,y} = 0$ for $x \neq y$.
- We write maps on the right. That is, given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $x \in X$ we denote the image of x under f by xf and the composite of f and g by $X \xrightarrow{fg} Z$.
- Let $X \xrightarrow{f} Y$ be a map. Given $X' \subseteq X$ and $Y' \subseteq Y$ such that $X'f \subseteq Y'$, we write $f|_{X'}^{Y'}$ for the restriction of f to X' and Y'. In the case of $Y = Y'$ we also write $f|_{X'} := f|_{X'}^{Y}$. In the case of $X = X'$ we also write $f|^{Y'} := f|_{X}^{Y'}$.
- Let $X' \subseteq X$ be a subset. We often write $X' \hookrightarrow X$ for the embedding.
- Let $a, b, z \in A$. We write $a \equiv_z b$ if there exists a $c \in A$ such that $a - b = cz$.
- By an A-module we understand a left A-module, if not specified otherwise.
- Let M be an A-module. We often write $1 = 1_M$ for the identity map id_M on M.
- Let M be an A-module. We often write $N \leq M$ to indicate that N is a submodule of M. Moreover, $N < M$ indicates that $n \leq M$ and $N \neq M$.

- Let $k, l \in \mathbb{N}$. Suppose given A-modules M_i for $i \in [1, k]$ and A-modules N_j for $j \in [1, l]$. Given $m_i \in M_i$ for $i \in [1, k]$ and $n_j \in N_j$ for $j \in [1, l]$ we often write $\big((m_i)_{i\in[1,k]}, (n_j)_{j\in[1,l]}\big)$ for the element $(m_1, \ldots, m_k, n_1, \ldots, n_l) \in M^{\oplus k} \oplus N^{\oplus l}$.

- Let M be an A-module. We write $\operatorname{rad}(M) := \bigcap\{N : N < M \text{ is maximal}\}$ for the radical of M.

- Let e and f be two idempotent elements of A. We identify along

$$\begin{aligned} \operatorname{Hom}_A(Ae, Af) &\xrightarrow{\sim} eAf \\ \varphi &\longmapsto e\varphi \\ (\mu e \mapsto \mu e\lambda f) &\longleftarrow e\lambda f. \end{aligned}$$

- Let $n \in \mathbb{N}$. The standard A-linear basis $(\mathrm{e}_i)_{i\in[1,n]}$ of $A^{\oplus n}$ consists of the tuples $\mathrm{e}_i \in A^{\oplus n}$ having the entry 1 at position i and entry 0 elsewhere.

- Let $n, m \in \mathbb{N}$. We denote the A-module of $n \times m$ matrices over A by $A^{n\times m}$. In the case of $n = m$ we denote the identity matrix of $A^{n\times n}$ by I_n.

- Let $n \in \mathbb{N}$. We identify A-linear maps $A^{\oplus n} \xrightarrow{f} A^{\oplus n}$ with their matrix representation in $A^{n\times n}$ with respect to the standard A-linear basis of $A^{\oplus n}$.

- Let $n, m \in \mathbb{N}$. The standard A-linear basis $(\mathrm{e}_{i,j})_{i\in[1,n],\, j\in[1,m]}$ of $A^{n\times m}$ consists of the $n \times m$ matrices $\mathrm{e}_{i,j}$ having the entry 1 at position (i, j) and entry 0 elsewhere.

- Let $n \in \mathbb{N}$ and $k_r \in \mathbb{N}$ for $r \in [1, n]$. The standard A-linear basis of $\prod\limits_{r\in[1,n]} A^{k_r\times k_r}$ consists of $(\mathrm{e}_{r;i,j})_{r\in[1,n],\, i,j\in[1,k_r]}$, where $\mathrm{e}_{r;i,j}$ is the tuple of matrices, whose matrix at position r has the entry 1 at position (i, j), and whose other matrix entries are 0. We choose the ordering

  $\mathrm{e}_{1;1,1}, \quad \mathrm{e}_{1;1,2}, \quad \ldots, \quad \mathrm{e}_{1;1,k_1}, \quad \mathrm{e}_{1;2,1}, \quad \ldots, \quad \mathrm{e}_{1;k_1,k_1}, \quad \mathrm{e}_{2;1,1}, \quad \ldots, \quad \mathrm{e}_{r;k_r,k_r}.$

- Let $n \in \mathbb{N}$ and M be an A-module. For $m_1, \ldots m_n \in M$ we write

$$_A\langle m_1, \ldots m_n\rangle := \left\{ \sum_{i\in[1,n]} \lambda_i m_i : \lambda_i \in A \text{ for } i \in [1, n] \right\} \subseteq M,$$

  for the submodule of M generated by these elements.

Chapter 1

Minimal projective resolutions

Let R be a commutative ring. Let Λ be an R-algebra.

We recall some basic facts on minimal projective resolutions.

1.1 Projective covers

Definition 1 Let M and X be Λ-modules.

(1) A Λ-module $N \leq M$ is *small* in M if $Z < M$ implies $N + Z < M$.

(2) An epimorphism $X \xrightarrow{f} M$ is *essential* if $\mathrm{Ker}(f) \leq X$ is small in X.

(3) A *projective cover* of M is a projective Λ-module P together with an essential epimorphism $P \xrightarrow{f} M$.

Lemma 2 Let M and X be Λ-modules and $X \xrightarrow{f} M$ an epimorphism. The following assertions (1,2) are equivalent.

(1) The epimorphism f is essential.

(2) Given a Λ-module Y and a Λ-linear map $Y \xrightarrow{g} X$ such that gf is an epimorphism, then g is an epimorphism.

Proof. Ad (1) $\Rightarrow$ (2). Let Y be a Λ-module. Let $Y \xrightarrow{g} X$ be Λ-linear such that gf is surjective. We show that $\mathrm{Im}(g) + \mathrm{Ker}(f) = X$.

For x in X we have $ygf = xf$ for a $y \in Y$ so that $(yg - x) \in \mathrm{Ker}(f)$. Hence

$$x \in \mathrm{Ker}(f) + \mathrm{Im}(g).$$

Thus $\mathrm{Im}(g) + \mathrm{Ker}(f) = X$.

Therefore, as $\mathrm{Ker}(f) \leq X$ is small in X, we have $\mathrm{Im}(g) = X$ so that g is surjective.

Ad (2) $\Rightarrow$ (1). Let $Z \leq X$ with $\mathrm{Ker}(f) + Z = X$. Then the composite

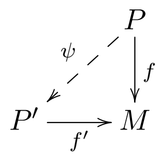
$$Z \hookrightarrow X \xrightarrow{f} M$$

is surjective so that by (2) the inclusion $Z \hookrightarrow X$ is surjective as well. Hence $Z = X$. Therefore $\mathrm{Ker}(f) \leq X$ is small, i.e. f is essential. ▫

Lemma 3 Let M be a Λ-module with a projective cover $P \xrightarrow{f} M$.

For every projective cover $P' \xrightarrow{f'} M$ there exists a Λ-linear isomorphism $P \xrightarrow{\psi} P'$ with $\psi f' = f$. That is, the projective cover is uniquely determined up to an isomorphism.

Proof. Let $P' \xrightarrow{f'} M$ be a projective cover. Since f' is surjective and P projective there exists a Λ-linear map $P \xrightarrow{\psi} P'$ such that $\psi f' = f$.



Since $f = \psi f'$ is surjective and f' essential, we obtain that ψ is surjective. Since P' is projective, there exists an injective Λ-linear map $P' \xrightarrow{\varphi} P$ with $\varphi\psi = \mathrm{id}_{P'}$.

We have that $\varphi f = \varphi\psi f' = f'$ is surjective so that φ is surjective since f is essential. In conclusion, we obtain that ψ is an isomorphism. ▫

1.2 Jacobson radical

Definition 4 The *Jacobson radical* $\mathrm{rad}(\Lambda)$ of Λ is the intersection of all maximal left ideals of Λ.

Lemma 5 The Jacobson radical $\mathrm{rad}(\Lambda)$ is a two-sided ideal in Λ.

Proof. Let $x \in \mathrm{rad}(\Lambda)$ and $\lambda \in \Lambda$. Let I be a maximal left ideal in Λ. We show that $x\lambda \in I$.

Assume that $x\lambda \notin I$. Let $I' := \{z \in \Lambda : z\lambda \in I\}$. Then I' is a left ideal in Λ. We show that I' is a maximal left ideal in Λ.

Since I is a maximal left ideal in Λ and since $x\lambda \notin I$, we have $I + {}_\Lambda\langle x\lambda\rangle = \Lambda$. Hence there exist $v \in I$ and $w \in \Lambda$ with $v + wx\lambda = 1$.

For $a \in I$ we have $awx\lambda = a - av \in I$ so that $awx \in I'$. Conversely, given $a' \in I'$ we have $a'\lambda \in I$. Therefore, we have the following well-defined Λ-linear maps.

$$\begin{aligned} \Lambda/I &\to \Lambda/I' \\ a + I &\mapsto awx + I' \\ a'\lambda + I &\leftarrow\!\shortmid a' + I' \end{aligned}$$

We have $1 - wx\lambda = v \in I$ so that $(1 - \lambda wx)\lambda = \lambda(1 - wx\lambda) \in I$, that is $1 - \lambda wx \in I'$. We obtain

$$\begin{aligned} a'\lambda wx + I' &= a'\lambda wx + a'(1 - \lambda wx) + I' = a' + I' \\ awx\lambda + I &= av + awx\lambda + I = a(v + wx\lambda) + I = a + I \end{aligned}$$

so that the maps above are mutually inverse.

Since I is a maximal left ideal in Λ, the module Λ/I is simple. By the isomorphism above, the module Λ/I' is simple as well. Therefore, I' is a maximal left ideal in Λ.

Hence $x \in \mathrm{rad}(\Lambda) \subseteq I'$ which *contradicts* $x\lambda \notin I$. ▫

Lemma 6 Let $x \in \mathrm{rad}(\Lambda)$. Then $1 - x$ has a left inverse in Λ.

Proof. Assume $1-x$ has no left inverse in Λ. Then ${}_\Lambda\langle 1 - x\rangle < \Lambda$. So there exists a maximal left ideal I in Λ such that $1 - x \in {}_\Lambda\langle 1 - x\rangle \leq I$. Moreover, we have $x \in \mathrm{rad}(\Lambda) \subseteq I$. We obtain

$$1 = (1 - x) + x \in I$$

so that $I = \Lambda$ in *contradiction* to I being a maximal left ideal in Λ. ▫

Lemma 7 (Nakayama)

Let M be a finitely generated Λ-module with $\mathrm{rad}(\Lambda)M = M$. Then $M = 0$.

Proof. Assume $M \neq 0$. Let s be the minimal number of generators of M. Then $s \in \mathbb{Z}_{\geq 1}$, since $M \neq 0$. Choose $m_1, \ldots, m_s \in M$ such that $M = {}_\Lambda\langle m_1, \ldots, m_s\rangle$.

Write $N := \left\{ \sum_{i\in[1,s]} a_i m_i : a_i \in \mathrm{rad}(\Lambda) \text{ for } i \in [1, s] \right\}$. We *claim* that $\mathrm{rad}(\Lambda)M = N$.

It suffices to show that $am \in N$ for $a \in \mathrm{rad}(\Lambda)$ and $m \in M$.

There exist $\lambda_1, \ldots, \lambda_s \in \Lambda$ with

$$m = \sum_{i\in[1,s]} \lambda_i m_i \,.$$

By Lemma 5, we have $a\lambda_i \in \mathrm{rad}(\Lambda)$ for $i \in [1, s]$ so that

$$am = \sum_{i\in[1,s]} a\lambda_i m_i \in N \,.$$

This proves the *claim.*

Since $m_1 \in M = \mathrm{rad}(\Lambda)M$, there exist $a_1, \ldots, a_s \in \mathrm{rad}(\Lambda)$ such that

$$m_1 = \sum_{i\in[1,s]} a_i m_i$$

by the claim above.

Since $a_1 \in \mathrm{rad}(\Lambda)$, the element $(1 - a_1)$ has a left inverse b in Λ by Lemma 6. We obtain

$$(1 - a_1)m_1 = \sum_{i\in[2,s]} a_i m_i$$

so that

$$m_1 = b\left(\sum_{i\in[2,s]} a_i m_i\right)$$

in *contradiction* to the minimality of s. ▫

Lemma 8 Let X be a finitely generated Λ-module. Then $\mathrm{rad}(\Lambda)X \leq X$ is small in X.

Proof. Let $\mathrm{rad}(\lambda)X + Y = X$ for a submodule $Y \leq X$. We obtain

$$\mathrm{rad}(\Lambda)\left(X\diagup Y\right) = (\mathrm{rad}(\Lambda)X + Y)\diagup Y = X\diagup Y$$

so that by Lemma 7 we have $X/Y = 0$. Hence $X = Y$. ▫

1.3 Uniqueness of minimal projective resolutions

Definition 9 Let M be a finitely generated Λ-module. A projective resolution

$$\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \longrightarrow 0 \longrightarrow \cdots$$

of M is a minimal projective resolution, if X_n is a finitely generated projective Λ-module and $\mathrm{Im}(d_n) \subseteq \mathrm{rad}(\Lambda)X_n$ for all $n \in \mathbb{Z}_{\geq 0}$.

Lemma 10 Let M be a finitely generated Λ-module. Let

$$\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \longrightarrow \cdots$$

be an augmented minimal projective resolution of M. Write $\tilde{d}_n := d_n|^{\mathrm{Im}(d_n)}$ for $n \geq 0$.

Then $X_0 \xrightarrow{\varepsilon} M$ is a projective cover of M and $X_n \xrightarrow{\tilde{d}_{n-1}} \mathrm{Im}(d_{n-1})$ a projective cover of $\mathrm{Im}(d_{n-1})$ for $n \geq 1$.

Proof. We show that $\mathrm{Ker}(\varepsilon) \leq X_0$ is small in X_0 and that $\mathrm{Ker}(d_{n-1}) \leq X_n$ is small in X_n for all $n \geq 1$; cf. Definition 1.(3,2).

Since $\mathrm{Ker}(d_{n-1}) = \mathrm{Im}(d_n)$ for $n \geq 1$ and $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(d_0)$, we have to show that $\mathrm{Im}(d_n)$ is small in X_n for $n \geq 0$.

Let $\mathrm{Im}(d_n) + Y = X_n$ for a submodule $Y \leq X_n$. We have $\mathrm{Im}(d_n) \subseteq \mathrm{rad}(\Lambda)X_n$ by minimality so that

$$X_n = \mathrm{Im}(d_n) + Y \subseteq \mathrm{rad}(\Lambda)X_n + Y \subseteq X_n\,.$$

Therefore $\mathrm{rad}(\Lambda)X_n + Y = X$. Since $\mathrm{rad}(\Lambda)X_n$ is small in X_n, we obtain $X = Y$; cf. Lemma 8. In conclusion, $\mathrm{Im}(d_n) \leq X_n$ is small in X_n. ▫

Lemma 11 Suppose given an augmented minimal projective resolution

$$\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \longrightarrow \cdots$$

of a finitely generated Λ-module M.

For every augmented minimal projective resolution

$$\cdots \longrightarrow Y_3 \xrightarrow{d'_2} Y_2 \xrightarrow{d'_1} Y_1 \xrightarrow{d'_0} Y_0 \xrightarrow{\varepsilon'} M \longrightarrow 0 \longrightarrow \cdots$$

of M, there exists an isomorphism of complexes f of the following form.

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X_3 & \xrightarrow{d_2} & X_2 & \xrightarrow{d_1} & X_1 & \xrightarrow{d_0} & X_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \wr\downarrow f_3 & & \wr\downarrow f_2 & & \wr\downarrow f_1 & & \wr\downarrow f_0 & & \downarrow 1 & & \downarrow & & \\
\cdots & \longrightarrow & Y_3 & \xrightarrow{d'_2} & Y_2 & \xrightarrow{d'_1} & Y_1 & \xrightarrow{d'_0} & Y_0 & \xrightarrow{\varepsilon'} & M & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

That is, the minimal projective resolution of M is uniquely determined up to an isomorphism of complexes.

Proof. Suppose given an augmented minimal projective resolution

$$\cdots \longrightarrow Y_3 \xrightarrow{d_2} Y_2 \xrightarrow{d_1} Y_1 \xrightarrow{d_0} Y_0 \xrightarrow{\varepsilon'} M \longrightarrow 0 \longrightarrow \cdots$$

of M. We inductively construct an isomorphism of complexes f.

We also write $\left(X_0 \xrightarrow{d_{-1}} X_{-1}\right) := \left(X_0 \xrightarrow{\varepsilon} M\right)$ and $\left(Y_0 \xrightarrow{d'_{-1}} Y_{-1}\right) := \left(Y_0 \xrightarrow{\varepsilon'} M\right)$.

By Lemma 10, $\left(X_0 \xrightarrow{d_{-1}} X_{-1}\right)$ and $\left(Y_0 \xrightarrow{d'_{-1}} Y_{-1}\right)$ are projective covers of M. Hence, by Lemma 3, there exists a Λ-linear isomorphism $X_0 \xrightarrow{f_0} Y_0$ such that the following diagram commutes.

$$\begin{array}{ccccc}
X_0 & \xrightarrow{d_{-1}} & X_{-1} & \longrightarrow & 0 \\
\wr\downarrow f_0 & & \wr\downarrow 1=:f_{-1} & & \downarrow \\
Y_0 & \xrightarrow{d'_{-1}} & Y_{-1} & \longrightarrow & 0
\end{array}$$

Let $n \geq 1$. Suppose given Λ-linear isomorphisms $X_k \xrightarrow{f_k} Y_k$ for $k \in [0, n-1]$ such that the following diagram commutes.

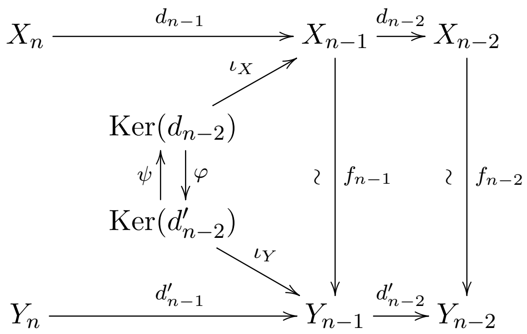
$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_n & \xrightarrow{d_{n-1}} & X_{n-1} & \xrightarrow{d_{n-2}} & X_{n-2} & \longrightarrow & \cdots \\
 & & & & \wr\downarrow f_{n-1} & & \wr\downarrow f_{n-2} & & \\
\cdots & \longrightarrow & Y_n & \xrightarrow{d'_{n-1}} & Y_{n-1} & \xrightarrow{d'_{n-2}} & Y_{n-2} & \longrightarrow & \cdots
\end{array}$$

We have the following Λ-linear embeddings.

$$\mathrm{Ker}(d_{n-2}) \xrightarrow{\iota_X} X_{n-1} \qquad\qquad \mathrm{Ker}(d'_{n-2}) \xrightarrow{\iota_Y} Y_{n-1}$$

Since $\iota_X f_{n-1} d'_{n-2} = \iota_X d_{n-2} f_{n-2} = 0$ we obtain a Λ-linear map $\mathrm{Ker}(d_{n-2}) \xrightarrow{\varphi} \mathrm{Ker}(d'_{n-2})$ with $\varphi\iota_Y = \iota_X f_{n-1}$.

Similarly, we have a Λ-linear map $\mathrm{Ker}(d'_{n-2}) \xrightarrow{\psi} \mathrm{Ker}(d_{n-2})$ with $\psi\iota_X = \iota_Y f_{n-1}^{-1}$ and obtain the following commutative diagram.



We have

$$\psi\varphi\iota_Y = \psi\iota_X f_{n-1} = \iota_Y f_{n-1}^{-1} f_{n-1} = \iota_Y$$

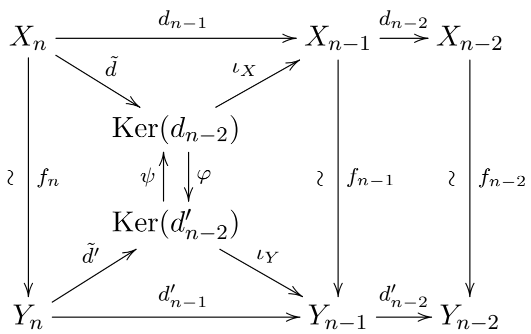
so that $\psi\varphi = \mathrm{id}_{\mathrm{Ker}(d'_{n-2})}$. Similarly, we have $\varphi\psi = \mathrm{id}_{\mathrm{Ker}(d_{n-2})}$. Hence φ is an isomorphism.

Write $\tilde{d} := d_{n-1}|^{\mathrm{Ker}(d_{n-2})}$ and $\tilde{d}' := d'_{n-1}|^{\mathrm{Ker}(d'_{n-2})}$. By Lemma 10, $X_n \xrightarrow{\tilde{d}} \mathrm{Ker}(d_{n-2})$ is a projective cover of $\mathrm{Ker}(d_{n-2})$ and $Y_n \xrightarrow{\tilde{d}'} \mathrm{Ker}(d'_{n-2})$ is a projective cover of $\mathrm{Ker}(d'_{n-2})$. Since $\tilde{d}\varphi$ is an epimorphism and since $\mathrm{Ker}(\tilde{d}\varphi) = \mathrm{Ker}(\tilde{d})$ is small in X_n, we have that $X_n \xrightarrow{\tilde{d}\varphi} \mathrm{Ker}(d'_{n-2})$ is a projective cover of $\mathrm{Ker}(d'_{n-2})$; cf. Definition 1.

By Lemma 3, there exists a Λ-linear isomorphism $X_n \xrightarrow{f_n} Y_n$ such that $f_n\tilde{d}' = \tilde{d}\varphi$. Consequently,

$$f_n d'_{n-1} = f_n\tilde{d}'\iota_Y = \tilde{d}\varphi\iota_Y = d_{n-1}f_{n-1}$$

so that the following diagram commutes.



In conclusion, we have constructed an isomorphism of complexes f of the following form.

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X_3 & \xrightarrow{d_2} & X_2 & \xrightarrow{d_1} & X_1 & \xrightarrow{d_0} & X_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & \wr\downarrow f_3 & & \wr\downarrow f_2 & & \wr\downarrow f_1 & & \wr\downarrow f_0 & & \downarrow 1 & & \downarrow & & \\
\cdots & \longrightarrow & Y_3 & \xrightarrow{d'_2} & Y_2 & \xrightarrow{d'_1} & Y_1 & \xrightarrow{d'_0} & Y_0 & \xrightarrow{\varepsilon'} & M & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

□

Chapter 2

Projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$

We aim to construct the minimal projective resolution

$$X = \left(\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \longrightarrow 0 \longrightarrow \cdots \right)$$

of the trivial module $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$; cf. Theorem 38 below.

To do so, we first consider the Wedderburn image of $\mathbb{Z}_{(2)}\mathrm{S}_4$ to find the projective indecomposable modules of $\mathbb{Z}_{(2)}\mathrm{S}_4$ and maps between them.

2.1 Wedderburn image of $\mathbb{Z}\mathrm{S}_4$

Let $\mathbb{Q}\mathrm{S}_4 \xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{3\times 3} \times \mathbb{Q}^{2\times 2}$ be the Wedderburn isomorphism. We consider its restriction to $\mathbb{Z}\mathrm{S}_4$, the Wedderburn embedding $\mathbb{Z}\mathrm{S}_4 \longrightarrow \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{2\times 2}$.

Remark 12 We have the following presentation of S_4.

$$\begin{aligned} \mathrm{S}_4 &\xleftarrow{\sim} \langle a, b : a^2, b^4, (ab)^3 \rangle \\ (1,2) &\leftarrow\!\shortmid a \\ (1,2,3,4) &\leftarrow\!\shortmid b \end{aligned}$$

Proof. Note that

$$\begin{aligned} ((1,2))^2 &= \mathrm{id} \\ (1,2,3,4)^4 &= \mathrm{id} \\ ((1,2)(1,2,3,4))^3 = (1,3,4)^3 &= \mathrm{id} \end{aligned}$$

so that

$$\begin{aligned} \mathrm{S}_4 &\stackrel{\rho}{\leftarrow} \langle a, b : a^2, b^4, (ab)^3 \rangle \\ (1,2) &\leftarrow\!\shortmid a \\ (1,2,3,4) &\leftarrow\!\shortmid b \end{aligned}$$

is a surjective group homomorphism. Using the computer algebra system Magma, cf. [6], we calculate $\left|\langle a, b : a^2, b^4, (ab)^3 \rangle\right| = 24 = |\mathrm{S}_4|$. Therefore ρ is bijective. ▫

Lemma 13 Let $\Gamma' := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{3\times 3} \times \mathbb{Z}^{2\times 2}$ and

$$\Lambda' := \left\{ \begin{array}{c} \left(\rho, \sigma, \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix}, \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \end{pmatrix}, \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \right) \in \Gamma' \ : \\ \\ \alpha_{1,1} \equiv_4 \beta_{1,1},\ \alpha_{1,2} \equiv_4 \beta_{1,2},\ \alpha_{1,3} \equiv_2 \beta_{1,3}, \\ \alpha_{2,1} \equiv_4 \beta_{2,1},\ \alpha_{2,2} \equiv_4 \beta_{2,2},\ \alpha_{2,3} \equiv_2 \beta_{2,3}, \\ \alpha_{3,1} \equiv_8 \beta_{3,1} \equiv_4 0,\ \alpha_{3,2} \equiv_8 \beta_{3,2} \equiv_4 0,\ \alpha_{3,3} \equiv_2 \beta_{3,3}, \\ \alpha_{1,1} + \beta_{1,1} \equiv_8 2\gamma_{1,1},\ \alpha_{1,2} + \beta_{1,2} \equiv_8 2\gamma_{1,2}, \\ \alpha_{2,1} + \beta_{2,1} \equiv_8 2\gamma_{2,1},\ \alpha_{2,2} + \beta_{2,2} \equiv_8 2\gamma_{2,2}, \\ \rho - \alpha_{3,3} \equiv_8 \sigma - \beta_{3,3} \equiv_4 0, \\ \\ \rho \equiv_3 \gamma_{2,2},\ \gamma_{1,2} \equiv_3 0,\ \sigma \equiv_3 \gamma_{1,1} \end{array} \right\}.$$

We have the isomorphism of $\mathbb{Z}$-algebras

$$\begin{aligned} \mathbb{Z}\mathrm{S}_4 &\xrightarrow[\sim]{\omega'} \Lambda' \\ (1,2) &\mapsto \left(1, -1, \begin{pmatrix} 11 & 24 & -2 \\ -5 & -11 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 5 & -24 \\ 1 & -5 \end{pmatrix} \right) \\ (1,2,3,4) &\mapsto \left(1, -1, \begin{pmatrix} -26 & -57 & -2 \\ 11 & 24 & 1 \\ 4 & 8 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 4 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -4 & 15 \\ -1 & 4 \end{pmatrix} \right). \end{aligned}$$

See also [**11,** Section 2.1.1, page 20].

Proof. By Remark 12 the group S_4 is generated by $(1,2)$ and $(1,2,3,4)$.

We define $\omega'_+ : \mathbb{Z}\mathrm{S}_4 \to \Gamma'$ by

$$\begin{aligned} (1,2) &\mapsto \left(1, -1, \begin{pmatrix} 11 & 24 & -2 \\ -5 & -11 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 5 & -24 \\ 1 & -5 \end{pmatrix} \right) \\ (1,2,3,4) &\mapsto \left(1, -1, \begin{pmatrix} -26 & -57 & -2 \\ 11 & 24 & 1 \\ 4 & 8 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 4 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -4 & 15 \\ -1 & 4 \end{pmatrix} \right). \end{aligned}$$

We have

$$\left(1,-1,\begin{pmatrix}11 & 24 & -2\\ -5 & -11 & 1\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}-1 & 0 & 0\\ -1 & 1 & -1\\ 0 & 0 & -1\end{pmatrix},\begin{pmatrix}5 & -24\\ 1 & -5\end{pmatrix}\right)^2 = \left(1,1,\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

$$\left(1,-1,\begin{pmatrix}-26 & -57 & -2\\ 11 & 24 & 1\\ 4 & 8 & 1\end{pmatrix},\begin{pmatrix}2 & -1 & 0\\ 3 & 0 & -1\\ 4 & 0 & -1\end{pmatrix},\begin{pmatrix}-4 & 15\\ -1 & 4\end{pmatrix}\right)^4 = \left(1,1,\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

and

$$\left(\left(1,-1,\begin{pmatrix}11 & 24 & -2\\ -5 & -11 & 1\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}-1 & 0 & 0\\ -1 & 1 & -1\\ 0 & 0 & -1\end{pmatrix},\begin{pmatrix}5 & -24\\ 1 & -5\end{pmatrix}\right)\left(1,-1,\begin{pmatrix}-26 & -57 & -2\\ 11 & 24 & 1\\ 4 & 8 & 1\end{pmatrix},\begin{pmatrix}2 & -1 & 0\\ 3 & 0 & -1\\ 4 & 0 & -1\end{pmatrix},\begin{pmatrix}-4 & 15\\ -1 & 4\end{pmatrix}\right)\right)^3$$
$$= \left(1,1,\begin{pmatrix}-30 & -67 & 0\\ 13 & 29 & 0\\ 4 & 8 & 1\end{pmatrix},\begin{pmatrix}-2 & 1 & 0\\ -3 & 1 & 0\\ -4 & 0 & 1\end{pmatrix},\begin{pmatrix}4 & -21\\ 1 & -5\end{pmatrix}\right)^3 = \left(1,1,\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

so that the images of the generators of S_4 fulfill the relations of Remark 12. Hence ω'_+ is a well-defined $\mathbb{Z}$-algebra homomorphism.

Now we calculate the images under ω'_+ for all elements in S_4.

Let

$$\begin{aligned}\mathcal{B} = \Big(&\mathrm{id},\\ &(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4),\\ &(1,2,3),\ (1,3,2),\ (1,2,4),\ (1,4,2),\ (1,3,4),\ (1,4,3),\ (2,3,4),\ (2,4,3),\\ &(1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3),\\ &(1,2,3,4),\ (1,4,3,2),\ (1,3,2,4),\ (1,4,2,3),\ (1,3,4,2),\ (1,2,4,3)\Big).\end{aligned}$$

Using the standard $\mathbb{Z}$-linear basis of Γ' and $\mathcal{B}$ as a $\mathbb{Z}$-linear basis for $\mathbb{Z}\mathrm{S}_4$, the map ω'_+ can be described with the following matrix.

$$A := \left(\begin{array}{cccccccccccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1\\
1 & 11 & 34 & -19 & -17 & 8 & -15 & 38 & -11 & -17 & 16 & -30 & 29 & -1 & -24 & -5 & 41 & -37 & -26 & -16 & -39 & 43 & 5 & 31\\
0 & 24 & 77 & -45 & -45 & 21 & -32 & 87 & -31 & -39 & 39 & -67 & 67 & 3 & -59 & -8 & 98 & -90 & -57 & -41 & -94 & 102 & 17 & 73\\
0 & -2 & 0 & 0 & 9 & -7 & 0 & -2 & 9 & -2 & -7 & 0 & 0 & -7 & 9 & -2 & -7 & 9 & -2 & 9 & 9 & -7 & -7 & -2\\
0 & -5 & -15 & 8 & 8 & -3 & 7 & -17 & 5 & 7 & -7 & 13 & -13 & 1 & 11 & 2 & -18 & 16 & 11 & 7 & 17 & -19 & -2 & -14\\
1 & -11 & -34 & 19 & 21 & -8 & 15 & -39 & 14 & 16 & -17 & 29 & -30 & 0 & 27 & 3 & -43 & 39 & 24 & 18 & 41 & -45 & -7 & -33\\
0 & 1 & 0 & 0 & -4 & 3 & 0 & 1 & -4 & 1 & 3 & 0 & 0 & 3 & -4 & 1 & 3 & -4 & 1 & -4 & -4 & 3 & 3 & 1\\
0 & 0 & -12 & 0 & 8 & 0 & 4 & -12 & 8 & 0 & 0 & 4 & -12 & 4 & 8 & 4 & -12 & 8 & 4 & 8 & 8 & -12 & 4 & -12\\
0 & 0 & -28 & 0 & 20 & 0 & 8 & -28 & 20 & 0 & 0 & 8 & -28 & 8 & 20 & 8 & -28 & 20 & 8 & 20 & 20 & -28 & 8 & -28\\
1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & -3 & -3 & 1 & 1 & 1\\
1 & -1 & -2 & 1 & -1 & 0 & 1 & 2 & 1 & -1 & 0 & -2 & 1 & -1 & 0 & -1 & 1 & -1 & 2 & 0 & 1 & -1 & 1 & -1\\
0 & 0 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 0 & -2 & 2 & -1 & -1 & -2 & 2 & 1 & 1\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & -1 & 0\\
0 & -1 & -3 & 0 & 0 & 1 & 3 & 3 & 1 & -1 & 1 & -3 & 3 & -3 & -1 & -2 & 2 & 0 & 3 & -1 & 1 & -3 & 2 & -2\\
1 & 1 & 2 & -1 & -3 & 0 & -1 & -3 & 2 & 0 & -1 & 1 & -2 & 0 & 3 & -1 & -3 & 3 & 0 & -2 & -3 & 3 & 1 & 3\\
0 & -1 & 0 & 0 & 2 & -1 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 1 & -2 & 1 & 1 & -2 & -1 & 2 & 2 & -1 & -1 & -1\\
0 & 0 & -4 & 0 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & -4 & 4 & -4 & 0 & -4 & 4 & 0 & 4 & 0 & 0 & -4 & 4 & -4\\
0 & 0 & 4 & 0 & -4 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & -4 & 0 & 4 & 0 & -4 & 4 & 0 & -4 & -4 & 4 & 0 & 4\\
1 & -1 & -1 & -1 & 3 & -1 & -1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & -1 & 3 & 3 & -1 & -1 & -1\\
1 & 5 & -4 & -1 & -1 & -4 & 5 & 4 & -5 & -5 & 4 & 4 & -5 & -5 & 4 & 1 & 1 & 1 & -4 & -4 & 5 & 5 & -1 & -1\\
0 & -24 & 15 & 9 & 9 & 15 & -24 & -21 & 21 & 21 & -21 & -21 & 21 & 21 & -21 & 0 & 0 & 0 & 15 & 15 & -24 & -24 & 9 & 9\\
0 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0\\
1 & -5 & 4 & 1 & 1 & 4 & -5 & -5 & 4 & 4 & -5 & -5 & 4 & 4 & -5 & 1 & 1 & 1 & 4 & 4 & -5 & -5 & 1 & 1
\end{array}\right)$$

Note that every image fulfills the congruences of Λ' so that the restriction

$$\omega'_{+}|^{\Lambda'}_{\mathbb{Z}\mathrm{S}_4} =: \omega' : \mathbb{Z}\mathrm{S}_4 \to \Lambda'$$

is a well-defined $\mathbb{Z}$-algebra homomorphism.

We have $\det(A) = -3^3 \cdot 2^{34} \neq 0$ so that ω'_{+} and its restriction ω' are injective.

We already know that $(\mathbb{Z}\mathrm{S}_4)\omega' \subseteq \Lambda'$. To show the equality we calculate the index of $(\mathbb{Z}\mathrm{S}_4)\omega'$ in Γ'.

$$[\Gamma' : (\mathbb{Z}\mathrm{S}_4)\omega'] = \left|\Gamma'/(\mathbb{Z}\mathrm{S}_4)\omega'\right| = |\det(A)| = 3^3 \cdot 2^{34}$$

By using a $\mathbb{Z}$-linear basis of Λ' and the standard $\mathbb{Z}$-linear basis of Γ', the embedding $\Lambda' \hookrightarrow \Gamma'$ can be described by the following matrix.

$$B := \left(\begin{array}{cccccccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&5&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&5&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&5&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&8&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&8&0&0&0&0&0\\
1&2&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&8&0&0&0&0\\
1&2&3&0&0&0&0&0&0&0&0&6&0&0&0&0&0&0&0&0&12&0&0&0\\
0&0&0&3&0&0&0&0&0&0&0&0&6&0&0&0&0&0&0&0&0&12&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&4&0\\
1&0&0&0&0&0&3&0&0&0&0&0&0&0&0&6&0&0&0&0&0&0&0&12
\end{array}\right)$$

We have $\det(B) = 3^3 \cdot 2^{34}$ and therefore

$$[\Gamma' : \Lambda'] = \left|\Gamma'/\Lambda'\right| = |\det(B)| = 3^3 \cdot 2^{34}.$$

Hence we obtain

$$[\Gamma' : \Lambda'] \cdot [\Lambda' : (\mathbb{Z}\mathrm{S}_4)\omega'] = [\Gamma' : (\mathbb{Z}\mathrm{S}_4)\omega'] = 3^3 \cdot 2^{34} = [\Gamma' : \Lambda']$$

so that the equality $(\mathbb{Z}\mathrm{S}_4)\omega' = \Lambda'$ holds.

In conclusion, $\omega' : \mathbb{Z}\mathrm{S}_4 \to \Lambda'$ is an isomorphism of $\mathbb{Z}$-algebras. ▫

Remark 14 Independently, we may calculate the index of $(\mathbb{Z}\mathrm{S}_4)\,\omega'$ in Γ' to be

$$[\Gamma' : (\mathbb{Z}\mathrm{S}_4)\omega'] = \left|\Gamma'/(\mathbb{Z}\mathrm{S}_4)\omega'\right| = \sqrt{\frac{24^{24}}{1^1 \cdot 1^1 \cdot 3^9 \cdot 3^9 \cdot 2^4}} = \left(\frac{3^{24} \cdot 2^{72}}{3^{18} \cdot 2^4}\right)^{1/2} = 3^3 \cdot 2^{34}\,;$$

cf. **[11,** Proposition 1.1.5].

2.2 Wedderburn image of $\mathbb{Z}_{(2)}\mathrm{S}_4$

We write $R := \mathbb{Z}_{(2)}$. We localize the isomorphism ω' of Lemma 13 at 2.

Corollary 15 Let $\Gamma := R \times R \times R^{3\times 3} \times R^{3\times 3} \times R^{2\times 2}$ and

$$\Lambda := \left\{ \begin{array}{c} \left(\rho, \sigma, \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix}, \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \end{pmatrix}, \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \right) \in \Gamma \; : \\ \\ \alpha_{1,1} \equiv_4 \beta_{1,1},\; \alpha_{1,2} \equiv_4 \beta_{1,2},\; \alpha_{1,3} \equiv_2 \beta_{1,3}, \\ \alpha_{2,1} \equiv_4 \beta_{2,1},\; \alpha_{2,2} \equiv_4 \beta_{2,2},\; \alpha_{2,3} \equiv_2 \beta_{2,3}, \\ \alpha_{3,1} \equiv_8 \beta_{3,1} \equiv_4 0,\; \alpha_{3,2} \equiv_8 \beta_{3,2} \equiv_4 0,\; \alpha_{3,3} \equiv_2 \beta_{3,3}, \\ \alpha_{1,1} + \beta_{1,1} \equiv_8 2\gamma_{1,1},\; \alpha_{1,2} + \beta_{1,2} \equiv_8 2\gamma_{1,2}, \\ \alpha_{2,1} + \beta_{2,1} \equiv_8 2\gamma_{2,1},\; \alpha_{2,2} + \beta_{2,2} \equiv_8 2\gamma_{2,2}, \\ \rho - \alpha_{3,3} \equiv_8 \sigma - \beta_{3,3} \equiv_4 0 \end{array} \right\}.$$

We have the isomorphism of R-algebras

$$\begin{array}{rcl} R\mathrm{S}_4 & \xrightarrow[\sim]{\omega} & \Lambda \\ (1,2) & \mapsto & \left(1, -1, \begin{pmatrix} 11 & 24 & -2 \\ -5 & -11 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 5 & -24 \\ 1 & -5 \end{pmatrix} \right) \\ (1,2,3,4) & \mapsto & \left(1, -1, \begin{pmatrix} -26 & -57 & -2 \\ 11 & 24 & 1 \\ 4 & 8 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 4 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -4 & 15 \\ -1 & 4 \end{pmatrix} \right). \end{array}$$

Remark 16

- Let M be an $R\mathrm{S}_4$-module.

  Given $\xi \in \Lambda$ and $x \in M$ we define

  $$\xi \cdot x := \xi\omega^{-1} \cdot x.$$

  Thus M becomes a Λ-module.

  We identify $R\mathrm{S}_4$-modules and Λ-modules in this way.

- Note that R with the action given by

  $$\begin{array}{rcl} \Lambda & \longrightarrow & R^{1\times 1} \\ (\rho, \sigma, N_1, N_2, N_3) & \longmapsto & \rho \end{array}$$

  is the trivial $R\mathrm{S}_4$-module. We also refer to this module as the trivial Λ-module.

2.3 Projective indecomposable modules

Lemma 17 Let

$$\mathcal{E}_0 := \left(1, 1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$\mathcal{E}_1 := \left(0,0,\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)$$
$$\mathcal{E}_2 := \left(0,0,\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right).$$

We have the orthogonal decomposition $1_\Lambda = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2$ into primitive idempotents of Λ. We obtain the Peirce decomposition

$$\Lambda = \mathcal{E}_0\Lambda\mathcal{E}_0 \oplus \mathcal{E}_0\Lambda\mathcal{E}_1 \oplus \mathcal{E}_0\Lambda\mathcal{E}_2 \oplus \mathcal{E}_1\Lambda\mathcal{E}_0 \oplus \mathcal{E}_1\Lambda\mathcal{E}_1 \oplus \mathcal{E}_1\Lambda\mathcal{E}_2 \oplus \mathcal{E}_2\Lambda\mathcal{E}_0 \oplus \mathcal{E}_2\Lambda\mathcal{E}_1 \oplus \mathcal{E}_2\Lambda\mathcal{E}_2\ .$$

Proof. Suppose $i \in [0,2]$. We show that $\mathcal{E}_i$ is a primitive idempotent.

Let $\Xi := \{(a_1,a_2,a_3,a_4) \in R\times R\times R\times R : a_1 \equiv_2 a_2 \equiv_2 a_3 \equiv_2 a_4\}$.

Claim. The only idempotents in Ξ are 0 and 1.

Suppose given an idempotent $(a_1,a_2,a_3,a_4) \in \Xi$, hence $a_i \in \{0_R, 1_R\}$ for $i \in [1,n]$. Suppose $(a_1,a_2,a_3,a_4) \notin \{0,1\}$ so that there exist $i,j \in [1,4]$ with $a_i \neq a_j$.

Then $a_i - a_j \in \{1,-1\}$ so that $a_i \not\equiv_2 a_j$ in contradiction to the congruences in Ξ. This proves the *claim*.

Suppose $\mathcal{E}_i = e_i + f_i \in \Lambda$ for orthogonal idempotents $e_i, f_i \in \Lambda$. We have

$$\mathcal{E}_i e_i \mathcal{E}_i = (e_i+f_i)e_i(e_i+f_i) = e_i$$

so that $e_i \in \mathcal{E}_i\Lambda\mathcal{E}_i$. Similarly we obtain $f_i \in \mathcal{E}_i\Lambda\mathcal{E}_i$. So it suffices to show that $\mathcal{E}_i$ is primitive in $\mathcal{E}_i\Lambda\mathcal{E}_i$.

We have

$$\mathcal{E}_0\Lambda\mathcal{E}_0 = \left\{ \begin{array}{c} u_{\rho,\sigma,\alpha,\beta} := \left(\rho,\sigma,\begin{pmatrix}0&0&0\\0&0&0\\0&0&\alpha\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&\beta\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix}\right) \in \Gamma\ : \\ \alpha \equiv_2 \beta,\ \rho-\alpha \equiv_8 \sigma-\beta \equiv_4 0 \end{array} \right\}$$

$$\mathcal{E}_1\Lambda\mathcal{E}_1 = \left\{ \begin{array}{c} v_{\alpha,\beta,\gamma} := \left(0,0,\begin{pmatrix}\alpha&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}\beta&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}\gamma&0\\0&0\end{pmatrix}\right) \in \Gamma\ : \\ \alpha \equiv_4 \beta,\ \alpha+\beta \equiv_8 2\gamma \end{array} \right\}$$

$$\mathcal{E}_2\Lambda\mathcal{E}_2 = \left\{ \begin{array}{c} w_{\alpha,\beta,\gamma} := \left(0,0,\begin{pmatrix}0&0&0\\0&\alpha&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&\beta&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0\\0&\gamma\end{pmatrix}\right) \in \Gamma\ : \\ \alpha \equiv_4 \beta,\ \alpha+\beta \equiv_8 2\gamma \end{array} \right\}$$

and the following injective homomorphisms of R-algebras.

$$\begin{array}{ccc} \mathcal{E}_0\Lambda\mathcal{E}_0 \to \Xi & \mathcal{E}_1\Lambda\mathcal{E}_1 \to \Xi & \mathcal{E}_2\Lambda\mathcal{E}_2 \to \Xi \\ u_{\rho,\sigma,\alpha,\beta} \mapsto (\rho,\sigma,\alpha,\beta) & v_{\alpha,\beta,\gamma} \mapsto (\alpha,\beta,\gamma,\gamma) & w_{\alpha,\beta,\gamma} \mapsto (\alpha,\beta,\gamma,\gamma) \end{array}$$

Since the image of $\mathcal{E}_i$ under the respective injective R-algebramorphism is primitive by the above claim, the same holds for $\mathcal{E}_i$. □

Definition 18 We denote the indecomposable projective Λ-modules belonging to the idempotents from Lemma 17 by

$$P := \Lambda\mathcal{E}_0 = \left\{ \begin{array}{c} \left(\rho,\sigma,\begin{pmatrix}0&0&\alpha_{1,3}\\0&0&\alpha_{2,3}\\0&0&\alpha_{3,3}\end{pmatrix},\begin{pmatrix}0&0&\beta_{1,3}\\0&0&\beta_{2,3}\\0&0&\beta_{3,3}\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix}\right)\in\Gamma\ : \\ \alpha_{1,3}\equiv_2\beta_{1,3},\ \alpha_{2,3}\equiv_2\beta_{2,3},\ \alpha_{3,3}\equiv_2\beta_{3,3}, \\ \rho-\alpha_{3,3}\equiv_8\sigma-\beta_{3,3}\equiv_4 0 \end{array} \right\}$$

$$Q := \Lambda\mathcal{E}_1 = \left\{ \begin{array}{c} \left(0,0,\begin{pmatrix}\alpha_{1,1}&0&0\\\alpha_{2,1}&0&0\\\alpha_{3,1}&0&0\end{pmatrix},\begin{pmatrix}\beta_{1,1}&0&0\\\beta_{2,1}&0&0\\\beta_{3,1}&0&0\end{pmatrix},\begin{pmatrix}\gamma_{1,1}&0\\\gamma_{2,1}&0\end{pmatrix}\right)\in\Gamma\ : \\ \alpha_{1,1}\equiv_4\beta_{1,1},\ \alpha_{2,1}\equiv_4\beta_{2,1},\ \alpha_{3,1}\equiv_8\beta_{3,1}\equiv_4 0 \\ \alpha_{1,1}+\beta_{1,1}\equiv_8 2\gamma_{1,1},\ \alpha_{2,1}+\beta_{2,1}\equiv_8 2\gamma_{2,1} \end{array} \right\}$$

$$\widetilde{Q} := \Lambda\mathcal{E}_2 = \left\{ \begin{array}{c} \left(0,0,\begin{pmatrix}0&\alpha_{1,2}&0\\0&\alpha_{2,2}&0\\0&\alpha_{3,2}&0\end{pmatrix},\begin{pmatrix}0&\beta_{1,2}&0\\0&\beta_{2,2}&0\\0&\beta_{3,2}&0\end{pmatrix},\begin{pmatrix}0&\gamma_{1,2}\\0&\gamma_{2,2}\end{pmatrix}\right)\in\Gamma\ : \\ \alpha_{1,2}\equiv_4\beta_{1,2},\ \alpha_{2,2}\equiv_4\beta_{2,2},\ \alpha_{3,2}\equiv_8\beta_{3,2}\equiv_4 0 \\ \alpha_{1,2}+\beta_{1,2}\equiv_8 2\gamma_{1,2},\ \alpha_{2,2}+\beta_{2,2}\equiv_8 2\gamma_{2,2} \end{array} \right\}.$$

By abuse of notation we often write

$$\left(\rho,\sigma,\begin{pmatrix}\alpha_{1,3}\\\alpha_{2,3}\\\alpha_{3,3}\end{pmatrix},\begin{pmatrix}\beta_{1,3}\\\beta_{2,3}\\\beta_{3,3}\end{pmatrix}\right) := \left(\rho,\sigma,\begin{pmatrix}0&0&\alpha_{1,3}\\0&0&\alpha_{2,3}\\0&0&\alpha_{3,3}\end{pmatrix},\begin{pmatrix}0&0&\beta_{1,3}\\0&0&\beta_{2,3}\\0&0&\beta_{3,3}\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix}\right)\in P$$

$$\left(\begin{pmatrix}\alpha_{1,1}\\\alpha_{2,1}\\\alpha_{3,1}\end{pmatrix},\begin{pmatrix}\beta_{1,1}\\\beta_{2,1}\\\beta_{3,1}\end{pmatrix},\begin{pmatrix}\gamma_{1,1}\\\gamma_{2,1}\end{pmatrix}\right) := \left(0,0,\begin{pmatrix}\alpha_{1,1}&0&0\\\alpha_{2,1}&0&0\\\alpha_{3,1}&0&0\end{pmatrix},\begin{pmatrix}\beta_{1,1}&0&0\\\beta_{2,1}&0&0\\\beta_{3,1}&0&0\end{pmatrix},\begin{pmatrix}\gamma_{1,1}&0\\\gamma_{2,1}&0\end{pmatrix}\right)\in Q\,.$$

Sometimes, we abbreviate further and write

$$\begin{array}{c} \alpha_i \text{ instead of } \alpha_{i,3},\ \text{ for } i\in[1,3] \\ \beta_i \text{ instead of } \beta_{i,3},\ \text{ for } i\in[1,3] \end{array}$$

for elements in P and

$$\begin{array}{c} \alpha_i \text{ instead of } \alpha_{i,1},\ \text{ for } i\in[1,3] \\ \beta_i \text{ instead of } \beta_{i,1},\ \text{ for } i\in[1,3] \\ \gamma_i \text{ instead of } \gamma_{i,1},\ \text{ for } i\in[1,2] \end{array}$$

for elements in Q, respectively. So

$$P = \left\{ \left(\rho,\sigma,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\in\Gamma\ :\ \begin{array}{c}\alpha_1\equiv_2\beta_1,\ \alpha_2\equiv_2\beta_2,\ \alpha_3\equiv_2\beta_3,\\ \rho-\alpha_3\equiv_8\sigma-\beta_3\equiv_4 0\end{array} \right\}$$

$$Q = \left\{ \left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\in\Gamma\ :\ \begin{array}{c}\alpha_1\equiv_4\beta_1,\ \alpha_2\equiv_4\beta_2,\ \alpha_3\equiv_8\beta_3\equiv_4 0\\ \alpha_1+\beta_1\equiv_8 2\gamma_1,\ \alpha_2+\beta_2\equiv_8 2\gamma_2\end{array} \right\}.$$

Lemma 19 We have $Q \simeq \widetilde{Q}$ as Λ-modules.

Proof. Let

$$p := \left(0, 0, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \quad q := \left(0, 0, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right).$$

We have that $p \in \mathcal{E}_1 \Lambda \mathcal{E}_2$ and $q \in \mathcal{E}_2 \Lambda \mathcal{E}_1$ so that we can define the following Λ-linear maps.

$$\begin{aligned} \varphi \ : \ \Lambda\mathcal{E}_1 &\to \Lambda\mathcal{E}_2 \\ \lambda\mathcal{E}_1 &\mapsto \lambda\mathcal{E}_1 p = \lambda p \mathcal{E}_2 \end{aligned} \qquad \begin{aligned} \psi \ : \ \Lambda\mathcal{E}_2 &\to \Lambda\mathcal{E}_1 \\ \lambda\mathcal{E}_2 &\mapsto \lambda\mathcal{E}_2 q = \lambda q \mathcal{E}_1 \end{aligned}$$

Since $p \cdot q = \mathcal{E}_1$, we obtain for $\lambda \in \Lambda$ that

$$(\lambda\mathcal{E}_1)\, \varphi\psi = (\lambda\mathcal{E}_1 p)\, \psi = \lambda\mathcal{E}_1 pq = \lambda\mathcal{E}_1$$

so that $\varphi\psi = \mathrm{id}_{\Lambda\mathcal{E}_1}$. Since $q \cdot p = \mathcal{E}_2$, we also obtain $\psi\varphi = \mathrm{id}_{\Lambda\mathcal{E}_2}$. $\square$

Lemma 20 We have the following R-linear basis of $P = \Lambda\mathcal{E}_0$.

$$b^0_{0,0} := \left(1, 1, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \in P \qquad b^1_{0,0} := \left(0, 0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \in P$$

$$b^0_{0,1} := \left(0, 2, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}\right) \in P \qquad b^1_{0,1} := \left(0, 0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right) \in P$$

$$b^0_{0,2} := \left(0, 0, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}\right) \in P \qquad b^2_{0,0} := \left(0, 0, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \in P$$

$$b^0_{0,3} := \left(0, 0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}\right) \in P \qquad b^2_{0,1} := \left(0, 0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right) \in P$$

We have the following R-linear basis of $Q = \Lambda\mathcal{E}_1$.

$$b^1_{1,0} := \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \in Q \qquad b^2_{1,0} := \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \in Q$$

$$b^1_{1,1} := \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) \in Q \qquad b^2_{1,1} := \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right) \in Q$$

$$b^1_{1,2} := \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}\right) \in Q \qquad b^2_{1,2} := \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}\right) \in Q$$

$$b^0_{1,0} := \left(\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \in Q \qquad b^0_{1,1} := \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \in Q$$

So $b^i_{j,k} \in \mathcal{E}_i \Lambda \mathcal{E}_j$ for all i, j, k.

Additionally, we define the following R-linear combinations of basis elements.

$$\widetilde{b}^0_{0,1} := 2b^0_{0,0} - b^0_{0,1} = \left(2, 0, \begin{pmatrix}0\\0\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) \in P \qquad \widetilde{b}^1_{0,0} := b^1_{0,0} - b^1_{0,1} = \left(0, 0, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}\right) \in P$$

$$\widetilde{b}^0_{0,2} := 4b^0_{0,0} - b^0_{0,2} = \left(4, 4, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) \in P \qquad \widetilde{b}^1_{0,1} := 2b^1_{0,0} - b^1_{0,1} = \left(0, 0, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) \in P$$

$$\widetilde{b}^0_{0,3} := 2b^0_{0,2} - b^0_{0,3} = \left(0, 0, \begin{pmatrix}0\\0\\8\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) \in P \qquad \widetilde{b}^1_{1,0} := 2b^1_{1,0} - b^1_{1,1} = \left(\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) \in Q$$

$$\widetilde{b}^0_{1,1} := 2b^0_{1,0} - b^0_{1,1} = \left(\begin{pmatrix}0\\0\\8\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) \in Q \qquad \widetilde{b}^1_{1,1} := 4b^1_{1,0} - b^1_{1,1} = \left(\begin{pmatrix}4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\right) \in Q$$

Note that we have used our abbreviation, so that e.g.

$$b^0_{0,1} = \left(0, 2, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\2\end{pmatrix}\right) = \left(0, 2, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&2\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}\right) ;$$

cf. Definition 18.

Remark 21 Suppose given a Λ-module M and $i, j \in \{0, 1\}$. We have

$$\begin{aligned} \mathrm{Hom}_\Lambda(\Lambda\mathcal{E}_i, M) &\xrightarrow{\sim} \mathcal{E}_i M \\ f &\longmapsto \mathcal{E}_i f \\ (\mu\mathcal{E}_i \mapsto \mu\mathcal{E}_i m) &\longleftarrow\!\shortmid \mathcal{E}_i m \,. \end{aligned}$$

In particular, we identify along

$$\begin{aligned} \mathrm{Hom}_\Lambda(\Lambda\mathcal{E}_i, \Lambda\mathcal{E}_j) &\xrightarrow{\sim} \mathcal{E}_i \Lambda\mathcal{E}_j \\ f &\longmapsto \mathcal{E}_i f \\ (\mu\mathcal{E}_i \mapsto \mu\mathcal{E}_i\lambda\mathcal{E}_j) &\longleftarrow\!\shortmid \mathcal{E}_i\lambda\mathcal{E}_j \,. \end{aligned}$$

This yields the following Λ-linear maps, for i, j, k as listed in Lemma 20.

$$\begin{aligned} \Lambda\mathcal{E}_i &\xrightarrow{b^i_{j,k}} \Lambda\mathcal{E}_j \\ \xi &\mapsto \xi b^i_{j,k} \end{aligned} \qquad\qquad \begin{aligned} \Lambda\mathcal{E}_i &\xrightarrow{\widetilde{b}^i_{j,k}} \Lambda\mathcal{E}_j \\ \xi &\mapsto \xi \widetilde{b}^i_{j,k} \end{aligned}$$

Lemma 22 We have the following multiplication table 1 for certain elements of Lemma 20.

Table 1

	$b^0_{0,0}$	$b^0_{0,1}$	$\widetilde{b}^0_{0,1}$	$b^0_{0,2}$	$\widetilde{b}^0_{0,2}$	$b^1_{0,0}$	$\widetilde{b}^1_{0,0}$	$b^1_{0,1}$	$\widetilde{b}^1_{0,1}$	$b^1_{1,0}$	$\widetilde{b}^1_{1,0}$	$b^1_{1,1}$	$b^0_{1,0}$	$b^0_{1,1}$	$\widetilde{b}^0_{1,1}$
$b^0_{0,0}$	$b^0_{0,0}$	$b^0_{0,1}$	$\widetilde{b}^0_{0,1}$	$b^0_{0,2}$	$\widetilde{b}^0_{0,2}$	0	0	0	0	0	0	0	$b^0_{1,0}$	$b^0_{1,1}$	$\widetilde{b}^0_{1,1}$
$b^0_{0,1}$	$b^0_{0,1}$	$2b^0_{0,1}$	0	$b^0_{0,3}$	$4b^0_{0,1} - b^0_{0,3}$	0	0	0	0	0	0	0	$b^0_{1,1}$	$2b^0_{1,1}$	0
$\widetilde{b}^0_{0,1}$	$\widetilde{b}^0_{0,1}$	0	$2\widetilde{b}^0_{0,1}$	$\widetilde{b}^0_{0,3}$	$4\widetilde{b}^0_{0,1} - \widetilde{b}^0_{0,3}$	0	0	0	0	0	0	0	$\widetilde{b}^0_{1,1}$	0	$2\widetilde{b}^0_{1,1}$
$b^0_{0,2}$	$b^0_{0,2}$	$b^0_{0,3}$	$\widetilde{b}^0_{0,3}$	$4b^0_{0,2}$	0	0	0	0	0	0	0	0	$4b^0_{1,0}$	$4b^0_{1,1}$	$4\widetilde{b}^0_{1,1}$
$\widetilde{b}^0_{0,2}$	$\widetilde{b}^0_{0,2}$	$4b^0_{0,1} - b^0_{0,3}$	$4\widetilde{b}^0_{0,1} - \widetilde{b}^0_{0,3}$	0	$4\widetilde{b}^0_{0,2}$	0	0	0	0	0	0	0	0	0	0
$b^1_{0,0}$	$b^1_{0,0}$	$b^1_{0,1}$	$\widetilde{b}^1_{0,1}$	$4b^1_{0,0}$	0	0	0	0	0	0	0	0	$4b^1_{1,0} - b^1_{1,2}$	$2b^1_{1,1} - b^1_{1,2}$	$2\widetilde{b}^1_{1,1} - b^1_{1,2}$
$\widetilde{b}^1_{0,0}$	$\widetilde{b}^1_{0,0}$	$-b^1_{0,1}$	$\widetilde{b}^1_{0,1}$	$4\widetilde{b}^1_{0,0}$	0	0	0	0	0	0	0	0	$2\widetilde{b}^1_{1,0}$	$b^1_{1,2} - 2b^1_{1,1}$	$2\widetilde{b}^1_{1,1} - b^1_{1,2}$
$b^1_{0,1}$	$b^1_{0,1}$	$2b^1_{0,1}$	0	$4b^1_{0,1}$	0	0	0	0	0	0	0	0	$2b^1_{1,1} - b^1_{1,2}$	$4b^1_{1,1} - 2b^1_{1,2}$	0
$\widetilde{b}^1_{0,1}$	$\widetilde{b}^1_{0,1}$	0	$2\widetilde{b}^1_{0,1}$	$4\widetilde{b}^1_{0,1}$	0	0	0	0	0	0	0	0	$2\widetilde{b}^1_{1,1} - b^1_{1,2}$	0	$4\widetilde{b}^1_{1,1} - 2b^1_{1,2}$
$b^1_{1,0}$	0	0	0	0	0	$b^1_{0,0}$	$\widetilde{b}^1_{0,0}$	$b^1_{0,1}$	$\widetilde{b}^1_{0,1}$	$b^1_{1,0}$	$\widetilde{b}^1_{1,0}$	$b^1_{1,1}$	0	0	0
$\widetilde{b}^1_{1,0}$	0	0	0	0	0	$2\widetilde{b}^1_{0,0}$	$2b^1_{0,0}$	$-2b^1_{0,1}$	$2\widetilde{b}^1_{0,1}$	$\widetilde{b}^1_{1,0}$	$4b^1_{1,0} - b^1_{1,2}$	$b^1_{1,2} - 2b^1_{1,1}$	0	0	0
$b^1_{1,1}$	0	0	0	0	0	$2b^1_{0,1}$	$-2b^1_{0,1}$	$4b^1_{0,1}$	0	$b^1_{1,1}$	$b^1_{1,2} - 2b^1_{1,1}$	$4b^1_{1,1} - b^1_{1,2}$	0	0	0
$b^0_{1,0}$	0	0	0	0	0	$b^0_{0,2}$	$b^0_{0,2} - b^0_{0,3}$	$b^0_{0,3}$	$\widetilde{b}^0_{0,3}$	$b^0_{1,0}$	$2b^0_{1,0} - 2b^0_{1,1}$	$2b^0_{1,1}$	0	0	0
$b^0_{1,1}$	0	0	0	0	0	$b^0_{0,3}$	$-b^0_{0,3}$	$2b^0_{0,3}$	0	$b^0_{1,1}$	$-2b^0_{1,1}$	$4b^0_{1,1}$	0	0	0
$\widetilde{b}^0_{1,1}$	0	0	0	0	0	$\widetilde{b}^0_{0,3}$	$\widetilde{b}^0_{0,3}$	0	$2\widetilde{b}^0_{0,3}$	$\widetilde{b}^0_{1,1}$	$2\widetilde{b}^0_{1,1}$	0	0	0	0

2.4 The objects of the projective resolution

We define projective modules X_k for $k \in \mathbb{Z}_{\geq 0}$ which shall appear as the projective objects of the projective resolution X.

$$X = \left(\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \longrightarrow 0 \longrightarrow \cdots \right)$$

Recall that $R = \mathbb{Z}_{(2)}$ and $\Gamma = R \times R \times R^{3\times 3} \times R^{3\times 3} \times R^{2\times 2}$, as well as

$$\Lambda = \left\{ \begin{array}{c} \left(\rho, \sigma, \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix}, \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \end{pmatrix}, \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \right) \in \Gamma \ : \\ \alpha_{1,1} \equiv_4 \beta_{1,1},\ \alpha_{1,2} \equiv_4 \beta_{1,2},\ \alpha_{1,3} \equiv_2 \beta_{1,3},\ \alpha_{2,1} \equiv_4 \beta_{2,1},\ \alpha_{2,2} \equiv_4 \beta_{2,2},\ \alpha_{2,3} \equiv_2 \beta_{2,3}, \\ \alpha_{3,1} \equiv_8 \beta_{3,1} \equiv_4 0,\ \alpha_{3,2} \equiv_8 \beta_{3,2} \equiv_4 0,\ \alpha_{3,3} \equiv_2 \beta_{3,3}, \\ \alpha_{1,1} + \beta_{1,1} \equiv_8 2\gamma_{1,1},\ \alpha_{1,2} + \beta_{1,2} \equiv_8 2\gamma_{1,2},\ \alpha_{2,1} + \beta_{2,1} \equiv_8 2\gamma_{2,1},\ \alpha_{2,2} + \beta_{2,2} \equiv_8 2\gamma_{2,2}, \\ \rho - \alpha_{3,3} \equiv_8 \sigma - \beta_{3,3} \equiv_4 0 \end{array} \right\}$$

and the following indecomposable projective modules of Λ; cf. Definition 18.

$$P = \left\{ \begin{array}{c} \left(\rho, \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) = \left(\rho, \sigma, \begin{pmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \beta_1 \\ 0 & 0 & \beta_2 \\ 0 & 0 & \beta_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \in \Gamma \ : \\ \alpha_1 \equiv_2 \beta_1,\ \alpha_2 \equiv_2 \beta_2,\ \alpha_3 \equiv_2 \beta_3, \\ \rho - \alpha_3 \equiv_8 \sigma - \beta_3 \equiv_4 0 \end{array} \right\}$$

$$Q = \left\{ \begin{array}{c} \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) = \left(0, 0, \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \beta_1 & 0 & 0 \\ \beta_2 & 0 & 0 \\ \beta_3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \gamma_1 & 0 \\ \gamma_2 & 0 \end{pmatrix} \right) \in \Gamma \ : \\ \alpha_1 \equiv_4 \beta_1,\ \alpha_2 \equiv_4 \beta_2,\ \alpha_3 \equiv_8 \beta_3 \equiv_4 0 \\ \alpha_1 + \beta_1 \equiv_8 2\gamma_1,\ \alpha_2 + \beta_2 \equiv_8 2\gamma_2 \end{array} \right\}$$

Definition 23 Let $k \in \mathbb{Z}$.

We write $\overline{\overline{k}}$ for the element in $[0, 2]$ such that $k \equiv_3 \overline{\overline{k}}$. We write $\overline{k}$ for the element in $[0, 5]$ such that $k \equiv_6 \overline{k}$.

Moreover, we denote by $\underline{\underline{x}}$ the unique element in $\mathbb{Z}$ with $x = 3\underline{\underline{x}} + \overline{\overline{x}}$.

Definition 24 Let

$$\begin{aligned} Y_0 &:= Q \oplus Q \\ Y_1 &:= P \oplus Q \\ Y_2 &:= Q \oplus P\,. \end{aligned}$$

We define the following projective Λ-modules for $l \in \mathbb{Z}_{\geq 0}$.

$$\begin{aligned} X_{4l} &:= \left(\bigoplus_{k \in [0, l-1]} P \oplus P \right) \oplus P \oplus \left(\bigoplus_{k \in [0, l-1]} Y_{\overline{\overline{k}}} \right) \\ X_{4l+1} &:= \left(\bigoplus_{k \in [0, l-1]} P \oplus P \right) \oplus P \oplus Q \oplus \left(\bigoplus_{k \in [0, l-1]} Y_{\overline{\overline{k+1}}} \right) \end{aligned}$$

$$X_{4l+2} := \left(\bigoplus_{k\in[0,l-1]} P \oplus P \right) \oplus P \oplus P \oplus Q \oplus \left(\bigoplus_{k\in[0,l-1]} Y_{\overline{\overline{k+2}}} \right)$$

$$X_{4l+3} := \left(\bigoplus_{k\in[0,l-1]} P \oplus P \right) \oplus P \oplus P \oplus Q \oplus P \oplus \left(\bigoplus_{k\in[0,l-1]} Y_{\overline{\overline{k}}} \right)$$

$$= \left(\bigoplus_{k\in[0,l]} P \oplus P \right) \oplus \left(\bigoplus_{k\in[0,l]} Y_{\overline{\overline{k+2}}} \right)$$

The necessary matrix calculations have been translated into calculations using projection and inclusion maps defined below, in order to avoid writing down large block matrices containing many zero blocks. However, these matrices will now appear as sum expressions.

Definition 25 Let $l \in \mathbb{Z}_{\geq 0}$. We define the following Λ-linear maps.

$$\begin{array}{lllcll}
\pi^+_{4l,k} & : X_{4l} \to P \oplus P\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & (v_k,w_k)\,, & \text{for } k\in[0,l-1] \\
\pi'_{4l} & : X_{4l} \to P\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & x & \\
\pi^-_{4l,k} & : X_{4l} \to Y_{\overline{\overline{k}}}\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & (y_k,z_k)\,, & \text{for } k\in[0,l-1]
\end{array}$$

$$\begin{array}{lllcll}
\iota^+_{4l,k} & : P \oplus P \to X_{4l}\,, & (v,w) & \mapsto & \big((\delta_{i,k}v,\delta_{i,k}w)_{i\in[0,l-1]}, 0, (0,0)_{i\in[0,l-1]}\big)\,, & \text{for } k\in[0,l-1] \\
\iota'_{4l} & : P \to X_{4l}\,, & x & \mapsto & \big((0,0)_{i\in[0,l-1]}, x, (0,0)_{i\in[0,l-1]}\big) & \\
\iota^-_{4l,k} & : Y_{\overline{\overline{k}}} \to X_{4l}\,, & (y,z) & \mapsto & \big((0,0)_{i\in[0,l-1]}, 0, (\delta_{i,k}y,\delta_{i,k}z)_{i\in[0,l-1]}\big)\,, & \text{for } k\in[0,l-1]
\end{array}$$

$$\begin{array}{lllcll}
\pi^+_{4l+1,k} & : X_{4l+1} \to P \oplus P\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x_1, x_2, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & (v_k,w_k)\,, & \text{for } k\in[0,l-1] \\
\pi'_{4l+1} & : X_{4l+1} \to P \oplus Q\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x_1, x_2, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & (x_1,x_2) & \\
\pi^-_{4l+1,k} & : X_{4l+1} \to Y_{\overline{\overline{k+1}}}\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x_1, x_2, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & (y_k,z_k)\,, & \text{for } k\in[0,l-1]
\end{array}$$

$$\begin{array}{lllcll}
\iota^+_{4l+1,k} & : P \oplus P \to X_{4l+1}\,, & (v,w) & \mapsto & \big((\delta_{i,k}v,\delta_{i,k}w)_{i\in[0,l-1]}, 0, 0, (0,0)_{i\in[0,l-1]}\big)\,, & \text{for } k\in[0,l-1] \\
\iota'_{4l+1} & : P \oplus Q \to X_{4l+1}\,, & (x_1,x_2) & \mapsto & \big((0,0)_{i\in[0,l-1]}, x_1, x_2, (0,0)_{i\in[0,l-1]}\big) & \\
\iota^-_{4l+1,k} & : Y_{\overline{\overline{k+1}}} \to X_{4l+1}\,, & (y,z) & \mapsto & \big((0,0)_{i\in[0,l-1]}, 0, 0, (\delta_{i,k}y,\delta_{i,k}z)_{i\in[0,l-1]}\big)\,, & \text{for } k\in[0,l-1]
\end{array}$$

$$\begin{array}{lllcll}
\pi^+_{4l+2,k} & : X_{4l+2} \to P \oplus P\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x_1, x_2, x_3, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & (v_k,w_k)\,, & \text{for } k\in[0,l-1] \\
\pi'_{4l+2} & : X_{4l+2} \to P\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x_1, x_2, x_3, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & x_1 & \\
\pi''_{4l+2} & : X_{4l+2} \to P \oplus Q\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x_1, x_2, x_3, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & (x_2,x_3) & \\
\pi^-_{4l+2,k} & : X_{4l+2} \to Y_{\overline{\overline{k+2}}}\,, & \big((v_i,w_i)_{i\in[0,l-1]}, x_1, x_2, x_3, (y_i,z_i)_{i\in[0,l-1]}\big) & \mapsto & (y_k,z_k)\,, & \text{for } k\in[0,l-1]
\end{array}$$

$$\begin{array}{lllcll}
\iota^+_{4l+2,k} & : P \oplus P \to X_{4l+2}\,, & (v,w) & \mapsto & \big((\delta_{i,k}v,\delta_{i,k}w)_{i\in[0,l-1]}, 0, 0, 0, (0,0)_{i\in[0,l-1]}\big)\,, & \text{for } k\in[0,l-1] \\
\iota'_{4l+2} & : P \to X_{4l+2}\,, & x_1 & \mapsto & \big((0,0)_{i\in[0,l-1]}, x_1, 0, 0, (0,0)_{i\in[0,l-1]}\big) & \\
\iota''_{4l+2} & : P \oplus Q \to X_{4l+2}\,, & (x_2,x_3) & \mapsto & \big((0,0)_{i\in[0,l-1]}, 0, x_2, x_3, (0,0)_{i\in[0,l-1]}\big) & \\
\iota^-_{4l+2,k} & : Y_{\overline{\overline{k+2}}} \to X_{4l+2}\,, & (y,z) & \mapsto & \big((0,0)_{i\in[0,l-1]}, 0, 0, 0, (\delta_{i,k}y,\delta_{i,k}z)_{i\in[0,l-1]}\big)\,, & \text{for } k\in[0,l-1]
\end{array}$$

$$\begin{array}{lllcll}
\pi^+_{4l+3,k} & : X_{4l+3} \to P \oplus P\,, & \big((v_i,w_i)_{i\in[0,l]}, (y_i,z_i)_{i\in[0,l]}\big) & \mapsto & (v_k,w_k)\,, & \text{for } k\in[0,l] \\
\pi^-_{4l+3,k} & : X_{4l+3} \to Y_{\overline{\overline{k+2}}}\,, & \big((v_i,w_i)_{i\in[0,l]}, (y_i,z_i)_{i\in[0,l]}\big) & \mapsto & (y_k,z_k)\,, & \text{for } k\in[0,l]
\end{array}$$

$$\begin{array}{lllcll}
\iota^+_{4l+3,k} & : P \oplus P \to X_{4l+3}\,, & (v,w) & \mapsto & \big((\delta_{i,k}v,\delta_{i,k}w)_{i\in[0,l]}, (0,0)_{i\in[0,l]}\big)\,, & \text{for } k\in[0,l] \\
\iota^-_{4l+3,k} & : Y_{\overline{\overline{k+2}}} \to X_{4l+3}\,, & (y,z) & \mapsto & \big((0,0)_{i\in[0,l]}, (\delta_{i,k}y,\delta_{i,k}z)_{i\in[0,l]}\big)\,, & \text{for } k\in[0,l]
\end{array}$$

Remark 26 Suppose given $l \in \mathbb{Z}_{\geq 0}$, $i \in [0,3]$ and $k_1, k_2 \in [0, l-1]$. The following equations hold.

$$\begin{array}{rll}
\iota^{+}_{4l+i,k_1} \cdot \pi^{+}_{4l+i,k_2} = 0 & \text{if } k_1 \neq k_2 & \iota^{+}_{4l+i,k_1} \cdot \pi^{+}_{4l+i,k_1} = 1 \\
\iota^{+}_{4l+i,k_1} \cdot \pi'_{4l+i} = 0 & \text{if } i \in [0,2] & \\
\iota^{+}_{4l+i,k_1} \cdot \pi^{-}_{4l+i,k_2} = 0 & & \\
\iota'_{4l+i} \cdot \pi^{+}_{4l+i,k_2} = 0 & \text{if } i \in [0,2] & \iota'_{4l+i} \cdot \pi'_{4l+i} = 1 \quad \text{if } i \in [0,2] \\
\iota'_{4l+i} \cdot \pi^{-}_{4l+i,k_2} = 0 & \text{if } i \in [0,2] & \\
\iota^{-}_{4l+i,k_1} \cdot \pi^{+}_{4l+i,k_2} = 0 & & \\
\iota^{-}_{4l+i,k_1} \cdot \pi'_{4l+i} = 0 & \text{if } i \in [0,2] & \\
\iota^{-}_{4l+i,k_1} \cdot \pi^{-}_{4l+i,k_2} = 0 & \text{if } k_1 \neq k_2 & \iota^{-}_{4l+i,k_1} \cdot \pi^{-}_{4l+i,k_1} = 1 \\
& & \\
\iota^{+}_{4l+2,k_1} \cdot \pi''_{4l+2} = 0 & & \iota''_{4l+3} \cdot \pi''_{4l+3} = 1 \\
\iota'_{4l+2} \cdot \pi''_{4l+2} = 0 & & \\
\iota^{-}_{4l+2,k_1} \cdot \pi''_{4l+2} = 0 & & \\
\iota''_{4l+2} \cdot \pi^{+}_{4l+2,k_2} = 0 & & \\
\iota''_{4l+2} \cdot \pi'_{4l+2} = 0 & & \\
\iota''_{4l+2} \cdot \pi^{-}_{4l+2,k_2} = 0 & & \\
\iota^{+}_{4l+3,l} \cdot \pi^{+}_{4l+3,k_2} = 0 & & \iota^{+}_{4l+3,l} \cdot \pi^{+}_{4l+3,l} = 1 \\
\iota^{+}_{4l+3,k_1} \cdot \pi^{+}_{4l+3,l} = 0 & & \\
\iota^{+}_{4l+3,l} \cdot \pi^{-}_{4l+3,k_2} = 0 & & \\
\iota^{+}_{4l+3,k_1} \cdot \pi^{-}_{4l+3,l} = 0 & & \\
\iota^{-}_{4l+3,l} \cdot \pi^{+}_{4l+3,k_2} = 0 & & \\
\iota^{-}_{4l+3,k_1} \cdot \pi^{+}_{4l+3,l} = 0 & & \\
\iota^{-}_{4l+3,l} \cdot \pi^{-}_{4l+3,k_2} = 0 & & \\
\iota^{-}_{4l+3,k_1} \cdot \pi^{-}_{4l+3,l} = 0 & & \iota^{-}_{4l+3,l} \cdot \pi^{-}_{4l+3,l} = 1
\end{array}$$

2.5 The differentials of the projective resolution

We shall define the differentials in the augmented projective resolution

$$X' = \Big(\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{\varepsilon} R \longrightarrow 0 \longrightarrow \cdots \Big);$$

cf. Theorem 38 below.

Definition 27 Let

$$A := \begin{pmatrix} b^0_{0,1} & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} \end{pmatrix} : P \oplus P \to P \oplus P \qquad \widetilde{A} := \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} : P \oplus P \to P \oplus P$$

$$B_0 := \begin{pmatrix} b^1_{1,1} & b^1_{0,1} \\ -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix} \; : \; Q \oplus Q \to Q \oplus P \qquad B_1 := \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} \; : \; P \oplus Q \to Q \oplus Q$$

$$B_2 := \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} \; : \; Q \oplus P \to P \oplus Q \qquad C := \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \; : \; P \oplus P \to P \oplus P$$

$$\widetilde{C} := \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \; : \; P \oplus Q \to P \oplus P \qquad C' := \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} \; : \; P \to P \oplus P$$

$$D_0 := \begin{pmatrix} 0 & 0 \\ \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix} \; : \; Q \oplus P \to Q \oplus P \qquad D'_0 := \begin{pmatrix} \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix} \; : \; P \to Q \oplus P$$

$$D_1 := \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} \; : \; Q \oplus Q \to Q \oplus Q \qquad \widetilde{D}_1 := \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} \; : \; P \oplus Q \to Q \oplus Q$$

$$D_2 := \begin{pmatrix} 2b^0_{0,1} & 0 \\ b^1_{0,1} & 0 \end{pmatrix} \; : \; P \oplus Q \to P \oplus Q \qquad D_3 := \begin{pmatrix} 0 & 0 \\ -b^0_{1,1} & -2b^0_{0,1} \end{pmatrix} \; : \; Q \oplus P \to Q \oplus P$$

$$D_4 := \begin{pmatrix} -2\widetilde{b}^1_{1,0} & 0 \\ \widetilde{b}^1_{1,0} & 0 \end{pmatrix} \; : \; Q \oplus Q \to Q \oplus Q \qquad D_5 := \begin{pmatrix} 2\widetilde{b}^0_{0,1} & 0 \\ \widetilde{b}^1_{0,1} & 0 \end{pmatrix} \; : \; P \oplus Q \to P \oplus Q$$

$$d'_1 := \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \end{pmatrix} \; : \; P \to P \oplus Q \qquad d''_1 := \begin{pmatrix} b^0_{0,2} & b^0_{1,1} \\ 2b^1_{0,0} & b^1_{1,1} \end{pmatrix} \; : \; P \oplus Q \to P \oplus Q$$

$$d'_2 := \begin{pmatrix} b^0_{0,1} \\ b^0_{0,2} \end{pmatrix} \; : \; P \oplus P \to P \qquad d''_2 := \begin{pmatrix} 0 & 0 \\ -\widetilde{b}^0_{0,1} & 0 \end{pmatrix} \; : \; P \oplus P \to P \oplus Q\,.$$

Definition 28 Let

$$\varepsilon \; : \; P \to R, \; \left(\rho, \sigma, \begin{pmatrix} \alpha_{3,1} \\ \alpha_{3,2} \\ \alpha_{3,3} \end{pmatrix}, \begin{pmatrix} \beta_{3,1} \\ \beta_{3,2} \\ \beta_{3,3} \end{pmatrix}\right) \mapsto \rho$$

$$d_0 := \begin{pmatrix} b^0_{0,1} \\ \widetilde{b}^1_{0,0} \end{pmatrix} \quad : \quad P \oplus Q \to P$$

$$d_1 := \begin{pmatrix} d'_1 \\ d''_1 \end{pmatrix} = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & b^0_{1,1} \\ 2b^1_{0,0} & b^1_{1,1} \end{pmatrix} \quad : \quad P \oplus P \oplus Q \to P \oplus Q\,.$$

For $l \in \mathbb{Z}_{\geq 1}$ we define the following Λ-linear differentials.

$$\begin{aligned} d_{4l} \;\; := \;\; & \left(\sum_{k\in[0,l-1]} \pi^+_{4l+1,k} \; A \; \iota^+_{4l,k} \right) + \left(\sum_{k\in[0,l-2]} \pi^+_{4l+1,k+1} \; C \; \iota^+_{4l,k} \right) \\ & + \left(\pi'_{4l+1} \; \widetilde{C} \; \iota^+_{4l,l-1} \right) + \left(\pi'_{4l+1} \; d_0 \; \iota'_{4l} \right) + \left(\pi'_{4l+1} \; \widetilde{D}_1 \; \iota^-_{4l,0} \right) \\ & + \left(\sum_{k\in[0,l-1]} \pi^-_{4l+1,k} \; B_{\overline{\overline{k+1}}} \; \iota^-_{4l,k} \right) + \left(\sum_{k\in[0,l-2]} \pi^-_{4l+1,k} \; D_{\overline{4k+5}} \; \iota^-_{4l,k+1} \right) \quad : \quad X_{4l+1} \longrightarrow X_{4l} \end{aligned}$$

$$
\begin{aligned}
d_{4l+1} \; := \; & \left(\sum_{k\in[0,l-1]} \pi^{+}_{4l+2,k} \; \widetilde{A} \; \iota^{+}_{4l+1,k} \right) + \left(\sum_{k\in[0,l-2]} \pi^{+}_{4l+2,k+1} \; C \; \iota^{+}_{4l+1,k} \right) \\
& + \left(\pi'_{4l+2} \; C' \; \iota^{+}_{4l+1,l-1} \right) + \left(\pi'_{4l+2} \; d'_1 \; \iota'_{4l+1} \right) \\
& + \left(\pi''_{4l+2} \; d''_1 \; \iota'_{4l+1} \right) + \left(\pi''_{4l+2} \; D_2 \; \iota^{-}_{4l+1,0} \right) \\
& + \left(\sum_{k\in[0,l-1]} \pi^{-}_{4l+2,k} \; B_{\overline{\overline{k+2}}} \; \iota^{-}_{4l+1,k} \right) + \left(\sum_{k\in[0,l-2]} \pi^{-}_{4l+2,k} \; D_{\overline{4k}} \; \iota^{-}_{4l+1,k+1} \right) \quad : \quad X_{4l+2} \longrightarrow X_{4l+1}
\end{aligned}
$$

For $l \in \mathbb{Z}_{\geq 0}$ we define the following Λ-linear differentials.

$$
\begin{aligned}
d_{4l+2} \; := \; & \left(\sum_{k\in[0,l-1]} \pi^{+}_{4l+3,k} \; A \; \iota^{+}_{4l+2,k} \right) + \left(\sum_{k\in[0,l-1]} \pi^{+}_{4l+3,k+1} \; C \; \iota^{+}_{4l+2,k} \right) \\
& + \left(\pi^{+}_{4l+3,l} \; d'_2 \; \iota'_{4l+2} \right) + \left(\pi^{+}_{4l+3,l} \; d''_2 \; \iota''_{4l+2} \right) + \left(\pi^{-}_{4l+3,0} \; B_2 \; \iota''_{4l+2} \right) \\
& + \left(\sum_{k\in[0,l-1]} \pi^{-}_{4l+3,k+1} \; B_{\overline{\overline{k}}} \; \iota^{-}_{4l+2,k} \right) + \left(\sum_{k\in[0,l-1]} \pi^{-}_{4l+3,k} \; D_{\overline{4k+3}} \; \iota^{-}_{4l+2,k} \right) \quad : \quad X_{4l+3} \longrightarrow X_{4l+2}
\end{aligned}
$$

$$
\begin{aligned}
d_{4l+3} \; := \; & \left(\sum_{k\in[0,l]} \pi^{+}_{4l+4,k} \; \widetilde{A} \; \iota^{+}_{4l+3,k} \right) + \left(\sum_{k\in[0,l-1]} \pi^{+}_{4l+4,k+1} \; C \; \iota^{+}_{4l+3,k} \right) \\
& + \left(\pi'_{4l+4} \; C' \; \iota^{+}_{4l+3,l} \right) + \left(\pi'_{4l+4} \; D'_0 \; \iota^{-}_{4l+3,0} \right) \\
& + \left(\sum_{k\in[0,l]} \pi^{-}_{4l+4,k} \; B_{\overline{\overline{k}}} \; \iota^{-}_{4l+3,k} \right) + \left(\sum_{k\in[0,l-1]} \pi^{-}_{4l+4,k} \; D_{\overline{4k+4}} \; \iota^{-}_{4l+3,k+1} \right) \quad : \quad X_{4l+4} \longrightarrow X_{4l+3}
\end{aligned}
$$

Remark 29 The first few differentials from Definition 28 can be written as follows. Note that the matrix of the differential d_k is part of the matrix of the differential d_{k+4} for $k \geq 0$. This is marked with a surrounding box.

$$
d_0 = \begin{pmatrix} b^0_{0,1} \\ \widetilde{b}^1_{0,0} \end{pmatrix} \; : \; P \oplus Q \to P
$$

$$
d_1 = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & b^0_{1,1} \\ 2b^1_{0,0} & b^1_{1,1} \end{pmatrix} = \begin{pmatrix} d'_1 \\ d''_1 \end{pmatrix} \; : \; P \oplus P \oplus Q \to P \oplus Q
$$

$$
d_2 = \begin{pmatrix} b^0_{0,1} & 0 & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 \\ 0 & -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ 0 & -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} = \begin{pmatrix} d'_2 & d''_2 \\ 0 & B_2 \end{pmatrix} \; : \; P \oplus P \oplus Q \oplus P \to P \oplus P \oplus Q
$$

$$
d_3 = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 & 0 & 0 \\ b^0_{0,2} & -b^0_{0,1} & 0 & 0 \\ 0 & \widetilde{b}^0_{0,2} & \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \\ 0 & 0 & b^1_{1,1} & b^1_{0,1} \\ 0 & 0 & -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix} = \begin{pmatrix} \widetilde{A} & 0 \\ C' & D'_0 \\ 0 & B_0 \end{pmatrix} \; : \; P \oplus P \oplus P \oplus Q \oplus Q \to P \oplus P \oplus Q \oplus P
$$

$$d_4 = \begin{pmatrix} b^0_{0,1} & 0 & 0 & 0 & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 & 0 & 0 \\ 0 & \widetilde{b}^0_{0,2} & b^0_{0,1} & 0 & 0 \\ 0 & 0 & \widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \\ 0 & 0 & 0 & b^0_{1,0} & b^0_{1,1} \\ 0 & 0 & 0 & 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ \widetilde{C} & \boxed{d_0} & \widetilde{D}_1 \\ 0 & 0 & B_1 \end{pmatrix} : \quad \begin{array}{l} P\oplus P\oplus P\oplus Q\oplus P\oplus Q \\ \quad \to P\oplus P\oplus P\oplus Q\oplus Q \end{array}$$

$$d_5 = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 \\ b^0_{0,2} & -b^0_{0,1} & 0 & 0 & 0 & 0 \\ 0 & \widetilde{b}^0_{0,2} & \widetilde{b}^0_{0,1} & 0 & 0 & 0 \\ 0 & 0 & b^0_{0,2} & b^0_{1,1} & 2b^0_{0,1} & 0 \\ 0 & 0 & 2b^1_{0,0} & b^1_{1,1} & b^1_{0,1} & 0 \\ 0 & 0 & 0 & 0 & -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ 0 & 0 & 0 & 0 & -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} = \begin{pmatrix} \widetilde{A} & 0 & 0 \\ C' & d'_1 & 0 \\ 0 & d''_1 & D_2 \\ 0 & 0 & B_2 \end{pmatrix} : \quad \begin{array}{l} P\oplus P\oplus P\oplus P\oplus Q\oplus Q\oplus P \\ \quad \to P\oplus P\oplus P\oplus Q\oplus P\oplus Q \end{array}$$

$$d_6 = \begin{pmatrix} b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{b}^0_{0,2} & b^0_{0,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} & 0 & 0 \\ 0 & 0 & 0 & -2b^0_{0,0} & b^0_{1,0} & -b^0_{1,1} & -2b^0_{0,1} \\ 0 & 0 & 0 & 0 & 0 & b^1_{1,1} & b^1_{0,1} \\ 0 & 0 & 0 & 0 & 0 & -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 \\ C & d'_2 & d''_2 & 0 \\ 0 & 0 & B_2 & D_3 \\ 0 & 0 & 0 & B_0 \end{pmatrix} : \quad \begin{array}{l} P\oplus P\oplus P\oplus P\oplus Q\oplus P\oplus Q\oplus Q \\ \quad \to P\oplus P\oplus P\oplus P\oplus Q\oplus Q\oplus P \end{array}$$

$$d_7 = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b^0_{0,2} & -b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{b}^0_{0,2} & \widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^0_{0,2} & -b^0_{0,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{b}^0_{0,2} & \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & b^1_{1,1} & b^1_{0,1} & -2\widetilde{b}^1_{1,0} & 0 \\ 0 & 0 & 0 & 0 & -2b^1_{1,0} & \widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b^0_{1,0} & b^0_{1,1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} = \begin{pmatrix} \widetilde{A} & 0 & 0 & 0 \\ C & \widetilde{A} & 0 & 0 \\ 0 & C' & D'_0 & 0 \\ 0 & 0 & B_0 & D_4 \\ 0 & 0 & 0 & B_1 \end{pmatrix} : \quad \begin{array}{l} P\oplus P\oplus P\oplus P\oplus P\oplus Q\oplus Q\oplus P\oplus Q \\ \quad \to P\oplus P\oplus P\oplus P\oplus Q\oplus P\oplus Q\oplus Q \end{array}$$

$$d_8 = \begin{pmatrix} b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{b}^0_{0,2} & b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{b}^0_{0,2} & b^0_{0,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b^0_{1,0} & b^0_{1,1} & 2\widetilde{b}^0_{0,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2b^1_{1,0} & b^1_{1,1} & \widetilde{b}^1_{0,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ C & A & 0 & 0 & 0 \\ 0 & \widetilde{C} & d_0 & \widetilde{D}_1 & 0 \\ 0 & 0 & 0 & B_1 & D_5 \\ 0 & 0 & 0 & 0 & B_2 \end{pmatrix} : \quad \begin{array}{l} P\oplus P\oplus P\oplus P\oplus P\oplus P\oplus Q\oplus P\oplus Q\oplus Q\oplus P \\ \quad \to P\oplus P\oplus P\oplus P\oplus P\oplus Q\oplus Q\oplus P\oplus Q \end{array}$$

$$d_9 = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b^0_{0,2} & -b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{b}^0_{0,2} & \widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^0_{0,2} & -b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{b}^0_{0,2} & \widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b^0_{0,2} & b^0_{1,1} & 2b^0_{0,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2b^1_{0,0} & b^1_{1,1} & b^1_{0,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2b^0_{0,0} & b^0_{1,0} & \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^1_{1,1} & b^1_{0,1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix} = \begin{pmatrix} \widetilde{A} & 0 & 0 & 0 & 0 \\ C & \widetilde{A} & 0 & 0 & 0 \\ 0 & C' & d'_1 & 0 & 0 \\ 0 & 0 & d''_1 & D_2 & 0 \\ 0 & 0 & 0 & B_2 & D_0 \\ 0 & 0 & 0 & 0 & B_0 \end{pmatrix} :$$

$$P{\oplus}P{\oplus}P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}Q{\oplus}P{\oplus}Q{\oplus}Q \to P{\oplus}P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}P{\oplus}Q{\oplus}Q{\oplus}P$$

$$d_{10} = \begin{pmatrix} b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widetilde{b}^0_{0,2} & b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{b}^0_{0,2} & b^0_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2b^0_{0,0} & b^0_{1,0} & -b^0_{1,1} & -2b^0_{0,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^1_{1,1} & b^1_{0,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2b^1_{1,0} & \widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^0_{1,0} & b^0_{1,1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ C & A & 0 & 0 & 0 & 0 \\ 0 & C & d'_2 & d''_2 & 0 & 0 \\ 0 & 0 & 0 & B_2 & D_3 & 0 \\ 0 & 0 & 0 & 0 & B_0 & D_1 \\ 0 & 0 & 0 & 0 & 0 & B_1 \end{pmatrix} :$$

$$P{\oplus}P{\oplus}P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}P{\oplus}Q{\oplus}Q{\oplus}P \to P{\oplus}P{\oplus}P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}Q{\oplus}P{\oplus}Q{\oplus}Q$$

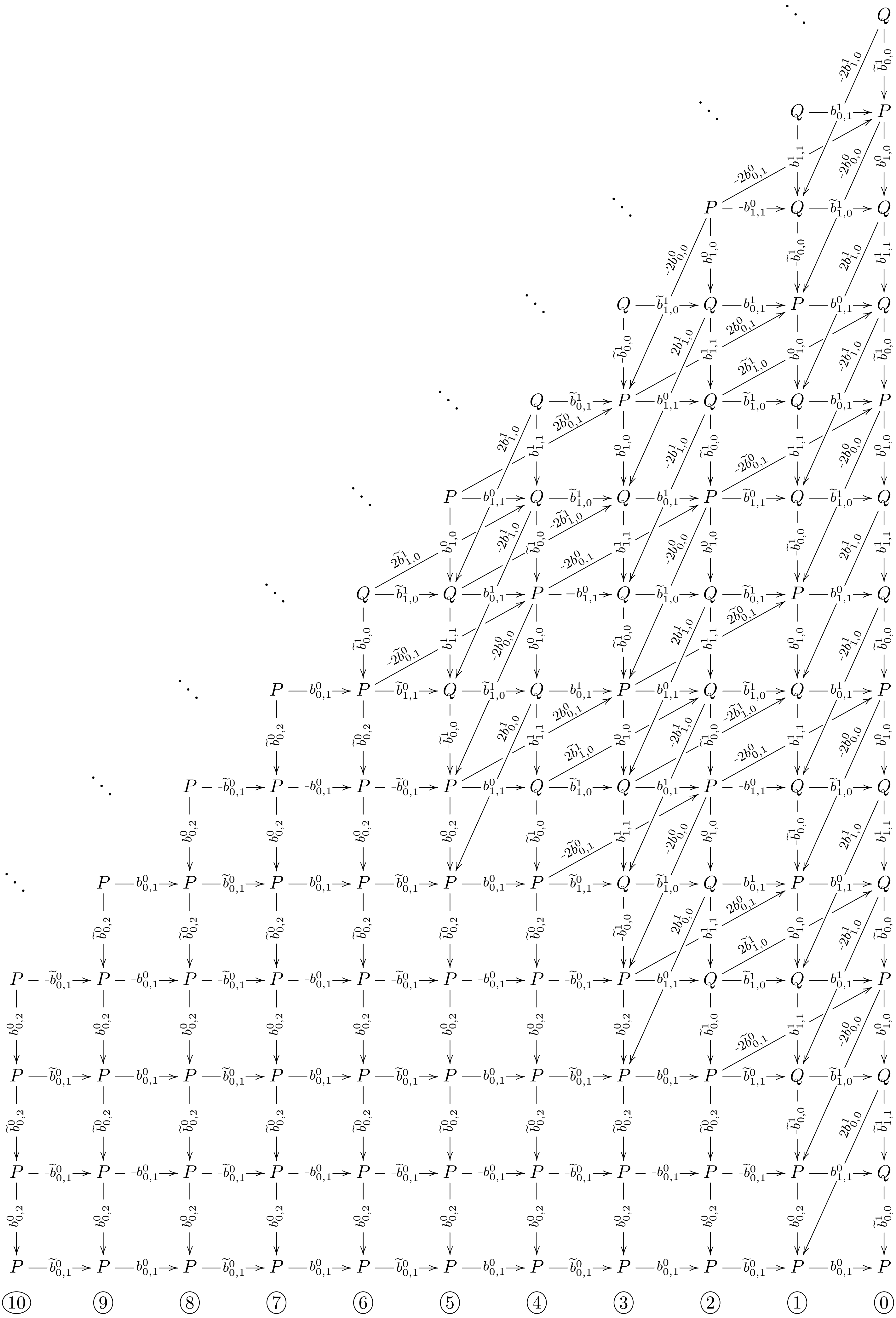
These differentials can be visualized in the following diagram.

The projective objects X_k of X appear as the direct sum over the modules in the kth diagonal. Moreover, the differentials d_k appear as matrices consisting of the maps between the $(k+1)$st and kth diagonal.

Every row and column of this diagram eventually becomes periodic. For instance, the diagonal maps from column number 2 to column number 0 eventually repeat with period length 6:

$$\begin{aligned} &0,\quad 0,\quad -2\widetilde{b}^0_{0,1},\quad 2\widetilde{b}^1_{1,0},\quad 0,\quad -2b^0_{0,1},\quad 0,\quad 0, \\ &\qquad\qquad -2\widetilde{b}^0_{0,1},\quad 2\widetilde{b}^1_{1,0},\quad 0,\quad -2b^0_{0,1},\quad \ldots \end{aligned}$$

Note that we do not claim commutativity of this diagram.


⑩ ⑨ ⑧ ⑦ ⑥ ⑤ ④ ③ ② ① ⓪

We now collect some auxiliary calculations which we shall use to verify that the augmented projective resolution X' is in fact a complex; cf. Theorem 38 below.

Lemma 30 The following assertions (i-xiv) hold.

(i) $\widetilde{A} \cdot A = 0$
$A \cdot \widetilde{A} = 0$

(ii) $C \cdot A + \widetilde{A} \cdot C = 0$
$C \cdot \widetilde{A} + A \cdot C = 0$

(iii) $C \cdot C = 0$
$C' \cdot C = 0$
$\widetilde{C} \cdot C = 0$

(iv) $C' \cdot A + d_1' \cdot \widetilde{C} = 0$
$d_1' \cdot \widetilde{D}_1 = 0$
$d_1'' \cdot \widetilde{C} = 0$
$d_1'' \cdot \widetilde{D}_1 + D_2 \cdot B_1 = 0$

(v) $C \cdot \widetilde{A} + d_2' \cdot C' = 0$
$d_2'' \cdot D_2 = 0$

(vi) $\widetilde{A} \cdot d_2' = 0$
$\widetilde{A} \cdot d_2'' = 0$
$C' \cdot d_2' = 0$
$C' \cdot d_2'' + D_0' \cdot B_2 = 0$
$D_0' \cdot D_3 = 0$

(vii) $\widetilde{C} \cdot \widetilde{A} + d_0 \cdot C' = 0$
$d_0 \cdot D_0' + \widetilde{D}_1 \cdot B_0 = 0$
$\widetilde{D}_1 \cdot D_4 = 0$

(viii) $d_0 \cdot \varepsilon = 0$

(ix) $d_1 \cdot d_0 = 0$
$d_2 \cdot d_1 = 0$

(x) $B_0 \cdot B_2 = 0$
$B_1 \cdot B_0 = 0$
$B_2 \cdot B_1 = 0$

(xi) $D_0 \cdot D_3 = 0$
$D_4 \cdot D_1 = 0$
$D_2 \cdot D_5 = 0$

(xii) $D_1 \cdot D_4 = 0$
$D_5 \cdot D_2 = 0$
$D_3 \cdot D_0 = 0$

(xiii) $B_0 \cdot D_3 + D_4 \cdot B_0 = 0$
$B_1 \cdot D_1 + D_2 \cdot B_1 = 0$
$B_2 \cdot D_5 + D_0 \cdot B_2 = 0$

(xiv) $B_0 \cdot D_0 + D_1 \cdot B_0 = 0$
$B_1 \cdot D_4 + D_5 \cdot B_1 = 0$
$B_2 \cdot D_2 + D_3 \cdot B_2 = 0$

Proof. We calculate as follows, using Lemma 22.

(i)
$$\widetilde{A} \cdot A = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} \begin{pmatrix} b^0_{0,1} & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} \end{pmatrix} = \begin{pmatrix} \widetilde{b}^0_{0,1} b^0_{0,1} & 0 \\ b^0_{0,2} b^0_{0,1} - b^0_{0,1} b^0_{0,2} & b^0_{0,1} \widetilde{b}^0_{0,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$A \cdot \widetilde{A} = \begin{pmatrix} b^0_{0,1} & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} \end{pmatrix} \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} = \begin{pmatrix} b^0_{0,1} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} \widetilde{b}^0_{0,1} - \widetilde{b}^0_{0,1} b^0_{0,2} & \widetilde{b}^0_{0,1} b^0_{0,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(ii)
$$C \cdot A + \widetilde{A} \cdot C = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b^0_{0,1} & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} \end{pmatrix} + \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \widetilde{b}^0_{0,2} b^0_{0,2} & -\widetilde{b}^0_{0,2} \widetilde{b}^0_{0,1} + \widetilde{b}^0_{0,1} \widetilde{b}^0_{0,2} \\ 0 & b^0_{0,2} \widetilde{b}^0_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$C \cdot \widetilde{A} + A \cdot C = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} + \begin{pmatrix} b^0_{0,1} & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \widetilde{b}^0_{0,2} b^0_{0,2} & -\widetilde{b}^0_{0,2} b^0_{0,1} + b^0_{0,1} \widetilde{b}^0_{0,2} \\ 0 & b^0_{0,2} \widetilde{b}^0_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(iii)
$$C \cdot C = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$C' \cdot C = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$
$$\widetilde{C} \cdot C = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(iv)
$$C' \cdot A + d'_1 \cdot \widetilde{C} = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} \begin{pmatrix} b^0_{0,1} & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} \end{pmatrix} + \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \widetilde{b}^0_{0,2} b^0_{0,2} & -\widetilde{b}^0_{0,2} \widetilde{b}^0_{0,1} + \widetilde{b}^0_{0,1} \widetilde{b}^0_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$
$$d'_1 \cdot \widetilde{D}_1 = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$
$$d''_1 \cdot \widetilde{C} = \begin{pmatrix} b^0_{0,2} & b^0_{1,1} \\ 2b^1_{0,0} & b^1_{1,1} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^0_{0,2} \widetilde{b}^0_{0,2} \\ 0 & 2b^1_{0,0} \widetilde{b}^0_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$d''_1 \cdot \widetilde{D}_1 + D_2 \cdot B_1 = \begin{pmatrix} b^0_{0,2} & b^0_{1,1} \\ 2b^1_{0,0} & b^1_{1,1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} + \begin{pmatrix} 2b^0_{0,1} & 0 \\ b^1_{0,1} & 0 \end{pmatrix} \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix}$$
$$= \begin{pmatrix} b^0_{1,1} \widetilde{b}^1_{1,0} + 2b^0_{0,1} b^0_{1,0} & 2b^0_{1,1} \widetilde{b}^1_{1,0} + 2b^0_{0,1} b^0_{1,1} \\ b^1_{1,1} \widetilde{b}^1_{1,0} + b^1_{0,1} b^0_{1,0} & 2b^1_{1,1} \widetilde{b}^1_{1,0} + b^1_{0,1} b^0_{1,1} \end{pmatrix}$$
$$= \begin{pmatrix} -2b^0_{1,1} + 2b^0_{1,1} & -4b^0_{1,1} + 4b^0_{1,1} \\ -2b^1_{1,1} + b^1_{1,2} + 2b^1_{1,1} - b^1_{1,2} & -4b^1_{1,1} + 2b^1_{1,2} + 4b^1_{1,1} - 2b^1_{1,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(v) $C \cdot \widetilde{A} + d_2' \cdot C' = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} + \begin{pmatrix} b^0_{0,1} \\ b^0_{0,2} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix}$

$$= \begin{pmatrix} \widetilde{b}^0_{0,2} b^0_{0,2} & -\widetilde{b}^0_{0,2} b^0_{0,1} + b^0_{0,1}\widetilde{b}^0_{0,2} \\ 0 & b^0_{0,2}\widetilde{b}^0_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$d_2'' \cdot D_2 = \begin{pmatrix} 0 & 0 \\ -\widetilde{b}^0_{0,1} & 0 \end{pmatrix} \begin{pmatrix} 2b^0_{0,1} & 0 \\ b^1_{0,1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2\widetilde{b}^0_{0,1} b^0_{0,1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(vi) $\widetilde{A} \cdot d_2' = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} \begin{pmatrix} b^0_{0,1} \\ b^0_{0,2} \end{pmatrix} = \begin{pmatrix} \widetilde{b}^0_{0,1} b^0_{0,1} \\ b^0_{0,2} b^0_{0,1} - b^0_{0,1} b^0_{0,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\widetilde{A} \cdot d_2'' = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\widetilde{b}^0_{0,1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b^0_{0,1}\widetilde{b}^0_{0,1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C' \cdot d_2' = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} \begin{pmatrix} b^0_{0,1} \\ b^0_{0,2} \end{pmatrix} = \widetilde{b}^0_{0,2} b^0_{0,2} = 0$$

$$C' \cdot d_2'' + D_0' \cdot B_2 = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\widetilde{b}^0_{0,1} & 0 \end{pmatrix} + \begin{pmatrix} \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix} \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix}$$

$$= \begin{pmatrix} -\widetilde{b}^0_{0,2}\widetilde{b}^0_{0,1} - \widetilde{b}^0_{1,1}\widetilde{b}^1_{0,0} + 4\widetilde{b}^0_{0,1} b^0_{0,0} & \widetilde{b}^0_{1,1}\widetilde{b}^1_{1,0} - 2\widetilde{b}^0_{0,1} b^0_{1,0} \end{pmatrix}$$

$$= \begin{pmatrix} -4\widetilde{b}^0_{0,1} + \widetilde{b}^0_{0,3} - \widetilde{b}^0_{0,3} + 4\widetilde{b}^0_{0,1} & 2\widetilde{b}^0_{1,1} - 2\widetilde{b}^0_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$D_0' \cdot D_3 = \begin{pmatrix} \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -b^0_{1,1} & -2b^0_{0,1} \end{pmatrix} = \begin{pmatrix} 2\widetilde{b}^0_{0,1} b^0_{1,1} & 4\widetilde{b}^0_{0,1} b^0_{0,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

(vii) $\widetilde{C} \cdot \widetilde{A} + d_0 \cdot C' = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} + \begin{pmatrix} b^0_{0,1} \\ \widetilde{b}^1_{0,0} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix}$

$$= \begin{pmatrix} \widetilde{b}^0_{0,2} b^0_{0,2} & -\widetilde{b}^0_{0,2} b^0_{0,1} + b^0_{0,1}\widetilde{b}^0_{0,2} \\ 0 & \widetilde{b}^1_{0,0}\widetilde{b}^0_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$d_0 \cdot D_0' + \widetilde{D}_1 \cdot B_0 = \begin{pmatrix} b^0_{0,1} \\ \widetilde{b}^1_{0,0} \end{pmatrix} \begin{pmatrix} \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} \begin{pmatrix} b^1_{1,1} & b^1_{0,1} \\ -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix}$$

$$= \begin{pmatrix} b^0_{0,1}\widetilde{b}^0_{1,1} & -2b^0_{0,1}\widetilde{b}^0_{0,1} \\ \widetilde{b}^1_{0,0}\widetilde{b}^0_{1,1} + \widetilde{b}^1_{1,0} b^1_{1,1} - 4\widetilde{b}^1_{1,0} b^1_{1,0} & -2\widetilde{b}^1_{0,0}\widetilde{b}^0_{0,1} + \widetilde{b}^1_{1,0} b^1_{0,1} + 2\widetilde{b}^1_{1,0}\widetilde{b}^1_{0,0} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 2\widetilde{b}^1_{1,1} - b^1_{1,2} + b^1_{1,2} - 2b^1_{1,1} - 4\widetilde{b}^1_{1,0} & -2\widetilde{b}^1_{0,1} - 2b^1_{0,1} + 4b^1_{0,0} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\widetilde{D}_1 \cdot D_4 = \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} \begin{pmatrix} -2\widetilde{b}^1_{1,0} & 0 \\ \widetilde{b}^1_{1,0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2\widetilde{b}^1_{1,0}\widetilde{b}^1_{1,0} + 2\widetilde{b}^1_{1,0}\widetilde{b}^1_{1,0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(viii) Suppose given $\left(\rho, \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right) \in P$ and $\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}\right) \in Q$. We have

$$\left(\left(\rho, \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right), \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}\right)\right) d_0 \cdot \varepsilon$$

$$= \left(0, 2\sigma, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\beta_1 \\ 2\beta_2 \\ 2\beta_3 \end{pmatrix}\right) \varepsilon + \left(0, 0, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} -\beta_1 \\ -\beta_2 \\ -\beta_3 \end{pmatrix}\right) \varepsilon = 0 + 0 = 0\,.$$

$$\text{(ix)}\quad d_1\cdot d_0=\begin{pmatrix}\widetilde{b}^0_{0,1} & 0\\ b^0_{0,2} & b^0_{1,1}\\ 2b^1_{0,0} & b^1_{1,1}\end{pmatrix}\begin{pmatrix}b^0_{0,1}\\ \widetilde{b}^1_{0,0}\end{pmatrix}=\begin{pmatrix}\widetilde{b}^0_{0,1}b^0_{0,1}\\ b^0_{0,2}b^0_{0,1}+b^0_{1,1}\widetilde{b}^1_{0,0}\\ 2b^1_{0,0}b^0_{0,1}+b^1_{1,1}\widetilde{b}^1_{0,0}\end{pmatrix}=\begin{pmatrix}0\\ b^0_{0,3}-b^0_{0,3}\\ 2b^1_{0,1}-2b^1_{0,1}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$d_2\cdot d_1=\begin{pmatrix}b^0_{0,1} & 0 & 0\\ b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0\\ 0 & -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0}\\ 0 & -2b^0_{0,0} & b^0_{1,0}\end{pmatrix}\begin{pmatrix}\widetilde{b}^0_{0,1} & 0\\ b^0_{0,2} & b^0_{1,1}\\ 2b^1_{0,0} & b^1_{1,1}\end{pmatrix}=\begin{pmatrix}b^0_{0,1}\widetilde{b}^0_{0,1} & 0\\ b^0_{0,2}\widetilde{b}^0_{0,1}-\widetilde{b}^0_{0,1}b^0_{0,2} & -\widetilde{b}^0_{0,1}b^0_{1,1}\\ -\widetilde{b}^1_{0,0}b^0_{0,2}+2\widetilde{b}^1_{1,0}b^1_{0,0} & -\widetilde{b}^1_{0,0}b^0_{1,1}+\widetilde{b}^1_{1,0}b^1_{1,1}\\ -2b^0_{0,0}b^0_{0,2}+2b^0_{1,0}b^1_{0,0} & -2b^0_{0,0}b^0_{1,1}+b^0_{1,0}b^1_{1,1}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0\\ 0 & 0\\ -4\widetilde{b}^1_{0,0}+4\widetilde{b}^1_{0,0} & 2b^1_{1,1}-b^1_{1,2}-2b^1_{1,1}+b^1_{1,2}\\ -2b^0_{0,2}+2b^0_{0,2} & -2b^0_{1,1}+2b^0_{1,1}\end{pmatrix}=\begin{pmatrix}0&0\\0&0\\0&0\\0&0\end{pmatrix}$$

$$\text{(x)}\quad B_0\cdot B_2=\begin{pmatrix}b^1_{1,1} & b^1_{0,1}\\ -2b^1_{1,0} & \widetilde{b}^1_{0,0}\end{pmatrix}\begin{pmatrix}-\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0}\\ -2b^0_{0,0} & b^0_{1,0}\end{pmatrix}=\begin{pmatrix}-b^1_{1,1}\widetilde{b}^1_{0,0}-2b^1_{0,1}b^0_{0,0} & b^1_{1,1}\widetilde{b}^1_{1,0}+b^1_{0,1}b^0_{1,0}\\ 2b^1_{1,0}\widetilde{b}^1_{0,0}-2\widetilde{b}^1_{0,0}b^0_{0,0} & -2b^1_{1,0}\widetilde{b}^1_{1,0}+\widetilde{b}^1_{0,0}b^0_{1,0}\end{pmatrix}$$

$$=\begin{pmatrix}2b^1_{0,1}-2b^1_{0,1} & -2b^1_{1,1}+b^1_{1,2}+2b^1_{1,1}-b^1_{1,2}\\ 2\widetilde{b}^1_{0,0}-2\widetilde{b}^1_{0,0} & -2\widetilde{b}^1_{1,0}+2\widetilde{b}^1_{1,0}\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$B_1\cdot B_0=\begin{pmatrix}b^0_{1,0} & b^0_{1,1}\\ 2b^1_{1,0} & b^1_{1,1}\end{pmatrix}\begin{pmatrix}b^1_{1,1} & b^1_{0,1}\\ -2b^1_{1,0} & \widetilde{b}^1_{0,0}\end{pmatrix}=\begin{pmatrix}b^0_{1,0}b^1_{1,1}-2b^0_{1,1}b^1_{1,0} & b^0_{1,0}b^1_{0,1}+b^0_{1,1}\widetilde{b}^1_{0,0}\\ 2b^1_{1,0}b^1_{1,1}-2b^1_{1,1}b^1_{1,0} & 2b^1_{1,0}b^1_{0,1}+b^1_{1,1}\widetilde{b}^1_{0,0}\end{pmatrix}$$

$$=\begin{pmatrix}2b^0_{1,1}-2b^0_{1,1} & b^0_{0,3}-b^0_{0,3}\\ 2b^1_{1,1}-2b^1_{1,1} & 2b^1_{0,1}-2b^1_{0,1}\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$B_2\cdot B_1=\begin{pmatrix}-\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0}\\ -2b^0_{0,0} & b^0_{1,0}\end{pmatrix}\begin{pmatrix}b^0_{1,0} & b^0_{1,1}\\ 2b^1_{1,0} & b^1_{1,1}\end{pmatrix}=\begin{pmatrix}-\widetilde{b}^1_{0,0}b^0_{1,0}+2\widetilde{b}^1_{1,0}b^1_{1,0} & -\widetilde{b}^1_{0,0}b^0_{1,1}+\widetilde{b}^1_{1,0}b^1_{1,1}\\ -2b^0_{0,0}b^0_{1,0}+2b^0_{1,0}b^1_{1,0} & -2b^0_{0,0}b^0_{1,1}+b^0_{1,0}b^1_{1,1}\end{pmatrix}$$

$$=\begin{pmatrix}-2\widetilde{b}^1_{1,0}+2\widetilde{b}^1_{1,0} & 2b^1_{1,1}-b^1_{1,2}+b^1_{1,2}-2b^1_{1,1}\\ -2b^0_{1,0}+2b^0_{1,0} & -2b^0_{1,1}+2b^0_{1,1}\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\text{(xi)}\quad D_0\cdot D_3=\begin{pmatrix}0 & 0\\ \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1}\end{pmatrix}\begin{pmatrix}0 & 0\\ -b^0_{1,1} & -2b^0_{0,1}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 2\widetilde{b}^0_{0,1}b^0_{1,1} & 4\widetilde{b}^0_{0,1}b^0_{0,1}\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$D_4\cdot D_1=\begin{pmatrix}-2\widetilde{b}^1_{1,0} & 0\\ \widetilde{b}^1_{1,0} & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0}\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$D_2\cdot D_5=\begin{pmatrix}2b^0_{0,1} & 0\\ b^1_{0,1} & 0\end{pmatrix}\begin{pmatrix}2\widetilde{b}^0_{0,1} & 0\\ \widetilde{b}^1_{0,1} & 0\end{pmatrix}=\begin{pmatrix}4b^0_{0,1}\widetilde{b}^0_{0,1} & 0\\ 2b^1_{0,1}\widetilde{b}^0_{0,1} & 0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\text{(xii)}\quad D_1\cdot D_4=\begin{pmatrix}0 & 0\\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0}\end{pmatrix}\begin{pmatrix}-2\widetilde{b}^1_{1,0} & 0\\ \widetilde{b}^1_{1,0} & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ -2\widetilde{b}^1_{1,0}\widetilde{b}^1_{1,0}+2\widetilde{b}^1_{1,0}\widetilde{b}^1_{1,0} & 0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$D_5\cdot D_2=\begin{pmatrix}2\widetilde{b}^0_{0,1} & 0\\ \widetilde{b}^1_{0,1} & 0\end{pmatrix}\begin{pmatrix}2b^0_{0,1} & 0\\ b^1_{0,1} & 0\end{pmatrix}=\begin{pmatrix}4\widetilde{b}^0_{0,1}b^0_{0,1} & 0\\ 2\widetilde{b}^1_{0,1}b^0_{0,1} & 0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$D_3\cdot D_0=\begin{pmatrix}0 & 0\\ -b^0_{1,1} & -2b^0_{0,1}\end{pmatrix}\begin{pmatrix}0 & 0\\ \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1}\end{pmatrix}=\begin{pmatrix}0 & 0\\ -2b^0_{0,1}\widetilde{b}^0_{1,1} & 4b^0_{0,1}\widetilde{b}^0_{0,1}\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\text{(xiii)}\quad B_0\cdot D_3+D_4\cdot B_0=\begin{pmatrix}b^1_{1,1} & b^1_{0,1}\\ -2b^1_{1,0} & \widetilde{b}^1_{0,0}\end{pmatrix}\begin{pmatrix}0 & 0\\ -b^0_{1,1} & -2b^0_{0,1}\end{pmatrix}+\begin{pmatrix}-2\widetilde{b}^1_{1,0} & 0\\ \widetilde{b}^1_{1,0} & 0\end{pmatrix}\begin{pmatrix}b^1_{1,1} & b^1_{0,1}\\ -2b^1_{1,0} & \widetilde{b}^1_{0,0}\end{pmatrix}$$

$$=\begin{pmatrix}-b^1_{0,1}b^0_{1,1}-2\widetilde{b}^1_{1,0}b^1_{1,1} & -2b^1_{0,1}b^0_{0,1}-2\widetilde{b}^1_{1,0}b^1_{0,1}\\ -\widetilde{b}^1_{0,0}b^0_{1,1}+\widetilde{b}^1_{1,0}b^1_{1,1} & -2\widetilde{b}^1_{0,0}b^0_{0,1}+\widetilde{b}^1_{1,0}b^1_{0,1}\end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} -4b^1_{1,1}+2b^1_{1,2}+4b^1_{1,1}-2b^1_{1,2} & -4b^1_{0,1}+4b^1_{0,1} \\ -b^1_{1,2}+2b^1_{1,1}+b^1_{1,2}-2b^1_{1,1} & 2b^1_{0,1}-2b^1_{0,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_1\cdot D_1 + D_2\cdot B_1 &= \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} + \begin{pmatrix} 2b^0_{0,1} & 0 \\ b^1_{0,1} & 0 \end{pmatrix} \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} \\
&= \begin{pmatrix} b^0_{1,1}\widetilde{b}^1_{1,0}+2b^0_{0,1}b^0_{1,0} & 2b^0_{1,1}\widetilde{b}^1_{1,0}+2b^0_{0,1}b^0_{1,1} \\ b^1_{1,1}\widetilde{b}^1_{1,0}+b^1_{0,1}b^0_{1,0} & 2b^1_{1,1}\widetilde{b}^1_{1,0}+b^1_{0,1}b^0_{1,1} \end{pmatrix} \\
&= \begin{pmatrix} -2b^0_{1,1}+2b^0_{1,1} & -4b^0_{1,1}+4b^0_{1,1} \\ -2b^1_{1,1}+b^1_{1,2}+2b^1_{1,1}-b^1_{1,2} & -4b^1_{1,1}+2b^1_{1,2}+4b^1_{1,1}-2b^1_{1,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_2\cdot D_5 + D_0\cdot B_2 &= \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} \begin{pmatrix} 2\widetilde{b}^0_{0,1} & 0 \\ \widetilde{b}^1_{0,1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix} \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} \\
&= \begin{pmatrix} -2\widetilde{b}^1_{0,0}\widetilde{b}^0_{0,1}+\widetilde{b}^1_{1,0}\widetilde{b}^1_{0,1} & 0 \\ -4b^0_{0,0}\widetilde{b}^0_{0,1}+b^0_{1,0}\widetilde{b}^1_{0,1}-\widetilde{b}^0_{1,1}\widetilde{b}^1_{0,0}+4\widetilde{b}^0_{0,1}b^0_{0,0} & \widetilde{b}^0_{1,1}\widetilde{b}^1_{1,0}-2\widetilde{b}^0_{0,1}b^0_{1,0} \end{pmatrix} \\
&= \begin{pmatrix} -2\widetilde{b}^1_{0,1}+2\widetilde{b}^1_{0,1} & 0 \\ -4\widetilde{b}^0_{0,1}+\widetilde{b}^0_{0,3}-\widetilde{b}^0_{0,3}+4\widetilde{b}^0_{0,1} & 2\widetilde{b}^0_{1,1}-2\widetilde{b}^0_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\text{(xiv)}\quad B_0\cdot D_0 + D_1\cdot B_0 &= \begin{pmatrix} b^1_{1,1} & b^1_{0,1} \\ -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} \begin{pmatrix} b^1_{1,1} & b^1_{0,1} \\ -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix} \\
&= \begin{pmatrix} b^1_{0,1}\widetilde{b}^0_{1,1} & -2b^1_{0,1}\widetilde{b}^0_{0,1} \\ \widetilde{b}^1_{0,0}\widetilde{b}^0_{1,1}+\widetilde{b}^1_{1,0}b^1_{1,1}-4\widetilde{b}^1_{1,0}b^1_{1,0} & -2\widetilde{b}^1_{0,0}\widetilde{b}^0_{0,1}+\widetilde{b}^1_{1,0}b^1_{0,1}+2\widetilde{b}^1_{1,0}\widetilde{b}^1_{0,0} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 2\widetilde{b}^1_{1,1}-b^1_{1,2}-2b^1_{1,1}+b^1_{1,2}-4\widetilde{b}^1_{1,0} & -2\widetilde{b}^1_{0,1}-2b^1_{0,1}+4b^1_{0,0} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_1\cdot D_4 + D_5\cdot B_1 &= \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} \begin{pmatrix} -2\widetilde{b}^1_{1,0} & 0 \\ \widetilde{b}^1_{1,0} & 0 \end{pmatrix} + \begin{pmatrix} 2\widetilde{b}^0_{0,1} & 0 \\ \widetilde{b}^1_{0,1} & 0 \end{pmatrix} \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} \\
&= \begin{pmatrix} -2b^0_{1,0}\widetilde{b}^1_{1,0}+b^0_{1,1}\widetilde{b}^1_{1,0}+2\widetilde{b}^0_{0,1}b^0_{1,0} & 2\widetilde{b}^0_{0,1}b^0_{1,1} \\ -4b^1_{1,0}\widetilde{b}^1_{1,0}+b^1_{1,1}\widetilde{b}^1_{1,0}+\widetilde{b}^1_{0,1}b^0_{1,0} & \widetilde{b}^1_{0,1}b^0_{1,1} \end{pmatrix} \\
&= \begin{pmatrix} -4b^0_{1,0}+4b^0_{1,1}-2b^0_{1,1}+2\widetilde{b}^0_{1,1} & 0 \\ -4\widetilde{b}^1_{1,0}+b^1_{1,2}-2b^1_{1,1}-b^1_{1,2}+2\widetilde{b}^1_{1,1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
B_2\cdot D_2 + D_3\cdot B_2 &= \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} \begin{pmatrix} 2b^0_{0,1} & 0 \\ b^1_{0,1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -b^0_{1,1} & -2b^0_{0,1} \end{pmatrix} \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} \\
&= \begin{pmatrix} -2\widetilde{b}^1_{0,0}b^0_{0,1}+\widetilde{b}^1_{1,0}b^1_{0,1} & 0 \\ -4b^0_{0,0}b^0_{0,1}+b^0_{1,0}b^1_{0,1}+b^0_{1,1}\widetilde{b}^1_{0,0}+4b^0_{0,1}b^0_{0,0} & -b^0_{1,1}\widetilde{b}^1_{1,0}-2b^0_{0,1}b^0_{1,0} \end{pmatrix} \\
&= \begin{pmatrix} 2b^1_{0,1}-2b^1_{0,1} & 0 \\ -4b^0_{0,1}+b^0_{0,3}-b^0_{0,3}+4b^0_{0,1} & 2b^0_{1,1}-2b^0_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

2.6 Homotopy maps

We define homotopy maps h_k for $k \in \mathbb{Z}_{\geq -1}$ which we shall use to check the acyclicity of the augmented projective resolution; cf. Theorem 38 below.

$$
\begin{array}{ccccccccccccc}
\cdots \to & X_3 & \xrightarrow{d_2} & X_2 & \xrightarrow{d_1} & X_1 & \xrightarrow{d_0} & X_0 & \xrightarrow{\varepsilon} & R & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow{\scriptstyle 1} & \overset{h_2}{\swarrow} & \downarrow{\scriptstyle 1} & \overset{h_1}{\swarrow} & \downarrow{\scriptstyle 1} & \overset{h_0}{\swarrow} & \downarrow{\scriptstyle 1} & \overset{h_{-1}}{\swarrow} & \downarrow{\scriptstyle 1} & \overset{h_{-2}}{\swarrow} & \downarrow & \\
\cdots \to & X_3 & \xrightarrow{d_2} & X_2 & \xrightarrow{d_1} & X_1 & \xrightarrow{d_0} & X_0 & \xrightarrow{\varepsilon} & R & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}
$$

Definition 31 We define the following R-linear maps.

$$g_{0,0} : Q \to P, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{4}\left(\alpha_3-\beta_3,\ 0,\ \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ \alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$g_{0,1} : Q \to Q, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \left(\begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\end{pmatrix}\right)$$

$$g_{0,2} : Q \to P, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{8}\left(\beta_3-\alpha_3,\ \beta_3-\alpha_3,\ \begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\ 2\gamma_2-\alpha_2-\beta_2\\ \beta_3-\alpha_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ \beta_3-\alpha_3\end{pmatrix}\right)$$

$$g_{0,3} : Q \to Q, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\end{pmatrix}\right)$$

$$g_{1,0} : P \to Q, \quad \left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \left(\begin{pmatrix}0\\ 0\\ \rho-\alpha_3\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \beta_3-\sigma\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix}\right)$$

$$g_{1,1} : P \to P, \quad \left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\alpha_3-\rho,\ \beta_3-\sigma,\ \begin{pmatrix}0\\ 0\\ \alpha_3-\rho\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \beta_3-\sigma\end{pmatrix}\right)$$

$$g_{1,2} : Q \to Q, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\begin{pmatrix}\alpha_1-\gamma_1\\ \alpha_2-\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}(\alpha_1-\beta_1)\\ \frac{1}{2}(\alpha_2-\beta_2)\end{pmatrix}\right)$$

$$g_{1,3} : Q \to P, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{4}\left(\alpha_3,\ \beta_3,\ \begin{pmatrix}0\\ 0\\ \alpha_3\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \beta_3\end{pmatrix}\right)$$

$$g_{2,0} : Q \to Q, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\ 2\gamma_2-\alpha_2-\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}(2\gamma_1-\alpha_1-\beta_1)\\ \frac{1}{2}(2\gamma_2-\alpha_2-\beta_2)\end{pmatrix}\right)$$

$$g_{2,1} : P \to Q, \quad \left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\rho+\sigma-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \rho-\alpha_3+\beta_3-\sigma\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\end{pmatrix}\right)$$

$$g_{2,2} : P \to Q, \quad \left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix}, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \rho-\alpha_3\end{pmatrix}, \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right)$$

$$g_{3,0} : Q \to P, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{4}\left(\alpha_3+\beta_3,\ 0,\ \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ \alpha_3+\beta_3\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)$$

$$g_{3,1} \;:\; Q \to P\,, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{8}\left(-\alpha_3-\beta_3,\ \alpha_3+\beta_3, \begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\ 2\gamma_2-\alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ \alpha_3+\beta_3\end{pmatrix}\right)$$

$$g_{3,2} \;:\; Q \to Q\,, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\begin{pmatrix}\gamma_1-\alpha_1\\ \gamma_2-\alpha_2\\ 0\end{pmatrix}, \begin{pmatrix}\gamma_1-\alpha_1\\ \gamma_2-\alpha_2\\ 0\end{pmatrix}, \begin{pmatrix}\gamma_1-\alpha_1\\ \gamma_2-\alpha_2\end{pmatrix}\right)$$

$$g_{5,0} \;:\; Q \to Q\,, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}(\alpha_1+\beta_1-2\gamma_1)\\ \frac{1}{2}(\alpha_2+\beta_2-2\gamma_2)\end{pmatrix}\right)$$

$$g_{5,1} \;:\; Q \to Q\,, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}(\alpha_1+\beta_1-2\gamma_1)\\ \frac{1}{2}(\alpha_2+\beta_2-2\gamma_2)\end{pmatrix}\right)$$

$$g_{5,2} \;:\; P \to Q\,, \quad \left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\rho+\beta_3-\sigma\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\rho+\beta_3-\sigma\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\end{pmatrix}\right)$$

$$g_{5,3} \;:\; P \to Q\,, \quad \left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix}, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix}, \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right)$$

$$g_{6,0} \;:\; P \to P\,, \quad \left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(\rho-\sigma,\ \sigma-\rho, \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \beta_3-\alpha_3\end{pmatrix}\right)$$

$$g_{6,1} \;:\; P \to P\,, \quad \left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(-\rho-\sigma,\ -\rho-\sigma, \begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)$$

$$g_{7,0} \;:\; Q \to P\,, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(0,\ 0, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\ 2\gamma_2-\alpha_2-\beta_2\\ 0\end{pmatrix}\right)$$

$$g_{8,0} \;:\; P \to P\,, \quad \left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \frac{1}{2}\left(-\alpha_3-\beta_3,\ -\alpha_3-\beta_3, \begin{pmatrix}-\alpha_1-\beta_1\\ -\alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)$$

$$g_{8,1} \;:\; Q \to P\,, \quad \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \longmapsto \frac{1}{4}\left(-\alpha_3-\beta_3,\ -\alpha_3-\beta_3, \begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\ 2\gamma_2-\alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)$$

$$g_{9,0} \;:\; P \to P\,, \quad \left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \longmapsto \frac{1}{4}\left(\rho-\alpha_3,\ \rho-\alpha_3, \begin{pmatrix}0\\ 0\\ \rho-\alpha_3\end{pmatrix}, \begin{pmatrix}0\\ 0\\ \rho-\alpha_3\end{pmatrix}\right)$$

Note that $g_{0,1} = -2g_{0,3}$ and $g_{5,1} = 2g_{5,0}$.

Definition 32 We define the following R-linear maps using Definition 31.

$$G := \begin{pmatrix} g_{6,0} & 0 \\ 0 & g_{6,1} \end{pmatrix} : P \oplus P \to P \oplus P \qquad \widetilde{G} := \begin{pmatrix} -g_{6,1} & 0 \\ 0 & -g_{6,0} \end{pmatrix} : P \oplus P \to P \oplus P$$

$$H_0 := \begin{pmatrix} g_{0,0} & g_{0,1} \\ g_{0,2} & g_{0,3} \end{pmatrix} : Q \oplus Q \to P \oplus Q \qquad H_1 := \begin{pmatrix} g_{1,0} & g_{1,1} \\ g_{1,2} & g_{1,3} \end{pmatrix} : P \oplus Q \to Q \oplus P$$

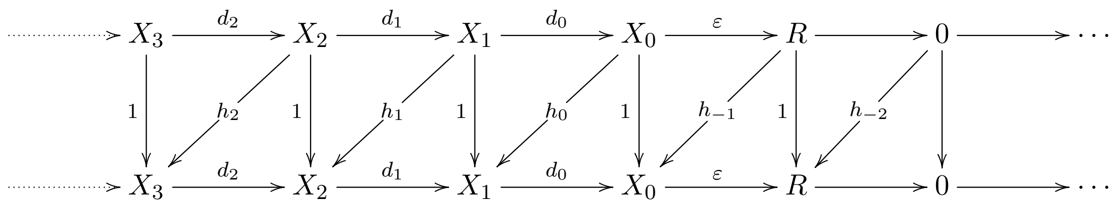
$$H_2 := \begin{pmatrix} g_{2,0} & 0 \\ g_{2,1} & g_{2,2} \end{pmatrix} : Q \oplus P \to Q \oplus Q \qquad H_3 := \begin{pmatrix} g_{3,0} & 0 \\ g_{3,1} & g_{3,2} \end{pmatrix} : Q \oplus Q \to P \oplus Q$$

$$H_1 = H_4 := \begin{pmatrix} g_{1,0} & g_{1,1} \\ g_{1,2} & g_{1,3} \end{pmatrix} : P \oplus Q \to Q \oplus P \qquad H_5 := \begin{pmatrix} g_{5,0} & g_{5,1} \\ g_{5,2} & g_{5,3} \end{pmatrix} : Q \oplus P \to Q \oplus Q$$

$$h'_0 := \begin{pmatrix} g_{6,0} & g_{2,2} \end{pmatrix} \;:\; P \to P \oplus Q \qquad h'_1 := \begin{pmatrix} -g_{6,1} \\ g_{7,0} \end{pmatrix} \;:\; P \oplus Q \to P$$

$$h''_1 := \begin{pmatrix} 0 & 0 \\ g_{3,1} & g_{3,2} \end{pmatrix} \;:\; P \oplus Q \to P \oplus Q \qquad h'_2 := \begin{pmatrix} g_{6,0} & 0 \end{pmatrix} \;:\; P \to P \oplus P$$

$$h''_2 := \begin{pmatrix} 0 & g_{8,0} \\ 0 & g_{8,1} \end{pmatrix} \;:\; P \oplus Q \to P \oplus P \qquad h'_3 := \begin{pmatrix} 0 \\ g_{9,0} \end{pmatrix} \;:\; P \oplus P \to P\,.$$



Definition 33 For $l \in \mathbb{Z}_{\geq 0}$ we define the following R-linear homotopy maps.

$$h_{-1} \quad : \quad R \to X_0,\; r \mapsto \left(r, r, \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \right)$$

$$\begin{aligned}
h_{4l} \quad &:= \quad \left(\sum_{k \in [0,l-1]} \pi^+_{4l,k}\, G\, \iota^+_{4l+1,k} \right) + \left(\sum_{k \in [0,l-1]} \pi^-_{4l,k}\, H_{\overline{4k}}\, \iota^-_{4l+1,k} \right) \\
&\quad + \left(\pi'_{4l}\, h'_0\, \iota'_{4l+1} \right) \;:\; X_{4l} \longrightarrow X_{4l+1} \\
h_{4l+1} \quad &:= \quad \left(\sum_{k \in [0,l-1]} \pi^+_{4l+1,k}\, \widetilde{G}\, \iota^+_{4l+2,k} \right) + \left(\sum_{k \in [0,l-1]} \pi^-_{4l+1,k}\, H_{\overline{4k+1}}\, \iota^-_{4l+2,k} \right) \\
&\quad + \left(\pi'_{4l+1}\, h'_1\, \iota'_{4l+2} \right) + \left(\pi'_{4l+1}\, h''_1\, \iota''_{4l+2} \right) \;:\; X_{4l+1} \longrightarrow X_{4l+2} \\
h_{4l+2} \quad &:= \quad \left(\sum_{k \in [0,l-1]} \pi^+_{4l+2,k}\, G\, \iota^+_{4l+3,k} \right) + \left(\sum_{k \in [0,l-1]} \pi^-_{4l+2,k}\, H_{\overline{4k+2}}\, \iota^-_{4l+3,k+1} \right) \\
&\quad + \left(\pi'_{4l+2}\, h'_2\, \iota^+_{4l+3,l} \right) + \left(\pi''_{4l+2}\, h''_2\, \iota^+_{4l+3,l} \right) + \left(\pi''_{4l+2}\, H_4\, \iota^-_{4l+3,0} \right) \;:\; X_{4l+2} \longrightarrow X_{4l+3} \\
h_{4l+3} \quad &:= \quad \left(\sum_{k \in [0,l]} \pi^+_{4l+3,k}\, \widetilde{G}\, \iota^+_{4l+4,k} \right) + \left(\sum_{k \in [0,l]} \pi^-_{4l+3,k}\, H_{\overline{4k+5}}\, \iota^-_{4l+4,k} \right) \\
&\quad + \left(\pi^+_{4l+3,l}\, h'_3\, \iota'_{4l+4} \right) \;:\; X_{4l+3} \longrightarrow X_{4l+4}
\end{aligned}$$

Remark 34 The first few homotopy maps from Definition 33 can be written as follows. Note that the matrix of the homotopy map h_k is part of the matrix of the homotopy map h_{k+4} for $k \geq 0$. This is marked with a surrounding box.

$$h_0 = \begin{pmatrix} g_{6,0} & g_{2,2} \end{pmatrix} = (h'_0) \;:\; P \to P \oplus Q$$

$$h_1 = \begin{pmatrix} -g_{6,1} & 0 & 0 \\ g_{7,0} & g_{3,1} & g_{3,2} \end{pmatrix} = \begin{pmatrix} h'_1 & h''_1 \end{pmatrix} \;:\; P{\oplus}Q \to P{\oplus}P{\oplus}Q$$

$$h_2 = \begin{pmatrix} g_{6,0} & 0 & 0 & 0 \\ 0 & g_{8,0} & g_{1,0} & g_{1,1} \\ 0 & g_{8,1} & g_{1,2} & g_{1,3} \end{pmatrix} = \begin{pmatrix} h_2' & 0 \\ h_2'' & H_4 \end{pmatrix} \; : \; P{\oplus}P{\oplus}Q \to P{\oplus}P{\oplus}Q{\oplus}P$$

$$h_3 = \begin{pmatrix} -g_{6,1} & 0 & 0 & 0 & 0 \\ 0 & -g_{6,0} & g_{9,0} & 0 & 0 \\ 0 & 0 & 0 & g_{5,0} & g_{5,1} \\ 0 & 0 & 0 & g_{5,2} & g_{5,3} \end{pmatrix} = \begin{pmatrix} \widetilde{G} & h_3' & 0 \\ 0 & 0 & H_5 \end{pmatrix} \; : \; P{\oplus}P{\oplus}Q{\oplus}P \to P{\oplus}P{\oplus}P{\oplus}Q{\oplus}Q$$

$$h_4 = \begin{pmatrix} g_{6,0} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{6,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{6,0} & g_{2,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{0,0} & g_{0,1} \\ 0 & 0 & 0 & 0 & g_{0,2} & g_{0,3} \end{pmatrix} = \begin{pmatrix} G & 0 & 0 \\ 0 & \boxed{h_0'} & 0 \\ 0 & 0 & H_0 \end{pmatrix} \; : \; P{\oplus}P{\oplus}P{\oplus}Q{\oplus}Q \to P{\oplus}P{\oplus}P{\oplus}Q{\oplus}P{\oplus}Q$$

$$h_5 = \begin{pmatrix} -g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g_{6,0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g_{6,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{7,0} & g_{3,1} & g_{3,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{1,0} & g_{1,1} \\ 0 & 0 & 0 & 0 & 0 & g_{1,2} & g_{1,3} \end{pmatrix} = \begin{pmatrix} \widetilde{G} & 0 & 0 & 0 \\ 0 & h_1' & h_1'' & 0 \\ 0 & 0 & 0 & H_1 \end{pmatrix} :$$

$$P{\oplus}P{\oplus}P{\oplus}Q{\oplus}P{\oplus}Q \to P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}Q{\oplus}P$$

$$h_6 = \begin{pmatrix} g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{6,0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{8,0} & g_{1,0} & g_{1,1} & 0 & 0 \\ 0 & 0 & 0 & g_{8,1} & g_{1,2} & g_{1,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_{2,0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_{2,1} & g_{2,2} \end{pmatrix} = \begin{pmatrix} G & 0 & 0 & 0 \\ 0 & h_2' & 0 & 0 \\ 0 & h_2'' & H_4 & 0 \\ 0 & 0 & 0 & H_2 \end{pmatrix} :$$

$$P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}Q{\oplus}P \to P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}P{\oplus}Q{\oplus}Q$$

$$h_7 = \begin{pmatrix} -g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g_{6,0} & g_{9,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{5,0} & g_{5,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{5,2} & g_{5,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{3,0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{3,1} & g_{3,2} \end{pmatrix} = \begin{pmatrix} \widetilde{G} & 0 & 0 & 0 & 0 \\ 0 & \widetilde{G} & h_3' & 0 & 0 \\ 0 & 0 & 0 & H_5 & 0 \\ 0 & 0 & 0 & 0 & H_3 \end{pmatrix} :$$

$$P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}P{\oplus}Q{\oplus}Q \to P{\oplus}P{\oplus}P{\oplus}P{\oplus}P{\oplus}Q{\oplus}Q{\oplus}P{\oplus}Q$$

$$
h_8 = \begin{pmatrix}
g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_{6,0} & g_{2,2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_{0,0} & g_{0,1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_{0,2} & g_{0,3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1,0} & g_{1,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1,2} & g_{1,3}
\end{pmatrix}
= \begin{pmatrix}
G & 0 & 0 & 0 & 0 \\
0 & G & 0 & 0 & 0 \\
0 & 0 & h'_0 & 0 & 0 \\
0 & 0 & 0 & H_0 & 0 \\
0 & 0 & 0 & 0 & H_4
\end{pmatrix} :
$$

$$
P \oplus P \oplus P \oplus P \oplus P \oplus Q \oplus Q \oplus P \oplus Q \rightarrow P \oplus P \oplus P \oplus P \oplus P \oplus Q \oplus P \oplus Q \oplus Q \oplus P
$$

$$
h_9 = \begin{pmatrix}
-g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_{7,0} & g_{3,1} & g_{3,2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1,0} & g_{1,1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1,2} & g_{1,3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{5,0} & g_{5,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{5,2} & g_{5,3}
\end{pmatrix}
= \begin{pmatrix}
\widetilde{G} & 0 & 0 & 0 & 0 & 0 \\
0 & \widetilde{G} & 0 & 0 & 0 & 0 \\
0 & 0 & h'_1 & h''_1 & 0 & 0 \\
0 & 0 & 0 & 0 & H_1 & 0 \\
0 & 0 & 0 & 0 & 0 & H_5
\end{pmatrix} :
$$

$$
P \oplus P \oplus P \oplus P \oplus P \oplus Q \oplus P \oplus Q \oplus Q \oplus P \longrightarrow P \oplus P \oplus P \oplus P \oplus P \oplus P \oplus Q \oplus Q \oplus P \oplus Q \oplus Q
$$

$$
h_{10} = \begin{pmatrix}
g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_{8,0} & g_{1,0} & g_{1,1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_{8,1} & g_{1,2} & g_{1,3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{2,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{2,1} & g_{2,2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{0,0} & g_{0,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{0,2} & g_{0,3}
\end{pmatrix}
= \begin{pmatrix}
G & 0 & 0 & 0 & 0 & 0 \\
0 & G & 0 & 0 & 0 & 0 \\
0 & 0 & h'_2 & 0 & 0 & 0 \\
0 & 0 & h''_2 & H_4 & 0 & 0 \\
0 & 0 & 0 & 0 & H_2 & 0 \\
0 & 0 & 0 & 0 & 0 & H_0
\end{pmatrix} :
$$

$$
P \oplus P \oplus P \oplus P \oplus P \oplus P \oplus Q \oplus Q \oplus P \oplus Q \oplus Q \longrightarrow P \oplus P \oplus P \oplus P \oplus P \oplus P \oplus Q \oplus P \oplus Q \oplus Q \oplus P \oplus Q
$$

We collect some auxiliary calculations which we shall use to verify that the augmented projective resolution X' is in fact acyclic; cf. Theorem 38 below.

Lemma 35 The following assertions (i-xiii) hold.

(i) $A \cdot G + \widetilde{G} \cdot \widetilde{A} = 1$
$\widetilde{A} \cdot \widetilde{G} + G \cdot A = 1$

(ii) $C \cdot G + \widetilde{G} \cdot C = 0$
$C \cdot \widetilde{G} + G \cdot C = 0$

(iii) $\widetilde{C} \cdot G + h_1' \cdot C' = 0$
$\widetilde{D}_1 \cdot H_0 + h_1'' \cdot D_2 = 0$

(iv) $C' \cdot \widetilde{G} + h_2' \cdot C = 0$
$h_2'' \cdot C = 0$

(v) $d_2' \cdot h_2' + d_2'' \cdot h_2'' + \widetilde{G} \cdot \widetilde{A} + h_3' \cdot C' = 1$
$d_2'' \cdot H_4 + h_3' \cdot D_0' = 0$
$B_2 \cdot h_2'' = 0$

(vi) $\widetilde{A} \cdot h_3' = 0$
$C' \cdot \widetilde{G} + h_0' \cdot \widetilde{C} = 0$
$C' \cdot h_3' + h_0' \cdot d_0 = 1$
$D_0' \cdot H_5 + h_0' \cdot \widetilde{D}_1 = 0$

(vii) $h_{-1} \cdot \varepsilon = 1$
$\varepsilon \cdot h_{-1} + h_0 \cdot d_0 = 1$

(viii) $d_0 \cdot h_0' + h_1 \cdot d_1 = 1$
$d_1 \cdot h_1 + h_2 \cdot d_2 = 1$

(ix) $B_0 \cdot H_2 + H_3 \cdot B_1 = 1$
$B_1 \cdot H_0 + H_1 \cdot B_2 = 1$
$B_2 \cdot H_4 + H_5 \cdot B_0 = 1$

(x) $B_0 \cdot H_5 + H_0 \cdot B_1 = 1$
$B_1 \cdot H_3 + H_4 \cdot B_2 = 1$
$B_2 \cdot H_1 + H_2 \cdot B_0 = 1$

(xi) $D_5 \cdot H_4 + H_1 \cdot D_0 = 0$
$D_3 \cdot H_2 + H_5 \cdot D_4 = 0$
$D_1 \cdot H_0 + H_3 \cdot D_2 = 0$

(xii) $D_0 \cdot H_5 + H_2 \cdot D_1 = 0$
$D_4 \cdot H_3 + H_0 \cdot D_5 = 0$
$D_2 \cdot H_1 + H_4 \cdot D_3 = 0$

Proof. We calculate as follows, using Definition 31. Some repetition occurs in this calculation, which we left in for sake of clarity.

(i)

- $$A \cdot G + \widetilde{G} \cdot \widetilde{A} = \begin{pmatrix} b^0_{0,1} & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1} \end{pmatrix} \begin{pmatrix} g_{6,0} & 0 \\ 0 & g_{6,1} \end{pmatrix} + \begin{pmatrix} -g_{6,1} & 0 \\ 0 & -g_{6,0} \end{pmatrix} \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix}$$
$$= \begin{pmatrix} b^0_{0,1} g_{6,0} - g_{6,1} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} g_{6,0} - g_{6,0} b^0_{0,2} & -\widetilde{b}^0_{0,1} g_{6,1} + g_{6,0} b^0_{0,1} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix},\ \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right) \in P$. We obtain the following.

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{0,1}g_{6,0}-g_{6,1}\widetilde{b}^0_{0,1}\right)$$
$$=\left(-\sigma,\ \sigma,\begin{pmatrix}\beta_1\\ \beta_2\\ -\beta_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)-\left(-\sigma-\rho,\ 0,\begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)=\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{0,2}g_{6,0}-g_{6,0}b^0_{0,2}\right)$$
$$=2\left(0,\ 0,\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \beta_3-\alpha_3\end{pmatrix}\right)-2\left(0,\ 0,\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \beta_3-\alpha_3\end{pmatrix}\right)=\left(0,\ 0,\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(-\widetilde{b}^0_{0,1}g_{6,1}+g_{6,0}b^0_{0,1}\right)$$
$$=-\left(-\rho,\ -\rho,\begin{pmatrix}-\alpha_1\\ -\alpha_2\\ -\alpha_3\end{pmatrix},\begin{pmatrix}\alpha_1\\ \alpha_2\\ -\alpha_3\end{pmatrix}\right)+\left(0,\ \sigma-\rho,\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \beta_3-\alpha_3\end{pmatrix}\right)=\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

- $\widetilde{A}\cdot\widetilde{G}+G\cdot A=\begin{pmatrix}\widetilde{b}^0_{0,1} & 0\\ b^0_{0,2} & -b^0_{0,1}\end{pmatrix}\begin{pmatrix}-g_{6,1} & 0\\ 0 & -g_{6,0}\end{pmatrix}+\begin{pmatrix}g_{6,0} & 0\\ 0 & g_{6,1}\end{pmatrix}\begin{pmatrix}b^0_{0,1} & 0\\ b^0_{0,2} & -\widetilde{b}^0_{0,1}\end{pmatrix}$

$$=\begin{pmatrix}-\widetilde{b}^0_{0,1}g_{6,1}+g_{6,0}b^0_{0,1} & 0\\ -b^0_{0,2}g_{6,1}+g_{6,1}b^0_{0,2} & b^0_{0,1}g_{6,0}-g_{6,1}\widetilde{b}^0_{0,1}\end{pmatrix}\overset{!}{=}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\in P$. We obtain the following; cf. also (i).

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(-\widetilde{b}^0_{0,1}g_{6,1}+g_{6,0}b^0_{0,1}\right)$$
$$=-\left(-\rho,\ -\rho,\begin{pmatrix}-\alpha_1\\ -\alpha_2\\ -\alpha_3\end{pmatrix},\begin{pmatrix}\alpha_1\\ \alpha_2\\ -\alpha_3\end{pmatrix}\right)+\left(0,\ \sigma-\rho,\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \beta_3-\alpha_3\end{pmatrix}\right)=\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(-b^0_{0,2}g_{6,1}+g_{6,1}b^0_{0,2}\right)$$
$$=-2\left(0,\ 0,\begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)+2\left(0,\ 0,\begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)=\left(0,\ 0,\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{0,1}g_{6,0}-g_{6,1}\widetilde{b}^0_{0,1}\right)$$
$$=\left(-\sigma,\ \sigma,\begin{pmatrix}\beta_1\\ \beta_2\\ -\beta_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)-\left(-\rho-\sigma,\ 0,\begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)=\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

(ii)

- $C \cdot G + \widetilde{G} \cdot C = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_{6,0} & 0 \\ 0 & g_{6,1} \end{pmatrix} + \begin{pmatrix} -g_{6,1} & 0 \\ 0 & -g_{6,0} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} g_{6,1} - g_{6,1} \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right) \in P$. We obtain the following.

$$\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right) \left(\widetilde{b}^0_{0,2} g_{6,1} - g_{6,1} \widetilde{b}^0_{0,2}\right)$$

$$= 2\left(-\rho - \sigma,\ -\rho - \sigma,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) - 2\left(-\rho - \sigma,\ -\rho - \sigma,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \left(0,\ 0,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

- $C \cdot \widetilde{G} + G \cdot C = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -g_{6,1} & 0 \\ 0 & -g_{6,0} \end{pmatrix} + \begin{pmatrix} g_{6,0} & 0 \\ 0 & g_{6,1} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -\widetilde{b}^0_{0,2} g_{6,0} + g_{6,0} \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right) \in P$. We obtain the following.

$$\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right) \left(-\widetilde{b}^0_{0,2} g_{6,0} + g_{6,0} \widetilde{b}^0_{0,2}\right)$$

$$= -2\left(\rho - \sigma,\ \sigma - \rho,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) + 2\left(\rho - \sigma,\ \sigma - \rho,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \left(0,\ 0,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

(iii)

- $\widetilde{C} \cdot G + h'_1 \cdot C' = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_{6,0} & 0 \\ 0 & g_{6,1} \end{pmatrix} + \begin{pmatrix} -g_{6,1} \\ g_{7,0} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} g_{6,1} - g_{6,1} \widetilde{b}^0_{0,2} \\ 0 & g_{7,0} \widetilde{b}^0_{0,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right) \in P$ and $\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}\right) \in Q$.

We obtain the following; cf. also (ii).

$$\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}\right) \left(\widetilde{b}^0_{0,2} g_{6,1} - g_{6,1} \widetilde{b}^0_{0,2}\right)$$

$$= 2\left(-\rho - \sigma,\ -\rho - \sigma,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) - 2\left(-\rho - \sigma,\ -\rho - \sigma,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \left(0,\ 0,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}\right) \left(g_{7,0} \widetilde{b}^0_{0,2}\right) = \left(0,\ 0,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

- $\widetilde{D}_1 \cdot H_0 + h_1'' \cdot D_2 = \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} \begin{pmatrix} g_{0,0} & g_{0,1} \\ g_{0,2} & g_{0,3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ g_{3,1} & g_{3,2} \end{pmatrix} \begin{pmatrix} 2b^0_{0,1} & 0 \\ b^1_{0,1} & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} g_{0,0} + 2\widetilde{b}^1_{1,0} g_{0,2} + 2g_{3,1} b^0_{0,1} + g_{3,2} b^1_{0,1} & \widetilde{b}^1_{1,0} g_{0,1} + 2\widetilde{b}^1_{1,0} g_{0,3} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Suppose given $\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \in Q$. We obtain the following.

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \left(\widetilde{b}^1_{1,0} g_{0,0} + 2\widetilde{b}^1_{1,0} g_{0,2} + 2g_{3,1} b^0_{0,1} + g_{3,2} b^1_{0,1} \right)$$

$$= \frac{1}{2} \left(\alpha_3 + \beta_3,\ 0,\ \begin{pmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \alpha_3 + \beta_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) + \frac{1}{2} \left(-\alpha_3 - \beta_3,\ -\alpha_3 - \beta_3,\ \begin{pmatrix} \beta_1 - \alpha_1 \\ \beta_2 - \alpha_2 \\ -\alpha_3 - \beta_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ -\alpha_3 - \beta_3 \end{pmatrix} \right)$$

$$+ \frac{1}{2} \left(0,\ \alpha_3 + \beta_3,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 + \beta_1 - 2\gamma_1 \\ \alpha_2 + \beta_2 - 2\gamma_2 \\ \alpha_3 + \beta_3 \end{pmatrix} \right) + \left(0,\ 0,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_1 - \alpha_1 \\ \gamma_2 - \alpha_2 \\ 0 \end{pmatrix} \right) = \left(0,\ 0,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \left(\widetilde{b}^1_{1,0} g_{0,1} + 2\widetilde{b}^1_{1,0} g_{0,3} \right)$$

$$= 2 \left(\begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right) + 2 \left(\begin{pmatrix} -\beta_1 \\ -\beta_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\beta_1 \\ -\beta_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\beta_1 \\ -\beta_2 \end{pmatrix} \right) = \left(0,\ 0,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

(iv)

- $C' \cdot \widetilde{G} + h_2' \cdot C = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} \begin{pmatrix} -g_{6,1} & 0 \\ 0 & -g_{6,0} \end{pmatrix} + \begin{pmatrix} g_{6,0} & 0 \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\widetilde{b}^0_{0,2} g_{6,0} + g_{6,0} \widetilde{b}^0_{0,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \end{pmatrix}$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \in P$. We obtain the following; cf. also (ii).

$$\left(\rho,\ \sigma,\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \left(-\widetilde{b}^0_{0,2} g_{6,0} + g_{6,0} \widetilde{b}^0_{0,2} \right)$$

$$= -2 \left(\rho - \sigma,\ \sigma - \rho,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) + 2 \left(\rho - \sigma,\ \sigma - \rho,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) = \left(0,\ 0,\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

- $h_2'' \cdot C = \begin{pmatrix} 0 & g_{8,0} \\ 0 & g_{8,1} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ holds.

(v)

- $d_2' \cdot h_2' + d_2'' \cdot h_2'' + \widetilde{G} \cdot \widetilde{A} + h_3' \cdot C'$

$$= \begin{pmatrix} b^0_{0,1} \\ b^0_{0,2} \end{pmatrix} \begin{pmatrix} g_{6,0} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\widetilde{b}^0_{0,1} & 0 \end{pmatrix} \begin{pmatrix} 0 & g_{8,0} \\ 0 & g_{8,1} \end{pmatrix} + \begin{pmatrix} -g_{6,1} & 0 \\ 0 & -g_{6,0} \end{pmatrix} \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} + \begin{pmatrix} 0 \\ g_{9,0} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix}$$

$$= \begin{pmatrix} b^0_{0,1} g_{6,0} - g_{6,1} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} g_{6,0} - g_{6,0} b^0_{0,2} & -\widetilde{b}^0_{0,1} g_{8,0} + g_{6,0} b^0_{0,1} + g_{9,0} \widetilde{b}^0_{0,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \in P$. We obtain the following; cf. also (i).

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{0,1}g_{6,0} - g_{6,1}\widetilde{b}^0_{0,1}\right)$$
$$= \left(-\sigma,\ \sigma,\ \begin{pmatrix}\beta_1\\ \beta_2\\ -\beta_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) - \left(-\rho-\sigma,\ 0,\ \begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right) = \left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{0,2}g_{6,0} - g_{6,0}b^0_{0,2}\right)$$
$$= 2\left(0,\ 0,\ \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\beta_3\end{pmatrix},\ \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \beta_3-\alpha_3\end{pmatrix}\right) - 2\left(0,\ 0,\ \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\beta_3\end{pmatrix},\ \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \beta_3-\alpha_3\end{pmatrix}\right) = \left(0,\ 0,\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(-\widetilde{b}^0_{0,1}g_{8,0} + g_{6,0}b^0_{0,1} + g_{9,0}\widetilde{b}^0_{0,2}\right)$$
$$= -\left(-\alpha_3,\ -\alpha_3,\ \begin{pmatrix}-\alpha_1\\ -\alpha_2\\ -\alpha_3\end{pmatrix},\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ -\alpha_3\end{pmatrix}\right) + \left(0,\ \sigma-\rho,\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \beta_3-\alpha_3\end{pmatrix}\right)$$
$$+ \left(\rho-\alpha_3,\ \rho-\alpha_3,\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right) = \left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

- $d''_2 \cdot H_4 + h'_3 \cdot D'_0 = \begin{pmatrix}0 & 0\\ -\widetilde{b}^0_{0,1} & 0\end{pmatrix}\begin{pmatrix}g_{1,0} & g_{1,1}\\ g_{1,2} & g_{1,3}\end{pmatrix} + \begin{pmatrix}0\\ g_{9,0}\end{pmatrix}\begin{pmatrix}\widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1}\end{pmatrix}$

$$= \begin{pmatrix}0 & 0\\ -\widetilde{b}^0_{0,1}g_{1,0} + g_{9,0}\widetilde{b}^0_{1,1} & -\widetilde{b}^0_{0,1}g_{1,1} - 2g_{9,0}\widetilde{b}^0_{0,1}\end{pmatrix} \stackrel{!}{=} \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \in P$. We obtain the following.

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(-\widetilde{b}^0_{0,1}g_{1,0} + g_{9,0}\widetilde{b}^0_{1,1}\right)$$
$$= -2\left(\begin{pmatrix}0\\ 0\\ \rho-\alpha_3\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\end{pmatrix}\right) + 2\left(\begin{pmatrix}0\\ 0\\ \rho-\alpha_3\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\end{pmatrix}\right) = \left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(-\widetilde{b}^0_{0,1}g_{1,1} - 2g_{9,0}\widetilde{b}^0_{0,1}\right)$$
$$= -\left(\alpha_3-\rho,\ 0,\ \begin{pmatrix}0\\ 0\\ \alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right) - \left(\rho-\alpha_3,\ 0,\ \begin{pmatrix}0\\ 0\\ \rho-\alpha_3\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right) = \left(0,\ 0,\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)$$

- $$B_2 \cdot h_2'' = \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} \begin{pmatrix} 0 & g_{8,0} \\ 0 & g_{8,1} \end{pmatrix} = \begin{pmatrix} 0 & -\widetilde{b}^1_{0,0}g_{8,0} + \widetilde{b}^1_{1,0}g_{8,1} \\ 0 & -2b^0_{0,0}g_{8,0} + b^0_{1,0}g_{8,1} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

  Suppose given $\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right) \in Q$ and $\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \in P$. We obtain the following.

  $$\begin{aligned}
  &\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right) \left(-\widetilde{b}^1_{0,0}g_{8,0} + \widetilde{b}^1_{1,0}g_{8,1}\right) \\
  &= -\frac{1}{2}\left(\beta_3-\alpha_3,\ \beta_3-\alpha_3, \begin{pmatrix}\beta_1-\alpha_1\\\beta_2-\alpha_2\\\beta_3-\alpha_3\end{pmatrix}, \begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\\\beta_3-\alpha_3\end{pmatrix}\right) + \frac{1}{2}\left(\beta_3-\alpha_3,\ \beta_3-\alpha_3, \begin{pmatrix}\beta_1-\alpha_1\\\beta_2-\alpha_2\\\beta_3-\alpha_3\end{pmatrix}, \begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\\\beta_3-\alpha_3\end{pmatrix}\right) \\
  &= \left(0,\ 0, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)
  \end{aligned}$$

  $$\begin{aligned}
  &\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \left(-2b^0_{0,0}g_{8,0} + b^0_{1,0}g_{8,1}\right) \\
  &= -\left(-\alpha_3-\beta_3,\ -\alpha_3-\beta_3, \begin{pmatrix}-\alpha_1-\beta_1\\-\alpha_2-\beta_2\\-\alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\\-\alpha_3-\beta_3\end{pmatrix}\right) \\
  &\quad + \left(-\alpha_3-\beta_3,\ -\alpha_3-\beta_3, \begin{pmatrix}-\alpha_1-\beta_1\\-\alpha_2-\beta_2\\-\alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\\-\alpha_3-\beta_3\end{pmatrix}\right) = \left(0,\ 0, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)
  \end{aligned}$$

(vi)

- $$\widetilde{A} \cdot h_3' = \begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1} \end{pmatrix} \begin{pmatrix} 0 \\ g_{9,0} \end{pmatrix} = \begin{pmatrix} 0 \\ -b^0_{0,1}g_{9,0} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  Suppose given $\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \in P$. We obtain the following.

  $$\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) (-b^0_{0,1}g_{9,0}) = \left(0,\ 0, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

- $$\begin{aligned}
  C' \cdot \widetilde{G} + h_0' \cdot \widetilde{C} &= \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} \begin{pmatrix} -g_{6,1} & 0 \\ 0 & -g_{6,0} \end{pmatrix} + \begin{pmatrix} g_{6,0} & g_{2,2} \end{pmatrix} \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0 \end{pmatrix} \\
  &= \begin{pmatrix} 0 & -\widetilde{b}^0_{0,2}g_{6,0} + g_{6,0}\widetilde{b}^0_{0,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \end{pmatrix}
  \end{aligned}$$

  Suppose given $\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \in P$. We obtain the following; cf. also (iv).

  $$\begin{aligned}
  &\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \left(-\widetilde{b}^0_{0,2}g_{6,0} + g_{6,0}\widetilde{b}^0_{0,2}\right) \\
  &= -2\left(\rho-\sigma,\ \sigma-\rho, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) + 2\left(\rho-\sigma,\ \sigma-\rho, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) = \left(0,\ 0, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)
  \end{aligned}$$

- $C' \cdot h'_3 + h'_0 \cdot d_0 = \begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \end{pmatrix} \begin{pmatrix} 0 \\ g_{9,0} \end{pmatrix} + \begin{pmatrix} g_{6,0} & g_{2,2} \end{pmatrix} \begin{pmatrix} b^0_{0,1} \\ \widetilde{b}^1_{0,0} \end{pmatrix} = \widetilde{b}^0_{0,2} g_{9,0} + g_{6,0} b^0_{0,1} + g_{2,2} \widetilde{b}^1_{0,0} \overset{!}{=} 1$

  Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \in P$. We obtain the following.

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \left(\widetilde{b}^0_{0,2} g_{9,0} + g_{6,0} b^0_{0,1} + g_{2,2} \widetilde{b}^1_{0,0}\right)$$
$$= \left(\rho,\ \rho,\ \begin{pmatrix}0\\0\\\rho\end{pmatrix},\ \begin{pmatrix}0\\0\\\rho\end{pmatrix}\right) + \left(0,\ \sigma-\rho,\ \begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\\\beta_3-\alpha_3\end{pmatrix}\right) + \left(0,\ 0,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}-\alpha_1\\-\alpha_2\\\alpha_3-\rho\end{pmatrix}\right)$$
$$= \left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)$$

- $D'_0 \cdot H_5 + h'_0 \cdot \widetilde{D}_1 = \begin{pmatrix} \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix} \begin{pmatrix} g_{5,0} & g_{5,1} \\ g_{5,2} & g_{5,3} \end{pmatrix} + \begin{pmatrix} g_{6,0} & g_{2,2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix}$

  $= \begin{pmatrix} \widetilde{b}^0_{1,1} g_{5,0} - 2\widetilde{b}^0_{0,1} g_{5,2} + g_{2,2}\widetilde{b}^1_{1,0} & \widetilde{b}^0_{1,1} g_{5,1} - 2\widetilde{b}^0_{0,1} g_{5,3} + 2g_{2,2}\widetilde{b}^1_{1,0} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \end{pmatrix}$

  Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \in P$. We obtain the following.

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \left(\widetilde{b}^0_{1,1} g_{5,0} - 2\widetilde{b}^0_{0,1} g_{5,2} + g_{2,2}\widetilde{b}^1_{1,0}\right)$$
$$= 2\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}2\alpha_1\\2\alpha_2\\0\end{pmatrix},\ \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\right) - 2\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\right) + 2\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}-\alpha_1\\-\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}0\\0\end{pmatrix}\right)$$
$$= \left(\begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\ \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \left(\widetilde{b}^0_{1,1} g_{5,1} - 2\widetilde{b}^0_{0,1} g_{5,3} + 2g_{2,2}\widetilde{b}^1_{1,0}\right)$$
$$= 4\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}2\alpha_1\\2\alpha_2\\0\end{pmatrix},\ \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\right) - 4\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\right) + 4\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}-\alpha_1\\-\alpha_2\\\alpha_3-\rho\end{pmatrix},\ \begin{pmatrix}0\\0\end{pmatrix}\right)$$
$$= \left(\begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\end{pmatrix}\right)$$

(vii)

- Suppose given $r \in R$. We have $rh_{-1}\varepsilon = \left(r, r, \begin{pmatrix}0\\0\\r\end{pmatrix} \begin{pmatrix}0\\0\\r\end{pmatrix}\right) \varepsilon = r\,.$

- $\varepsilon \cdot h_{-1} + h_0 \cdot d_0 = \varepsilon \cdot h_{-1} + \begin{pmatrix} g_{6,0} & g_{2,2} \end{pmatrix} \begin{pmatrix} b^0_{0,1} \\ \widetilde{b}^1_{0,0} \end{pmatrix} \overset{!}{=} 1$

Given $\left(\rho,\sigma,\begin{pmatrix}\alpha_{1,3}\\ \alpha_{2,3}\\ \alpha_{3,3}\end{pmatrix},\begin{pmatrix}\beta_{1,3}\\ \beta_{2,3}\\ \beta_{3,3}\end{pmatrix}\right)\in P$ we have

$$\left(\rho,\sigma,\begin{pmatrix}\alpha_{1,3}\\ \alpha_{2,3}\\ \alpha_{3,3}\end{pmatrix},\begin{pmatrix}\beta_{1,3}\\ \beta_{2,3}\\ \beta_{3,3}\end{pmatrix}\right)(\varepsilon h_{-1}+h_0 d_0)$$

$$= \rho h_{-1} + \left(\frac{1}{2}\left(\rho-\sigma,\ \sigma-\rho,\begin{pmatrix}\alpha_{1,3}+\beta_{1,3}\\ \alpha_{2,3}+\beta_{2,3}\\ \alpha_{3,3}-\beta_{3,3}\end{pmatrix},\begin{pmatrix}\alpha_{1,3}+\beta_{1,3}\\ \alpha_{2,3}+\beta_{2,3}\\ \beta_{3,3}-\alpha_{3,3}\end{pmatrix}\right),\left(\begin{pmatrix}\alpha_{1,3}\\ \alpha_{2,3}\\ \alpha_{3,3}-\rho\end{pmatrix},\begin{pmatrix}\alpha_{1,3}\\ \alpha_{2,3}\\ \rho-\alpha_{3,3}\end{pmatrix},\begin{pmatrix}\alpha_{1,3}\\ \alpha_{2,3}\end{pmatrix}\right)\right)d_0$$

$$= \left(\rho,\rho,\begin{pmatrix}0\\0\\ \rho\end{pmatrix},\begin{pmatrix}0\\0\\ \rho\end{pmatrix}\right)+\left(0,\ \sigma-\rho,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha_{1,3}+\beta_{1,3}\\ \alpha_{2,3}+\beta_{2,3}\\ \beta_{3,3}-\alpha_{3,3}\end{pmatrix}\right)+\left(0,\ 0,\begin{pmatrix}\alpha_{1,3}\\ \alpha_{2,3}\\ \alpha_{3,3}-\rho\end{pmatrix},\begin{pmatrix}-\alpha_{1,3}\\ -\alpha_{2,3}\\ \alpha_{3,3}-\rho\end{pmatrix}\right)$$

$$= \left(\rho,\sigma,\begin{pmatrix}\alpha_{1,3}\\ \alpha_{2,3}\\ \alpha_{3,3}\end{pmatrix},\begin{pmatrix}\beta_{1,3}\\ \beta_{2,3}\\ \beta_{3,3}\end{pmatrix}\right).$$

(viii)

- $$d_0\cdot h'_0 + h_1\cdot d_1 = \begin{pmatrix} b^0_{0,1}\\ \widetilde{b}^1_{0,0}\end{pmatrix}\begin{pmatrix} g_{6,0} & g_{2,2}\end{pmatrix}+\begin{pmatrix}-g_{6,1} & 0 & 0\\ g_{7,0} & g_{3,1} & g_{3,2}\end{pmatrix}\begin{pmatrix}\widetilde{b}^0_{0,1} & 0\\ b^0_{0,2} & b^0_{1,1}\\ 2b^1_{0,0} & b^1_{1,1}\end{pmatrix}$$
$$= \begin{pmatrix} b^0_{0,1}g_{6,0}-g_{6,1}\widetilde{b}^0_{0,1} & b^0_{0,1}g_{2,2}\\ \widetilde{b}^1_{0,0}g_{6,0}+g_{7,0}\widetilde{b}^0_{0,1}+g_{3,1}b^0_{0,2}+2g_{3,2}b^1_{0,0} & \widetilde{b}^1_{0,0}g_{2,2}+g_{3,1}b^0_{1,1}+g_{3,2}b^1_{1,1}\end{pmatrix}\overset{!}{=}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

  Suppose given $\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\in P$ and $\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)\in Q$.

  We obtain the following; cf. also (v).

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{0,1}g_{6,0}-g_{6,1}\widetilde{b}^0_{0,1}\right)$$

$$= \left(-\sigma,\ \sigma,\begin{pmatrix}\beta_1\\ \beta_2\\ -\beta_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)-\left(-\rho-\sigma,\ 0,\begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)=\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{0,1}g_{2,2}\right)=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)\left(\widetilde{b}^1_{0,0}g_{6,0}+g_{7,0}\widetilde{b}^0_{0,1}+g_{3,1}b^0_{0,2}+2g_{3,2}b^1_{0,0}\right)$$

$$= \frac{1}{2}\left(0,\ 0,\begin{pmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\\ \alpha_3+\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)+\left(0,\ 0,\begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ 0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$
$$+\ \frac{1}{2}\left(0,\ 0,\begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\ 2\gamma_2-\alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ \alpha_3+\beta_3\end{pmatrix}\right)+\left(0,\ 0,\begin{pmatrix}\gamma_1-\alpha_1\\ \gamma_2-\alpha_2\\ 0\end{pmatrix},\begin{pmatrix}\gamma_1-\alpha_1\\ \gamma_2-\alpha_2\\ 0\end{pmatrix}\right)$$

$$= \left(0,\ 0, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(\widetilde{b}^1_{0,0}g_{2,2} + g_{3,1}b^0_{1,1} + g_{3,2}b^1_{1,1}\right)$$

$$= \left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\alpha_1\\\alpha_2\\-\alpha_3\end{pmatrix}, \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\right) + \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\\alpha_2+\beta_2-2\gamma_2\\\alpha_3+\beta_3\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) + \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\gamma_1-2\alpha_1\\2\gamma_2-2\alpha_2\\0\end{pmatrix}, \begin{pmatrix}\gamma_1-\alpha_1\\\gamma_2-\alpha_2\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)$$

- $d_1 \cdot h_1 + h_2 \cdot d_2$

$$= \begin{pmatrix}\widetilde{b}^0_{0,1} & 0\\ b^0_{0,2} & b^0_{1,1}\\ 2b^1_{0,0} & b^1_{1,1}\end{pmatrix}\begin{pmatrix}-g_{6,1} & 0 & 0\\ g_{7,0} & g_{3,1} & g_{3,2}\end{pmatrix} + \begin{pmatrix}g_{6,0} & 0 & 0 & 0\\ 0 & g_{8,0} & g_{1,0} & g_{1,1}\\ 0 & g_{8,1} & g_{1,2} & g_{1,3}\end{pmatrix}\begin{pmatrix}b^0_{0,1} & 0 & 0\\ b^0_{0,2} & -\widetilde{b}^0_{0,1} & 0\\ 0 & -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0}\\ 0 & -2b^0_{0,0} & b^0_{1,0}\end{pmatrix}$$

$$= \begin{pmatrix}-\widetilde{b}^0_{0,1}g_{6,1} + g_{6,0}b^0_{0,1} & 0 & 0\\ -b^0_{0,2}g_{6,1} + b^0_{1,1}g_{7,0} + g_{8,0}b^0_{0,2} & b^0_{1,1}g_{3,1} - g_{8,0}\widetilde{b}^0_{0,1} - g_{1,0}\widetilde{b}^1_{0,0} - 2g_{1,1}b^0_{0,0} & b^0_{1,1}g_{3,2} + g_{1,0}\widetilde{b}^1_{1,0} + g_{1,1}b^0_{1,0}\\ -2b^1_{0,0}g_{6,1} + b^1_{1,1}g_{7,0} + g_{8,1}b^0_{0,2} & b^1_{1,1}g_{3,1} - g_{8,1}\widetilde{b}^0_{0,1} - g_{1,2}\widetilde{b}^1_{0,0} - 2g_{1,3}b^0_{0,0} & b^1_{1,1}g_{3,2} + g_{1,2}\widetilde{b}^1_{1,0} + g_{1,3}b^0_{1,0}\end{pmatrix}$$

$$\stackrel{!}{=} \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \in P$ and $\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right) \in Q$.

We obtain the following; cf. also (i).

$$\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\left(-\widetilde{b}^0_{0,1}g_{6,1} + g_{6,0}b^0_{0,1}\right)$$

$$= -\left(-\rho,\ -\rho, \begin{pmatrix}-\alpha_1\\-\alpha_2\\-\alpha_3\end{pmatrix}, \begin{pmatrix}\alpha_1\\\alpha_2\\-\alpha_3\end{pmatrix}\right) + \left(0,\ \sigma-\rho, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\\\beta_3-\alpha_3\end{pmatrix}\right) = \left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\left(-b^0_{0,2}g_{6,1} + b^0_{1,1}g_{7,0} + g_{8,0}b^0_{0,2}\right)$$

$$= -2\left(0,\ 0, \begin{pmatrix}\beta_1-\alpha_1\\\beta_2-\alpha_2\\-\alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\\-\alpha_3-\beta_3\end{pmatrix}\right) + 2\left(0,\ 0, \begin{pmatrix}2\beta_1\\2\beta_2\\0\end{pmatrix}, \begin{pmatrix}-2\beta_1\\-2\beta_2\\0\end{pmatrix}\right)$$

$$+ 2\left(0,\ 0, \begin{pmatrix}-\alpha_1-\beta_1\\-\alpha_2-\beta_2\\-\alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\\-\alpha_3-\beta_3\end{pmatrix}\right) = \left(0,\ 0, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{1,1}g_{3,1}-g_{8,0}\widetilde{b}^0_{0,1}-g_{1,0}\widetilde{b}^1_{0,0}-2g_{1,1}b^0_{0,0}\right)$$

$$=\left(-\beta_3,\ \beta_3,\begin{pmatrix}-\beta_1\\ -\beta_2\\ -\beta_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)-\left(-\alpha_3-\beta_3,\ 0,\begin{pmatrix}-\alpha_1-\beta_1\\ -\alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)-\left(0,\ 0,\begin{pmatrix}0\\ 0\\ \rho-\alpha_3\end{pmatrix},\begin{pmatrix}0\\ 0\\ \sigma-\beta_3\end{pmatrix}\right)$$

$$-\left(\alpha_3-\rho,\ \beta_3-\sigma,\begin{pmatrix}0\\ 0\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}0\\ 0\\ \beta_3-\sigma\end{pmatrix}\right)=\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(b^0_{1,1}g_{3,2}+g_{1,0}\widetilde{b}^1_{1,0}+g_{1,1}b^0_{1,0}\right)$$

$$=\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\end{pmatrix}\right)+2\left(\begin{pmatrix}0\\ 0\\ \rho-\alpha_3\end{pmatrix},\begin{pmatrix}0\\ 0\\ \sigma-\beta_3\end{pmatrix},\begin{pmatrix}0\\ 0\end{pmatrix}\right)+2\left(\begin{pmatrix}0\\ 0\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}0\\ 0\\ \beta_3-\sigma\end{pmatrix},\begin{pmatrix}0\\ 0\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)(-2b^1_{0,0}g_{6,1}+b^1_{1,1}g_{7,0}+g_{8,1}b^0_{0,2})$$

$$=-\left(0,\ 0,\begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)+2\left(0,\ 0,\begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ 0\end{pmatrix},\begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\\ 0\end{pmatrix}\right)$$

$$+\left(0,\ 0,\begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\ 2\gamma_2-\alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)=\left(0,\ 0,\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)\left(b^1_{1,1}g_{3,1}-g_{8,1}\widetilde{b}^0_{0,1}-g_{1,2}\widetilde{b}^1_{0,0}-2g_{1,3}b^0_{0,0}\right)$$

$$=\frac{1}{2}\left(-\beta_3,\ \beta_3,\begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\\ -\beta_3\end{pmatrix},\begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ \beta_3\end{pmatrix}\right)-\frac{1}{2}\left(-\alpha_3-\beta_3,\ 0,\begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\ 2\gamma_2-\alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)$$

$$-\frac{1}{2}\left(0,\ 0,\begin{pmatrix}\alpha_1-\gamma_1\\ \alpha_2-\gamma_2\\ 0\end{pmatrix},\begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ 0\end{pmatrix}\right)-\frac{1}{2}\left(\alpha_3,\ \beta_3,\begin{pmatrix}0\\ 0\\ \alpha_3\end{pmatrix},\begin{pmatrix}0\\ 0\\ \beta_3\end{pmatrix}\right)=\left(0,\ 0,\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)\left(b^1_{1,1}g_{3,2}+g_{1,2}\widetilde{b}^1_{1,0}+g_{1,3}b^0_{1,0}\right)$$

$$=\left(\begin{pmatrix}\gamma_1\\ \gamma_2\\ 0\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\\ 0\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)+\left(\begin{pmatrix}\alpha_1-\gamma_1\\ \alpha_2-\gamma_2\\ 0\end{pmatrix},\begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ 0\end{pmatrix},\begin{pmatrix}0\\ 0\end{pmatrix}\right)+\left(\begin{pmatrix}0\\ 0\\ \alpha_3\end{pmatrix},\begin{pmatrix}0\\ 0\\ \beta_3\end{pmatrix},\begin{pmatrix}0\\ 0\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)$$

(ix)

- $B_0 \cdot H_2 + H_3 \cdot B_1 = \begin{pmatrix} b^1_{1,1} & b^1_{0,1} \\ -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix} \begin{pmatrix} g_{2,0} & 0 \\ g_{2,1} & g_{2,2} \end{pmatrix} + \begin{pmatrix} g_{3,0} & 0 \\ g_{3,1} & g_{3,2} \end{pmatrix} \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix}$

$$= \begin{pmatrix} b^1_{1,1}g_{2,0} + b^1_{0,1}g_{2,1} + g_{3,0}b^0_{1,0} & b^1_{0,1}g_{2,2} + g_{3,0}b^0_{1,1} \\ -2b^1_{1,0}g_{2,0} + \widetilde{b}^1_{0,0}g_{2,1} + g_{3,1}b^0_{1,0} + 2g_{3,2}b^1_{1,0} & \widetilde{b}^1_{0,0}g_{2,2} + g_{3,1}b^0_{1,1} + g_{3,2}b^1_{1,1} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Suppose given $\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \in Q$. We obtain the following.

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) (b^1_{1,1}g_{2,0} + b^1_{0,1}g_{2,1} + g_{3,0}b^0_{1,0})$$

$$= \left(\begin{pmatrix} 2\gamma_1 - 2\beta_1 \\ 2\gamma_2 - 2\beta_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_1 - \beta_1 \\ \gamma_2 - \beta_2 \end{pmatrix} \right) + \left(\begin{pmatrix} \beta_1 \\ \beta_2 \\ -\beta_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right)$$

$$+ \left(\begin{pmatrix} \alpha_1 + \beta_1 - 2\gamma_1 \\ \alpha_2 + \beta_2 - 2\gamma_2 \\ \alpha_3 + \beta_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) (b^1_{0,1}g_{2,2} + g_{3,0}b^0_{1,1})$$

$$= \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \left(-2b^1_{1,0}g_{2,0} + \widetilde{b}^1_{0,0}g_{2,1} + g_{3,1}b^0_{1,0} + 2g_{3,2}b^1_{1,0}\right)$$

$$= -\frac{1}{2} \left(\begin{pmatrix} 4\gamma_1 - 2\alpha_1 - 2\beta_1 \\ 4\gamma_2 - 2\alpha_2 - 2\beta_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\gamma_1 - \alpha_1 - \beta_1 \\ 2\gamma_2 - \alpha_2 - \beta_2 \end{pmatrix} \right) + \frac{1}{2} \left(\begin{pmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \alpha_3 + \beta_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ -\alpha_3 - \beta_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \end{pmatrix} \right)$$

$$+ \frac{1}{2} \left(\begin{pmatrix} 2\gamma_1 - \alpha_1 - \beta_1 \\ 2\gamma_2 - \alpha_2 - \beta_2 \\ -\alpha_3 - \beta_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 + \beta_1 - 2\gamma_1 \\ \alpha_2 + \beta_2 - 2\gamma_2 \\ \alpha_3 + \beta_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + \left(\begin{pmatrix} \gamma_1 - \alpha_1 \\ \gamma_2 - \alpha_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_1 - \alpha_1 \\ \gamma_2 - \alpha_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_1 - \alpha_1 \\ \gamma_2 - \alpha_2 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \left(\widetilde{b}^1_{0,0}g_{2,2} + g_{3,1}b^0_{1,1} + g_{3,2}b^1_{1,1}\right)$$

$$= \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ -\alpha_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right) + \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 + \beta_1 - 2\gamma_1 \\ \alpha_2 + \beta_2 - 2\gamma_2 \\ \alpha_3 + \beta_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$+ \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\gamma_1 - 2\alpha_1 \\ 2\gamma_2 - 2\alpha_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_1 - \alpha_1 \\ \gamma_2 - \alpha_2 \end{pmatrix} \right) = \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right)$$

- $B_1 \cdot H_0 + H_1 \cdot B_2 = \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} \begin{pmatrix} g_{0,0} & g_{0,1} \\ g_{0,2} & g_{0,3} \end{pmatrix} + \begin{pmatrix} g_{1,0} & g_{1,1} \\ g_{1,2} & g_{1,3} \end{pmatrix} \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix}$

$$= \begin{pmatrix} b^0_{1,0}g_{0,0} + b^0_{1,1}g_{0,2} - g_{1,0}\widetilde{b}^1_{0,0} - 2g_{1,1}b^0_{0,0} & b^0_{1,0}g_{0,1} + b^0_{1,1}g_{0,3} + g_{1,0}\widetilde{b}^1_{1,0} + g_{1,1}b^0_{1,0} \\ 2b^1_{1,0}g_{0,0} + b^1_{1,1}g_{0,2} - g_{1,2}\widetilde{b}^1_{0,0} - 2g_{1,3}b^0_{0,0} & 2b^1_{1,0}g_{0,1} + b^1_{1,1}g_{0,3} + g_{1,2}\widetilde{b}^1_{1,0} + g_{1,3}b^0_{1,0} \end{pmatrix}$$

$$\overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \in P$ and $\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \in Q$. We obtain the following.

$$\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \left(b^0_{1,0}g_{0,0} + b^0_{1,1}g_{0,2} - g_{1,0}\widetilde{b}^1_{0,0} - 2g_{1,1}b^0_{0,0}\right)$$

$$= \left(\alpha_3 - \beta_3,\ 0, \begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) + \left(\beta_3,\ \beta_3, \begin{pmatrix}-\beta_1\\ -\beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) - \left(0,\ 0, \begin{pmatrix}0\\0\\ \rho-\alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\\ \sigma-\beta_3\end{pmatrix}\right)$$

$$- \left(\alpha_3 - \rho,\ \beta_3 - \sigma, \begin{pmatrix}0\\0\\ \alpha_3-\rho\end{pmatrix}, \begin{pmatrix}0\\0\\ \beta_3-\sigma\end{pmatrix}\right) = \left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \left(b^0_{1,0}g_{0,1} + b^0_{1,1}g_{0,3} + g_{1,0}\widetilde{b}^1_{1,0} + g_{1,1}b^0_{1,0}\right)$$

$$= 4\left(\begin{pmatrix}-\beta_1\\ -\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}-\beta_1\\ -\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}-\beta_1\\ -\beta_2\end{pmatrix}\right) + 4\left(\begin{pmatrix}\beta_1\\ \beta_2\\ 0\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ 0\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\right) + 2\left(\begin{pmatrix}0\\0\\ \rho-\alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\\ \sigma-\beta_3\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$+ 2\left(\begin{pmatrix}0\\0\\ \alpha_3-\rho\end{pmatrix}, \begin{pmatrix}0\\0\\ \beta_3-\sigma\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) = \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \left(2b^1_{1,0}g_{0,0} + b^1_{1,1}g_{0,2} - g_{1,2}\widetilde{b}^1_{0,0} - 2g_{1,3}b^0_{0,0}\right)$$

$$= \frac{1}{2}\left(\alpha_3 - \beta_3,\ 0, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ \alpha_3-\beta_3\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) + \frac{1}{2}\left(\beta_3,\ \beta_3, \begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ \beta_3\end{pmatrix}\right)$$

$$- \frac{1}{2}\left(0,\ 0, \begin{pmatrix}\alpha_1-\gamma_1\\ \alpha_2-\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ 0\end{pmatrix}\right) - \frac{1}{2}\left(\alpha_3,\ \beta_3, \begin{pmatrix}0\\0\\ \alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\\ \beta_3\end{pmatrix}\right) = \left(0,\ 0, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \left(2b^1_{1,0}g_{0,1} + b^1_{1,1}g_{0,3} + g_{1,2}\widetilde{b}^1_{1,0} + g_{1,3}b^0_{1,0}\right)$$

$$= 2\left(\begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}\gamma_1-\beta_1\\ \gamma_2-\beta_2\end{pmatrix}\right) + \left(\begin{pmatrix}2\beta_1-\gamma_1\\ 2\beta_2-\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}2\beta_1-\gamma_1\\ 2\beta_2-\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}2\beta_1-\gamma_1\\ 2\beta_2-\gamma_2\end{pmatrix}\right)$$

$$+ \left(\begin{pmatrix}\alpha_1-\gamma_1\\ \alpha_2-\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}\beta_1-\gamma_1\\ \beta_2-\gamma_2\\ 0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) + \left(\begin{pmatrix}0\\0\\ \alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\\ \beta_3\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) = \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)$$

- $B_2 \cdot H_4 + H_5 \cdot B_0 = \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix} \begin{pmatrix} g_{1,0} & g_{1,1} \\ g_{1,2} & g_{1,3} \end{pmatrix} + \begin{pmatrix} g_{5,0} & g_{5,1} \\ g_{5,2} & g_{5,3} \end{pmatrix} \begin{pmatrix} b^1_{1,1} & b^1_{0,1} \\ -2b^1_{1,0} & \widetilde{b}^1_{0,0} \end{pmatrix}$

$$= \begin{pmatrix} -\widetilde{b}^1_{0,0}g_{1,0} + \widetilde{b}^1_{1,0}g_{1,2} + g_{5,0}b^1_{1,1} - 2g_{5,1}b^1_{1,0} & -\widetilde{b}^1_{0,0}g_{1,1} + \widetilde{b}^1_{1,0}g_{1,3} + g_{5,0}b^1_{0,1} + g_{5,1}\widetilde{b}^1_{0,0} \\ -2b^0_{0,0}g_{1,0} + b^0_{1,0}g_{1,2} + g_{5,2}b^1_{1,1} - 2g_{5,3}b^1_{1,0} & -2b^0_{0,0}g_{1,1} + b^0_{1,0}g_{1,3} + g_{5,2}b^1_{0,1} + g_{5,3}\widetilde{b}^1_{0,0} \end{pmatrix}$$

$$\overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \in P$ and $\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right) \in Q$. We obtain the following.

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right) \left(-\widetilde{b}^1_{0,0}g_{1,0} + \widetilde{b}^1_{1,0}g_{1,2} + g_{5,0}b^1_{1,1} - 2g_{5,1}b^1_{1,0}\right)$$

$$= -\left(\begin{pmatrix}0\\0\\-\alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\\-\beta_3\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) + \left(\begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}(\alpha_1+\beta_1)\\\frac{1}{2}(\alpha_2+\beta_2)\end{pmatrix}\right)$$

$$+ \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\alpha_1+2\beta_1-4\gamma_1\\2\alpha_2+2\beta_2-4\gamma_2\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}(\alpha_1+\beta_1-2\gamma_1)\\\frac{1}{2}(\alpha_2+\beta_2-2\gamma_2)\end{pmatrix}\right) - \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\alpha_1+2\beta_1-4\gamma_1\\2\alpha_2+2\beta_2-4\gamma_2\\0\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\\alpha_2+\beta_2-2\gamma_2\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right) \left(-\widetilde{b}^1_{0,0}g_{1,1} + \widetilde{b}^1_{1,0}g_{1,3} + g_{5,0}b^1_{0,1} + g_{5,1}\widetilde{b}^1_{0,0}\right)$$

$$= -\frac{1}{2}\left(\alpha_3,\ -\beta_3,\ \begin{pmatrix}0\\0\\\alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\\-\beta_3\end{pmatrix}\right) + \frac{1}{2}\left(\alpha_3,\ -\beta_3,\ \begin{pmatrix}0\\0\\\alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\\-\beta_3\end{pmatrix}\right)$$

$$+ \left(0,\ 0,\ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\\alpha_2+\beta_2-2\gamma_2\\0\end{pmatrix}\right) + \left(0,\ 0,\ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\2\gamma_2-\alpha_2-\beta_2\\0\end{pmatrix}\right) = \left(0,\ 0,\ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right) \left(-2b^0_{0,0}g_{1,0} + b^0_{1,0}g_{1,2} + g_{5,2}b^1_{1,1} - 2g_{5,3}b^1_{1,0}\right)$$

$$= -2\left(\begin{pmatrix}0\\0\\\rho-\alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\\\beta_3-\sigma\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) + \left(\begin{pmatrix}2\alpha_1\\2\alpha_2\\0\end{pmatrix}, \begin{pmatrix}-2\beta_1\\-2\beta_2\\0\end{pmatrix}, \begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\end{pmatrix}\right)$$

$$+ \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\alpha_1+2\beta_1\\2\alpha_2+2\beta_2\\2\alpha_3-2\rho+2\beta_3-2\sigma\end{pmatrix}, \begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\end{pmatrix}\right) - 2\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix}, \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix}, \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\left(-2b^0_{0,0}g_{1,1}+b^0_{1,0}g_{1,3}+g_{5,2}b^1_{0,1}+g_{5,3}\widetilde{b}^1_{0,0}\right)$$

$$=-\left(\alpha_3-\rho,\ \beta_3-\sigma,\begin{pmatrix}0\\0\\\alpha_3-\rho\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3-\sigma\end{pmatrix}\right)+\left(\alpha_3,\ \beta_3,\begin{pmatrix}0\\0\\\alpha_3\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3\end{pmatrix}\right)$$

$$+\left(0,\ 0,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\\\alpha_3-\rho+\beta_3-\sigma\end{pmatrix}\right)+\left(0,\ 0,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\begin{pmatrix}-\alpha_1\\-\alpha_2\\\rho-\alpha_3\end{pmatrix}\right)=\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)$$

(x)

- $B_0\cdot H_5+H_0\cdot B_1=\begin{pmatrix}b^1_{1,1}&b^1_{0,1}\\-2b^1_{1,0}&\widetilde{b}^1_{0,0}\end{pmatrix}\begin{pmatrix}g_{5,0}&g_{5,1}\\g_{5,2}&g_{5,3}\end{pmatrix}+\begin{pmatrix}g_{0,0}&g_{0,1}\\g_{0,2}&g_{0,3}\end{pmatrix}\begin{pmatrix}b^0_{1,0}&b^0_{1,1}\\2b^1_{1,0}&b^1_{1,1}\end{pmatrix}$

$$=\begin{pmatrix}b^1_{1,1}g_{5,0}+b^1_{0,1}g_{5,2}+g_{0,0}b^0_{1,0}+2g_{0,1}b^1_{1,0} & b^1_{1,1}g_{5,1}+b^1_{0,1}g_{5,3}+g_{0,0}b^0_{1,1}+g_{0,1}b^1_{1,1}\\-2b^1_{1,0}g_{5,0}+\widetilde{b}^1_{0,0}g_{5,2}+g_{0,2}b^0_{1,0}+2g_{0,3}b^1_{1,0} & -2b^1_{1,0}g_{5,1}+\widetilde{b}^1_{0,0}g_{5,3}+g_{0,2}b^0_{1,1}+g_{0,3}b^1_{1,1}\end{pmatrix}$$

$$\overset{!}{=}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Suppose given $\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\in Q$. We obtain the following.

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(b^1_{1,1}g_{5,0}+b^1_{0,1}g_{5,2}+g_{0,0}b^0_{1,0}+2g_{0,1}b^1_{1,0}\right)$$

$$=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\beta_1-2\gamma_1\\2\beta_2-2\gamma_2\\0\end{pmatrix},\begin{pmatrix}\beta_1-\gamma_1\\\beta_2-\gamma_2\end{pmatrix}\right)+\left(\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix}\right)+\left(\begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\\alpha_2+\beta_2-2\gamma_2\\\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$+2\left(\begin{pmatrix}\gamma_1-\beta_1\\\gamma_2-\beta_2\\0\end{pmatrix},\begin{pmatrix}\gamma_1-\beta_1\\\gamma_2-\beta_2\\0\end{pmatrix},\begin{pmatrix}\gamma_1-\beta_1\\\gamma_2-\beta_2\end{pmatrix}\right)=\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(b^1_{1,1}g_{5,1}+b^1_{0,1}g_{5,3}+g_{0,0}b^0_{1,1}+g_{0,1}b^1_{1,1}\right)$$

$$=2\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\beta_1-2\gamma_1\\2\beta_2-2\gamma_2\\0\end{pmatrix},\begin{pmatrix}\beta_1-\gamma_1\\\beta_2-\gamma_2\end{pmatrix}\right)+\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)+\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$+2\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\gamma_1-2\beta_1\\2\gamma_2-2\beta_2\\0\end{pmatrix},\begin{pmatrix}\gamma_1-\beta_1\\\gamma_2-\beta_2\end{pmatrix}\right)=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(-2b^1_{1,0}g_{5,0}+\widetilde{b}^1_{0,0}g_{5,2}+g_{0,2}b^0_{1,0}+2g_{0,3}b^1_{1,0}\right)$$

$$=-\frac{1}{2}\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\alpha_1+2\beta_1-4\gamma_1\\2\alpha_2+2\beta_2-4\gamma_2\\0\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\\alpha_2+\beta_2-2\gamma_2\end{pmatrix}\right)+\frac{1}{2}\left(\begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\\\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\\\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\end{pmatrix}\right)$$

$$+\frac{1}{2}\left(\begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\2\gamma_2-\alpha_2-\beta_2\\\beta_3-\alpha_3\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\\alpha_2+\beta_2-2\gamma_2\\\beta_3-\alpha_3\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)+\left(\begin{pmatrix}\beta_1-\gamma_1\\\beta_2-\gamma_2\\0\end{pmatrix},\begin{pmatrix}\beta_1-\gamma_1\\\beta_2-\gamma_2\\0\end{pmatrix},\begin{pmatrix}\beta_1-\gamma_1\\\beta_2-\gamma_2\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \left(-2b^1_{1,0} g_{5,1} + \widetilde{b}^1_{0,0} g_{5,3} + g_{0,2} b^0_{1,1} + g_{0,3} b^1_{1,1} \right)$$

$$= - \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\alpha_1 + 2\beta_1 - 4\gamma_1 \\ 2\alpha_2 + 2\beta_2 - 4\gamma_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 + \beta_1 - 2\gamma_1 \\ \alpha_2 + \beta_2 - 2\gamma_2 \end{pmatrix} \right) + \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right)$$

$$+ \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 + \beta_1 - 2\gamma_1 \\ \alpha_2 + \beta_2 - 2\gamma_2 \\ \beta_3 - \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\beta_1 - 2\gamma_1 \\ 2\beta_2 - 2\gamma_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta_1 - \gamma_1 \\ \beta_2 - \gamma_2 \end{pmatrix} \right) = \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right)$$

- $$B_1 \cdot H_3 + H_4 \cdot B_2 = \begin{pmatrix} b^0_{1,0} & b^0_{1,1} \\ 2b^1_{1,0} & b^1_{1,1} \end{pmatrix} \begin{pmatrix} g_{3,0} & 0 \\ g_{3,1} & g_{3,2} \end{pmatrix} + \begin{pmatrix} g_{1,0} & g_{1,1} \\ g_{1,2} & g_{1,3} \end{pmatrix} \begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0} \end{pmatrix}$$

$$= \begin{pmatrix} b^0_{1,0} g_{3,0} + b^0_{1,1} g_{3,1} - g_{1,0} \widetilde{b}^1_{0,0} - 2g_{1,1} b^0_{0,0} & b^0_{1,1} g_{3,2} + g_{1,0} \widetilde{b}^1_{1,0} + g_{1,1} b^0_{1,0} \\ 2b^1_{1,0} g_{3,0} + b^1_{1,1} g_{3,1} - g_{1,2} \widetilde{b}^1_{0,0} - 2g_{1,3} b^0_{0,0} & b^1_{1,1} g_{3,2} + g_{1,2} \widetilde{b}^1_{1,0} + g_{1,3} b^0_{1,0} \end{pmatrix}$$

$$\overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \in P$ and $\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \in Q$. We obtain the following.

$$\left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \left(b^0_{1,0} g_{3,0} + b^0_{1,1} g_{3,1} - g_{1,0} \widetilde{b}^1_{0,0} - 2g_{1,1} b^0_{0,0} \right)$$

$$= \left(\alpha_3 + \beta_3,\ 0, \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) + \left(-\beta_3,\ \beta_3, \begin{pmatrix} -\beta_1 \\ -\beta_2 \\ -\beta_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) - \left(0,\ 0, \begin{pmatrix} 0 \\ 0 \\ \rho - \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sigma - \beta_3 \end{pmatrix} \right)$$

$$- \left(\alpha_3 - \rho,\ \beta_3 - \sigma, \begin{pmatrix} 0 \\ 0 \\ \alpha_3 - \rho \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \beta_3 - \sigma \end{pmatrix} \right) = \left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right)$$

$$\left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \left(b^0_{1,1} g_{3,2} + g_{1,0} \widetilde{b}^1_{1,0} + g_{1,1} b^0_{1,0} \right)$$

$$= \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + 2 \left(\begin{pmatrix} 0 \\ 0 \\ \rho - \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sigma - \beta_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + 2 \left(\begin{pmatrix} 0 \\ 0 \\ \alpha_3 - \rho \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \beta_3 - \sigma \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \left(2b^1_{1,0} g_{3,0} + b^1_{1,1} g_{3,1} - g_{1,2} \widetilde{b}^1_{0,0} - 2g_{1,3} b^0_{0,0} \right)$$

$$= \frac{1}{2} \left(\alpha_3 + \beta_3,\ 0, \begin{pmatrix} \alpha_1 + \beta_1 - 2\gamma_1 \\ \alpha_2 + \beta_2 - 2\gamma_2 \\ \alpha_3 + \beta_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) + \frac{1}{2} \left(-\beta_3,\ \beta_3, \begin{pmatrix} \gamma_1 - \beta_1 \\ \gamma_2 - \beta_2 \\ -\beta_3 \end{pmatrix}, \begin{pmatrix} \beta_1 - \gamma_1 \\ \beta_2 - \gamma_2 \\ \beta_3 \end{pmatrix} \right)$$

$$- \frac{1}{2} \left(0,\ 0, \begin{pmatrix} \alpha_1 - \gamma_1 \\ \alpha_2 - \gamma_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta_1 - \gamma_1 \\ \beta_2 - \gamma_2 \\ 0 \end{pmatrix} \right) - \frac{1}{2} \left(\alpha_3,\ \beta_3, \begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix} \right) = \left(0,\ 0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(b^1_{1,1}g_{3,2}+g_{1,2}\widetilde{b}^1_{1,0}+g_{1,3}b^0_{1,0}\right)$$

$$=\left(\begin{pmatrix}\gamma_1\\\gamma_2\\0\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\\0\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)+\left(\begin{pmatrix}\alpha_1-\gamma_1\\\alpha_2-\gamma_2\\0\end{pmatrix},\begin{pmatrix}\beta_1-\gamma_1\\\beta_2-\gamma_2\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)+\left(\begin{pmatrix}0\\0\\\alpha_3\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)$$

- $B_2\cdot H_1+H_2\cdot B_0=\begin{pmatrix}-\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0}\\-2b^0_{0,0} & b^0_{1,0}\end{pmatrix}\begin{pmatrix}g_{1,0} & g_{1,1}\\g_{1,2} & g_{1,3}\end{pmatrix}+\begin{pmatrix}g_{2,0} & 0\\g_{2,1} & g_{2,2}\end{pmatrix}\begin{pmatrix}b^1_{1,1} & b^1_{0,1}\\-2b^1_{1,0} & \widetilde{b}^1_{0,0}\end{pmatrix}$

$$=\begin{pmatrix}-\widetilde{b}^1_{0,0}g_{1,0}+\widetilde{b}^1_{1,0}g_{1,2}+g_{2,0}b^1_{1,1} & -\widetilde{b}^1_{0,0}g_{1,1}+\widetilde{b}^1_{1,0}g_{1,3}+g_{2,0}b^1_{0,1}\\-2b^0_{0,0}g_{1,0}+b^0_{1,0}g_{1,2}+g_{2,1}b^1_{1,1}-2g_{2,2}b^1_{1,0} & -2b^0_{0,0}g_{1,1}+b^0_{1,0}g_{1,3}+g_{2,1}b^1_{0,1}+g_{2,2}\widetilde{b}^1_{0,0}\end{pmatrix}$$

$$\overset{!}{=}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\in P$ and $\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\in Q$. We obtain the following.

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(-\widetilde{b}^1_{0,0}g_{1,0}+\widetilde{b}^1_{1,0}g_{1,2}+g_{2,0}b^1_{1,1}\right)$$

$$=-\left(\begin{pmatrix}0\\0\\-\alpha_3\end{pmatrix},\begin{pmatrix}0\\0\\-\beta_3\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)+\left(\begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\0\end{pmatrix},\begin{pmatrix}\frac{1}{2}(\alpha_1+\beta_1)\\\frac{1}{2}(\alpha_2+\beta_2)\end{pmatrix}\right)$$

$$+\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}\frac{1}{2}(2\gamma_1-\alpha_1-\beta_1)\\\frac{1}{2}(2\gamma_2-\alpha_2-\beta_2)\end{pmatrix}\right)=\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(-\widetilde{b}^1_{0,0}g_{1,1}+\widetilde{b}^1_{1,0}g_{1,3}+g_{2,0}b^1_{0,1}\right)$$

$$=-\frac{1}{2}\left(\alpha_3,\ -\beta_3,\begin{pmatrix}0\\0\\\alpha_3\end{pmatrix},\begin{pmatrix}0\\0\\-\beta_3\end{pmatrix}\right)+\frac{1}{2}\left(\alpha_3,\ -\beta_3,\begin{pmatrix}0\\0\\\alpha_3\end{pmatrix},\begin{pmatrix}0\\0\\-\beta_3\end{pmatrix}\right)+\left(0,\ 0,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$=\left(0,\ 0,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\left(-2b^0_{0,0}g_{1,0}+b^0_{1,0}g_{1,2}+g_{2,1}b^1_{1,1}-2g_{2,2}b^1_{1,0}\right)$$

$$=-2\left(\begin{pmatrix}0\\0\\\rho-\alpha_3\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3-\sigma\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)+2\left(\begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix},\begin{pmatrix}-\beta_1\\-\beta_2\\0\end{pmatrix},\begin{pmatrix}\frac{1}{2}(\alpha_1-\beta_1)\\\frac{1}{2}(\alpha_2-\beta_2)\end{pmatrix}\right)$$

$$+2\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1\\\alpha_2+\beta_2\\\rho-\alpha_3+\beta_3-\sigma\end{pmatrix},\begin{pmatrix}\frac{1}{2}(\alpha_1+\beta_1)\\\frac{1}{2}(\alpha_2+\beta_2)\end{pmatrix}\right)-2\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3-\rho\end{pmatrix},\begin{pmatrix}\alpha_1\\\alpha_2\\\rho-\alpha_3\end{pmatrix},\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \left(-2b^0_{0,0} g_{1,1} + b^0_{1,0} g_{1,3} + g_{2,1} b^1_{0,1} + g_{2,2} \widetilde{b}^1_{0,0} \right)$$

$$= - \left(\alpha_3 - \rho,\ \beta_3 - \sigma, \begin{pmatrix} 0 \\ 0 \\ \alpha_3 - \rho \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \beta_3 - \sigma \end{pmatrix} \right) + \left(\alpha_3,\ \beta_3, \begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix} \right)$$

$$+ \left(0,\ 0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \rho - \alpha_3 + \beta_3 - \sigma \end{pmatrix} \right) + \left(0,\ 0, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 - \rho \end{pmatrix}, \begin{pmatrix} -\alpha_1 \\ -\alpha_2 \\ \alpha_3 - \rho \end{pmatrix} \right) = \left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right)$$

(xi)

- $$D_5 \cdot H_4 + H_1 \cdot D_0 = \begin{pmatrix} 2\widetilde{b}^0_{0,1} & 0 \\ \widetilde{b}^1_{0,1} & 0 \end{pmatrix} \begin{pmatrix} g_{1,0} & g_{1,1} \\ g_{1,2} & g_{1,3} \end{pmatrix} + \begin{pmatrix} g_{1,0} & g_{1,1} \\ g_{1,2} & g_{1,3} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1} \end{pmatrix}$$
$$= \begin{pmatrix} 2\widetilde{b}^0_{0,1} g_{1,0} + g_{1,1} \widetilde{b}^0_{1,1} & 2\widetilde{b}^0_{0,1} g_{1,1} - 2 g_{1,1} \widetilde{b}^0_{0,1} \\ \widetilde{b}^1_{0,1} g_{1,0} + g_{1,3} \widetilde{b}^0_{1,1} & \widetilde{b}^1_{0,1} g_{1,1} - 2 g_{1,3} \widetilde{b}^0_{0,1} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \in P$ and $\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \in Q$. We obtain the following.

$$\left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \left(2\widetilde{b}^0_{0,1} g_{1,0} + g_{1,1} \widetilde{b}^0_{1,1} \right)$$

$$= 4 \left(\begin{pmatrix} 0 \\ 0 \\ \rho - \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + 4 \left(\begin{pmatrix} 0 \\ 0 \\ \alpha_3 - \rho \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\rho,\ \sigma, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right) \left(2\widetilde{b}^0_{0,1} g_{1,1} - 2 g_{1,1} \widetilde{b}^0_{0,1} \right)$$

$$= 2 \left(\alpha_3 - \rho,\ 0, \begin{pmatrix} 0 \\ 0 \\ \alpha_3 - \rho \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) - 2 \left(\alpha_3 - \rho,\ 0, \begin{pmatrix} 0 \\ 0 \\ \alpha_3 - \rho \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) = \left(0,\ 0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \left(\widetilde{b}^1_{0,1} g_{1,0} + g_{1,3} \widetilde{b}^0_{1,1} \right)$$

$$= 2 \left(\begin{pmatrix} 0 \\ 0 \\ -\alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + 2 \left(\begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \left(\widetilde{b}^1_{0,1} g_{1,1} - 2 g_{1,3} \widetilde{b}^0_{0,1} \right)$$

$$= \left(\alpha_3,\ 0, \begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) - \left(\alpha_3,\ 0, \begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) = \left(0,\ 0, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

- $D_3 \cdot H_2 + H_5 \cdot D_4 = \begin{pmatrix} 0 & 0 \\ -b^0_{1,1} & -2b^0_{0,1} \end{pmatrix} \begin{pmatrix} g_{2,0} & 0 \\ g_{2,1} & g_{2,2} \end{pmatrix} + \begin{pmatrix} g_{5,0} & g_{5,1} \\ g_{5,2} & g_{5,3} \end{pmatrix} \begin{pmatrix} -2\widetilde{b}^1_{1,0} & 0 \\ \widetilde{b}^1_{1,0} & 0 \end{pmatrix}$

$$= \begin{pmatrix} -2g_{5,0}\widetilde{b}^1_{1,0} + g_{5,1}\widetilde{b}^1_{1,0} & 0 \\ -b^0_{1,1}g_{2,0} - 2b^0_{0,1}g_{2,1} - 2g_{5,2}\widetilde{b}^1_{1,0} + g_{5,3}\widetilde{b}^1_{1,0} & -2b^0_{0,1}g_{2,2} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \in P$ and $\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \in Q$. We obtain the following.

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \left(-2g_{5,0}\widetilde{b}^1_{1,0} + g_{5,1}\widetilde{b}^1_{1,0}\right)$$

$$= -2\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\gamma_1 - \alpha_1 - \beta_1\\ 2\gamma_2 - \alpha_2 - \beta_2\\ 0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) + 2\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\gamma_1 - \alpha_1 - \beta_1\\ 2\gamma_2 - \alpha_2 - \beta_2\\ 0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) = \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \left(-b^0_{1,1}g_{2,0} - 2b^0_{0,1}g_{2,1} - 2g_{5,2}\widetilde{b}^1_{1,0} + g_{5,3}\widetilde{b}^1_{1,0}\right)$$

$$= -2\left(\begin{pmatrix}-2\beta_1\\ -2\beta_2\\ 0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}-\beta_1\\ -\beta_2\end{pmatrix}\right) - 2\left(\begin{pmatrix}\beta_1\\ \beta_2\\ \sigma - \beta_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3 - \sigma\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\right)$$

$$- 2\left(\begin{pmatrix}\alpha_1 + \beta_1\\ \alpha_2 + \beta_2\\ \alpha_3 - \rho + \beta_3 - \sigma\end{pmatrix}, \begin{pmatrix}-\alpha_1 - \beta_1\\ -\alpha_2 - \beta_2\\ \rho - \alpha_3 + \sigma - \beta_3\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) + 2\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3 - \rho\end{pmatrix}, \begin{pmatrix}-\alpha_1\\ -\alpha_2\\ \rho - \alpha_3\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$= \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \left(-2b^0_{0,1}g_{2,2}\right) = \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)$$

- $D_1 \cdot H_0 + H_3 \cdot D_2 = \begin{pmatrix} 0 & 0 \\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0} \end{pmatrix} \begin{pmatrix} g_{0,0} & g_{0,1} \\ g_{0,2} & g_{0,3} \end{pmatrix} + \begin{pmatrix} g_{3,0} & 0 \\ g_{3,1} & g_{3,2} \end{pmatrix} \begin{pmatrix} 2b^0_{0,1} & 0 \\ b^1_{0,1} & 0 \end{pmatrix}$

$$= \begin{pmatrix} 2g_{3,0}b^0_{0,1} & 0 \\ \widetilde{b}^1_{1,0}g_{0,0} + 2\widetilde{b}^1_{1,0}g_{0,2} + 2g_{3,1}b^0_{0,1} + g_{3,2}b^1_{0,1} & \widetilde{b}^1_{1,0}g_{0,1} + 2\widetilde{b}^1_{1,0}g_{0,3} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Suppose given $\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \in Q$. We obtain the following; cf. also (iii).

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \left(2g_{3,0}b^0_{0,1}\right) = \left(0,\ 0,\ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \left(\widetilde{b}^1_{1,0}g_{0,0} + 2\widetilde{b}^1_{1,0}g_{0,2} + 2g_{3,1}b^0_{0,1} + g_{3,2}b^1_{0,1}\right)$$

$$= \frac{1}{2}\left(\alpha_3+\beta_3,\ 0,\ \begin{pmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\\ \alpha_3+\beta_3\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)+\frac{1}{2}\left(-\alpha_3-\beta_3,\ -\alpha_3-\beta_3,\ \begin{pmatrix}\beta_1-\alpha_1\\ \beta_2-\alpha_2\\ -\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\\ -\alpha_3-\beta_3\end{pmatrix}\right)$$

$$+\frac{1}{2}\left(0,\ \alpha_3+\beta_3,\ \begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\ \alpha_2+\beta_2-2\gamma_2\\ \alpha_3+\beta_3\end{pmatrix}\right)+\left(0,\ 0,\ \begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}\gamma_1-\alpha_1\\ \gamma_2-\alpha_2\\ 0\end{pmatrix}\right)=\left(0,\ 0,\ \begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)\left(\widetilde{b}^1_{1,0}g_{0,1}+2\widetilde{b}^1_{1,0}g_{0,3}\right)$$

$$=2\left(\begin{pmatrix}\beta_1\\ \beta_2\\ 0\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ 0\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\right)+2\left(\begin{pmatrix}-\beta_1\\ -\beta_2\\ 0\end{pmatrix},\begin{pmatrix}-\beta_1\\ -\beta_2\\ 0\end{pmatrix},\begin{pmatrix}-\beta_1\\ -\beta_2\end{pmatrix}\right)=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

(xii)

- $$D_0\cdot H_5+H_2\cdot D_1=\begin{pmatrix}0 & 0\\ \widetilde{b}^0_{1,1} & -2\widetilde{b}^0_{0,1}\end{pmatrix}\begin{pmatrix}g_{5,0} & g_{5,1}\\ g_{5,2} & g_{5,3}\end{pmatrix}+\begin{pmatrix}g_{2,0} & 0\\ g_{2,1} & g_{2,2}\end{pmatrix}\begin{pmatrix}0 & 0\\ \widetilde{b}^1_{1,0} & 2\widetilde{b}^1_{1,0}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0\\ \widetilde{b}^0_{1,1}g_{5,0}-2\widetilde{b}^0_{0,1}g_{5,2}+g_{2,2}\widetilde{b}^1_{1,0} & \widetilde{b}^0_{1,1}g_{5,1}-2\widetilde{b}^0_{0,1}g_{5,3}+2g_{2,2}\widetilde{b}^1_{1,0}\end{pmatrix}\overset{!}{=}\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\in P$. We obtain the following; cf. also (vi).

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(\widetilde{b}^0_{1,1}g_{5,0}-2\widetilde{b}^0_{0,1}g_{5,2}+g_{2,2}\widetilde{b}^1_{1,0}\right)$$

$$=2\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\alpha_1\\ 2\alpha_2\\ 0\end{pmatrix},\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right)-2\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right)+2\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}-\alpha_1\\ -\alpha_2\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\ \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right)\left(\widetilde{b}^0_{1,1}g_{5,1}-2\widetilde{b}^0_{0,1}g_{5,3}+2g_{2,2}\widetilde{b}^1_{1,0}\right)$$

$$=4\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\alpha_1\\ 2\alpha_2\\ 0\end{pmatrix},\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right)-4\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right)+4\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}-\alpha_1\\ -\alpha_2\\ \alpha_3-\rho\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

- $$D_4\cdot H_3+H_0\cdot D_5=\begin{pmatrix}-2\widetilde{b}^1_{1,0} & 0\\ \widetilde{b}^1_{1,0} & 0\end{pmatrix}\begin{pmatrix}g_{3,0} & 0\\ g_{3,1} & g_{3,2}\end{pmatrix}+\begin{pmatrix}g_{0,0} & g_{0,1}\\ g_{0,2} & g_{0,3}\end{pmatrix}\begin{pmatrix}2\widetilde{b}^0_{0,1} & 0\\ \widetilde{b}^1_{0,1} & 0\end{pmatrix}$$

$$=\begin{pmatrix}-2\widetilde{b}^1_{1,0}g_{3,0}+2g_{0,0}\widetilde{b}^0_{0,1}+g_{0,1}\widetilde{b}^1_{0,1} & 0\\ \widetilde{b}^1_{1,0}g_{3,0}+2g_{0,2}\widetilde{b}^0_{0,1}+g_{0,3}\widetilde{b}^1_{0,1} & 0\end{pmatrix}\overset{!}{=}\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Suppose given $\left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)\in Q$. We obtain the following.

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(-2\widetilde{b}^1_{1,0}g_{3,0}+2g_{0,0}\widetilde{b}^0_{0,1}+g_{0,1}\widetilde{b}^1_{0,1}\right)$$
$$=-\left(\alpha_3-\beta_3,\ 0,\begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\\\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)+\left(\alpha_3-\beta_3,\ 0,\begin{pmatrix}\alpha_1+\beta_1-2\gamma_1\\\alpha_2+\beta_2-2\gamma_2\\\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$
$$+\left(0,\ 0,\begin{pmatrix}2\gamma_1-2\beta_1\\2\gamma_2-2\beta_2\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)=\left(0,\ 0,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(\widetilde{b}^1_{1,0}g_{3,0}+2g_{0,2}\widetilde{b}^0_{0,1}+g_{0,3}\widetilde{b}^1_{0,1}\right)$$
$$=\frac{1}{2}\left(\alpha_3-\beta_3,\ 0,\begin{pmatrix}\alpha_1-\beta_1\\\alpha_2-\beta_2\\\alpha_3-\beta_3\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)+\frac{1}{2}\left(\beta_3-\alpha_3,\ 0,\begin{pmatrix}2\gamma_1-\alpha_1-\beta_1\\2\gamma_2-\alpha_2-\beta_2\\\beta_3-\alpha_3\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$
$$+\left(0,\ 0,\begin{pmatrix}\beta_1-\gamma_1\\\beta_2-\gamma_2\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)=\left(0,\ 0,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

- $$D_2\cdot H_1+H_4\cdot D_3=\begin{pmatrix}2b^0_{0,1}&0\\b^1_{0,1}&0\end{pmatrix}\begin{pmatrix}g_{1,0}&g_{1,1}\\g_{1,2}&g_{1,3}\end{pmatrix}+\begin{pmatrix}g_{1,0}&g_{1,1}\\g_{1,2}&g_{1,3}\end{pmatrix}\begin{pmatrix}0&0\\-b^0_{1,1}&-2b^0_{0,1}\end{pmatrix}$$
$$=\begin{pmatrix}2b^0_{0,1}g_{1,0}-g_{1,1}b^0_{1,1}&2b^0_{0,1}g_{1,1}-2g_{1,1}b^0_{0,1}\\b^1_{0,1}g_{1,0}-g_{1,3}b^0_{1,1}&b^1_{0,1}g_{1,1}-2g_{1,3}b^0_{0,1}\end{pmatrix}\overset{!}{=}\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Suppose given $\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\in P$ and $\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\in Q$. We obtain the following.

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\left(2b^0_{0,1}g_{1,0}-g_{1,1}b^0_{1,1}\right)$$
$$=4\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3-\sigma\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)-4\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3-\sigma\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\rho,\ \sigma,\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\right)\left(2b^0_{0,1}g_{1,1}-2g_{1,1}b^0_{0,1}\right)$$
$$=2\left(0,\ \beta_3-\sigma,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3-\sigma\end{pmatrix}\right)-2\left(0,\ \beta_3-\sigma,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3-\sigma\end{pmatrix}\right)=\left(0,\ 0,\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(b^1_{0,1}g_{1,0}-g_{1,3}b^0_{1,1}\right)$$
$$=2\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)-2\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\beta_3\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix},\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\right)\left(b^1_{0,1}g_{1,1}-2g_{1,3}b^0_{0,1}\right)$$

$$= \left(0,\ \beta_3,\ \begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\\beta_3\end{pmatrix}\right) - \left(0,\ \beta_3,\ \begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\\beta_3\end{pmatrix}\right) = \left(0,\ 0,\ \begin{pmatrix}0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\end{pmatrix}\right)$$

□

2.7 Radical of the projective indecomposable modules

We calculate the radical of the projective modules P and Q which shall be used to verify that X is in fact minimal.

Lemma 36 Recall that $\Gamma = R \times R \times R^{3\times 3} \times R^{3\times 3} \times R^{2\times 2}$ and that

$$\Lambda = \left\{ \begin{array}{c} \left(\rho, \sigma, \begin{pmatrix}\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3}\\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3}\\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3}\end{pmatrix}, \begin{pmatrix}\beta_{1,1} & \beta_{1,2} & \beta_{1,3}\\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3}\\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3}\end{pmatrix}, \begin{pmatrix}\gamma_{1,1} & \gamma_{1,2}\\ \gamma_{2,1} & \gamma_{2,2}\end{pmatrix}\right) \in \Gamma \ : \\ \alpha_{1,1} \equiv_4 \beta_{1,1},\ \alpha_{1,2} \equiv_4 \beta_{1,2},\ \alpha_{1,3} \equiv_2 \beta_{1,3}, \\ \alpha_{2,1} \equiv_4 \beta_{2,1},\ \alpha_{2,2} \equiv_4 \beta_{2,2},\ \alpha_{2,3} \equiv_2 \beta_{2,3}, \\ \alpha_{3,1} \equiv_8 \beta_{3,1} \equiv_4 0,\ \alpha_{3,2} \equiv_8 \beta_{3,2} \equiv_4 0,\ \alpha_{3,3} \equiv_2 \beta_{3,3}, \\ \alpha_{1,1} + \beta_{1,1} \equiv_8 2\gamma_{1,1},\ \alpha_{1,2} + \beta_{1,2} \equiv_8 2\gamma_{1,2}, \\ \alpha_{2,1} + \beta_{2,1} \equiv_8 2\gamma_{2,1},\ \alpha_{2,2} + \beta_{2,2} \equiv_8 2\gamma_{2,2}, \\ \rho - \alpha_{3,3} \equiv_8 \sigma - \beta_{3,3} \equiv_4 0 \end{array} \right\};$$

cf. Corollary 15.

We have the following.

$$\mathrm{rad}(\Lambda) = \left\{ \begin{array}{c} \left(\rho, \sigma, \begin{pmatrix}\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3}\\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3}\\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3}\end{pmatrix}, \begin{pmatrix}\beta_{1,1} & \beta_{1,2} & \beta_{1,3}\\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3}\\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3}\end{pmatrix}, \begin{pmatrix}\gamma_{1,1} & \gamma_{1,2}\\ \gamma_{2,1} & \gamma_{2,2}\end{pmatrix}\right) \in \Gamma \ : \\ \rho \equiv_2 \sigma \equiv_2 0, \\ \alpha_{1,1} \equiv_4 \beta_{1,1} \equiv_2 0,\ \alpha_{1,2} \equiv_4 \beta_{1,2} \equiv_2 0,\ \alpha_{1,3} \equiv_2 \beta_{1,3}, \\ \alpha_{2,1} \equiv_4 \beta_{2,1} \equiv_2 0,\ \alpha_{2,2} \equiv_4 \beta_{2,2} \equiv_2 0,\ \alpha_{2,3} \equiv_2 \beta_{2,3}, \\ \alpha_{3,1} \equiv_8 \beta_{3,1} \equiv_4 0,\ \alpha_{3,2} \equiv_8 \beta_{3,2} \equiv_4 0,\ \alpha_{3,3} \equiv_2 \beta_{3,3} \equiv_2 0, \\ \alpha_{1,1} + \beta_{1,1} \equiv_8 2\gamma_{1,1} \equiv_4 0,\ \alpha_{1,2} + \beta_{1,2} \equiv_8 2\gamma_{1,2} \equiv_4 0, \\ \alpha_{2,1} + \beta_{2,1} \equiv_8 2\gamma_{2,1} \equiv_4 0,\ \alpha_{2,2} + \beta_{2,2} \equiv_8 2\gamma_{2,2} \equiv_4 0, \\ \rho - \alpha_{3,3} \equiv_8 \sigma - \beta_{3,3} \equiv_4 0 \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\rho, \sigma, \begin{pmatrix}\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3}\\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3}\\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3}\end{pmatrix}, \begin{pmatrix}\beta_{1,1} & \beta_{1,2} & \beta_{1,3}\\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3}\\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3}\end{pmatrix}, \begin{pmatrix}\gamma_{1,1} & \gamma_{1,2}\\ \gamma_{2,1} & \gamma_{2,2}\end{pmatrix}\right) \in \Lambda \ : \\ \rho \equiv_2 \sigma \equiv_2 0, \\ \alpha_{1,1} \equiv_2 \alpha_{1,2} \equiv_2 \alpha_{2,1} \equiv_2 \alpha_{2,2} \equiv_2 \alpha_{3,3} \equiv_2 0, \\ \beta_{1,1} \equiv_2 \beta_{1,2} \equiv_2 \beta_{2,1} \equiv_2 \beta_{2,2} \equiv_2 \beta_{3,3} \equiv_2 0, \\ \gamma_{1,1} \equiv_2 \gamma_{1,2} \equiv_2 \gamma_{2,1} \equiv_2 \gamma_{2,2} \equiv_2 0 \end{array} \right\}$$

Proof. Note that R is a discrete valuation ring with maximal ideal $2R$ and that Λ is stable as an R-order by [**12,** Remark 208] in the sense of the definition given in [**12,** Definition 207].

By using an R-linear basis of Λ and the standard R-linear basis of Γ, the embedding $\Lambda \hookrightarrow \Gamma$ can be described by the following matrix.

$$B := \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&1&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&8&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&8&0&0&0&0&0\\
1&2&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&8&0&0&0&0\\
1&2&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&4&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&4&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&4&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&4
\end{pmatrix}$$

Recall that $\mathbb{Q} := \mathrm{frac}(R)$. We define

$$\eta_1 := \left(1, 0, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}\right)$$

$$\eta_2 := \left(0, 1, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}\right)$$

$$\eta_3 := \left(0, 0, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}\right)$$

$$\eta_4 := \left(0, 0, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}\right)$$

$$\eta_5 := \left(0, 0, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$$

so that

$$\left(1, 1, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}\right) = \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5$$

is an orthogonal decomposition of $1_{\mathbb{Q}\Lambda}$ into primitive central idempotents.

By **[12,** Proposition 222] we obtain

$$(*) \qquad \mathrm{rad}(\Lambda) = \Lambda \cap \bigoplus_{i \in [1,5]} \mathrm{rad}(\eta_i \Lambda)$$

as R-submodules of $\mathbb{Q}\Lambda$.

For the first two summands we consider the following isomorphisms of R-algebras

$$\begin{aligned} \eta_1\Lambda &\xrightarrow{\sim} R, & \eta_1(\rho,\sigma,A_1,A_2,A_3) &\mapsto \rho \\ \eta_2\Lambda &\xrightarrow{\sim} R, & \eta_2(\rho,\sigma,A_1,A_2,A_3) &\mapsto \sigma \end{aligned}$$

as can be verified using the matrix B.

Since $\mathrm{rad}(R) = 2R$, we get

$$\begin{aligned} \mathrm{rad}(\eta_1\Lambda) &= 2\eta_1\Lambda \\ \mathrm{rad}(\eta_2\Lambda) &= 2\eta_2\Lambda. \end{aligned}$$

For the next two summands we consider the following isomorphisms of R-modules

$$\eta_3\Lambda \xrightarrow{\sim} \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} \in R^{3\times 3} : x_{3,1} \equiv_4 x_{3,2} \equiv_4 0 \right\} =: \Xi, \qquad \eta_2(\rho,\sigma,A_1,A_2,A_3) \mapsto A_1$$

$$\eta_4\Lambda \xrightarrow{\sim} \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} \in R^{3\times 3} : x_{3,1} \equiv_4 x_{3,2} \equiv_4 0 \right\} = \Xi, \qquad \eta_3(\rho,\sigma,A_1,A_2,A_3) \mapsto A_2$$

as can be verified using the matrix B.

We define the following idempotents in Ξ.

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $1_\Xi = e_1 + e_2 + e_3$ is an orthogonal decomposition into primitive idempotents.

We have the following mutually inverse isomorphisms of right Ξ-modules.

$$\begin{aligned} e_1\Xi &\xleftrightarrow{\sim} e_2\Xi \\ e_1x &\longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e_1x = e_2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \\ e_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e_2x &\longleftarrow e_2x. \end{aligned}$$

Since

$$e_3 \notin e_3\Xi e_2 \cdot e_2\Xi e_3\,,$$

there exist no morphisms of right Ξ-modules $\varphi \in \mathrm{Hom}_\Xi(e_2\Xi, e_3\Xi) \simeq e_3\Xi e_2$ and $\psi \in \mathrm{Hom}_\Xi(e_3\Xi, e_2\Xi) \simeq e_2\Xi e_3$ with $\psi\varphi = \mathrm{id}_{e_3\Xi}$. Hence $e_2\Xi \not\simeq e_3\Xi$.

By [**12,** Remark 208] the R-order Ξ is stable so that we obtain with [**12,** Proposition 217] that

$$\mathrm{rad}(\Xi) = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} \in R^{3\times 3} : \begin{array}{c} x_{1,1} \equiv_2 x_{1,2} \equiv_2 x_{2,1} \equiv_2 x_{2,2} \equiv_2 0, \\ x_{3,3} \equiv_2 0, \\ x_{3,1} \equiv_4 x_{3,2} \equiv_4 0 \end{array} \right\}$$

since $\mathrm{rad}(e_i \Xi e_j) = 2 e_i \Xi e_j$ for $(i,j) \in \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$; concerning the cases $(i,j) \in \{(1,2),(2,1)\}$, cf. the definition in [**12,** Lemma 185].

For the last summand we consider the following isomorphism of R-algebras

$$\eta_5\Lambda \xrightarrow{\sim} R^{2\times 2}, \qquad \eta_5(\rho,\sigma,A_1,A_2,A_3) \mapsto A_3$$

as can be verified using the matrix B.

Since $\mathrm{rad}(R^{2\times 2}) = 2R^{2\times 2}$, we obtain $\mathrm{rad}(\eta_5\Lambda) = 2\eta_5\Lambda$.

Using the respective isomorphisms for $\eta_3\Lambda$ and $\eta_4\Lambda$ and $(*)$ we obtain

$$\begin{aligned}\mathrm{rad}(\Lambda) &= \Lambda \cap \bigoplus_{i\in[1,5]} \mathrm{rad}(\eta_i\Lambda)\\ &= \left\{ \begin{array}{c} \left(\rho,\sigma, \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3}\\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3}\\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3}\end{pmatrix}, \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3}\\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3}\\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3}\end{pmatrix}, \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2}\\ \gamma_{2,1} & \gamma_{2,2}\end{pmatrix}\right) \in \Lambda \;: \\ \rho \equiv_2 \sigma \equiv_2 0,\\ \alpha_{1,1} \equiv_2 \alpha_{1,2} \equiv_2 \alpha_{2,1} \equiv_2 \alpha_{2,2} \equiv_2 \alpha_{3,3} \equiv_2 0,\\ \beta_{1,1} \equiv_2 \beta_{1,2} \equiv_2 \beta_{2,1} \equiv_2 \beta_{2,2} \equiv_2 \beta_{3,3} \equiv_2 0,\\ \gamma_{1,1} \equiv_2 \gamma_{1,2} \equiv_2 \gamma_{2,1} \equiv_2 \gamma_{2,2} \equiv_2 0 \end{array}\right\}.\end{aligned}$$

□

Corollary 37 Recall that $\Gamma := R \times R \times R^{3\times 3} \times R^{3\times 3} \times R^{2\times 2}$.

By [**7,** §9.3, Proposition 6] and Lemma 36 we have the following.

$$\begin{aligned}\mathrm{rad}(P) = \mathrm{rad}(\Lambda)\mathcal{E}_0 &= \left\{ \left(\rho,\sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \in \Gamma \;:\; \begin{array}{c} \rho \equiv_2 \sigma \equiv_2 0,\\ \alpha_1 \equiv_2 \beta_1,\ \alpha_2 \equiv_2 \beta_2,\ \alpha_3 \equiv_2 \beta_3 \equiv_2 0,\\ \rho - \alpha_3 \equiv_8 \sigma - \beta_3 \equiv_4 0 \end{array} \right\}\\ &= \left\{ \left(\rho,\sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \in P \;:\; \begin{array}{c} \rho \equiv_2 \sigma \equiv_2 0,\\ \alpha_3 \equiv_2 \beta_3 \equiv_2 0 \end{array} \right\}\\ \mathrm{rad}(Q) = \mathrm{rad}(\Lambda)\mathcal{E}_1 &= \left\{ \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \in \Gamma \;:\; \begin{array}{c} \alpha_1 \equiv_4 \beta_1 \equiv_2 0,\ \alpha_2 \equiv_4 \beta_2 \equiv_2 0,\ \alpha_3 \equiv_8 \beta_3 \equiv_4 0\\ \alpha_1 + \beta_1 \equiv_8 2\gamma_1 \equiv_4 0,\ \alpha_2 + \beta_2 \equiv_8 2\gamma_2 \equiv_4 0, \end{array} \right\}\\ &= \left\{ \left(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right) \in Q \;:\; \begin{array}{c} \alpha_1 \equiv_2 \alpha_2 \equiv_2 \beta_1 \equiv_2 \beta_2 \equiv_2 0,\\ \gamma_1 \equiv_2 \gamma_2 \equiv_2 0 \end{array} \right\}\end{aligned}$$

2.8 Projective resolution of $\mathbb{Z}_{(2)}$ over $\mathbb{Z}_{(2)}\mathrm{S}_4$

Theorem 38 Recall that $R = \mathbb{Z}_{(2)}$ and $R\mathrm{S}_4 \underset{\sim}{\xrightarrow{\omega}} \Lambda$; cf. Corollary 15.

Consider the following sequences of Λ-modules from Definition 24 with the differentials from Definition 28.

$$X := \left(\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \longrightarrow 0 \longrightarrow \cdots\right)$$

$$X' := \left(\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{\varepsilon} R \longrightarrow 0 \longrightarrow \cdots\right)$$

Then X is a minimal projective resolution of the trivial Λ-module R.
Moreover, X' is an augmented minimal projective resolution of the trivial Λ-module R.

Note that a minimal projective resolution of R is uniquely determined up to isomorphism; cf. Lemma 11.

Proof. In part (I) of this proof, we shall verify that X' is a complex by showing that the composite of two successive differentials is the zero map. For this we have to consider the following cases.

$$\begin{array}{ll} d_0 \cdot \varepsilon = 0 & \\ d_1 \cdot d_0 = 0 & \\ d_2 \cdot d_1 = 0 & \\ d_{4l+1} \cdot d_{4l} = 0\ , & \text{for } l \geq 1 \\ d_{4l+2} \cdot d_{4l+1} = 0\ , & \text{for } l \geq 1 \\ d_{4l+3} \cdot d_{4l+2} = 0\ , & \text{for } l \geq 0 \\ d_{4l+4} \cdot d_{4l+3} = 0\ , & \text{for } l \geq 0 \end{array}$$

In part (II) we shall verify that X' is acyclic using the homotopy defined in Definition 33, so that X' is an augmented projective resolution of R. For this we have to show the following equations.

$$\begin{array}{ll} h_{-1}\, \varepsilon = 1 & \\ \varepsilon\, h_{-1} + h_0\, d_0 = 1 & \\ d_0\, h_0 + h_1\, d_1 = 1 & \\ d_1\, h_1 + h_2\, d_2 = 1 & \\ d_{4l}\, h_{4l} + h_{4l+1}\, d_{4l+1} = 1\ , & \text{for } l \geq 1 \\ d_{4l+1}\, h_{4l+1} + h_{4l+2}\, d_{4l+2} = 1\ , & \text{for } l \geq 1 \\ d_{4l+2}\, h_{4l+2} + h_{4l+3}\, d_{4l+3} = 1\ , & \text{for } l \geq 0 \\ d_{4l+3}\, h_{4l+3} + h_{4l+4}\, d_{4l+4} = 1\ , & \text{for } l \geq 0 \end{array}$$

In part (III) we shall verify that X is minimal.

Throughout the proof we will use Remark 26 without further comment.

(I) *Differential condition*

In the following calculations, we will refer to the assertions of Lemma 30 using the corresponding numbers (i-xiv).

By (viii) and (ix) we have that

$$\begin{aligned} d_0 \cdot \varepsilon &= 0 \\ d_1 \cdot d_0 &= 0 \\ d_2 \cdot d_1 &= 0\,. \end{aligned}$$

Let $l \geq 1$. We want to show that $d_{4l+1} \cdot d_{4l} = 0$.

$$\begin{aligned} d_{4l+1} \cdot d_{4l} = &\Big(\sum_{k \in [0,l-1]} \pi^+_{4l+2,k}\, \widetilde{A}\, \iota^+_{4l+1,k} \Big) \Big(\sum_{k \in [0,l-1]} \pi^+_{4l+1,k}\, A\, \iota^+_{4l,k} \Big) \\ &+ \Big(\sum_{k \in [0,l-1]} \pi^+_{4l+2,k}\, \widetilde{A}\, \iota^+_{4l+1,k} \Big) \Big(\sum_{k \in [0,l-2]} \pi^+_{4l+1,k+1}\, C\, \iota^+_{4l,k} \Big) \end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{k\in[0,l-2]} \pi^+_{4l+2,k+1}\, C\, \iota^+_{4l+1,k}\Big)\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+1,k}\, A\, \iota^+_{4l,k}\Big)\\
&+\Big(\sum_{k\in[0,l-2]} \pi^+_{4l+2,k+1}\, C\, \iota^+_{4l+1,k}\Big)\Big(\sum_{k\in[0,l-2]} \pi^+_{4l+1,k+1}\, C\, \iota^+_{4l,k}\Big)\\
&+\Big(\pi'_{4l+2}\, C'\, \iota^+_{4l+1,l-1}\Big)\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+1,k}\, A\, \iota^+_{4l,k}\Big)\\
&+\Big(\pi'_{4l+2}\, C'\, \iota^+_{4l+1,l-1}\Big)\Big(\sum_{k\in[0,l-2]} \pi^+_{4l+1,k+1}\, C\, \iota^+_{4l,k}\Big)\\
&+\Big(\pi'_{4l+2}\, d'_1 \iota'_{4l+1}\Big)\Big(\pi'_{4l+1}\, \widetilde{C}\, \iota^+_{4l,l-1}\Big)\\
&+\Big(\pi'_{4l+2}\, d'_1 \iota'_{4l+1}\Big)\Big(\pi'_{4l+1}\, d_0\, \iota'_{4l}\Big)\\
&+\Big(\pi'_{4l+2}\, d'_1 \iota'_{4l+1}\Big)\Big(\pi'_{4l+1}\, \widetilde{D}_1\, \iota^-_{4l,0}\Big)\\
&+\Big(\pi''_{4l+2}\, d''_1 \iota'_{4l+1}\Big)\Big(\pi'_{4l+1}\, \widetilde{C}\, \iota^+_{4l,l-1}\Big)\\
&+\Big(\pi''_{4l+2}\, d''_1 \iota'_{4l+1}\Big)\Big(\pi'_{4l+1}\, d_0\, \iota'_{4l}\Big)\\
&+\Big(\pi''_{4l+2}\, d''_1 \iota'_{4l+1}\Big)\Big(\pi'_{4l+1}\, \widetilde{D}_1\, \iota^-_{4l,0}\Big)\\
&+\Big(\pi''_{4l+2}\, D_2\, \iota^-_{4l+1,0}\Big)\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+1,k}\, B_{\overline{\overline{k+1}}}\, \iota^-_{4l,k}\Big)\\
&+\Big(\pi''_{4l+2}\, D_2\, \iota^-_{4l+1,0}\Big)\Big(\sum_{k\in[0,l-2]} \pi^-_{4l+1,k}\, D_{\overline{4k+5}}\, \iota^-_{4l,k+1}\Big)\\
&+\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+2,k}\, B_{\overline{\overline{k+2}}}\, \iota^-_{4l+1,k}\Big)\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+1,k}\, B_{\overline{\overline{k+1}}}\, \iota^-_{4l,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+2,k}\, B_{\overline{\overline{k+2}}}\, \iota^-_{4l+1,k}\Big)\Big(\sum_{k\in[0,l-2]} \pi^-_{4l+1,k}\, D_{\overline{4k+5}}\, \iota^-_{4l,k+1}\Big)\\
&+\Big(\sum_{k\in[0,l-2]} \pi^-_{4l+2,k}\, D_{\overline{4k}}\, \iota^-_{4l+1,k+1}\Big)\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+1,k}\, B_{\overline{\overline{k+1}}}\, \iota^-_{4l,k}\Big)\\
&+\Big(\sum_{k\in[0,l-2]} \pi^-_{4l+2,k}\, D_{\overline{4k}}\, \iota^-_{4l+1,k+1}\Big)\Big(\sum_{k\in[0,l-2]} \pi^-_{4l+1,k}\, D_{\overline{4k+5}}\, \iota^-_{4l,k+1}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&=\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+2,k}\, \underbrace{(\widetilde{A}\cdot A)}_{=0;\ \text{cf. (i)}}\, \iota^+_{4l,k}\Big)+\Big(\sum_{k\in[0,l-2]} \pi^+_{4l+2,k+1}\, \underbrace{(C\cdot A+\widetilde{A}\cdot C)}_{=0;\ \text{cf. (ii)}}\, \iota^+_{4l,k}\Big)\\
&+\Big(\sum_{k\in[0,l-3]} \pi^+_{4l+2,k+2}\, \underbrace{(C\cdot C)}_{=0;\ \text{cf. (iii)}}\, \iota^+_{4l,k}\Big)+\Big(\pi'_{4l+2}\, \underbrace{(C'\cdot A+d'_1\cdot \widetilde{C})}_{=0;\ \text{cf. (iv)}}\, \iota^+_{4l,l-1}\Big)\\
&+\Big(\pi'_{4l+2}\, \underbrace{(C'\cdot C)}_{=0;\ \text{cf. (iii)}}\, \iota^+_{4l,l-2}\Big)+\Big(\underbrace{\begin{pmatrix}\pi'_{4l+2} & \pi''_{4l+2}\end{pmatrix}\begin{pmatrix} d'_1\\ d''_1\end{pmatrix}\cdot d_0}_{=0;\ \text{cf. (viii)}}\, \iota'_{4l}\Big)
\end{aligned}
$$

$$+\Big(\pi'_{4l+2}\ \underbrace{(d'_1\cdot\widetilde{D}_1)}_{=0;\text{ cf. (iv)}}\ \iota^-_{4l,0}\Big)+\Big(\pi''_{4l+2}\ \underbrace{(d''_1\cdot\widetilde{C})}_{=0;\text{ cf. (iv)}}\ \iota^+_{4l,l-1}\Big)$$

$$+\Big(\pi''_{4l+2}\ \underbrace{(d''_1\cdot\widetilde{D}_1+D_2\cdot B_1)}_{=0;\text{ cf. (iv)}}\ \iota^-_{4l,0}\Big)+\Big(\pi''_{4l+2}\ \underbrace{(D_2\cdot D_5)}_{=0;\text{ cf. (xi)}}\ \iota^-_{4l,1}\Big)$$

$$+\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+2,k}\ \underbrace{(B_{\overline{\overline{k+2}}}\cdot B_{\overline{\overline{k+1}}})}_{=0;\text{ cf. (x)}}\ \iota^-_{4l,k}\Big)+\Big(\sum_{k\in[0,l-3]}\pi^-_{4l+2,k}\ \underbrace{(D_{\overline{4k}}\cdot D_{\overline{4k+3}})}_{=0;\text{ cf. (xi)}}\ \iota^-_{4l,k+2}\Big)$$

$$+\Big(\sum_{k\in[0,l-2]}\pi^-_{4l+2,k}\ \underbrace{(B_{\overline{\overline{k+2}}}\cdot D_{\overline{4k+5}}+D_{\overline{4k}}\cdot B_{\overline{\overline{k+2}}})}_{=0;\text{ cf. (xiii)}}\ \iota^-_{4l,k+1}\Big)$$

$$=0$$

Let $l\geq 1$. We want to show that $d_{4l+2}\cdot d_{4l+1}=0$.

$$\begin{aligned}
d_{4l+2}\cdot d_{4l+1}=&\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+3,k}\ A\ \iota^+_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+2,k}\ \widetilde{A}\ \iota^+_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+3,k}\ A\ \iota^+_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-2]}\pi^+_{4l+2,k+1}\ C\ \iota^+_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+3,k+1}\ C\ \iota^+_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+2,k}\ \widetilde{A}\ \iota^+_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+3,k+1}\ C\ \iota^+_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-2]}\pi^+_{4l+2,k+1}\ C\ \iota^+_{4l+1,k}\Big)\\
&+\Big(\pi^+_{4l+3,l}\ d'_2\ \iota'_{4l+2}\Big)\Big(\pi'_{4l+2}\ C'\ \iota^+_{4l+1,l-1}\Big)\\
&+\Big(\pi^+_{4l+3,l}\ d'_2\ \iota'_{4l+2}\Big)\Big(\pi'_{4l+2}\ d'_1\ \iota'_{4l+1}\Big)\\
&+\Big(\pi^+_{4l+3,l}\ d''_2\ \iota''_{4l+2}\Big)\Big(\pi''_{4l+2}\ d''_1\ \iota'_{4l+1}\Big)\\
&+\Big(\pi^+_{4l+3,l}\ d''_2\ \iota''_{4l+2}\Big)\Big(\pi''_{4l+2}\ D_2\ \iota^-_{4l+1,0}\Big)\\
&+\Big(\pi^-_{4l+3,0}\ B_2\ \iota''_{4l+2}\Big)\Big(\pi''_{4l+2}\ d''_1\ \iota'_{4l+1}\Big)\\
&+\Big(\pi^-_{4l+3,0}\ B_2\ \iota''_{4l+2}\Big)\Big(\pi''_{4l+2}\ D_2\ \iota^-_{4l+1,0}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+3,k+1}\ B_{\overline{\overline{k}}}\ \iota^-_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+2,k}\ B_{\overline{\overline{k+2}}}\ \iota^-_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+3,k+1}\ B_{\overline{\overline{k}}}\ \iota^-_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-2]}\pi^-_{4l+2,k}\ D_{\overline{4k}}\ \iota^-_{4l+1,k+1}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+3,k}\ D_{\overline{4k+3}}\ \iota^-_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+2,k}\ B_{\overline{\overline{k+2}}}\ \iota^-_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+3,k}\ D_{\overline{4k+3}}\ \iota^-_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-2]}\pi^-_{4l+2,k}\ D_{\overline{4k}}\ \iota^-_{4l+1,k+1}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+3,k} \underbrace{(A \cdot \widetilde{A})}_{=0;\text{ cf. (i)}} \iota^+_{4l+1,k} \Big) + \Big(\sum_{k\in[0,l-2]} \pi^+_{4l+3,k+1} \underbrace{(A \cdot C + C \cdot \widetilde{A})}_{=0;\text{ cf. (ii)}} \iota^+_{4l+1,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-2]} \pi^+_{4l+3,k+2} \underbrace{(C \cdot C)}_{=0;\text{ cf. (iii)}} \iota^+_{4l+1,k} \Big) + \Big(\pi^+_{4l+3,l} \underbrace{(C \cdot \widetilde{A} + d_2' \cdot C')}_{=0;\text{ cf. (v)}} \iota^+_{4l+1,l-1} \Big) \\
&+ \Big(\underbrace{\begin{pmatrix} \pi^+_{4l+3,l} & \pi^-_{4l+3,0} \end{pmatrix} \begin{pmatrix} d_2' & d_2'' \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} d_1' \\ d_1'' \end{pmatrix}}_{=0;\text{ cf. (ix)}} \iota'_{4l+1} \Big) + \Big(\pi^+_{4l+3,l} \underbrace{(d_2'' \cdot D_2)}_{=0;\text{ cf. (v)}} \iota^-_{4l+1,0} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+3,k+1} \underbrace{(B_{\overline{\overline{k}}} \cdot B_{\overline{\overline{k+2}}})}_{=0;\text{ cf. (x)}} \iota^-_{4l+1,k} \Big) + \Big(\sum_{k\in[0,l-2]} \pi^-_{4l+3,k} \underbrace{(D_{\overline{4k+3}} \cdot D_{\overline{4k}})}_{=0;\text{ cf. (xii)}} \iota^-_{4l+1,k+1} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+3,k} \underbrace{(B_{\overline{\overline{k+2}}} \cdot D_{\overline{4k+2}} + D_{\overline{4k+3}} \cdot B_{\overline{\overline{k+2}}})}_{=0;\text{ cf. (xiv)}} \iota^-_{4l+1,k} \Big) \\
&= 0
\end{aligned}
$$

Let $l \geq 0$. We want to show that $d_{4l+3} \cdot d_{4l+2} = 0$.

$$
\begin{aligned}
d_{4l+3} \cdot d_{4l+2} &= \Big(\sum_{k\in[0,l]} \pi^+_{4l+4,k} \widetilde{A} \iota^+_{4l+3,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+3,k} A \iota^+_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l]} \pi^+_{4l+4,k} \widetilde{A} \iota^+_{4l+3,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+3,k+1} C \iota^+_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l]} \pi^+_{4l+4,k} \widetilde{A} \iota^+_{4l+3,k} \Big) \Big(\pi^+_{4l+3,l} d_2' \iota'_{4l+2} \Big) \\
&+ \Big(\sum_{k\in[0,l]} \pi^+_{4l+4,k} \widetilde{A} \iota^+_{4l+3,k} \Big) \Big(\pi^+_{4l+3,l} d_2'' \iota''_{4l+2} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+4,k+1} C \iota^+_{4l+3,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+3,k} A \iota^+_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+4,k+1} C \iota^+_{4l+3,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+3,k+1} C \iota^+_{4l+2,k} \Big) \\
&+ \Big(\pi'_{4l+4} C' \iota^+_{4l+3,l} \Big) \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+3,k+1} C \iota^+_{4l+2,k} \Big) \\
&+ \Big(\pi'_{4l+4} C' \iota^+_{4l+3,l} \Big) \Big(\pi^+_{4l+3,l} d_2' \iota'_{4l+2} \Big) \\
&+ \Big(\pi'_{4l+4} C' \iota^+_{4l+3,l} \Big) \Big(\pi^+_{4l+3,l} d_2'' \iota''_{4l+2} \Big) \\
&+ \Big(\pi'_{4l+4} D_0' \iota^-_{4l+3,0} \Big) \Big(\pi^-_{4l+3,0} B_2 \iota''_{4l+2} \Big) \\
&+ \Big(\pi'_{4l+4} D_0' \iota^-_{4l+3,0} \Big) \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+3,k} D_{\overline{4k+3}} \iota^-_{4l+2,k} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(\sum_{k\in[0,l]} \pi^-_{4l+4,k}\ B_{\overline{\overline{k}}}\ \iota^-_{4l+3,k} \Big) \Big(\pi^-_{4l+3,0}\ B_2\ \iota''_{4l+2} \Big) \\
&+ \Big(\sum_{k\in[0,l]} \pi^-_{4l+4,k}\ B_{\overline{\overline{k}}}\ \iota^-_{4l+3,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+3,k+1}\ B_{\overline{\overline{k}}}\ \iota^-_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l]} \pi^-_{4l+4,k}\ B_{\overline{\overline{k}}}\ \iota^-_{4l+3,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+3,k}\ D_{\overline{4k+3}}\ \iota^-_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+4,k}\ D_{\overline{4k+4}}\ \iota^-_{4l+3,k+1} \Big) \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+3,k+1}\ B_{\overline{\overline{k}}}\ \iota^-_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+4,k}\ D_{\overline{4k+4}}\ \iota^-_{4l+3,k+1} \Big) \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+3,k}\ D_{\overline{4k+3}}\ \iota^-_{4l+2,k} \Big) \\
&= \Big(\sum_{k\in[0,l-1]} \pi^+_{4l+4,k}\ \underbrace{(\widetilde{A}\cdot A)}_{=0;\text{ cf. (i)}}\ \iota^+_{4l+2,k} \Big) + \Big(\sum_{k\in[0,l-2]} \pi^+_{4l+4,k+1}\ \underbrace{(\widetilde{A}\cdot C + C\cdot A)}_{=0;\text{ cf. (ii)}}\ \iota^+_{4l+2,k} \Big) \\
&+ \Big(\pi^+_{4l+4,l}\ \underbrace{(\widetilde{A}\cdot d'_2)}_{=0;\text{ cf. (vi)}}\ \iota'_{4l+2} \Big) + \Big(\pi^+_{4l+4,l}\ \underbrace{(\widetilde{A}\cdot d''_2)}_{=0;\text{ cf. (vi)}}\ \iota''_{4l+2} \Big) \\
&+ \Big(\sum_{k\in[0,l-2]} \pi^+_{4l+4,k+2}\ \underbrace{(C\cdot C)}_{=0;\text{ cf. (iii)}}\ \iota^+_{4l+2,k} \Big) + \Big(\pi'_{4l+4}\ \underbrace{(C'\cdot C)}_{=0;\text{ cf. (iii)}}\ \iota^+_{4l+2,l-1} \Big) \\
&+ \Big(\pi'_{4l+4}\ \underbrace{(C'\cdot d'_2)}_{=0;\text{ cf. (vi)}}\ \iota'_{4l+2} \Big) + \Big(\pi'_{4l+4}\ \underbrace{(C'\cdot d''_2 + D'_0\cdot B_2)}_{=0;\text{ cf. (vi)}}\ \iota''_{4l+2} \Big) \\
&+ \Big(\pi'_{4l+4}\ \underbrace{(D'_0\cdot D_3)}_{=0;\text{ cf. (vi)}}\ \iota^-_{4l+2,0} \Big) + \Big(\pi^-_{4l+4,0}\ \underbrace{(B_0\cdot B_2)}_{=0;\text{ cf. (x)}}\ \iota''_{4l+2} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+4,k+1}\ \underbrace{(B_{\overline{\overline{k+1}}}\cdot B_{\overline{\overline{k}}})}_{=0;\text{ cf. (x)}}\ \iota^-_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^-_{4l+4,k}\ \underbrace{(B_{\overline{\overline{k}}}\cdot D_{\overline{4k+3}} + D_{\overline{4k+4}}\cdot B_{\overline{\overline{k}}})}_{=0;\text{ cf. (xiii)}}\ \iota^-_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-2]} \pi^-_{4l+4,k}\ \underbrace{(D_{\overline{4k+4}}\cdot D_{\overline{4k+1}})}_{=0;\text{ cf. (xi)}}\ \iota^-_{4l+2,k+1} \Big) \\
&= 0
\end{aligned}
$$

Let $l \geq 0$. We want to show that $d_{4l+4} \cdot d_{4l+3} = 0$.

Note that

$$
d_{4l+4} = \left(\sum_{k\in[0,l]} \pi^+_{4l+5,k}\ A\ \iota^+_{4l+4,k} \right) + \left(\sum_{k\in[0,l-1]} \pi^+_{4l+5,k+1}\ C\ \iota^+_{4l+4,k} \right)
$$

$$+\left(\pi'_{4l+5}\;\widetilde{C}\;\iota^{+}_{4l+4,l}\right)+\left(\pi'_{4l+5}\;d_0\;\iota'_{4l+4}\right)+\left(\pi'_{4l+5}\;\widetilde{D}_1\;\iota^{-}_{4l+4,0}\right)$$
$$+\left(\sum_{k\in[0,l]}\pi^{-}_{4l+5,k}\;B_{\overline{\overline{k+1}}}\;\iota^{-}_{4l+4,k}\right)+\left(\sum_{k\in[0,l-1]}\pi^{-}_{4l+5,k}\;D_{\overline{4k+5}}\;\iota^{-}_{4l+4,k+1}\right).$$

$$\begin{aligned}
d_{4l+4}\cdot d_{4l+3} &= \Big(\sum_{k\in[0,l]}\pi^{+}_{4l+5,k}\;A\;\iota^{+}_{4l+4,k}\Big)\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+4,k}\;\widetilde{A}\;\iota^{+}_{4l+3,k}\Big)\\
&+\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+5,k}\;A\;\iota^{+}_{4l+4,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+4,k+1}\;C\;\iota^{+}_{4l+3,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+5,k+1}\;C\;\iota^{+}_{4l+4,k}\Big)\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+4,k}\;\widetilde{A}\;\iota^{+}_{4l+3,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+5,k+1}\;C\;\iota^{+}_{4l+4,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+4,k+1}\;C\;\iota^{+}_{4l+3,k}\Big)\\
&+\Big(\pi'_{4l+5}\;\widetilde{C}\;\iota^{+}_{4l+4,l}\Big)\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+4,k}\;\widetilde{A}\;\iota^{+}_{4l+3,k}\Big)\\
&+\Big(\pi'_{4l+5}\;\widetilde{C}\;\iota^{+}_{4l+4,l}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+4,k+1}\;C\;\iota^{+}_{4l+3,k}\Big)\\
&+\Big(\pi'_{4l+5}\;d_0\;\iota'_{4l+4}\Big)\Big(\pi'_{4l+4}\;C'\;\iota^{+}_{4l+3,l}\Big)\\
&+\Big(\pi'_{4l+5}\;d_0\;\iota'_{4l+4}\Big)\Big(\pi'_{4l+4}\;D'_0\;\iota^{-}_{4l+3,0}\Big)\\
&+\Big(\pi'_{4l+5}\;\widetilde{D}_1\;\iota^{-}_{4l+4,0}\Big)\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+4,k}\;B_{\overline{\overline{k}}}\;\iota^{-}_{4l+3,k}\Big)\\
&+\Big(\pi'_{4l+5}\;\widetilde{D}_1\;\iota^{-}_{4l+4,0}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+4,k}\;D_{\overline{4k+4}}\;\iota^{-}_{4l+3,k+1}\Big)\\
&+\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+5,k}\;B_{\overline{\overline{k+1}}}\;\iota^{-}_{4l+4,k}\Big)\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+4,k}\;B_{\overline{\overline{k}}}\;\iota^{-}_{4l+3,k}\Big)\\
&+\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+5,k}\;B_{\overline{\overline{k+1}}}\;\iota^{-}_{4l+4,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+4,k}\;D_{\overline{4k+4}}\;\iota^{-}_{4l+3,k+1}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+5,k}\;D_{\overline{4k+5}}\;\iota^{-}_{4l+4,k+1}\Big)\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+4,k}\;B_{\overline{\overline{k}}}\;\iota^{-}_{4l+3,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+5,k}\;D_{\overline{4k+5}}\;\iota^{-}_{4l+4,k+1}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+4,k}\;D_{\overline{4k+4}}\;\iota^{-}_{4l+3,k+1}\Big)\\
&=\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+5,k}\;\underbrace{(A\cdot\widetilde{A})}_{=0;\text{ cf. (i)}}\;\iota^{+}_{4l+3,k}\Big)+\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+5,k+1}\;\underbrace{(A\cdot C+C\cdot\widetilde{A})}_{=0;\text{ cf. (ii)}}\;\iota^{+}_{4l+3,k}\Big)\\
&+\Big(\sum_{k\in[0,l-2]}\pi^{+}_{4l+5,k+2}\;\underbrace{(C\cdot C)}_{=0;\text{ cf. (iii)}}\;\iota^{+}_{4l+3,k}\Big)+\Big(\pi'_{4l+5}\;\underbrace{(\widetilde{C}\cdot\widetilde{A}+d_0\cdot C')}_{=0;\text{ cf. (vii)}}\;\iota^{+}_{4l+3,l}\Big)
\end{aligned}$$

$$+\Big(\pi'_{4l+5}\ \underbrace{(\widetilde{C}\cdot C)}_{=0;\ \text{cf. (iii)}}\ \iota^{+}_{4l+3,l-1}\Big)+\Big(\pi'_{4l+5}\ \underbrace{(d_0\cdot D'_0+\widetilde{D}_1\cdot B_0)}_{=0;\ \text{cf. (vii)}}\ \iota^{-}_{4l+3,0}\Big)$$

$$+\Big(\pi'_{4l+5}\ \underbrace{(\widetilde{D}_1\cdot D_4)}_{=0;\ \text{cf. (vii)}}\ \iota^{-}_{4l+3,1}\Big)+\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+5,k}\ \underbrace{(B_{\overline{\overline{k+1}}}\cdot B_{\overline{\overline{k}}})}_{=0;\ \text{cf. (x)}}\ \iota^{-}_{4l+3,k}\Big)$$

$$+\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+5,k}\ \underbrace{(B_{\overline{\overline{k+1}}}\cdot D_{\overline{4k+4}}+D_{\overline{4k+5}}\cdot B_{\overline{\overline{k+1}}})}_{=0;\ \text{cf. (xiv)}}\ \iota^{-}_{4l+3,k+1}\Big)$$

$$+\Big(\sum_{k\in[0,l-2]}\pi^{-}_{4l+5,k}\ \underbrace{(D_{\overline{4k+5}}\cdot D_{\overline{4k+2}})}_{=0;\ \text{cf. (xii)}}\ \iota^{-}_{4l+3,k+2}\Big)$$

$$=0$$

(II) *Homotopy condition*

In the following calculations we will refer to the assertions of Lemma 35 using the corresponding numbers (i-xii).

By part (vii) and (viii) we have the following.

$$\begin{aligned} h_{-1}\,\varepsilon &= 1\\ \varepsilon\,h_{-1}+h_0\,d_0 &= 1\\ d_0h_0+d_1h_1 &= 1\\ d_1h_1+h_2d_2 &= 1 \end{aligned}$$

Let $l\geq 1$. We want to show that $d_{4l}\,h_{4l}+h_{4l+1}\,d_{4l+1}=1$.

$$\begin{aligned}
&d_{4l}\,h_{4l}+h_{4l+1}\,d_{4l+1}\\
&=\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+1,k}\ A\ \iota^{+}_{4l,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l,k}\ G\ \iota^{+}_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-2]}\pi^{+}_{4l+1,k+1}\ C\ \iota^{+}_{4l,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l,k}\ G\ \iota^{+}_{4l+1,k}\Big)\\
&+\Big(\pi'_{4l+1}\ \widetilde{C}\ \iota^{+}_{4l,l-1}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l,k}\ G\ \iota^{+}_{4l+1,k}\Big)\\
&+\Big(\pi'_{4l+1}\ d_0\ \iota'_{4l}\Big)\Big(\pi'_{4l}\ h'_0\ \iota'_{4l+1}\Big)\\
&+\Big(\pi'_{4l+1}\ \widetilde{D}_1\ \iota^{-}_{4l,0}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l,k}\ H_{\overline{4k}}\ \iota^{-}_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+1,k}\ B_{\overline{\overline{k+1}}}\ \iota^{-}_{4l,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l,k}\ H_{\overline{4k}}\ \iota^{-}_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-2]}\pi^{-}_{4l+1,k}\ D_{\overline{4k+5}}\ \iota^{-}_{4l,k+1}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l,k}\ H_{\overline{4k}}\ \iota^{-}_{4l+1,k}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+1,k}\, \widetilde{G}\, \iota^+_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+2,k}\, \widetilde{A}\, \iota^+_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+1,k}\, \widetilde{G}\, \iota^+_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-2]} \pi^+_{4l+2,k+1}\, C\, \iota^+_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+1,k}\, H_{\overline{4k+1}}\, \iota^-_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+2,k}\, B_{\overline{\overline{k+2}}}\, \iota^-_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+1,k}\, H_{\overline{4k+1}}\, \iota^-_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-2]} \pi^-_{4l+2,k}\, D_{\overline{4k}}\, \iota^-_{4l+1,k+1}\Big)\\
&+\Big(\pi'_{4l+1}\, h'_1\, \iota'_{4l+2}\Big)\Big(\pi'_{4l+2}\, C'\, \iota^+_{4l+1,l-1}\Big)\\
&+\Big(\pi'_{4l+1}\, h'_1\, \iota'_{4l+2}\Big)\Big(\pi'_{4l+2}\, d'_1\, \iota'_{4l+1}\Big)\\
&+\Big(\pi'_{4l+1}\, h''_1\, \iota''_{4l+2}\Big)\Big(\pi''_{4l+2}\, d''_1\, \iota'_{4l+1}\Big)\\
&+\Big(\pi'_{4l+1}\, h''_1\, \iota''_{4l+2}\Big)\Big(\pi''_{4l+2}\, D_2\, \iota^-_{4l+1,0}\Big)\\
&=\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+1,k}\, \underbrace{(A\cdot G+\widetilde{G}\cdot\widetilde{A})}_{=1;\text{ cf. (i)}}\, \iota^+_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-2]} \pi^+_{4l+1,k+1}\, \underbrace{(C\cdot G+\widetilde{G}\cdot C)}_{=0;\text{ cf. (ii)}}\, \iota^+_{4l+1,k}\Big)\\
&+\Big(\pi'_{4l+1}\, \underbrace{(\widetilde{C}\cdot G+h'_1\cdot C')}_{=0;\text{ cf. (iii)}}\, \iota^+_{4l+1,l-1}\Big)\\
&+\Big(\pi'_{4l+1}\, \underbrace{\Big(d_0\cdot h'_0+\begin{pmatrix}h'_1 & h''_1\end{pmatrix}\begin{pmatrix}d'_1\\ d''_1\end{pmatrix}\Big)}_{=1;\text{ cf. (vii)}}\, \iota'_{4l+1}\Big)\\
&+\Big(\pi'_{4l+1}\, \underbrace{(\widetilde{D}_1\cdot H_0+h''_1\cdot D_2)}_{=0;\text{ cf. (iii)}}\, \iota^-_{4l+1,0}\Big)\\
&+\Big(\sum_{k\in[0,l-1]} \pi^-_{4l+1,k}\, \underbrace{(B_{\overline{\overline{k+1}}}\cdot H_{\overline{4k}}+H_{\overline{4k+1}}\cdot B_{\overline{\overline{k+2}}})}_{=1;\text{ cf. (ix)}}\, \iota^-_{4l+1,k}\Big)\\
&+\Big(\sum_{k\in[0,l-2]} \pi^-_{4l+1,k}\, \underbrace{(D_{\overline{4k+5}}\cdot H_{\overline{4k+4}}+H_{\overline{4k+1}}\cdot D_{\overline{4k}})}_{=0;\text{ cf. (xi)}}\, \iota^-_{4l+1,k+1}\Big)\\
&=1
\end{aligned}
$$

Let $l \geq 1$. We want to show that $d_{4l+1}\, h_{4l+1} + h_{4l+2}\, d_{4l+2} = 1$.

$$
\begin{aligned}
&d_{4l+1}\, h_{4l+1} + h_{4l+2}\, d_{4l+2}\\
&=\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+2,k}\, \widetilde{A}\, \iota^+_{4l+1,k}\Big)\Big(\sum_{k\in[0,l-1]} \pi^+_{4l+1,k}\, \widetilde{G}\, \iota^+_{4l+2,k}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(\sum_{k\in[0,l-2]} \pi^{+}_{4l+2,k+1}\, C\, \iota^{+}_{4l+1,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+1,k}\, \widetilde{G}\, \iota^{+}_{4l+2,k} \Big) \\
&+ \Big(\pi'_{4l+2}\, C'\, \iota^{+}_{4l+1,l-1} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+1,k}\, \widetilde{G}\, \iota^{+}_{4l+2,k} \Big) \\
&+ \Big(\pi'_{4l+2}\, d'_1\, \iota'_{4l+1} \Big) \Big(\pi'_{4l+1}\, h'_1\, \iota'_{4l+2} \Big) \\
&+ \Big(\pi'_{4l+2}\, d'_1\, \iota'_{4l+1} \Big) \Big(\pi'_{4l+1}\, h''_1\, \iota''_{4l+2} \Big) \\
&+ \Big(\pi''_{4l+2}\, d''_1\, \iota'_{4l+1} \Big) \Big(\pi'_{4l+1}\, h'_1\, \iota'_{4l+2} \Big) \\
&+ \Big(\pi''_{4l+2}\, d''_1\, \iota'_{4l+1} \Big) \Big(\pi'_{4l+1}\, h''_1\, \iota''_{4l+2} \Big) \\
&+ \Big(\pi''_{4l+2}\, D_2\, \iota^{-}_{4l+1,0} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+1,k}\, H_{\overline{4k+1}}\, \iota^{-}_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+2,k}\, B_{\overline{\overline{k+2}}}\, \iota^{-}_{4l+1,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+1,k}\, H_{\overline{4k+1}}\, \iota^{-}_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-2]} \pi^{-}_{4l+2,k}\, D_{\overline{4k}}\, \iota^{-}_{4l+1,k+1} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+1,k}\, H_{\overline{4k+1}}\, \iota^{-}_{4l+2,k} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+2,k}\, G\, \iota^{+}_{4l+3,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+3,k}\, A\, \iota^{+}_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+2,k}\, G\, \iota^{+}_{4l+3,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+3,k+1}\, C\, \iota^{+}_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+2,k}\, H_{\overline{4k+2}}\, \iota^{-}_{4l+3,k+1} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+3,k+1}\, B_{\overline{\overline{k}}}\, \iota^{-}_{4l+2,k} \Big) \\
&+ \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+2,k}\, H_{\overline{4k+2}}\, \iota^{-}_{4l+3,k+1} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+3,k}\, D_{\overline{4k+3}}\, \iota^{-}_{4l+2,k} \Big) \\
&+ \Big(\pi'_{4l+2}\, h'_2\, \iota^{+}_{4l+3,l} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+3,k+1}\, C\, \iota^{+}_{4l+2,k} \Big) \\
&+ \Big(\pi'_{4l+2}\, h'_2\, \iota^{+}_{4l+3,l} \Big) \Big(\pi^{+}_{4l+3,l}\, d'_2\, \iota'_{4l+2} \Big) \\
&+ \Big(\pi'_{4l+2}\, h'_2\, \iota^{+}_{4l+3,l} \Big) \Big(\pi^{+}_{4l+3,l}\, d''_2\, \iota''_{4l+2} \Big) \\
&+ \Big(\pi''_{4l+2}\, h''_2\, \iota^{+}_{4l+3,l} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+3,k+1}\, C\, \iota^{+}_{4l+2,k} \Big) \\
&+ \Big(\pi''_{4l+2}\, h''_2\, \iota^{+}_{4l+3,l} \Big) \Big(\pi^{+}_{4l+3,l}\, d'_2\, \iota'_{4l+2} \Big) \\
&+ \Big(\pi''_{4l+2}\, h''_2\, \iota^{+}_{4l+3,l} \Big) \Big(\pi^{+}_{4l+3,l}\, d''_2\, \iota''_{4l+2} \Big) \\
&+ \Big(\pi''_{4l+2}\, H_4\, \iota^{-}_{4l+3,0} \Big) \Big(\pi^{-}_{4l+3,0}\, B_2\, \iota''_{4l+2} \Big) \\
&+ \Big(\pi''_{4l+2}\, H_4\, \iota^{-}_{4l+3,0} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+3,k}\, D_{\overline{4k+3}}\, \iota^{-}_{4l+2,k} \Big)
\end{aligned}
$$

$$= \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+2,k} \underbrace{(\widetilde{A}\cdot\widetilde{G} + G\cdot A)}_{=1;\text{ cf. (i)}} \iota^{+}_{4l+2,k} \Big)$$

$$+ \Big(\sum_{k\in[0,l-2]} \pi^{+}_{4l+2,k+1} \underbrace{(C\cdot\widetilde{G} + G\cdot C)}_{=0;\text{ cf. (ii)}} \iota^{+}_{4l+2,k} \Big)$$

$$+ \Big(\pi'_{4l+2} \underbrace{(C'\cdot\widetilde{G} + h'_2\cdot C)}_{=0;\text{ cf. (iv)}} \iota^{+}_{4l+2,l-1} \Big)$$

$$+ \Big(\begin{pmatrix} \pi'_{4l+2} & \pi''_{4l+2} \end{pmatrix} \underbrace{\Big(\begin{pmatrix} d'_1 \\ d''_1 \end{pmatrix} \begin{pmatrix} h'_1 & h''_1 \end{pmatrix} + \begin{pmatrix} h'_2 & 0 \\ h''_2 & H_4 \end{pmatrix} \begin{pmatrix} d'_2 & d''_2 \\ 0 & B_2 \end{pmatrix} \Big)}_{=1;\text{ cf. (viii)}} \begin{pmatrix} \iota'_{4l+2} \\ \iota''_{4l+2} \end{pmatrix} \Big)$$

$$+ \Big(\pi''_{4l+2} \underbrace{(h''_2\cdot C)}_{=0;\text{ cf. (iv)}} \iota^{+}_{4l+2,l-1} \Big)$$

$$+ \Big(\pi''_{4l+2} \underbrace{(D_2\cdot H_1 + H_4\cdot D_3)}_{=0;\text{ cf. (xii)}} \iota^{-}_{4l+2,0} \Big)$$

$$+ \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+2,k} \underbrace{(B_{\overline{\overline{k+2}}}\cdot H_{\overline{4k+1}} + H_{\overline{4k+2}}\cdot B_{\overline{\overline{k}}})}_{=1;\text{ cf. (x)}} \iota^{-}_{4l+2,k} \Big)$$

$$+ \Big(\sum_{k\in[0,l-2]} \pi^{-}_{4l+2,k} \underbrace{(D_{\overline{4k}}\cdot H_{\overline{4k+5}} + H_{\overline{4k+2}}\cdot D_{\overline{4k+7}})}_{=0;\text{ cf. (xii)}} \iota^{-}_{4l+2,k+1} \Big)$$

$$= 1$$

Let $l \geq 0$. We want to show that $d_{4l+2}\,h_{4l+2} + h_{4l+3}\,d_{4l+3} = 1$.

$$\begin{aligned}
& d_{4l+2}\,h_{4l+2} + h_{4l+3}\,d_{4l+3} \\
& = \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+3,k}\, A\, \iota^{+}_{4l+2,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+2,k}\, G\, \iota^{+}_{4l+3,k} \Big) \\
& + \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+3,k+1}\, C\, \iota^{+}_{4l+2,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+2,k}\, G\, \iota^{+}_{4l+3,k} \Big) \\
& + \Big(\pi^{+}_{4l+3,l}\, d'_2\, \iota'_{4l+2} \Big) \Big(\pi'_{4l+2}\, h'_2\, \iota^{+}_{4l+3,l} \Big) \\
& + \Big(\pi^{+}_{4l+3,l}\, d''_2\, \iota''_{4l+2} \Big) \Big(\pi''_{4l+2}\, h''_2\, \iota^{+}_{4l+3,l} \Big) \\
& + \Big(\pi^{+}_{4l+3,l}\, d''_2\, \iota''_{4l+2} \Big) \Big(\pi''_{4l+2}\, H_4\, \iota^{-}_{4l+3,0} \Big) \\
& + \Big(\pi^{-}_{4l+3,0}\, B_2\, \iota''_{4l+2} \Big) \Big(\pi''_{4l+2}\, h''_2\, \iota^{+}_{4l+3,l} \Big) \\
& + \Big(\pi^{-}_{4l+3,0}\, B_2\, \iota''_{4l+2} \Big) \Big(\pi''_{4l+2}\, H_4\, \iota^{-}_{4l+3,0} \Big) \\
& + \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+3,k+1}\, B_{\overline{\overline{k}}}\, \iota^{-}_{4l+2,k} \Big) \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+2,k}\, H_{\overline{4k+2}}\, \iota^{-}_{4l+3,k+1} \Big)
\end{aligned}$$

$$+\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+3,k}\;D_{\overline{4k+3}}\;\iota^-_{4l+2,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+2,k}\;H_{\overline{4k+2}}\;\iota^-_{4l+3,k+1}\Big)$$

$$+\Big(\sum_{k\in[0,l]}\pi^+_{4l+3,k}\;\widetilde{G}\;\iota^+_{4l+4,k}\Big)\Big(\sum_{k\in[0,l]}\pi^+_{4l+4,k}\;\widetilde{A}\;\iota^+_{4l+3,k}\Big)$$
$$+\Big(\sum_{k\in[0,l]}\pi^+_{4l+3,k}\;\widetilde{G}\;\iota^+_{4l+4,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+4,k+1}\;C\;\iota^+_{4l+3,k}\Big)$$
$$+\Big(\sum_{k\in[0,l]}\pi^-_{4l+3,k}\;H_{\overline{4k+5}}\;\iota^-_{4l+4,k}\Big)\Big(\sum_{k\in[0,l]}\pi^-_{4l+4,k}\;B_{\overline{\overline{k}}}\;\iota^-_{4l+3,k}\Big)$$
$$+\Big(\sum_{k\in[0,l]}\pi^-_{4l+3,k}\;H_{\overline{4k+5}}\;\iota^-_{4l+4,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+4,k}\;D_{\overline{4k+4}}\;\iota^-_{4l+3,k+1}\Big)$$
$$+\Big(\pi^+_{4l+3,l}\;h'_3\;\iota'_{4l+4}\Big)\Big(\pi'_{4l+4}\;C'\;\iota^+_{4l+3,l}\Big)$$
$$+\Big(\pi^+_{4l+3,l}\;h'_3\;\iota'_{4l+4}\Big)\Big(\pi'_{4l+4}\;D'_0\;\iota^-_{4l+3,0}\Big)$$

$$=\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+3,k}\;\underbrace{(A\cdot G+\widetilde{G}\cdot\widetilde{A})}_{=1;\text{ cf. (i)}}\;\iota^+_{4l+3,k}\Big)$$

$$+\Big(\sum_{k\in[0,l-1]}\pi^+_{4l+3,k+1}\;\underbrace{(C\cdot G+\widetilde{G}\cdot C)}_{=0;\text{ cf. (ii)}}\;\iota^+_{4l+3,k}\Big)$$

$$+\Big(\pi^+_{4l+3,l}\;\underbrace{(d'_2\cdot h'_2+d''_2\cdot h''_2+\widetilde{G}\cdot\widetilde{A}+h'_3\cdot C')}_{=1;\text{ cf. (v)}}\;\iota^+_{4l+3,l}\Big)$$

$$+\Big(\pi^+_{4l+3,l}\;\underbrace{(d''_2\cdot H_4+h'_3\cdot D'_0)}_{=0;\text{ cf. (v)}}\;\iota^-_{4l+3,0}\Big)$$

$$+\Big(\pi^-_{4l+3,0}\;\underbrace{(B_2\cdot h''_2)}_{=0;\text{ cf. (v)}}\;\iota^+_{4l+3,l}\Big)$$

$$+\Big(\sum_{k\in[0,l]}\pi^-_{4l+3,k}\;\underbrace{(B_{\overline{\overline{k+2}}}\cdot H_{\overline{4k+4}}+H_{\overline{4k+5}}\cdot B_{\overline{\overline{k}}})}_{=1;\text{ cf. (ix)}}\;\iota^-_{4l+3,k}\Big)$$

$$+\Big(\sum_{k\in[0,l-1]}\pi^-_{4l+3,k}\;\underbrace{(D_{\overline{4k+3}}\cdot H_{\overline{4k+2}}+H_{\overline{4k+5}}\cdot D_{\overline{4k+4}})}_{=0;\text{ cf. (xi)}}\;\iota^-_{4l+3,k+1}\Big)$$

$$=1$$

Let $l\geq 0$. We want to show that $d_{4l+3}\,h_{4l+3}+h_{4l+4}\,d_{4l+4}=1$.

Note that

$$d_{4l+4}=\left(\sum_{k\in[0,l]}\pi^+_{4l+5,k}\;A\;\iota^+_{4l+4,k}\right)+\left(\sum_{k\in[0,l-1]}\pi^+_{4l+5,k+1}\;C\;\iota^+_{4l+4,k}\right)$$

$$
\begin{aligned}
&+\Big(\pi'_{4l+5}\;\widetilde{C}\;\iota^{+}_{4l+4,l}\Big)+\big(\pi'_{4l+5}\;d_0\;\iota'_{4l+4}\big)+\Big(\pi'_{4l+5}\;\widetilde{D}_1\;\iota^{-}_{4l+4,0}\Big)\\
&+\left(\sum_{k\in[0,l]}\pi^{-}_{4l+5,k}\;B_{\overline{\overline{k+1}}}\;\iota^{-}_{4l+4,k}\right)+\left(\sum_{k\in[0,l-1]}\pi^{-}_{4l+5,k}\;D_{\overline{4k+5}}\;\iota^{-}_{4l+4,k+1}\right)
\end{aligned}
$$

$$
h_{4l+4}=\left(\sum_{k\in[0,l]}\pi^{+}_{4l+4,k}\;G\;\iota^{+}_{4l+5,k}\right)+\left(\sum_{k\in[0,l]}\pi^{-}_{4l+4,k}\;H_{\overline{4k}}\;\iota^{-}_{4l+5,k}\right)+\big(\pi'_{4l+4}\;h'_0\;\iota'_{4l+5}\big)\,.
$$

$$
\begin{aligned}
&d_{4l+3}\,h_{4l+3}+h_{4l+4}\,d_{4l+4}\\
&=\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+4,k}\;\widetilde{A}\;\iota^{+}_{4l+3,k}\Big)\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+3,k}\;\widetilde{G}\;\iota^{+}_{4l+4,k}\Big)\\
&+\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+4,k}\;\widetilde{A}\;\iota^{+}_{4l+3,k}\Big)\Big(\pi^{+}_{4l+3,l}\;h'_3\;\iota'_{4l+4}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+4,k+1}\;C\;\iota^{+}_{4l+3,k}\Big)\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+3,k}\;\widetilde{G}\;\iota^{+}_{4l+4,k}\Big)\\
&+\Big(\pi'_{4l+4}\;C'\;\iota^{+}_{4l+3,l}\Big)\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+3,k}\;\widetilde{G}\;\iota^{+}_{4l+4,k}\Big)\\
&+\Big(\pi'_{4l+4}\;C'\;\iota^{+}_{4l+3,l}\Big)\Big(\pi^{+}_{4l+3,l}\;h'_3\;\iota'_{4l+4}\Big)\\
&+\Big(\pi'_{4l+4}\;D'_0\;\iota^{-}_{4l+3,0}\Big)\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+3,k}\;H_{\overline{4k+5}}\;\iota^{-}_{4l+4,k}\Big)\\
&+\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+4,k}\;B_{\overline{\overline{k}}}\;\iota^{-}_{4l+3,k}\Big)\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+3,k}\;H_{\overline{4k+5}}\;\iota^{-}_{4l+4,k}\Big)\\
&+\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+4,k}\;D_{\overline{4k+4}}\;\iota^{-}_{4l+3,k+1}\Big)\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+3,k}\;H_{\overline{4k+5}}\;\iota^{-}_{4l+4,k}\Big)\\
&\\
&+\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+4,k}\;G\;\iota^{+}_{4l+5,k}\Big)\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+5,k}\;A\;\iota^{+}_{4l+4,k}\Big)\\
&+\Big(\sum_{k\in[0,l]}\pi^{+}_{4l+4,k}\;G\;\iota^{+}_{4l+5,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{+}_{4l+5,k+1}\;C\;\iota^{+}_{4l+4,k}\Big)\\
&+\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+4,k}\;H_{\overline{4k}}\;\iota^{-}_{4l+5,k}\Big)\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+5,k}\;B_{\overline{\overline{k+1}}}\;\iota^{-}_{4l+4,k}\Big)\\
&+\Big(\sum_{k\in[0,l]}\pi^{-}_{4l+4,k}\;H_{\overline{4k}}\;\iota^{-}_{4l+5,k}\Big)\Big(\sum_{k\in[0,l-1]}\pi^{-}_{4l+5,k}\;D_{\overline{4k+5}}\;\iota^{-}_{4l+4,k+1}\Big)\\
&+\Big(\pi'_{4l+4}\;h'_0\;\iota'_{4l+5}\Big)\Big(\pi'_{4l+5}\;\widetilde{C}\;\iota^{+}_{4l+4,l}\Big)\\
&+\Big(\pi'_{4l+4}\;h'_0\;\iota'_{4l+5}\Big)\Big(\pi'_{4l+5}\;d_0\;\iota'_{4l+4}\Big)\\
&+\Big(\pi'_{4l+4}\;h'_0\;\iota'_{4l+5}\Big)\Big(\pi'_{4l+5}\;\widetilde{D}_1\;\iota^{-}_{4l+4,0}\Big)
\end{aligned}
$$

$$= \Big(\sum_{k\in[0,l]} \pi^{+}_{4l+4,k} \underbrace{(\widetilde{A}\cdot\widetilde{G} + G\cdot A)}_{=1;\ \text{cf. (i)}} \iota^{+}_{4l+4,k} \Big)$$

$$+ \Big(\pi^{+}_{4l+4,l} \underbrace{(\widetilde{A}\cdot h'_3)}_{=0;\ \text{cf. (vi)}} \iota'_{4l+4} \Big)$$

$$+ \Big(\sum_{k\in[0,l-1]} \pi^{+}_{4l+4,k+1} \underbrace{(C\cdot\widetilde{G} + G\cdot C)}_{=0;\ \text{cf. (ii)}} \iota^{+}_{4l+4,k} \Big)$$

$$+ \Big(\pi'_{4l+4} \underbrace{(C'\cdot\widetilde{G} + h'_0\cdot\widetilde{C})}_{=0;\ \text{cf. (vi)}} \iota^{+}_{4l+4,l} \Big)$$

$$+ \Big(\pi'_{4l+4} \underbrace{(C'\cdot h'_3 + h'_0\cdot d_0)}_{=1;\ \text{cf. (vi)}} \iota'_{4l+4} \Big)$$

$$+ \Big(\pi'_{4l+4} \underbrace{(D'_0\cdot H_5 + h'_0\cdot\widetilde{D}_1)}_{=0;\ \text{cf. (vi)}} \iota^{-}_{4l+4,0} \Big)$$

$$+ \Big(\sum_{k\in[0,l]} \pi^{-}_{4l+4,k} \underbrace{(B_{\overline{\overline{k}}}\cdot H_{\overline{4k+5}} + H_{\overline{4k}}\cdot B_{\overline{\overline{k+1}}})}_{=1;\ \text{cf. (x)}} \iota^{-}_{4l+4,k} \Big)$$

$$+ \Big(\sum_{k\in[0,l-1]} \pi^{-}_{4l+4,k} \underbrace{(D_{\overline{4k+4}}\cdot H_{\overline{4k+9}} + H_{\overline{4k}}\cdot D_{\overline{4k+5}})}_{=0;\ \text{cf. (xii)}} \iota^{-}_{4l+4,k+1} \Big)$$

$$= 1$$

(III) *Minimality*

We aim to proof that the projective resolution X of the trivial Λ-module R is minimal; cf. Definition 9.

We have that X_n is a finitely generated projective Λ-module for all $n \in \mathbb{Z}_{\geq 0}$. We show that $X_{n+1}d_n \subseteq \mathrm{rad}(\Lambda)X_n$ for all $n \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{B} := \{2b^0_{0,0},\, b^0_{0,1},\, b^0_{0,2},\, 2b^1_{0,0},\, b^1_{0,1},\, 2b^1_{1,0},\, b^1_{1,1},\, b^0_{1,0},\, b^0_{1,1},\, \widetilde{b}^0_{0,1},\, \widetilde{b}^0_{0,2},\, \widetilde{b}^1_{0,0},\, \widetilde{b}^1_{0,1},\, \widetilde{b}^1_{1,0},\, \widetilde{b}^0_{1,1}\}$.

Then $\mathcal{B}$ is the set of all Λ-linear maps appearing as constituents of the differentials of X; cf. Definition 28 and Definition 27.

Note that for $b \in \mathcal{B}$ we have $(\Lambda\mathcal{E}_i)2b \subseteq \mathrm{rad}(\Lambda)\mathcal{E}_j$, since $2\Lambda \subseteq \mathrm{rad}(\Lambda)$ so that by [**7,** §9.1, Corollary 2] it suffices to show that

$$(\Lambda\mathcal{E}_i)b \subseteq \mathrm{rad}(\Lambda)\mathcal{E}_j$$

for all $b \in \mathcal{B}$, where $\Lambda\mathcal{E}_i$ is the source of b and $\Lambda\mathcal{E}_j$ is the target of b.

We have

$$\mathrm{rad}(\Lambda)\mathcal{E}_0 = \left\{ \left(\rho, \sigma, \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}\right) \in P \;:\; \begin{matrix}\rho \equiv_2 \sigma \equiv_2 0,\\ \alpha_3 \equiv_2 \beta_3 \equiv_2 0\end{matrix} \right\}$$

$$\operatorname{rad}(\Lambda)\mathcal{E}_1 = \left\{ \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \in Q \ : \ \begin{array}{c} \alpha_1 \equiv_2 \alpha_2 \equiv_2 \beta_1 \equiv_2 \beta_2 \equiv_2 0, \\ \gamma_1 \equiv_2 \gamma_2 \equiv_2 0 \end{array} \right\};$$

cf. Corollary 37.

A direct verification using the lists in Lemma 20 yields that $b \in \mathcal{E}_i \operatorname{rad}(\Lambda)\mathcal{E}_j$ for all $b \in \mathcal{B}$, where $\Lambda\mathcal{E}_i$ is the source of b and $\Lambda\mathcal{E}_j$ is the target of b. Therefore, we obtain

$$(\Lambda\mathcal{E}_i)b \subseteq \Lambda\mathcal{E}_i \operatorname{rad}(\Lambda)\mathcal{E}_j \subseteq \operatorname{rad}(\Lambda)\mathcal{E}_j$$

and the projective resolution X is minimal.

$\square$

Chapter 3

Cohomology of S_4 over $\mathbb{Z}_{(2)}$

Recall that $R = \mathbb{Z}_{(2)}$ and $R\mathrm{S}_4 \xrightarrow[\sim]{\omega} \Lambda$; cf. Corollary 15.

Suppose given $x \in \mathbb{Z}$. Recall that $\underline{\underline{x}}$ is the unique element in $\mathbb{Z}$ with $x = 3\underline{\underline{x}} + \overline{\overline{x}}$, where $\overline{\overline{x}} \in [0, 2]$; cf. Definition 23.

We aim to calculate the cohomology of S_4 over $\mathbb{Z}_{(2)}$ using the minimal projective resolution X of Theorem 38; cf. Theorem 54 below.

3.1 Preparations

Definition 39 Let M, N be Λ-modules and $\psi \in \mathrm{Hom}_\Lambda(M, N)$.

We write $M^* := \mathrm{Hom}_\Lambda(M, R)$, where R is the trivial Λ-module; cf. Remark 16. Furthermore, we have the R-linear map $\psi^* = \mathrm{Hom}_\Lambda(\psi, R)$ with

$$N^* \xrightarrow{\psi^*} M^*,\ f \mapsto \psi f\,.$$

If ψ is written in the form $\psi = \rho^\diamond$, where $\diamond$ is a symbol, we often write $\psi^* = (\rho^\diamond)^* =: \rho^{\diamond,*}$, e.g. $(\iota^+_{1,0})^* =: \iota^{+,*}_{1,0}$.

Remark 40

(1) For Λ-modules M, N, there exist the following projection and inclusion maps.

$$M \xrightarrow{\iota_1} M \oplus N \xrightarrow{\pi_1} M$$
$$N \xrightarrow{\iota_2} M \oplus N \xrightarrow{\pi_2} N$$

We have the following mutually inverse isomorphisms of R-modules.

$$\psi_{M,N} : (M \oplus N)^* \xrightarrow[\sim]{\begin{pmatrix} \iota_1^* & \iota_2^* \end{pmatrix}} M^* \oplus N^*$$

$$(M \oplus N)^* \xleftarrow[\sim]{\begin{pmatrix} \pi_1^* \\ \pi_2^* \end{pmatrix}} M^* \oplus N^*$$

(2) Let U, V, X, Y be Λ-modules and $U \oplus V \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} X \oplus Y$ be a Λ-linear map. The following diagram commutes.

$$\begin{array}{ccc} (U \oplus V)^* & \xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^*} & (X \oplus Y)^* \\ {\scriptstyle \psi_{U,V}} \downarrow \wr & & \wr \downarrow {\scriptstyle \psi_{X,Y}} \\ U^* \oplus V^* & \xleftarrow{\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}} & X^* \oplus Y^* \end{array}$$

Proof. *Ad* (1). We have

$$\begin{pmatrix} \iota_1^* & \iota_2^* \end{pmatrix} \begin{pmatrix} \pi_1^* \\ \pi_2^* \end{pmatrix} = \iota_1^* \pi_1^* + \iota_2^* \pi_2^* = (\pi_1 \iota_1 + \pi_2 \iota_2)^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^* = \mathrm{id}_{(M \oplus N)^*}$$

$$\begin{pmatrix} \pi_1^* \\ \pi_2^* \end{pmatrix} \begin{pmatrix} \iota_1^* & \iota_2^* \end{pmatrix} = \begin{pmatrix} \pi_1^* \iota_1^* & \pi_1^* \iota_2^* \\ \pi_2^* \iota_1^* & \pi_2^* \iota_2^* \end{pmatrix} = \begin{pmatrix} (\iota_1 \pi_1)^* & (\iota_2 \pi_1)^* \\ (\iota_1 \pi_2)^* & (\iota_2 \pi_2)^* \end{pmatrix} = \begin{pmatrix} 1^* & 0^* \\ 0^* & 1^* \end{pmatrix} = \mathrm{id}_{M^* \oplus N^*}$$

so that $\begin{pmatrix} \pi_1^* \\ \pi_2^* \end{pmatrix}$ and $\begin{pmatrix} \iota_1^* & \iota_2^* \end{pmatrix}$ are mutually inverse isomorphisms.

Ad (2). We have

$$\begin{aligned} \begin{pmatrix} \iota_1^* & \iota_2^* \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \begin{pmatrix} \pi_1^* \\ \pi_2^* \end{pmatrix} &= \iota_1^* a^* \pi_1^* + \iota_2^* b^* \pi_1^* + \iota_1^* c^* \pi_2^* + \iota_2^* d^* \pi_2^* \\ &= (\pi_1 a \iota_1 + \pi_1 b \iota_2 + \pi_2 c \iota_1 + \pi_2 d \iota_2)^* \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \end{aligned}$$

so that

$$\begin{pmatrix} \iota_1^* & \iota_2^* \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} \pi_1^* \\ \pi_2^* \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} \iota_1^* & \iota_2^* \end{pmatrix},$$

i.e. the diagram commutes. □

Remark 41 Let J be a finite set.

Suppose given Λ-modules M_j and R-modules N_j and R-linear isomorphisms $M_j^* \xrightarrow[\sim]{\eta_j} N_j$ for $j \in J$.

Suppose given $I \subseteq J$ such that $M_j^* = 0$ for $j \in J \backslash I$.

We write $M := \bigoplus_{j \in J} M_j$ and $N := \bigoplus_{i \in I} N_i$.

Consider the following projection and inclusion maps for $i, k \in J$.

$$M_i \xrightarrow{\iota_i} M \xrightarrow{\pi_k} M_k \qquad\qquad N_i \xrightarrow{\iota_i'} N \xrightarrow{\pi_k'} N_k$$

We have the following mutually inverse R-linear isomorphisms.

$$\eta := \sum_{i\in I} \iota_i^* \, \eta_i \, \iota_i' \qquad : \quad M^* \to N$$
$$\eta^{-1} = \sum_{i\in I} \pi_i' \, \eta_i^{-1} \pi_i^* \quad : \quad N \to M^*$$

Proof. Write $\tilde{\eta} := \sum_{i\in I} \pi_i' \, \eta_i^{-1} \pi_i^*$. Note that $(\pi_j \iota_j)^* = 0$ for $j \in J \backslash I$. We have the following.

$$\eta\,\tilde{\eta} = \sum_{i,k\in I} \iota_i^* \, \eta_i \, \iota_i' \, \pi_k' \, \eta_k^{-1} \, \pi_k^* = \sum_{i\in I} \iota_i^* \, \eta_i \, \eta_i^{-1} \, \pi_i^* = \sum_{i\in I} \iota_i^* \, \pi_i^* = \left(\sum_{i\in I} \pi_i \, \iota_i\right)^* = \left(\sum_{i\in J} \pi_i \, \iota_i\right)^* = \mathrm{id}_{M^*}$$
$$\tilde{\eta}\,\eta = \sum_{i,k\in I} \pi_i' \, \eta_i^{-1} \, \pi_i^* \, \iota_k^* \, \eta_k \, \iota_k' = \sum_{i,k\in I} \pi_i' \, \eta_i^{-1} \, (\iota_k \, \pi_i)^* \, \eta_k \, \iota_k' = \sum_{i\in I} \pi_i' \, \eta_i^{-1} \, \eta_i \, \iota_i' = \sum_{i\in I} \pi_i' \, \iota_i' = \mathrm{id}_N$$

□

Remark 42 Suppose given $x, k \in \mathbb{Z}$. Recall that $\underline{\underline{x}}$ is the unique element in $\mathbb{Z}$ with $x = 3\underline{\underline{x}} + \overline{\overline{x}}$, where $\overline{\overline{x}} \in [0, 2]$.

The following equations hold.

(i) $\underline{\underline{x + 3k}} = \underline{\underline{x}} + k$

(ii) $\underline{\underline{-x}} = -(\underline{\underline{2 + x}})$

(iii) $\underline{\underline{2x}} = (x - 1) - (\underline{\underline{x - 1}})$

(iv) $\underline{\underline{2x}} = 2\underline{\underline{x}}$, for $x \equiv_3 0$

(v) $\underline{\underline{2x - 2}} = 2\underline{\underline{x}} - 1$, for $x \equiv_3 0$

(vi) $\underline{\underline{x}} + \underline{\underline{x - 1}} + \underline{\underline{x - 2}} = x - 2$

Proof.

(i) We have $x + 3k = 3(\underline{\underline{x + 3k}}) + \overline{\overline{x + 3k}} = 3(\underline{\underline{x + 3k}}) + \overline{\overline{x}} = 3(\underline{\underline{x + 3k}}) + x - 3\underline{\underline{x}}$, so that $k = \underline{\underline{x + 3k}} - \underline{\underline{x}}$.

(ii) We have $3(\underline{\underline{-x}}) + 3(\underline{\underline{2 + x}}) = -x - (\overline{\overline{-x}}) + (2 + x) - (\overline{\overline{2 + x}}) = -(\overline{\overline{-x}}) - (\overline{\overline{2 + x}}) + 2$ so that $\underline{\underline{-x}} = -(\underline{\underline{2 + x}})$ if and only if $\overline{\overline{-x}} + \overline{\overline{2 + x}} = 2$. However, this holds for $x \in [0, 2]$ and the left hand side is periodic modulo 3.

(iii) $\underline{\underline{2x}} = \underline{\underline{-x + 3x}} \overset{\text{(i)}}{=} \underline{\underline{-x}} + x \overset{\text{(ii)}}{=} -(\underline{\underline{2 + x}}) + x = -(\underline{\underline{x - 1 + 3}}) + x \overset{\text{(i)}}{=} -(\underline{\underline{x - 1}}) + x - 1$

(iv) Let $x =: 3y$. We have $\underline{\underline{2x}} \overset{\text{(i)}}{=} \underline{\underline{6y}} = 2y \overset{\text{(i)}}{=} 2(\underline{\underline{3y}}) = 2\underline{\underline{x}}$.

(v) Let $x =: 3y$. We have $\underline{\underline{2x - 2}} = \underline{\underline{6y - 2}} \overset{\text{(i)}}{=} 2y + \underline{\underline{-2}} = 2y - 1 \overset{\text{(i)}}{=} \underline{\underline{2x}} - 1$.

(vi) We have

$$\begin{aligned} 3(\underline{\underline{x}} + \underline{\underline{x - 1}} + \underline{\underline{x - 2}}) &= x - \overline{\overline{x}} + x - 1 - (\overline{\overline{x - 1}}) + x - 2 - (\overline{\overline{x - 2}}) \\ &= 3x - 3 - (\overline{\overline{x}} + \overline{\overline{x - 1}} + \overline{\overline{x - 2}}) = 3x - 6 \end{aligned}$$

so that $\underline{\underline{x}} + \underline{\underline{x - 1}} + \underline{\underline{x - 2}} = x - 2$. □

Remark 43 Let $l \in \mathbb{Z}_{\geq 0}$.

(1) The following equations hold.

$$\begin{aligned} |\{k \in [0, l-1] : k \equiv_3 0\}| &= \underline{\underline{\underline{l+2}}} \\ |\{k \in [0, l-1] : k \equiv_3 1\}| &= \underline{\underline{\underline{l+1}}} \\ |\{k \in [0, l-1] : k \equiv_3 2\}| &= \underline{\underline{\underline{l}}} \end{aligned}$$

(2) We have the following monotone bijections.

$$\begin{aligned} \{k \in [0, l-1] : k \not\equiv_3 0\} &\xrightarrow[\sim]{\mu_0} [2l+2, 3l - (\underline{\underline{\underline{l-1}}})] \\ k &\longmapsto 2l+2+\underline{\underline{\underline{2k-1}}} \\ \{k \in [0, l-1] : k \not\equiv_3 2\} &\xrightarrow[\sim]{\mu_1} [2l+2, 3l - (\underline{\underline{\underline{l-3}}})] \\ k &\longmapsto 2l+2+\underline{\underline{\underline{2k+1}}} \\ \{k \in [0, l-1] : k \not\equiv_3 1\} &\xrightarrow[\sim]{\mu_2} [2l+3, 3l - (\underline{\underline{\underline{l-5}}})] \\ k &\longmapsto 2l+3+\underline{\underline{\underline{2k}}} \\ \{k \in [0, l] : k \not\equiv_3 1\} &\xrightarrow[\sim]{\mu_3} [2l+3, 3l - (\underline{\underline{\underline{l-7}}})] \\ k &\longmapsto 2l+3+\underline{\underline{\underline{2k}}} \end{aligned}$$

Proof. *Ad* (1). We have

$$|\{k \in [0, l-1] : k \equiv_3 2\}| = |\{k \in [1, l] : k \equiv_3 0\}| = \underline{\underline{\underline{l}}}.$$

Using this, we obtain the following for $i \in [0, 1]$.

$$\begin{aligned} |\{k \in [0, l-1] : k \equiv_3 i\}| &= |\{k \in [2-i, (l-1)+(2-i)] : k \equiv_3 2\}| \\ &= |\{k \in [0, (l+2-i)-1] : k \equiv_3 2\}| = \underline{\underline{\underline{l+2-i}}} \end{aligned}$$

Ad (2). We will use the equations (i, iii) of Remark 42. We obtain the following.

$$2l+2+\underline{\underline{\underline{2k-1}}} \overset{\text{(i)}}{=} 2l+1+\underline{\underline{\underline{2(k+1)}}} \overset{\text{(iii)}}{=} 2l+1+k-\underline{\underline{\underline{k}}}$$

$$2l+2+\underline{\underline{\underline{2k+1}}} \overset{\text{(i)}}{=} 2l+3+\underline{\underline{\underline{2(k-1)}}} \overset{\text{(iii)}}{=} 2l+3+(k-2)-(\underline{\underline{\underline{k-2}}}) = 2l+1+k-(\underline{\underline{\underline{k-2}}})$$

$$2l+3+\underline{\underline{\underline{2k}}} \overset{\text{(iii)}}{=} 2l+2+k-(\underline{\underline{\underline{k-1}}}) \overset{\text{(i)}}{=} 2l+1+k-(\underline{\underline{\underline{k-4}}})$$

Hence, μ_i is well-defined and strictly monotone and therefore injective for $i \in [0, 3]$.

By part (1) we have

$$|[2l+2, 3l-(\underline{\underline{\underline{l-1}}})]| = 3l-(\underline{\underline{\underline{l-1}}})-(2l+2)+1 \overset{\text{(i)}}{=} l-(\underline{\underline{\underline{l+2}}}) = |\{k \in [0, l-1] : k \not\equiv_3 0\}|$$

$$|[2l+2, 3l-(\underline{\underline{\underline{l-3}}})]| = 3l-(\underline{\underline{\underline{l-3}}})-(2l+2)+1 \overset{\text{(i)}}{=} l-\underline{\underline{\underline{l}}} = |\{k \in [0, l-1] : k \not\equiv_3 2\}|$$

$$|[2l+3, 3l-(\underline{\underline{\underline{l-5}}})]| = 3l-(\underline{\underline{\underline{l-5}}})-(2l+3)+1 \overset{\text{(i)}}{=} l-(\underline{\underline{\underline{l+1}}}) = |\{k \in [0, l-1] : k \not\equiv_3 1\}|$$

$$|[2l+3, 3l-(\underline{\underline{\underline{l-7}}})]| = 3l-(\underline{\underline{\underline{l-7}}})-(2l+3)+1 \overset{\text{(i)}}{=} l+1-(\underline{\underline{\underline{l+2}}}) = |\{k \in [0, l] : k \not\equiv_3 1\}|$$

so that μ_i is bijective for $i \in [0, 3]$. □

Remark 44 Suppose given $x \in \mathbb{Z}$. The following inequalities hold.

(1) $2\underline{\underline{x}} + \underline{\underline{x+1}} \leq x$

(2) $2(\underline{\underline{x+1}}) + \underline{\underline{x}} \leq x$

Proof. Ad (1). We have

$$\begin{array}{rrcl} & 2\underline{\underline{x}} + \underline{\underline{x+1}} & \leq & x \\ \Leftrightarrow & 2\underline{\underline{x}} + \underline{\underline{x+1}} & \leq & 3\underline{\underline{x}} + \overline{\overline{x}} \\ \Leftrightarrow & \underline{\underline{x+1}} & \leq & \underline{\underline{x}} + \overline{\overline{x}} \\ \Leftrightarrow & \underline{\underline{x+1}} - \underline{\underline{x}} & \leq & \overline{\overline{x}} \end{array}$$

where $\underline{\underline{x+1}} - \underline{\underline{x}} = 1$ if $\overline{\overline{x}} = 2$ and $\underline{\underline{x+1}} - \underline{\underline{x}} = 0$ if $\overline{\overline{x}} \in [0,1]$.
Ad (2). We have

$$\begin{array}{rrcl} & 2(\underline{\underline{x+1}}) + \underline{\underline{x}} & \leq & x \\ \Leftrightarrow & 2(\underline{\underline{x+1}}) + \underline{\underline{x}} & \leq & 3\underline{\underline{x}} + \overline{\overline{x}} \\ \Leftrightarrow & 2(\underline{\underline{x+1}}) - 2\underline{\underline{x}} & \leq & \overline{\overline{x}} \end{array}$$

where $2(\underline{\underline{x+1}}) - 2\underline{\underline{x}} = 2$ if $\overline{\overline{x}} = 2$ and $2(\underline{\underline{x+1}}) - 2\underline{\underline{x}} = 0$ if $\overline{\overline{x}} \in [0,1]$. $\square$

3.2 Calculation of the cohomology groups

Remark 45 We have the following isomorphisms of R-modules.

$$\begin{array}{lcl} P^* \xrightarrow[\sim]{\varphi} \mathcal{E}_0 R = R & \qquad & Q^* \xrightarrow[\sim]{} 0 \\ f \mapsto \mathcal{E}_0 f & & \end{array}$$

Furthermore, the following diagrams commute.

$$\begin{array}{ccc} P^* & \xrightarrow{(2b^0_{0,0})^*} & P^* \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ R & \xrightarrow{2} & R \end{array} \qquad \begin{array}{ccc} P^* & \xrightarrow{(b^0_{0,1})^*} & P^* \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ R & \xrightarrow{0} & R \end{array} \qquad \begin{array}{ccc} P^* & \xrightarrow{(b^0_{0,2})^*} & P^* \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ R & \xrightarrow{0} & R \end{array}$$

$$\begin{array}{ccc} P^* & \xrightarrow{(\widetilde{b}^0_{0,1})^*} & P^* \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ R & \xrightarrow{2} & R \end{array} \qquad \begin{array}{ccc} P^* & \xrightarrow{(\widetilde{b}^0_{0,2})^*} & P^* \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ R & \xrightarrow{4} & R \end{array}$$

Proof. Recall that $P = \Lambda\mathcal{E}_0$ and $Q = \Lambda\mathcal{E}_1$; cf. Definition 18.
By Remark 21 we have for $i \in \{0,1\}$

$$\begin{aligned}(\Lambda\mathcal{E}_i)^* = \operatorname{Hom}_\Lambda(\Lambda\mathcal{E}_i, R) &\xrightarrow{\sim} \mathcal{E}_i R \\ f &\mapsto \mathcal{E}_i f \\ m_\xi &\leftarrow\!\shortmid \xi,\end{aligned}$$

where $m_\xi : \Lambda\mathcal{E}_i \to R$ is given, using multiplication on the trivial module R, by

$$x m_\xi := x \cdot \xi = \rho\xi$$

for $\xi \in \mathcal{E}_i R$ and $x = (\rho, \sigma, A_1, A_2, A_3) \in \Lambda\mathcal{E}_i$; cf. Remark 16.

Note that we have $\mathcal{E}_0 R = R$ and $\mathcal{E}_1 R = 0$.

Suppose given $\lambda = (\rho, \sigma, A_1, A_2, A_3) \in \mathcal{E}_0\Lambda\mathcal{E}_0 = \operatorname{Hom}_\Lambda(P, P)$. We show that the following diagram of R-linear maps commutes.

$$\begin{array}{ccc} P^* & \xrightarrow{\lambda^*} & P^* \\ {\scriptstyle\varphi}\downarrow{\scriptstyle\wr} & & {\scriptstyle\wr}\downarrow{\scriptstyle\varphi} \\ R & \xrightarrow{\rho} & R \end{array}$$

For $r \in R$ we have

$$r\varphi^{-1}\lambda^*\varphi = m_r\lambda^*\varphi = (\lambda m_r)\varphi = \mathcal{E}_0\lambda m_r = \lambda m_r = \lambda \cdot r = \rho r = r\rho\,.$$

□

Definition 46 We define the following isomorphisms of R-modules; cf. Remark 40 and Remark 45.

$$\begin{array}{rcll} \chi_{P,P} & := & \psi_{P,P}\begin{pmatrix}\varphi & 0 \\ 0 & \varphi\end{pmatrix} & : \ (P \oplus P)^* \xrightarrow{\sim} R \oplus R \\ \chi_1 := \chi_{P,Q} & := & \psi_{P,Q}\begin{pmatrix}\varphi \\ 0\end{pmatrix} & : \ (P \oplus Q)^* \xrightarrow{\sim} R \\ \chi_2 := \chi_{Q,P} & := & \psi_{Q,P}\begin{pmatrix}0 \\ \varphi\end{pmatrix} & : \ (Q \oplus P)^* \xrightarrow{\sim} R \\ \chi_P & := & \varphi & : \ P^* \xrightarrow{\sim} R \end{array}$$

Remark 47 The following diagrams with the R-linear maps from Definition 27 commute; cf. Remark 40, Remark 45 and Definition 46.

We will denote the isomorphic replacement of a morphism of the form F^* by $F^\star$.

$$\begin{array}{ccccc}
 & (P\oplus P)^* & \xrightarrow{A^*=\begin{pmatrix} b^0_{0,1} & 0 \\ b^0_{0,2} & -\widetilde{b}^0_{0,1}\end{pmatrix}^*} & (P\oplus P)^* & \\
 & \wr\downarrow \psi_{P,P} & & \psi_{P,P}\downarrow\wr & \\
\chi_{P,P} & P^*\oplus P^* & \xrightarrow{\begin{pmatrix} (b^0_{0,1})^* & (b^0_{0,2})^* \\ 0 & (-\widetilde{b}^0_{0,1})^*\end{pmatrix}} & P^*\oplus P^* & \chi_{P,P} \\
 & \wr\downarrow \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix} & & \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix}\downarrow\wr & \\
 & R\oplus R & \xrightarrow{A^\star:=\begin{pmatrix} 0 & 0 \\ 0 & -2\end{pmatrix}} & R\oplus R &
\end{array}$$

$$\begin{array}{ccccc}
 & (P\oplus P)^* & \xrightarrow{\widetilde{A}^*=\begin{pmatrix} \widetilde{b}^0_{0,1} & 0 \\ b^0_{0,2} & -b^0_{0,1}\end{pmatrix}^*} & (P\oplus P)^* & \\
 & \wr\downarrow \psi_{P,P} & & \psi_{P,P}\downarrow\wr & \\
\chi_{P,P} & P^*\oplus P^* & \xrightarrow{\begin{pmatrix} (\widetilde{b}^0_{0,1})^* & (b^0_{0,2})^* \\ 0 & (-b^0_{0,1})^*\end{pmatrix}} & P^*\oplus P^* & \chi_{P,P} \\
 & \wr\downarrow \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix} & & \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix}\downarrow\wr & \\
 & R\oplus R & \xrightarrow{\widetilde{A}^\star:=\begin{pmatrix} 2 & 0 \\ 0 & 0\end{pmatrix}} & R\oplus R &
\end{array}$$

$$\begin{array}{ccc}
(Q\oplus P)^* & \xrightarrow{B_0^*} & (Q\oplus Q)^* \\
\psi_{Q,P}\downarrow\wr & & \wr\downarrow\psi_{Q,Q} \\
Q^*\oplus P^* & \longrightarrow & Q^*\oplus Q^* \\
\begin{pmatrix}0\\ \varphi\end{pmatrix}\downarrow\wr & & \wr\downarrow\begin{pmatrix}0\\ 0\end{pmatrix} \\
R & \longrightarrow & 0
\end{array}$$

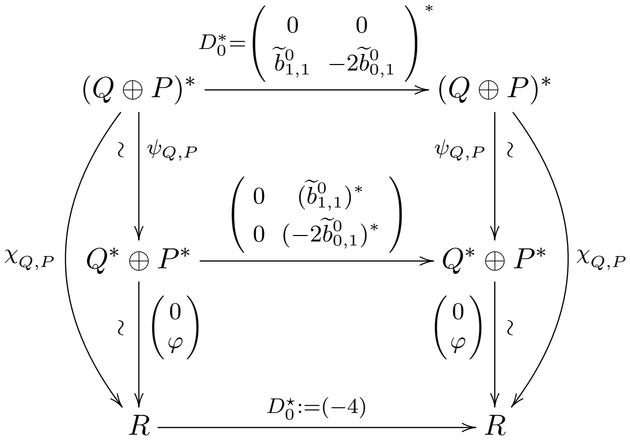
$$\begin{array}{ccc}
(Q\oplus Q)^* & \xrightarrow{B_1^*} & (P\oplus Q)^* \\
\psi_{Q,Q}\downarrow\wr & & \wr\downarrow\psi_{P,Q} \\
Q^*\oplus Q^* & \longrightarrow & P^*\oplus Q^* \\
\begin{pmatrix}0\\ 0\end{pmatrix}\downarrow\wr & & \wr\downarrow\begin{pmatrix}\varphi\\ 0\end{pmatrix} \\
0 & \longrightarrow & R
\end{array}$$

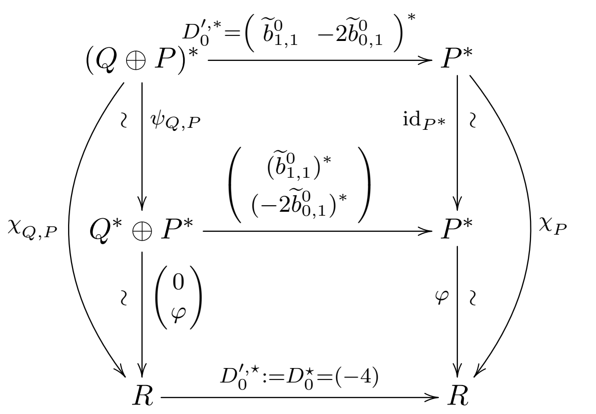
$$\begin{array}{ccccc}
 & (P\oplus Q)^* & \xrightarrow{B_2^*=\begin{pmatrix} -\widetilde{b}^1_{0,0} & \widetilde{b}^1_{1,0} \\ -2b^0_{0,0} & b^0_{1,0}\end{pmatrix}^*} & (Q\oplus P)^* & \\
 & \wr\downarrow \psi_{P,Q} & & \psi_{Q,P}\downarrow\wr & \\
\chi_{P,Q} & P^*\oplus Q^* & \xrightarrow{\begin{pmatrix} (-\widetilde{b}^1_{0,0})^* & (-2b^0_{0,0})^* \\ (\widetilde{b}^1_{1,0})^* & (b^0_{1,0})^*\end{pmatrix}} & Q^*\oplus P^* & \chi_{Q,P} \\
 & \wr\downarrow \begin{pmatrix}\varphi\\ 0\end{pmatrix} & & \begin{pmatrix}0\\ \varphi\end{pmatrix}\downarrow\wr & \\
 & R & \xrightarrow{B_2^\star:=(-2)} & R &
\end{array}$$

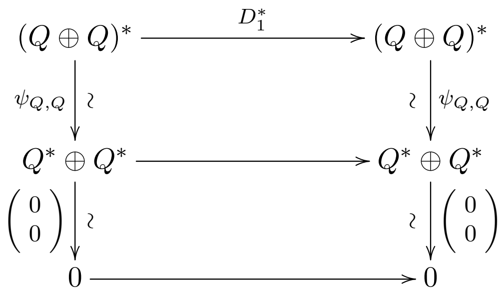
$$\begin{array}{ccccc}
 & (P\oplus P)^* & \xrightarrow{C^*=\begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0\end{pmatrix}^*} & (P\oplus P)^* & \\
 & \wr\downarrow \psi_{P,P} & & \psi_{P,P}\downarrow\wr & \\
\chi_{P,P} & P^*\oplus P^* & \xrightarrow{\begin{pmatrix} 0 & 0 \\ (\widetilde{b}^0_{0,2})^* & 0\end{pmatrix}} & P^*\oplus P^* & \chi_{P,P} \\
 & \wr\downarrow \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix} & & \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix}\downarrow\wr & \\
 & R\oplus R & \xrightarrow{C^\star:=\begin{pmatrix} 0 & 0 \\ 4 & 0\end{pmatrix}} & R\oplus R &
\end{array}$$

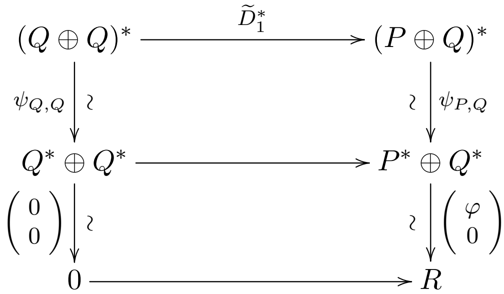
$$\begin{array}{ccccc}
 & (P\oplus P)^* & \xrightarrow{\widetilde{C}^*=\begin{pmatrix} 0 & \widetilde{b}^0_{0,2} \\ 0 & 0\end{pmatrix}^*} & (P\oplus Q)^* & \\
 & \wr\downarrow \psi_{P,P} & & \psi_{P,Q}\downarrow\wr & \\
\chi_{P,P} & P^*\oplus P^* & \xrightarrow{\begin{pmatrix} 0 & 0 \\ (\widetilde{b}^0_{0,2})^* & 0\end{pmatrix}} & P^*\oplus Q^* & \chi_{P,Q} \\
 & \wr\downarrow \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix} & & \begin{pmatrix}\varphi\\ 0\end{pmatrix}\downarrow\wr & \\
 & R\oplus R & \xrightarrow{\widetilde{C}^\star:=\begin{pmatrix} 0 \\ 4\end{pmatrix}} & R &
\end{array}$$

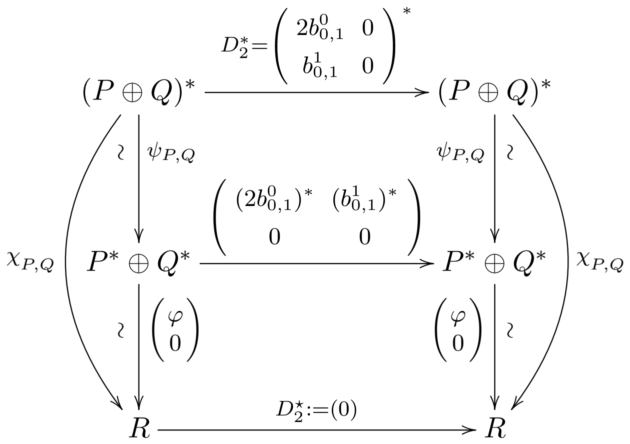
$$\begin{array}{ccccc}
 & (P\oplus P)^* & \xrightarrow{C'^{,*}=\begin{pmatrix} 0 & \widetilde{b}^0_{0,2}\end{pmatrix}^*} & P^* & \\
 & \wr\downarrow \psi_{P,P} & & \mathrm{id}_{P^*}\downarrow\wr & \\
\chi_{P,P} & P^*\oplus P^* & \xrightarrow{\begin{pmatrix} 0 \\ (\widetilde{b}^0_{0,2})^*\end{pmatrix}} & P^* & \chi_{P} \\
 & \wr\downarrow \begin{pmatrix}\varphi & 0\\ 0 & \varphi\end{pmatrix} & & \varphi\downarrow\wr & \\
 & R\oplus R & \xrightarrow{C'^{,\star}:=\begin{pmatrix} 0 \\ 4\end{pmatrix}} & R &
\end{array}$$

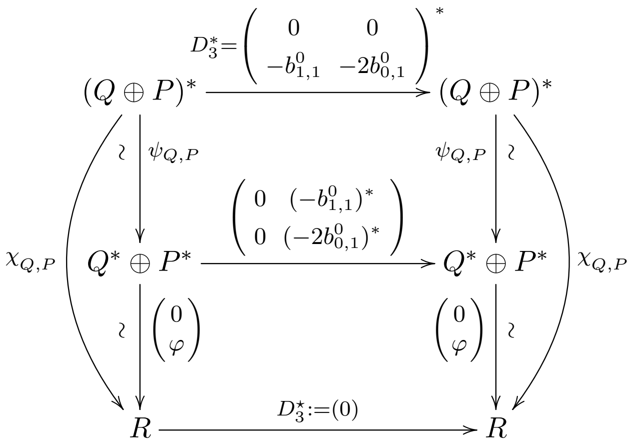

$D_0^*=\begin{pmatrix}0&0\\ \widetilde{b}^0_{1,1}&-2\widetilde{b}^0_{0,1}\end{pmatrix}^*$
$(Q\oplus P)^*$
$(Q\oplus P)^*$
$\psi_{Q,P}$
$\psi_{Q,P}$
$\begin{pmatrix}0&(\widetilde{b}^0_{1,1})^*\\ 0&(-2\widetilde{b}^0_{0,1})^*\end{pmatrix}$
$\chi_{Q,P}$
$Q^*\oplus P^*$
$Q^*\oplus P^*$
$\chi_{Q,P}$
$\begin{pmatrix}0\\ \varphi\end{pmatrix}$
$\begin{pmatrix}0\\ \varphi\end{pmatrix}$
R
$D_0^\star:=(-4)$
R

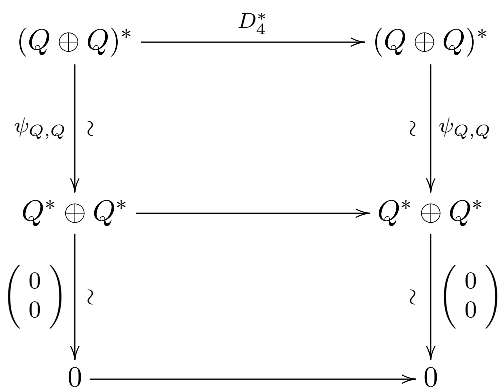

$D_0'^{,*}=\begin{pmatrix}\widetilde{b}^0_{1,1}&-2\widetilde{b}^0_{0,1}\end{pmatrix}^*$
$(Q\oplus P)^*$
P^*
$\psi_{Q,P}$
id_{P^*}
$\begin{pmatrix}(\widetilde{b}^0_{1,1})^*\\ (-2\widetilde{b}^0_{0,1})^*\end{pmatrix}$
$\chi_{Q,P}$
$Q^*\oplus P^*$
P^*
χ_P
$\begin{pmatrix}0\\ \varphi\end{pmatrix}$
φ
R
$D_0'^{,\star}:=D_0^\star=(-4)$
R

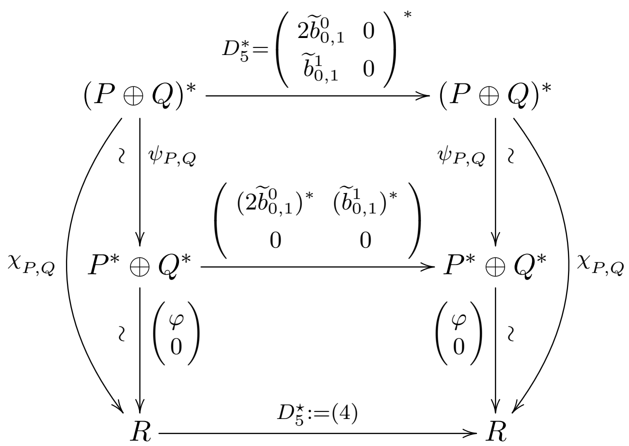

$(Q\oplus Q)^*$
D_1^*
$(Q\oplus Q)^*$
$\psi_{Q,Q}$
$\psi_{Q,Q}$
$Q^*\oplus Q^*$
$Q^*\oplus Q^*$
$\begin{pmatrix}0\\ 0\end{pmatrix}$
$\begin{pmatrix}0\\ 0\end{pmatrix}$
0
0

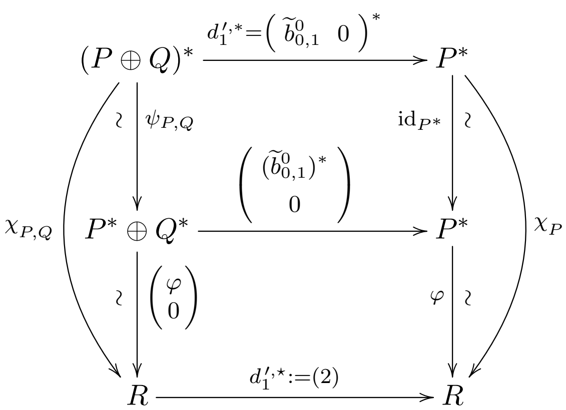

$(Q\oplus Q)^*$
$\widetilde{D}_1^*$
$(P\oplus Q)^*$
$\psi_{Q,Q}$
$\psi_{P,Q}$
$Q^*\oplus Q^*$
$P^*\oplus Q^*$
$\begin{pmatrix}0\\ 0\end{pmatrix}$
$\begin{pmatrix}\varphi\\ 0\end{pmatrix}$
0
R

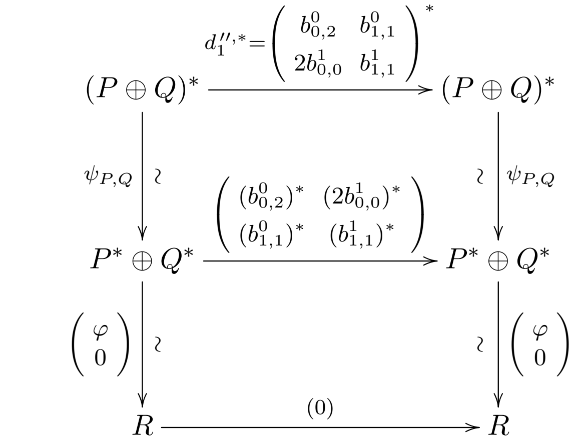

$D_2^*=\begin{pmatrix}2b^0_{0,1}&0\\ b^1_{0,1}&0\end{pmatrix}^*$
$(P\oplus Q)^*$
$(P\oplus Q)^*$
$\psi_{P,Q}$
$\psi_{P,Q}$
$\begin{pmatrix}(2b^0_{0,1})^*&(b^1_{0,1})^*\\ 0&0\end{pmatrix}$
$\chi_{P,Q}$
$P^*\oplus Q^*$
$P^*\oplus Q^*$
$\chi_{P,Q}$
$\begin{pmatrix}\varphi\\ 0\end{pmatrix}$
$\begin{pmatrix}\varphi\\ 0\end{pmatrix}$
R
$D_2^\star:=(0)$
R

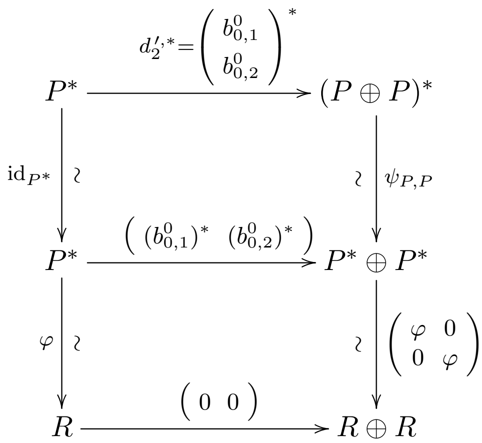

$D_3^*=\begin{pmatrix}0&0\\ -b^0_{1,1}&-2b^0_{0,1}\end{pmatrix}^*$
$(Q\oplus P)^*$
$(Q\oplus P)^*$
$\psi_{Q,P}$
$\psi_{Q,P}$
$\begin{pmatrix}0&(-b^0_{1,1})^*\\ 0&(-2b^0_{0,1})^*\end{pmatrix}$
$\chi_{Q,P}$
$Q^*\oplus P^*$
$Q^*\oplus P^*$
$\chi_{Q,P}$
$\begin{pmatrix}0\\ \varphi\end{pmatrix}$
$\begin{pmatrix}0\\ \varphi\end{pmatrix}$
R
$D_3^\star:=(0)$
R

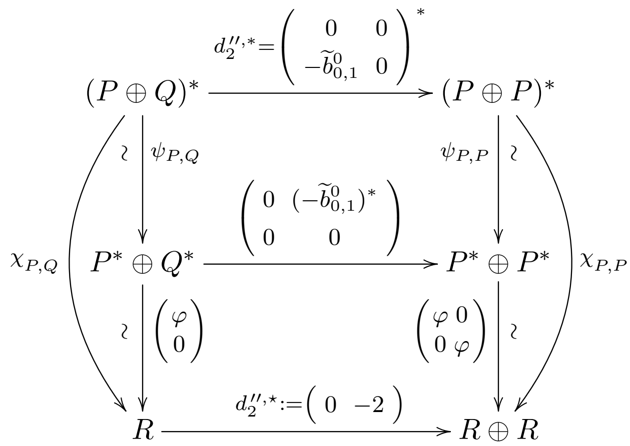

$(Q\oplus Q)^*$
D_4^*
$(Q\oplus Q)^*$
$\psi_{Q,Q}$
$\psi_{Q,Q}$
$Q^*\oplus Q^*$
$Q^*\oplus Q^*$
$\begin{pmatrix}0\\ 0\end{pmatrix}$
$\begin{pmatrix}0\\ 0\end{pmatrix}$
0
0

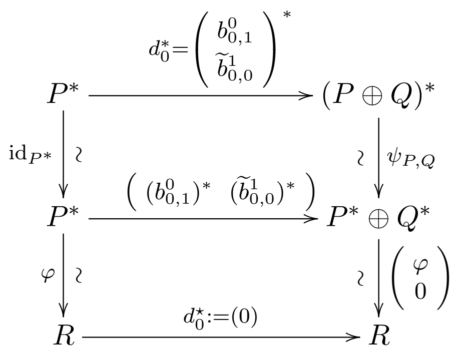

$D_5^*=\begin{pmatrix}2\widetilde{b}^0_{0,1}&0\\ \widetilde{b}^1_{0,1}&0\end{pmatrix}^*$
$(P\oplus Q)^*$
$(P\oplus Q)^*$
$\psi_{P,Q}$
$\psi_{P,Q}$
$\begin{pmatrix}(2\widetilde{b}^0_{0,1})^*&(\widetilde{b}^1_{0,1})^*\\ 0&0\end{pmatrix}$
$\chi_{P,Q}$
$P^*\oplus Q^*$
$P^*\oplus Q^*$
$\chi_{P,Q}$
$\begin{pmatrix}\varphi\\ 0\end{pmatrix}$
$\begin{pmatrix}\varphi\\ 0\end{pmatrix}$
R
$D_5^\star:=(4)$
R











Definition 48 Given $m \in \mathbb{Z}_{\geq 1}$ and $1 \leq k_1 \leq k_2 \leq m$, we define the following R-linear maps.

$$\pi^m_{k_1,k_2} \quad : \quad R^{\oplus m} \to R^{\oplus(k_2+1-k_1)} \quad , \quad (x_i)_{i\in[1,m]} \mapsto (x_i)_{i\in[k_1,k_2]}$$

$$\iota^m_{k_1,k_2} \quad : \quad R^{\oplus(k_2+1-k_1)} \to R^{\oplus m} \quad , \quad (x_i)_{i\in[k_1,k_2]} \mapsto \big((0)_{i\in[1,k_1-1]}, (x_i)_{i\in[k_1,k_2]}, (0)_{i\in[k_2+1,m]}\big)$$

Furthermore, we sometimes abbreviate for $1 \leq k \leq m$ as follows.

$$\pi^m_k := \pi^m_{k,k} \quad : \quad R^{\oplus m} \to R$$

$$\iota^m_k := \iota^m_{k,k} \quad : \quad R \to R^{\oplus m}$$

Remark 49 Suppose given $l \in \mathbb{Z}_{\geq 0}$. For $i \in [0,3]$ let

$$m_{4l+i} := 3l - (\underline{\underline{l - 2i - 1}})\,.$$

We can abbreviate as follows; cf. Remark 42.(i) and Remark 43.(1).

$$\Big(\bigoplus_{k\in[0,l-1]} R\oplus R\Big)\oplus R\oplus\Big(\bigoplus_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0}} R\Big) \overset{(1)}{=} R^{\oplus 2l+1+l-(\underline{\underline{l+2}})} \overset{(\mathrm{i})}{=} R^{\oplus 3l-(\underline{\underline{l-1}})} = R^{\oplus m_{4l}}$$

$$\Big(\bigoplus_{k\in[0,l-1]} R\oplus R\Big)\oplus R\oplus\Big(\bigoplus_{\substack{k\in[0,l-1]\\ k\not\equiv_3 2}} R\Big) \overset{(1)}{=} R^{\oplus 2l+1+l-\underline{\underline{l}}} \overset{(\mathrm{i})}{=} R^{\oplus 3l-(\underline{\underline{l-3}})} = R^{\oplus m_{4l+1}}$$

$$\Big(\bigoplus_{k\in[0,l-1]} R\oplus R\Big)\oplus R\oplus R\oplus\Big(\bigoplus_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1}} R\Big) \overset{(1)}{=} R^{\oplus 2l+2+l-(\underline{\underline{l+1}})} \overset{(\mathrm{i})}{=} R^{\oplus 3l-(\underline{\underline{l-5}})} = R^{\oplus m_{4l+2}}$$

$$\Big(\bigoplus_{k\in[0,l]} R\oplus R\Big)\oplus\Big(\bigoplus_{\substack{k\in[0,l]\\ k\not\equiv_3 1}} R\Big) \overset{(1)}{=} R^{\oplus 2(l+1)+(l+1)-(\underline{\underline{l+2}})} \overset{(\mathrm{i})}{=} R^{\oplus 3l-(\underline{\underline{l-7}})} = R^{\oplus m_{4l+3}}$$

Definition 50 For $l \in \mathbb{Z}_{\geq 0}$ we define the following R-linear isomorphisms; cf. Definitions 25, 46 and 48 and Remarks 41 and 49. Note that $Y_0^* = 0$ by Definition 24 and Remark 45.

Using Remark 43.(2) we obtain in each case the existence of the maps $\pi_j^{m_{4l+i}}$ and $\iota_j^{m_{4l+i}}$ that occur in the defining formulas.

$$X_{4l}^* = \bigg(\Big(\bigoplus_{k\in[0,l-1]} P\oplus P\Big)\oplus P\oplus\Big(\bigoplus_{k\in[0,l-1]} Y_{\overline{\overline{k}}}\Big)\bigg)^* \xrightarrow[\sim]{\chi_{4l}^{\circledast}} R^{\oplus m_{4l}}$$

$$\chi_{4l}^{\circledast} := \Big(\sum_{k\in[0,l-1]} \iota_{4l,k}^{+,*}\,\chi_{P,P}\,\iota_{2k+1,2k+2}^{m_{4l}}\Big) + \Big(\iota_{4l}^{\prime,*}\,\chi_P\,\iota_{2l+1}^{m_{4l}}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0}} \iota_{4l,k}^{-,*}\,\chi_{\overline{\overline{k}}}\,\iota_{2l+2+\underline{\underline{\underline{2k-1}}}}^{m_{4l}}\Big)$$

$$X_{4l+1}^* = \bigg(\Big(\bigoplus_{k\in[0,l-1]} P\oplus P\Big)\oplus P\oplus Q\oplus\Big(\bigoplus_{k\in[0,l-1]} Y_{\overline{\overline{\overline{k+1}}}}\Big)\bigg)^* \xrightarrow[\sim]{\chi_{4l+1}^{\circledast}} R^{\oplus m_{4l+1}}$$

$$\chi_{4l+1}^{\circledast} := \Big(\sum_{k\in[0,l-1]} \iota_{4l+1,k}^{+,*}\,\chi_{P,P}\,\iota_{2k+1,2k+2}^{m_{4l+1}}\Big) + \Big(\iota_{4l+1}^{\prime,*}\,\chi_{P,Q}\,\iota_{2l+1}^{m_{4l+1}}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 2}} \iota_{4l+1,k}^{-,*}\,\chi_{\overline{\overline{\overline{k+1}}}}\,\iota_{2l+2+\underline{\underline{\underline{2k+1}}}}^{m_{4l+1}}\Big)$$

$$X_{4l+2}^* = \bigg(\Big(\bigoplus_{k\in[0,l-1]} P\oplus P\Big)\oplus P\oplus P\oplus Q\oplus\Big(\bigoplus_{k\in[0,l-1]} Y_{\overline{\overline{\overline{k+2}}}}\Big)\bigg)^* \xrightarrow[\sim]{\chi_{4l+2}^{\circledast}} R^{\oplus m_{4l+2}}$$

$$\chi_{4l+2}^{\circledast} := \Big(\sum_{k\in[0,l-1]} \iota_{4l+2,k}^{+,*}\,\chi_{P,P}\,\iota_{2k+1,2k+2}^{m_{4l+2}}\Big) + \Big(\iota_{4l+2}^{\prime,*}\,\chi_P\,\iota_{2l+1}^{m_{4l+2}}\Big) + \Big(\iota_{4l+2}^{\prime\prime,*}\,\chi_{P,Q}\,\iota_{2l+2}^{m_{4l+2}}\Big)$$
$$+ \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1}} \iota_{4l+2,k}^{-,*}\,\chi_{\overline{\overline{\overline{k+2}}}}\,\iota_{2l+3+\underline{\underline{\underline{2k}}}}^{m_{4l+2}}\Big)$$

$$X_{4l+3}^* = \bigg(\Big(\bigoplus_{k\in[0,l]} P\oplus P\Big)\oplus\Big(\bigoplus_{k\in[0,l]} Y_{\overline{\overline{\overline{k+2}}}}\Big)\bigg)^* \xrightarrow[\sim]{\chi_{4l+3}^{\circledast}} R^{\oplus m_{4l+3}}$$

$$\chi^{\circledast}_{4l+3} :=\Big(\sum_{k\in[0,l]} \iota^{+,*}_{4l+3,k}\, \chi_{P,P}\, \iota^{m_{4l+3}}_{2k+1,2k+2}\Big) + \Big(\sum_{\substack{k\in[0,l]\\ k\not\equiv_3 1}} \iota^{-,*}_{4l+3,k}\, \chi_{\overline{\overline{k+2}}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{\underline{2k}}}}\Big)$$

Lemma 51 Let $l \in \mathbb{Z}_{\geq 0}$.

We define the following R-linear maps; cf. Remark 47.

Using Remark 42.(i) and Remark 44 we obtain in each case the existence of the maps $\pi^{m_{4l+i}}_j$ and $\iota^{m_{4l+i}}_j$ that occur in the defining formulas.

$$R^{\oplus m_{4l}} \xrightarrow{d^\star_{4l}} R^{\oplus m_{4l+1}}$$

$$d^\star_0 = 0\,; \quad \text{cf. Remark 47}$$

$$\begin{aligned} d^\star_{4l} :=& \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+1,2k+2}\, A^\star\, \iota^{m_{4l+1}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-2]} \pi^{m_{4l}}_{2k+1,2k+2}\, C^\star\, \iota^{m_{4l+1}}_{2k+3,2k+4}\Big) \\ &+ \Big(\pi^{m_{4l}}_{2l-1,2l}\, \widetilde{C}^\star\, \iota^{m_{4l+1}}_{2l+1}\Big) \\ &+ \Big(\sum_{k\in[0,\underline{\underline{\underline{l-2}}}]} \pi^{m_{4l}}_{2l+2+2k}\, B^\star_2\, \iota^{m_{4l+1}}_{2l+3+2k}\Big) + \Big(\sum_{k\in[0,\underline{\underline{\underline{l-2}}}]} \pi^{m_{4l}}_{2l+2+2k}\, D^\star_5\, \iota^{m_{4l+1}}_{2l+2+2k}\Big), \qquad \text{for } l \geq 1 \end{aligned}$$

$$R^{\oplus m_{4l+1}} \xrightarrow{d^\star_{4l+1}} R^{\oplus m_{4l+2}}$$

$$d^\star_1 := \pi^1_1\, d'^{,\star}_1\, \iota^2_1$$

$$\begin{aligned} d^\star_{4l+1} :=& \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, \widetilde{A}^\star\, \iota^{m_{4l+2}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-2]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, C^\star\, \iota^{m_{4l+2}}_{2k+3,2k+4}\Big) \\ &+ \Big(\pi^{m_{4l+1}}_{2l-1,2l}\, C'^{,\star}\, \iota^{m_{4l+2}}_{2l+1}\Big) + \Big(\pi^{m_{4l+1}}_{2l+1}\, d'^{,\star}_1\, \iota^{m_{4l+2}}_{2l+1}\Big) \\ &+ \Big(\sum_{k\in[0,\underline{\underline{\underline{l-1}}}]} \pi^{m_{4l+1}}_{2l+2+2k}\, B^\star_2\, \iota^{m_{4l+2}}_{2l+3+2k}\Big) + \Big(\sum_{k\in[0,\underline{\underline{\underline{l-2}}}]} \pi^{m_{4l+1}}_{2l+3+2k}\, D^\star_0\, \iota^{m_{4l+2}}_{2l+3+2k}\Big), \qquad \text{for } l \geq 1 \end{aligned}$$

$$R^{\oplus m_{4l+2}} \xrightarrow{d^\star_{4l+2}} R^{\oplus m_{4l+3}}$$

$$\begin{aligned} d^\star_{4l+2} :=& \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, A^\star\, \iota^{m_{4l+3}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, C^\star\, \iota^{m_{4l+3}}_{2k+3,2k+4}\Big) \\ &+ \Big(\pi^{m_{4l+2}}_{2l+2}\, d''^{,\star}_2\, \iota^{m_{4l+3}}_{2l+1,2l+2}\Big) \\ &+ \Big(\sum_{k\in[0,\underline{\underline{l}}]} \pi^{m_{4l+2}}_{2l+2+2k}\, B^\star_2\, \iota^{m_{4l+3}}_{2l+3+2k}\Big) + \Big(\sum_{k\in[1,\underline{\underline{l}}]} \pi^{m_{4l+2}}_{2l+2+2k}\, D^\star_5\, \iota^{m_{4l+3}}_{2l+2+2k}\Big) \end{aligned}$$

$$R^{\oplus m_{4l+3}} \xrightarrow{d^\star_{4l+3}} R^{\oplus m_{4l+4}}$$

$$\begin{aligned} d^\star_{4l+3} :=& \Big(\sum_{k\in[0,l]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, \widetilde{A}^\star\, \iota^{m_{4l+4}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, C^\star\, \iota^{m_{4l+4}}_{2k+3,2k+4}\Big) \\ &+ \Big(\pi^{m_{4l+3}}_{2l+1,2l+2}\, C'^{,\star}\, \iota^{m_{4l+4}}_{2l+3}\Big) \\ &+ \Big(\sum_{k\in[0,\underline{\underline{\underline{l-2}}}]} \pi^{m_{4l+3}}_{2l+4+2k}\, B^\star_2\, \iota^{m_{4l+4}}_{2l+5+2k}\Big) + \Big(\sum_{k\in[0,\underline{\underline{l}}]} \pi^{m_{4l+3}}_{2l+3+2k}\, D^\star_0\, \iota^{m_{4l+4}}_{2l+3+2k}\Big) \end{aligned}$$

The following diagram commutes for all $n \in \mathbb{Z}_{\geq 0}$.

$$\begin{array}{ccc} X_n^* & \xrightarrow{\ d_n^*\ } & X_{n+1}^* \\ {\scriptstyle \chi_n^{\circledast}}\big\downarrow \wr & & \wr \big\downarrow {\scriptstyle \chi_{n+1}^{\circledast}} \\ R^{\oplus m_n} & \xrightarrow{\ d_n^{\star}\ } & R^{\oplus m_{n+1}} \end{array}$$

Proof. Throughout the proof we will use Definition 28 and Remarks 26, 41 and 47 without further comment. In particular, we will use $d_0^* = 0$, $d_1^{\prime\prime,*} = 0$, $d_2^{\prime,*} = 0$, $\widetilde{D}_1^* = 0$ and $D_2^* = 0$ as follows from Remark 47.

Consider the case $n = 4l$. We have the following for $l = 0$.

$$(\chi_0^{\circledast})^{-1}\, d_0^*\, \chi_1^{\circledast} = \chi_P^{-1}\, d_0^*\, \chi_{P,Q} = 0 = d_0^{\star}$$

Let $l \geq 1$. We calculate as follows.

$$\begin{aligned}
&(\chi_{4l}^{\circledast})^{-1}\, d_{4l}^*\, \chi_{4l+1}^{\circledast} \\
&= \Bigg(\Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, \pi^{+,*}_{4l,k}\Big) + \Big(\pi^{m_{4l}}_{2l+1}\, \chi_P^{-1}\, \pi'^{,*}_{4l}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k-1}}}}\, \chi^{-1}_{\overline{\overline{k}}}\, \pi^{-,*}_{4l,k}\Big)\Bigg) \\
&\quad \Bigg(\Big(\sum_{k\in[0,l-1]} \iota^{+,*}_{4l,k}\, A^*\, \pi^{+,*}_{4l+1,k}\Big) + \Big(\sum_{k\in[0,l-2]} \iota^{+,*}_{4l,k}\, C^*\, \pi^{+,*}_{4l+1,k+1}\Big) + \Big(\iota^{+,*}_{4l,l-1}\, \widetilde{C}^*\, \pi'^{,*}_{4l+1}\Big) + \Big(\iota'^{,*}_{4l}\, d_0^*\, \pi'^{,*}_{4l+1}\Big) \\
&\qquad + \Big(\iota^{-,*}_{4l,0}\, \widetilde{D}_1^*\, \pi'^{,*}_{4l+1}\Big) + \Big(\sum_{k\in[0,l-1]} \iota^{-,*}_{4l,k}\, B^*_{\overline{\overline{\overline{k+1}}}}\, \pi^{-,*}_{4l+1,k}\Big) + \Big(\sum_{k\in[0,l-2]} \iota^{-,*}_{4l,k+1}\, D^*_{\overline{\overline{\overline{4k+5}}}}\, \pi^{-,*}_{4l+1,k}\Big)\Bigg) \\
&\quad \Bigg(\Big(\sum_{k\in[0,l-1]} \iota^{+,*}_{4l+1,k}\, \chi_{P,P}\, \iota^{m_{4l+1}}_{2k+1,2k+2}\Big) + \Big(\iota'^{,*}_{4l+1}\, \chi_{P,Q}\, \iota^{m_{4l+1}}_{2l+1}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 2}} \iota^{-,*}_{4l+1,k}\, \chi_{\overline{\overline{\overline{k+1}}}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\Big)\Bigg)
\end{aligned}$$

$$\begin{aligned}
&= \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, A^*\, \chi_{P,P}\, \iota^{m_{4l+1}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-2]} \pi^{m_{4l}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, C^*\, \chi_{P,P}\, \iota^{m_{4l+1}}_{2k+3,2k+4}\Big) \\
&\quad + \Big(\pi^{m_{4l}}_{2l-1,2l}\, \chi^{-1}_{P,P}\, \widetilde{C}^*\, \chi_{P,Q}\, \iota^{m_{4l+1}}_{2l+1}\Big) \\
&\quad + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0,\ k\not\equiv_3 2}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k-1}}}}\, \chi^{-1}_{\overline{\overline{k}}}\, B^*_{\overline{\overline{\overline{k+1}}}}\, \chi_{\overline{\overline{\overline{k+1}}}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\Big) + \Big(\sum_{\substack{k\in[0,l-2]\\ k\not\equiv_3 2}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, \chi^{-1}_{\overline{\overline{\overline{k+1}}}}\, D^*_{\overline{\overline{\overline{4k+5}}}}\, \chi_{\overline{\overline{\overline{k+1}}}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\Big)
\end{aligned}$$

$$\begin{aligned}
&= \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+1,2k+2}\, A^{\star}\, \iota^{m_{4l+1}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-2]} \pi^{m_{4l}}_{2k+1,2k+2}\, C^{\star}\, \iota^{m_{4l+1}}_{2k+3,2k+4}\Big) \\
&\quad + \Big(\pi^{m_{4l}}_{2l-1,2l}\, \widetilde{C}^{\star}\, \iota^{m_{4l+1}}_{2l+1}\Big) \\
&\quad + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0,\ k\not\equiv_3 2}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k-1}}}}\, B^{\star}_{\overline{\overline{\overline{k+1}}}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\Big) + \Big(\sum_{\substack{k\in[0,l-2]\\ k\not\equiv_3 2}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, D^{\star}_{\overline{\overline{\overline{4k+5}}}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\Big)
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{(E1)}}{=} \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+1,2k+2}\, A^{\star}\, \iota^{m_{4l+1}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-2]} \pi^{m_{4l}}_{2k+1,2k+2}\, C^{\star}\, \iota^{m_{4l+1}}_{2k+3,2k+4}\Big) \\
&\quad + \Big(\pi^{m_{4l}}_{2l-1,2l}\, \widetilde{C}^{\star}\, \iota^{m_{4l+1}}_{2l+1}\Big) \\
&\quad + \Big(\sum_{k\in[0,\underline{\underline{\underline{l-2}}}]} \pi^{m_{4l}}_{2l+2+2k}\, B_2^{\star}\, \iota^{m_{4l+1}}_{2l+3+2k}\Big) + \Big(\sum_{k\in[0,\underline{\underline{\underline{l-2}}}]} \pi^{m_{4l}}_{2l+2+2k}\, D_5^{\star}\, \iota^{m_{4l+1}}_{2l+2+2k}\Big)
\end{aligned}$$

For equation (E1) we use Remark 42.(i, iv, v) to obtain the following.

$$\Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0,\ k\not\equiv_3 2}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k-1}}}}\, B^{\star}_{\overline{\overline{\overline{k+1}}}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\Big)$$

$$= \Big(\sum_{\substack{k\in[0,l-1]\\ k\equiv_3 1}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k-1}}}}\, B^\star_2\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}} \Big) = \Big(\sum_{\substack{k\in[-1,l-2]\\ k\equiv_3 0}} \pi^{m_{4l}}_{2l+3+\underline{\underline{\underline{2k-2}}}}\, B^\star_2\, \iota^{m_{4l+1}}_{2l+3+\underline{\underline{2k}}} \Big)$$

$$\overset{k'=\underline{\underline{k}}}{=} \Big(\sum_{k'\in[0,\underline{\underline{l-2}}\,]} \pi^{m_{4l}}_{2l+2+2k'}\, B^\star_2\, \iota^{m_{4l+1}}_{2l+3+2k'} \Big)$$

$$\Big(\sum_{\substack{k\in[0,l-2]\\ k\not\equiv_3 2}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, D^\star_{\overline{4k+5}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}} \Big)$$

$$= \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 0}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, D^\star_{\overline{4k+5}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}} \Big) + \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 1}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, D^\star_{\overline{4k+5}}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}} \Big)$$

$$= \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 0}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, D^\star_5\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}} \Big) + \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 1}} \pi^{m_{4l}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, \underbrace{D^\star_3}_{=0}\, \iota^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}} \Big)$$

$$= \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 0}} \pi^{m_{4l}}_{2l+3+\underline{\underline{\underline{2k-2}}}}\, D^\star_5\, \iota^{m_{4l+1}}_{2l+3+\underline{\underline{\underline{2k-2}}}} \Big) \overset{k'=\underline{\underline{k}}}{=} \Big(\sum_{k'\in[0,\underline{\underline{l-2}}\,]} \pi^{m_{4l}}_{2l+2+2k'}\, D^\star_5\, \iota^{m_{4l+1}}_{2l+2+2k'} \Big)$$

Consider the case $n = 4l+1$. We have the following for $l = 0$.

$$(\chi^{\circledast}_1)^{-1}\, d^*_1 \chi^{\circledast}_2 = \Big(\pi^{m_1}_1\, \chi^{-1}_{P,Q}\, \pi'^{,*}_1 \Big) \Big(\iota'^{,*}_1\, d'^{,*}_1\, \pi'^{,*}_2 + \iota'^{,*}_1\, d''^{,*}_1\, \pi''^{,*}_2 \Big) \Big(\iota'^{,*}_2\, \chi_P\, \iota^{m_2}_1 + \iota''^{,*}_2\, \chi_{P,Q}\, \iota^{m_2}_2 \Big)$$
$$= \Big(\pi^1_1\, \chi^{-1}_{P,Q}\, d'^{,*}_1\, \chi_P\, \iota^2_1 \Big) = \pi^1_1\, d'^{,\star}_1\, \iota^2_1$$

Let $l \geq 1$. We calculate as follows.

$$(\chi^{\circledast}_{4l+1})^{-1}\, d^*_{4l+1}\, \chi^{\circledast}_{4l+2}$$
$$= \bigg(\Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, \pi^{+,*}_{4l+1,k} \Big) + \Big(\pi^{m_{4l+1}}_{2l+1}\, \chi^{-1}_{P,Q}\, \pi'^{,*}_{4l+1} \Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 2}} \pi^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, \chi_{\overline{\overline{\overline{k+1}}}}^{-1}\, \pi^{-,*}_{4l+1,k} \Big) \bigg)$$
$$\bigg(\Big(\sum_{k\in[0,l-1]} \iota^{+,*}_{4l+1,k}\, \widetilde{A}^*\, \pi^{+,*}_{4l+2,k} \Big) + \Big(\sum_{k\in[0,l-2]} \iota^{+,*}_{4l+1,k}\, C^*\, \pi^{+,*}_{4l+2,k+1} \Big) + \Big(\iota^{+,*}_{4l+1,l-1}\, C'^{,*}\, \pi'^{,*}_{4l+2} \Big) + \Big(\iota'^{,*}_{4l+1}\, d'^{,*}_1\, \pi'^{,*}_{4l+2} \Big)$$
$$+ \Big(\iota'^{,*}_{4l+1}\, d''^{,*}_1\, \pi''^{,*}_{4l+2} \Big) + \Big(\iota^{-,*}_{4l+1,0}\, D^*_2\, \pi''^{,*}_{4l+2} \Big) + \Big(\sum_{k\in[0,l-1]} \iota^{-,*}_{4l+1,k}\, B^*_{\overline{\overline{\overline{k+2}}}}\, \pi^{-,*}_{4l+2,k} \Big) + \Big(\sum_{k\in[0,l-2]} \iota^{-,*}_{4l+1,k+1}\, D^*_{\overline{4k}}\, \pi^{-,*}_{4l+2,k} \Big) \bigg)$$
$$\bigg(\Big(\sum_{k\in[0,l-1]} \iota^{+,*}_{4l+2,k}\, \chi_{P,P}\, \iota^{m_{4l+2}}_{2k+1,2k+2} \Big) + \Big(\iota'^{,*}_{4l+2}\, \chi_P\, \iota^{m_{4l+2}}_{2l+1} \Big) + \Big(\iota''^{,*}_{4l+2}\, \chi_{P,Q}\, \iota^{m_{4l+2}}_{2l+2} \Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1}} \iota^{-,*}_{4l+2,k}\, \chi_{\overline{\overline{\overline{k+2}}}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}} \Big) \bigg)$$

$$= \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, \widetilde{A}^*\, \chi_{P,P}\, \iota^{m_{4l+2}}_{2k+1,2k+2} \Big) + \Big(\sum_{k\in[0,l-2]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, C^*\, \chi_{P,P}\, \iota^{m_{4l+2}}_{2k+3,2k+4} \Big)$$
$$+ \Big(\pi^{m_{4l+1}}_{2l-1,2l}\, \chi^{-1}_{P,P}\, C'^{,*}\, \chi_P\, \iota^{m_{4l+2}}_{2l+1} \Big) + \Big(\pi^{m_{4l+1}}_{2l+1}\, \chi^{-1}_{P,Q}\, d'^{,*}_1\, \chi_P\, \iota^{m_{4l+2}}_{2l+1} \Big)$$
$$+ \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1,\ k\not\equiv_3 2}} \pi^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, \chi_{\overline{\overline{\overline{k+1}}}}^{-1}\, B^*_{\overline{\overline{\overline{k+2}}}}\, \chi_{\overline{\overline{\overline{k+2}}}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}} \Big) + \Big(\sum_{\substack{k\in[0,l-2]\\ k\not\equiv_3 1}} \pi^{m_{4l+1}}_{2l+3+\underline{\underline{2k}}}\, \chi_{\overline{\overline{\overline{k+2}}}}^{-1}\, D^*_{\overline{4k}}\, \chi_{\overline{\overline{\overline{k+2}}}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}} \Big)$$

$$= \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, \widetilde{A}^\star\, \iota^{m_{4l+2}}_{2k+1,2k+2} \Big) + \Big(\sum_{k\in[0,l-2]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, C^\star\, \iota^{m_{4l+2}}_{2k+3,2k+4} \Big)$$
$$+ \Big(\pi^{m_{4l+1}}_{2l-1,2l}\, C'^{,\star}\, \iota^{m_{4l+2}}_{2l+1} \Big) + \Big(\pi^{m_{4l+1}}_{2l+1}\, d'^{,\star}_1\, \iota^{m_{4l+2}}_{2l+1} \Big)$$
$$+ \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1,\ k\not\equiv_3 2}} \pi^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, B^\star_{\overline{\overline{\overline{k+2}}}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}} \Big) + \Big(\sum_{\substack{k\in[0,l-2]\\ k\not\equiv_3 1}} \pi^{m_{4l+1}}_{2l+3+\underline{\underline{2k}}}\, D^\star_{\overline{4k}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}} \Big)$$

$$
\begin{aligned}
&\overset{(\mathrm{E2})}{=} \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, \widetilde{A}^{\star}\, \iota^{m_{4l+2}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-2]} \pi^{m_{4l+1}}_{2k+1,2k+2}\, C^{\star}\, \iota^{m_{4l+2}}_{2k+3,2k+4}\Big) \\
&\quad + \Big(\pi^{m_{4l+1}}_{2l-1,2l}\, C'^{,\star}\, \iota^{m_{4l+2}}_{2l+1}\Big) + \Big(\pi^{m_{4l+1}}_{2l+1}\, d_1'^{,\star}\, \iota^{m_{4l+2}}_{2l+1}\Big) \\
&\quad + \Big(\sum_{k\in[0,\underline{\underline{\underline{l-1}}}\,]} \pi^{m_{4l+1}}_{2l+2+2k}\, B^{\star}_2\, \iota^{m_{4l+2}}_{2l+3+2k}\Big) + \Big(\sum_{k\in[0,\underline{\underline{\underline{l-2}}}\,]} \pi^{m_{4l+1}}_{2l+3+2k}\, D^{\star}_0\, \iota^{m_{4l+2}}_{2l+3+2k}\Big)
\end{aligned}
$$

For equation (E2) we use Remark 42.(i, iv, v) to obtain the following.

$$
\begin{aligned}
&\Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1,\ k\not\equiv_3 2}} \pi^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, B^{\star}_{\overline{\overline{k+2}}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\Big) \\
&\qquad = \Big(\sum_{\substack{k\in[0,l-1]\\ k\equiv_3 0}} \pi^{m_{4l+1}}_{2l+2+\underline{\underline{\underline{2k+1}}}}\, B^{\star}_2\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\Big) = \Big(\sum_{\substack{k\in[0,l-1]\\ k\equiv_3 0}} \pi^{m_{4l+1}}_{2l+3+\underline{\underline{\underline{2k-2}}}}\, B^{\star}_2\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\Big) \\
&\qquad \overset{k'=\underline{\underline{k}}}{=} \Big(\sum_{k'\in[0,\underline{\underline{\underline{l-1}}}\,]} \pi^{m_{4l+1}}_{2l+2+2k'}\, B^{\star}_2\, \iota^{m_{4l+2}}_{2l+3+2k'}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\sum_{\substack{k\in[0,l-2]\\ k\not\equiv_3 1}} \pi^{m_{4l+1}}_{2l+3+\underline{\underline{2k}}}\, D^{\star}_{\overline{4k}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\Big) \\
&\qquad = \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 0}} \pi^{m_{4l+1}}_{2l+3+\underline{\underline{2k}}}\, D^{\star}_{\overline{4k}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\Big) + \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 2}} \pi^{m_{4l+1}}_{2l+3+\underline{\underline{2k}}}\, D^{\star}_{\overline{4k}}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\Big) \\
&\qquad = \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 0}} \pi^{m_{4l+1}}_{2l+3+\underline{\underline{2k}}}\, D^{\star}_0\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\Big) + \Big(\sum_{\substack{k\in[0,l-2]\\ k\equiv_3 2}} \pi^{m_{4l+1}}_{2l+3+\underline{\underline{2k}}}\, \underbrace{D^{\star}_2}_{=0}\, \iota^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\Big) \\
&\qquad \overset{k'=\underline{\underline{k}}}{=} \Big(\sum_{k'\in[0,\underline{\underline{\underline{l-2}}}\,]} \pi^{m_{4l+1}}_{2l+3+2k'}\, D^{\star}_0\, \iota^{m_{4l+2}}_{2l+3+2k'}\Big)
\end{aligned}
$$

Consider the case $n = 4l+2$. We calculate as follows.

$$
\begin{aligned}
&(\chi^{\circledast}_{4l+2})^{-1}\, d^*_{4l+2}\, \chi^{\circledast}_{4l+3} \\
&= \Bigg(\Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, \pi^{+,*}_{4l+2,k}\Big) + \Big(\pi^{m_{4l+2}}_{2l+1}\, \chi^{-1}_{P}\, \pi'^{,*}_{4l+2}\Big) + \Big(\pi^{m_{4l+2}}_{2l+2}\, \chi^{-1}_{P,Q}\, \pi''^{,*}_{4l+2}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, \chi^{-1}_{\overline{\overline{k+2}}}\, \pi^{-,*}_{4l+2,k}\Big)\Bigg) \\
&\quad \Bigg(\Big(\sum_{k\in[0,l-1]} \iota^{+,*}_{4l+2,k}\, A^*\, \pi^{+,*}_{4l+3,k}\Big) + \Big(\sum_{k\in[0,l-1]} \iota^{+,*}_{4l+2,k}\, C^*\, \pi^{+,*}_{4l+3,k+1}\Big) + \Big(\iota'^{,*}_{4l+2}\, d_2'^{,*}\, \pi^{+,*}_{4l+3,l}\Big) \\
&\quad + \Big(\iota''^{,*}_{4l+2}\, d_2''^{,*}\, \pi^{+,*}_{4l+3,l}\Big) + \Big(\iota''^{,*}_{4l+2}\, B^*_2\, \pi^{-,*}_{4l+3,0}\Big) + \Big(\sum_{k\in[0,l-1]} \iota^{-,*}_{4l+2,k}\, B^*_{\overline{\overline{k}}}\, \pi^{-,*}_{4l+3,k+1}\Big) + \Big(\sum_{k\in[0,l-1]} \iota^{-,*}_{4l+2,k}\, D^*_{\overline{4k+3}}\, \pi^{-,*}_{4l+3,k}\Big)\Bigg) \\
&\quad \Bigg(\Big(\sum_{k\in[0,l]} \iota^{+,*}_{4l+3,k}\, \chi_{P,P}\, \iota^{m_{4l+3}}_{2k+1,2k+2}\Big) + \Big(\sum_{\substack{k\in[0,l]\\ k\not\equiv_3 1}} \iota^{-,*}_{4l+3,k}\, \chi_{\overline{\overline{k+2}}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big)\Bigg) \\
&= \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, A^*\, \chi_{P,P}\, \iota^{m_{4l+3}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, C^*\, \chi_{P,P}\, \iota^{m_{4l+3}}_{2k+3,2k+4}\Big) \\
&\quad + \Big(\pi^{m_{4l+2}}_{2l+2}\, \chi^{-1}_{P,Q}\, d_2''^{,*}\, \chi_{P,P}\, \iota^{m_{4l+3}}_{2l+1,2l+2}\Big) + \Big(\pi^{m_{4l+2}}_{2l+2}\, \chi^{-1}_{P,Q}\, B^*_2\, \chi_{Q,P}\, \iota^{m_{4l+3}}_{2l+3}\Big) \\
&\quad + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0,\ k\not\equiv_3 1}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, \chi^{-1}_{\overline{\overline{k+2}}}\, B^*_{\overline{\overline{k}}}\, \chi_{\overline{\overline{k}}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{\underline{2k+2}}}}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, \chi^{-1}_{\overline{\overline{k+2}}}\, D^*_{\overline{4k+3}}\, \chi_{\overline{\overline{k+2}}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, A^\star\, \iota^{m_{4l+3}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, C^\star\, \iota^{m_{4l+3}}_{2k+3,2k+4}\Big)\\
&\quad + \Big(\pi^{m_{4l+2}}_{2l+2}\, d_2^{\prime\prime,\star}\, \iota^{m_{4l+3}}_{2l+1,2l+2}\Big) + \Big(\pi^{m_{4l+2}}_{2l+2}\, B_2^\star\, \iota^{m_{4l+3}}_{2l+3}\Big)\\
&\quad + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0,\ k\not\equiv_3 1}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, B^\star_{\underline{\underline{k}}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, D^\star_{\overline{4k+3}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big)\\
&\overset{(E3)}{=} \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, A^\star\, \iota^{m_{4l+3}}_{2k+1,2k+2}\Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+1,2k+2}\, C^\star\, \iota^{m_{4l+3}}_{2k+3,2k+4}\Big)\\
&\quad + \Big(\pi^{m_{4l+2}}_{2l+2}\, d_2^{\prime\prime,\star}\, \iota^{m_{4l+3}}_{2l+1,2l+2}\Big) + \Big(\sum_{k\in[0,\underline{l}]} \pi^{m_{4l+2}}_{2l+2+2k}\, B_2^\star\, \iota^{m_{4l+3}}_{2l+3+2k}\Big) + \Big(\sum_{k\in[1,\underline{l}]} \pi^{m_{4l+2}}_{2l+2+2k}\, D_5^\star\, \iota^{m_{4l+3}}_{2l+2+2k}\Big)
\end{aligned}
$$

For equation (E3) we use Remark 42, (i, iv, v) to obtain the following.

$$
\begin{aligned}
&\Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 0,\ k\not\equiv_3 1}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, B^\star_{\underline{\underline{k}}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\Big) + \Big(\pi^{m_{4l+2}}_{2l+2}\, B_2^\star\, \iota^{m_{4l+3}}_{2l+3}\Big)\\
&\qquad = \Big(\sum_{\substack{k\in[1,l]\\ k\equiv_3 0}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k-2}}}\, B^\star_{\overline{\overline{k-1}}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big) + \Big(\pi^{m_{4l+2}}_{2l+2}\, B_2^\star\, \iota^{m_{4l+3}}_{2l+3}\Big)\\
&\qquad = \Big(\sum_{\substack{k\in[0,l]\\ k\equiv_3 0}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k-2}}}\, B_2^\star\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big) \overset{k'=\underline{k}}{=} \Big(\sum_{k'\in[0,\underline{l}]} \pi^{m_{4l+2}}_{2l+2+2k'}\, B_2^\star\, \iota^{m_{4l+3}}_{2l+3+2k'}\Big)\\
&\Big(\sum_{\substack{k\in[0,l-1]\\ k\not\equiv_3 1}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, D^\star_{\overline{4k+3}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big)\\
&\qquad = \Big(\sum_{\substack{k\in[0,l-1]\\ k\equiv_3 0}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, D^\star_{\overline{4k+3}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\equiv_3 2}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, D^\star_{\overline{4k+3}}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big)\\
&\qquad = \Big(\sum_{\substack{k\in[0,l-1]\\ k\equiv_3 0}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, \underbrace{D_3^\star}_{=0}\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big) + \Big(\sum_{\substack{k\in[0,l-1]\\ k\equiv_3 2}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k}}}\, D_5^\star\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\Big)\\
&\qquad = \Big(\sum_{\substack{k\in[1,l]\\ k\equiv_3 0}} \pi^{m_{4l+2}}_{2l+3+\underline{\underline{2k-2}}}\, D_5^\star\, \iota^{m_{4l+3}}_{2l+3+\underline{\underline{2k-2}}}\Big) \overset{k'=\underline{k}}{=} \Big(\sum_{k'\in[1,\underline{l}]} \pi^{m_{4l+2}}_{2l+2+2k'}\, D_5^\star\, \iota^{m_{4l+3}}_{2l+2+2k'}\Big)
\end{aligned}
$$

Consider the case $n = 4l+3$. We calculate as follows.

$$
\begin{aligned}
&(\chi^{\circledast}_{4l+3})^{-1}\, d^*_{4l+3}\, \chi^{\circledast}_{4l+4}\\
&\quad = \Big(\Big(\sum_{k\in[0,l]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, \pi^{+,*}_{4l+3,k}\Big) + \Big(\sum_{\substack{k\in[0,l]\\ k\not\equiv_3 1}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\, \chi^{-1}_{\overline{\overline{k+2}}}\, \pi^{-,*}_{4l+3,k}\Big)\Big)\\
&\qquad \Big(\Big(\sum_{k\in[0,l]} \iota^{+,*}_{4l+3,k}\, \widetilde{A}^*\, \pi^{+,*}_{4l+4,k}\Big) + \Big(\sum_{k\in[0,l-1]} \iota^{+,*}_{4l+3,k}\, C^*\, \pi^{+,*}_{4l+4,k+1}\Big) + \Big(\iota^{+,*}_{4l+3,l}\, C^{\prime,*}\, \pi^{\prime,*}_{4l+4}\Big)\\
&\qquad + \Big(\iota^{-,*}_{4l+3,0}\, D_0^{\prime *}\, \pi^{\prime,*}_{4l+4}\Big) + \Big(\sum_{k\in[0,l]} \iota^{-,*}_{4l+3,k}\, B^*_{\underline{\underline{k}}}\, \pi^{-,*}_{4l+4,k}\Big) + \Big(\sum_{k\in[0,l-1]} \iota^{-,*}_{4l+3,k+1}\, D^*_{\overline{4k+4}}\, \pi^{-,*}_{4l+4,k}\Big)\Big)\\
&\qquad \Big(\Big(\sum_{k\in[0,l]} \iota^{+,*}_{4l+4,k}\, \chi_{P,P}\, \iota^{m_{4l+4}}_{2k+1,2k+2}\Big) + \Big(\iota^{\prime,*}_{4l+4}\, \chi_P\, \iota^{m_{4l+4}}_{2l+3}\Big) + \Big(\sum_{\substack{k\in[0,l]\\ k\not\equiv_3 0}} \iota^{-,*}_{4l+4,k}\, \chi_{\underline{\underline{k}}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\sum_{k\in[0,l]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, \widetilde{A}^{*}\, \chi_{P,P}\, \iota^{m_{4l+4}}_{2k+1,2k+2} \Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, \chi^{-1}_{P,P}\, C^{*}\, \chi_{P,P}\, \iota^{m_{4l+4}}_{2k+3,2k+4} \Big) \\
&\quad + \Big(\pi^{m_{4l+3}}_{2l+1,2l+2}\, \chi^{-1}_{P,P}\, C'^{,*}\, \chi_{P}\, \iota^{m_{4l+4}}_{2l+3} \Big) + \Big(\pi^{m_{4l+3}}_{2l+3}\, \chi^{-1}_{Q,P}\, D'^{*}_{0}\, \chi_{P}\, \iota^{m_{4l+4}}_{2l+3} \Big) \\
&\quad + \Big(\sum_{\substack{k\in[0,l] \\ k\not\equiv_3 0,\ k\not\equiv_3 1}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\, \chi^{-1}_{\overline{\overline{k+2}}}\, B^{*}_{\overline{\overline{k}}}\, \chi_{\overline{\overline{k}}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) + \Big(\sum_{\substack{k\in[0,l-1] \\ k\not\equiv_3 0}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\, \chi^{-1}_{\overline{\overline{k}}}\, D^{*}_{\overline{4k+4}}\, \chi_{\overline{\overline{k}}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) \\[1em]
&= \Big(\sum_{k\in[0,l]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, \widetilde{A}^{\star}\, \iota^{m_{4l+4}}_{2k+1,2k+2} \Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, C^{\star}\, \iota^{m_{4l+4}}_{2k+3,2k+4} \Big) \\
&\quad + \Big(\pi^{m_{4l+3}}_{2l+1,2l+2}\, C'^{,\star}\, \iota^{m_{4l+4}}_{2l+3} \Big) + \Big(\pi^{m_{4l+3}}_{2l+3}\, D^{\star}_{0}\, \iota^{m_{4l+4}}_{2l+3} \Big) \\
&\quad + \Big(\sum_{\substack{k\in[0,l] \\ k\not\equiv_3 0,\ k\not\equiv_3 1}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\, B^{\star}_{\overline{\overline{k}}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) + \Big(\sum_{\substack{k\in[0,l-1] \\ k\not\equiv_3 0}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\, D^{\star}_{\overline{4k+4}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) \\[1em]
&\overset{\text{(E4)}}{=} \Big(\sum_{k\in[0,l]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, \widetilde{A}^{\star}\, \iota^{m_{4l+4}}_{2k+1,2k+2} \Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+1,2k+2}\, C^{\star}\, \iota^{m_{4l+4}}_{2k+3,2k+4} \Big) \\
&\quad + \Big(\pi^{m_{4l+3}}_{2l+1,2l+2}\, C'^{,\star}\, \iota^{m_{4l+4}}_{2l+3} \Big) \\
&\quad + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+3}}_{2l+4+2k}\, B^{\star}_{2}\, \iota^{m_{4l+4}}_{2l+5+2k} \Big) + \Big(\sum_{k\in[0,\underline{l}]} \pi^{m_{4l+3}}_{2l+3+2k}\, D^{\star}_{0}\, \iota^{m_{4l+4}}_{2l+3+2k} \Big)
\end{aligned}
$$

For equation (E4) we use Remark 42.(i, iv, v) to obtain the following.

$$
\begin{aligned}
&\Big(\sum_{\substack{k\in[0,l] \\ k\not\equiv_3 0,\ k\not\equiv_3 1}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\, B^{\star}_{\overline{\overline{k}}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) \\
&\qquad = \Big(\sum_{\substack{k\in[0,l] \\ k\equiv_3 2}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\, B^{\star}_{2}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) = \Big(\sum_{\substack{k\in[-2,l-2] \\ k\equiv_3 0}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+4}}}\, B^{\star}_{2}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k+3}}} \Big) \\
&\qquad = \Big(\sum_{\substack{k\in[-2,l-2] \\ k\equiv_3 0}} \pi^{m_{4l+3}}_{2l+5+\underline{\underline{2k-2}}}\, B^{\star}_{2}\, \iota^{m_{4l+4}}_{2l+5+\underline{\underline{2k}}} \Big) \overset{k'=\underline{\underline{k}}}{=} \Big(\sum_{k'\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+3}}_{2l+4+2k'}\, B^{\star}_{2}\, \iota^{m_{4l+4}}_{2l+5+2k'} \Big) \\[1em]
&\Big(\sum_{\substack{k\in[0,l-1] \\ k\not\equiv_3 0}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\, D^{\star}_{\overline{4k+4}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) + \Big(\pi^{m_{4l+3}}_{2l+3}\, D^{\star}_{0}\, \iota^{m_{4l+4}}_{2l+3} \Big) \\
&\qquad = \Big(\sum_{\substack{k\in[-1,l-1] \\ k\not\equiv_3 0}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\, D^{\star}_{\overline{4k+4}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) \\
&\qquad = \Big(\sum_{\substack{k\in[-1,l-1] \\ k\equiv_3 1}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\, D^{\star}_{\overline{4k+4}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) + \Big(\sum_{\substack{k\in[-1,l-1] \\ k\equiv_3 2}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\, D^{\star}_{\overline{4k+4}}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) \\
&\qquad = \Big(\sum_{\substack{k\in[-1,l-1] \\ k\equiv_3 1}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\, \underbrace{D^{\star}_{2}}_{=0}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) + \Big(\sum_{\substack{k\in[-1,l-1] \\ k\equiv_3 2}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k+2}}}\, D^{\star}_{0}\, \iota^{m_{4l+4}}_{2l+4+\underline{\underline{2k-1}}} \Big) \\
&\qquad = \Big(\sum_{\substack{k\in[0,l] \\ k\equiv_3 0}} \pi^{m_{4l+3}}_{2l+3+\underline{\underline{2k}}}\, D^{\star}_{0}\, \iota^{m_{4l+4}}_{2l+3+\underline{\underline{2k}}} \Big) \overset{k'=\underline{\underline{k}}}{=} \Big(\sum_{k'\in[0,\underline{l}]} \pi^{m_{4l+3}}_{2l+3+2k'}\, D^{\star}_{0}\, \iota^{m_{4l+4}}_{2l+3+2k'} \Big)
\end{aligned}
$$

□

Remark 52 The maps from Lemma 51 can be rewritten using Remark 47.

Recall that given an integer z, we write (z) for the (1×1)-matrix with entry z.

For $l \in \mathbb{Z}_{\geq 0}$ we have the following.

$$\begin{aligned} d^\star_{4l} =& \Big(\sum_{k \in [0,l-1]} \pi^{m_{4l}}_{2k+2}\ (-2)\ \iota^{m_{4l+1}}_{2k+2} \Big) + \Big(\sum_{k \in [0,l-1]} \pi^{m_{4l}}_{2k+2}\ (4)\ \iota^{m_{4l+1}}_{2k+3} \Big) \\ &+ \Big(\sum_{k \in [0,\underline{\underline{l-2}}\,]} \pi^{m_{4l}}_{2l+2+2k}\ (-2)\ \iota^{m_{4l+1}}_{2l+3+2k} \Big) + \Big(\sum_{k \in [0,\underline{\underline{l-2}}\,]} \pi^{m_{4l}}_{2l+2+2k}\ (4)\ \iota^{m_{4l+1}}_{2l+2+2k} \Big) : \\ & R^{\oplus m_{4l}} \longrightarrow R^{\oplus m_{4l+1}} \end{aligned}$$

$$\begin{aligned} d^\star_{4l+1} =& \Big(\sum_{k \in [0,l]} \pi^{m_{4l+1}}_{2k+1}\ (2)\ \iota^{m_{4l+2}}_{2k+1} \Big) + \Big(\sum_{k \in [0,l-1]} \pi^{m_{4l+1}}_{2k+2}\ (4)\ \iota^{m_{4l+2}}_{2k+3} \Big) \\ &+ \Big(\sum_{k \in [0,\underline{\underline{l-1}}\,]} \pi^{m_{4l+1}}_{2l+2+2k}\ (-2)\ \iota^{m_{4l+2}}_{2l+3+2k} \Big) + \Big(\sum_{k \in [0,\underline{\underline{l-2}}\,]} \pi^{m_{4l+1}}_{2l+3+2k}\ (-4)\ \iota^{m_{4l+2}}_{2l+3+2k} \Big) : \\ & R^{\oplus m_{4l+1}} \longrightarrow R^{\oplus m_{4l+2}} \end{aligned}$$

$$\begin{aligned} d^\star_{4l+2} =& \Big(\sum_{k \in [0,l]} \pi^{m_{4l+2}}_{2k+2}\ (-2)\ \iota^{m_{4l+3}}_{2k+2} \Big) + \Big(\sum_{k \in [0,l-1]} \pi^{m_{4l+2}}_{2k+2}\ (4)\ \iota^{m_{4l+3}}_{2k+3} \Big) \\ &+ \Big(\sum_{k \in [0,\underline{l}\,]} \pi^{m_{4l+2}}_{2l+2+2k}\ (-2)\ \iota^{m_{4l+3}}_{2l+3+2k} \Big) + \Big(\sum_{k \in [1,\underline{l}\,]} \pi^{m_{4l+2}}_{2l+2+2k}\ (4)\ \iota^{m_{4l+3}}_{2l+2+2k} \Big) : \\ & R^{\oplus m_{4l+2}} \longrightarrow R^{\oplus m_{4l+3}} \end{aligned}$$

$$\begin{aligned} d^\star_{4l+3} =& \Big(\sum_{k \in [0,l]} \pi^{m_{4l+3}}_{2k+1}\ (2)\ \iota^{m_{4l+4}}_{2k+1} \Big) + \Big(\sum_{k \in [0,l]} \pi^{m_{4l+3}}_{2k+2}\ (4)\ \iota^{m_{4l+4}}_{2k+3} \Big) \\ &+ \Big(\sum_{k \in [0,\underline{\underline{l-2}}\,]} \pi^{m_{4l+3}}_{2l+4+2k}\ (-2)\ \iota^{m_{4l+4}}_{2l+5+2k} \Big) + \Big(\sum_{k \in [0,\underline{l}\,]} \pi^{m_{4l+3}}_{2l+3+2k}\ (-4)\ \iota^{m_{4l+4}}_{2l+3+2k} \Big) : \\ & R^{\oplus m_{4l+3}} \longrightarrow R^{\oplus m_{4l+4}} \end{aligned}$$

Lemma 53 Let $l \in \mathbb{Z}_{\geq 0}$. There exist R-linear isomorphisms $R^{\oplus m_{4l+1}} \xrightarrow{T_{4l}} R^{\oplus m_{4l+1}}$ and $R^{\oplus m_{4l+3}} \xrightarrow{T_{4l+2}} R^{\oplus m_{4l+3}}$ such that the following holds.

$$\begin{aligned} \operatorname{Ker}(d^\star_{4l} \cdot T_{4l}) &= \langle \mathrm{e}_{2k-1} : k \in [1, l+1+\underline{l}\,] \rangle \\ \operatorname{Im}(d^\star_{4l} \cdot T_{4l}) &= \langle 2\,\mathrm{e}_{2k} : k \in [1, l] \rangle \oplus \langle 2\,\mathrm{e}_{2k+1} : k \in [l+1, l+1+\underline{\underline{l-2}}\,] \rangle \end{aligned}$$

$$\begin{aligned} \operatorname{Ker}(T^{-1}_{4l} \cdot d^\star_{4l+1}) &= \langle \mathrm{e}_{2k} : k \in [1, l] \rangle \oplus \langle \mathrm{e}_{2k+1} : k \in [l+1, l+1+\underline{\underline{l-2}}\,] \rangle \\ \operatorname{Im}(T^{-1}_{4l} \cdot d^\star_{4l+1}) &= \langle 2\,\mathrm{e}_{2k-1} : k \in [1, l+2+\underline{\underline{l-1}}\,] \rangle \end{aligned}$$

$$\begin{aligned} \operatorname{Ker}(d^\star_{4l+2} \cdot T_{4l+2}) &= \langle \mathrm{e}_{2k-1} : k \in [1, l+2+\underline{\underline{l-1}}\,] \rangle \\ \operatorname{Im}(d^\star_{4l+2} \cdot T_{4l+2}) &= \langle 2\,\mathrm{e}_{2k} : k \in [1, l+1] \rangle \oplus \langle 2\,\mathrm{e}_{2k+1} : k \in [l+2, l+1+\underline{l}\,] \rangle \end{aligned}$$

$$\begin{aligned} \operatorname{Ker}(T^{-1}_{4l+2} \cdot d^\star_{4l+3}) &= \langle \mathrm{e}_{2k} : k \in [1, l+1] \rangle \oplus \langle \mathrm{e}_{2k+1} : k \in [l+2, l+1+\underline{l}\,] \rangle \\ \operatorname{Im}(T^{-1}_{4l+2} \cdot d^\star_{4l+3}) &= \langle 2\,\mathrm{e}_{2k-1} : k \in [1, l+3+\underline{\underline{l-2}}\,] \backslash \{l+2\} \rangle \oplus \langle 4\,\mathrm{e}_{2l+3} \rangle \end{aligned}$$

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & R^{\oplus m_{4l}} & \xrightarrow{d^\star_{4l}} & R^{\oplus m_{4l+1}} & \xrightarrow{d^\star_{4l+1}} & R^{\oplus m_{4l+2}} & \xrightarrow{d^\star_{4l+2}} & R^{\oplus m_{4l+3}} & \xrightarrow{d^\star_{4l+3}} & R^{\oplus m_{4l+4}} & \longrightarrow \cdots \\
& \downarrow 1 & & T_{4l} \downarrow \wr & & \downarrow 1 & & T_{4l+2} \downarrow \wr & & \downarrow 1 & \\
\cdots \longrightarrow & R^{\oplus m_{4l}} & \xrightarrow{d^\star_{4l}\cdot T_{4l}} & R^{\oplus m_{4l+1}} & \xrightarrow{T_{4l}^{-1}\cdot d^\star_{4l+1}} & R^{\oplus m_{4l+2}} & \xrightarrow{d^\star_{4l+2}\cdot T_{4l+2}} & R^{\oplus m_{4l+3}} & \xrightarrow{T_{4l+2}^{-1}\cdot d^\star_{4l+3}} & R^{\oplus m_{4l+4}} & \longrightarrow \cdots
\end{array}$$

Proof. Using Remark 42.(i) and Remark 44, we see that the occurring standard basis elements actually lie in the source respectively in the target of the map under consideration.

We define the following R-linear maps.

$$R^{\oplus m_{4l+1}} \xrightarrow{T_{4l}} R^{\oplus m_{4l+1}}$$

$$T_{4l} := \mathrm{I}_{m_{4l+1}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2}\ (2)\ \iota^{m_{4l+1}}_{2k+3} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+1}}_{2l+3+2k}\ (2)\ \iota^{m_{4l+1}}_{2l+2+2k} \Big)$$

$$R^{\oplus m_{4l+1}} \xrightarrow{Q_{4l+1}} R^{\oplus m_{4l+1}}$$

$$Q_{4l+1} := \mathrm{I}_{m_{4l+1}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2}\ (-2)\ \iota^{m_{4l+1}}_{2k+3} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+1}}_{2l+3+2k}\ (-2)\ \iota^{m_{4l+1}}_{2l+2+2k} \Big)$$

$$R^{\oplus m_{4l+3}} \xrightarrow{T_{4l+2}} R^{\oplus m_{4l+3}}$$

$$\begin{aligned} T_{4l+2} := \mathrm{I}_{m_{4l+3}} &+ \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+2}\ (2)\ \iota^{m_{4l+3}}_{2k+3} \Big) + \Big(\pi^{m_{4l+3}}_{2l+2}\ (-1)\ \iota^{m_{4l+3}}_{2l+3} \Big) \\ &+ \Big(\sum_{k\in[1,\underline{l}]} \pi^{m_{4l+3}}_{2l+3+2k}\ (2)\ \iota^{m_{4l+3}}_{2l+2+2k} \Big) \end{aligned}$$

$$R^{\oplus m_{4l+3}} \xrightarrow{Q_{4l+3}} R^{\oplus m_{4l+3}}$$

$$\begin{aligned} Q_{4l+3} := \mathrm{I}_{m_{4l+3}} &+ \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+2}\ (-2)\ \iota^{m_{4l+3}}_{2k+3} \Big) + \Big(\pi^{m_{4l+3}}_{2l+2}\ (1)\ \iota^{m_{4l+3}}_{2l+3} \Big) \\ &+ \Big(\sum_{k\in[1,\underline{l}]} \pi^{m_{4l+3}}_{2l+3+2k}\ (-2)\ \iota^{m_{4l+3}}_{2l+2+2k} \Big) \end{aligned}$$

Using Remark 42.(i) and Remark 44, we see that these maps are well-defined.

Recall that for $m \geq 0$, endomorphisms φ and ψ of $R^{\oplus m}$ are already mutually inverse, if $\varphi\psi = \mathrm{I}_m$.

We show that T_{4l} and Q_{4l+1} are mutually inverse isomorphisms.

$$\begin{aligned} T_{4l}\cdot Q_{4l+1} = &\Big(\mathrm{I}_{m_{4l+1}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2}\ (2)\ \iota^{m_{4l+1}}_{2k+3} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+1}}_{2l+3+2k}\ (2)\ \iota^{m_{4l+1}}_{2l+2+2k} \Big) \Big) \\ &\Big(\mathrm{I}_{m_{4l+1}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2}\ (-2)\ \iota^{m_{4l+1}}_{2k+3} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+1}}_{2l+3+2k}\ (-2)\ \iota^{m_{4l+1}}_{2l+2+2k} \Big) \Big) \end{aligned}$$

$$= \mathrm{I}_{m_{4l+1}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2} \; \big((2)+(-2)\big) \; \iota^{m_{4l+1}}_{2k+3} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+1}}_{2l+3+2k} \; \big((2)+(-2)\big) \; \iota^{m_{4l+1}}_{2l+2+2k} \Big)$$

$$= \mathrm{I}_{m_{4l+1}}$$

We show that T_{4l+2} and Q_{4l+3} are mutually inverse isomorphisms.

$$T_{4l+2} \cdot Q_{4l+3}$$

$$= \Big(\mathrm{I}_{m_{4l+3}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+2} \; (2) \; \iota^{m_{4l+3}}_{2k+3} \Big) + \Big(\pi^{m_{4l+3}}_{2l+2} \; (-1) \; \iota^{m_{4l+3}}_{2l+3} \Big) + \Big(\sum_{k\in[1,\underline{l}]} \pi^{m_{4l+3}}_{2l+3+2k} \; (2) \; \iota^{m_{4l+3}}_{2l+2+2k} \Big) \Big)$$

$$\Big(\mathrm{I}_{m_{4l+3}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+2} \; (-2) \; \iota^{m_{4l+3}}_{2k+3} \Big) + \Big(\pi^{m_{4l+3}}_{2l+2} \; (1) \; \iota^{m_{4l+3}}_{2l+3} \Big) + \Big(\sum_{k\in[1,\underline{l}]} \pi^{m_{4l+3}}_{2l+3+2k} \; (-2) \; \iota^{m_{4l+3}}_{2l+2+2k} \Big) \Big)$$

$$= \mathrm{I}_{m_{4l+3}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+2} \; \big((2)+(-2)\big) \; \iota^{m_{4l+3}}_{2k+3} \Big) + \Big(\pi^{m_{4l+3}}_{2l+2} \; \big((-1)+(1)\big) \; \iota^{m_{4l+3}}_{2l+3} \Big)$$

$$+ \Big(\sum_{k\in[1,\underline{l}]} \pi^{m_{4l+3}}_{2l+3+2k} \; \big((2)+(-2)\big) \; \iota^{m_{4l+3}}_{2l+2+2k} \Big)$$

$$= \mathrm{I}_{m_{4l+3}}$$

Consider the map $d^{\star}_{4l}$. We calculate as follows.

$$d^{\star}_{4l} \cdot T_{4l} = \Big(\Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+2} \; (-2) \; \iota^{m_{4l+1}}_{2k+2} \Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+2} \; (4) \; \iota^{m_{4l+1}}_{2k+3} \Big)$$

$$+ \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l}}_{2l+2+2k} \; (-2) \; \iota^{m_{4l+1}}_{2l+3+2k} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l}}_{2l+2+2k} \; (4) \; \iota^{m_{4l+1}}_{2l+2+2k} \Big) \Big)$$

$$\Big(\mathrm{I}_{m_{4l+1}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2} \; (2) \; \iota^{m_{4l+1}}_{2k+3} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+1}}_{2l+3+2k} \; (2) \; \iota^{m_{4l+1}}_{2l+2+2k} \Big) \Big)$$

$$= \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+2} \; (-2) \; \iota^{m_{4l+1}}_{2k+2} \Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l}}_{2k+2} \; \big((-2)\cdot(2)+(4)\big) \; \iota^{m_{4l+1}}_{2k+3} \Big)$$

$$+ \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l}}_{2l+2+2k} \; (-2) \; \iota^{m_{4l+1}}_{2l+3+2k} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l}}_{2l+2+2k} \; \big((-2)\cdot(2)+(4)\big) \; \iota^{m_{4l+1}}_{2l+2+2k} \Big)$$

$$= \Big(\sum_{k\in[1,l]} \pi^{m_{4l}}_{2k} \; (-2) \; \iota^{m_{4l+1}}_{2k} \Big) + \Big(\sum_{k\in[l+1,l+1+\underline{\underline{l-2}}]} \pi^{m_{4l}}_{2k} \; (-2) \; \iota^{m_{4l+1}}_{2k+1} \Big)$$

We obtain the following.

$$\mathrm{Ker}(d^{\star}_{4l} \cdot T_{4l}) = \langle \mathrm{e}_{2k-1} : k \in [1, l+1+\underline{\underline{l-2}}] \rangle \oplus \langle \mathrm{e}_k : k \in [2l+3+2(\underline{\underline{l-2}}), m_{4l}] \rangle$$
$$\mathrm{Im}(d^{\star}_{4l} \cdot T_{4l}) = \langle 2\,\mathrm{e}_{2k} : k \in [1,l] \rangle \oplus \langle 2\,\mathrm{e}_{2k+1} : k \in [l+1, l+1+\underline{\underline{l-2}}] \rangle$$

Using Remark 42.(vi) we have

$$m_{4l} - 2l - 3 - 2(\underline{\underline{l-2}}) = 3l - (\underline{\underline{l-1}}) - 2l - 3 - 2(\underline{\underline{l-2}}) = l - 3 - (\underline{\underline{l-1}}) - 2(\underline{\underline{l-2}})$$
$$= -1 + \underline{l} + \underline{\underline{l-1}} + \underline{\underline{l-2}} - (\underline{\underline{l-1}}) - 2(\underline{\underline{l-2}}) = -1 + \underline{l} - (\underline{\underline{l-2}})$$

so that $|\{\mathrm{e}_k : k \in [2l+3+2(\underline{\underline{l-2}}), m_{4l}\,]\}| = \underline{l} - (\underline{\underline{l-2}}) \in \{0,1\}$.

Note that $\mathrm{e}_{2l+3+2(\underline{\underline{l-2}})} = \mathrm{e}_{2k-1}$ for $k = l+2+\underline{\underline{l-2}}$. Hence

$$\begin{aligned}\mathrm{Ker}(d^{\star}_{4l} \cdot T_{4l}) &= \langle \mathrm{e}_{2k-1} : k \in [1, l+1+\underline{\underline{l-2}} + \underline{l} - (\underline{\underline{l-2}})]\rangle \\ &= \langle \mathrm{e}_{2k-1} : k \in [1, l+1+\underline{l}\,]\rangle.\end{aligned}$$

Consider the map $d^{\star}_{4l+1}$. We calculate as follows.

$$\begin{aligned}T_{4l}^{-1} \cdot d^{\star}_{4l+1} =& \Bigg(\mathrm{I}_{m_{4l+1}} + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2}\ (-2)\ \iota^{m_{4l+1}}_{2k+3} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}\,]} \pi^{m_{4l+1}}_{2l+3+2k}\ (-2)\ \iota^{m_{4l+1}}_{2l+2+2k} \Big) \Bigg) \\ & \Bigg(\Big(\sum_{k\in[0,l]} \pi^{m_{4l+1}}_{2k+1}\ (2)\ \iota^{m_{4l+2}}_{2k+1} \Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2}\ (4)\ \iota^{m_{4l+2}}_{2k+3} \Big) \\ & + \Big(\sum_{k\in[0,\underline{\underline{l-1}}\,]} \pi^{m_{4l+1}}_{2l+2+2k}\ (-2)\ \iota^{m_{4l+2}}_{2l+3+2k} \Big) + \Big(\sum_{k\in[0,\underline{\underline{l-2}}\,]} \pi^{m_{4l+1}}_{2l+3+2k}\ (-4)\ \iota^{m_{4l+2}}_{2l+3+2k} \Big) \Bigg) \\ =& \Big(\sum_{k\in[0,l]} \pi^{m_{4l+1}}_{2k+1}\ (2)\ \iota^{m_{4l+2}}_{2k+1} \Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+1}}_{2k+2}\ \big((-2)\cdot(2)+(4)\big)\ \iota^{m_{4l+2}}_{2k+3} \Big) \\ & + \Big(\sum_{k\in[0,\underline{\underline{l-1}}\,]} \pi^{m_{4l+1}}_{2l+2+2k}\ (-2)\ \iota^{m_{4l+2}}_{2l+3+2k} \Big) \\ & + \Big(\sum_{k\in[0,\underline{\underline{l-2}}\,]} \pi^{m_{4l+1}}_{2l+3+2k}\ \big((-2)\cdot(-2)+(-4)\big)\ \iota^{m_{4l+2}}_{2l+3+2k} \Big) \\ =& \Big(\sum_{k\in[1,l+1]} \pi^{m_{4l+1}}_{2k-1}\ (2)\ \iota^{m_{4l+2}}_{2k-1} \Big) + \Big(\sum_{k\in[l+2,l+2+\underline{\underline{l-1}}\,]} \pi^{m_{4l+1}}_{2k-2}\ (-2)\ \iota^{m_{4l+2}}_{2k-1} \Big)\end{aligned}$$

We obtain the following.

$$\begin{aligned}\mathrm{Ker}(T_{4l}^{-1} \cdot d^{\star}_{4l+1}) &= \langle \mathrm{e}_{2k} : k \in [1,l]\rangle \oplus \langle \mathrm{e}_{2k+1} : k \in [l+1, l+\underline{\underline{l-1}}\,]\rangle \\ &\quad \oplus \langle \mathrm{e}_k : k \in [2l+3+2(\underline{\underline{l-1}}), m_{4l+1}\,]\rangle \\ \mathrm{Im}(T_{4l}^{-1} \cdot d^{\star}_{4l+1}) &= \langle 2\,\mathrm{e}_{2k-1} : k \in [1, l+2+\underline{\underline{l-1}}\,]\rangle\end{aligned}$$

Using Remark 42.(i,vi) we have

$$\begin{aligned}m_{4l+1} - 2l - 3 - 2(\underline{\underline{l-1}}) &= 3l - (\underline{\underline{l-3}}) - 2l - 3 - 2(\underline{\underline{l-1}}) = l - 2 - \underline{l} - 2(\underline{\underline{l-1}}) \\ &= \underline{l} + \underline{\underline{l-1}} + \underline{\underline{l-2}} - \underline{l} - 2(\underline{\underline{l-1}}) = \underline{\underline{l-2}} - (\underline{\underline{l-1}})\end{aligned}$$

so that $|\{\mathrm{e}_k : k \in [2l+3+2(\underline{\underline{l-1}}), m_{4l+1}\,]\}| = 1 + \underline{\underline{l-2}} - (\underline{\underline{l-1}}) \in \{0,1\}$.

Note that $\mathrm{e}_{2l+3+2(\underline{\underline{l-1}})} = \mathrm{e}_{2k+1}$ for $k = l+1+\underline{\underline{l-1}}$. Hence

$$\begin{aligned}\mathrm{Ker}(T_{4l}^{-1} \cdot d^{\star}_{4l+1}) &= \langle \mathrm{e}_{2k} : k \in [1,l]\rangle \oplus \langle \mathrm{e}_{2k+1} : k \in [l+1, l+\underline{\underline{l-1}}+1+\underline{\underline{l-2}} - (\underline{\underline{l-1}})]\rangle \\ &= \langle \mathrm{e}_{2k} : k \in [1,l]\rangle \oplus \langle \mathrm{e}_{2k+1} : k \in [l+1, l+1+\underline{\underline{l-2}}\,]\rangle.\end{aligned}$$

Consider the map $d^{\star}_{4l+2}$. We calculate as follows.

$$d^{\star}_{4l+2} \cdot T_{4l+2} = \Bigg(\Big(\sum_{k\in[0,l]} \pi^{m_{4l+2}}_{2k+2}\ (-2)\ \iota^{m_{4l+3}}_{2k+2} \Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+2}\ (4)\ \iota^{m_{4l+3}}_{2k+3} \Big)$$

$$
\begin{aligned}
&+\Big(\sum_{k\in[0,\underline{\underline{l}}\,]} \pi^{m_{4l+2}}_{2l+2+2k}\ (-2)\ \iota^{m_{4l+3}}_{2l+3+2k}\Big)+\Big(\sum_{k\in[1,\underline{\underline{l}}\,]} \pi^{m_{4l+2}}_{2l+2+2k}\ (4)\ \iota^{m_{4l+3}}_{2l+2+2k}\Big)\Big)\\
&\Big(\mathrm{I}_{m_{4l+3}}+\Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+2}\ (2)\ \iota^{m_{4l+3}}_{2k+3}\Big)+\Big(\pi^{m_{4l+3}}_{2l+2}\ (-1)\ \iota^{m_{4l+3}}_{2l+3}\Big)\\
&+\Big(\sum_{k\in[1,\underline{\underline{l}}\,]} \pi^{m_{4l+3}}_{2l+3+2k}\ (2)\ \iota^{m_{4l+3}}_{2l+2+2k}\Big)\Big)\\
=&\Big(\sum_{k\in[0,l]} \pi^{m_{4l+2}}_{2k+2}\ (-2)\ \iota^{m_{4l+3}}_{2k+2}\Big)+\Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+2}}_{2k+2}\ \big((-2)\cdot(2)+(4)\big)\ \iota^{m_{4l+3}}_{2k+3}\Big)\\
&+\Big(\pi^{m_{4l+2}}_{2l+2}\ \big((-2)\cdot(-1)+(-2)\big)\ \iota^{m_{4l+3}}_{2l+3}\Big)\\
&+\Big(\sum_{k\in[1,\underline{\underline{l}}\,]} \pi^{m_{4l+2}}_{2l+2+2k}\ (-2)\ \iota^{m_{4l+3}}_{2l+3+2k}\Big)+\Big(\sum_{k\in[1,\underline{\underline{l}}\,]} \pi^{m_{4l+2}}_{2l+2+2k}\ \big((-2)\cdot(2)+(4)\big)\ \iota^{m_{4l+3}}_{2l+2+2k}\Big)\\
=&\Big(\sum_{k\in[1,l+1]} \pi^{m_{4l+2}}_{2k}\ (-2)\ \iota^{m_{4l+3}}_{2k}\Big)+\Big(\sum_{k\in[l+2,l+1+\underline{\underline{l}}\,]} \pi^{m_{4l+2}}_{2k}\ (-2)\ \iota^{m_{4l+3}}_{2k+1}\Big)
\end{aligned}
$$

We obtain the following.

$$
\begin{aligned}
\mathrm{Ker}(d^\star_{4l+2}\cdot T_{4l+2}) &= \langle \mathrm{e}_{2k-1} : k\in[1,l+1+\underline{\underline{l}}\,]\rangle \oplus \langle \mathrm{e}_k : k\in[2l+3+2\underline{\underline{l}}, m_{4l+2}\,]\rangle\\
\mathrm{Im}(d^\star_{4l+2}\cdot T_{4l+2}) &= \langle 2\,\mathrm{e}_{2k} : k\in[1,l+1]\rangle \oplus \langle 2\,\mathrm{e}_{2k+1} : k\in[l+2,l+1+\underline{\underline{l}}\,]\rangle
\end{aligned}
$$

Using Remark 42.(vi) we have

$$
\begin{aligned}
m_{4l+2}-2l-3-2\underline{\underline{l}} &= 3l-(\underline{\underline{l-5}})-2l-3-2\underline{\underline{l}} = l-2-(\underline{\underline{l-2}})-2\underline{\underline{l}}\\
&= \underline{\underline{l}}+\underline{\underline{l-1}}+\underline{\underline{l-2}}-(\underline{\underline{l-2}})-2\underline{\underline{l}} = \underline{\underline{l-1}}-\underline{\underline{l}}
\end{aligned}
$$

so that $|\{\mathrm{e}_k : k\in[2l+3+2\underline{\underline{l}}, m_{4l+2}\,]\}| = 1+\underline{\underline{l-1}}-\underline{\underline{l}}\in\{0,1\}$.
Note that $\mathrm{e}_{2l+3+2\underline{\underline{l}}} = \mathrm{e}_{2k-1}$ for $k = l+2+\underline{\underline{l}}$. Hence

$$
\begin{aligned}
\mathrm{Ker}(d^\star_{4l+2}\cdot T_{4l+2}) &= \langle \mathrm{e}_{2k-1} : k\in[1,l+1+\underline{\underline{l}}+1+\underline{\underline{l-1}}-\underline{\underline{l}}\,]\rangle\\
&= \langle \mathrm{e}_{2k-1} : k\in[1,l+2+\underline{\underline{l-1}}\,]\rangle.
\end{aligned}
$$

Consider the map $d^\star_{4l+3}$. We calculate as follows.

$$
\begin{aligned}
T^{-1}_{4l+2}\cdot d^\star_{4l+3} =&\Big(\mathrm{I}_{m_{4l+3}}+\Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+2}\ (-2)\ \iota^{m_{4l+3}}_{2k+3}\Big)+\Big(\pi^{m_{4l+3}}_{2l+2}\ (1)\ \iota^{m_{4l+3}}_{2l+3}\Big)\\
&+\Big(\sum_{k\in[1,\underline{\underline{l}}\,]} \pi^{m_{4l+3}}_{2l+3+2k}\ (-2)\ \iota^{m_{4l+3}}_{2l+2+2k}\Big)\Big)\\
&\Big(\Big(\sum_{k\in[0,l]} \pi^{m_{4l+3}}_{2k+1}\ (2)\ \iota^{m_{4l+4}}_{2k+1}\Big)+\Big(\sum_{k\in[0,l]} \pi^{m_{4l+3}}_{2k+2}\ (4)\ \iota^{m_{4l+4}}_{2k+3}\Big)\\
&+\Big(\sum_{k\in[0,\underline{\underline{l-2}}\,]} \pi^{m_{4l+3}}_{2l+4+2k}\ (-2)\ \iota^{m_{4l+4}}_{2l+5+2k}\Big)+\Big(\sum_{k\in[0,\underline{\underline{l}}\,]} \pi^{m_{4l+3}}_{2l+3+2k}\ (-4)\ \iota^{m_{4l+4}}_{2l+3+2k}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&\Bigg(\Big(\sum_{k\in[0,l]} \pi^{m_{4l+3}}_{2k+1}\ (2)\ \iota^{m_{4l+4}}_{2k+1}\Big) + \Big(\sum_{k\in[0,l-1]} \pi^{m_{4l+3}}_{2k+2}\ \big((-2)\cdot(2)+(4)\big)\ \iota^{m_{4l+4}}_{2k+3}\Big)\\
&+\Big(\pi^{m_{4l+3}}_{2l+2}\ \big((1)\cdot(-4)+(4)\big)\ \iota^{m_{4l+4}}_{2l+3}\Big) + \Big(\pi^{m_{4l+3}}_{2l+3}\ (-4)\ \iota^{m_{4l+4}}_{2l+3}\Big)\\
&+\Big(\sum_{k\in[0,\underline{\underline{l-2}}]} \pi^{m_{4l+3}}_{2l+4+2k}\ (-2)\ \iota^{m_{4l+4}}_{2l+5+2k}\Big) + \Big(\sum_{k\in[1,\underline{\underline{l}}]} \pi^{m_{4l+3}}_{2l+3+2k}\ \big((-2)\cdot(-2)+(-4)\big)\ \iota^{m_{4l+4}}_{2l+3+2k}\Big)\Bigg)\\
=&\Big(\sum_{k\in[1,l+1]} \pi^{m_{4l+3}}_{2k-1}\ (2)\ \iota^{m_{4l+4}}_{2k-1}\Big) + \Big(\pi^{m_{4l+3}}_{2l+3}\ (-4)\ \iota^{m_{4l+4}}_{2l+3}\Big)\\
&\quad+\Big(\sum_{k\in[l+2,l+2+\underline{\underline{l-2}}]} \pi^{m_{4l+3}}_{2k}\ (-2)\ \iota^{m_{4l+4}}_{2k+1}\Big)
\end{aligned}
$$

We obtain the following.

$$
\begin{aligned}
\mathrm{Ker}(T^{-1}_{4l+2}\cdot d^{\star}_{4l+3}) &= \langle \mathrm{e}_{2k} : k\in[1,l+1]\rangle \oplus \langle \mathrm{e}_{2k+1} : k\in[l+2,l+1+\underline{\underline{l-2}}\,]\rangle\\
&\qquad \oplus \langle \mathrm{e}_{k} : k\in[2l+5+2(\underline{\underline{l-2}}), m_{4l+3}\,]\rangle\\
\mathrm{Im}(T^{-1}_{4l+2}\cdot d^{\star}_{4l+3}) &= \langle 2\,\mathrm{e}_{2k-1} : k\in[1,l+3+\underline{\underline{l-2}}\,]\backslash\{l+2\}\rangle \oplus \langle 4\,\mathrm{e}_{2l+3}\rangle
\end{aligned}
$$

Using Remark 42.(i,vi) we have

$$
\begin{aligned}
m_{4l+3}-2l-5-2(\underline{\underline{l-2}}) &= 3l-(\underline{\underline{l-7}})-2l-5-2(\underline{\underline{l-2}}) = l-3-(\underline{\underline{l-1}})-2(\underline{\underline{l-2}})\\
&= -1+\underline{\underline{l}}+\underline{\underline{l-1}}+\underline{\underline{l-2}}-(\underline{\underline{l-1}})-2(\underline{\underline{l-2}}) = -1+\underline{\underline{l}}-(\underline{\underline{l-2}})
\end{aligned}
$$

so that $|\{\mathrm{e}_k : k\in[2l+5+2(\underline{\underline{l-2}}), m_{4l+3}\,]\}| = \underline{\underline{l}}-(\underline{\underline{l-2}})\in\{0,1\}$.

Note that $\mathrm{e}_{2l+5+2(\underline{\underline{l-2}})} = \mathrm{e}_{2k+1}$ for $k = l+2+\underline{\underline{l-2}}$. Hence

$$
\begin{aligned}
\mathrm{Ker}(T^{-1}_{4l+2}\cdot d^{\star}_{4l+3}) &= \langle \mathrm{e}_{2k} : k\in[1,l+1]\rangle \oplus \langle \mathrm{e}_{2k+1} : k\in[l+2,l+1+\underline{\underline{l-2}}+\underline{\underline{l}}-(\underline{\underline{l-2}})]\rangle\\
&= \langle \mathrm{e}_{2k} : k\in[1,l+1]\rangle \oplus \langle \mathrm{e}_{2k+1} : k\in[l+2,l+1+\underline{\underline{l}}\,]\rangle.
\end{aligned}
$$

□

Theorem 54 Recall that $R=\mathbb{Z}_{(2)}$. Recall that we write $\underline{\underline{x}}$ for the unique element in $\mathbb{Z}$ with $x = 3\underline{\underline{x}}+\overline{\overline{x}}$, where $\overline{\overline{x}}\in[0,2]$.

For $l\in\mathbb{Z}_{\geq 0}$ we have the following.

$$
\begin{array}{rcllll}
\mathrm{H}^{0}(\mathrm{S}_4)_{(2)} & \simeq & R & & & \\
\mathrm{H}^{4l}(\mathrm{S}_4)_{(2)} & \simeq & (R/2R)^{\oplus l+\underline{\underline{l}}}\ \oplus\ (R/4R) & \simeq & (\mathbb{Z}/2\mathbb{Z})^{\oplus l+\underline{\underline{l}}}\ \oplus\ (\mathbb{Z}/4\mathbb{Z}) & \text{if } l\geq 1\\
\mathrm{H}^{4l+1}(\mathrm{S}_4)_{(2)} & \simeq & (R/2R)^{\oplus l+1+\underline{\underline{l-2}}} & \simeq & (\mathbb{Z}/2\mathbb{Z})^{\oplus l+1+\underline{\underline{l-2}}} & \\
\mathrm{H}^{4l+2}(\mathrm{S}_4)_{(2)} & \simeq & (R/2R)^{\oplus l+2+\underline{\underline{l-1}}} & \simeq & (\mathbb{Z}/2\mathbb{Z})^{\oplus l+2+\underline{\underline{l-1}}} & \\
\mathrm{H}^{4l+3}(\mathrm{S}_4)_{(2)} & \simeq & (R/2R)^{\oplus l+1+\underline{\underline{l}}} & \simeq & (\mathbb{Z}/2\mathbb{Z})^{\oplus l+1+\underline{\underline{l}}} &
\end{array}
$$

Note that $\mathrm{H}^{k}(\mathrm{S}_4)_{(2)}$ is the 2-part of the finite abelian group $\mathrm{H}^{k}(\mathrm{S}_4)$ for $k\in\mathbb{Z}_{\geq 1}$.

Proof. By **[16,** Proposition 3.3.10] we have $\mathrm{H}^{k}(\mathrm{S}_4;\mathbb{Z}_{(2)})\simeq \mathrm{H}^{k}(\mathrm{S}_4)_{(2)}$ for $k\in\mathbb{Z}_{\geq 0}$.

We have $d^*_{-1} = 0$ and $d^\star_0 = 0$ so that

$$\mathrm{H}^0(\mathrm{S}_4)_{(2)} \simeq \mathrm{Ker}(d^\star_0) \simeq R^{\oplus m_0} \simeq R;$$

cf. Lemma 51.

Note that for $l \geq 1$ we have the following; cf. Lemma 53.

$$\begin{aligned}\mathrm{Im}(T^{-1}_{4l-2} \cdot d^\star_{4l-1}) &= \langle 2\,\mathrm{e}_{2k-1} : k \in [1, l+2+\underline{\underline{l-3}}]\backslash\{l+1\}\rangle \oplus \langle 4\,\mathrm{e}_{2l+1}\rangle \\ &= \langle 2\,\mathrm{e}_{2k-1} : k \in [1, l+1+\underline{l}]\backslash\{l+1\}\rangle \oplus \langle 4\,\mathrm{e}_{2l+1}\rangle\end{aligned}$$

The result follows from Lemma 51 and Lemma 53. ▫

3.3 Comparison with the calculation of $\mathrm{H}^*(S_4)$ by C. B. Thomas

At the prime 2, we may compare our results of Theorem 54 with the calculation of $\mathrm{H}^*(S_4)$ by C. B. Thomas in [**15**].

The comparison was done using the computer algebra system Magma; cf. [**6**].

```
Magma Code:

P<a, n, z, e> := PolynomialRing(Integers(), [2, 3, 4, 4]);

// Generate list of all monomials with degree m ////////////
ListMonomials := function(m)
 return Isetseq(MonomialsOfWeightedDegree(P, m));
end function;

// Determine coefficient matrix for the degree m ///////////
CoefMatrix := function(m)

 D := [j : j in [1..((m-2) div 6)]];          // List of indices j for which the last relation
                                              //    has to be considered

 PDeg_m := ListMonomials(m);                  // Generate lists of monomials such that
 if m ge 2 then                               //    the degree is m after multiplication with
   PDeg_a := ListMonomials(m-2);              //    a,n,z,e respectively
   else  PDeg_a := [];                        // Generate an empty list if m is too small
  end if;
 if m ge 3 then
   PDeg_n := ListMonomials(m-3);
   else  PDeg_n := [];
  end if;
 if m ge 4 then
   PDeg_z := ListMonomials(m-4);
   else  PDeg_z := [];
  end if;
 if m ge 4 then
   PDeg_e := ListMonomials(m-4);
   else  PDeg_e := [];
 end if;
```

```
 CSeq:=                                                            // Generate list of matrix elements
 [[MonomialCoefficient(2*a * x,y) : y in PDeg_m] : x in PDeg_a] cat //   of the Coefficient matrix
 [[MonomialCoefficient(2*n * x,y) : y in PDeg_m] : x in PDeg_n] cat //   omitting the last relation
 [[MonomialCoefficient(4*z * x,y) : y in PDeg_m] : x in PDeg_z] cat
 [[MonomialCoefficient(3*e * x,y) : y in PDeg_m] : x in PDeg_e];

 for j in D do                                             // Complement the list CSeq with the
  PDeg_j := ListMonomials(m-2-6*j);                        //    last relation for each j in D
  CSeq cat:= [[MonomialCoefficient((a*n^(2*j) - a^(j+1)*(z + a^2)^j) * x,y) : y in PDeg_m] : x in PDeg_j];
 end for;

 return Matrix(IntegerRing(),CSeq);                       // Return coefficient matrix with
                                                          //     entries from CSeq
end function;


// Determine the cohomology with the coefficient matrix /////////////
Cohomology_Thomas := function(m)

 S,S1,S2:=SmithForm(CoefMatrix(m));

 return [x : x in Diagonal(S) | x ne 1];   // Returned list [a_1,..,a_n] means
                                           //    H^m(S_4) = (Z/a_1 Z) + .. + (Z/a_n Z)
end function;


// Determine the 2-part of the cohomology /////////////
Cohomology_Thomas_at_2 := function(m, Result_Thomas)

 return [2^(Valuation(x,2)) : x in Result_Thomas[m]];

end function;


// Output and comparison: /////////////

n := 27;              // Upper bound for output is 4*n+4
Result_Thomas := [];  // List with results

// Print cohomology according to Thomas and Theorem 54        //
//  and compare 2-part according to Thomas with Theorem 54    //
for l in [0..n] do
 Result_Thomas cat:= [Cohomology_Thomas(i) : i in [4*l+1..4*l+4]];
 print 4*l+1 , "\n",
        "Thomas:    ", Result_Thomas[4*l+1] ,
        " Theorem 54:", [2 : j in [1..l+1+((l-2) div 3)]] ,
        " Comparison:",
          Cohomology_Thomas_at_2(4*l+1,Result_Thomas) eq [2 : j in [1..l+1+((l-2) div 3)]];
 print 4*l+2 , "\n",
        "Thomas:    ", Result_Thomas[4*l+2] ,
        " Theorem 54:", [2 : j in [1..l+2+((l-1) div 3)]] ,
        " Comparison:",
          Cohomology_Thomas_at_2(4*l+2,Result_Thomas) eq [2 : j in [1..l+2+((l-1) div 3)]];
 print 4*l+3 , "\n",
        "Thomas:    ", Result_Thomas[4*l+3] ,
        " Theorem 54:", [2 : j in [1..l+1+(l div 3)]] ,
        " Comparison:",
          Cohomology_Thomas_at_2(4*l+3,Result_Thomas) eq [2 : j in [1..l+1+(l div 3)]];
```

```
 print 4*l+4 , "\n",
        "Thomas:    ", Result_Thomas[4*l+4] ,
        " Theorem 54:", [2 : j in [1..l+1+((l+1) div 3)]] cat [4] ,
        " Comparison:",
          Cohomology_Thomas_at_2(4*l+4,Result_Thomas) eq [2 : j in [1..l+1+((l+1) div 3)]] cat [4];
end for;
```

Remark 55 Using this program, we could calculate $\mathrm{H}^k(S_4)_{(2)}$ via Thomas in [**15,** Theorem 4] for $k \in [0, 115]$. In these cases, the calculation of Thomas coincided with our results in Theorem 54.

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Zusammenfassung

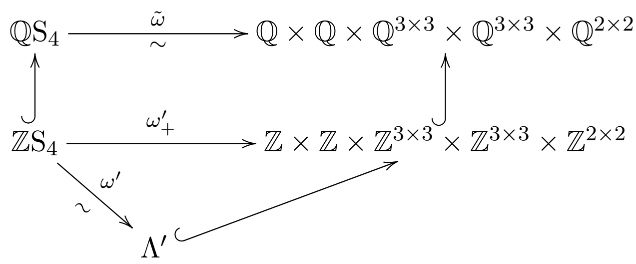
Wir betrachten den trivialen Modul $\mathbb{Z}_{(2)}$ über dem Gruppenring $\mathbb{Z}_{(2)}\mathrm{S}_4$.

Das Ziel ist die Konstruktion einer minimalen projektiven Auflösung X von $\mathbb{Z}_{(2)}$ über $\mathbb{Z}_{(2)}\mathrm{S}_4$ mit regelmäßigem Verhalten.

$$X = \left(\cdots \longrightarrow X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \longrightarrow 0 \longrightarrow \cdots \right)$$

Dazu betrachten wir zuerst das Wedderburn-Bild von $\mathbb{Z}_{(2)}\mathrm{S}_4$.

Sei $\tilde{\omega} : \mathbb{Q}\mathrm{S}_4 \xrightarrow{\sim} \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^{3\times 3}\times\mathbb{Q}^{3\times 3}\times\mathbb{Q}^{2\times 2}$ der Wedderburn-Isomorphismus. Wir schränken $\tilde{\omega}$ auf $\mathbb{Z}\mathrm{S}_4$ ein und erhalten die Wedderburn-Einbettung ω'_+. Erneutes Einschränken auf das Bild Λ' von ω'_+ liefert einen Isomorphismus ω' von $\mathbb{Z}$-Algebren.



Nach Lokalisation an 2 erhalten wir einen Isomorphismus $\mathbb{Z}_{(2)}\mathrm{S}_4 \xrightarrow[\sim]{\omega} \Lambda := \Lambda'_{(2)}$ von $\mathbb{Z}_{(2)}$-Algebren.

Wir beschreiben Λ durch Kongruenzen zwischen Matrixeinträgen und können so die unzerlegbar projektiven $\mathbb{Z}_{(2)}\mathrm{S}_4$-Moduln P und Q ablesen.

Wir geben eine geschlossene Formel für die projektiv Terme X_k von X an, als direkte Summen dieser Moduln P und Q.

Ebenso geben wir eine geschlossene Formel für die Differentiale d_k von X an, als Matrizen mit $\mathbb{Z}_{(2)}\mathrm{S}_4$-linearen Abbildungen als Einträgen. Diese Abbildungen zwischen P und Q definieren wir als Multiplikation mit ausgewählten Elementen von Λ.

Dann ist X mit den Termen X_k und den Differentialen d_k die minimale projektive Auflösung von $\mathbb{Z}_{(2)}$ über $\mathbb{Z}_{(2)}\mathrm{S}_4$. Um dies zu zeigen, konstruieren wir eine $\mathbb{Z}_{(2)}$-lineare Homotopie der Identität auf X.

Als minimale projektive Auflösung ist X eindeutig bis auf Isomorphie. Sowohl die projektive Auflösung als auch die Homotopie-Abbildungen zeigen ein regelmäßiges Verhalten.

Als Anwendung bestimmen wir die Kohomologiegruppen von S_4 über $\mathbb{Z}_{(2)}$ unter Verwendung der projektiven Auflösung X. Durch einen Vergleich mit dem Ergebnis von C. B. Thomas in **[15]** konnte das Resultat für $\mathrm{H}^n(\mathrm{S}_4)_{(2)}$ mit $n \in [0, 115]$ bestätigt werden.

Hiermit versichere ich,

(1) dass ich meine Arbeit selbstständig verfasst habe,

(2) dass ich keine anderen als die angegebenen Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,

(3) dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und

(4) dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, im September 2017 Sebastian Nitsche