Descriptions of some double Burnside rings

Master's thesis

in partial fulfillment of the requirements for the degree of
Master of Science in Mathematics

Universität Stuttgart

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December 2017

Contents

Chapter 0

Introduction

Let G, H, P and K be finite groups. Let $\mathcal{L}_G$ denote a chosen system of representatives for the conjugacy classes of subgroups of G. I.e. for each subgroup $U \leq G$, there exists a unique subgroup $V \in \mathcal{L}_G$ such that U is conjugate to V in G.

For a ring B and a commutative ring R we write $B_R := R \underset{\mathbf{Z}}{\otimes} B$.

0.1 Context and aims

Biset functors have been introduced by Bouc in order to strengthen and to conceptually simplify Mackey functors [2, §1.4]. They are functors on the category $\mathsf{Burnside}$ with values in modules over the chosen ground ring. As objects, the category $\mathsf{Burnside}$ has finite groups. Morphisms between H and G are constructed using (H, G)-bisets.

Biset functors yield classical Mackey functors when restricted to the non-full subcategory $\mathsf{Burnside}^\Delta$ [2, §1.2.5].

The endomorphism rings in the category $\mathsf{Burnside}$ are called *double Burnside rings.* I.e. for each finite group G, we have the double Burnside ring $\mathrm{B}_{\mathbf{Z}}(G, G)$ of G.

As a subring, $\mathrm{B}_{\mathbf{Z}}(G, G)$ contains the endomorphism ring of G as an object of the subcategory $\mathsf{Burnside}^\Delta$, written $\mathrm{B}^\Delta_{\mathbf{Z}}(G, G)$ and called *bifree double Burnside ring* of G.

In turn, $\mathrm{B}^\Delta_{\mathbf{Z}}(G, G)$ contains the classical Burnside ring $\mathrm{B}_{\mathbf{Z}}(G)$ as a subring.

$$\mathrm{B}_{\mathbf{Z}}(G) \hookrightarrow \mathrm{B}^\Delta_{\mathbf{Z}}(G, G) \hookrightarrow \mathrm{B}_{\mathbf{Z}}(G, G)$$

After scalar extension from $\mathbf{Z}$ to $\mathbf{Q}$ we obtain the semisimple $\mathbf{Q}$-Algebra $\mathrm{B}^\Delta_{\mathbf{Q}}(G, G)$, as shown by Boltje and Danz [1, Theorem 5.5]. The $\mathbf{Q}$-Algebra $\mathrm{B}_{\mathbf{Q}}(G, G)$, however, is not semisimple in general.

We shall give an account of the result of Boltje and Danz.

We shall calculate the rings $\mathrm{B}^\Delta_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$, $\mathrm{B}^\Delta_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$ and $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$.

0.2 Bisets

An (H, G)-*biset* X is a finite set X together with a left H-action and a right G-action that commute with each other, i.e.

$$(h \cdot x) \cdot g = h \cdot (x \cdot g) =: h \cdot x \cdot g$$

for $h \in H$, $g \in G$ and $x \in X$, cf. Definition 14.

Every (H, G)-biset X can be regarded as a left $(H \times G)$-set by setting

$$(h, g)x := hxg^{-1}$$

for $(h, g) \in H \times G$ and $x \in X$.

Likewise, every left $(H \times G)$-set Y can be regarded as an (H, G)-biset by setting

$$h \cdot y \cdot g := (h, g^{-1})y$$

for $h \in H$, $g \in G$ and $y \in Y$, cf. Remark 17.

A *morphism of* (H, G)-*bisets* X, Y is the same as a left $(H \times G)$-map. So a map $f : X \to Y$ is a morphism of (H, G)-bisets if and only if $f(h \cdot x \cdot g) = h \cdot f(x) \cdot g$ for $h \in H$, $g \in G$ and $x \in X$, cf. Definition 18.

The (H, G)-bisets together with the morphisms of (H, G)-bisets form the category of finite (H, G)-bisets.

Let M be an (H, G)-biset and N be a (G, K)-biset. The group G acts on $M \times N$ via

$$g \cdot (m, n) := (mg^{-1}, gn)$$

for $g \in G$, $m \in M$ and $n \in N$, cf. Remark 29.

We call the set of G-orbits on the cartesian product $M \times N$ the *tensor product* $M \underset{G}{\times} N$ of M and N. Then $M \underset{G}{\times} N$ is an (H, K)-biset, cf. Remark 29.

The G-orbit of the element $(m, n) \in M \times N$ is denoted by $m \underset{G}{\times} n \in M \underset{G}{\times} N$, cf. Definition 30.

0.3 The category $\mathsf{Burnside}$

In this setting, we consider the double Burnside group $\mathrm{B}_{\mathbf{Z}}(H, G)$ which is the Grothendieck group of the category of finite (H, G)-bisets.

More precisely, the *double Burnside group* $\mathrm{B}_{\mathbf{Z}}(H, G)$ is defined as the factor group of the free abelian group

$$\mathcal{F}_{H,G} := \mathbf{Z}(\{[M]_{\cong} : M \text{ is a finite } (H, G)\text{-biset}\})$$

on the set of isomorphism classes $[M]_{\cong}$ of finite (H, G)-sets M modulo the subgroup

$$\mathcal{U}_{H,G} := \langle [M \sqcup N]_{\cong} - [M]_{\cong} - [N]_{\cong} : \ M, N \text{ are finite } (H, G)\text{-bisets} \rangle \leq \mathcal{F}_{H,G} \ .$$

I.e. $\mathrm{B}_{\mathbf{Z}}(H,G) := \mathcal{F}_{H,G}/\mathcal{U}_{H,G}$, cf. Definition 23. We write $[M] := [M]_{\cong} + \mathcal{U}$ for M a finite (H,G)-biset.

The set

$$\{[M] : \ M \text{ is a finite transitive } (H,G)\text{-biset}\}$$

is a $\mathbf{Z}$-linear basis of $\mathrm{B}_{\mathbf{Z}}(H,G)$.

In particular, the set $\{[(H \times G)/L] : L \in \mathcal{L}_{H\times G}\}$ is a $\mathbf{Z}$-linear basis of $\mathrm{B}_{\mathbf{Z}}(H,G)$, cf. Lemma 28.

We have a preadditive category $\mathsf{Burnside}$. Its objects are finite groups. The set of morphisms from H to G is given by $\mathrm{B}_{\mathbf{Z}}(H,G)$. Composition is defined by

$$\begin{array}{rcl} (\underset{G}{\cdot}) : \mathrm{B}_{\mathbf{Z}}(H,G) \times \mathrm{B}_{\mathbf{Z}}(G,K) & \to & \mathrm{B}_{\mathbf{Z}}(H,K) \\ ([M],[N]) & \mapsto & [M \underset{G}{\times} N] \ , \end{array}$$

cf. Remark 37, Remark 38.

In case of $G = H$, the double Burnside group $\mathrm{B}_{\mathbf{Z}}(G,G)$ is a ring, namely the endomorphism ring of G in $\mathsf{Burnside}$, cf. Remark 39. We call $\mathrm{B}_{\mathbf{Z}}(G,G)$ the double Burnside ring of G.

Its identity element is given by

$$\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G,G)} = [G] = [(G \times G)/\Delta(G)] \ ,$$

where $\Delta(G) = \{(g,g) \in G \times G : g \in G\}$, cf. Remark 35.

0.4 The bifree double Burnside ring

Let L be a subgroup of $H \times G$. In particular, L is a subset of $H \times G$, i.e. a relation between H and G.

We call L a *twisted diagonal subgroup* if, considered as a relation, it is left unique and right unique, cf. Definition 47, Definition 48.

We denote by $\Delta_{H\times G}$ the set of all twisted diagonal subgroups of $H \times G$.

Every twisted diagonal subgroup of $H \times G$ is of the form

$$\Delta(U,\alpha,V) := \{(\alpha(v),v) \in H \times G : v \in V\} \leq H \times G$$

for $U \leq H$ and $V \leq G$ isomorphic subgroups via $\alpha : V \xrightarrow{\sim} U$, cf. Lemma 49.

We define the *bifree double Burnside group* $\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$ to be the abelian subgroup of $\mathrm{B}_{\mathbf{Z}}(H,G)$ spanned by the basis elements $[(H \times G)/L]$ with $L \in \Delta_{H\times G} \cap \mathcal{L}_{H\times G}$, cf. Definition 64.

Note that $\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G,G)} = [(G \times G)/\Delta(G)] \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$.

We obtain a preadditive subcategory $\mathsf{Burnside}^{\Delta}$ of $\mathsf{Burnside}$ as the composition

$$(\ \underset{G}{\cdot}\ ) : \mathrm{B}_{\mathbf{Z}}(H,G) \times \mathrm{B}_{\mathbf{Z}}(G,K) \to \mathrm{B}_{\mathbf{Z}}(H,K),\ \ ([X],[Y]) \mapsto [X \underset{G}{\times} Y]$$

restricts,

$$\begin{tikzcd}
\mathrm{B}_{\mathbf{Z}}(H,G) \times \mathrm{B}_{\mathbf{Z}}(G,K) \arrow[r, "(\ \underset{G}{\cdot}\ )"] & \mathrm{B}_{\mathbf{Z}}(H,K) \\
\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) \times \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,K) \arrow[u, hook] \arrow[r, "(\ \underset{G}{\cdot}\ ):=(\ \underset{G}{\cdot}\ )|^{\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,K)}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)\times\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,K)}"] & \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,K) \arrow[u, hook]
\end{tikzcd} ,$$

cf. Remark 65, Remark 66.

As a consequence, in case of $G = H$, the bifree double Burnside group $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ is a subring of the double Burnside ring $\mathrm{B}_{\mathbf{Z}}(G,G)$. We call $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ the *bifree double Burnside ring* of G.

0.5 The classical Burnside ring inside the bifree double Burnside ring

We can embed the Burnside ring $\mathrm{B}_{\mathbf{Z}}(G)$ in the bifree double Burnside ring via the injective ring morphism

$$\begin{array}{rcl} \delta : \mathrm{B}_{\mathbf{Z}}(G) & \to & \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G) \\ {[G/U]} & \mapsto & [G \times (G/U)] = [(G \times G)/\Delta(U, \mathrm{id}, U)] \ , \end{array}$$

cf. Lemma 73, Lemma 71.

The ring morphism δ is an isomorphism if and only if every twisted diagonal subgroup of $G \times G$ is conjugate to a twisted diagonal subgroup of the form

$$\Delta(W, \mathrm{id}, W) \text{ for some } W \leq G$$

as $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ has the $\mathbf{Z}$-linear basis

$$\{[(G \times G)/L] : L \in \Delta_{G\times G} \cap \mathcal{L}_{G\times G}\} \ .$$

This is rarely the case. For instance, it holds if $G = \mathrm{S}_3$, cf. Example 75.

0.6 The Wedderburn embedding of Boltje and Danz

Let $\mathcal{T}$ denote a set of representatives of isomorphism classes of finite groups. We denote by $\mathcal{T}_G$ a set of representatives of isomorphism classes of subgroups of G, cf. Notation 96.

For $T \in \mathcal{T}$ let $\mathrm{Inj}(T,G)$ denote the set of injective group morphisms from T to G, cf. Definition 99.

Let $\bar{\lambda} := \lambda|^{\lambda(T)}$ for $\lambda \in \mathrm{Inj}(T,G)$. Note that $\bar{\lambda}$ is an isomorphism.

The set $\mathrm{Inj}(T,G)$ is a $(G, \mathrm{Aut}(T))$-biset via

$$x \cdot \lambda \cdot \omega := \kappa_x^G \circ \lambda \circ \omega$$

for $x \in G$, $\lambda \in \mathrm{Inj}(T,G)$ and $\omega \in \mathrm{Aut}(T)$, cf. Remark 100. Here κ_x^G is left-conjugation with x in G.

Let $\overline{\mathrm{Inj}}(T,G)$ be the set of G-orbits of $\mathrm{Inj}(T,G)$. We denote by $[\lambda]$ the G-orbit of an element $\lambda \in \mathrm{Inj}(T,G)$, cf. Definition 101. So,

$$\overline{\mathrm{Inj}}(T,G) = \{[\lambda] : \ \lambda \in \mathrm{Inj}(T,G)\} \ .$$

Moreover, $\overline{\mathrm{Inj}}(T,G)$ is a right $\mathrm{Out}(T)$-set, cf. Remark 102.

Consequently, $\mathbf{Z}\overline{\mathrm{Inj}}(T,G)$ is a right $\mathbf{Z}\,\mathrm{Out}(T)$-module.

Boltje and Danz ([1, Theorem 5.5]) have obtained the injective ring morphism

$$\begin{array}{rcl} \sigma_{G,G}^{\Delta} : \mathrm{B}_{\mathbf{Z}}^{\Delta}(G,G) & \to & \prod\limits_{T \in \mathcal{T}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) \\ {[X]} & \mapsto & \left([\mu] \mapsto \sum\limits_{[\lambda] \in \overline{\mathrm{Inj}}(T,G)} \dfrac{|\,\mathrm{Fix}_{\Delta(\lambda(T), \bar{\lambda}\circ\bar{\mu}^{-1}, \mu(T))}(X)|}{|\,\mathrm{C}_G(\lambda(T))|} [\lambda] \right)_{T \in \mathcal{T}} \end{array} ,$$

with $\mathbf{Q} \underset{\mathbf{Z}}{\otimes} \sigma_{G,G}^{\Delta}$ an isomorphism.

More precisely, the image of $\sigma_{G,G}^{\Delta}$ has finite index of order

$$\prod_{\Delta(U,\alpha,V) \in \mathcal{L}_{G\times G}^{\Delta}} \frac{[\mathrm{N}_{G\times G}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\,\mathrm{C}_G(U)|} \ .$$

In particular, the ring $\mathrm{B}_{\mathbf{Q}}^{\Delta}(G,G) = \mathbf{Q} \underset{\mathbf{Z}}{\otimes} \mathrm{B}_{\mathbf{Z}}^{\Delta}(G,G)$ is semisimple.

We give an account of this result, cf. Section 2.3, Theorem 108. We proceed as follows. Let

$$I_{G,G}^{\Delta} := \{(U,\alpha,V) : U, V \le G, U \overset{\alpha}{\underset{\sim}{\leftarrow}} V\} \ .$$

We denote by $A_{\mathbf{Z}}^{\Delta}(G,G) := \mathbf{Z}I_{G,G}^{\Delta}$ the free abelian group with $\mathbf{Z}$-linear basis $I_{G,G}^{\Delta}$. We define the *ghost ring* $\tilde{\mathrm{B}}_{\mathbf{Z}}^{\Delta}(G,G)$ of $\mathrm{B}_{\mathbf{Z}}^{\Delta}(G,G)$ by

$$\tilde{\mathrm{B}}_{\mathbf{Z}}^{\Delta}(G,G) := \mathrm{Fix}_{G\times G}(A_{\mathbf{Z}}^{\Delta}(G,G)) \ ,$$

cf. Definition 83, Lemma 93(4).

We have an injective ring morphism, called *mark homomorphism*, mapping

$$\begin{array}{rcl} \mathfrak{m}_{G,G}^{\Delta} : \mathrm{B}_{\mathbf{Z}}^{\Delta}(G,G) & \to & \tilde{\mathrm{B}}_{\mathbf{Z}}^{\Delta}(G,G) \\ {[X]} & \mapsto & \sum\limits_{(U,\alpha,V) \in I_{G,G}^{\Delta}} \dfrac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|}{|\,\mathrm{C}_G(U)|} (U,\alpha,V) \ , \end{array}$$

cf. Lemma 94.

Given $T \in \mathcal{T}_G$, we let

$$\begin{array}{rcl} I^{\Delta}_{G,G,T} & := & \{(U,\alpha,V) \in I^{\Delta}_{G,G} : U \cong T \cong V\} \\ A^{\Delta}_{\mathbf{Z},T}(G,G) & := & \mathbf{Z} I^{\Delta}_{G,G,T} \\ \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(G,G) & := & \mathrm{Fix}_{G\times G}(A^{\Delta}_{\mathbf{Z},T}(G,G)) \ , \end{array}$$

cf. Definition 97, Lemma 98. So, $\mathfrak{m}^{\Delta}_{G,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G) \to \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G) \cong \prod_{T\in\mathcal{T}_G} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(G,G)$.

Finally, we obtain the ring isomorphisms

$$\begin{array}{rcl} \tau^{\Delta}_{G,G,T} : \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(G,G) & \stackrel{\sim}{\to} & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) \\ a = \sum\limits_{(U,\alpha,V)\in I^{\Delta}_{G,G,T}} a_{(U,\alpha,V)}(U,\alpha,V) & \mapsto & \left([\mu] \mapsto \sum\limits_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}[\lambda] \right) \ , \end{array}$$

yielding the ring isomorphism

$$\tau^{\Delta}_{G,G} := \prod_{T\in\mathcal{T}} \tau^{\Delta}_{G,G,T} : \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G) = \prod_{T\in\mathcal{T}} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(G,G) \stackrel{\sim}{\to} \prod_{T\in\mathcal{T}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) \ ,$$

cf. Lemma 107. Now, let

$$\sigma^{\Delta}_{G,G} := \tau^{\Delta}_{G,G} \circ \mathfrak{m}^{\Delta}_{G,G} \ .$$

So the following diagram commutes.

$$\begin{tikzcd} & \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G) \arrow[dl, "\mathfrak{m}^{\Delta}_{G,G}"'] \arrow[dr, "\sigma^{\Delta}_{G,G}"] & \\ \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G) = \prod\limits_{T\in\mathcal{T}} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(G,G) \arrow[rr, "\tau^{\Delta}_{G,G} = \prod_{T\in\mathcal{T}} \tau^{\Delta}_{G,G,T}"] & & \prod\limits_{T\in\mathcal{T}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) \ . \end{tikzcd}$$

0.6.1 The bifree double Burnside ring $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$

In case of $G = \mathrm{S}_3$ we already know that $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) \cong \mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3)$, cf. Example 75.

The injective ring morphism

$$\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3}} \prod_{T\in\mathcal{T}_{\mathrm{S}_3}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) \stackrel{\text{identify}}{=} \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z} \ ,$$

yields the description

$$\begin{array}{rcl} \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) & \stackrel{\sim}{\to} & \{(x_1,x_2,x_3,x_4) \in \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z} : x_1 \equiv_2 x_2,\ x_3 \equiv_2 x_4,\ x_1 \equiv_3 x_3\} \\ & = & \left(\begin{tikzcd} \mathbf{Z} \arrow[dash, rr, bend left, "3"] \arrow[dash, r, "2"] & \mathbf{Z} & \mathbf{Z} \arrow[dash, r, "2"] & \mathbf{Z} \end{tikzcd} \right) \ , \end{array}$$

cf. Example 109.

0.6.2 The bifree double Burnside ring $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$

Let $G = \mathrm{S}_4$.

We may replace $\mathcal{T}$ by

$$\mathcal{T}_{\mathrm{S}_4} = \{1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_3, \mathrm{D}_8, \mathrm{A}_4, \mathrm{S}_4\}$$

in Theorem 108(5), cf. Notation 96.

We may identify

$$\begin{array}{lclcl}
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(1)}(\mathbf{Z}\overline{\mathrm{Inj}}(1, \mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(1, \mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_4, \mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_4, \mathrm{S}_4)) & = & \mathbf{Z}^{2\times 2}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_3)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3, \mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(1, \mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_4)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_4, \mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_4, \mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{S}_3)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_3, \mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_3, \mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{A}_4)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{A}_4, \mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{A}_4, \mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{S}_4)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_4, \mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_4, \mathrm{S}_4)) & = & \mathbf{Z}\ ,
\end{array}$$

cf. Corollary 120.

Moreover, we have an isomorphism

$$\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4)) \ \xrightarrow{\sim}\ \{(a,b) \in \mathbf{Z} \times \mathbf{Z} : a \equiv_2 b\} \subseteq \mathbf{Z} \times \mathbf{Z}\ ,$$

cf. Lemma 121.

Furthermore, we have an isomorphism

$$\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2 \times \mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)) \ \xrightarrow{\sim}\ \left\{\left(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right)\right) : a,b,c,d,e \in \mathbf{Z},\ c \equiv_3 0,\ a \equiv_3 e\right\} \subseteq \mathbf{Z} \times \mathbf{Z}^{2\times 2}\ ,$$

cf. Lemma 122.

Let $\Xi := \mathbf{Z} \times \mathbf{Z}^{2\times 2} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

Thus, we may give a description of the image of the composite embedding

$$\begin{tikzcd}
\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4) \arrow[r, "\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4}"] \arrow[dr, "\tilde{\sigma}^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4}"'] & \prod\limits_{T \in \mathcal{T}_{\mathrm{S}_4}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T, G)) \arrow[d] \\
 & \Xi\ .
\end{tikzcd}$$

Let $x := \left(1, \left(\begin{smallmatrix} 3 & 2 \\ 1 & 1 \end{smallmatrix}\right), 1, 1, 1, \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right), 1, 1, 1, 1, 1\right) \in \Xi$. To decrease the complexity of the congruences, we consider the altered ring morphism σ' given by

$$\sigma'(b) := x \cdot \tilde{\sigma}^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4}(b) \cdot x^{-1}$$

for $b \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$.

The element x was found using the computer algebra system Magma [6].

So letting

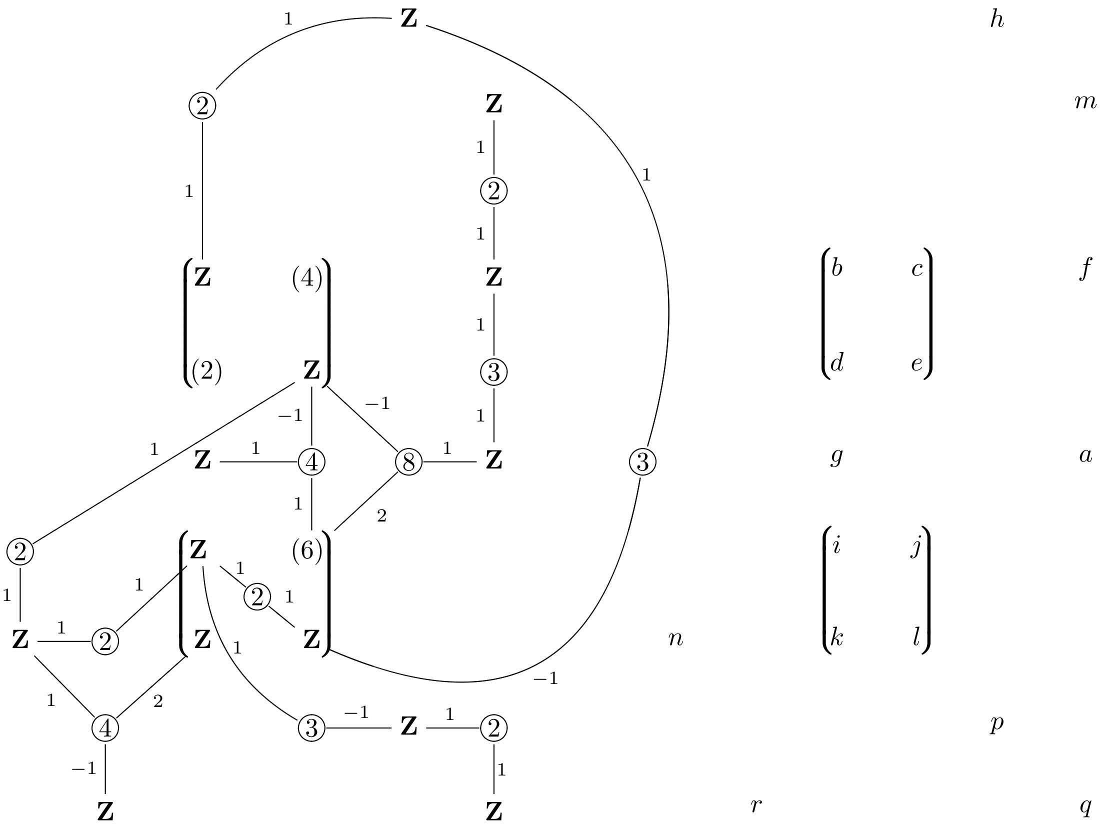
$$\Lambda := \left\{ \begin{array}{rlllll} (a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f, g, h, \left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right), m, n, r, p, q) \in \Xi : e - a & \equiv_8 & 2j & \equiv_4 & 0 & \\ e - g & \equiv_4 & j & & & \\ n - r & \equiv_4 & 2k & & & \\ c & \equiv_4 & 0 & & & \\ d & \equiv_2 & 0 & & & \\ n & \equiv_2 & l & \equiv_2 & e & \equiv_2 \; i \\ h & \equiv_2 & b & & & \\ p & \equiv_2 & q & & & \\ m & \equiv_2 & f & & & \\ a & \equiv_3 & f & & & \\ p & \equiv_3 & i & & & \\ h & \equiv_3 & l & & & \\ j & \equiv_3 & 0 & & & \end{array} \right\}$$

we obtain the

Theorem (cf. Theorem 126). We have a ring isomorphism

$$\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4) \xrightarrow{\sim} \Lambda \subseteq \Xi \; .$$

More symbolically written, we may describe Λ as follows. The letters to the right are the key to this picture.



Herein  means $t \cdot x + u \cdot y + v \cdot z \equiv_s 0$, etc.

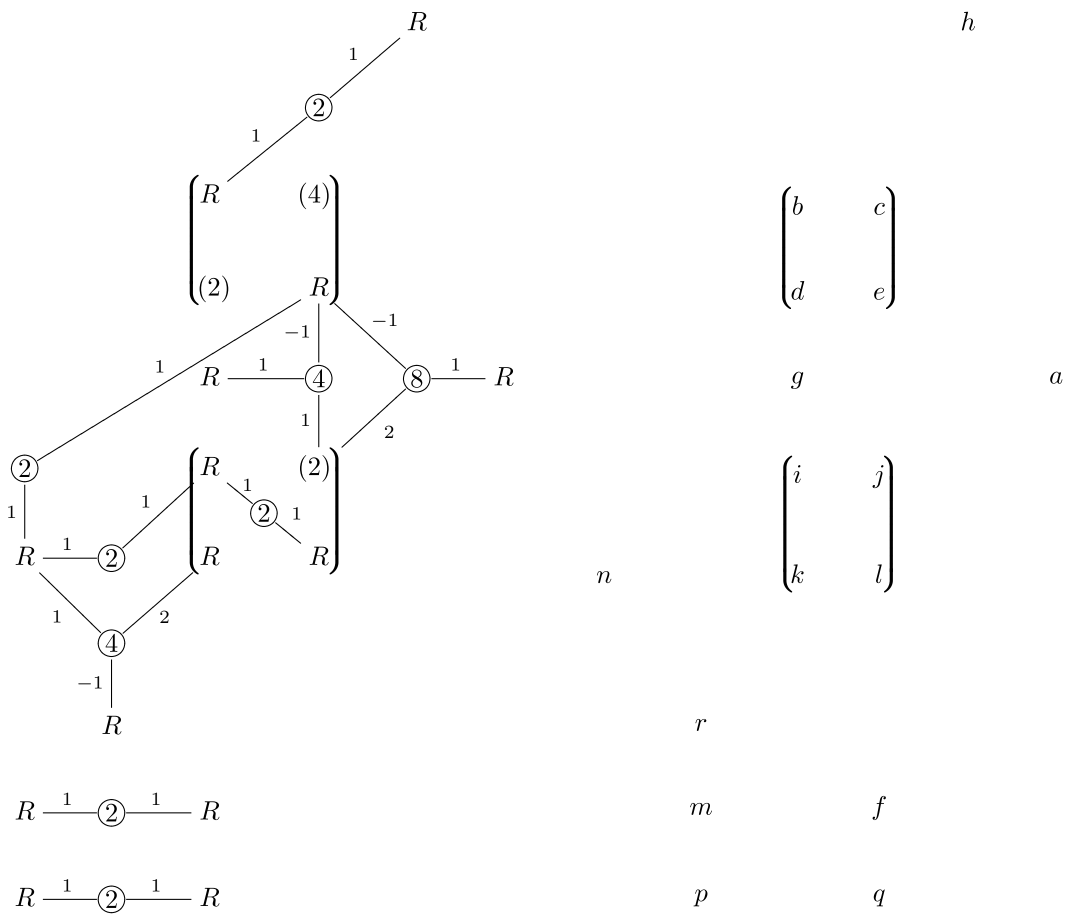
0.6.2.1 The bifree double Burnside rings $\mathrm{B}^{\Delta}_{\mathbf{Z}_{(2)}}(\mathrm{S}_4,\mathrm{S}_4)$ and $\mathrm{B}^{\Delta}_{\mathbf{F}_2}(\mathrm{S}_4,\mathrm{S}_4)$

Let $R := \mathbf{Z}_{(2)}$. By localization at (2), we obtain

$$\Lambda_{(2)} = \left\{ \begin{array}{rrlll} (a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f, g, h, \left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right), m, n, r, p, q) \in \Xi_{(2)} : e - a & \equiv_8 & 2j & \equiv_4 & 0 \\ e - g & \equiv_4 & j & & \\ n - r & \equiv_4 & 2k & & \\ c & \equiv_4 & 0 & & \\ d & \equiv_2 & 0 & & \\ n & \equiv_2 & l & \equiv_2 \; e & \equiv_2 \; i \\ h & \equiv_2 & b & & \\ p & \equiv_2 & q & & \\ m & \equiv_2 & f & & \end{array} \right\},$$

cf. Corollary 128.

More symbolically written, we may describe $\Lambda_{(2)}$ as follows. The letters to the right are the key to this picture.



Moreover, we write $\mathrm{B}^{\Delta}_{\mathbf{Z}_{(2)}}(\mathrm{S}_4,\mathrm{S}_4)$ as path algebra modulo relations, cf. Proposition 130.

We have isomorphisms of R-algebras

$$\mathrm{B}^{\Delta}_{\mathbf{Z}_{(2)}}(\mathrm{S}_4,\mathrm{S}_4) \xrightarrow{\sim} \Lambda_{(2)} \;\xrightarrow{\sim}\; R\left[\begin{tikzcd} \mathrm{e}_3 \arrow[loop left, "\nu_1"] \end{tikzcd}\right]/I' \times R\left[\begin{tikzcd} \mathrm{e}_2 \arrow[loop left, "\nu_2"] \end{tikzcd}\right]/I'' \times R\left[\begin{tikzcd} \mathrm{e}_1 \arrow[loop left, "\beta_1"] \arrow[r, bend left, "\alpha_1"] & \mathrm{e}_4 \arrow[l, bend left, "\alpha_2"] \arrow[loop above, "\gamma_1"] \arrow[loop right, "\gamma_2"] \arrow[loop below, "\gamma_8"] \end{tikzcd}\right]/I'''$$

where $I' = (\nu_1^2 - 2\nu_1)$, $I'' = (\nu_2^2 - 2\nu_2)$ and $I''' =$

$$\left(\begin{array}{cccc} \gamma_2\alpha_2 - 2\alpha_2, & \gamma_8\alpha_2, & \gamma_1\alpha_2, & \gamma_2\gamma_1 - \gamma_1\gamma_2, \\ \gamma_1^2 - 3\gamma_1\gamma_2 + 4\gamma_2 - \alpha_2\alpha_1, & \gamma_2^2 - 2\gamma_2, & \gamma_1\gamma_8^2, & \gamma_2\gamma_8, \\ \alpha_2\beta_1 - 2\alpha_2, & \gamma_8\gamma_1\gamma_8 - 2\gamma_8 + \gamma_8^2, & \gamma_8^3 - 2\gamma_8^2, & \gamma_8\gamma_2, \\ \gamma_1\gamma_8\gamma_1 - 2\gamma_1 + \gamma_1\gamma_2, & \alpha_1\gamma_1 & \alpha_1\gamma_2 - 2\alpha_1, & \gamma_1\gamma_2\gamma_1 - 6\gamma_1\gamma_2 - 2\alpha_2\alpha_1 + 8\gamma_2, \\ \alpha_1\gamma_8, & \beta_1\alpha_1 - 2\alpha_1, & \alpha_1\alpha_2 - 4\beta_1, & \beta_1^2 - 2\beta_1 \end{array}\right).$$

As a consequence, we obtain isomorphisms of $\mathbf{F}_2$-algebras

$$\mathrm{B}^{\Delta}_{\mathbf{F}_2}(\mathrm{S}_4,\mathrm{S}_4)) \cong \Lambda/2\Lambda \cong \mathbf{F}_2\left[\begin{tikzcd} \mathrm{e}_3 \arrow[loop left, "\nu_1"] \end{tikzcd}\right]/\overline{I'} \times \mathbf{F}_2\left[\begin{tikzcd} \mathrm{e}_2 \arrow[loop left, "\nu_2"] \end{tikzcd}\right]/\overline{I''} \times \mathbf{F}_2\left[\begin{tikzcd} \mathrm{e}_1 \arrow[loop left, "\beta_1"] \arrow[r, bend left, "\alpha_1"] & \mathrm{e}_4 \arrow[l, bend left, "\alpha_2"] \arrow[loop above, "\gamma_1"] \arrow[loop right, "\gamma_2"] \arrow[loop below, "\gamma_8"] \end{tikzcd}\right]/\overline{I'''}$$

where $\overline{I'} = (\nu_1^2)$, $\overline{I''} = (\nu_2^2)$ and

$$\overline{I'''} = \left(\begin{array}{cccc} \gamma_2\alpha_2, & \gamma_8\alpha_2, & \gamma_1\alpha_2, & \gamma_2\gamma_1 - \gamma_1\gamma_2, \\ \gamma_1^2 - \gamma_1\gamma_2 - \alpha_2\alpha_1, & \gamma_2^2, & \gamma_1\gamma_8^2, & \gamma_2\gamma_8, \\ \alpha_2\beta_1, & \gamma_8\gamma_1\gamma_8 + \gamma_8^2, & \gamma_8^3, & \gamma_8\gamma_2, \\ \gamma_1\gamma_8\gamma_1 + \gamma_1\gamma_2, & \alpha_1\gamma_1 & \alpha_1\gamma_2, & \gamma_1\gamma_2\gamma_1, \\ \alpha_1\gamma_8, & \beta_1\alpha_1, & \alpha_1\alpha_2, & \beta_1^2 \end{array}\right),$$

cf. Corollary 131.

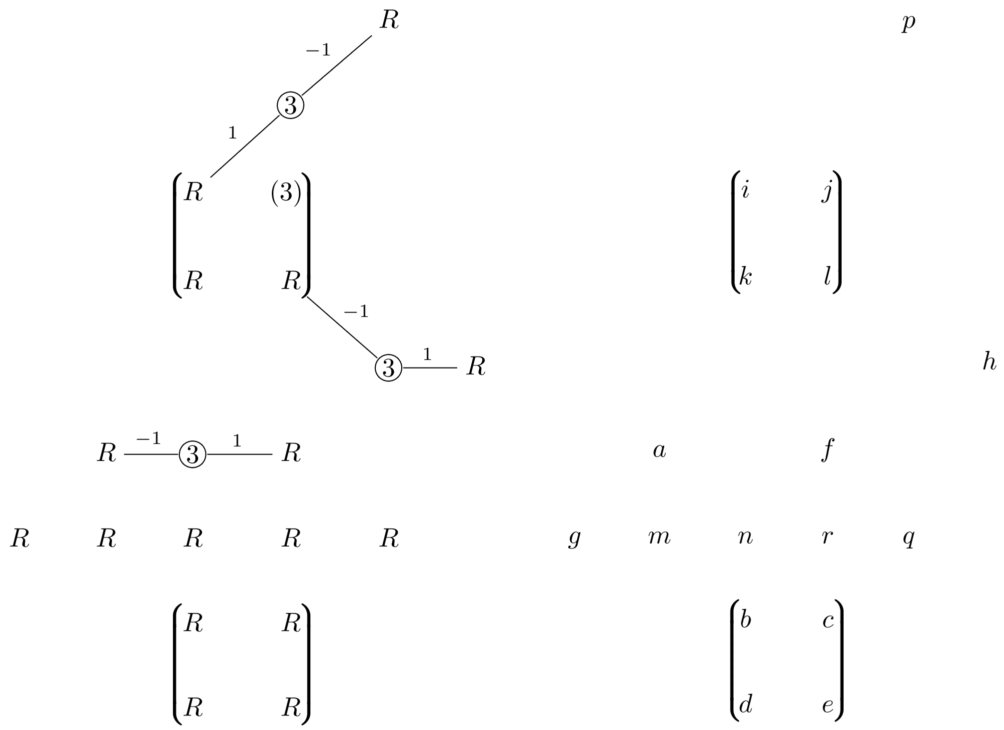
0.6.2.2 The bifree double Burnside rings $\mathrm{B}^{\Delta}_{\mathbf{Z}_{(3)}}(\mathrm{S}_4,\mathrm{S}_4)$ and $\mathrm{B}^{\Delta}_{\mathbf{F}_3}(\mathrm{S}_4,\mathrm{S}_4)$

Write $R := \mathbf{Z}_{(3)}$. By localization at (3), we obtain

$$\Lambda_{(3)} = \left\{\begin{array}{rcl} (a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f, g, h, \left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right), m, n, r, p, q) \in \Xi_{(3)} : a & \equiv_3 & f \\ p & \equiv_3 & i \\ h & \equiv_3 & l \\ j & \equiv_3 & 0 \end{array}\right\},$$

cf. Corollary 132.

More symbolically written, we may describe $\Lambda_{(3)}$ as follows. The letters to the right are the key to this picture.



Moreover, we write $\mathrm{B}^{\Delta}_{\mathbf{Z}_{(3)}}(\mathrm{S}_4, \mathrm{S}_4)$ as path algebra modulo relations, cf. Proposition 134.

We have isomorphisms of R-algebras

$$\mathrm{B}^{\Delta}_{\mathbf{Z}_{(3)}}(\mathrm{S}_4, \mathrm{S}_4) \xrightarrow{\sim} \Lambda_{(3)} \quad \xrightarrow{\sim} \quad \frac{R\left[\beta_1 \circlearrowright \mathrm{e}_1\right]}{(\beta_1^2 - 3\beta_1)} \times \frac{R\left[e_5 \underset{\beta_4}{\overset{\alpha_3}{\rightleftarrows}} e_6 \right]}{(\alpha_3\beta_4\alpha_3 - 3\alpha_3, \beta_4\alpha_3\beta_4 - 3\beta_4)} \times R^{2\times 2} \times R \times R \times R \times R \times R \;.$$

As a consequence, we obtain isomorphisms of $\mathbf{F}_3$-algebras

$$\mathrm{B}^{\Delta}_{\mathbf{F}_3}(\mathrm{S}_4, \mathrm{S}_4) \xrightarrow{\sim} \Lambda/3\Lambda \quad \xrightarrow{\sim} \quad \frac{\mathbf{F}_3\left[\beta_1 \circlearrowright \mathrm{e}_1\right]}{(\beta_1^2)} \times \frac{\mathbf{F}_3\left[e_5 \underset{\beta_4}{\overset{\alpha_3}{\rightleftarrows}} e_6 \right]}{(\alpha_3\beta_4\alpha_3, \beta_4\alpha_3\beta_4)} \times \mathbf{F}_3^{2\times 2} \times \mathbf{F}_3 \times \mathbf{F}_3 \times \mathbf{F}_3 \times \mathbf{F}_3 \times \mathbf{F}_3 \;,$$

cf. Corollary 135.

0.7 The double Burnside ring $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$

0.7.1 $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$

The double Burnside ring $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3) = \mathbf{Q} \underset{\mathbf{Z}}{\otimes} \mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ is not semisimple, thus not isomorphic to a direct product of matrix rings over $\mathbf{Q}$.

As a substitute, we construct a *Peirce composition*, i.e. a tuple of $\mathbf{Q}$-vector spaces $(A_{i,j})_{i,j\in[1,r]}$ and a tuple of $\mathbf{Q}$-bilinear maps $(\alpha_{i,j,k} : A_{i,j} \times A_{j,k} \to A_{i,k})_{i,j,k\in[1,r]}$ such that the following conditions hold.

(1) We have

$$\alpha_{i,k,l}(\alpha_{i,j,k}(a_{i,j}, a'_{j,k}), a''_{k,l}) = \alpha_{i,j,l}(a_{i,j}, \alpha_{j,k,l}(a'_{j,k}, {}'' a_{k,l}))$$

for $i, j, k, l \in [1, r]$ and $a_{i,j} \in A_{i,j}$, $a'_{j,k} \in A_{j,k}$, $a''_{k,l} \in A_{k,l}$.

(2) For $j \in [1, r]$ there exists $e_j \in A_{j,j}$ fulfilling

$$\begin{aligned} \alpha_{j,j,k}(e_j, a_{j,k}) &= a_{j,k} \\ \text{and } \alpha_{i,j,j}(a_{i,j}, e_j) &= a_{i,j} \end{aligned}$$

for $i, k \in [1, r]$, $a_{i,j} \in A_{i,j}$, $a_{j,k} \in A_{j,k}$.

Letting multiplication be defined by $a_{i,j} \cdot a'_{j,k} := \alpha_{i,j,k}(a_{i,j}, a'_{j,k})$ for $i, j, k \in [1, r]$ and $a_{i,j} \in A_{i,j}$, $a'_{j,k} \in A_{j,k}$, this defines a $\mathbf{Q}$-algebra $A := \bigoplus_{i,j\in[1,r]} A_{i,j}$, cf. Definition 152, Lemma 156.

Alternatively, a Peirce composition $((A_{i,j})_{i,j\in[1,r]}, (\alpha_{i,j,k})_{i,j,k\in[1,r]})$ can be regarded as a $\mathbf{Q}$-linear preadditive category $\mathcal{X}$ with objects $[1, r]$, morphisms ${}_{\mathcal{X}}(i, j) = A_{i,j}$ for $i, j \in [1, r]$ and composition given by the maps $\alpha_{i,j,k}$, cf. Remark 153.

We shall construct this Peirce composition in such a way that $\bigoplus_{i,j\in[1,r]} A_{i,j}$ is isomorphic to $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$ as $\mathbf{Q}$-algebras.

Write $\mathbf{Q}[\overline{\eta}, \overline{\xi}] := \mathbf{Q}[\eta, \xi]/(\eta^2, \eta\xi, \xi^2)$.

We define the tuple of $\mathbf{Q}$-vector spaces

$$\begin{array}{llll} (A_{1,1}\ , & A_{1,2}\ , & A_{1,3}\ , & A_{1,4}\ , \\ \ A_{2,1}\ , & A_{2,2}\ , & A_{2,3}\ , & A_{2,4}\ , \\ \ A_{3,1}\ , & A_{3,2}\ , & A_{3,3}\ , & A_{3,4}\ , \\ \ A_{4,1}\ , & A_{4,2}\ , & A_{4,3}\ , & A_{4,4}) \end{array} := \begin{array}{llll} (\mathbf{Q}^{3\times 3}\ , & 0\ , & 0\ , & \mathbf{Q}^{3\times 1}\ , \\ \ 0\ , & \mathbf{Q}\ , & 0\ , & \mathbf{Q}\ , \\ \ 0\ , & 0\ , & \mathbf{Q}\ , & 0\ , \\ \ \mathbf{Q}^{1\times 3}\ , & \mathbf{Q}\ , & 0\ , & \mathbf{Q}[\overline{\eta}, \overline{\xi}])\ . \end{array}$$

The multiplication maps are given by

$$\begin{array}{cccl} A_{i,j} \times A_{j,k} & \xrightarrow{\alpha_{i,j,k}} & A_{i,k} & \\ (u, v) & \mapsto & 0 & \text{if } (i, j), (j, k) \text{ or } (i, k) \text{ is contained in} \\ & & & \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 3)\}\ , \end{array}$$

$$
\begin{array}{rcl}
A_{1,1}\times A_{1,1} & \xrightarrow{\alpha_{1,1,1}} & A_{1,1} \\
(X,Y) & \mapsto & XY \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{1,1}\times A_{1,4} & \xrightarrow{\alpha_{1,1,4}} & A_{1,4} \\
(X,u) & \mapsto & Xu \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{1,4}\times A_{4,4} & \xrightarrow{\alpha_{1,4,4}} & A_{1,4} \\
(u, a+b\overline{\eta}+c\overline{\xi}) & \mapsto & ua \ ,
\end{array}
$$

$$
\begin{array}{rcl}
A_{2,2}\times A_{2,2} & \xrightarrow{\alpha_{2,2,2}} & A_{2,2} \\
(u,v) & \mapsto & uv \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{2,2}\times A_{2,4} & \xrightarrow{\alpha_{2,2,4}} & A_{2,4} \\
(u,v) & \mapsto & uv \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{2,4}\times A_{4,4} & \xrightarrow{\alpha_{2,4,4}} & A_{2,4} \\
(u, a+b\overline{\eta}+c\overline{\xi}) & \mapsto & ua \ ,
\end{array}
$$

$$
\begin{array}{rcl}
A_{2,4}\times A_{4,2} & \xrightarrow{\alpha_{2,4,2}} & A_{2,2} \\
(u,v) & \mapsto & 0 \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{3,3}\times A_{3,3} & \xrightarrow{\alpha_{3,3,3}} & A_{3,3} \\
(u,v) & \mapsto & uv \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{4,1}\times A_{1,1} & \xrightarrow{\alpha_{4,1,1}} & A_{4,1} \\
(v,X) & \mapsto & vX \ ,
\end{array}
$$

$$
\begin{array}{rcl}
A_{4,1}\times A_{1,4} & \xrightarrow{\alpha_{4,1,4}} & A_{4,4} \\
(u,v) & \mapsto & uv\overline{\eta} \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{4,2}\times A_{2,2} & \xrightarrow{\alpha_{4,2,2}} & A_{4,2} \\
(u,v) & \mapsto & uv \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{4,2}\times A_{2,4} & \xrightarrow{\alpha_{4,2,4}} & A_{4,4} \\
(u,v) & \mapsto & uv(\overline{\xi}-12\overline{\eta}) \ ,
\end{array}
$$

$$
\begin{array}{rcl}
A_{4,4}\times A_{4,1} & \xrightarrow{\alpha_{4,4,1}} & A_{4,1} \\
(a+b\overline{\eta}+c\overline{\xi}, v) & \mapsto & av \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{4,4}\times A_{4,2} & \xrightarrow{\alpha_{4,4,2}} & A_{4,2} \\
(a+b\overline{\eta}+c\overline{\xi}, v) & \mapsto & av \ ,
\end{array}
\qquad
\begin{array}{rcl}
A_{4,4}\times A_{4,4} & \xrightarrow{\alpha_{4,4,4}} & A_{4,4} \\
(x,y) & \mapsto & xy \ ,
\end{array}
$$

cf. Section 4.3.2.

We obtain an isomorphism of $\mathbf{Q}$-algebras

$$A = \bigoplus_{i,j\in[1,4]} A_{i,j} \overset{\gamma}{\underset{\sim}{\rightarrow}} \mathrm{B}_{\mathbf{Q}} \ ,$$

cf. Proposition 140.

Elements of A are written as (4×4)-block matrices with block entries in the respective summands.

0.7.1.1 $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ as path algebra modulo relations

The $\mathbf{Q}$-algebra $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ is Morita equivalent to the path algebra modulo relations

$$
\mathbf{Q}\left[\begin{tikzcd} \mathrm{e}_3 & \mathrm{e}_2 \arrow[r, bend left, "\sigma"] & \mathrm{e}_4 \arrow[l, bend left, "\vartheta"] \arrow[r, bend right, "\rho"'] & \mathrm{e}_1 \arrow[l, bend right, "\pi"'] \end{tikzcd}\right] / I
$$

where $I := (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$, cf. Proposition 141.

0.7.2 The image of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$

Let

$$
\begin{array}{llll}
(A_{\mathbf{Z},1,1} & , A_{\mathbf{Z},1,2} & , A_{\mathbf{Z},1,3} & , A_{\mathbf{Z},1,4} \\
A_{\mathbf{Z},2,1} & , A_{\mathbf{Z},2,2} & , A_{\mathbf{Z},2,3} & , A_{\mathbf{Z},2,4} \\
A_{\mathbf{Z},3,1} & , A_{\mathbf{Z},3,2} & , A_{\mathbf{Z},3,3} & , A_{\mathbf{Z},3,4} \\
A_{\mathbf{Z},4,1} & , A_{\mathbf{Z},4,2} & , A_{\mathbf{Z},4,3} & , A_{\mathbf{Z},4,4})
\end{array}
:=
\begin{array}{llll}
(\mathbf{Z}^{3\times 3} & , 0 & , 0 & , \mathbf{Z}^{3\times 1} \\
0 & , \mathbf{Z} & , 0 & , \mathbf{Z} \\
0 & , 0 & , \mathbf{Z} & , 0 \\
\mathbf{Z}^{1\times 3} & , \mathbf{Z} & , 0 & , \mathbf{Z}[\overline{\eta},\overline{\xi}])
\end{array} .
$$

Define the subring

$$A_{\mathbf{Z}} := \bigoplus_{i,j\in[1,4]} A_{\mathbf{Z},i,j} \subseteq A\ .$$

Then $A = (A_{\mathbf{Z}})_{\mathbf{Q}}$.

The subring $A_{\mathbf{Z}}$ is a $\mathbf{Z}$-order, but not a maximal $\mathbf{Z}$-order in A. Moreover, it is not contained in a maximal $\mathbf{Z}$-order in A, cf. Remark 143. Nonetheless, we may use it to describe an isomorphic copy of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$.

We give a description of the image of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ in $A_{\mathbf{Z}}$ via congruences of tuple entries. Therefor, we firstly construct the element

$$x_1 \ := \left[\begin{array}{ccc|c|c|c} 0 & -2 & 0 & 0 & 0 & 0 \\ 6 & 6 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \in A$$

to ensure that the image of the map $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3) \to A$, $y \mapsto x_1^{-1} \cdot \gamma^{-1}(y) \cdot x_1$ lies in $A_{\mathbf{Z}}$. Afterwards, we conjugate with the elements

$$x_2 := \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \in A \text{ and } x_3 := \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \in A$$

in order to simplify the system of congruences describing the image of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$, cf. Section 4.4.1.

Write $x := x_1 \cdot x_2 \cdot x_3$. We consider the altered ring morphism

$$\delta : \mathrm{B}_{\mathbf{Z}} \to A_{\mathbf{Z}},\ y \mapsto x^{-1} \cdot \gamma^{-1}(y) \cdot x$$

$$\begin{tikzcd} \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3) \arrow[rr, "\gamma^{-1}", "\sim"'] & & A \arrow[rr, "\text{conjugation with } x", "\sim"'] & & A \\ \mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3) \arrow[u, hook] \arrow[rrrr, "\delta"] & & & & A_{\mathbf{Z}} \arrow[u, hook] \end{tikzcd}\ .$$

Restricting δ to $\delta|^{\mathrm{im}(\delta)}$ this yields an isomorphism of rings

$$\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) \xrightarrow{\sim} \Lambda := \left\{ \left[\begin{array}{ccc|c|c|c} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ \hline 0 & 0 & 0 & u & 0 & v \\ \hline 0 & 0 & 0 & 0 & w & 0 \\ \hline x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{array}\right] \in A_{\mathbf{Z}} : \begin{array}{r} 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\ x_1 \equiv_4 0 \\ x_2 \equiv_4 0 \\ x_3 \equiv_4 0 \\ y \equiv_2 0 \\ t_1 \equiv_2 0 \\ t_2 \equiv_2 0 \\ t_3 \equiv_2 0 \\ v \equiv_2 0 \\ \\ x_1 \equiv_3 0 \\ x_2 \equiv_3 0 \\ x_3 \equiv_3 0 \\ z_2 \equiv_3 0 \end{array} \right\},$$

cf. Proposition 142.

More symbolically written, we have

$$\Lambda = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\ 0 & 0 & 0 & \mathbf{Z} & 0 & (2) & & \\ 0 & 0 & 0 & 0 & \mathbf{Z} & 0 & & 8 \\ (12) & (12) & (12) & (2) & 0 & \mathbf{Z} & +(12)\overline{\eta} & +(4)\overline{\xi} \end{pmatrix}.$$

Herein

$$\begin{array}{ccccc} a & \overset{t}{\text{—}} & s & \overset{v}{\text{—}} & b \\ & & |\,u & & \\ & & c & & \end{array}$$

means $t \cdot a + u \cdot c + v \cdot b \equiv_s 0$.

0.7.2.1 The double Burnside rings $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3,\mathrm{S}_3)$ and $\mathrm{B}_{\mathbf{F}_2}(\mathrm{S}_3,\mathrm{S}_3)$

Let $R := \mathbf{Z}_{(2)}$. By localization at (2), we obtain

$$\Lambda_{(2)} = \left\{ \left[\begin{array}{ccc|c|c|c} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ \hline 0 & 0 & 0 & u & 0 & v \\ \hline 0 & 0 & 0 & 0 & w & 0 \\ \hline x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{array}\right] \in A_{\mathbf{Z}_{(2)}} : \begin{array}{r} 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\ t_1 \equiv_2 0 \\ t_2 \equiv_2 0 \\ t_3 \equiv_2 0 \\ v \equiv_2 0 \\ x_1 \equiv_4 0 \\ x_2 \equiv_4 0 \\ x_3 \equiv_4 0 \\ y \equiv_2 0 \end{array} \right\} .$$

In particular, we have $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3,\mathrm{S}_3) \cong \Lambda_{(2)}$ as R-algebras, cf. Corollary 144.
More symbolically written, we have

$$\Lambda_{(2)} = \begin{pmatrix} R & R & R & 0 & 0 & (2) & & \\ R & R & R & 0 & 0 & (2) & & \\ R & R & R & 0 & 0 & (2) & & \\ 0 & 0 & 0 & R & 0 & (2) & & \\ 0 & 0 & 0 & 0 & R & 0 & & 8 \\ (4) & (4) & (4) & (2) & 0 & R & +(4)\overline{\eta} & +(4)\overline{\xi} \end{pmatrix} .$$

We deduce that $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3,\mathrm{S}_3)$ is Morita equivalent to the path algebra modulo relations

$$R\left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_5 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[loop right, "\tilde{\tau}_7"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right] / I ,$$

where

$$I := \begin{pmatrix} \tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_2\tilde{\tau}_7 - 2\tilde{\tau}_2 \\ \tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_4\tilde{\tau}_7 - 2\tilde{\tau}_4 \\ \tilde{\tau}_7\tilde{\tau}_1 - 2\tilde{\tau}_1 & , & \tilde{\tau}_7\tilde{\tau}_3 - 2\tilde{\tau}_3 & , & \tilde{\tau}_7^2 - 2\tilde{\tau}_7 - \tilde{\tau}_1\tilde{\tau}_2 \end{pmatrix} ,$$

cf. Proposition 146.

As a consequence, $\mathrm{B}_{\mathbf{F}_2}(\mathrm{S}_3,\mathrm{S}_3)$ is Morita equivalent to

$$\mathbf{F}_2\left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[rr, bend left, "\tilde{\tau}_2"] & & \tilde{\mathrm{e}}_5 \arrow[ll, bend left, "\tilde{\tau}_1"] \arrow[loop right, "\tilde{\tau}_7"] \arrow[rr, bend right, "\tilde{\tau}_3"'] & & \tilde{\mathrm{e}}_4 \arrow[ll, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right] \Big/ \begin{pmatrix} \tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_2\tilde{\tau}_7 \\ \tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_4\tilde{\tau}_7 \\ \tilde{\tau}_7\tilde{\tau}_1 & , & \tilde{\tau}_7\tilde{\tau}_3 & , & \tilde{\tau}_7^2-\tilde{\tau}_1\tilde{\tau}_2 \end{pmatrix},$$

cf. Corollary 147.

0.7.2.2 The double Burnside rings $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3,\mathrm{S}_3)$ and $\mathrm{B}_{\mathbf{F}_3}(\mathrm{S}_3,\mathrm{S}_3)$

Let $R := \mathbf{Z}_{(3)}$. By localization at (3), we obtain

$$\Lambda_{(3)} := \left\{ \left[\begin{array}{ccc|c|c|c} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ \hline 0 & 0 & 0 & u & 0 & v \\ \hline 0 & 0 & 0 & 0 & w & 0 \\ \hline x_1 & x_2 & x_3 & y & 0 & z_1+z_2\overline{\eta}+z_3\overline{\xi} \end{array}\right] \in A_{\mathbf{Z}_{(3)}} : \begin{array}{r} x_1 \equiv_3 0 \\ x_2 \equiv_3 0 \\ x_3 \equiv_3 0 \\ z_2 \equiv_3 0 \end{array} \right\}.$$

In particular, we have $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3,\mathrm{S}_3) \cong \Lambda_{(3)}$ as R-algebras, cf. Corollary 148.
More symbolically written, we have

$$\Lambda_{(3)} = \begin{pmatrix} R & R & R & 0 & 0 & R \\ R & R & R & 0 & 0 & R \\ R & R & R & 0 & 0 & R \\ 0 & 0 & 0 & R & 0 & R \\ 0 & 0 & 0 & 0 & R & 0 \\ (3) & (3) & (3) & R & 0 & R+(3)\overline{\eta}+R\overline{\xi} \end{pmatrix}.$$

We deduce that $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3,\mathrm{S}_3)$ is Morita equivalent to the path algebra modulo relations

$$R\left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[rr, bend left, "\tilde{\tau}_2"] & & \tilde{\mathrm{e}}_6 \arrow[ll, bend left, "\tilde{\tau}_1"] \arrow[rr, bend right, "\tilde{\tau}_3"'] & & \tilde{\mathrm{e}}_4 \arrow[ll, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right] /I\ ,$$

where $I := (\tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_2\tilde{\tau}_1,\ \tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_3)$, cf. Proposition 150.

As a consequence, $\mathrm{B}_{\mathbf{F}_3}(\mathrm{S}_3, \mathrm{S}_3)$ is Morita equivalent to

$$\mathbf{F}_3 \left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_6 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd} \right] \Big/ (\tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_2\tilde{\tau}_1,\ \tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_3)\ ,$$

cf. Corollary 151.

0.8 Acknowledgments

I would like to thank my advisor Matthias Künzer, who has supported me with his knowledge and patience. I thank him for the numerous constructive conversations we had, his comments vastly helped improving the quality of this text.

0.9 Conventions

Let R be a commutative ring. Let B be a ring. Let H, G be finite groups. Let V be a subgroup of G.

- Morphisms will be written on the left, i.e $(\xrightarrow{a}\xrightarrow{b}) = (\xrightarrow{b\circ a})$.
- Given elements i, j of some set I, we let $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.
- For $a, b \in \mathbf{Z}$ we denote by $[a, b] := \{z \in \mathbf{Z} : a \leq z \leq b\}$ the integral interval.
- Write $B_R := R \underset{\mathbf{Z}}{\otimes} B$.
- Write $R^\times := R \setminus \{0\}$.
- We denote by $\mathrm{U}(R) := \{r \in R :$ there exists $s \in R$ such that $rs = 1_R\}$ the group of units of R.
- Suppose $x, y, z \in R$. We write $x \equiv_z y$ if there exists $a \in R$ such that $x - y = az$.
- Given $n, m \geq 1$, we denote by $R^{n\times m}$ the R-module of $n \times m$ matrices over R.
- The standard R-linear basis $(\mathrm{e}_{i,j})_{i\in[1,n],j\in[1,m]}$ of $R^{n\times m}$ consists of the $n \times m$ matrices with entry 1 at position (i, j) and entry 0 elsewhere. In case of $m = 1$ we write $\mathrm{e}_{i,1} =: \mathrm{e}_i$ and $(\mathrm{e}_i)_{i\in[1,n]}$ is an R-linear basis of $R^{n\times 1}$.
- Let $n \in \mathbf{N}$ and $m_r \in \mathbf{N}$ for $r \in [1, n]$. The standard R-linear basis of $\bigoplus_{r\in[1,n]} R^{m_r\times m_r}$ is given by $(e_{i,j;r})_{i,j\in[1,m_r],r\in[1,n]}$, where $e_{i,j;r}$ is the tuple of matrices whose matrix at position r has the entry 1 at position (i, j), and whose other matrix entries are 0.

- Let X be a set. We denote by S_X the symmetric group on X, consisting of the bijections from X to X. Multiplication is given by $(\circ)$ and $1_{\mathrm{S}_X} = \mathrm{id}_X$. For $n \geq 1$ then we abbreviate $\mathrm{S}_n := \mathrm{S}_{[1,n]}$.

- Let M be a finite G-set. Consider the $\mathbf{Z}G$ permutation module $\mathbf{Z}M$. In particular, $\mathbf{Z}M$ is a G-set and we may consider

$$\mathrm{Fix}_G(\mathbf{Z}M) := \{\sum_{m\in M} z_m m : \sum_{m\in M} z_m gm = \sum_{m\in M} z_m m\} \ .$$

- Given a set X consisting of elements $x_1, \ldots, x_n$, we often identify

$$\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}X) = \mathbf{Z}^{n\times n} \ ,$$

  where a $\mathbf{Z}$-linear endomorphism is identified with its representing matrix with respect to the $\mathbf{Z}$-linear basis $(x_1, \ldots, x_n)$ of X.

- A category $\mathcal{X}$ is called preadditive if the following conditions hold.
  (1) For $A, B \in \mathrm{Ob}\,\mathcal{X}$ the set ${}_{\mathcal{X}}(A, B)$ carries the structure of an abelian group, written additively.
  (2) For $A, B, C, D \in \mathrm{Ob}\,\mathcal{X}$ and $f \in {}_{\mathcal{X}}(C, D)$, $g, g' \in {}_{\mathcal{X}}(B, C)$ and $h \in {}_{\mathcal{X}}(A, B)$ we have $f \circ (g + g') \circ h = f \circ g \circ h + f \circ g' \circ h$.

- Let I be a set and let X_i be an abelian group for $i \in I$. We have the exterior disjoint union

$$\bigsqcup_{i\in I} X_i := \{(x, i) : i \in I,\ x \in X_i\} \ .$$

- Let I be a set and let X_i be a set for $i \in I$. We denote by π_j the projection $\bigoplus_{i\in I} X_i \to X_j$, $(x_i)_{i\in I} \mapsto x_j$ for $j \in I$.

- We denote by ι_V the embedding $V \hookrightarrow G$, $v \mapsto v$.

- We denote by $[G : V] := |G/V|$ the index of V in G.

- For $g, x \in G$ we write ${}^g x := g \cdot x \cdot g^{-1}$ and ${}^g V := \{{}^g v : v \in V\}$. Moreover, we write $x^g := g^{-1} \cdot x \cdot g$ and $V^g := \{v^g : v \in V\}$.

- Let $U := {}^y V$ for some $y \in G$. We have the isomorphism

$$\kappa_y^{U,V} : V \to U,\ v \mapsto {}^y v \ .$$

  In case of $U = V$ we set $\kappa_y^V := \kappa_y^{V,V}$.

  We write $\mathrm{Inn}_G(V) := \{\kappa_g^V : g \in \mathrm{N}_G(V)\}$.

  We often write $\mathrm{Inn}(G) := \mathrm{Inn}_G(G)$, cf. Notation 43.

- We denote by $\mathrm{Aut}(G)$ the automorphism group of G.

  We denote by $\mathrm{Out}(G) := \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ the outer automorphism group of G.

- Let $U \leq H$ and $V \leq G$ be such that $U \cong V$. We write $\mathrm{Isom}(U,V) := \{U \xleftarrow[\sim]{f} V :$ f is an isomorphism of groups $\}$, cf. Remark 45.

- Let $\mathcal{L}_G$ denote a chosen system of representatives for the conjugacy classes of subgroups of G. I.e. for each subgroup $U \leq G$, there exists a unique subgroup $W \in \mathcal{L}_G$ such that U is conjugate in G to W.

- Let $\Delta_{H\times G}$ denote the set of twisted diagonal subgroups of $H \times G$, cf. Definition 48. Let $\mathcal{L}^{\Delta}_{H\times G} = \mathcal{L}_{H\times G} \cap \Delta_{H\times G}$.

- We often write $R[\overline{\eta},\overline{\xi}] := R[\eta,\xi]/(\eta^2,\eta\xi,\xi^2)$, where $\overline{\eta} := \eta + (\eta^2,\eta\xi,\xi^2)$ and $\overline{\xi} :=$ $\xi + (\eta^2,\eta\xi,\xi^2)$. We have the R-linear standard basis $(1,\overline{\eta},\overline{\xi})$ of $R[\overline{\eta},\overline{\xi}]$.

- Let R be an integral domain and $p \in R$ a prime element. Then we denote by $R_{(p)}$ the localization at (p), that is, $R_{(p)} := \{\frac{r}{s} : r \in R,\ s \in R \setminus (p)\} \subseteq \mathrm{frac}(R)$.

- By an *R-order*, we understand a finitely generated free R-algebra.

- Suppose given $r \in \mathbf{Z}_{\geq 0}$. Suppose given R-modules $A_{i,j}$ for $i,j \in [1,r]$.

  Let $A := \bigoplus_{i,j\in[1,r]} A_{i,j}$. We write

$$A = \bigoplus_{i,j\in[1,r]} A_{i,j} =: \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,r} \\ A_{2,1} & A_{2,2} & \dots & A_{2,r} \\ \vdots & \vdots & \dots & \vdots \\ A_{r,1} & A_{r,2} & \dots & A_{r,r} \end{bmatrix}.$$

  Accordingly, elements of A are written as matrices with entries in the respective summands.

Chapter 1

The double Burnside ring

Let K, H, G, P, Q be finite groups.

Let $\mathcal{L}_G$ denote a chosen system of representatives for the conjugacy classes of subgroups of G. I.e. for each subgroup $U \leq G$, there exists a unique subgroup $V \in \mathcal{L}_G$ such that U is conjugate in G to V.

For a ring B and a commutative ring R we write $B_R := R \underset{\mathbf{Z}}{\otimes} B$.

1.1 Preliminaries on G-sets

Definition 1.

(1) A *left G-set* is a pair (X, s) consisting of a set X and a group morphism $s : G \to \mathrm{S}_X$, called the *action* of G on X. We write $s(g)(x) := g \cdot x = gx$ for $g \in G$, $x \in X$.

A *morphism of left G-sets X, Y* or a *left G-map* is a map $f : X \to Y$ such that $f(g \cdot x) = g \cdot f(x)$, for $g \in G$ and $x \in X$.

An isomorphism of left G-sets is a bijective left G-map.

(2) A *right G-set* is a pair (X, s) consisting of a set X and a group morphism $s : G^{\mathrm{op}} \to \mathrm{S}_X$, called the *action* of G on X. We write $s(g)(x) := x \cdot g = xg$ for $g \in G$, $x \in X$.

A *morphism of right G-sets X, Y* or a *right G-map* is a map $f : X \to Y$ such that $f(x \cdot g) = f(x) \cdot g$, for $g \in G$ and $x \in X$.

An isomorphism of right G-sets is a bijective right G-map.

Remark 2. *To construct a left G-set structure on a set X, it is also possible to equip X with a map $G \times X \to X$, $(g, x) \mapsto g \cdot x = gx$ such that $(1, 2)$ hold.*

(1) *We have $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ for $g, h \in G$ and $x \in X$.*

(2) *We have $1_G \cdot x = x$ for $x \in X$.*

Similarly, one can construct a right G-set structure on a set X by establishing a map $X \times G \to X$, $(x, g) \to x \cdot g = xg$ such that $(1', 2')$ hold.

$(1')$ *We have $x \cdot (g \cdot h) = (x \cdot g) \cdot h$ for $g, h \in G$ and $x \in X$.*

$(2')$ *We have $x \cdot 1_G = x$ for $x \in X$.*

Proof. We consider the case of a left G-set.

If (X, s) is a left G-set in sense of Definition 1, we obtain a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x := s(g)(x) \ . \end{array}$$

Then

$$(gh) \cdot x = s(gh)(x) = (s(g) \circ s(h))(x) = s(g)(s(h)(x)) = g \cdot (h \cdot x)$$

and

$$1_G \cdot x = s(1_G)(x) = \mathrm{id}_X \, x = x \ .$$

Conversely, suppose given a map $G \times X \to X$, $(g, x) \mapsto g \cdot x$ fullfiling (1,2). Then define

$$s : G \to \mathrm{S}_X \ \text{ by } g \mapsto X \to X, \ x \mapsto g \cdot x \ .$$

This is a welldefined map as the maps $(x \mapsto g \cdot x)$ and $(x \mapsto g^{-1} \cdot x)$ are mutually inverse. In fact, $g \cdot (g^{-1} \cdot x) = (g \cdot g^{-1}) \cdot x = 1_G \cdot x = x$ and $g^{-1} \cdot (g \cdot x) = (g^{-1} \cdot g) \cdot x = 1_G \cdot x = x$.

Moreover, it is a group morphism as for $g, h \in G$, we have

$$(s(g) \circ s(h))(x) = g \cdot (h \cdot x) = (gh) \cdot x = s(gh)(x)$$

for $x \in X$, thus $s(g) \circ s(h) = s(gh)$. □

Remark 3. *Suppose given a set I and left G-sets X_i for $i \in I$. Then $\bigsqcup_{i \in I} X_i$ is a left G-set via*

$$g \cdot (x_i, i) := (g \cdot x_i, i)$$

for $i \in I$, $x_i \in X_i$ and $g \in G$.

Proof. We have

$$1_G \cdot (x_i, i) = (1_G \cdot x_i, i) = (x_i, i)$$

for $i \in I$, $x_i \in X_i$.

Moreover, we have

$$g \cdot (h \cdot (x_i, i)) = g \cdot (h \cdot x_i, i) = (g \cdot (h \cdot x_i), i) = ((g \cdot h) \cdot x_i, i) = (g \cdot h) \cdot (x_i, i)$$

for $i \in I$, $x_i \in X_i$ and $g, h \in G$.

So, by Remark 2, $\bigsqcup_{i \in I} X_i$ is a left G-set. □

Definition 4. Suppose given a left G-set X. We call a subset $X' \subseteq X$ *sub-G-set* if for $x' \in X'$ and $g \in G$ we have $gx' \in X'$.

Definition 5. Let V be a subgroup of G and X be a left G-set.

(1) We denote by
$$\mathrm{Fix}_V(X) := \{x \in X : vx = x \text{ for } v \in V\}$$
the set of *fixed points of X under the action of V*. We often call $x \in \mathrm{Fix}_V(X)$ a *V-fixed point.*

(2) We denote by
$$\mathrm{Stab}_V(x) := \{v \in V : vx = x\}$$
the *stabilizer of $x \in X$ under the action of V.*

Remark 6. Let V be a subgroup of G. We consider the left G-set G/V. For $gV \in G/V$, we have
$$\mathrm{Stab}_G(gV) = \{\tilde{g} \in G : \tilde{g}gV = gV\} = \{\tilde{g} \in G : g^{-1}\tilde{g}g \in V\} = gVg^{-1} \ .$$

Lemma 7. *Let $V \leq G$, $g \in G$ and X be a left G-set. Then*
$$\mathrm{Fix}_{{}^gV}(X) = g\,\mathrm{Fix}_V(X) \ .$$

Proof. We have $x \in \mathrm{Fix}_{{}^gV}(X)$ if and only if ${}^gv \cdot x = x$ for $v \in V$, i.e. if $vg^{-1}x = g^{-1}x$ for $v \in V$, i.e. if $g^{-1}x \in \mathrm{Fix}_V(X)$, i.e. if $x \in g\,\mathrm{Fix}_V(X)$. □

Lemma 8. *Let U, V be subgroups of G.*

The left G-sets G/U and G/V are isomorphic if and only if there exists $x \in G$ such that $U = {}^xV$.

Proof. We have the bijection
$$\begin{array}{rcl} \{G/U \xrightarrow{f} G/V : f \text{ is a left } G\text{-map}\} & \overset{\sim}{\longleftrightarrow} & \{gV \in G/V : U \leq C_G(gV)\} \\ f & \mapsto & f(1U) \\ (hU \mapsto hgV) & \leftarrow\!\shortmid & gV \end{array}$$

and
$$\begin{array}{rcl} \{gV \in G/V : U \leq \mathrm{C}_G(gV)\} & = & \{gV \in G/V : U \leq \{x \in G : xgV = gV\}\} \\ & = & \{gV \in G/V : U \leq \{x \in G : {}^{g^{-1}}x \in V\}\} \\ & = & \{gV \in G/V : U \leq {}^gV\} \ . \end{array}$$

Suppose that $U = {}^xV$ for some $x \in G$. Then it follows that $|U| = |V|$ and that
$$xV \in \{gV \in G/V : U \leq {}^gV\} \neq \emptyset \ .$$

So, $f : G/U \to G/V : hU \mapsto hxV$ is a left G-map.

As G/V is a transitive left G-set, every left G-map $f : G/U \to G/V$ is surjective. Because of $|U| = |V|$ we conclude that f is an isomorphism.

Conversely suppose that $f : G/U \to G/V$ is an isomorphism of left G-sets.

Then $|U| = |V|$ and $\{gV \in G/V : U \leq {}^gV\} \neq \emptyset$. Therefore, there exists $x \in G$ such that $U \leq {}^xV$. Because of $|U| = |V|$ it follows that $U = {}^xV$. □

Lemma 9. *Let X, Y be isomorphic finite left G-sets and let $\varphi : Y \xrightarrow{\sim} X$ be an isomorphism of left G-sets. Suppose given $\sum_{y \in Y} b_y \varphi(y) \in \mathbf{Z}X$.*

The element $\sum_{y \in Y} b_y \varphi(y)$ is a G-fixed point if and only if $b_{gy} = b_y$ for $y \in Y$ and $g \in G$.

Proof. Suppose that $\sum_{y \in Y} b_y \varphi(y)$ is a G-fixed point.

Write $\sum_{y \in Y} b_y \varphi(y) =: \sum_{x \in X} a_x x$, i.e. $a_x = b_{\varphi^{-1}(x)}$.

As $\sum_{x \in X} a_x x$ is a G-fixed point, we have for $g \in G$ that

$$\sum_{x \in X} a_x x = g \cdot \sum_{x \in X} a_x x = \sum_{x \in X} a_x \cdot g \cdot x = \sum_{x \in X} a_{g^{-1}x} x$$

i.e. $a_x = a_{gx}$ for $x \in X$.

So, for $x \in X$ we have $b_{\varphi^{-1}(x)} = a_x = a_{gx} = b_{\varphi^{-1}(gx)} = b_{g\varphi^{-1}(x)}$. Thus, for $y \in Y$ we have $b_{gy} = b_y$.

Conversely, suppose that $b_{gy} = b_y$ for $y \in Y$ and $g \in G$.

Then, $\sum_{y \in Y} b_y \varphi(y) = \sum_{y \in Y} b_{g^{-1}y} \varphi(y) = \sum_{y \in Y} b_y \varphi(gy) = \sum_{y \in Y} b_y \cdot g \cdot \varphi(y) = g \cdot \sum_{y \in Y} b_y \varphi(y)$.

So, $\sum_{y \in Y} b_y \varphi(y)$ is a G-fixed point. □

Definition 10. Let U, V be subgroups of G. U is *subconjugate* to V if there exists $x \in G$ such that

$$x^{-1}Ux \subseteq V \ .$$

We write $U \leq_G V$.

Remark 11. *The set $\mathcal{L}_G$ together with $(\leq_G)$ is a partially ordered set.*

Proof. Let $A, B, C \in \mathcal{L}_G$.

As $A^{1_G} = A$, the relation $(\leq_G)$ is reflexive.

Suppose that $A \leq_G B$ and $B \leq_G C$. Then there exist $x, y \in G$ such that $x^{-1}Ax \subseteq B$ and $y^{-1}By \subseteq C$. So, for $a \in A$ we have

$$y^{-1} \underbrace{x^{-1}ax}_{\in B} y = (xy)^{-1}axy \in C \ .$$

It follows that $A \leq_G C$. So $(\leq_G)$ is transitive.

Suppose that $A \leq_G B$ and $B \leq_G A$. Then there exist $x, y \in G$ such that $x^{-1}Ax \subseteq B$ and $y^{-1}By \subseteq A$. So,

$$x^{-1}Ax \subseteq B \subseteq yAy^{-1} \ .$$

As $|x^{-1}Ax| = |yAy^{-1}|$, it follows that $x^{-1}Ax = B = yAy^{-1}$. Since $\mathcal{L}_G$ is a set of representatives of conjugacy classes of subgroups of G, it follows that $A = B$. So, $(\leq_G)$ is antisymmetric. $\square$

Lemma 12. *Let $U, V \leq G$ be subgroups. Then $(1, 2, 3)$ hold.*

(1) *We have*

$$\mathrm{Fix}_V(G/U) \ = \ \{gU \in G/U : V^g \subseteq U\} \ .$$

(2) *We have*

$$\mathrm{Fix}_V(G/U) \ = \ \emptyset$$

if $V \nleq_G U$.

(3) *We have*

$$\mathrm{Fix}_U(G/U) \ = \ N_G(U)/U \ .$$

Proof. We have

$$\begin{aligned} \mathrm{Fix}_V(G/U) &= \{gU \in G/U : vgU = gU \ \forall v \in V\} \\ &= \{gU \in G/U : g^{-1}vg \in U \ \forall v \in V\} \\ &= \{gU \in G/U : V^g \subseteq U\}, \end{aligned}$$

which shows (1).

Thus, $\mathrm{Fix}_V(G/U) = \emptyset$ if $V \nleq_G U$, which shows (2).

Moreover,

$$\begin{aligned} \mathrm{Fix}_U(G/U) &= \{gU \in G/U : U^g \subseteq U\} \\ &= \{gU \in G/U : U^g = U\} \\ &= N_G(U)/U, \end{aligned}$$

which shows (3). $\square$

Lemma 13. cf. e.g. ([2, Theorem 2.4.5])

Let X and Y be finite left G-sets. Then $X \cong Y$ if and only if for each subgroup V of G, we have $|\mathrm{Fix}_V(X)| = |\mathrm{Fix}_V(Y)|$.

Proof. Suppose given an isomorphism of left G-sets $f : X \xrightarrow{\sim} Y$. We have for $x \in \mathrm{Fix}_V(X)$ that $vx = x$ for $v \in V$ and therefore, as f is a left G-map,

$$f(x) = f(vx) = vf(x) \ .$$

So, $f(x) \in \mathrm{Fix}_V(Y)$. Hence, $|\mathrm{Fix}_V(Y)| \geq |\mathrm{Fix}_V(X)|$. Likewise, $|\mathrm{Fix}_V(Y)| \leq |\mathrm{Fix}_V(X)|$.

Conversely, suppose that for each subgroup V of G, we have $|\operatorname{Fix}_V(X)| = |\operatorname{Fix}_V(Y)|$. Let $X = \bigsqcup_{i\in[1,k]} X_i$ be the decomposition into disjoint G-orbits. For $U \in \mathcal{L}_G$ set

$$a_U(X) := |\{i \in [1,k] :\ X_i \cong G/U\}| \ .$$

Then, by Lemma 8,

$$|\operatorname{Fix}_U(X)| = \sum_{U\in\mathcal{L}_G} |\operatorname{Fix}_U(G/U)| \cdot a_U(X) \ .$$

Likewise,

$$|\operatorname{Fix}_U(Y)| = \sum_{U\in\mathcal{L}_G} |\operatorname{Fix}_U(G/U)| \cdot a_U(Y) \ .$$

We have

$$0 = |\operatorname{Fix}_V(X)| - |\operatorname{Fix}_V(Y)| = \sum_{U\in\mathcal{L}_G} |\operatorname{Fix}_V(G/U)| \cdot (a_U(X) - a_U(Y)) \ .$$

Choose a total ordering $(\hat{\leq}_G)$ on $\mathcal{L}_G$ containing $(\leq_G)$, cf. Definition 10, Remark 11.

We consider the matrix $M := (|\operatorname{Fix}_V(G/U)|)_{V,U\in\mathcal{L}_G} \in \mathbf{Q}^{|\mathcal{L}_G|\times|\mathcal{L}_G|}$.

The matrix M is upper triangular as for $U, V \in \mathcal{L}_G$ such that $U \hat{<}_G V$, we have $V \hat{\nleq}_G U$, thus $V \nleq_G U$ and so

$$|\operatorname{Fix}_V(G/U)| = 0 \ ,$$

cf. Lemma 12(2). By Lemma 12(3), we get for $U \in \mathcal{L}_G$ that

$$\operatorname{Fix}_U(G/U) \quad = \quad \mathrm{N}_G(U)/U \ .$$

Thus, the matrix M has nonzero diagonal entries $[\mathrm{N}_G(U) : U] \geq 1$. So, $M \in \mathrm{GL}_{|\mathcal{L}_G|}(\mathbf{Q})$ and it follows that $a_U(X) = a_U(Y)$ for $U \in \mathcal{L}_G$. Thus, $X \cong Y$. □

1.2 Bisets

Definition 14. An (H,G)-*biset* X is a left H-set and a right G-set such that the action of H and the action of G commute, i.e.

$$(h \cdot x) \cdot g = h \cdot (x \cdot g) \text{ for } h \in H,\ g \in G,\ x \in X.$$

Remark 15. *To construct an (H,G)-biset structure on a set X, it is also possible to equip X with a map $H \times X \times G \to X$, $(h,x,g) \mapsto h \cdot x \cdot g$ such that (1,2) hold.*

(1) *We have $1_H \cdot x \cdot 1_G = x$ for $x \in X$.*

(2) *We have $h(\tilde{h} \cdot x \cdot g)\tilde{g} = (h\tilde{h}) \cdot x \cdot (g\tilde{g})$ for $h, \tilde{h} \in H, g, \tilde{g} \in G, x \in X$.*

Proof. Suppose given a map $H \times X \times G \to X$ satisfiying (1,2). Write

$$hx := hx1_G \text{ and } xg := 1_H xg$$

for $h \in H, g \in G, x \in X$.

Then X is a left H-set as

$$\begin{array}{rcl} 1_H x & = & 1_H x 1_G = x \\ h(\tilde{h}x) & = & h(\tilde{h}x1_G)1_G \\ & = & (h\tilde{h})x(1_G 1_G) \\ & = & (h\tilde{h})x \ , \end{array}$$

cf. Remark 2.

Similarly, X is a right G-set i.e. $x1_G = x$ and $(xg)\tilde{g} = x(g\tilde{g})$ for $x \in X$ and $g\tilde{g} \in G$.

Moreover,

$$(hx)g = 1_H(hx1_G)g = (1_H h)x(1_G g) = (h1_H)x(g1_G) = h(1_H xg)1_G = h(xg)$$

for $h, \tilde{h} \in H, g, \tilde{g} \in G, x \in X$. □

Example 16. Suppose given subgroups U, V of G. Then the set G is a (U, V)-biset via left and right multiplication in G.

From now on, we often omit brackets in products of biset elements with group elements.

Remark 17.

(1) *Every (H, G)-biset X can be regarded as a left $(H \times G)$-set by setting*

$$(h, g)x := hxg^{-1}$$

for $(h, g) \in H \times G$, $x \in X$.

(2) *Every left $(H \times G)$-set Y can be regarded as an (H, G)-biset by setting*

$$h \cdot y \cdot g = hyg := (h, g^{-1})y$$

for $h \in H$, $g \in G$, $y \in Y$.

Proof. Ad (1). We have for $(h, g), (\tilde{h}, \tilde{g}) \in H \times G$, $x \in X$ that

$$(h, g)((\tilde{h}, \tilde{g}) \cdot x) = (h, g)(\tilde{h}x\tilde{g}^{-1}) = (h\tilde{h})x(\tilde{g}^{-1}g^{-1}) = (h\tilde{h}, g\tilde{g}) \cdot x = ((h, g)(\tilde{h}, \tilde{g}))x$$

and

$$(1_H, 1_G) \cdot x = 1_H x 1_G^{-1} = x \ .$$

So, by Remark 2, we have that X is a left $(H \times G)$-set.

Ad (2). Suppose given $h, \tilde{h} \in H$, $g, \tilde{g} \in G$. Then Y is an (H, G)-biset as

$$1_H y 1_G = (1_H, 1_G) \cdot y = y$$

and

$$h \cdot (\tilde{h} \cdot y \cdot g) \cdot \tilde{g} = h((\tilde{h}, g^{-1})y)\tilde{g} = (h, \tilde{g}^{-1})((\tilde{h}, g^{-1})y) = (h\tilde{h}, (g\tilde{g})^{-1})y = (h\tilde{h}) \cdot y \cdot (g\tilde{g}) \ .$$

So, by Remark 15, we have that Y is an (H, G)-biset. □

From now on, we freely identify (H,G)-bisets and left $(H\times G)$-sets using Remark 17.

Definition 18.

(1) A *morphism of* (H,G)*-bisets* X,Y or an (H,G)*-bimap* is defined to be a left $(H\times G)$-map. I.e. a map $f: X\to Y$ is an (H,G)-bimap if

$$f(h\cdot x\cdot g) = f((h,g^{-1})x) = (h,g^{-1})f(x) = h\cdot f(x)\cdot g$$

for $h\in H$, $g\in G$ and $x\in X$.

An *isomorphism of* (H,G)*-bisets* is a bijective (H,G)-bimap.

Given an (H,G)-biset M, we denote its isomorphism class by $[M]_{\cong}$.

(2) Suppose given an (H,G)-biset X. We call a subset $X'\subseteq X$ *sub-biset* if it is a left sub-$(H\times G)$-set, cf. Definition 4.

(3) An (H,G)-biset X is called *transitive* if it is transitive as a left $(H\times G)$-set.

Remark 19. Let L be a subgroup of $H\times G$. Then $(H\times G)/L$ is a transitive (H,G)-biset for the action defined by

$$h\cdot(a,b)L\cdot g = (ha, g^{-1}b)L$$

for $(h,g)\in H\times G$, $(a,b)L\in(H\times G)/L$, cf. Remark 17.

In particular, any transitive (H,G)-biset is isomorphic to $(H\times G)/L$ for some subgroup L of $H\times G$, cf. Definition 18(3).

Remark 20. *Suppose given an* (H,G)*-biset* X*. Let* $X^{\mathrm{op}} := X$ *as sets. We write* x^{op} *if we view the element* $x\in X$ *as an element in* X^{op}*.*

Then X^{op} *becomes a* (G,H)*-biset by setting*

$$g\cdot x^{\mathrm{op}}\cdot h := (h^{-1}\cdot x\cdot g^{-1})^{\mathrm{op}}$$

for $x^{\mathrm{op}}\in X^{\mathrm{op}}$*,* $h\in H$*, and* $g\in G$*. We call* X^{op} *the* opposite biset *of* X*.*

Proof. We have $1_G\cdot x^{\mathrm{op}}\cdot 1_H = (1_H\cdot x\cdot 1_G)^{\mathrm{op}} = x^{\mathrm{op}}$.

Suppose given $g,\tilde{g}\in G$, $h,\tilde{h}\in H$ and $x^{\mathrm{op}}\in X^{\mathrm{op}}$. Then

$$g\cdot(\tilde{g}\cdot x^{\mathrm{op}}\cdot h)\cdot\tilde{h} = g((h^{-1}\cdot x\cdot\tilde{g}^{-1})^{\mathrm{op}})\tilde{h} = (\tilde{h}^{-1}h^{-1}\cdot x\cdot\tilde{g}^{-1}g^{-1})^{\mathrm{op}} = ((h\tilde{h})^{-1}\cdot x\cdot(g\tilde{g})^{-1})^{\mathrm{op}} = (g\tilde{g})\cdot x^{\mathrm{op}}(h\tilde{h})\,.$$

So, by Remark 15, we have that X^{op} is a (G,H)-biset. □

Remark 21. Let $L\le H\times G$ be a subgroup and X be an (H,G)-biset.

We obtain

$$\mathrm{Fix}_L(X) = \{x\in X : (h,g)x = x \text{ for } (h,g)\in L\} = \{x\in X : hx = xg \text{ for } (h,g)\in L\}\ ,$$

the set of *fixed points of* X *under the action of* L.

We obtain

$$\mathrm{Stab}_L(x) = \{(h,g)\in L : (h,g)x = x\} = \{(h,g)\in L : hx = xg\}\le L\ ,$$

the *stabilizer of* x *under the action of* L, cf. Definition 5.

Definition 22. Let X be an (H, G)-biset. Let $x \in X$.

(1) We denote by
$$\mathrm{Stab}_H^{\mathrm{left}}(x) := \{h \in H : hx = x\} \leq H$$
the *stabilizer of x under the action of H.*

(2) We denote by
$$\mathrm{Stab}_G^{\mathrm{right}}(x) := \{g \in G : xg = x\} \leq G$$
the *stabilizer of x under the action of G.*

1.3 The category Burnside and its endomorphism rings

1.3.1 Definition of the category Burnside

Definition 23. The *double Burnside group* $\mathrm{B}_{\mathbf{Z}}(H, G)$ is defined as the factor group of the free abelian group

$$\mathcal{F}_{H,G} := \mathbf{Z}(\{[M]_{\cong} : M \text{ is a finite } (H, G)\text{-biset}\})$$

on the set of isomorphism classes $[M]_{\cong}$ of finite (H, G)-bisets M modulo the subgroup

$$\mathcal{U}_{H,G} := \langle [M \sqcup N]_{\cong} - [M]_{\cong} - [N]_{\cong} : \ M, N \text{ are finite } (H, G)\text{-bisets}\rangle \leq \mathcal{F}_{H,G} \ .$$

I.e. $\mathrm{B}_{\mathbf{Z}}(H, G) := \mathcal{F}_{H,G}/\mathcal{U}_{H,G}$. We write $[M] := [M]_{\cong} + \mathcal{U}$ for a finite (H, G)-biset M.

So, $[M \sqcup N] = [M] + [N]$ in $\mathrm{B}_{\mathbf{Z}}(H, G)$.

The group $\mathrm{B}_{\mathbf{Z}}(H, G)$ is often called the Grothendieck group of the category of finite (H, G)-bisets.

Lemma 24. *Let X and Y be finite (H, G)-bisets. Let $X = \sqcup_{i \in I} X_i$ and $Y = \sqcup_{j \in J} Y_j$ be the decompositions into disjoint $(H \times G)$-orbits.*

$$\text{If } X \cong Y, \text{ then we have } \sum_{i \in I} [X_i]_{\cong} = \sum_{j \in J} [Y_j]_{\cong} \text{ in } \mathcal{F}_{H,G} \ .$$

Proof. Suppose given an isomorphism $f : X \to Y$ of (H, G)-bisets. Then $Y = \sqcup_{i \in I} f(X_i)$ is a decomposition into disjoint $(H \times G)$-orbits as $f(X_i)$ is transitive if X_i is transitive. Therefore, there exists a bijection $\sigma : I \to J$ such that $X_i \cong f(X_i) = Y_{\sigma(i)}$ for $i \in I$. Then

$$\sum_{j \in J} [Y_j]_{\cong} = \sum_{i \in I} [Y_{\sigma(i)}]_{\cong} = \sum_{i \in I} [X_i]_{\cong} \ .$$

$\square$

Remark 25 (universal property of $\mathrm{B}_{\mathbf{Z}}(H,G)$)**.** *Let T be an abelian group. Suppose given a map*

$$\mathcal{F}^0_{H,G} := \{[M]_{\cong} : M \text{ is a finite } (H,G)\text{-biset}\} \xrightarrow{f} T$$

such that we always have $f([M \sqcup N]_{\cong}) = f([M]_{\cong}) + f([N]_{\cong})$.

Then there exists a unique $\mathbf{Z}$-linear map

$$\mathrm{B}_{\mathbf{Z}}(H,G) \xrightarrow{\hat{f}} T \text{ such that } \hat{f}([M]) = f([M]_{\cong})$$

for each finite (H,G)-biset M.

$$\begin{tikzcd} {[M]} \\ {[M]_{\cong}} \arrow[u, mapsto] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{B}_{\mathbf{Z}}(H,G) \arrow[r, dotted, "\exists!\hat{f}"] & T \\ \mathcal{F}^0_{H,G} \arrow[u] \arrow[ur, "f"'] \end{tikzcd}$$

Proof. Uniqueness of $\hat{f}$ follows by $\mathrm{B}_{\mathbf{Z}}(H,G) = {}_{\mathbf{Z}}\langle [M] : M \text{ is a finite } (H,G)\text{-biset}\rangle$.

To prove existence, note that f induces a $\mathbf{Z}$-linear map $f' : \mathcal{F}_{H,G} \to T$ such that $f'([M]_{\cong}) = f([M]_{\cong})$ for each finite (H,G)-biset M. Then we always have

$$\begin{array}{rcl} f'([M \sqcup N]_{\cong} - [M]_{\cong} - [N]_{\cong}) & = & f'([M \sqcup N]_{\cong}) - f'([M]_{\cong}) - f'([N]_{\cong}) \\ & = & f([M \sqcup N]_{\cong}) - f([M]_{\cong}) - f([N]_{\cong}) \\ & = & 0 \ . \end{array}$$

So, $f'(\mathcal{U}_{H,G}) = 0$. Hence, there exists a $\mathbf{Z}$-linear map $\hat{f} : \mathrm{B}_{\mathbf{Z}}(H,G) \to T$ such that $\hat{f}(\xi + \mathcal{U}_{H,G}) = f'(\xi)$ for $\xi \in \mathcal{F}_{H,G}$. In particular, we always have

$$\hat{f}([M]) = \hat{f}([M]_{\cong} + \mathcal{U}_{H,G}) = f'([M]_{\cong}) = f([M]_{\cong}) \ .$$

□

Remark 26. Let L be a subgroup of $H \times G$. We have the well-defined map

$$\begin{array}{rcl} \mathcal{F}^0_{H,G} & \xrightarrow{\mathrm{fix}^0_L} & \mathbf{Z} \\ {[M]_{\cong}} & \mapsto & |\operatorname{Fix}_L(M)| \ , \end{array}$$

cf. Lemma 13. Moreover,

$$\begin{array}{rcl} \mathrm{fix}^0_L([M \sqcup N]_{\cong}) & = & |\operatorname{Fix}_L(M \sqcup N)| = |\operatorname{Fix}_L(M) \sqcup \operatorname{Fix}_L(N)| \\ & = & |\operatorname{Fix}_L(M)| + |\operatorname{Fix}_L(N)| = \mathrm{fix}^0_L([M]_{\cong}) + \mathrm{fix}^0_L([N]_{\cong}) \ . \end{array}$$

So, by Remark 25, there exists a $\mathbf{Z}$-linear map

$$\begin{array}{rcl} \mathrm{B}_{\mathbf{Z}}(H,G) & \xrightarrow{\mathrm{fix}_L} & \mathbf{Z} \\ {[M]} & \mapsto & \mathrm{fix}_L([M]) = \mathrm{fix}^0_L([M]_{\cong}) = |\operatorname{Fix}_L(M)| \ . \end{array}$$

Lemma 27. *Let M, N be finite (H,G)-bisets. Then $[M]=[N]$ if and only if $M \cong N$.*

Proof. Suppose that $M \cong N$. Then $[M]=[N]$ as follows by construction in Definition 23.

Conversely, suppose that $[M]=[N]$. For $L \leq H \times G$, we may use the map $\mathrm{B}_{\mathbf{Z}}(H,G) \xrightarrow{\mathrm{fix}_L} \mathbf{Z}$ from Remark 26 to conclude that

$$|\operatorname{Fix}_L(M)| = \operatorname{fix}_L([M]) = \operatorname{fix}_L([N]) = |\operatorname{Fix}_L(N)| \ .$$

Therefore, $M \cong N$ by Lemma 13. $\square$

Lemma 28. *The set*

$$\{[M] : \ M \textit{ is a finite transitive } (H,G)\textit{-biset}\}$$

is a $\mathbf{Z}$-linear basis of $\mathrm{B}_{\mathbf{Z}}(H,G)$.

In particular, $([(H \times G)/L] : L \in \mathcal{L}_{H \times G})$ is a $\mathbf{Z}$-linear basis of $\mathrm{B}_{\mathbf{Z}}(H,G)$, cf. Lemma 8, Lemma 27.

Note that transitive (H,G)-bisets are finite, cf. Remark 19.

Proof. Let $\mathrm{Tr}_{H,G} := \{[X]_{\cong} : \ X \text{ is a transitive } (H,G)\text{-biset } \} \subseteq \mathcal{F}^0_{H,G}$. Consider the free abelian group $\mathbf{Z}\,\mathrm{Tr}_{H,G} \leq \mathcal{F}_{H,G}$. By the universal property of $\mathbf{Z}\,\mathrm{Tr}_{H,G}$ there exists a $\mathbf{Z}$-linear map

$$\begin{array}{rcl} \mathbf{Z}\,\mathrm{Tr}_{H,G} & \xrightarrow{\psi} & \mathrm{B}_{\mathbf{Z}}(H,G) \\ {[M]_{\cong}} & \mapsto & [M] \ . \end{array}$$

Let

$$\begin{array}{rcl} \varphi : \mathcal{F}^0_{H,G} & \to & \mathbf{Z}\,\mathrm{Tr}_{H,G} \\ {[X]_{\cong}} & \mapsto & \sum\limits_{i \in I} [X_i]_{\cong} \ , \end{array}$$

where $X = \sqcup_{i \in I} X_i$ is the decomposition into disjoint $(H \times G)$-orbits.

By Lemma 24 the map φ is well-defined.

Furthermore, we have

$$\varphi([X \sqcup Y]_{\cong}) = \varphi([X]_{\cong}) + \varphi([Y]_{\cong})$$

for all (H,G)-bisets X, Y.

By the universal property of $\mathrm{B}_{\mathbf{Z}}(H,G)$, cf. Remark 25, there exists a unique $\mathbf{Z}$-linear map

$$\bar{\varphi} : \mathrm{B}_{\mathbf{Z}}(H,G) \to \mathbf{Z}\,\mathrm{Tr}_{H,G}$$

such that $\bar{\varphi}([X]) = \varphi([X]_{\cong}) = \sum\limits_{i \in I} [X_i]_{\cong}$.

So,

$$\begin{array}{rcl} \mathbf{Z}\,\mathrm{Tr}_{H,G} & \overset{\sim}{\leftrightarrow} & \mathrm{B}_{\mathbf{Z}}(H,G) \\ [M]_{\cong} & \overset{\psi}{\mapsto} & [M] \\ \sum_{i\in I}[X_i]_{\cong} & \overset{\bar{\varphi}}{\leftarrow\!\shortmid} & [X]\ . \end{array}$$

We have

$$(\bar{\varphi}\circ\psi)([M]_{\cong}) = \bar{\varphi}([M]) = \varphi([M]_{\cong}) = [M]_{\cong}$$

since M is transitive.

We have

$$(\psi\circ\bar{\varphi})([X]) = (\psi\circ\bar{\varphi})([\sqcup_{i\in I}X_i]) = \psi(\sum_{i\in I}[X_i]_{\cong}) = \sum_{i\in I}\psi([X_i]_{\cong}) = \sum_{i\in I}[X_i] = [\sqcup_{i\in I}X_i] = [X]\ .$$

Hence, $\bar{\varphi}$ and ψ are mutually inverse.

In particular, ψ maps the $\mathbf{Z}$-linear basis $\{[M]_{\cong} : M$ is a finite transitive (H,G)-biset$\}$ of $\mathbf{Z}\,\mathrm{Tr}_{H,G}$ to a $\mathbf{Z}$-linear basis of $\mathrm{B}_{\mathbf{Z}}(H,G)$, as claimed in the Lemma. □

Remark 29. *Let M be an (H,G)-biset and N be a (G,P)-biset. Then the cartesian product $M\times N$ is a left G-set where the action of G is given by*

$$g\cdot(m,n) = (m\cdot g^{-1}, g\cdot n)$$

for $(m,n)\in M\times N$ and $g\in G$.

Proof. As M is a right G-set and N is a left G-set it follows for $g,g'\in G$ and $m\in M$, $n\in N$ that

$$\begin{array}{rcl} 1_G\cdot(m,n) & = & (m\cdot 1_G, 1_G\cdot n) = (m,n) \\ g\cdot(g'\cdot(m,n)) & = & g\cdot(m\cdot g'^{-1}, g'\cdot n) = (m\cdot g'^{-1}g^{-1}, gg'\cdot n) = (g\cdot g')\cdot(m,n)\ . \end{array}$$

So, $M\times N$ is a left G-set by Remark 2. □

Definition 30. Let M be an (H,G)-biset and N be a (G,P)-biset.

We call the set of G-orbits on the cartesian product $M\times N$, cf. Remark 29, the *tensor product* $M\underset{G}{\times}N$ of M and N over G.

The G-orbit of the element $(m,n)\in M\times N$ is denoted by $m\underset{G}{\times}n\in M\underset{G}{\times}N$.

Remark 31. *Let M be an (H,G)-biset and N be a (G,P)-biset.*

Note that $mg^{-1}\underset{G}{\times}gn = m\underset{G}{\times}n$ for $(m,n)\in M\times N$ and $g\in G$ as $g\cdot(m,n) = (m\cdot g^{-1}, g\cdot n)$.

The tensor product $M\underset{G}{\times}N$ is an (H,P)-biset via

$$h\cdot(m\underset{G}{\times}n)\cdot p = (h,p^{-1})(m\underset{G}{\times}n) := (h\cdot m)\underset{G}{\times}(n\cdot p)$$

for $m\underset{G}{\times}n\in M\underset{G}{\times}N$ and $(h,p)\in H\times P$.

Proof. Because of $g \cdot (h \cdot m, n \cdot p) = (h \cdot m \cdot g^{-1}, g \cdot n \cdot p)$ for $g \in G$, $m \underset{G}{\times} n \in M \underset{G}{\times} N$ and $(h, p) \in H \times P$ we have

$$(h \cdot m) \underset{G}{\times} (n \cdot p) = (h \cdot m \cdot g^{-1}) \underset{G}{\times} (g \cdot n \cdot p) \ ,$$

so the map $H \times (M \underset{G}{\times} N) \times P \to M \underset{G}{\times} N$, $(h, m \underset{G}{\times} n, p) \mapsto (h \cdot m) \underset{G}{\times} (n \cdot p)$ is well-defined.

Moreover, we have

$$1_H \cdot (m \underset{G}{\times} n) \cdot 1_P = (1_H \cdot m) \underset{G}{\times} (n \cdot 1_P) = m \underset{G}{\times} n$$

for $m \underset{G}{\times} n \in M \underset{G}{\times} N$ and

$$h(\tilde{h} \cdot (m \underset{G}{\times} n) \cdot p)\tilde{p} = h(\tilde{h}m \underset{G}{\times} np)\tilde{p} = (h(\tilde{h}m)) \underset{G}{\times} ((np)\tilde{p}) = ((h\tilde{h})m) \underset{G}{\times} (n(p\tilde{p})) = (h\tilde{h}) \cdot (m \underset{G}{\times} n) \cdot (p\tilde{p})$$

for $h, \tilde{h} \in H$, $p, \tilde{p} \in P$ and $m \underset{G}{\times} n \in M \underset{G}{\times} N$.

So, using this map $M \underset{G}{\times} N$ becomes an (H, P)-biset, cf. Remark 15.

□

Remark 32. We have the surjective map $\tau := \tau_{M,H,N} : M \times N \to M \underset{G}{\times} N$, $(m, n) \mapsto m \underset{G}{\times} n$. In particular, for finite M, N the set $M \underset{G}{\times} N$ is finite.

Lemma 33. *Let M be a finite (H, G)-biset, N be a finite (G, P)-biset and L be finite a (P, Q)-biset. Then there are mutually inverse isomorphisms of (H, Q)-bisets*

$$\begin{array}{rcl} M \underset{G}{\times} (N \underset{P}{\times} L) & \leftrightarrow & (M \underset{G}{\times} N) \underset{P}{\times} L \\ m \underset{G}{\times} (n \underset{P}{\times} \ell) & \overset{\varphi}{\mapsto} & (m \underset{G}{\times} n) \underset{P}{\times} \ell \\ m \underset{G}{\times} (n \underset{P}{\times} \ell) & \overset{\psi}{\leftarrow\!\shortmid} & (m \underset{G}{\times} n) \underset{P}{\times} \ell \ . \end{array}$$

Proof. We have $m \underset{G}{\times} (n \underset{P}{\times} \ell) = \tilde{m} \underset{G}{\times} (\tilde{n} \underset{P}{\times} \tilde{\ell})$ for $m \underset{G}{\times} (n \underset{P}{\times} \ell), \tilde{m} \underset{G}{\times} (\tilde{n} \underset{P}{\times} \tilde{\ell}) \in M \underset{G}{\times} (N \underset{P}{\times} L)$ if and only if there exists $g \in G$ such that

$$\tilde{m} = mg \text{ and } \tilde{n} \underset{P}{\times} \tilde{\ell} = g^{-1}(n \underset{P}{\times} \ell) = (g^{-1}n) \underset{P}{\times} \ell \ .$$

I.e. if and only if there exists $g \in G$ such that

$$\tilde{m} = mg$$

and there exists $p \in P$ such that

$$\tilde{n} = g^{-1}np \text{ and } \tilde{\ell} = p^{-1}\ell \ .$$

Likewise, we have $(m \underset{G}{\times} n) \underset{P}{\times} \ell = (\tilde{m} \underset{G}{\times} \tilde{n}) \underset{P}{\times} \tilde{\ell}$ for $(m \underset{G}{\times} n) \underset{P}{\times} \ell, (\tilde{m} \underset{G}{\times} \tilde{n}) \underset{P}{\times} \tilde{\ell} \in (M \underset{G}{\times} N) \underset{P}{\times} L$ if and only if there exist $g \in G$ and $p \in P$ such that $\tilde{m} = mg$, $\tilde{n} = g^{-1}np$ and $\tilde{\ell} = p^{-1}\ell$.

Thus, φ and ψ are well-defined.

For $h \in H$ and $q \in Q$ we obtain

$$\begin{aligned}
\varphi(h \cdot (m \underset{G}{\times} (n \underset{P}{\times} \ell)) \cdot q) &= \varphi(hm \underset{G}{\times} (n \underset{P}{\times} \ell)q) \\
&= \varphi(hm \underset{G}{\times} (n \underset{P}{\times} \ell q)) \\
&= (hm \underset{G}{\times} n) \underset{P}{\times} \ell q \\
&= h(m \underset{G}{\times} n) \underset{P}{\times} \ell q \\
&= h \cdot ((m \underset{G}{\times} n) \underset{P}{\times} \ell) \cdot q \\
&= h \cdot \varphi(m \underset{G}{\times} (n \underset{P}{\times} \ell)) \cdot q\ .
\end{aligned}$$

So, φ is an (H, Q)-bimap.

Moreover, $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$ hold. Thus, φ and ψ are mutually inverse isomorphisms of (H, Q)-bisets. $\square$

Lemma 34. *Let M, M_1, M_2 be (H, G)-bisets and N, N_1, N_2 be (G, P)-bisets.*

(1) *The set $M_1 \sqcup M_2$ is an (H, G)-biset via $h \cdot (m_i, i) \cdot g = (h \cdot m_i \cdot g, i)$ for $i \in \{1, 2\}$, $m_i \in M_i$, $h \in H$ and $g \in G$.*

(2) *There are mutually inverse isomorphisms of (H, P)-bisets*

$$\begin{array}{rcl}
(M_1 \sqcup M_2) \underset{G}{\times} N & \overset{\sim}{\leftrightarrow} & (M_1 \underset{G}{\times} N) \sqcup (M_2 \underset{G}{\times} N) \\
(m_i, i) \underset{G}{\times} n & \overset{\varphi}{\mapsto} & (m_i \underset{G}{\times} n, i) \\
(m_i, i) \underset{G}{\times} n & \overset{\varphi'}{\hookleftarrow} & (m_i \underset{G}{\times} n, i)
\end{array}$$

and mutually inverse isomorphisms of (H, P)-bisets

$$\begin{array}{rcl}
M \underset{G}{\times} (N_1 \sqcup N_2) & \overset{\sim}{\leftrightarrow} & (M \underset{G}{\times} N_1) \sqcup (M \underset{G}{\times} N_2) \\
m \underset{G}{\times} (n_i, i) & \overset{\psi}{\mapsto} & (m \underset{G}{\times} n_i, i) \\
m \underset{G}{\times} (n_i, i) & \overset{\psi'}{\hookleftarrow} & (m \underset{G}{\times} n_i, i)\ ,
\end{array}$$

where $i \in \{1, 2\}$.

Proof. *Ad* (1). See Remark 3.

Ad (2). We show that φ and φ' are mutually inverse isomorphisms of (H, P)-bisets

We have

$$(m_i, i) \underset{G}{\times} n = (\tilde{m}_j, j) \underset{G}{\times} \tilde{n}$$

for $(m_i, i) \underset{G}{\times} n, (\tilde{m}_j, j) \underset{G}{\times} \tilde{n} \in (M_1 \sqcup M_2) \underset{G}{\times} N$ if and only if $i = j$ and there exists $g \in G$ such that $(\tilde{m}_i, i) = (m_i, i) \cdot g = (m_i \cdot g, i)$ and $\tilde{n} = g^{-1} \cdot n$.

Likewise,

$$(\tilde{m}_i \underset{G}{\times} \tilde{n}, i) = (m_j \underset{G}{\times} n, j)$$

for $(\tilde{m}_i \underset{G}{\times} \tilde{n}, i), (m_j \underset{G}{\times} n, j) \in (M_1 \underset{G}{\times} N) \sqcup (M_2 \underset{G}{\times} N)$ if and only if $i = j$ and there exists $g \in G$ such that $\tilde{m}_i = m_i \cdot g$ and $\tilde{n} = g^{-1} \cdot n$.

So, φ and φ' are well-defined maps.

Suppose given $h \in H$ and $p \in P$. Then

$$\begin{aligned} \varphi(h \cdot ((m_i, i) \underset{G}{\times} n) \cdot p) &= \varphi((h \cdot m_i, i) \underset{G}{\times} (n \cdot p)) = ((h \cdot m_i) \underset{G}{\times} (n \cdot p), i) = (h \cdot (m_i \underset{G}{\times} n) \cdot p, i) \\ &= h \cdot (m_i \underset{G}{\times} n, i) \cdot p = h \cdot \varphi((m_i, i) \underset{G}{\times} n) \cdot p \ . \end{aligned}$$

So, φ is an (H, P)-bimap.

Moreover, φ and φ' are mutually inverse. $\square$

Remark 35. *For a (G, G)-biset X one has the isomorphisms of (G, G)-bisets*

$$\begin{array}{rcl} G \underset{G}{\times} X & \overset{\sim}{\longleftrightarrow} & X \\ g \underset{G}{\times} x & \overset{\varphi}{\mapsto} & gx \\ 1_G \underset{G}{\times} x & \overset{\psi}{\leftarrow\!\shortmid} & x \end{array}$$

and

$$\begin{array}{rcl} X \underset{G}{\times} G & \to & X \\ x \underset{G}{\times} g & \mapsto & xg \\ x \underset{G}{\times} 1_G & \leftarrow\!\shortmid & x \ . \end{array}$$

Proof. We show that φ and ψ are mutually inverse isomorphisms of (G, G)-bisets.

For $g \in G$ we have $g \underset{G}{\times} x = g\tilde{g} \underset{G}{\times} \tilde{g}^{-1}x$, but also $gx = g\tilde{g}\tilde{g}^{-1}x$. So, φ is well-defined.

Because of

$$\varphi(a \cdot (g \underset{G}{\times} x) \cdot b) = \varphi((ag) \underset{G}{\times} (xb)) = agxb = a \cdot \varphi(g \underset{G}{\times} x) \cdot b$$

for $a, b \in G$ the map φ is a (G, G)-bimap.

Moreover,

$$(\varphi \circ \psi)(x) = \varphi(1_G \underset{G}{\times} x) = x$$

$$(\psi \circ \varphi)(g \underset{G}{\times} x) = \psi(gx) = 1_G \underset{G}{\times} gx = g \underset{G}{\times} x \ .$$

So, φ and ψ are mutually inverse. $\square$

Lemma 36. *Suppose given isomorphic (H,G)-bisets M and $\tilde{M}$ via $\varphi : M \xrightarrow{\sim} \tilde{M}$. Suppose given isomorphic (G,P)-bisets N and $\tilde{N}$ via $\psi : N \xrightarrow{\sim} \tilde{N}$.*

Then we have the mutually inverse isomorphisms of (H,P)-bisets

$$\begin{array}{rcl} M \underset{G}{\times} N & \overset{\sim}{\leftrightarrow} & \tilde{M} \underset{G}{\times} \tilde{N} \\ m \underset{G}{\times} n & \overset{\eta}{\mapsto} & \varphi(m) \underset{G}{\times} \psi(n) \\ \varphi^{-1}(\tilde{m}) \underset{G}{\times} \psi^{-1}(\tilde{n}) & \overset{\xi}{\leftarrow\!\shortmid} & \tilde{m} \underset{G}{\times} \tilde{n} \ . \end{array}$$

Proof. Suppose given $m \underset{G}{\times} n,\ m' \underset{G}{\times} n' \in M \underset{G}{\times} N$ with $m \underset{G}{\times} n = m' \underset{G}{\times} n'$. Then there exists $g \in G$ such that $m' = mg$ and $n' = g^{-1}n$. But then

$$\varphi(m') \underset{G}{\times} \psi(n') = \varphi(mg) \underset{G}{\times} \psi(g^{-1}n) = \varphi(m)g \underset{G}{\times} g^{-1}\psi(n) = \varphi(m) \underset{G}{\times} \psi(n) \ .$$

So, the map η is well-defined.

Analogously, the map ξ is well-defined.

Suppose given $h \in H$ and $p \in P$. We have

$$\begin{array}{rclcl} \eta(h \cdot (m \underset{G}{\times} n) \cdot p) & = & \eta((hm) \underset{G}{\times} (np)) & = & \varphi(hm) \underset{G}{\times} \psi(np) \\ & = & (h\varphi(m)) \underset{G}{\times} (\psi(n)p) & = & h \cdot (\varphi(m) \underset{G}{\times} \psi(n)) \cdot p = h \cdot \eta(m \underset{G}{\times} n) \cdot p \ . \end{array}$$

Hence, η is an (H,P)-bimap.

Moreover, $\eta \circ \xi = \mathrm{id}$ and $\xi \circ \eta = \mathrm{id}$ hold. So, η and ξ are isomorphisms of (H,P)-bisets. $\square$

Remark 37. *Let M be an (H,G)-biset and N be a (G,P)-biset. Then $M \underset{G}{\times} N$ is an (H,P)-biset, cf. Remark 31.*

We obtain the bilinear map

$$\begin{array}{rcl} (\ \underset{G}{\cdot}\ ) : \mathrm{B}_{\mathbf{Z}}(H,G) \times \mathrm{B}_{\mathbf{Z}}(G,P) & \to & \mathrm{B}_{\mathbf{Z}}(H,P) \\ ([M],[N]) & \mapsto & [M] \underset{G}{\cdot} [N] := [M \underset{G}{\times} N] \ . \end{array}$$

Proof. By Lemma 27 and Lemma 36, the map $(\ \underset{G}{\cdot}\ )$ is well-defined.

We claim that it is $\mathbf{Z}$-bilinear. In fact, for (H,G)-bisets M_1, M_2 and a (G,P)-biset N we have

$$\begin{array}{rcl} ([M_1] + [M_2]) \underset{G}{\cdot} [N] & = & [M_1 \sqcup M_2] \underset{G}{\cdot} [N] \\ & = & [(M_1 \sqcup M_2) \underset{G}{\times} N] \\ & \overset{\text{R.34(2)}}{=} & [(M_1 \underset{G}{\times} N) \sqcup (M_2 \underset{G}{\times} N)] \\ & = & [M_1 \underset{G}{\times} N] + [M_2 \underset{G}{\times} N] \ . \end{array}$$

Likewise, we have for an (H,G)-biset M and (G,P)-bisets N_1, N_2 that

$$[M] \underset{G}{\cdot} ([N_1] + [N_2]) = [M \underset{G}{\times} N_1] + [M \underset{G}{\times} N_2] \ .$$

$\square$

Remark 38. We have a preadditive category $\mathsf{Burnside}$. Its objects are finite groups. The abelian group of morphisms from H to G is given by $\mathrm{B}_{\mathbf{Z}}(H,G)$. Composition

$$(\underset{G}{\cdot}) : \mathrm{B}_{\mathbf{Z}}(H,G) \times \mathrm{B}_{\mathbf{Z}}(G,P) \to \mathrm{B}_{\mathbf{Z}}(H,P)$$

is defined in Remark 37. By Lemma 33 the composition is associative. By Remark 35 the identity element on G is given by $\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G,G)} := [G]$, cf. Example 16.

Remark 39. The *double Burnside ring* $\mathrm{B}_{\mathbf{Z}}(G,G)$ is the endomorphism ring of G in $\mathsf{Burnside}$, cf. Remark 38. Its multiplication is given by

$$[M] \underset{G}{\cdot} [N] = [M \underset{G}{\times} N]$$

for finite (G,G)-bisets M and N.

The identity element $\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G,G)}$ is given by $[G]$.

1.3.2 A Mackey formula for composition in $\mathsf{Burnside}$

Notation 40. Let L be a subgroup of $H \times G$. Let M be a subgroup of $K \times H$.

We denote by

$$\begin{array}{rcl} \mathrm{p}_1 & : & H \times G \to H, \quad (h,g) \mapsto h \\ \mathrm{p}_2 & : & K \times H \to H, \quad (k,h) \mapsto h \end{array}$$

the canonical projections. In particular,

$$\begin{array}{rcl} \mathrm{p}_1(L) & = & \{h \in H : (h,g) \in L \text{ for some } g \in G\} \leq H \\ \mathrm{p}_2(M) & = & \{h \in H : (k,h) \in M \text{ for some } k \in K\} \leq H \ . \end{array}$$

Lemma 41. *For subgroups M of $K \times H$ and L of $H \times G$ we set*

$$M * L := \{(k,g) \in K \times G : \text{there exists } h \in H \text{ such that } (k,h) \in M \text{ and } (h,g) \in L\} \ .$$

Then $M * L$ is a subgroup of $K \times G$.

Proof. We have $(1_K, 1_G) \in M * L$ as $(1_K, 1_H) \in M$ and $(1_H, 1_G) \in L$.

Let $(k,g) \in M * L$. Then there exists $h \in H$ such that $(k,h) \in M$ and $(h,g) \in L$.

Because of $M \leq K \times H$ and $L \leq H \times G$, we have $(k^{-1}, h^{-1}) \in M$ and $(h^{-1}, g^{-1}) \in L$.

As $h^{-1} \in H$ it follows $(k,g)^{-1} = (k^{-1}, g^{-1}) \in M * L$.

Let $(k,g), (k',g') \in M * L$. Then there exist $h, h' \in H$ such that $(k,h), (k',h') \in M$ and $(h,g), (h',g') \in L$.

Because of $M \leq K \times H$ and $L \leq H \times G$, we have $(k,h) \cdot (k',h') = (kk', hh') \in M$ and $(h,g) \cdot (h',g') = (hh', gg') \in L$. As $hh' \in H$, it follows that $(kk', gg') \in M * L$. $\square$

Lemma 42. ([2, Lemma 2.3.24]) *For subgroups M of $K \times H$ and L of $H \times G$ we have*

$$[(K \times H)/M] \cdot_H [(H \times G)/L] = \sum_{h \in [\mathrm{p}_2(M)\backslash H/\mathrm{p}_1(L)]} [(K \times G)/(M *^{(h,1)} L)] \ ,$$

where $[\mathrm{p}_2(M)\backslash H/\mathrm{p}_1(L)]$ denotes an arbitrarily choosen set of double coset representatives. Concerning $M *^{(h,1)} L$ see Lemma 41. See Remark 19 concerning the bisets involved.

Proof. Set $V := (K \times H)/M$ and $U := (H \times G)/L$. Let $(K \times G)\backslash(V \underset{H}{\times} U)$ be the set of $(K \times G)$-orbits of $V \underset{H}{\times} U$.

Claim. We have the bijection of sets

$$\begin{array}{rcl} (K \times G)\backslash(V \underset{H}{\times} U) & \overset{\sim}{\longleftrightarrow} & \mathrm{p}_2(M)\backslash H/\mathrm{p}_1(L) \\ (K \times G)((k,h)M \underset{H}{\times} (h',g)L) & \overset{\varphi}{\mapsto} & \mathrm{p}_2(M)h^{-1}h'\,\mathrm{p}_1(L) \\ (K \times G)((1_K,1_H)M \underset{H}{\times} (h,1_G)L) & \overset{\psi}{\hookleftarrow} & \mathrm{p}_2(M)h\,\mathrm{p}_1(L) \ . \end{array}$$

Step 1 The map φ is well-defined. We consider

$$\begin{tikzcd} (v,u) \arrow[d, mapsto] \\ v \underset{H}{\times} u \arrow[d, mapsto] \\ (K \times G)\backslash(v \underset{H}{\times} u) \end{tikzcd} \qquad \begin{tikzcd} V \times U \arrow[d] \arrow[drr, "\varphi_0"] & & \\ V \underset{H}{\times} U \arrow[d] \arrow[rr, "\varphi_1"] & & \mathrm{p}_2(M)\backslash H/\mathrm{p}_1(L) \ . \\ (K \times G)\backslash(V \underset{H}{\times} U) \arrow[urr, "\varphi"] & & \end{tikzcd}$$

We need that the map

$$\begin{array}{rcl} \varphi_0 : V \times U & \to & \mathrm{p}_2(M)\backslash H/\mathrm{p}_1(L) \\ ((k,h)M, (h',g)L) & \mapsto & \mathrm{p}_2(M)h^{-1}h'\,\mathrm{p}_1(L) \end{array}$$

is well-defined. Suppose given

$$(\tilde{k},\tilde{h})M = (k,h)M, \text{ i.e. } (k^{-1}\tilde{k}, h^{-1}\tilde{h}) \in M$$

and

$$(\tilde{h}',\tilde{g})L = (h',g)L, \text{ i.e. } (h'^{-1}\tilde{h}', g^{-1}\tilde{g}) \in L \ .$$

We need to show that

$$\mathrm{p}_2(M) \cdot h^{-1}h' \cdot \mathrm{p}_1(L) \overset{!}{=} \mathrm{p}_2(M) \cdot \tilde{h}^{-1}\tilde{h}' \cdot \mathrm{p}_1(L)$$

i.e. that there exist $m \in M$, $\ell \in L$ such that

$$\mathrm{p}_2(m) \cdot h^{-1}h' \cdot \mathrm{p}_1(\ell) \overset{!}{=} \tilde{h}^{-1}\tilde{h}' \ .$$

Choose

$$\begin{array}{rcl} m & = & (k^{-1}\tilde{k}, h^{-1}\tilde{h})^{-1} \in M \\ \ell & = & (h'^{-1}\tilde{h}', g^{-1}\tilde{g}) \in L \ . \end{array}$$

Then we obtain $\mathrm{p}_2(m) = (h^{-1}\tilde{h})^{-1}$, $\mathrm{p}_1(\ell) = h'^{-1}\tilde{h}'$ and therefore

$$\mathrm{p}_2(m) h^{-1}h' \,\mathrm{p}_1(\ell) = (h^{-1}\tilde{h})^{-1}(h^{-1}h')(h'^{-1}\tilde{h}') = \tilde{h}^{-1}\tilde{h}' \ .$$

So the map φ_0 is well-defined.

We need that the map

$$\begin{array}{rcl} \varphi_1 : V \underset{H}{\times} U & \to & \mathrm{p}_2(M)\backslash H/\,\mathrm{p}_1(L) \\ (k,h)M \underset{H}{\times} (h',g)L & \mapsto & \mathrm{p}_2(M)h^{-1}h'\,\mathrm{p}_1(L) \end{array}$$

is well-defined. We need to show that

$$\varphi_0(v\tilde{h}, u) \stackrel{!}{=} \varphi_0(v, \tilde{h}u)$$

for $\tilde{h} \in H$, $(v,u) \in V \times U$.

We have, writing $v = (k,h)M$ and $u = (h',g)L$ for some $k \in K$, $h, h' \in H$, $g \in G$, that

$$\begin{array}{rcl} \varphi_0((v\tilde{h}, u)) = \varphi_0(((k,h)M \cdot \tilde{h}, (h',g)L)) & = & \varphi_0(((k, \tilde{h}^{-1}h)M, (h',g)L)) \\ & = & \mathrm{p}_2(M)h^{-1}\tilde{h}h'\,\mathrm{p}_1(L) \\ & = & \varphi_0(((k,h)M, (\tilde{h}h', g)L)) \\ & = & \varphi_0(((k,h)M, \tilde{h} \cdot (h',g)L)) \\ & = & \varphi_0((v, \tilde{h}u)) \ . \end{array}$$

So, the map φ_1 is well-defined.

To show that φ is well-defined, we need that for $(\tilde{k}, \tilde{g}) \in K \times G$ and $(v,u) \in V \times U$

$$\varphi_1((\tilde{k}, \tilde{g}) \cdot (v \underset{H}{\times} u)) \stackrel{!}{=} \varphi_1(v \underset{H}{\times} u)$$

i.e. that

$$\varphi_1((\tilde{k}, \tilde{g}) \cdot ((k,h)M \underset{H}{\times} (h',g)L)) \stackrel{!}{=} \varphi_1((k,h)M \underset{H}{\times} (h',g)L) \ ,$$

writing $v = (k,h)M$ and $u = (h',g)L$ for some $k \in K$, $h, h' \in H$, $g \in G$.

We have, cf. Remark 19, Remark 31,

$$\begin{array}{rcl} \varphi_1((\tilde{k}, \tilde{g}) \cdot ((k,h)M \underset{H}{\times} (h',g)L)) & = & \varphi_1((\tilde{k} \cdot (k,h)M) \underset{H}{\times} ((h',g)L \cdot \tilde{g}^{-1})) \\ & = & \varphi_1((\tilde{k}k, h)M \underset{H}{\times} (h', \tilde{g}g)L) \\ & = & \mathrm{p}_2(M)h^{-1}h'\,\mathrm{p}_1(L) \\ & = & \varphi_1((k,h)M \underset{H}{\times} (h',g)L) \ . \end{array}$$

So, the map φ is well-defined.

Step 2 The map ψ is well-defined. We consider

$$\begin{tikzcd} h \arrow[d, mapsto] \\ \mathrm{p}_2(M)h\,\mathrm{p}_1(L) \end{tikzcd} \qquad \begin{tikzcd} H \arrow[d] \arrow[drr, "\psi_0"] & & \\ \mathrm{p}_2(M)\backslash H/\mathrm{p}_1(L) \arrow[rr, "\psi"] & & (K\times G)\backslash(V \underset{H}{\times} U)\ . \end{tikzcd}$$

The map

$$\begin{array}{rcl} \psi_0 : H & \to & (K\times G)\backslash(V \underset{H}{\times} U) \\ h & \mapsto & (K\times G)((1_K,1_H)M \underset{H}{\times} (h,1_G)L) \end{array}$$

is well-defined.

To show that ψ is well-defined, we need that for $m \in M$, $\ell \in L$ we have

$$\psi_0(h) \overset{!}{=} \psi_0(\mathrm{p}_2(m)h\,\mathrm{p}_1(\ell))\ .$$

We have

$$\begin{array}{rcl} \psi_0(\mathrm{p}_2(m)h\,\mathrm{p}_1(\ell)) & = & (K\times G)((1_K,1_H)M \underset{H}{\times} (\mathrm{p}_2(m)h\,\mathrm{p}_1(\ell),1_G)L) \\ & = & (K\times G)((1_K,1_H)M \underset{H}{\times} \mathrm{p}_2(m)\cdot((h,1_G)(\mathrm{p}_1(\ell),1_G))L) \\ & = & (K\times G)((1_K,1_H)M\cdot \mathrm{p}_2(m) \underset{H}{\times} ((h,1_G)(\mathrm{p}_1(\ell),1_G))L) \\ & = & (K\times G)((1_K,\mathrm{p}_2(m)^{-1})M \underset{H}{\times} ((h,1_G)(\mathrm{p}_1(\ell),1_G))L) \\ & = & (K\times G)(\underbrace{\mathrm{p}_1(m)^{-1}}_{\in K}\cdot(1_K,\mathrm{p}_2(m)^{-1})M \underset{H}{\times} ((h,1_G)(\mathrm{p}_1(\ell),1_G))L\cdot\underbrace{\mathrm{p}_2(\ell)^{-1}}_{\in G}) \\ & = & (K\times G)(\underbrace{(\mathrm{p}_1(m^{-1}),\mathrm{p}_2(m^{-1}))}_{m^{-1}}M \underset{H}{\times} ((h,1_G)\underbrace{(\mathrm{p}_1(\ell),\mathrm{p}_2(\ell))}_{\ell})L) \\ & = & (K\times G)((1_K,1_H)M \underset{H}{\times} (h,1_G)L) \\ & = & \psi_0(h)\ . \end{array}$$

So, ψ is well-defined.

Step 3 The maps φ and ψ are mutually inverse.

We have

$$\begin{array}{rcl} (\psi\circ\varphi)((K\times G)((k,h)M \underset{H}{\times} (h',g)L)) & = & \psi(\mathrm{p}_2(M)h^{-1}h'\,\mathrm{p}_1(L)) \\ & = & (K\times G)((1_K,1_H)M \underset{H}{\times} (h^{-1}h',1_G)L) \\ & = & (K\times G)(k\cdot(1_K,1_H)M \underset{H}{\times} h^{-1}(h',1_G)L\cdot g^{-1}) \\ & = & (K\times G)(k\cdot(1_K,1_H)M\cdot h^{-1} \underset{H}{\times} (h',1_G)L\cdot g^{-1}) \\ & = & (K\times G)((k,h)M \underset{H}{\times} (h',g)L) \end{array}$$

and

$$\begin{aligned}(\varphi \circ \psi)(\mathrm{p}_2(M)h\,\mathrm{p}_1(L)) &= \varphi((K\times G)((1_K,1_H)M \underset{H}{\times} (h,1_G)L)) \\ &= \mathrm{p}_2(M)h\,\mathrm{p}_1(L)\ .\end{aligned}$$

So, we have $\psi\circ\varphi = \mathrm{id}_{(K\times G)\backslash(V\underset{H}{\times}U)}$ and $\varphi\circ\psi = \mathrm{id}_{\mathrm{p}_2(M)\backslash H/\,\mathrm{p}_1(L)}$.

This proves the *claim*.

We obtain

$$\begin{aligned} V\underset{H}{\times}U &= \bigsqcup_{x\in(K\times G)\backslash(V\underset{H}{\times}U)} x \\ &= \bigsqcup_{y\in \mathrm{p}_2(M)\backslash H/\,\mathrm{p}_1(L)} \psi(y) \\ &= \bigsqcup_{y\in[\mathrm{p}_2(M)\backslash H/\,\mathrm{p}_1(L)]} (K\times G)((1_K,1_H)M \underset{H}{\times} (y,1_G)L)\ .\end{aligned}$$

As (K,G)-bisets, i.e. as left $(K\times G)$-sets, we have

$$(K\times G)((1_K,1_H)M \underset{H}{\times} (y,1_G)L) \;\cong\; (K\times G)/\operatorname{Stab}_{K\times G}((1_K,1_H)M \underset{H}{\times} (y,1_G)L)\ .$$

Moreover,

$$\begin{aligned} &\operatorname{Stab}_{K\times G}((1_K,1_H)M \underset{H}{\times} (y,1_G)L) \\ =\;& \{(k,g)\in K\times G : (k,g)((1_K,1_H)M \underset{H}{\times} (y,1_G)L) = (1_K,1_H)M \underset{H}{\times} (y,1_G)L\} \\ =\;& \{(k,g)\in K\times G : k\cdot(1_K,1_H)M \underset{H}{\times} (y,1_G)L\cdot g^{-1} = (1_K,1_H)M \underset{H}{\times} (y,1_G)L\} \\ =\;& \{(k,g)\in K\times G : (k,1_H)M \underset{H}{\times} (y,g)L = (1_K,1_H)M \underset{H}{\times} (y,1_G)L\} \\ =\;& \{(k,g)\in K\times G : \text{there exists } h\in H \text{ such that} \\ & \qquad (k,1_H)M\cdot h^{-1} = (1_K,1_H)M \text{ and } h\cdot(y,g)L = (y,1_G)L\} \\ =\;& \{(k,g)\in K\times G : \text{there exists } h\in H \text{ such that } (k,h)\in M \text{ and } (y^{-1}hy,g)\in L\} \\ =\;& \{(k,g)\in K\times G : \text{there exists } h\in H \text{ such that } (k,h)\in M \text{ and } (h,g)\in {}^{(y,1)}L\} \\ =\;& M * {}^{(y,1)}L\ ,\end{aligned}$$

cf. Remark 21, Remark 31, Remark 19, Lemma 41. $\square$

Chapter 2

The bifree double Burnside ring

Let K, H, G, P, Q be finite groups.

Let $\mathcal{L}_G$ denote a chosen system of representatives for the conjugacy classes of subgroups of G. I.e. for each subgroup $U \leq G$, there exists a unique subgroup $V \in \mathcal{L}_G$ such that U is conjugate in G to V.

Recall the set of twisted diagonal subgroups $\Delta_{H\times G}$ of $H \times G$, cf. Definition 48. Let $\mathcal{L}^{\Delta}_{H\times G} := \mathcal{L}_{H\times G} \cap \Delta_{H\times G}$.

For a ring B and a commutative ring R we write $B_R := R \underset{\mathbf{Z}}{\otimes} B$.

2.1 Definition and elementary properties

Notation 43. Let V be a subgroup of G and $U := {}^yV$ for some $y \in G$. We have the isomorphism

$$\kappa_y^{U,V} : V \to U,\, v \mapsto {}^yv \ .$$

In case of $U = V$ we set $\kappa_y^V := \kappa_y^{V,V}$.

We write $\mathrm{Inn}_G(V) := \{\kappa_g^V : g \in \mathrm{N}_G(V)\}$.

Remark 44. *The set* $\mathrm{Inn}_G(V)$ *is a subgroup of* $\mathrm{Aut}(V)$.

Proof. The map

$$\begin{array}{ccc} \mathrm{N}_G(V) & \xrightarrow{f} & \mathrm{Aut}(V) \\ g & \mapsto & \kappa_g^V \end{array}$$

is a group morphism as for $g, \tilde{g} \in \mathrm{N}_G(V)$, $v \in V$ we obtain

$$(\kappa_g^V \circ \kappa_{\tilde{g}}^V)(v) = {}^{g\tilde{g}}v = \kappa_{g\tilde{g}}^V(v) \ .$$

Then $\mathrm{im}(f) = \mathrm{Inn}_G(V) \leq \mathrm{Aut}(V)$. $\square$

Remark 45. *Let $U \leq H$ and $V \leq G$ be isomorphic subgroups. The set*

$$\mathrm{Isom}(V, U) := \{V \xleftarrow[\sim]{f} U : f \text{ is an isomorphism of groups}\}$$

is an $(\mathrm{Inn}_G(V), \mathrm{Inn}_H(U))$*-biset via*

$$\begin{array}{ccccccc} \mathrm{Inn}_G(V) & \times & \mathrm{Isom}(V,U) & \times & \mathrm{Inn}_H(U) & \to & \mathrm{Isom}(V,U) \\ (\kappa_g^V & , & f & , & \kappa_h^U) & \mapsto & \kappa_g^V \circ f \circ \kappa_h^U \ . \end{array}$$

Proof. For $f \in \mathrm{Isom}(V, U)$ we have

$$\mathrm{id}_V \circ f \circ \mathrm{id}_U = f \ .$$

Moreover, we have for $g, \tilde{g} \in \mathrm{N}_G(V)$ and for $h, \tilde{h} \in \mathrm{N}_H(U)$ that

$$\kappa_g^V \circ (\kappa_{\tilde{g}}^V \circ f \circ \kappa_h^U) \circ \kappa_{\tilde{h}}^U = (\kappa_g^V \circ \kappa_{\tilde{g}}^V) \circ f \circ (\kappa_h^U \circ \kappa_{\tilde{h}}^U) \ ,$$

cf. Notation 43.

So, $\mathrm{Isom}(V, U)$ is an $(\mathrm{Inn}_G(V), \mathrm{Inn}_H(U))$-biset, cf. Remark 15. □

Lemma 46. *Let U, V be isomorphic subgroups of G. Suppose that* $\mathrm{Inn}_G(U) = \mathrm{Aut}(U)$*. Then* (1,2) *hold.*

(1) $\mathrm{Isom}(U, V)$ *is a transitive* $(\mathrm{Inn}_G(U), \mathrm{Inn}_G(V))$*-biset.*

(2) $\mathrm{Isom}(V, U)$ *is a transitive* $(\mathrm{Inn}_G(V), \mathrm{Inn}_G(U))$*-biset.*

Proof. Ad (1). Let $\varphi, \psi \in \mathrm{Isom}(U, V)$.

As $\mathrm{Inn}_G(U) = \mathrm{Aut}(U)$ it suffices to show that there exists $\alpha \in \mathrm{Aut}(U)$ such that $\alpha \circ \varphi = \psi$. As φ is an isomorphism, we may set $\alpha = \psi \circ \varphi^{-1} \in \mathrm{Aut}(U)$.

Ad (2). Analogously, for $\rho, \tau \in \mathrm{Isom}(V, U)$, there exists $\beta \in \mathrm{Aut}(U)$ such that $\tau = \rho \circ \beta$, namely $\beta = \rho^{-1} \circ \tau$. □

Definition 47. Let L be a subgroup of $H \times G$. For $h \in H$, $g \in G$, we set the *right fibre* to be

$${}_hL := \{\tilde{g} \in G : (h, \tilde{g}) \in L\}$$

and the *left fibre* to be

$$L_g := \{\tilde{h} \in H : (\tilde{h}, g) \in L\} \ .$$

(1) We call L *right unique* if $|{}_hL| \leq 1$ for $h \in H$.

(2) We call L *left unique* if $|L_g| \leq 1$ for $g \in G$.

(3) We call L *right total* if $|{}_hL| \geq 1$ for $h \in H$.

(4) We call L *left total* if $|L_g| \geq 1$ for $g \in G$.

Definition 48. Let L be a subgroup of $H \times G$.

We call L a *twisted diagonal subgroup* if it is left unique and right unique.

I.e. L is a twisted diagonal subgroup if $|{}_hL| \leq 1$ for $h \in H$ and $|L_g| \leq 1$ for $g \in G$, cf. Definition 47.

We denote by $\Delta_{H \times G}$ the set of all twisted diagonal subgroups of $H \times G$.

Lemma 49.

(1) *Let $U \leq H$ and $V \leq G$ be isomorphic subgroups and $\alpha \in \mathrm{Isom}(U, V)$. Then*

$$\Delta(U, \alpha, V) := \{(\alpha(v), v) \in H \times G : v \in V\} \leq H \times G$$

is a twisted diagonal subgroup.

In case of $G = H$ we set $\Delta(U) := \Delta(U, \mathrm{id}, U)$.

(2) *For every twisted diagonal subgroup $L \leq H \times G$ there exist isomorphic subgroups $U \leq H$, $V \leq G$ and an isomorphism $\alpha : V \xrightarrow{\sim} U$ such that*

$$L = \Delta(U, \alpha, V) \ .$$

Proof. *Ad* (1). Write $L := \Delta(U, \alpha, V)$. Let $(\alpha(v), v), (\alpha(\tilde{v}), \tilde{v}) \in L$. Then

$$(\alpha(v), v) \cdot (\alpha(\tilde{v}), \tilde{v}) = (\alpha(v)\alpha(\tilde{v}), v\tilde{v}) = (\alpha(v\tilde{v}), v\tilde{v}) \in L$$

and

$$(\alpha(v), v)^{-1} = (\alpha(v)^{-1}, v^{-1}) = (\alpha(v^{-1}), v^{-1}) \in L \ .$$

So, L is a subgroup.

For $h \in H$ we obtain

$$\begin{aligned} {}_hL = \{g \in G : (h, g) \in L\} &= \{g \in G : (h, g) \in H \times G, g \in V, h = \alpha(g)\} \\ &= \begin{cases} \emptyset \text{ if } & h \notin U \\ \{\alpha^{-1}(h)\} \text{ if } & h \in U \ . \end{cases} \end{aligned}$$

So, $|{}_hL| \leq 1$.

For $g \in G$, we obtain

$$\begin{aligned} L_g = \{h \in H : (h, g) \in L\} &= \{h \in H : (h, g) \in H \times G, g \in V, h = \alpha(g)\} \\ &= \begin{cases} \emptyset \text{ if } & g \notin V \\ \{\alpha(g)\} \text{ if } & g \in V \ . \end{cases} \end{aligned}$$

So, $|L_g| \leq 1$.

Ad (2). Set

$$\begin{aligned} U &:= \{h \in H : {}_hL \neq \emptyset\} \subseteq H \\ V &:= \{g \in G : L_g \neq \emptyset\} \subseteq G \ . \end{aligned}$$

We show that U is a subgroup of H. As $(1_H, 1_G) \in L$, we have $1_H \in {}_{1_H}L$ and therefore $1_H \in U$. Suppose given $u, \tilde{u} \in U$. To show that $u\tilde{u}^{-1} \in U$, we have to show that

$${}_{u\tilde{u}^{-1}}L = \{x \in G : (u\tilde{u}^{-1}, x) \in L\} \overset{!}{\neq} \emptyset \ .$$

Since $u, \tilde{u} \in U$ there exist $g, \tilde{g} \in G$ such that $(u, g), (\tilde{u}, \tilde{g}) \in L$. As L is a subgroup we have

$$(u\tilde{u}^{-1}, g\tilde{g}^{-1}) = (u, g)(\tilde{u}, \tilde{g})^{-1} \in L$$

and therefore $g\tilde{g}^{-1} \in {}_{u\tilde{u}^{-1}}L$.

Likewise, V is a subgroup of G.

We need to show that there exists an isomorphism $\alpha : V \xrightarrow{\sim} U$. Suppose given $v \in V$ and $u \in U$. Then $L_v \neq \emptyset$ and ${}_uL \neq \emptyset$. Since $L \leq H \times G$ is a twisted diagonal subgroup it follows that $|L_v| = 1$ and $|{}_uL| = 1$.

Set

$$L_v =: \{\alpha_0(v)\} \ , \qquad {}_uL =: \{\beta_0(u)\} \ ,$$

so that $\alpha_0 : V \to H$ and $\beta_0 : U \to G$. We have $(\alpha_0(v), v), (u, \beta_0(u)) \in L$ and therefore $v \in {}_{\alpha_0(v)}L \neq \emptyset$ and $u \in L_{\beta_0(u)} \neq \emptyset$. So, $\alpha_0(v) \in U$ and $\beta_0(u) \in V$.

Set

$$\alpha := \alpha_0|^U : V \to U, \ \beta := \beta_0|^V : U \to V \ .$$

We show that the map $\alpha : V \to U$ is a group morphism. Suppose given $v, \tilde{v} \in V$. Then

$$(\alpha(v), v), (\alpha(\tilde{v}), \tilde{v}) \in L \text{ and therefore } (\alpha(v)\alpha(\tilde{v}), v\tilde{v}) \in L \ .$$

So $\alpha(v)\alpha(\tilde{v}) \in L_{v\tilde{v}} = \{\alpha(v\tilde{v})\}$. Hence, $\alpha(v)\alpha(\tilde{v}) = \alpha(v\tilde{v})$.

We now show that α and β are mutually inverse.

For $v \in V$ we have $(\alpha(v), v) \in L$ and $(\alpha(v), \beta(\alpha(v))) \in L$. Because of $|{}_{\alpha(v)}L| = 1$ it follows that $v = \beta(\alpha(v))$. Therefore, $\beta \circ \alpha = \mathrm{id}_V$.

For $u \in U$ we have $(u, \beta(u)) \in L$ and $(\alpha(\beta(u)), \beta(u)) \in L$. Because of $|L_{\beta(u)}| = 1$ it follows that $u = \alpha(\beta(u))$. So, $\alpha \circ \beta = \mathrm{id}_U$.

It remains to show that

$$L \overset{!}{=} \Delta(U, \alpha, V) \ .$$

As every element in $\Delta(U, \alpha, V)$ is of the form $(\alpha(v), v)$ for $v \in V$, it lies in L. So, $\Delta(U, \alpha, V) \subseteq L$. Conversely, let $(h, g) \in L$. Then $h \in L_g$ and therefore $\emptyset \neq L_g$. So, $g \in V$ and $h = \alpha(g)$. Hence,

$$(h, g) = (\alpha(g), g) \in \Delta(U, \alpha, V) \ .$$

So, $L \subseteq \Delta(U, \alpha, V)$.

Altogether, $L = \Delta(U, \alpha, V)$. $\square$

Example 50. Recall that $\Delta(G) = \Delta(G, \mathrm{id}, G) \leq G \times G$, cf. Lemma 49.
For $g \in G$, we have

$$ {}_g\Delta(G) = \{g' \in G : (g, g') \in \Delta(G)\} = \{g\} = \{g' \in G : (g', g) \in \Delta(G)\} = \Delta(G)_g \ . $$

So, the subgroup $\Delta(G) \leq G \times G$ is left total, right total, left unique and right unique.
This confirms that it is a twisted diagonal subgroup, cf. Lemma 49(1).

Lemma 51. *The (G, G)-bisets G and $(G \times G)/\Delta(G)$ are isomorphic via*

$$ \begin{array}{rcl} G & \overset{\sim}{\longleftrightarrow} & (G \times G)/\Delta(G) \\ g & \overset{\varphi}{\mapsto} & (g, 1_G)\Delta(G) \\ gh^{-1} & \overset{\psi}{\leftarrow\!\shortmid} & (g, h)\Delta(G) \ , \end{array} $$

cf. Example 16, Remark 19.
In particular, $\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G,G)} = [G] = [(G \times G)/\Delta(G)]$*, cf. Example 50.*

Proof. The map φ is a (G, G)-bimap as for $g \in G$ we get, given $a, b \in G$

$$ \begin{array}{rcl} \varphi(a \cdot g \cdot b) & = & (agb, 1_G)\Delta(G) = (ag, b^{-1})(b, b)\Delta(G) = (ag, b^{-1})\Delta(G) = a \cdot ((g, 1_G)\Delta(G)) \cdot b \\ & = & a \cdot \varphi(g) \cdot b \ . \end{array} $$

We show that the map ψ is well-defined.
For $(g, h), (a, b) \in G \times G$ with $(g, h)\Delta(G) = (a, b)\Delta(G)$ it follows that

$$ (a, b)^{-1}(g, h) = (a^{-1}g, b^{-1}h) \in \Delta(G) $$

and therefore

$$ a^{-1}g = b^{-1}h \ , \text{ i.e. } gh^{-1} = ab^{-1} \ . $$

So, the map ψ is well-defined.
We have for $g, h \in G$

$$ \psi(\varphi(g)) = \psi((g, 1)\Delta(G)) = g $$

and

$$ \varphi(\psi((g, h)\Delta(G))) = \varphi(gh^{-1}) = (gh^{-1}, 1_G)\Delta(G) = (gh^{-1}, 1_G)(h, h)\Delta(G) = (g, h)\Delta(G) \ . $$

So, φ and ψ are mutually inverse. $\square$

Remark 52. The conjugate of a twisted diagonal subgroup in $H \times G$ is again a twisted diagonal subgroup.

More precisely, using Lemma 49, we obtain the following. Let $U \leq H$ and $V \leq G$ be isomorphic subgroups and $\alpha \in \mathrm{Isom}(U, V)$.

For $(x, y) \in H \times G$, we have

$$\begin{aligned} {}^{(x,y)}\Delta(U, \alpha, V) &= \{({}^{x}\alpha(v), {}^{y}v) : v \in V\} \\ &= \{{}^{x}\alpha({}^{y^{-1}}v), v) : v \in {}^{y}V\} \\ &= \Delta({}^{x}U, \kappa_{x}^{{}^{x}U,U} \circ \alpha \circ \kappa_{y^{-1}}^{V,{}^{y}V}, {}^{y}V) \ . \end{aligned}$$

So, $\Delta_{H \times G}$ becomes an $(H \times G)$-set via conjugation, as an $(H \times G)$-subset of the set of subgroups of $H \times G$.

Lemma 53. *Suppose given a subgroup L of $H \times G$. Suppose given $(h', g') \in H \times G$.*

(1) *If L is right unique, then ${}^{(h',g')}L$ is right unique.*

(2) *If L is left unique, then ${}^{(h',g')}L$ is left unique.*

(3) *If L is right total, then ${}^{(h',g')}L$ is right total.*

(4) *If L is left total, then ${}^{(h',g')}L$ is left total.*

In particular, this shows again that if L is a twisted diagonal subgroup, then ${}^{(h',g')}L$ is a twisted diagonal subgroup, cf. Remark 52.

Proof. Suppose given $h \in H$ and $g \in G$. Then

$$\begin{aligned} {}_{h}\left({}^{(h',g')}L\right) &= \{\tilde{g} \in G : (h, \tilde{g}) \in {}^{(h',g')}L\} \\ &= \{\tilde{g} \in G : (h'^{-1}, g'^{-1})(h, \tilde{g})(h', g') \in L\} \\ &= \{\tilde{g} \in G : (h'^{-1}hh', g'^{-1}\tilde{g}g') \in L\} \\ &= \{\tilde{g} \in G : g'^{-1}\tilde{g}g' \in {}_{h'^{-1}hh'}L\} \\ &= {}^{g'}\left({}_{h'^{-1}hh'}L\right) \end{aligned}$$

and

$$\begin{aligned} \left({}^{(h',g')}L\right)_{g} &= \{\tilde{h} \in H : (\tilde{h}, g) \in {}^{(h',g')}L\} \\ &= \{\tilde{h} \in H : (h'^{-1}, g'^{-1})(\tilde{h}, g)(h', g') \in L\} \\ &= \{\tilde{h} \in H : (h'^{-1}\tilde{h}h', g'^{-1}gg') \in L\} \\ &= \{\tilde{h} \in H : h'^{-1}\tilde{h}h' \in L_{g'^{-1}gg'}\} \\ &= {}^{h'}\left(L_{g'^{-1}gg'}\right) \ . \end{aligned}$$

Since the notations in Definition 47 are defined via cardialities of left respectively right fibres, the assertions (1-4) follow. $\square$

Lemma 54. *Suppose given subgroups M of $K \times H$ and L of $H \times G$. Recall that*

$$M*L = \{(k,g) \in K\times G : \textit{there exists } h \in H \textit{ such that } (k,h) \in M \textit{ and } (h,g) \in L\} \leq K\times G\ ,$$

cf. Lemma 41.

(1) *If M and L are right unique, then $M * L$ is right unique.*

(2) *If M and L are left unique, then $M * L$ is left unique.*

(3) *If M and L are right total, then $M * L$ is right total.*

(4) *If M and L are left total, then $M * L$ is left total.*

In particular, if M and L are twisted diagonal subgroups, then $M * L$ is a twisted diagonal subgroup.

> These assertions are true for relations in general. Nonetheless, we prove them only in our particular case of subgroups.

Proof. We consider

$$\begin{array}{rcl} M * L & = & \{(k,g) \in K \times G : \text{there exists } h \in H \text{ such that } (k,h) \in M \text{ and } (h,g) \in L\} \\ & = & \{(k,g) \in K \times G : \text{there exists } h \in H \text{ such that } h \in {}_kM \text{ and } g \in {}_hL\} \\ & = & \{(k,g) \in K \times G : \text{there exists } h \in H \text{ such that } h \in L_g \text{ and } k \in M_h\}\ . \end{array}$$

Ad (1). Let M and L be right unique, i.e. $|{}_kM| \leq 1$ for $k \in K$ and $|{}_hL| \leq 1$ for $h \in H$.

For $k \in K$, we have

$${}_k(M * L) = \{g \in G : (k,g) \in M * L\} = \left\{ \begin{array}{rl} \emptyset & \text{if} \quad |{}_kM| = 0 \\ {}_hL & \text{if} \quad |{}_kM| = 1 \text{ and } {}_kM =: \{h\}\ . \end{array} \right.$$

Thus, $|{}_k(M * L)| \leq 1$.

Ad (2). Let M and L be left unique, i.e. $|M_h| \leq 1$ for $h \in H$ and $|L_g| \leq 1$ for $g \in G$.

For $g \in G$, we have

$$(M * L)_g = \{k \in K : (k,g) \in M * L\} = \left\{ \begin{array}{rl} \emptyset & \text{if} \quad |L_g| = 0 \\ M_h & \text{if} \quad |L_g| = 1 \text{ and } L_g =: \{h\}\ . \end{array} \right.$$

Thus, $|(M * L)_g| \leq 1$.

Ad (3). Let M and L be right total, i.e. $|{}_kM| \geq 1$ for $k \in K$ and $|{}_hL| \geq 1$ for $h \in H$.

For $k \in K$, we have

$${}_k(M * L) = \{g \in G : (k,g) \in M * L\} = \bigcup_{h \in {}_kM} {}_hL\ .$$

Thus, $|_k(M * L)| \geq 1$.

Ad (4). Let M and L be left total, i.e. $|M_h| \geq 1$ for $h \in H$ and $|L_g| \geq 1$ for $g \in G$.

For $g \in G$, we have

$$(M * L)_g = \{k \in K : (k, g) \in M * L\} = \bigcup_{h \in\ L_g} M_h \ .$$

Thus, $|(M * L)_g| \geq 1$. $\square$

Definition 55. An (H, G)-biset X is called

(1) *left-free* if $\mathrm{Stab}_H^{\mathrm{left}}(x) = \{h \in H : hx = x\} = \{1_H\}$ for $x \in X$, cf. Definition 22.

(2) *right-free* if $\mathrm{Stab}_G^{\mathrm{right}}(x) = \{g \in G : xg = x\} = \{1_G\}$ for $x \in X$, cf. Definition 22.

(3) *bifree* if it is left- and right-free.

Remark 56. Let X_1, X_2 be (H, G)-bisets. Consider the disjoint union $X_1 \sqcup X_2$, which is an (H, G)-biset, cf. Lemma 34(1). Then $\mathrm{Stab}_H^{\mathrm{left}}((x_i, i)) = \mathrm{Stab}_H^{\mathrm{left}}(x_i)$ for $i \in \{1, 2\}$, $x_i \in X_i$. So, $X_1 \sqcup X_2$ is bifree if and only if X_1 and X_2 are bifree.

Remark 57. Let X be a bifree (H, G)-biset. Let $H' \leq H$ and $G' \leq G$ be subgroups. Let $X' \subseteq X$ be an (H', G')-sub-biset. Then X' is a bifree (H', G')-biset.

Remark 58. Suppose given an (H, G)-biset X and a left-free (G, P)-biset Y, i.e. $\mathrm{Stab}_H^{\mathrm{left}}(y) = \{g \in G : gy = y\} = \{1_G\}$ for $y \in Y$, i.e. the map $G \to Y$, $g \mapsto g \cdot y$ is injective for $y \in Y$. Then

$$|X \times Y| = |G| \cdot |X \underset{G}{\times} Y| \ ,$$

as for $x \underset{G}{\times} y \in X \underset{G}{\times} Y$ we have $(x \underset{G}{\times} y) = (xg^{-1} \underset{G}{\times} gy)$ for $g \in G$ and therefore the equivalence class $x \underset{G}{\times} y$ consists of $|G|$ elements.

Lemma 59. *An (H, G)-biset X is bifree if and only if* $\mathrm{Stab}_{H \times G}(x)$ *is a twisted diagonal subgroup of $H \times G$ for all $x \in X$.*

Proof. Let X be bifree. Suppose given $x \in X$. We need to show that $|_h(\mathrm{Stab}_{H \times G}(x))| \overset{!}{\leq} 1$ for $h \in H$ and that $|(\mathrm{Stab}_{H \times G}(x))_g| \overset{!}{\leq} 1$ for $g \in G$, cf. Definition 48.

For $h \in H$, we have

$$_h(\mathrm{Stab}_{H \times G}(x)) = \{\tilde{g} \in G : (h, \tilde{g}) \in \mathrm{Stab}_{H \times G}\}(x) = \{\tilde{g} \in G : hx = x\tilde{g}\} \ .$$

If $|_h(\mathrm{Stab}_{H \times G}(x))| = 0$ then the statement is true.

Consider the case $|_h(\mathrm{Stab}_{H \times G}(x))| > 0$. Then there exists $g \in G$ such that $hx = xg$.

Assume that $|_h(\mathrm{Stab}_{H \times G}(x))| > 1$. Then there exists $\tilde{g} \in G$ such that $g \neq \tilde{g}$ and $hx = x\tilde{g}$. Then $xg = hx = x\tilde{g}$ and therefore $x = x\tilde{g}g^{-1}$. As $\mathrm{Stab}_G^{\mathrm{right}}(x) = \{g \in G : xg = x\} = \{1_G\}$, it follows that $g = \tilde{g}$. *Contradiction.*

So, $|_h(\mathrm{Stab}_{H\times G}(x))| \leq 1$. Analogously it follows that $|(\mathrm{Stab}_{H\times G}(x))_g|\leq 1$ for $g \in G$.

Conversely, suppose that $\mathrm{Stab}_{H\times G}(x)$ is a twisted diagonal subgroup of $H \times G$ for all $x \in X$, cf. Definition 48. We have

$$1 \geq |_{1_H}\mathrm{Stab}_{H\times G}(x)| = |\{\tilde{g} \in G : (1_H, \tilde{g}) \in \mathrm{Stab}_{H\times G}(x)\}| = |\{\tilde{g} \in G : x = x\tilde{g}\}| = |\operatorname{Stab}_G^{\mathrm{right}}(x)| \,.$$

As $1_G \in \mathrm{Stab}_G^{\mathrm{right}}(x)$ it follows that $\mathrm{Stab}_G^{\mathrm{right}}(x) = \{1_G\}$ for $x \in X$.

We have

$$1 \geq |\mathrm{Stab}_{H\times G}(x)_{1_G}| = |\{\tilde{h} \in H : (\tilde{h}, 1_G) \in \mathrm{Stab}_{H\times G}(x)\}| = |\{\tilde{h} \in H : \tilde{h}x = x\}| = |\operatorname{Stab}_H^{\mathrm{left}}(x)| \,.$$

As $1_H \in \mathrm{Stab}_H^{\mathrm{left}}(x)$ it follows that $\mathrm{Stab}_H^{\mathrm{left}}(x) = \{1_H\}$ for $x \in X$.

So, X is bifree. $\square$

Lemma 60. *Suppose given a subgroup L of $H \times G$.*

Then L is a twisted diagonal subgroup of $H \times G$ if and only if $(H \times G)/L$ is bifree.

Proof. Suppose that L is a twisted diagonal subgroup of $H \times G$.

By Lemma 59 it suffices to show that for $(\tilde{h}, \tilde{g})L \in (H \times G)/L$, we have that $\mathrm{Stab}_{H\times G}((\tilde{h}, \tilde{g})L)$ is a twisted diagonal subgroup.

But, $\mathrm{Stab}_{H\times G}((\tilde{h}, \tilde{g})L) = {}^{(\tilde{h},\tilde{g})}L$, cf. Remark 6.

By Lemma 53 the conjugate of a twisted diagonal subgroup is a twisted diagonal subgroup. So, it follows that $\mathrm{Stab}_{H\times G}((\tilde{h}, \tilde{g})L)$ is a twisted diagonal subgroup. Thus, $(H \times G)/L$ is bifree.

Conversely, suppose that $(H \times G)/L$ is bifree. Then it follows that

$$L = {}^{(1_H,1_G)}L = \mathrm{Stab}_{H\times G}((1_H, 1_G)L) \;,$$

is a twisted diagonal subgroup, cf. Remark 6, Lemma 59. $\square$

Remark 61. Recall that each transitive bifree (H, G)-biset M is up to isomorphism of the form $(H \times G)/L$ for some subgroup $L \leq H \times G$, cf. Remark 19.

By Lemma 60 the (H, G)-biset M is bifree if and only if L is a twisted diagonal subgroup.

Lemma 62. *Let X be a bifree (H, G)-biset and Y be a bifree (G, P)-biset. Then*

$$\text{the } (H, P)\text{-biset } X \underset{G}{\times} Y \text{ is bifree },$$

cf. Remark 31.

Proof. We show that for $x \in X$ and $y \in Y$

$$\mathrm{Stab}_H^{\mathrm{left}}(x \underset{G}{\times} y) \overset{!}{=} \{1_H\} \;.$$

We have $\mathrm{Stab}_H^{\mathrm{left}}(x) = \{1_H\}$ and $\mathrm{Stab}_G^{\mathrm{left}}(y) = \{1_G\}$.

Suppose given $h \in H$ such that $h(x \underset{G}{\times} y) = x \underset{G}{\times} y$. We need to show that $h \overset{!}{=} 1_H$. We have

$$(hx) \underset{G}{\times} y = h(x \underset{G}{\times} y) = x \underset{G}{\times} y \ ,$$

i.e. there exists $g \in G$ such that

$$(hx, y) = (xg, g^{-1}y) \ ,$$

i.e. $hx = xg$ and $y = g^{-1}y$. However, $\mathrm{Stab}_G^{\mathrm{left}}(y) = \{1_G\}$ and therefore it follows that $g = 1_G$ and $hx = x$. Since $\mathrm{Stab}_H^{\mathrm{left}}(x) = \{1_H\}$, we have $h = 1_H$.

Analogously we obtain $\mathrm{Stab}_P^{\mathrm{right}}(x \underset{G}{\times} y) = \{1_P\}$. □

Remark 63. Suppose given twisted diagonal subgroups M of $H \times G$ and L of $G \times P$. By Lemma 42, Definition 23 and Remark 37 we have

$$[((H \times G)/M) \underset{G}{\times} ((G \times P)/L)] = [\ \textstyle\bigsqcup_{h \in [\mathrm{p}_2(M)\backslash G/\,\mathrm{p}_1(L)]} (H \times P)/(M * {}^{(h,1)}L)\ ] \ ,$$

where $[\mathrm{p}_2(M)\backslash G/\,\mathrm{p}_1(L)]$ denotes an arbitrarily chosen set of double coset representatives.

As $(H \times G)/M$ and $(G \times P)/L$ are bifree, cf. Lemma 60, so is $(H \times P)/(M * {}^{(h,1)}L)$, cf. Lemma 62, Lemma 27, Remark 56.

Hence, $M * {}^{(h,1)}L$ is a twisted diagonal subgroup of $H \times P$, cf. Lemma 60.

Alternatively, by Lemma 53 and Lemma 54 we have that $M * {}^{(h,1)}L$ is a twisted diagonal subgroup of $H \times P$.

Definition 64. We define *the bifree double Burnside group*

$$\begin{array}{rcl}
\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) & := & {}_{\mathbf{Z}}\langle [M] :\ M \text{ is a bifree finite } (H,G)\text{-biset}\rangle \\
& = & {}_{\mathbf{Z}}\langle [M] :\ M \text{ is a bifree finite transitive } (H,G)\text{-biset}\rangle \\
& = & {}_{\mathbf{Z}}\langle [(H \times G)/L] :\ L \in \mathcal{L}^{\Delta}_{H \times G}\rangle \\
& \leq & \mathrm{B}_{\mathbf{Z}}(H,G) \ ,
\end{array}$$

cf. Remark 61.

We call the basis $([(H\times G)/L] :\ L \in \mathcal{L}^{\Delta}_{H\times G})$ the *standard basis* of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$, cf. Lemma 28, Lemma 8, Lemma 27.

Remark 65. By Lemma 62, or by Lemma 42, Lemma 54 and Lemma 53, we have for $M \in \mathcal{L}^{\Delta}_{H\times G}$ and $L \in \mathcal{L}^{\Delta}_{G\times P}$ that

$$[(H \times G)/M] \underset{G}{\cdot} [(G \times P)/L] \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,P) \ .$$

Therefore, the composition map

$$\begin{array}{rcl}
(\ \underset{G}{\cdot}\ ) : \mathrm{B}_{\mathbf{Z}}(H,G) \times \mathrm{B}_{\mathbf{Z}}(G,P) & \to & \mathrm{B}_{\mathbf{Z}}(H,P) \\
([X],[Y]) & \mapsto & [X \underset{G}{\times} Y]
\end{array}$$

from Remark 37 restricts to

$$\begin{aligned}(\,\underset{G}{\cdot}\,) := (\,\underset{G}{\cdot}\,)|^{\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,P)}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)\times\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,P)} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)\times\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,P) &\to \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,P)\\ ([X],[Y]) &\mapsto [X\underset{G}{\times}Y]\,.\end{aligned}$$

We have

$$\begin{tikzcd} \mathrm{B}_{\mathbf{Z}}(H,G)\times\mathrm{B}_{\mathbf{Z}}(G,P) \arrow[rr, "(\,\underset{G}{\cdot}\,)"] & & \mathrm{B}_{\mathbf{Z}}(H,P) \\ \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)\times\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,P) \arrow[u, hook] \arrow[rr, "(\,\underset{G}{\cdot}\,):=(\,\underset{G}{\cdot}\,)|^{\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,P)}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)\times\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,P)}"] & & \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,P) \arrow[u, hook] \end{tikzcd}\,.$$

Remark 66. As the composition map $(\,\underset{G}{\cdot}\,) : \mathrm{B}_{\mathbf{Z}}(H,G)\times\mathrm{B}_{\mathbf{Z}}(G,P)\to\mathrm{B}_{\mathbf{Z}}(H,P)$, $([X],[Y])\mapsto[X\underset{G}{\times}Y]$ restricts to $\mathrm{B}^{\Delta}_{\mathbf{Z}}$ by Remark 65 and since $\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G,G)} = [G]\in\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ by Example 50 and Lemma 51, we have a non-full preadditive subcategory $\mathsf{Burnside}^{\Delta}$ of $\mathsf{Burnside}$, cf. Remark 38.

The objects of $\mathsf{Burnside}^{\Delta}$ are finite groups, the abelian group of morphisms from H to G is given by $\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$ and composition is given by $(\,\cdot\,) : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)\times\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,P)\to\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,P)$.

Definition 67. The *bifree double Burnside ring* $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ is the endomorphism ring of G in $\mathsf{Burnside}^{\Delta}$, cf. Remark 66. So, $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ is a subring of $\mathrm{B}_{\mathbf{Z}}(G,G)$. We have the $\mathbf{Z}$-linear basis

$$([(G\times G)/L] : \; L\in\mathcal{L}^{\Delta}_{G\times G})$$

of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$, cf. Definition 64.

Moreover, $\mathrm{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)} = \mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G,G)} = [(G\times G)/\Delta(G)]\in\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$, cf. Lemma 51 and Example 50.

Remark 68. Analogously to Definition 67 one can define the following subrings of $\mathrm{B}_{\mathbf{Z}}(G,G)$.

(1) $R_1 := {}_{\mathbf{Z}}\langle[(G\times G)/L] : \; L$ is a left unique subgroup of $G\times G\rangle$

(2) $R_2 := {}_{\mathbf{Z}}\langle[(G\times G)/L] : \; L$ is a right unique subgroup of $G\times G\rangle$

(3) $R_3 := {}_{\mathbf{Z}}\langle[(G\times G)/L] : \; L$ is a left total subgroup of $G\times G\rangle$

(4) $R_4 := {}_{\mathbf{Z}}\langle[(G\times G)/L] : \; L$ is a right total subgroup of $G\times G\rangle$

Cf. Lemma 42, Lemma 54, Lemma 53, Lemma 51 and Example 50.

We have e.g. $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G) = R_1\cap R_2$, cf. Definition 48, Definition 67.

2.2 The Burnside ring inside the bifree double Burnside ring

Lemma 69. *Let X be a left G-set. The set $G \times X$ is a (G,G)-biset for the action defined by*

$$a \cdot (g,x) \cdot b := (agb, b^{-1}x)$$

for $a, b, g \in G$, $x \in X$.

Proof. We have for $a, \tilde{a}, b, \tilde{b}, g \in G$ and $x \in X$

$$1_G \cdot (g,x) \cdot 1_G = (1_G \cdot g \cdot 1_G, 1_G \cdot x) = (g,x)$$

and

$$\begin{array}{rcl} a \cdot (\tilde{a} \cdot (g,x) \cdot b) \cdot \tilde{b} & = & a \cdot (\tilde{a} \cdot g \cdot b, b^{-1} \cdot x) \cdot \tilde{b} \\ & = & (a \cdot \tilde{a} \cdot g \cdot b \cdot \tilde{b}, \tilde{b}^{-1} \cdot b^{-1} \cdot x) \\ & = & (a\tilde{a} \cdot g \cdot b\tilde{b}, (b\tilde{b})^{-1} \cdot x) \\ & = & (a\tilde{a}) \cdot (g,x) \cdot (b\tilde{b}) \ . \end{array}$$

By Remark 15, the set $G \times X$ is a (G,G)-biset. □

Lemma 70. *Suppose given left G-sets X_1, X_2. Then we have mutually inverse isomorphisms of (G,G)-bisets*

$$\begin{array}{rcl} G \times (X_1 \sqcup X_2) & \overset{\sim}{\leftrightarrow} & (G \times X_1) \sqcup (G \times X_2) \\ (g,(x,i)) & \overset{\varphi}{\mapsto} & ((g,x),i) \\ (g,(x,i)) & \overset{\varphi'}{\leftarrow\!\shortmid} & ((g,x),i) \ , \end{array}$$

for $i \in \{1,2\}$.

Proof. Both maps are well-defined and mutually inverse.

We show that φ is a (G,G)-bimap. Suppose given $a, b \in G$. Then

$$\begin{array}{rcl} \varphi(a \cdot (g,(x,i)) \cdot b) & = & \varphi((agb, b^{-1}(x,i))) = \varphi((agb,(b^{-1}x,i))) \\ & = & ((agb, b^{-1}x),i) = ((a \cdot (g,x) \cdot b),i) \\ & = & a \cdot ((g,x),i) \cdot b = a \cdot \varphi((g,(x,i))) \cdot b \end{array}$$

for $i \in \{1,2\}$ and $x \in X_i$. □

Lemma 71. *Suppose given $U \le G$. Consider the left G-set G/U. Consider the (G,G)-biset $G \times (G/U)$ as in Lemma 69. Consider the (G,G)-biset $(G \times G)/\Delta(U)$ as in Remark 19.*

We have mutually inverse isomorphisms of (G,G)-bisets

$$\begin{array}{rcl} G \times (G/U) & \overset{\sim}{\leftrightarrow} & (G \times G)/\Delta(U) \\ (g,hU) & \overset{\varphi}{\mapsto} & (gh,h)\Delta(U) \\ (uv^{-1},vU) & \overset{\psi}{\leftarrow\!\shortmid} & (u,v)\Delta(U) \ . \end{array}$$

Proof. We need to show that the map φ is a well-defined (G,G)-bimap. Suppose given $g, \tilde{g}, h, \tilde{h} \in G$ such that $(g, hU) = (\tilde{g}, \tilde{h}U)$, i.e. such that $g = \tilde{g}$ and $\tilde{h}^{-1}h \in U$. We obtain

$$(\tilde{g}\tilde{h}, \tilde{h})^{-1}(gh, h) = (\tilde{h}^{-1}\tilde{g}^{-1}gh, \tilde{h}^{-1}h) = (\tilde{h}^{-1}h, \tilde{h}^{-1}h) \in \Delta(U) \ .$$

Suppose given $a, b, g, h \in G$. Then

$$\varphi(a(g, hU)b) = \varphi((agb, b^{-1}hU)) = (agbb^{-1}h, b^{-1}h)\Delta(U) = a(gh, h)\Delta(U)b = a\varphi(g, hU)b \ .$$

So, φ is a well-defined (G,G)-bimap.

We show that ψ is a well defined map.

Suppose given $g, \tilde{g}, h, \tilde{h} \in G$ such that $(g, h)\Delta(U) = (\tilde{g}, \tilde{h})\Delta(U)$. Then

$$(g^{-1}\tilde{g}, h^{-1}\tilde{h}) \in \Delta(U)$$

and therefore $g^{-1}\tilde{g} = h^{-1}\tilde{h} \in U$, i.e. $\tilde{g}\tilde{h}^{-1} = gh^{-1}$ and $h^{-1}\tilde{h} \in U$. So,

$$(gh^{-1}, hU) = (\tilde{g}\tilde{h}^{-1}, \tilde{h}U) \ .$$

Hence, the map ψ is well-defined.

Moreover, the maps φ, ψ are mutually inverse. $\square$

Remark 72. We denote by $\mathrm{B}_{\mathbf{Z}}(G)$ the Burnside ring of G. Recall that $\mathrm{B}_{\mathbf{Z}}(G)$ is the abelian group freely generated by the isomorphism classes of finite left G-sets, modulo the relations

$$[X \sqcup Y] = [X] + [Y] \text{ for left } G\text{-sets } X, Y.$$

Multiplication is defined by

$$[X] \cdot [Y] = [X \times Y] \text{ for left } G\text{-sets } X, Y.$$

It has the $\mathbf{Z}$-linear basis $([G/U] : U \in \mathcal{L}_G)$. Moreover, $\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G)} = [G/G]$.

For more details on $\mathrm{B}_{\mathbf{Z}}(G)$ see [3, §80].

Lemma 73. ([2, cf. Lemma 2.5.8]) *We have the injective ring morphism*

$$\begin{array}{rcl} \delta : \mathrm{B}_{\mathbf{Z}}(G) & \to & \mathrm{B}^{\Delta}_{\mathbf{Z}}(G, G) \\ {[X]} & \mapsto & [G \times X] \end{array}$$

where X is a finite left G-set. Recall that then $G \times X$ is a finite (G,G)-biset, cf. Lemma 69.

Proof. We show that δ is well-defined. Suppose given finite left G-sets X and X' such that $[X] = [X']$, i.e. $X \cong X'$. Let $\alpha : X \xrightarrow{\sim} X'$ be an isomorphism of left G-sets. We have to show that $[G \times X] = [G \times X']$. We have mutually inverse isomorphisms of (G,G)-bisets

$$\begin{array}{rcl} G \times X & \overset{\sim}{\leftrightarrow} & G \times X' \\ (g, x) & \overset{\varphi}{\mapsto} & (g, \alpha(x)) \\ (g, \alpha^{-1}(x')) & \overset{\psi}{\hookleftarrow} & (g, x') \ . \end{array}$$

In fact, φ is a (G,G)-bimap as for $a, b, \in G$ we have

$$\varphi(a{\cdot}(g,x){\cdot}b) = \varphi((agb, b^{-1}x)) = (agb, \alpha(b^{-1}x)) = (agb, b^{-1}\alpha(x)) = a{\cdot}(g,\alpha(x)){\cdot}b = a{\cdot}\varphi(g,x){\cdot}b\,.$$

So, $[G \times X] = [G \times X']$.

Moreover, for finite left G-sets X and X' we have $G \times (X \sqcup X') \cong (G \times X) \sqcup (G \times X')$ and therefore $[G \times (X \sqcup X')] = [G \times X] + [G \times X']$. So, δ is well-defined by the universal property of $\mathrm{B}_{\mathbf{Z}}(G)$.

To obtain that δ is injective, we show that different basis elements are mapped to a $\mathbf{Z}$-linearly independent set.

We have for $U \leq G$ that $G \times (G/U) \cong (G \times G)/\Delta(U)$, cf. Lemma 71.

Suppose given $[G/U], [G/V] \in \mathrm{B}_{\mathbf{Z}}(G)$. We have

$$\begin{array}{rcl} G/U \cong G/V & \overset{\text{L.8}}{\Leftrightarrow} & \text{there exists } g \in G \text{ with } {}^{g}U = V \\ & \Leftrightarrow & \text{there exists } (g, g') \in G \times G \text{ with } {}^{g}U = V \text{ and } {}^{g'}U = V \\ & \xLeftrightarrow{\text{L.8, R.52}} & (G \times G)/\Delta(U) \cong (G \times G)/\Delta(V)\ . \end{array}$$

The $\mathbf{Z}$-linear basis $([G/U] : U \in \mathcal{L}_G)$ of $\mathrm{B}_{\mathbf{Z}}(G)$ is mapped to $([G \times G)/\Delta(U)] : U \in \mathcal{L}_G)$, where $\Delta(U)$ and $\Delta(V)$ are not conjugate in $G \times G$ for $U, V \in \mathcal{L}_G$ with $U \neq V$.

Picking $\Delta(U)$ for $U \in \mathcal{L}_G$ as representatives for their conjugacy classes in $G \times G$, we see that the tuple $([G \times G)/\Delta(U)] : U \in \mathcal{L}_G)$ can be extended to a $\mathbf{Z}$-linear basis of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ and is therefore $\mathbf{Z}$-linearly independent, cf. Definition 64.

Hence, δ is injective.

To show that δ is a ring morphism, we need that $\delta([X] \cdot [Y]) = \delta([X]) \underset{G}{\cdot} \delta([Y])$ for finite left G-sets X and Y. I.e. that

$$G \times (X \times Y) \cong (G \times X) \underset{G}{\times} (G \times Y)\ .$$

We consider the maps

$$\begin{array}{rcl} G \times (X \times Y) & \leftrightarrow & (G \times X) \underset{G}{\times} (G \times Y) \\ (g, (x,y)) & \overset{\alpha}{\mapsto} & (g,x) \underset{G}{\times} (1_G, y) \\ (gh, (h^{-1}x, y)) & \overset{\beta}{\leftarrow\!\shortmid} & (g,x) \underset{G}{\times} (h,y)\ . \end{array}$$

The map α is a (G,G)-bimap as for $a, b \in G$ and $(g,(x,y)) \in G \times (X \times Y)$ we have

$$
\begin{aligned}
\alpha(a\cdot(g,(x,y))\cdot b) &= \alpha((agb,(b^{-1}x,b^{-1}y)))\\
&= (agb,b^{-1}x)\underset{G}{\times}(1_G,b^{-1}y)\\
&= (ag,x)b\underset{G}{\times}(1_G,b^{-1}y)\\
&= (ag,x)\underset{G}{\times}b(1_G,b^{-1}y)\\
&= (ag,x)\underset{G}{\times}(b,b^{-1}y)\\
&= ((ag,x)\underset{G}{\times}(1_G,y)b)\\
&= a((g,x)\underset{G}{\times}(1_G,y))b\\
&= a\cdot\alpha(g,x,y)\cdot b\ .
\end{aligned}
$$

We show that the map β is well-defined. For $a\in G$ and $(g,x)\in G\times X$ and $(h,y)\in G\times Y$ we have $((g,x)a\underset{G}{\times}a^{-1}(h,y))=(g,x)\underset{G}{\times}(h,y)$. Moreover, we have $(g,x)a=(ga,a^{-1}x)$, $a^{-1}(h,y)=(a^{-1}h,y)$ and $(gh,(a^{-1}h)^{-1}a^{-1}x,y)=(gh,(h)^{-1}x,y)$.

Moreover, $\beta\circ\alpha$ and $\alpha\circ\beta$ are the identity maps as

$$(\beta\circ\alpha)((g,x,y))=\beta((g,x)\underset{G}{\times}(1_G,y))=(g,x,y)$$

for $(g,(x,y))\in G\times(X\times Y)$ and

$$
\begin{aligned}
(\alpha\circ\beta)(((g,x)\underset{G}{\times}(h,y))) &= \alpha((gh,(h^{-1}x,y)))\\
&= (gh,h^{-1}x)\underset{G}{\times}(1_G,y)\\
&= (g,x)h\underset{G}{\times}(1_G,y)\\
&= (g,x)\underset{G}{\times}h(1_G,y)\\
&= (g,x)\underset{G}{\times}(h,y)
\end{aligned}
$$

for $(g,x)\in G\times X$ and $(h,y)\in G\times Y$.

Furthermore $\delta(\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G)})=\delta([G/G])=[G\times(G/G)]=[(G\times G)/\Delta(G)]=\mathrm{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)}$. □

Corollary 74. Consider the injective ring morphism

$$
\begin{aligned}
\delta:\mathrm{B}_{\mathbf{Z}}(G) &\to \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)\\
[G/U] &\mapsto [G\times(G/U)]=[(G\times G)/\Delta(U)]
\end{aligned}
$$

from Lemma 73. The ring morphism is also surjective if and only if every twisted diagonal subgroup of $G\times G$ is conjugate to a twisted diagonal subgroup of the form

$$\Delta(W)\ \text{for}\ W\leq G\ ,$$

as $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ has the $\mathbf{Z}$-linear basis $([(G\times G)/L]:L\in\mathcal{L}^{\Delta}_{G\times G})$.

Example 75. The group S_3 has the subgroups $U_0:=1$, $U_1:=\langle(1,2)\rangle$, $U_2:=\langle(1,3)\rangle$, $U_3:=\langle(2,3)\rangle$, $U_4:=\langle(1,2,3)\rangle$, S_3.

To obtain a basis of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$ we firstly determine representatives of conjugacy classes of twisted diagonal subgroups of $\mathrm{S}_3\times\mathrm{S}_3$.

We have the isomorphisms

$$\begin{array}{rcccccccl}
\mathrm{id} & : & U_i & \to & U_i\ , & & & & i\in[0,4]\\
\alpha_{1,2} & : & U_1 & \to & U_2\ , & (1,2) & \mapsto & (1,3) & \\
\alpha_{1,3} & : & U_1 & \to & U_3\ , & (1,2) & \mapsto & (2,3) & \\
\alpha_{2,1} & : & U_2 & \to & U_1\ , & (1,3) & \mapsto & (1,2) & \\
\alpha_{2,3} & : & U_2 & \to & U_3\ , & (1,3) & \mapsto & (2,3) & \\
\alpha_{3,1} & : & U_3 & \to & U_1\ , & (2,3) & \mapsto & (1,2) & \\
\alpha_{3,2} & : & U_3 & \to & U_2\ , & (2,3) & \mapsto & (1,3) & \\
\alpha_{4,4} & : & U_4 & \to & U_4\ , & (1,2,3) & \mapsto & (1,3,2) & \\
\kappa_x^{\mathrm{S}_3} & : & \mathrm{S}_3 & \to & \mathrm{S}_3\ , & g & \mapsto & {}^x g, & x\in\mathrm{S}_3\ .
\end{array}$$

We have

$$\begin{array}{rcl}
\Delta(1) & = & \{(\mathrm{id},\mathrm{id})\}\\[1em]
\Delta(U_1) & = & \langle((1,2),(1,2))\rangle \;=\; {}^{((2,3),(2,3))}\Delta(U_2) \;=\; {}^{((1,3),(1,3))}\Delta(U_3)\\
& = & {}^{((2,3),\mathrm{id})}\Delta(U_2,\alpha_{1,2},U_1) \;=\; {}^{((1,3),\mathrm{id})}\Delta(U_3,\alpha_{1,3},U_1)\\
& = & {}^{(\mathrm{id},(2,3))}\Delta(U_1,\alpha_{2,1},U_2) \;=\; {}^{((1,3),(2,3))}\Delta(U_3,\alpha_{2,3},U_2)\\
& = & {}^{(\mathrm{id},(1,3))}\Delta(U_1,\alpha_{3,1},U_3) \;=\; {}^{((2,3),(1,3))}\Delta(U_2,\alpha_{3,2},U_3)\\[1em]
\Delta(U_4) & = & \langle((1,2,3),(1,2,3))\rangle \;=\; {}^{((1,2),\mathrm{id})}\Delta(U_4,\alpha_{4,4},U_4)\\[1em]
\Delta(\mathrm{S}_3) & = & \langle((1,2),(1,2)),((1,2,3),(1,2,3))\rangle \;=\; {}^{(\mathrm{id},x)}\Delta(\mathrm{S}_3,\kappa_x^{\mathrm{S}_3},\mathrm{S}_3)\ , \qquad x\in\mathrm{S}_3\ .
\end{array}$$

So, we obtain the $\mathbf{Z}$-linear basis

$$\{[(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(1)],\ [(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_1)],\ [(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_4)],\ [(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3)]\}$$

of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$, cf. Definition 64.

Note that every twisted diagonal subgroup of $\mathrm{S}_3\times\mathrm{S}_3$ is conjugate to a twisted diagonal subgroup of the form $\Delta(W)$ for $W\le\mathrm{S}_3$.

Hence, the injective ring morphism

$$\begin{array}{rcl}
\delta:\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3) & \to & \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)\\
{[\mathrm{S}_3/U]} & \mapsto & [\mathrm{S}_3\times(\mathrm{S}_3/U)]=[(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U)]
\end{array}$$

from Lemma 73 is also surjective, cf. Corollary 74.

Thus,

$$\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3)\cong\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)\ .$$

Remark 76. The Burnside ring $\mathrm{B}_{\mathbf{Z}}(G)$ is commutative whereas the bifree double Burnside ring $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ is not commutative in general, cf. $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4)$ in Chapter 3 below.

So, the injective ring morphism δ from Lemma 73 is not surjective in general.

2.3 The Wedderburn embedding of Boltje and Danz

In [1] Boltje and Danz give a description of the bifree double Burnside ring $\mathrm{B}^{\Delta}_{\mathbf{Q}}(G,G)$ via a direct product of endomorphism rings of permutation modules over outer automorphism groups of subgroups of G. In this chapter we give an account of this result, cf. Theorem 108 below.

2.3.1 Preparations

Lemma 77. *Let $U \le H$ and $V \le G$ be isomorphic subgroups and $\alpha \in \mathrm{Isom}(U,V)$, cf. Remark 45.*

Let X be a bifree (H,G)-biset, cf. Definition 55. Then

$$\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \subseteq X$$

is a $(\mathrm{C}_H(U),\mathrm{C}_G(V))$-sub-biset of X, cf. Definition 18(2).

As such,

$$\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$$

is a bifree $(\mathrm{C}_H(U),\mathrm{C}_G(V))$-biset.

Proof. We need to show that for $x \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$, $h \in \mathrm{C}_H(U)$ and $g \in \mathrm{C}_G(V)$ we have that $hxg \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$ since then $\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$ is a bifree $(\mathrm{C}_H(U),\mathrm{C}_G(V))$-biset by Remark 57.

We have $\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) = \{x \in X : \alpha(v)x = xv \text{ for } v \in V\}$, cf. Remark 21.

We obtain, as $h \in \mathrm{C}_H(U)$ and $g \in \mathrm{C}_G(V)$, that

$$\alpha(v)\cdot h\cdot x\cdot g = h\cdot \alpha(v)\cdot x\cdot g = h\cdot x\cdot v\cdot g = h\cdot x\cdot g\cdot v\ .$$

Thus, $hxg \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$. □

Lemma 78. *Let $U \le H$, $V \le G$ and $W \le P$. Suppose given isomorphisms $\beta \in \mathrm{Isom}(V,W)$ and $\alpha \in \mathrm{Isom}(U,V)$. Let X be a bifree (H,G)-biset and Y be a bifree (G,P)-biset.*

We have an injective morphism of bifree $(\mathrm{C}_H(U),\mathrm{C}_P(W))$-bisets

$$\begin{array}{rcl}\bar\mu = \bar\mu_{\alpha,\beta} : \mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y) & \to & \mathrm{Fix}_{\Delta(U,\alpha\circ\beta,W)}(X \underset{G}{\times} Y) \\ x \underset{\mathrm{C}_G(V)}{\times} y & \mapsto & x \underset{G}{\times} y\ .\end{array}$$

Proof. By Lemma 62 and Lemma 77 we have that $\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$ and $\mathrm{Fix}_{\Delta(U,\alpha\circ\beta,W)}(X \underset{G}{\times} Y)$ are bifree $(\mathrm{C}_H(U), \mathrm{C}_P(W))$-bisets.

We need to show that $\bar{\mu}$ is well-defined. Let $x \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$, i.e. $\alpha(v)x = xv$ for $v \in V$. Let $y \in \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$, i.e. $\beta(w)y = yw$ for $w \in W$.

We have for $w \in W$ that

$$\alpha(\beta(w))(x \underset{G}{\times} y) = x\beta(w) \underset{G}{\times} y = x \underset{G}{\times} \beta(w)y = (x \underset{G}{\times} y)w$$

and therefore $x \underset{G}{\times} y \in \mathrm{Fix}_{\Delta(U,\alpha\circ\beta,W)}(X \underset{G}{\times} Y)$, cf. Remark 31.

Since $xg \underset{G}{\times} y = x \underset{G}{\times} gy$ for $g \in \mathrm{C}_G(V)$, the map $\bar{\mu}$ is well-defined.

Moreover, we have for $h \in \mathrm{C}_H(U)$ and $p \in \mathrm{C}_P(W)$ that

$$\bar{\mu}(h \cdot (x \underset{\mathrm{C}_G(V)}{\times} y) \cdot p) = \bar{\mu}(hx \underset{\mathrm{C}_G(V)}{\times} yp) = hx \underset{G}{\times} yp = h \cdot (x \underset{G}{\times} y) \cdot p = h \cdot \bar{\mu}(x \underset{\mathrm{C}_G(V)}{\times} y) \cdot p \ .$$

So, $\bar{\mu}$ is a $(\mathrm{C}_H(U), \mathrm{C}_P(W))$-bimap.

It remains to show that $\bar{\mu}$ is injective.

Let $x, \tilde{x} \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$ and $y, \tilde{y} \in \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$ such that $x \underset{G}{\times} y = \tilde{x} \underset{G}{\times} \tilde{y}$. Then there exists $g \in G$ such that $\tilde{x} = xg^{-1}$ and $\tilde{y} = gy$, cf. Remark 31. Moreover, as $y, \tilde{y} \in \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$ we have, on the one hand

$$\tilde{y}w = \beta(w)\tilde{y} = \beta(w)gy$$

and on the other hand

$$\tilde{y}w = gyw = g\beta(w)y$$

for $w \in W$.

So, since Y is bifree, we obtain $g^{-1}\beta(w)^{-1}g\beta(w) = 1_G$, i.e. $g\beta(w)g^{-1} = \beta(w)$ for $w \in W$. Thus, $g \in \mathrm{C}_G(V)$. Therefore, we have $x \underset{\mathrm{C}_G(V)}{\times} y = \tilde{x} \underset{\mathrm{C}_G(V)}{\times} \tilde{y}$. So, $\bar{\mu}$ is injective. $\square$

Definition 79. Let $U \leq H$ and $W \leq P$ be isomorphic subgroups and $\gamma \in \mathrm{Isom}(U, W)$. We define

$$\Gamma_G(U, \gamma, W) := \{(\alpha, V, \beta) : V \leq G,\ \alpha \in \mathrm{Isom}(U, V),\ \beta \in \mathrm{Isom}(V, W),\ \alpha \circ \beta = \gamma\} \ .$$

I.e. $\Gamma_G(U, \gamma, W)$ consists of all *factorisations of* γ *via subgroups of* G.

The group G acts on $\Gamma_G(U, \gamma, W)$ by

$$g(\alpha, V, \beta) := (\alpha \circ \kappa_{g^{-1}}^{V,{}^gV}, {}^gV, \kappa_g^{{}^gV,V} \circ \beta)$$

for $g \in G$, $(\alpha, V, \beta) \in \Gamma_G(U, \gamma, W)$, cf. Notation 43. Note that

$$\mathrm{Stab}_G((\alpha, V, \beta)) = \{g \in G : g(\alpha, V, \beta) = (\alpha, V, \beta)\} = \{g \in G : \kappa_g^{{}^gV,V} = \mathrm{id}_V\} = \mathrm{C}_G(V) \ .$$

Lemma 80. *Let $U \leq H$ and $W \leq P$ be isomorphic subgroups and $\gamma \in \mathrm{Isom}(U, W)$.*

Let X be a bifree (H, G)-biset and Y be a bifree (G, P) biset.

Let $(\alpha, V, \beta), (\tilde{\alpha}, \tilde{V}, \tilde{\beta}) \in \Gamma_G(U, \gamma, W)$ such that (α, V, β) and $(\tilde{\alpha}, \tilde{V}, \tilde{\beta})$ lie in the same G-orbit of $\Gamma_G(U, \gamma, W)$. Choose $g \in G$ such that

$$(\tilde{\alpha}, \tilde{V}, \tilde{\beta}) = g(\alpha, V, \beta) = (\alpha \circ \kappa_{g^{-1}}^{V, {}^gV}, {}^gV, \kappa_g^{{}^gV, V} \circ \beta) \ .$$

(1) *The map*

$$\begin{array}{rcl} \mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \times \mathrm{Fix}_{\Delta(V,\beta,W)}(Y) & \xrightarrow{\varphi_g} & \mathrm{Fix}_{\Delta(U,\tilde{\alpha},\tilde{V})}(X) \times \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y) \\ (x, y) & \mapsto & (xg^{-1}, gy) \end{array}$$

is bijective.

(2) *We have an isomorphism of $(\mathrm{C}_H(U), \mathrm{C}_P(W))$-bisets*

$$\begin{array}{rcl} \mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y) & \xrightarrow{\bar{\varphi}_g} & \mathrm{Fix}_{\Delta(U,\tilde{\alpha},\tilde{V})}(X) \underset{\mathrm{C}_G(\tilde{V})}{\times} \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y) \\ x \underset{\mathrm{C}_G(V)}{\times} y & \mapsto & xg^{-1} \underset{\mathrm{C}_G(\tilde{V})}{\times} gy \ , \end{array}$$

cf. Lemma 77, Remark 31.

(3) *We have $\bar{\mu}_{\tilde{\alpha},\tilde{\beta}} \circ \bar{\varphi}_g = \bar{\mu}_{\alpha,\beta}$, cf. Lemma 78.*

Proof. Ad (1). The map φ_g is well-defined, as for $x \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$ we have

$$\alpha(v)x = xv$$

for $v \in V$ and therefore

$$\tilde{\alpha}({}^gv)xg^{-1} = (\alpha \circ \kappa_{g^{-1}}^{V,{}^gV})(gvg^{-1})xg^{-1} = \alpha(v)xg^{-1} = xvg^{-1} = xg^{-1} \cdot {}^gv \ .$$

So, $xg^{-1} \in \mathrm{Fix}_{\Delta(U,\tilde{\alpha},\tilde{V})}(X)$. Moreover, for $y \in \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$ we have

$$\beta(w)y = yw$$

for $w \in W$ and therefore

$$\tilde{\beta}(w)gy = (\kappa_g^{{}^gV,V} \circ \beta)(w)gy = g\beta(w)g^{-1}gy = gyw \ .$$

So, $gy \in \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y)$.

Analogously, the map

$$\begin{array}{rcl} \mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \times \mathrm{Fix}_{\Delta(V,\beta,W)}(Y) & \xleftarrow{\varphi'_g} & \mathrm{Fix}_{\Delta(U,\tilde{\alpha},\tilde{V})}(X) \times \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y) \\ (xg, g^{-1}y) & \leftarrow\!\shortmid & (x, y) \end{array}$$

is well-defined.

Furthermore, we always have

$$(\varphi'_g \circ \varphi_g)(x,y) = \varphi'_g(xg^{-1}, gy) = (x,y)$$

and

$$(\varphi_g \circ \varphi'_g)(x,y) = \varphi_g(xg, g^{-1}y) = (x,y)\ .$$

So, $\varphi'_g \circ \varphi_g = \mathrm{id}$ and $\varphi_g \circ \varphi'_g = \mathrm{id}$.

Ad (2). To show that $\bar{\varphi}_g$ is well-defined, it suffices to show that for $g' \in \mathrm{C}_G(V)$ we have $(\tau \circ \varphi_g)((x\tilde{g}, y)) \stackrel{!}{=} (\tau \circ \varphi_g)((x, \tilde{g}y))$, cf. Definition 30, Remark 32.

We obtain for $g' \in \mathrm{C}_G(V)$

$$\begin{aligned}
(\tau \circ \varphi_g)((xg', y)) &= xg'g^{-1} \underset{\mathrm{C}_G({}^gV)}{\times} gy \\
&= xg^{-1} \cdot {}^gg' \underset{{}^g\,\mathrm{C}_G(V)}{\times} gy \\
&= xg^{-1} \underset{{}^g\,\mathrm{C}_G(V)}{\times} {}^gg'gy \\
&= xg^{-1} \underset{{}^g\,\mathrm{C}_G(V)}{\times} gg'y \\
&= (\tau \circ \varphi_g)((x, g'y))\ .
\end{aligned}$$

Moreover, we have for $h \in \mathrm{C}_H(U)$, $p \in \mathrm{C}_P(W)$ that

$$\bar{\varphi}_g(h{\cdot}(x \underset{\mathrm{C}_G(V)}{\times} y){\cdot}p) = \bar{\varphi}_g(hx \underset{\mathrm{C}_G(V)}{\times} yp) = hxg^{-1} \underset{\mathrm{C}_G(\tilde{V})}{\times} gyp = h{\cdot}(xg^{-1} \underset{\mathrm{C}_G(\tilde{V})}{\times} gy){\cdot}p = h{\cdot}\bar{\varphi}_g(x \underset{\mathrm{C}_G(V)}{\times} y){\cdot}p\ .$$

So, $\bar{\varphi}_g$ is a map of $(\mathrm{C}_H(U), \mathrm{C}_P(W))$-bisets.

Analogously, the map

$$\begin{array}{ccc}
\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y) & \xleftarrow{\bar{\varphi}'_g} & \mathrm{Fix}_{\Delta(U,\tilde{\alpha},\tilde{V})}(X) \underset{\mathrm{C}_G(\tilde{V})}{\times} \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y) \\
xg \underset{\mathrm{C}_G(V)}{\times} g^{-1}y & \longleftarrow\!\shortmid & x \underset{\mathrm{C}_G(\tilde{V})}{\times} y
\end{array}$$

is a well-defined $(\mathrm{C}_H(U), \mathrm{C}_P(W))$-bimap.

So, $\bar{\varphi}'_g \circ \bar{\varphi}_g \circ \tau = \bar{\varphi}'_g \circ \tau \circ \varphi_g = \tau \circ \varphi'_g \circ \varphi_g = \tau$ and hence, by surjectivity of τ,

$$\bar{\varphi}'_g \circ \bar{\varphi}_g = \mathrm{id}\ .$$

Likewise, it follows that $\bar{\varphi}_g \circ \bar{\varphi}'_g = \mathrm{id}$. So $\bar{\varphi}_g$ is an isomorphism of $(\mathrm{C}_H(U), \mathrm{C}_P(W))$-bisets.

Ad (3). Since $\tilde{\alpha} \circ \tilde{\beta} = \gamma = \alpha \circ \beta$ we always have

$$(\bar{\mu}_{\tilde{\alpha},\tilde{\beta}} \circ \bar{\varphi}_g)(x \underset{\mathrm{C}_G(V)}{\times} y) = \bar{\mu}_{\tilde{\alpha},\tilde{\beta}}(xg^{-1} \underset{\mathrm{C}_G(\tilde{V})}{\times} gy) = xg^{-1} \underset{G}{\times} gy = x \underset{G}{\times} y = \bar{\mu}_{\alpha,\beta}(x \underset{\mathrm{C}_G(V)}{\times} y)\ .$$

$\square$

Remark 81. *Note that in Lemma 80 the map $\bar{\varphi}_g$ does not depend on the choice of $g \in G$. I.e. for $g' \in G$ such that $g'(\alpha, V, \beta) = (\tilde{\alpha}, \tilde{V}, \tilde{\beta}) = g(\alpha, V, \beta)$ we have $\bar{\varphi}_g = \bar{\varphi}_{g'}$.*

Proof. Since $g'(\alpha, V, \beta) = g(\alpha, V, \beta)$ we have $g^{-1}g' \in \mathrm{Stab}_G((\alpha, V, \beta)) = \mathrm{C}_G(V)$, i.e. there exists $c \in \mathrm{C}_G(V)$ such that $g' = gc$, cf. Definition 79. We always have

$$\begin{aligned}
\bar{\varphi}_{g'}(x \underset{\mathrm{C}_G(V)}{\times} y) &= xg'^{-1} \underset{\mathrm{C}_G(\tilde{V})}{\times} g'y \\
&= xc^{-1}g^{-1} \underset{\mathrm{C}_G(\tilde{V})}{\times} gcy \\
&= xg^{-1} \cdot {}^g(c^{-1}) \underset{{}^g\,\mathrm{C}_G(V)}{\times} gcy \\
&= xg^{-1} \underset{{}^g\,\mathrm{C}_G(V)}{\times} {}^g(c^{-1})gcy \\
&= xg^{-1} \underset{{}^g\,\mathrm{C}_G(V)}{\times} gy \\
&= \bar{\varphi}_g(x \underset{\mathrm{C}_G(V)}{\times} y)\ .
\end{aligned}$$

Thus, $\bar{\varphi}_g = \bar{\varphi}_{g'}$. □

Lemma 82. *Let $U \leq H$ and $W \leq P$ be isomorphic subgroups and $\gamma \in \mathrm{Isom}(U, W)$. Let X be a bifree (H, G)-biset and Y be a bifree (G, P) biset.*

(1) *Let $\dot{\Gamma}_G(U, \gamma, W) \subseteq \Gamma_G(U, \gamma, W)$ be a set of representatives of the G-orbits of $\Gamma_G(U, \gamma, W)$, cf. Definition 79. Then the maps $\bar{\mu}_{\alpha,\beta}$, cf. Lemma 78, induce an isomorphism of $(\mathrm{C}_H(U), \mathrm{C}_P(W))$-bisets, i.e.*

$$\begin{aligned}
\hat{\mu} : \coprod_{(\alpha,V,\beta)\in\dot{\Gamma}_G(U,\gamma,W)} \mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y) &\rightarrow \mathrm{Fix}_{\Delta(U,\gamma,W)}(X \underset{G}{\times} Y) \\
(x \underset{\mathrm{C}_G(V)}{\times} y, (\alpha, V, \beta)) &\mapsto x \underset{G}{\times} y\ .
\end{aligned}$$

(2) *Suppose X and Y to be finite. Then*

$$\begin{aligned}
&|\,\mathrm{Fix}_{\Delta(U,\gamma,W)}(X \underset{G}{\times} Y)| \\
= &\sum_{V\leq G} |G|^{-1} \sum_{\substack{(\alpha,\beta)\in\mathrm{Isom}(U,V)\times\mathrm{Isom}(V,W) \\ \alpha\circ\beta=\gamma}} |\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)| \cdot |\,\mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| \\
= &\sum_{V\in\mathcal{L}_G} |\,\mathrm{N}_G(V)|^{-1} \sum_{\substack{(\alpha,\beta)\in\mathrm{Isom}(U,V)\times\mathrm{Isom}(V,W) \\ \alpha\circ\beta=\gamma}} |\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)| \cdot |\,\mathrm{Fix}_{\Delta(V,\beta,W)}(Y)|\ .
\end{aligned}$$

Note that $\mathrm{Isom}(U, V) \times \mathrm{Isom}(V, W) = \emptyset$ if V is not isomorphic to U and to W.

Proof. Ad (1). By the universal property of the coproduct, $\hat{\mu}$ is well-defined and a map of $(\mathrm{C}_H(U), \mathrm{C}_P(W))$-bisets.

We *claim* that $\hat{\mu}$ is injective.

Let $(\alpha, V, \beta), (\tilde{\alpha}, \tilde{V}, \tilde{\beta}) \in \dot{\Gamma}_G(U, \gamma, W)$. Suppose given $x \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$, $\tilde{x} \in \mathrm{Fix}_{\Delta(U,\tilde{\alpha},\tilde{V})}(X)$, $y \in \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$ and $\tilde{y} \in \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y)$ such that

$$\hat{\mu}(x \underset{\mathrm{C}_G(V)}{\times} y, (\alpha, V, \beta)) = x \underset{G}{\times} y = \tilde{x} \underset{G}{\times} \tilde{y} = \hat{\mu}(x \underset{\mathrm{C}_G(\tilde{V})}{\times} y, (\tilde{\alpha}, \tilde{V}, \tilde{\beta})) \ .$$

As the restriction of $\hat{\mu}$ to each participant of the disjoint union is injective, cf. Lemma 78, it suffices to show that $(\alpha, V, \beta) = (\tilde{\alpha}, \tilde{V}, \tilde{\beta})$. It suffices to show that (α, V, β) and $(\tilde{\alpha}, \tilde{V}, \tilde{\beta})$ lie in the same G-orbit, i.e. that there exists $g \in G$ such that

$$g(\alpha, V, \beta) = (\alpha \circ \kappa_{g^{-1}}^{V,{}^gV}, {}^gV, \kappa_g^{{}^gV,V} \circ \beta) \overset{!}{=} (\tilde{\alpha}, \tilde{V}, \tilde{\beta}) \ ,$$

cf. Definition 79.

Since $x \underset{G}{\times} y = \tilde{x} \underset{G}{\times} \tilde{y}$ there exists $g \in G$ such that

$$\tilde{x} = xg^{-1} \text{ and } \tilde{y} = gy \ .$$

As $y \in \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$ and $\tilde{y} \in \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y)$ we have for $w \in W$ that

$$\beta(w)y = yw \text{ and } \tilde{\beta}(w)gy = \tilde{\beta}(w)\tilde{y} = \tilde{y}w = gyw$$

and therefore $g^{-1}\tilde{\beta}(w)gy = yw = \beta(w)y$.

Since Y is bifree, we obtain $\beta(w)^{-1}g^{-1}\tilde{\beta}(w)g = 1_G$ for $w \in W$, i.e. $\tilde{\beta} = \kappa_g^{{}^gV,V} \circ \beta$. Thus, $\tilde{V} = \tilde{\beta}(W) = (\kappa_g^{{}^gV,V} \circ \beta)(W) = {}^gV$. It remains to show that $\tilde{\alpha} = \alpha \circ \kappa_{g^{-1}}^{V,{}^gV}$. We have

$$\tilde{\alpha} = \tilde{\alpha} \circ \tilde{\beta} \circ \tilde{\beta}^{-1} = \gamma \circ \tilde{\beta}^{-1} = \alpha \circ \beta \circ \tilde{\beta}^{-1} = \alpha \circ \kappa_{g^{-1}}^{V,{}^gV} \circ \tilde{\beta} \circ \tilde{\beta}^{-1} = \alpha \circ \kappa_{g^{-1}}^{V,{}^gV} \ .$$

This proves the *claim*.

We *claim* that $\hat{\mu}$ is surjective.

Suppose given $x \in X$ and $y \in Y$ such that $x \underset{G}{\times} y \in \mathrm{Fix}_{\Delta(U,\gamma,W)}(X \underset{G}{\times} Y)$. Then we have

$$x \underset{G}{\times} yw = \gamma(w)x \underset{G}{\times} y$$

for $w \in W$. Therefore, for $w \in W$ we may choose an element $g_w \in G$ such that

$$xg_w = \gamma(w)x \text{ and } g_w y = yw \ .$$

Let $\tilde{V} := \{g_w : w \in W\}$. Consider the map $\tilde{\beta} : W \to \tilde{V}$ $w \mapsto g_w$. For $w, w' \in W$ we get

$$\tilde{\beta}(ww')y = g_{ww'}y = yww' = \tilde{\beta}(w)\tilde{\beta}(w')y \ .$$

Since Y is bifree, $\tilde{\beta}(ww') = \tilde{\beta}(w)\tilde{\beta}(w')$ ensues. So, $\tilde{V} \leq G$ and $\tilde{\beta}$ is a surjective group morphism.

We show that $\tilde{\beta}$ is injective. Given $w \in W$ with $\tilde{\beta}(w) = 1_G$, we obtain

$$y = \tilde{\beta}(w)y = g_w y = yw \ ,$$

hence $w = 1_P$ since Y is bifree. So, $\tilde{\beta}$ is injective.

Let $\tilde{\alpha} := \gamma \circ \tilde{\beta}^{-1} : \tilde{V} \xrightarrow{\sim} U$. As γ and $\tilde{\beta}$ are isomorphisms so is $\tilde{\alpha}$.

So, $(\tilde{\alpha}, \tilde{V}, \tilde{\beta}) \in \Gamma_G(U, \gamma, W)$.

As $\tilde{\beta}(w)y = g_w y = yw$ for $w \in W$ it follows that $y \in \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y)$.

Since $x\tilde{\beta}(w) = xg_w = \gamma(w)x$ for $w \in W$, we have

$$x\tilde{v} = x\tilde{\beta}(\tilde{\beta}^{-1}(\tilde{v})) = \gamma(\tilde{\beta}^{-1}(\tilde{v}))x = \tilde{\alpha}(\tilde{v})x$$

for $\tilde{v} \in \tilde{V}$, i.e. $x \in \mathrm{Fix}_{\Delta(U,\tilde{\alpha},\tilde{V})}(X)$.

As $(\tilde{\alpha}, \tilde{V}, \tilde{\beta}) \in \Gamma_G(U, \gamma, W)$, there exist $g \in G$ and $(\alpha, V, \beta) \in \dot{\Gamma}_G(U, \gamma, W)$ such that

$$(\tilde{\alpha}, \tilde{V}, \tilde{\beta}) = g(\alpha, V, \beta) = (\alpha \circ \kappa_{g^{-1}}^{V,{}^gV}, {}^gV, \kappa_g^{{}^gV,V} \circ \beta) \ .$$

We have

$$xg \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \ ,$$

as for $v \in V$

$$\alpha(v) \cdot (x \cdot g) = (\tilde{\alpha} \circ \kappa_g^{{}^gV,V})(v) \cdot x \cdot g = \tilde{\alpha}({}^gv) \cdot x \cdot g = x \cdot {}^gv \cdot g = (x \cdot g) \cdot v \ .$$

Moreover, $g^{-1}yw = g^{-1}\tilde{\beta}(w)y = g^{-1}\tilde{\beta}(w)gg^{-1}y = (\kappa_{g^{-1}}^{V,{}^gV} \circ \tilde{\beta})(w)g^{-1}y = \beta(w)g^{-1}y$ for $w \in W$. So,

$$g^{-1}y \in \mathrm{Fix}_{\Delta(V,\beta,W)}(Y) \ .$$

Thus, $xg \underset{\mathrm{C}_G(V)}{\times} g^{-1}y \in \mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$ and so

$$\hat{\mu}(xg \underset{\mathrm{C}_G({}^gV)}{\times} g^{-1}y, (\alpha, V, \beta)) = xg \underset{G}{\times} g^{-1}y = x \underset{G}{\times} y \ .$$

This proves the *claim*.

Ad (2). As $\hat{\mu}$ is bijective, we have

$$\begin{aligned} |\mathrm{Fix}_{\Delta(U,\gamma,W)}(X \underset{G}{\times} Y)| &= |\bigsqcup_{(\alpha,V,\beta)\in\dot{\Gamma}_G(U,\gamma,W)} \mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| \\ &= \sum_{(\alpha,V,\beta)\in\dot{\Gamma}_G(U,\gamma,W)} |\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| \ . \end{aligned}$$

Given (α, V, β) and $(\tilde{\alpha}, \tilde{V}, \tilde{\beta})$ that lie in the same G-orbit of $\Gamma_G(U, \gamma, W)$, we have by Lemma 80(2) that

$$|\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| = |\mathrm{Fix}_{\Delta(U,\tilde{\alpha},\tilde{V})}(X) \underset{\mathrm{C}_G(\tilde{V})}{\times} \mathrm{Fix}_{\Delta(\tilde{V},\tilde{\beta},W)}(Y)| \ .$$

The G-orbit of $(\alpha, V, \beta) \in \Gamma_G(U, \gamma, W)$ has length $\dfrac{|G|}{|\mathrm{Stab}_G((\alpha, V, \beta))|} = \dfrac{|G|}{|\mathrm{C}_G(V)|}$, cf. Definition 79. Hence,

$$\begin{aligned} &\sum_{(\alpha,V,\beta)\in\dot{\Gamma}_G(U,\gamma,W)} |\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| \\ = &\sum_{(\alpha,V,\beta)\in\Gamma_G(U,\gamma,W)} \frac{|\mathrm{C}_G(V)|}{|G|} |\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| \ . \end{aligned}$$

As Y is bifree, $\mathrm{Fix}_{\Delta(V,\beta,W)}(Y)$ is left-free over $\mathrm{C}_G(V)$, cf. Lemma 77. Thus, we have

$$|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| = |\,\mathrm{C}_G(V)|^{-1}|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \times \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)|\ ,$$

by Remark 58. So,

$$\begin{aligned}
&\sum_{(\alpha,V,\beta)\in\Gamma_G(U,\gamma,W)} \frac{|\,\mathrm{C}_G(V)|}{|G|}|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \underset{\mathrm{C}_G(V)}{\times} \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| \\
=\ &\sum_{(\alpha,V,\beta)\in\Gamma_G(U,\gamma,W)} |G|^{-1}|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X) \times \mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| \\
=\ &\sum_{V\le G} |G|^{-1} \sum_{\substack{(\alpha,\beta)\in\mathrm{Isom}(U,V)\times\mathrm{Isom}(V,W)\\ \alpha\circ\beta=\gamma}} |\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|\cdot|\,\mathrm{Fix}_{\Delta(V,\beta,W)}(Y)|\ .
\end{aligned}$$

By Lemma 80(1) we have, for $g\in G$ and $(\alpha',V,\beta')\in\Gamma_G(U,\gamma,W)$, that

$$|\,\mathrm{Fix}_{\Delta(U,\alpha',V)}(X) \times \mathrm{Fix}_{\Delta(V,\beta',W)}(Y)| = |\,\mathrm{Fix}_{\Delta(U,\alpha'\circ\kappa^{V,{}^gV}_{g^{-1}},{}^gV)}(X) \times \mathrm{Fix}_{\Delta({}^gV,\kappa^{{}^gV,V}_g\circ\beta',W)}(Y)|\ .$$

Therefore, we get

$$\begin{aligned}
&\sum_{V\le G} |G|^{-1} \sum_{\substack{(\alpha,\beta)\in\mathrm{Isom}(U,V)\times\mathrm{Isom}(V,W)\\ \alpha\circ\beta=\gamma}} |\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|\cdot|\,\mathrm{Fix}_{\Delta(V,\beta,W)}(Y)| \\
=\ & |G|^{-1}\sum_{V\in\mathcal{L}_{\mathcal{G}}}\sum_{g\in G}|\,\mathrm{N}_G(V)|^{-1} \sum_{\substack{(\alpha,\beta)\in\mathrm{Isom}(U,{}^gV)\times\mathrm{Isom}({}^gV,W)\\ \alpha\circ\beta=\gamma}} |\,\mathrm{Fix}_{\Delta(U,\alpha,{}^gV)}(X)|\cdot|\,\mathrm{Fix}_{\Delta({}^gV,\beta,W)}(Y)| \\
=\ & |G|^{-1}\sum_{V\in\mathcal{L}_{\mathcal{G}}}\sum_{g\in G}|\,\mathrm{N}_G(V)|^{-1} \sum_{\substack{(\alpha',\beta')\in\mathrm{Isom}(U,V)\times\mathrm{Isom}(V,W)\\ \alpha'\circ\beta'=\gamma}} |\,\mathrm{Fix}_{\Delta(U,\alpha'\circ\kappa^{V,{}^gV}_{g^{-1}},{}^gV)}(X)|\cdot|\,\mathrm{Fix}_{\Delta({}^gV,\kappa^{{}^gV,V}_g\circ\beta',W)}(Y)| \\
\overset{\mathrm{L.80(1)}}{=}\ & |G|^{-1}\sum_{V\in\mathcal{L}_G}\frac{|G|}{|\,\mathrm{N}_G(V)|} \sum_{\substack{(\alpha',\beta')\in\mathrm{Isom}(U,V)\times\mathrm{Isom}(V,W)\\ \alpha'\circ\beta'=\gamma}} |\,\mathrm{Fix}_{\Delta(U,\alpha',V)}(X)|\cdot|\,\mathrm{Fix}_{\Delta(V,\beta',W)}(Y)| \\
=\ & \sum_{V\in\mathcal{L}_{\mathcal{G}}}|\,\mathrm{N}_G(V)|^{-1} \sum_{\substack{(\alpha,\beta)\in\mathrm{Isom}(U,V)\times\mathrm{Isom}(V,W)\\ \alpha\circ\beta=\gamma}} |\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|\cdot|\,\mathrm{Fix}_{\Delta(V,\beta,W)}(Y)|\ .
\end{aligned}$$

□

2.3.2 The ghost ring

Definition 83.

(1) Let $I_{H,G}^{\Delta}$ denote the set of triples (U,α,V), where $U \leq H$, $V \leq G$ and $\alpha : V \xrightarrow{\sim} U$ is an isomorphism

$$I_{H,G}^{\Delta} = \{(U,\alpha,V) : \; U \leq H, \; V \leq G, \; \alpha \in \operatorname{Isom}(U,V)\} \; .$$

The group $H \times G$ acts on $I_{H,G}^{\Delta}$ as follows. For $(U,\alpha,V) \in I_{H,G}^{\Delta}$ and $(h,g) \in H \times G$, we have

$$^{(h,g)}(U,\alpha,V) := ({}^hU, \kappa_h^{{}^hU,U} \circ \alpha \circ \kappa_{g^{-1}}^{V,{}^gV}, {}^gV) \; .$$

In particular, for $(T,\mathrm{id},T) \in I_{G,G}^{\Delta}$ and $g \in G$ we have $^{(g,g)}(T,\mathrm{id},T) = ({}^gT,\mathrm{id},{}^gT)$.

(2) We denote by $A_{\mathbf{Z}}^{\Delta}(H,G) := \mathbf{Z}I_{H,G}^{\Delta}$ the free abelian group with $\mathbf{Z}$-linear basis $I_{H,G}^{\Delta}$. It is a permutation module over $\mathbf{Z}(H \times G)$, cf. (1).

(3) We define the *ghost group* $\tilde{\mathrm{B}}_{\mathbf{Z}}^{\Delta}(H,G)$ of $\mathrm{B}_{\mathbf{Z}}^{\Delta}(H,G)$ by

$$\tilde{\mathrm{B}}_{\mathbf{Z}}^{\Delta}(H,G) := \mathrm{Fix}_{H\times G}(A_{\mathbf{Z}}^{\Delta}(H,G)) \; .$$

For $(U,\alpha,V) \in I_{H,G}^{\Delta}$ let $[U,\alpha,V]_{H\times G} := \{^{(h,g)}(U,\alpha,V) : (h,g) \in H \times G\}$ denote its $(H \times G)$-orbit. The orbit sums

$$[U,\alpha,V]_{H\times G}^{+} := \sum_{(\tilde{U},\tilde{\alpha},\tilde{V}) \in [U,\alpha,V]_{H\times G}} (\tilde{U},\tilde{\alpha},\tilde{V})$$

form a $\mathbf{Z}$-linear basis of $\tilde{\mathrm{B}}_{\mathbf{Z}}^{\Delta}(H,G)$. We call this basis the *standard* basis of $\tilde{\mathrm{B}}_{\mathbf{Z}}^{\Delta}(H,G)$.

Lemma 84. *We have an isomorphism of* (H,G)*-bisets*

$$\begin{array}{rcl} f : I_{H,G}^{\Delta} & \xrightarrow{\sim} & \Delta_{H\times G} \\ (U,\alpha,V) & \mapsto & \Delta(U,\alpha,V) \; , \end{array}$$

cf. Definition 83(1), Remark 52.

Proof. Suppose given $U,U' \leq H$, $V,V' \leq G$ and isomorphisms $V \xrightarrow{\alpha} U$, $V' \xrightarrow{\alpha'} V'$. We *claim* that

$$\Delta(U,\alpha,V) = \Delta(U',\alpha',V') \Leftrightarrow (U,\alpha,V) = (U',\alpha',V') \; .$$

Ad $\Leftarrow$. This holds by construction.

Ad $\Rightarrow$. Suppose that $\Delta(U,\alpha,V) = \Delta(U',\alpha',V')$.

We have

$$\mathrm{p}_1(\Delta(U,\alpha,V)) = \{h \in H : \text{ there exists } g \in G \text{ such that } (h,g) \in \Delta(U,\alpha,V)\} \subseteq U \; ,$$

cf. Notation 40.

Moreover, for $u \in U$ we have $u = \mathrm{p}_1(u, \alpha^{-1}(u)) \in \mathrm{p}_1(\Delta(U, \alpha, V))$.

So, $U = \mathrm{p}_1(\Delta(U, \alpha, V)) = \mathrm{p}_1(\Delta(U', \alpha', V')) = U'$. Analogously, we have $V = V'$.

It remains to show that for $v \in V$ we have $\alpha(v) = \alpha'(v)$. But,

$$(\alpha(v), v) \in \Delta(U, \alpha, V) = \Delta(U, \alpha', V)$$

and therefore $\alpha(v) = \alpha'(v)$.

This proves the *claim*.

By construction, f is surjective. By the claim, f is injective. So, f is bijective.

Suppose given $(h, g) \in H \times G$. Then

$$\begin{aligned} f({}^{(h,g)}(U, \alpha, V)) &= f(({}^hU, \kappa_h^{{}^hU,U} \circ \alpha \circ \kappa_{g^{-1}}^{V,{}^gV}, {}^gV)) \\ &= \Delta({}^hU, \kappa_h^{{}^hU,U} \circ \alpha \circ \kappa_{g^{-1}}^{V,{}^gV}, {}^gV) \\ &= {}^{(h,g)}\Delta(U, \alpha, V) \\ &= {}^{(h,g)}f((U, \alpha, V)) \ , \end{aligned}$$

cf. Remark 52, Definition 83(1). So, f is an isomorphism of (H, G)-bisets. □

The next Lemma provides a system of representatives of conjugacy classes of twisted diagonal subgroups.

Lemma 85. *Let U_i for $i \in [1, k]$ be representatives for the conjugacy classes of subgroups of G. Given $i, j \in [1, k]$, we let $\gamma_s^{i,j}$ for $s \in [1, m_{i,j}]$ be representatives for the $(\mathrm{Inn}_G(U_i), \mathrm{Inn}_G(U_j))$-orbits of $\mathrm{Isom}(U_i, U_j)$. Then for each twisted diagonal subgroup $L \leq G \times G$ there exists a unique pair $(i, j) \in [1, k] \times [1, k]$ and a unique $s \in [1, m_{i,j}]$ such that L is conjugate in $G \times G$ to $\Delta(U_i, \gamma_s^{i,j}, U_j)$.*

Proof. Existence. By Lemma 49(2) and Remark 52, L is conjugate to $\Delta(U_i, \alpha, U_j)$ for some $i, j \in [1, k]$ and some $\alpha \in \mathrm{Isom}(U_i, U_j)$. By Remark 52 again, $\Delta(U_i, \alpha, U_j)$ is conjugate to $\Delta(U_i, \gamma_s^{i,j}, U_j)$ for some $s \in [1, m_{i,j}]$.

Uniqueness. Suppose given $i, j \in [1, k]$ such that $U_i \cong U_j$ and $\gamma \in \mathrm{Isom}(U_i, U_j)$. Suppose given $\tilde{i}, \tilde{j} \in [1, k]$ such that $U_{\tilde{i}} \cong U_{\tilde{j}}$ and $\tilde{\gamma} \in \mathrm{Isom}(U_{\tilde{i}}, U_{\tilde{j}})$. Suppose given $(x, y) \in G \times G$ such that

$$\Delta(U_{\tilde{i}}, \tilde{\gamma}, U_{\tilde{j}}) = {}^{(x,y)}\Delta(U_i, \gamma, U_j) \ .$$

We have to show that $i \overset{!}{=} \tilde{i}$, that $j \overset{!}{=} \tilde{j}$ and that γ and $\tilde{\gamma}$ lie in the same $(\mathrm{Inn}_G(U_i), \mathrm{Inn}_G(U_j))$-orbits of $\mathrm{Isom}(U_i, U_j)$.

By Remark 52, we have

$$\Delta(U_{\tilde{i}}, \tilde{\gamma}, U_{\tilde{j}}) = {}^{(x,y)}\Delta(U_i, \gamma, U_j) = \Delta({}^xU_i, \kappa_x^{{}^xU_i,U_i} \circ \gamma \circ \kappa_{y^{-1}}^{U_j,{}^yU_j}, {}^yU_j) \ .$$

By Lemma 84, we get $(U_{\tilde{i}}, \tilde{\gamma}, U_{\tilde{j}}) = ({}^xU_i, \kappa_x^{{}^xU_i,U_i} \circ \gamma \circ \kappa_{y^{-1}}^{U_j,{}^yU_j}, {}^yU_j)$.

Hence, $i = \tilde{i}$, $j = \tilde{j}$, $x \in \mathrm{N}_G(U_i)$ and $y \in \mathrm{N}_G(U_j)$.

So, $\tilde{\gamma} = \kappa_x^{U_i,U_i} \circ \gamma \circ \kappa_{y^{-1}}^{U_j,U_j}$. This shows the assertion, cf. Notation 43. □

2.3.3 The mark homomorphism

Lemma 86 (mark homomorphism)**.** *We have the* $\mathbf{Z}$*-linear map*

$$\begin{array}{rcl} \mathfrak{m}^{\Delta}_{H,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) & \to & \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G) \\ [X] & \mapsto & \displaystyle\sum_{(U,\alpha,V)\in I^{\Delta}_{H,G}} \frac{\mathrm{fix}_{\Delta(U,\alpha,V)}([X])}{|\,\mathrm{C}_H(U)|}(U,\alpha,V) \\ & = & \displaystyle\sum_{(U,\alpha,V)\in I^{\Delta}_{H,G}} \frac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|}{|\,\mathrm{C}_H(U)|}(U,\alpha,V)\ , \end{array}$$

cf. Remark 26.

Proof. First of all, $|\,\mathrm{C}_H(U)|$ divides $|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|$ for a finite bifree (H,G)-biset X, since $\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)$ is a left-free $\mathrm{C}_H(U)$-set, cf. Lemma 77.

Therefore, $\mathfrak{m}^{\Delta}_{H,G}$ maps to $A^{\Delta}_{\mathbf{Z}}(H,G)$.

It remains to show that $\mathfrak{m}^{\Delta}_{H,G}([X]) \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G) = \mathrm{Fix}_{H\times G}(A^{\Delta}_{\mathbf{Z}}(H,G))$ for a finite bifree (H,G)-biset X. Therefore, it suffices to show that for $(h,g) \in H \times G$ the coefficients of (U,α,V) and ${}^{(h,g)}(U,\alpha,V)$ that appear in $\mathfrak{m}^{\Delta}_{H,G}([X])$ are equal, cf. Definition 83(1).

We have

$$|\,\mathrm{C}_H({}^hU)| = |\,{}^h\,\mathrm{C}_H(U)\,| = |\,\mathrm{C}_H(U)|$$

and

$$\begin{array}{rcl} |\,\mathrm{Fix}_{\Delta({}^hU,\kappa_h^{{}^hU,U}\circ\alpha\circ\kappa_{g^{-1}}^{V,{}^gV},{}^gV)}| & = & |\,\mathrm{Fix}_{{}^{(h,g)}\Delta(U,\alpha,V)}(X)| \\ & = & |(h,g)\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)| \\ & = & |\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|\ , \end{array}$$

cf. Lemma 7. So, the coefficients are equal.

Thus, $\mathfrak{m}^{\Delta}_{H,G}$ is a well-defined $\mathbf{Z}$-linear map. □

Definition 87. Let $A^{\Delta}_{\mathbf{Q}}(H,G) := \mathbf{Q} \otimes_{\mathbf{Z}} A^{\Delta}_{\mathbf{Z}}(H,G)$, which has $I^{\Delta}_{H,G}$ as a $\mathbf{Q}$-linear basis. We define the $\mathbf{Q}$-bilinear map

$$\begin{array}{rcl} (-)\underset{G}{\cdot}(=) : A^{\Delta}_{\mathbf{Q}}(H,G) \times A^{\Delta}_{\mathbf{Q}}(G,P) & \to & A^{\Delta}_{\mathbf{Q}}(H,P) \\ ((U,\alpha,V),(V',\beta,W)) & \mapsto & (U,\alpha,V)\underset{G}{\cdot}(V',\beta,W) := \begin{cases} 0 & \text{if } V \neq V' \\ \dfrac{|\,\mathrm{C}_G(V)|}{|G|}(U,\alpha\circ\beta,W) & \text{if } V = V'\ , \end{cases} \end{array}$$

where $(U,\alpha,V) \in I^{\Delta}_{H,G}$ and $(V',\beta,W) \in I^{\Delta}_{G,P}$.

Remark 88. *We have*

$$(\xi \underset{G}{\cdot} \xi') \underset{P}{\cdot} \xi'' = \xi \underset{G}{\cdot} (\xi' \underset{P}{\cdot} \xi'')$$

for $\xi \in A^{\Delta}_{\mathbf{Q}}(H,G)$*,* $\xi' \in A^{\Delta}_{\mathbf{Q}}(G,P)$*,* $\xi'' \in A^{\Delta}_{\mathbf{Q}}(P,Q)$*.*

Proof. We have for $(U, \alpha, V) \in I^{\Delta}_{H,G}$, $(V', \beta, W) \in I^{\Delta}_{G,P}$ and $(W', \gamma, T) \in I^{\Delta}_{P,Q}$

$$
\begin{aligned}
& ((U,\alpha,V) \underset{G}{\cdot} (V',\beta,W)) \underset{P}{\cdot} (W',\gamma,T) \\
= & \left\{ \begin{array}{rl} 0 & \text{if } V \neq V' \\ \dfrac{|\,\mathrm{C}_G(V)|}{|G|}(U,\alpha\circ\beta,W) & \text{if } V = V' \end{array} \right\} \underset{P}{\cdot} (W',\gamma,T) \\
= & \left\{ \begin{array}{rl} 0 & \text{if } V \neq V' \text{ or } W \neq W' \\ \dfrac{|\,\mathrm{C}_G(V)|}{|G|}\dfrac{|\,\mathrm{C}_P(W)|}{|P|}(U,\alpha\circ\beta\circ\gamma,T) & \text{if } V = V' \text{ and } W = W' \end{array} \right. \\
= & (U,\alpha,V) \underset{G}{\cdot} \left\{ \begin{array}{rl} 0 & \text{if } W \neq W' \\ \dfrac{|\,\mathrm{C}_P(W)|}{|P|}(V',\beta\circ\gamma,T) & \text{if } W = W' \end{array} \right\} \\
= & (U,\alpha,V) \underset{G}{\cdot} ((V',\beta,W) \underset{P}{\cdot} (W',\gamma,T)) \,.
\end{aligned}
$$

□

Lemma 89. *Let $L \leq H \times G$ be a subgroup. Suppose given systems of coset representatives as follows.*

$$
\begin{array}{rll}
\mathcal{A}_1 \subseteq H & \textit{for} & H/L_{1_G} = H/\{h \in H : (h, 1_G) \in L\} \\
\mathcal{B}_1 \subseteq G & \textit{for} & G/\,\mathrm{p}_2(L) = G/\{g \in G : \textit{ there exists } h \in H \textit{ such that } (h,g) \in L\} \\
\\
\mathcal{A}_2 \subseteq H & \textit{for} & H/\,\mathrm{p}_1(L) = H/\{h \in H : \textit{ there exists } g \in G \textit{ such that } (h,g) \in L\} \\
\mathcal{B}_2 \subseteq G & \textit{for} & G/_{1_H}L = G/\{g \in G : (1_H, g) \in L\}
\end{array}
$$

Then $\mathcal{A}_1 \times \mathcal{B}_1$ and $\mathcal{A}_2 \times \mathcal{B}_2$ are systems of coset representatives for $(H \times G)/L$.

Proof. We show that $\mathcal{A}_1 \times \mathcal{B}_1$ represents $(H \times G)/L$.

Suppose given $(h, g) \in H \times G$. We need to show that there exists $(a, b) \in \mathcal{A}_1 \times \mathcal{B}_1$ such that $(a, b)L = (h, g)L$.

As $\mathcal{B}_1$ represents $G/\,\mathrm{p}_2(L)$ there exists $b \in \mathcal{B}_1$ and $v \in \mathrm{p}_2(L)$ such that

$$g = bv \ .$$

Since $v \in \mathrm{p}_2(L)$ there exists $u \in H$ such that $(u, v) \in L$.

Moreover, as $\mathcal{A}_1$ represents H/L_{1_G} there exists $a \in \mathcal{A}_1$ and $\ell \in L_{1_G}$ such that

$$hu^{-1} = a\ell \ .$$

Now,

$$(a, b)^{-1}(h, g) = (a, b)^{-1}(a\ell u, bv) = (\ell, 1_G)(u, v) \in L \ .$$

So, $(a,b)L = (h,g)L$.

It remains to show that for $(a,b),(a',b') \in \mathcal{A}_1 \times \mathcal{B}_1$ with $(a,b)L = (a',b')L$ it follows that $(a,b) = (a',b')$.

But, as $(a'^{-1}a, b'^{-1}b) \in L$ it follows that $b'^{-1}b \in \mathrm{p}_2(L)$, i.e. that $b\,\mathrm{p}_2(L) = b'\,\mathrm{p}_2(L)$ and therefore that $b = b'$.

Hence, $(a'^{-1}a, 1_G) \in L$ and therefore $a'^{-1}a \in L_{1_G}$. So, $aL_{1_G} = a'L_{1_G}$ and it follows that $a = a'$.

Analogously, $\mathcal{A}_2 \times \mathcal{B}_2$ is a system of representatives for $(H \times G)/L$. □

Lemma 90. *Let $C \le N \le G$ be subgroups. Suppose given systems of coset representatives as follows.*

$$\begin{array}{lcl} \mathcal{B}_1 \subseteq N & \textit{for} & N/C \\ \mathcal{B}_2 \subseteq G & \textit{for} & G/N \end{array}$$

Set $\mathcal{B}_3 := \{gx : \ g \in \mathcal{B}_2, \ x \in \mathcal{B}_1\}$.

(1) *The set $\mathcal{B}_3$ represents G/C.*

(2) *The map*

$$\begin{array}{rcl} f : \mathcal{B}_2 \times \mathcal{B}_1 & \to & \mathcal{B}_3 \\ (g,x) & \mapsto & gx \end{array}$$

is bijective.

(3) *Suppose given $n \in N$. Then $n\mathcal{B}_1 := \{nx : \ x \in \mathcal{B}_1\}$ represents N/C.*

(4) *Suppose that $C \trianglelefteq N$. Suppose given $n \in N$. Then $\mathcal{B}_1 n := \{xn : \ x \in \mathcal{B}_1\}$ represents N/C.*

(5) *Write $x_2^{-1}\mathcal{B}_1 := \{x_2^{-1}x : \ x \in \mathcal{B}_1\}$ for $x_2 \in \mathcal{B}_1$. We have the mutually inverse bijections*

$$\begin{array}{rcl} \mathcal{B}_1 \times \mathcal{B}_1 & \overset{\sim}{\leftrightarrow} & \bigsqcup_{x_2 \in \mathcal{B}_1} x_2^{-1}\mathcal{B}_1 \\ (x_1,x_2) & \mapsto & (x_2^{-1}x_1, x_2) \\ (x_2x', x_2) & \leftarrow\!\shortmid & (x', x_2) \ , \end{array}$$

where the target is an exterior disjoint union.

Proof. Claim. Suppose given $(b_2,b_1),(b_2',b_1') \in \mathcal{B}_2 \times \mathcal{B}_1$ with $b_2b_1C = b_2'b_1'C$. Then $b_1 = b_1'$ and $b_2 = b_2'$.

We have $\mathcal{B}_1 \subseteq N$. So,

$$b_2'^{-1}b_2 = \underbrace{b_1'}_{\in N}\underbrace{(b_1'^{-1}b_2'^{-1}b_2b_1)}_{\in C \le N}\underbrace{b_1^{-1}}_{\in N} \in N \ .$$

Hence, $b_2'N = b_2N$ and therefore it follows that $b_2' = b_2$.

Moreover, $b_1'^{-1}b_1 = b_1'^{-1}b_2'^{-1}b_2b_1 \in C$. Hence, $b_1'C = b_1C$ and therefore $b_1' = b_1$.

This proves the *claim*.

Ad (1). Suppose given $g \in G$. We need to show that there exists $(b_2, b_1) \in \mathcal{B}_2 \times \mathcal{B}_1$ such that $b_2 b_1 C = gC$.

As $\mathcal{B}_2$ represents G/N there exists $b_2 \in \mathcal{B}_2$ and $n \in N$ such that

$$g = b_2 n \ .$$

As $\mathcal{B}_1$ represents N/C there exists $b_1 \in \mathcal{B}_1$ and $c \in C$ such that

$$n = b_1 c \ .$$

So, $g = b_2 n = b_2 b_1 c$. Hence, $gC = b_2 b_1 C$.

It remains to show that for $(b_2, b_1), (b_2', b_1') \in \mathcal{B}_2 \times \mathcal{B}_1$ with $b_2 b_1 C = b_2' b_1' C$ it follows that $b_2 b_1 = b_2' b_1'$. This follows by the claim. So, $\mathcal{B}_3$ represents G/C.

Ad (2). By construction, the map f is well-defined and surjective. By the claim f is injective.

Ad (3). We show that $n\mathcal{B}_1$ represents N/C. Suppose given $y \in N$. We need to show that there exist $nb_1 \in n\mathcal{B}_1$ with $nb_1 C = yC$. But, as $n^{-1}y \in N$ and $\mathcal{B}_1$ represents N/C, there exists $b_1 \in \mathcal{B}_1$ and $c \in C$ such that $n^{-1}y = b_1 c$, i.e. $y = nb_1 c$.

Suppose given $nb_1, nb_1' \in n\mathcal{B}_1$ with $nb_1 C = nb_1' C$. Then $b_1 C = b_1' C$. As $\mathcal{B}_1$ represents N/C we have $b_1 = b_1'$ and therefore $nb_1 = nb_1'$.

Ad (4). We show that $\mathcal{B}_1 n$ represents N/C. Suppose given $y \in N$. We need to show that there exist $b_1 n \in \mathcal{B}_1 n$ with $b_1 nC = yC$. But, as $yn^{-1} \in N$ and $\mathcal{B}_1$ represents N/C, there exists $b_1 \in \mathcal{B}_1$ and $c \in C$ such that $yn^{-1} = b_1 c$ i.e. $y = b_1 c \cdot n = b_1 n \underbrace{n^{-1} c n}_{\in C}$.

Suppose given $b_1 n, b_1' n \in \mathcal{B}_1 n$ with $b_1 nC = b_1' nC$. Then

$$n^{-1} b_1'^{-1} b_1 n \in C \ , \text{ and so } b_1'^{-1} b_1 \in nCn^{-1} = C \ .$$

Thus, $b_1' C = b_1 C$. As $\mathcal{B}_1$ represents N/C we have $b_1' = b_1$. So, we have $b_1 n = b_1' n$. ☐

Remark 91. *Suppose given* $(U, \alpha, V) \in I_{H,G}^{\Delta}$. *Let* $L := \mathrm{N}_{H \times G}(\Delta(U, \alpha, V)) \leq H \times G$. *We have*

$$\begin{array}{rcl} L & = & \{(h,g) \in H \times G : h \in \mathrm{N}_H(U),\ g \in \mathrm{N}_G(V),\ \kappa_h^U \circ \alpha \circ \kappa_{g^{-1}}^V = \alpha\} \ , \\ \mathrm{p}_1(L) & = & \{h \in \mathrm{N}_H(U) : \textit{ there exists } g \in \mathrm{N}_G(V) \textit{ such that } \kappa_h^U = \alpha \circ \kappa_g^V \circ \alpha^{-1}\} \ , \\ \mathrm{p}_2(L) & = & \{g \in \mathrm{N}_G(V) : \textit{ there exists } h \in \mathrm{N}_H(U) \textit{ such that } \kappa_g^V = \alpha^{-1} \circ \kappa_h^U \circ \alpha\} \ , \\ L_{1_G} & = & \mathrm{C}_H(U) \ , \\ {}_{1_H}L & = & \mathrm{C}_G(V) \ . \end{array}$$

Proof. We have

$$\begin{array}{rcl} L = \mathrm{N}_{H \times G}(\Delta(U, \alpha, V)) & = & \{(h,g) \in H \times G : {}^{(h,g)}\Delta(U, \alpha, V) = \Delta(U, \alpha, V)\} \\ & \overset{\text{L.84}}{=} & \{(h,g) \in H \times G : \Delta({}^h U, \kappa_h^{{}^h U, U} \circ \alpha \circ \kappa_{g^{-1}}^{V, {}^g V}, {}^g V) = \Delta(U, \alpha, V)\} \\ & = & \{(h,g) \in H \times G : h \in \mathrm{N}_H(U),\ g \in \mathrm{N}_G(V),\ \kappa_h^U \circ \alpha \circ \kappa_{g^{-1}}^V = \alpha\} \ . \end{array}$$

Now,

$$\begin{array}{rcl}
\mathrm{p}_1(L) & = & \{h \in H : \text{ there exists } g \in G \text{ such that } (h,g) \in L\} \\
& = & \{h \in \mathrm{N}_H(U) : \text{ there exists } g \in \mathrm{N}_G(V) \text{ such that } \kappa_h^U \circ \alpha \circ \kappa_{g^{-1}}^V = \alpha\} \\
& = & \{h \in \mathrm{N}_H(U) : \text{ there exists } g \in \mathrm{N}_G(V) \text{ such that } \kappa_h^U = \alpha \circ \kappa_g^V \circ \alpha^{-1}\}
\end{array}$$

$$\begin{array}{rcl}
\mathrm{p}_2(L) & = & \{g \in G : \text{ there exists } h \in H \text{ such that } (h,g) \in L\} \\
& = & \{g \in \mathrm{N}_G(V) : \text{ there exists } h \in \mathrm{N}_H(U) \text{ such that } \kappa_h^U \circ \alpha \circ \kappa_{g^{-1}}^V = \alpha\} \\
& = & \{g \in \mathrm{N}_G(V) : \text{ there exists } h \in \mathrm{N}_H(U) \text{ such that } \kappa_g^V = \alpha^{-1} \circ \kappa_h^U \circ \alpha\}
\end{array}$$

$$\begin{array}{rcl}
L_{1_G} & = & \{h \in H : (h, 1_G) \in L\} \\
& = & \{h \in N_H(U) : \kappa_h^U \circ \alpha = \alpha\} \\
& = & \{h \in N_H(U) : huh^{-1} = u \text{ for } u \in U\} \\
& = & \mathrm{C}_H(U)
\end{array}$$

$$\begin{array}{rcl}
{}_{1_H}L & = & \{g \in G : (1_H, g) \in L\} \\
& = & \{g \in \mathrm{N}_G(V) : \alpha \circ \kappa_{g^{-1}}^V = \alpha\} \\
& = & \{g \in \mathrm{N}_G(V) : g^{-1}vg = v \text{ for } v \in V\} \\
& = & \mathrm{C}_G(V)\ .
\end{array}$$

□

Remark 92. *Suppose given a subgroup $V \leq G$. Then*

$$\frac{|\operatorname{Fix}_{\Delta(V)}((G \times G)/\Delta(V))|}{|\operatorname{C}_G(V)|} = \frac{[\mathrm{N}_{G\times G}(\Delta(V)) : \Delta(V)]}{|\operatorname{C}_G(V)|} = \frac{|\operatorname{N}_G(V)|}{|V|}\ .$$

Proof. The first equality follows by Lemma 12(3). Consider the second equality. We have

$$\begin{array}{rcl}
\mathrm{N}_{G\times G}(\Delta(V)) & \overset{\mathrm{R.91}}{=} & \{(g,h) \in \mathrm{N}_G(V) \times \mathrm{N}_G(V) : {}^g v = {}^h v \text{ for all } v \in V\} \\
& = & \{(g,h) \in \mathrm{N}_G(V) \times \mathrm{N}_G(V) : h^{-1}g \in \mathrm{C}_G(V)\}\ .
\end{array}$$

Thus, we have a bijection

$$\begin{array}{rcl}
\mathrm{N}_{G\times G}(\Delta(V)) & \overset{\sim}{\leftrightarrow} & \mathrm{C}_G(V) \times \mathrm{N}_G(V) \\
(g,h) & \mapsto & (h^{-1}g, h) \\
(hg, h) & \leftarrow\!\shortmid & (g,h)\ .
\end{array}$$

Hence,

$$\begin{array}{rcl}
\dfrac{[\mathrm{N}_{G\times G}(\Delta(V)) : \Delta(V)]}{|\operatorname{C}_G(V)|} & = & \dfrac{|\operatorname{N}_G(V)| \cdot |\operatorname{C}_G(V)|}{|\Delta(V)| \cdot |\operatorname{C}_G(V)|} \\[2ex]
& = & \dfrac{|\operatorname{N}_G(V)|}{|V|}\ .
\end{array}$$

□

Lemma 93. *Let $(U,\alpha,V) \in I^{\Delta}_{H,G}$ and $(V',\beta,W) \in I^{\Delta}_{G,P}$. Let $\mathcal{A} \subseteq H$ represent $H/\,\mathrm{p}_1(\mathrm{N}_{H\times G}(\Delta(U,\alpha,V)))$. Let $\mathcal{B}_{\mathrm{N,C}} \subseteq \mathrm{N}_G(V)$ represent $\mathrm{N}_G(V)/\,\mathrm{C}_G(V)$. Let $\mathcal{C} \subseteq P$ represent $P/\,\mathrm{p}_2(\mathrm{N}_{G\times P}(\Delta(V',\beta,W)))$.*

We make use of the bilinear map $(-)\underset{G}{\cdot}(=)$ introduced in Definition 87. Recall the definition of the standard basis consisting of orbit sums in Definition 83(3).

(1) *Suppose that V and V' are not conjugate in G. Then*

$$[U,\alpha,V]^{+}_{H\times G}\underset{G}{\cdot}[V',\beta,W]^{+}_{G\times P} = 0\ .$$

(2) *Suppose that $V = V'$.*
Then

$$[U,\alpha,V]^{+}_{H\times G}\underset{G}{\cdot}[V,\beta,W]^{+}_{G\times P} = \sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(h,p)}(U,\alpha\circ\kappa^{V}_{g^{-1}}\circ\beta,W)\ .$$

Note that if $V = {}^{x}V'$ for some $x \in G$ then

$$[V',\beta,W]^{+}_{G\times P} = [{}^{x^{-1}}V,\beta,W]^{+}_{G\times P} = [V,\kappa^{V,{}^{x^{-1}}V}_{x}\circ\beta,W]^{+}_{G\times P}\ .$$

(3) *We have $[U,\alpha,V]^{+}_{H\times G}\underset{G}{\cdot}[V',\beta,W]^{+}_{G\times P} \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,P)$.*

(4) *For convenience, we suppose that $V = V'$.*
Write

$$\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)} := \sum_{T\in\mathcal{L}_G}[T,\mathrm{id}_T,T]^{+}_{G\times G}\ .$$

Then

$$[U,\alpha,V]^{+}_{H\times G}\cdot\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)} = [U,\alpha,V]^{+}_{H\times G}$$

and

$$\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}\cdot[V,\beta,W]^{+}_{G\times P} = [V,\beta,W]^{+}_{G\times P}\ .$$

Finally, $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)$ is a ring with identity element $\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}$, called the ghost ring *of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$.*

Proof. Let $\mathcal{B}_{G,\mathrm{N}}$ represent $G/\,\mathrm{N}_G(V)$. Let $\mathcal{B}_{G,\mathrm{C}} := \{gx : g\in\mathcal{B}_{G,\mathrm{N}},\ x\in\mathcal{B}_{\mathrm{N,C}}\}$. Then $\mathcal{B}_{G,\mathrm{C}}$ represents $G/\,\mathrm{C}_G(V)$, cf. Lemma 90(1).

Claim I. Suppose that $V = V'$. We have

$$[U,\alpha,V]^{+}_{H\times G} = \sum_{(h,g_1)\in\mathcal{A}\times\mathcal{B}_{G,\mathrm{C}}}({}^{h}U,\kappa^{{}^{h}U,U}_{h}\circ\alpha\circ\kappa^{V,{}^{g_1}V}_{g_1^{-1}},{}^{g_1}V)$$

and

$$[V,\beta,W]^{+}_{G\times P} = \sum_{(g_2,p)\in\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}}({}^{g_2}V,\kappa^{{}^{g_2}V,V}_{g_2}\circ\beta\circ\kappa^{W,{}^{p}W}_{p^{-1}},{}^{p}W)\ .$$

The set $\mathcal{A} \times \mathcal{B}_{G,\mathrm{C}}$ represents $(H \times G)/\operatorname{Stab}_{H\times G}((U,\alpha,V)) = (H\times G)/\operatorname{N}_{H\times G}(\Delta(U,\alpha,V))$, cf. Lemma 84, Lemma 89, Remark 91.

Moreover, $\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}$ represents $(G\times P)/\operatorname{Stab}_{G\times P}((V,\beta,W)) = (G\times P)/\operatorname{N}_{G\times P}(\Delta(V,\beta,W))$, cf. Lemma 84, Lemma 89, Remark 91.

Now,

$$\begin{aligned} [U,\alpha,V]^+_{H\times G} &= \sum_{(\tilde{U},\tilde{\alpha},\tilde{V})\in[(U,\alpha,V)]_{H\times G}} (\tilde{U},\tilde{\alpha},\tilde{V}) \\ &= \sum_{(h,g_1)\in\mathcal{A}\times\mathcal{B}_{G,\mathrm{C}}} {}^{(h,g_1)}(U,\alpha,V) \\ &= \sum_{(h,g_1)\in\mathcal{A}\times\mathcal{B}_{G,\mathrm{C}}} ({}^hU, \kappa_h^{{}^hU,U}\circ\alpha\circ\kappa_{g_1^{-1}}^{V,{}^{g_1}V}, {}^{g_1}V) \end{aligned}$$

and

$$\begin{aligned} [V,\beta,W]^+_{G\times P} &= \sum_{(\tilde{V},\tilde{\beta},\tilde{W})\in[(V,\beta,W)]_{G\times P}} (\tilde{V},\tilde{\beta},\tilde{W}) \\ &= \sum_{(g_2,p)\in\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}} {}^{(g_2,p)}(V,\beta,W) \\ &= \sum_{(g_2,p)\in\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}} ({}^{g_2}V, \kappa_{g_2}^{{}^{g_2}V,V}\circ\beta\circ\kappa_{p^{-1}}^{W,{}^pW}, {}^pW)\ . \end{aligned}$$

This proves *Claim* I.

Claim II. The sum

$$\sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(h,p)}(U,\alpha\circ\kappa_{g^{-1}}^V\circ\beta,W) = \sum_{(h,p)\in\mathcal{A}\times\mathcal{C}}\ \sum_{g\in\mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U,\alpha\circ\kappa_{g^{-1}}^V\circ\beta,W)$$

does not depend on the choices of $\mathcal{A}$, $\mathcal{B}_{\mathrm{N,C}}$ and $\mathcal{C}$.

Suppose that $\mathcal{B}'_{\mathrm{N,C}}$ represents $\operatorname{N}_G(V)/\operatorname{C}_G(V)$. It suffices to show that for $(h,p)\in\mathcal{A}\times\mathcal{C}$ we have

$$\sum_{g\in\mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U,\alpha\circ\kappa_{g^{-1}}^V\circ\beta,W) \stackrel{!}{=} \sum_{g'\in\mathcal{B}'_{\mathrm{N,C}}} {}^{(h,p)}(U,\alpha\circ\kappa_{g'^{-1}}^V\circ\beta,W)\ .$$

We have a bijection $\varphi_1 : \mathcal{B}'_{\mathrm{N,C}} \xrightarrow{\sim} \mathcal{B}_{\mathrm{N,C}}$ characterised by $\varphi_1(g')C = g'C$ for $g'\in\mathcal{B}'_{\mathrm{N,C}}$.

Suppose given $g'\in\mathcal{B}'_{\mathrm{N,C}}$. Write $g := \varphi_1(g')\in\mathcal{B}_{\mathrm{N,C}}$. So, there exists $c\in\operatorname{C}_G(V)$ such that $g' = gc$. But then for $v\in V$

$$\kappa_{g'^{-1}}^V(v) = \kappa_{(gc)^{-1}}^V(v) = c^{-1}g^{-1}vgc = g^{-1}vg = \kappa_{g^{-1}}^V(v)\ .$$

Hence, $\kappa_{g^{-1}}^V = \kappa_{g'^{-1}}^V$. Thus, the sum $\sum\limits_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(h,p)}(U,\alpha\circ\kappa_{g^{-1}}^V\circ\beta,W)$ does not depend on the choice of $\mathcal{B}_{\mathrm{N,C}}$.

Suppose that $\mathcal{C}'$ represents $P/\operatorname{p}_2(\operatorname{N}_{G\times P}(\Delta(V,\beta,W)))$.

We have a bijection $\varphi_2 : \mathcal{C}' \xrightarrow{\sim} \mathcal{C}$ characterised by $\varphi_2(p')C = p'C$ for $p'\in\mathcal{C}'$.

Suppose given $p'\in\mathcal{C}'$. Write $p := \varphi_2(p')\in\mathcal{C}$. So, there exists $x\in\operatorname{p}_2(\operatorname{N}_{G\times P}(\Delta(V,\beta,W)))$ such that $p' = px$. Suppose given $h\in\mathcal{A}$. It suffices to show that

$$\sum_{g\in\mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U,\alpha\circ\kappa_{g^{-1}}^V\circ\beta,W) \stackrel{!}{=} \sum_{g\in\mathcal{B}_{\mathrm{N,C}}} {}^{(h,px)}(U,\alpha\circ\kappa_{g^{-1}}^V\circ\beta,W)\ .$$

By Remark 91 we have $x \in \mathrm{N}_P(W)$ and there exists $y \in \mathrm{N}_G(V)$ such that $\kappa_x^W = \beta^{-1} \circ \kappa_y^V \circ \beta$. So,

$$\begin{aligned}
\sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,px)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W) &= \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta \circ \kappa_{x^{-1}}^{W,{}^xW}, {}^xW) \\
&= \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \kappa_{y^{-1}}^V \circ \beta, W) \\
&= \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{(yg)^{-1}}^V \circ \beta, W) \, .
\end{aligned}$$

Let $y\mathcal{B}_{\mathrm{N,C}} := \{yx : \ x \in \mathcal{B}_{\mathrm{N,C}}\}$. Then $y\mathcal{B}_{\mathrm{N,C}}$ represents $\mathrm{N}_G(V)/\,\mathrm{C}_G(V)$, cf. Lemma 90(3). We have seen above that $\sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W)$ does not depend on the choice of $\mathcal{B}_{\mathrm{N,C}}$. So, we get

$$\begin{aligned}
\sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{(yg)^{-1}}^V \circ \beta, W) &= \sum_{\tilde{g} \in y\mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{\tilde{g}^{-1}}^V \circ \beta, W) \\
&= \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W) \, .
\end{aligned}$$

Thus, the sum $\sum_{(h,g,p) \in \mathcal{A} \times \mathcal{B}_{\mathrm{N,C}} \times \mathcal{C}} {}^{(h,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W)$ does not depend on the choice of $\mathcal{C}$.

Suppose that $\mathcal{A}'$ represents $H/\,\mathrm{p}_1(\mathrm{N}_{H \times G}(\Delta(U, \alpha, V)))$.

We have a bijection $\varphi_3 : \mathcal{A}' \xrightarrow{\sim} \mathcal{A}$ characterised by $\varphi_3(h')\mathcal{A} = h'\mathcal{A}$ for $h' \in \mathcal{A}'$.

Suppose given $h' \in \mathcal{A}'$. Write $h := \varphi_3(h') \in \mathcal{A}$. So, there exists $x \in \mathrm{p}_1(\mathrm{N}_{H \times G}(\Delta(U, \alpha, V)))$ such that $h' = hx$. Suppose given $p \in \mathcal{C}$. It suffices to show that

$$\sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W) \stackrel{!}{=} \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(hx,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W) \, .$$

By Remark 91 we have $x \in \mathrm{N}_H(U)$ and there exists $y \in \mathrm{N}_G(V)$ such that $\kappa_x^U = \alpha \circ \kappa_y^V \circ \alpha^{-1}$. So,

$$\begin{aligned}
\sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(hx,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W) &= \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}({}^xU, \kappa_x^{{}^hU,U} \circ \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W) \\
&= \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_y^V \circ \kappa_{g^{-1}}^V \circ \beta, W) \\
&= \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{(gy^{-1})^{-1}}^V \circ \beta, W) \, .
\end{aligned}$$

Let $\mathcal{B}_{\mathrm{N,C}}y^{-1} := \{xy^{-1} : \ x \in \mathcal{B}_{\mathrm{N,C}}\}$. Then $\mathcal{B}_{\mathrm{N,C}}y^{-1}$ represents $\mathrm{N}_G(V)/\,\mathrm{C}_G(V)$, cf. Lemma 90(4). So, as $\sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W)$ does not depend on the choice of $\mathcal{B}_{\mathrm{N,C}}$, as we have seen above, we get

$$\begin{aligned}
\sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{(gy^{-1})^{-1}}^V \circ \beta, W) &= \sum_{\tilde{g} \in \mathcal{B}_{\mathrm{N,C}}y^{-1}} {}^{(h,p)}(U, \alpha \circ \kappa_{\tilde{g}^{-1}}^V \circ \beta, W) \\
&= \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa_{g^{-1}}^V \circ \beta, W) \, .
\end{aligned}$$

Thus, $\sum\limits_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(h,p)}(U,\alpha\circ\kappa^{V}_{g^{-1}}\circ\beta,W)$ does not depend on the choice of $\mathcal{A}$.

This proves *Claim* II.

Ad (1). Each summand in $[U,\alpha,V]^{+}_{H\times G}$ is of the form

$$^{(h,g)}(U,\alpha,V) = ({}^{h}U,\kappa_{h}^{{}^{h}U,U}\circ\alpha\circ\kappa_{g^{-1}}^{V,{}^{g}V},{}^{g}V)$$

for some $(h,g)\in H\times G$. Likewise, each summand in $[V',\beta,W]^{+}_{G\times P}$ is of the form

$$^{(g',p)}(V',\beta,W) = ({}^{g'}V',\kappa_{g'}^{{}^{g'}V',V'}\circ\beta\circ\kappa_{p^{-1}}^{W,{}^{p}W},{}^{p}W)$$

for some $(g',p)\in G\times P$. If V and V' are not conjugate in G, we have for $g,g'\in G$ that ${}^{g}V\neq{}^{g'}V'$ and therefore

$$\begin{aligned}[U,\alpha,V]^{+}_{H\times G}\underset{G}{\cdot}[V',\beta,W]^{+}_{G\times P} &= \left(\sum_{(U,\alpha,V)\in[(U,\alpha,V)]_{H\times G}}(U,\alpha,V)\right)\underset{G}{\cdot}\left(\sum_{(V',\beta,W)\in[(V',\beta,W)]_{G\times P}}(V',\beta,W)\right)\\ &= 0\ ,\end{aligned}$$

cf. Definition 87.

Ad (2). We have $[U,\alpha,V]^{+}_{H\times G} = \sum\limits_{(h,g_1)\in\mathcal{A}\times\mathcal{B}_{G,\mathrm{C}}}({}^{h}U,\kappa_{h}^{{}^{h}U,U}\circ\alpha\circ\kappa_{g_1^{-1}}^{V,{}^{g_1}V},{}^{g_1}V)$ and $[V,\beta,W]^{+}_{G\times P} = \sum\limits_{(g_2,p)\in\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}}({}^{g_2}V,\kappa_{g_2}^{{}^{g_2}V,V}\circ\beta\circ\kappa_{p^{-1}}^{W,{}^{p}W},{}^{p}W)$, cf. Claim I.

By Definition 87, we have

$$\begin{aligned}[U,\alpha,V]^{+}_{H\times G}\underset{G}{\cdot}[V,\beta,W]^{+}_{G\times P} &= \sum_{\substack{(h,g_1)\in\mathcal{A}\times\mathcal{B}_{G,\mathrm{C}}\\(g_2,p)\in\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}\\g_1^{-1}g_2\in\mathrm{N}_G(V)}}\frac{|\,\mathrm{C}_G(V)|}{|G|}({}^{h}U,\kappa_{h}^{{}^{h}U,U}\circ\alpha\circ\kappa^{V}_{g_1^{-1}g_2}\circ\beta\circ\kappa_{p^{-1}}^{W,{}^{p}W},{}^{p}W)\\ &= \sum_{\substack{(h,g_1)\in\mathcal{A}\times\mathcal{B}_{G,\mathrm{C}}\\(g_2,p)\in\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}\\g_1^{-1}g_2\in\mathrm{N}_G(V)}}\frac{|\,\mathrm{C}_G(V)|}{|G|}{}^{(h,p)}(U,\alpha\circ\kappa^{V}_{g_1^{-1}g_2}\circ\beta,W)\ .\end{aligned}$$

Using the bijection

$$\begin{aligned}\mathcal{B}_{G,\mathrm{N}}\times\mathcal{B}_{\mathrm{N,C}} &\to \mathcal{B}_{G,\mathrm{C}}\\ (g,x) &\mapsto gx\ ,\end{aligned}$$

cf. Lemma 90(2), we obtain

$$\begin{aligned}&\sum_{\substack{(h,g_1)\in\mathcal{A}\times\mathcal{B}_{G,\mathrm{C}}\\(g_2,p)\in\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}\\g_1^{-1}g_2\in\mathrm{N}_G(V)}}\frac{|\,\mathrm{C}_G(V)|}{|G|}{}^{(h,p)}(U,\alpha\circ\kappa^{V}_{g_1^{-1}g_2}\circ\beta,W)\\ &= \sum_{\substack{(h,p)\in\mathcal{A}\times\mathcal{C}\\(\tilde g_1,x_1,\tilde g_2,x_2)\in\mathcal{B}_{G,\mathrm{N}}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{B}_{G,\mathrm{N}}\times\mathcal{B}_{\mathrm{N,C}}\\(\tilde g_1x_1)^{-1}(\tilde g_2x_2)\in\mathrm{N}_G(V)}}\frac{|\,\mathrm{C}_G(V)|}{|G|}{}^{(h,p)}(U,\alpha\circ\kappa^{V}_{(\tilde g_1x_1)^{-1}(\tilde g_2x_2)}\circ\beta,W)\ .\end{aligned}$$

For $(\tilde{g}_1, x_1, \tilde{g}_2, x_2) \in \mathcal{B}_{G,\mathrm{N}} \times \mathcal{B}_{\mathrm{N,C}} \times \mathcal{B}_{G,\mathrm{N}} \times \mathcal{B}_{\mathrm{N,C}}$ we have

$$(\tilde{g}_1 x_1)^{-1}(\tilde{g}_2 x_2) \in \mathrm{N}_G(V) \Leftrightarrow \tilde{g}_1 x_1 \, \mathrm{N}_G(V) = \tilde{g}_2 x_2 \, \mathrm{N}_G(V) \Leftrightarrow \tilde{g}_1 \, \mathrm{N}_G(V) = \tilde{g}_2 \, \mathrm{N}_G(V) \Leftrightarrow \tilde{g}_1 = \tilde{g}_2 \, .$$

Therefore, we get

$$\begin{aligned}
& \sum_{\substack{(h,p) \in \mathcal{A} \times \mathcal{C} \\ (\tilde{g}_1, x_1, \tilde{g}_2, x_2) \in \mathcal{B}_{G,\mathrm{N}} \times \mathcal{B}_{\mathrm{N,C}} \times \mathcal{B}_{G,\mathrm{N}} \times \mathcal{B}_{\mathrm{N,C}} \\ (\tilde{g}_1 x_1)^{-1}(\tilde{g}_2 x_2) \in \mathrm{N}_G(V)}} \frac{|\,\mathrm{C}_G(V)|}{|G|} {}^{(h,p)}(U, \alpha \circ \kappa^V_{(\tilde{g}_1 x_1)^{-1}(\tilde{g}_2 x_2)} \circ \beta, W) \\
= & \sum_{\substack{(h,p) \in \mathcal{A} \times \mathcal{C} \\ (\tilde{g}_1, x_1, x_2) \in \mathcal{B}_{G,\mathrm{N}} \times \mathcal{B}_{\mathrm{N,C}} \times \mathcal{B}_{G,\mathrm{N}}}} \frac{|\,\mathrm{C}_G(V)|}{|G|} {}^{(h,p)}(U, \alpha \circ \kappa^V_{x_1{}^{-1} x_2} \circ \beta, W) \\
= & \sum_{(h,p) \in \mathcal{A} \times \mathcal{C}} \; \sum_{g \in \mathcal{B}_{G,\mathrm{N}}} \; \sum_{(x_1, x_2) \in \mathcal{B}_{\mathrm{N,C}} \times \mathcal{B}_{\mathrm{N,C}}} \frac{|\,\mathrm{C}_G(V)|}{|G|} {}^{(h,p)}(U, \alpha \circ \kappa^V_{x_1{}^{-1} x_2} \circ \beta, W) \\
= & \sum_{(h,p) \in \mathcal{A} \times \mathcal{C}} \; \sum_{(x_1, x_2) \in \mathcal{B}_{\mathrm{N,C}} \times \mathcal{B}_{\mathrm{N,C}}} \frac{|\,\mathrm{C}_G(V)|}{|\,\mathrm{N}_G(V)|} {}^{(h,p)}(U, \alpha \circ \kappa^V_{x_1{}^{-1} x_2} \circ \beta, W) \, .
\end{aligned}$$

As for every $x_2 \in \mathcal{B}_{\mathrm{N,C}}$ the set $x_2^{-1} \mathcal{B}_{\mathrm{N,C}} := \{x_2^{-1} x_1 : \; x_1 \in \mathcal{B}_{\mathrm{N,C}}\}$ is again a system of representatives for $\mathrm{N}_G(V)/\,\mathrm{C}_G(V)$, cf. Lemma 90(3), and, as we may use the bijection

$$\begin{array}{rcl}
\mathcal{B}_{\mathrm{N,C}} \times \mathcal{B}_{\mathrm{N,C}} & \to & \bigsqcup\limits_{x_2 \in \mathcal{B}_{\mathrm{N,C}}} x_2^{-1} \mathcal{B}_{\mathrm{N,C}} \\
(x_1, x_2) & \mapsto & (x_2^{-1} x_1, x_2) \; ,
\end{array}$$

cf. Lemma 90(5), we get

$$\begin{aligned}
& \sum_{(h,p) \in \mathcal{A} \times \mathcal{C}} \; \sum_{(x_1, x_2) \in \mathcal{B}_{\mathrm{N,C}} \times \mathcal{B}_{\mathrm{N,C}}} \frac{|\,\mathrm{C}_G(V)|}{|\,\mathrm{N}_G(V)|} {}^{(h,p)}(U, \alpha \circ \kappa^V_{x_1{}^{-1} x_2} \circ \beta, W) \\
= \quad & \sum_{(h,p) \in \mathcal{A} \times \mathcal{C}} \; \sum_{(x_2, x_2^{-1} x_1) \in \bigsqcup\limits_{x_2 \in \mathcal{B}_{\mathrm{N,C}}} x_2^{-1} \mathcal{B}_{\mathrm{N,C}}} \frac{|\,\mathrm{C}_G(V)|}{|\,\mathrm{N}_G(V)|} {}^{(h,p)}(U, \alpha \circ \kappa^V_{x_1{}^{-1} x_2} \circ \beta, W) \\
\overset{x' := x_2^{-1} x_1}{=} \quad & \sum_{(h,p) \in \mathcal{A} \times \mathcal{C}} \; \sum_{x_2 \in \mathcal{B}_{\mathrm{N,C}}} \; \sum_{x' \in x_2^{-1} \mathcal{B}_{\mathrm{N,C}}} \frac{|\,\mathrm{C}_G(V)|}{|\,\mathrm{N}_G(V)|} {}^{(h,p)}(U, \alpha \circ \kappa^V_{x'^{-1}} \circ \beta, W) \\
\overset{\text{Claim II}}{=} \quad & \sum_{(h,p) \in \mathcal{A} \times \mathcal{C}} \; \sum_{x' \in \mathcal{B}_{\mathrm{N,C}}} \frac{|\,\mathrm{C}_G(V)|}{|\,\mathrm{N}_G(V)|} \frac{|\,\mathrm{N}_G(V)|}{|\,\mathrm{C}_G(V)|} {}^{(h,p)}(U, \alpha \circ \kappa^V_{x'^{-1}} \circ \beta, W) \\
= \quad & \sum_{(h,p) \in \mathcal{A} \times \mathcal{C}} \; \sum_{g \in \mathcal{B}_{\mathrm{N,C}}} {}^{(h,p)}(U, \alpha \circ \kappa^V_{g^{-1}} \circ \beta, W) \, .
\end{aligned}$$

Thus,

$$[U, \alpha, V]^+_{H \times G} \underset{G}{\cdot} [V, \beta, W]^+_{G \times P} = \sum_{(h,g,p) \in \mathcal{A} \times \mathcal{B}_{\mathrm{N,C}} \times \mathcal{C}} {}^{(h,p)}(U, \alpha \circ \kappa^V_{g^{-1}} \circ \beta, W) \, .$$

Ad (3). Recall that $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,P) = \mathrm{Fix}_{H\times P}(A^{\Delta}(H,P))$, cf. Definition 83(3). Suppose that V and V' are not conjugate in G. Then, by (1), we have

$$[U,\alpha,V]^{+}_{H\times G} \underset{G}{\cdot} [V',\beta,W]^{+}_{G\times P} = 0 \in \tilde{\mathrm{B}}_{\mathbf{Z}}(H,P) \ .$$

Suppose that $V = {}^{g}V'$ for $g \in G$. We may assume $V = V'$, cf. (2).

By (2) we have

$$[U,\alpha,V]^{+}_{H\times G} \underset{G}{\cdot} [V,\beta,W]^{+}_{G\times P} = \sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}\times\mathcal{C}} {}^{(h,p)}(U,\alpha\circ\kappa^{V}_{g^{-1}}\circ\beta,W) \in A^{\Delta}_{\mathbf{Z}}(H,P) \ .$$

Therefore, it suffices to show that it is a fixed point under $H \times P$.

Suppose given $(h',p') \in H \times P$. Then

$$\begin{aligned} & {}^{(h',p')}\left(\sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(h,p)}(U,\alpha\circ\kappa^{V}_{g^{-1}}\circ\beta,W)\right) \\ = \ & \sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(h'h,p'p)}(U,\alpha\circ\kappa^{V}_{g^{-1}}\circ\beta,W) \\ = \ & \sum_{(\tilde{h},g,\tilde{p})\in h'\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times p'\mathcal{C}} {}^{(\tilde{h},\tilde{p})}(U,\alpha\circ\kappa^{V}_{g^{-1}}\circ\beta,W) \\ \overset{\text{Claim II}}{=} \ & \sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(h,p)}(U,\alpha\circ\kappa^{V}_{g^{-1}}\circ\beta,W) \ , \end{aligned}$$

as the set $h'\mathcal{A}$ is again a system of representatives for $H/\,\mathrm{p}_1(\mathrm{N}_{H\times G}(\Delta(U,\alpha,V)))$ and the set $p'\mathcal{C}$ is again a system of representatives for $P/\,\mathrm{p}_2(\mathrm{N}_{G\times P}(\Delta(V,\beta,W)))$, cf. Lemma 90(3).

Ad (4). We show that $[U,\alpha,V]^{+}_{H\times G}\cdot \mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)} \overset{!}{=} [U,\alpha,V]^{+}_{H\times G}$. By (1) and Definition 83(1) we obtain

$$\begin{aligned} [U,\alpha,V]^{+}_{H\times G} \underset{G}{\cdot} \sum_{T\in\mathcal{L}_G} [T,\mathrm{id},T]^{+}_{G\times G} &= \sum_{T\in\mathcal{L}_G} [U,\alpha,V]^{+}_{H\times G} \underset{G}{\cdot} [T,\mathrm{id},T]^{+}_{G\times G} \\ &= [U,\alpha,V]^{+}_{H\times G} \underset{G}{\cdot} [V,\mathrm{id},V]^{+}_{G\times G} \ . \end{aligned}$$

Note that $G/\,\mathrm{p}_2(\mathrm{N}_{G\times G}(\Delta(V,\mathrm{id},V))) = G/\,\mathrm{N}_G(V)$, cf. Remark 91. So, by (2) it follows that

$$\begin{aligned} [U,\alpha,V]^{+}_{H\times G} \underset{G}{\cdot} [V,\mathrm{id},V]^{+}_{G\times G} &= \sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{B}_{G,\mathrm{N}}} {}^{(h,p)}(U,\alpha\circ\kappa^{V}_{g^{-1}},V) \\ &= \sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{B}_{G,\mathrm{N}}} {}^{(h,1_G)}(U,\alpha\circ\kappa^{V,{}^{p}V}_{(pg)^{-1}},{}^{p}V) \ . \end{aligned}$$

Using the bijection

$$\begin{aligned} \mathcal{B}_{G,\mathrm{N}}\times\mathcal{B}_{\mathrm{N,C}} &\to \mathcal{B}_{G,\mathrm{C}} \\ (g,x) &\mapsto gx \ , \end{aligned}$$

cf. Lemma 90(2), we obtain

$$\begin{aligned}\sum_{(h,g,p)\in\mathcal{A}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{B}_{G,\mathrm{N}}} {}^{(h,1_G)}(U,\alpha\circ\kappa^{V,{}^gV}_{(pg)^{-1}},{}^pV) &= \sum_{(h,g')\in\mathcal{A}\times\mathcal{B}_{G,\mathrm{C}}} ({}^hU,\kappa^{{}^hU,U}_h\circ\alpha\circ\kappa^{V,{}^{g'}V}_{g'^{-1}},{}^{g'}V)\\ &\overset{\text{Claim I}}{=} [U,\alpha,V]^+_{H\times G}\ .\end{aligned}$$

We show that $\mathrm{id}_{\tilde{\mathrm{B}}^\Delta_{\mathbf{Z}}(G,G)}\cdot[V,\beta,W]^+_{G\times P}\overset{!}{=}[V,\beta,W]^+_{G\times P}$. By (1) and Definition 83(1) we obtain

$$\sum_{T\in\mathcal{L}_G}[T,\mathrm{id},T]^+_{G\times G}\underset{G}{\cdot}[V,\beta,W]^+_{G\times P} = [V,\mathrm{id},V]^+_{G\times G}\underset{G}{\cdot}[V,\beta,W]^+_{G\times P}\ .$$

Note that $G/\,\mathrm{p}_1(\mathrm{N}_{G\times G}(\Delta(V,\mathrm{id},V)))=G/N_G(V)$, cf. Remark 91. So, by (2) it follows that

$$\begin{aligned}[V,\mathrm{id},V]^+_{G\times G}\underset{G}{\cdot}[V,\beta,W]^+_{G\times P} &= \sum_{(h,g,p)\in\mathcal{B}_{G,\mathrm{N}}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(h,p)}(V,\kappa^V_{g^{-1}}\circ\beta,W)\\ &= \sum_{(h,g,p)\in\mathcal{B}_{G,\mathrm{N}}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(1_G,p)}({}^hV,\kappa^{{}^hV,V}_{hg^{-1}}\circ\beta,W)\ .\end{aligned}$$

Since $\mathrm{C}_G(V)\trianglelefteq\mathrm{N}_G(V)$, the set $\{x^{-1}:x\in\mathcal{B}_{\mathrm{N,C}}\}$ represents $\mathrm{N}_G(V)/\,\mathrm{C}_G(V)$. So, we have the bijection

$$\begin{aligned}\mathcal{B}_{G,\mathrm{N}}\times\mathcal{B}_{\mathrm{N,C}} &\to \mathcal{B}_{G,\mathrm{C}}\\ (g,x) &\mapsto gx^{-1}\ ,\end{aligned}$$

cf. Lemma 90(2). We obtain

$$\begin{aligned}\sum_{(h,g,p)\in\mathcal{B}_{G,\mathrm{N}}\times\mathcal{B}_{\mathrm{N,C}}\times\mathcal{C}} {}^{(1_G,p)}({}^hV,\kappa^{{}^hV,V}_{hg^{-1}}\circ\beta,W) &= \sum_{(g',p)\in\mathcal{B}_{G,\mathrm{C}}\times\mathcal{C}} ({}^{g'}V,\kappa^{{}^{g'}V,V}_{g'}\circ\beta\circ\kappa^{W,{}^pW}_{p^{-1}},{}^pW)\\ &\overset{\text{Claim I}}{=} [V,\beta,W]^+_{G\times P}\ .\end{aligned}$$

Moreover, by (3) we have for $G=H=P$ that

$$[U,\alpha,V]^+_{G\times G}\underset{G}{\cdot}[V,\beta,W]^+_{G\times G}\in\tilde{\mathrm{B}}^\Delta_{\mathbf{Z}}(G,G)\ .$$

The multiplication $(\,\underset{G}{\cdot}\,)$ on $\tilde{\mathrm{B}}^\Delta_{\mathbf{Z}}(G,G)$ is associative by Remark 88.

So, $\tilde{\mathrm{B}}^\Delta_{\mathbf{Z}}(G,G)$ is a ring with identity element $\mathrm{id}_{\tilde{\mathrm{B}}^\Delta_{\mathbf{Z}}(G,G)}=\sum_{T\in\mathcal{L}_G}[T,\mathrm{id}_T,T]^+_{G\times G}$. □

Lemma 94. *Recall the* $\mathbf{Z}$*-linear map*

$$\begin{aligned}\mathfrak{m}^\Delta_{H,G}:\mathrm{B}^\Delta_{\mathbf{Z}}(H,G) &\to \tilde{\mathrm{B}}^\Delta_{\mathbf{Z}}(H,G)\\ [X] &\mapsto \sum_{(U,\alpha,V)\in I^\Delta_{H,G}}\frac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|}{|\,\mathrm{C}_H(U)|}(U,\alpha,V)\ ,\end{aligned}$$

cf. Lemma 86.

(1) *For* $a \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$ *and* $b \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,P)$ *we have*

$$\mathfrak{m}^{\Delta}_{H,P}(a \underset{G}{\cdot} b) = \mathfrak{m}^{\Delta}_{H,G}(a) \underset{G}{\cdot} \mathfrak{m}^{\Delta}_{G,P}(b) \ .$$

Moreover, $\mathfrak{m}^{\Delta}_{G,G}(\mathrm{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)}) = \mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}$ *, cf. Definition 67, Lemma 93(4).*

(2) *The map* $\mathfrak{m}^{\Delta}_{H,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) \to \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G)$ *is an injective group morphism with finite cokernel of order*

$$\prod_{\Delta(U,\alpha,V) \in \mathcal{L}^{\Delta}_{H \times G}} \frac{[\mathrm{N}_{H \times G}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\,\mathrm{C}_H(U)|} \ ,$$

which divides $|H \times G|^{|\mathcal{L}^{\Delta}_{H\times G}|}$*.*

(3) *We obtain a* $\mathbf{Q}$*-module isomorphism*

$$\mathbf{Q} \otimes \mathfrak{m}^{\Delta}_{H,G} : \mathrm{B}^{\Delta}_{\mathbf{Q}}(H,G) \to \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Q}}(H,G) \ .$$

(4) *The map* $\mathfrak{m}^{\Delta}_{G,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G) \to \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)$ *is an injective ring morphism with finite cokernel of order*

$$\prod_{\Delta(U,\alpha,V) \in \mathcal{L}^{\Delta}_{G \times G}} \frac{[\mathrm{N}_{G \times G}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\,\mathrm{C}_G(U)|} \ ,$$

which divides $|G \times G|^{|\mathcal{L}^{\Delta}_{G\times G}|}$*.*

We obtain a $\mathbf{Q}$*-algebra isomorphism*

$$\mathbf{Q} \otimes \mathfrak{m}^{\Delta}_{G,G} : \mathrm{B}^{\Delta}_{\mathbf{Q}}(G,G) \to \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Q}}(G,G) \ .$$

Proof. *Ad* (1). Without loss of generality we may assume $a = [X]$ and $b = [Y]$ for some bifree (H,G)-biset X and some bifree (G,P)-biset Y.

We have

$$\begin{aligned}
&\mathfrak{m}^{\Delta}_{H,G}(a) \underset{G}{\cdot} \mathfrak{m}^{\Delta}_{G,P}(b) \\
&= \left(\sum_{(U,\alpha,V) \in I^{\Delta}_{H,G}} \frac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|}{|\,\mathrm{C}_H(U)|} (U,\alpha,V) \right) \underset{G}{\cdot} \left(\sum_{(V',\beta,W) \in I^{\Delta}_{G,P}} \frac{|\,\mathrm{Fix}_{\Delta(V',\beta,W)}(Y)|}{|\,\mathrm{C}_G(V')|} (V',\beta,W) \right) \\
&\overset{\text{D.87}}{=} \sum_{\substack{(U,\alpha,V) \in I^{\Delta}_{H,G} \\ (V',\beta,W) \in I^{\Delta}_{G,P} \\ V = V'}} \frac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)| \cdot |\,\mathrm{Fix}_{\Delta(V,\beta,W)}(Y)|}{|\,\mathrm{C}_H(U)| \cdot |\,\mathrm{C}_G(V)|} \frac{|\,\mathrm{C}_G(V)|}{|G|} (U, \alpha \circ \beta, W) \\
&= \sum_{\substack{(U,\alpha,V) \in I^{\Delta}_{H,G} \\ (V',\beta,W) \in I^{\Delta}_{G,P} \\ V = V'}} \frac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)| \cdot |\,\mathrm{Fix}_{\Delta(V,\beta,W)}(Y)|}{|\,\mathrm{C}_H(U)| \cdot |G|} (U, \alpha \circ \beta, W) \ .
\end{aligned}$$

Suppose given a basis element $(U,\gamma,W) \in I^{\Delta}_{H,P}$ of $A^{\Delta}_{\mathbf{Z}}(H,P)$. Now, the coefficient of $\mathfrak{m}^{\Delta}_{H,G}(a) \cdot_G \mathfrak{m}^{\Delta}_{G,P}(b)$ at (U,γ,W) equals

$$\sum_{(\alpha,V,\beta)\in\Gamma_G(U,\gamma,W)} \frac{|\operatorname{Fix}_{\Delta(U,\alpha,V)}(X)| \cdot |\operatorname{Fix}_{\Delta(V,\beta,W)}(Y)|}{|\operatorname{C}_H(U)| \cdot |G|} \;,$$

cf. Definition 79.

We have

$$\mathfrak{m}^{\Delta}_{H,P}(a \underset{G}{\cdot} b) = \mathfrak{m}^{\Delta}_{H,P}([X \underset{G}{\times} Y]) = \sum_{(U,\gamma,W)\in I^{\Delta}_{H,P}} \frac{|\operatorname{Fix}_{\Delta(U,\gamma,W)}(X \underset{G}{\times} Y)|}{|\operatorname{C}_H(U)|}(U,\gamma,W) \;,$$

cf. Remark 65. So, the coefficient at the basis element (U,γ,W) equals

$$\frac{|\operatorname{Fix}_{\Delta(U,\gamma,W)}(X \underset{G}{\times} Y)|}{|\operatorname{C}_H(U)|} \;.$$

By Lemma 82(2) these two numbers are equal.

We have

$$\begin{array}{rcl} \mathfrak{m}^{\Delta}_{G,G}(\operatorname{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)}) & \overset{\mathrm{D.67}}{=} & \mathfrak{m}^{\Delta}_{G,G}([(G\times G)/\Delta(G))]) \\ & = & \displaystyle\sum_{(U,\alpha,V)\in I^{\Delta}_{G,G}} \frac{|\operatorname{Fix}_{\Delta(U,\alpha,V)}((G\times G)/\Delta(G))|}{|\operatorname{C}_G(U)|}(U,\alpha,V) \;. \end{array}$$

We have $\operatorname{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)} = \sum_{T\in\mathcal{L}_G} [T,\operatorname{id}_T,T]^{+}_{G\times G} = \sum_{T\in\mathcal{L}_G} \sum_{(U,\alpha,V)\in[T,\operatorname{id}_T,T]_{G\times G}} (U,\alpha,V)$.

The coefficient of $\operatorname{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}$ at (U,α,V) equals 1 if there exists $(g,g') \in G\times G$ such that $(U,\alpha,V) = {}^{(g,g')}(T,\operatorname{id}_T,T) = ({}^gT, \kappa^{{}^gT,{}^{g'}T}_{gg'^{-1}}, {}^{g'}T)$, it equals 0 if no such $(g,g')\in G\times G$ exists. Note that such an element $(g,g')\in G\times G$ exists if and only if $\alpha = \kappa^{{}^{\tilde g}V,V}_{\tilde g}$ for some $\tilde g \in G$.

We have

$$\begin{array}{rl} & \operatorname{Fix}_{\Delta(U,\alpha,V)}((G\times G)/\Delta(G)) \\ = & \{(g,g')\Delta(G) \in (G\times G)/\Delta(G) :\; (\alpha(v),v)(g,g')\Delta(G) = (g,g')\Delta \;\text{ for } v\in V\} \\ = & \{(g,g')\Delta(G) \in (G\times G)/\Delta(G) :\; (g^{-1}\alpha(v)g, g'^{-1}vg') \in \Delta(G) \text{ for } v\in V\} \\ = & \{(g,g')\Delta(G) \in (G\times G)/\Delta(G) :\; g^{-1}\alpha(v)g = g'^{-1}vg' \text{ for } v\in V\} \\ = & \{(gg'^{-1},1_G)\Delta(G) \in (G\times G)/\Delta(G) :\; \alpha(v) = {}^{gg'^{-1}}v \text{ for } v\in V\} \;. \end{array}$$

Hence, the coefficient of $\mathfrak{m}^{\Delta}_{G,G}(\operatorname{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)})$ at (U,α,V) is 0 if α is not given by conjugation in G.

Suppose that $\alpha = \kappa^{{}^{\tilde g}V,V}_{\tilde g}$ for some $\tilde g\in G$.

Then

$$\begin{array}{rcl} |\operatorname{Fix}_{\Delta(U,\alpha,V)}((G\times G)/\Delta(G))| & = & |\{g\in G :\; {}^{\tilde g}v = {}^{g}v \text{ for } v\in V\}| \\ & = & |\{g\in G :\; \tilde g^{-1}g \in \operatorname{C}_G(V)\}| \;. \end{array}$$

So,

$$|\operatorname{Fix}_{\Delta(U,\alpha,V)}((G\times G)/\Delta(G))| = |\tilde{g}\,\mathrm{C}_G(V)| = |{}^{\tilde{g}}\,\mathrm{C}_G(V)| = |\,\mathrm{C}_G({}^{\tilde{g}}V)| = |\,\mathrm{C}_G(U)|\ .$$

Therefore, if α is given by conjugation in G, the coefficient of $\mathfrak{m}^{\Delta}_{G,G}(\mathrm{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)})$ at (U,α,V) is 1.

Thus, $\mathfrak{m}^{\Delta}_{G,G}(\mathrm{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)}) = \mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}$.

Ad (2). Let ℓ be the number of $(H\times G)$-orbits of the (H,G)-biset $I^{\Delta}_{(H,G)}$, cf. Definition 83(1). Let (U_i,α_i,V_i) for $i\in[1,\ell]$ be the orbit representatives therein. Note that $\ell = |\mathcal{L}^{\Delta}_{H\times G}|$, cf. Lemma 84.

Choose the numbering of the representatives in such a way that for $i,j\in[1,\ell]$ such that $\Delta(U_i,\alpha_i,V_i)\leq_{H\times G}\Delta(U_j,\alpha_j,V_j)$, we have $i\leq j$, cf. Definition 10, Remark 11 and Lemma 84.

Recall the standard basis $\{[(H\times G)/\Delta(U_i,\alpha_i,V_i)] : i\in[1,\ell]\}$ of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$, cf. Definition 64 and the standard basis $\{[U_i,\alpha_i,V_i]^{+}_{H\times G} :\ i\in[1,\ell]\}$ of $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G)$, cf. Definition 83(3).

As recalled above, we have for $i\in[1,\ell]$

$$\begin{aligned}\mathfrak{m}^{\Delta}_{H,G}([(H\times G)/\Delta(U_i,\alpha_i,V_i)]) &= \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G}} \frac{|\operatorname{Fix}_{\Delta(U,\alpha,V)}((H\times G)/\Delta(U_i,\alpha_i,V_i))|}{|\,\mathrm{C}_H(U)|}(U,\alpha,V)\\ &= \sum_{j\in[1,\ell]} \frac{|\operatorname{Fix}_{\Delta(U_j,\alpha_j,V_j)}((H\times G)/\Delta(U_i,\alpha_i,V_i))|}{|\,\mathrm{C}_H(U_j)|}[U_j,\alpha_j,V_j]^{+}_{H\times G}\ ,\end{aligned}$$

the latter equation following from the fact that $\mathfrak{m}^{\Delta}_{H,G}$ actually maps to $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G)$, cf. Lemma 86.

Let $M\in\mathbf{Z}^{\ell\times\ell}$ be the representing matrix of $\mathfrak{m}^{\Delta}_{H,G}$ with respect to the standard basis of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$ and the standard basis of $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G)$ numbered as explained above.

Then M is an upper triangular matrix as for $1\leq i<j\leq\ell$, we have $j\not\leq i$, whence $\Delta(U_j,\alpha_j,V_j)\not\leq_{H\times G}\Delta(U_i,\alpha_i,V_i)$ and so

$$|\operatorname{Fix}_{\Delta(U_j,\alpha_j,V_j)}((H\times G)/\Delta(U_i,\alpha_i,V_i))| = 0\ ,\ \text{cf. Lemma 12(2).}$$

By Lemma 12(3) we get that, for $i\in[1,\ell]$ and $L := \Delta(U_i,\alpha_i,V_i)$,

$$\operatorname{Fix}_L((H\times G)/L) = \mathrm{N}_{H\times G}(L)/L \;=\; \{(h,g)L\in(H\times G)/L : L^{(h,g)}\subseteq L\}\ .$$

So, the matrix M has diagonal entries $\dfrac{[\mathrm{N}_{H\times G}(\Delta(U_i,\alpha_i,V_i)) : \Delta(U_i,\alpha_i,V_i)]}{|\,\mathrm{C}_H(U_i)|}$ for $i\in[1,\ell]$, cf. Lemma 12(3).

So, the map $\mathfrak{m}^{\Delta}_{H,G}$ is injective. The cokernel is of order

$$\begin{aligned}\det(M) &= \prod_{i\in[1,\ell]} \frac{[\mathrm{N}_{H\times G}(\Delta(U_i,\alpha_i,V_I)) : \Delta(U_i,\alpha_i,V_i)]}{|\,\mathrm{C}_H(U_i)|}\\ &= \prod_{\Delta(U,\alpha,V)\in\mathcal{L}^{\Delta}_{H\times G}} \frac{[\mathrm{N}_{H\times G}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\,\mathrm{C}_H(U)|}\ .\end{aligned}$$

Ad (3). This follows from (2).

Ad (4). Follows by (1), (2) and (3). $\square$

Example 95. In case of $G = \mathrm{S}_4$, using Magma [6], we obtain the representing matrix

$$M = \begin{pmatrix}
24 & 12 & 12 & 12 & 12 & 8 & 6 & 6 & 6 & 6 & 6 & 6 & 4 & 3 & 3 & 2 & 1 \\
0 & 4 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 3 & 2 & 1 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

of $\mathfrak{m}^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4}$.

2.3.4 An isomorphism for the ghost ring

Notation 96. Let $\mathcal{T}$ denote a set of representatives of isomorphism classes of finite groups. We denote by $\mathcal{T}_G$ a set of representatives of isomorphism classes of subgroups of G.

Definition 97. Recall that $I^{\Delta}_{H,G} = \{(U,\alpha,V) :\ U \leq H,\ V \leq G,\ \alpha \in \mathrm{Isom}(U,V)\}$, cf. Definition 83(2).

(1) Write $I^{\Delta}_{H,G,T} := \{(U,\alpha,V) \in I^{\Delta}_{H,G} : U \cong T \cong V\}$ for $T \in \mathcal{T}$. Then we have a disjoint decomposition
$$I^{\Delta}_{H,G} = \bigsqcup_{T\in\mathcal{T}} I^{\Delta}_{H,G,T}$$
into (H,G)-sub-bisets, cf. Definition 18(2).

To see this, suppose given $T \in \mathcal{T}$. We have for $(U,\alpha,V) \in I^{\Delta}_{H,G,T}$ and $(h,g) \in H \times G$ that
$${}^{(h,g)}(U,\alpha,V) = ({}^hU, \kappa_h^{{}^hU,U} \circ \alpha \circ \kappa_{g^{-1}}^{V,{}^gV}, {}^gV)\ .$$

So, ${}^{(h,g)}(U,\alpha,V) \in I^{\Delta}_{H,G,T}$ as ${}^hU \cong U \cong T \cong V \cong {}^gV$.

Thus, the set $I^{\Delta}_{H,G,T}$ is an (H,G)-sub-biset of $I^{\Delta}_{H,G}$ for $T \in \mathcal{T}$.

(2) Given $T \in \mathcal{T}$, we denote by $A^{\Delta}_{\mathbf{Z},T}(H,G) := \mathbf{Z} I^{\Delta}_{H,G,T}$ the $\mathbf{Z}$-span of the subset $I^{\Delta}_{H,G,T}$ of $I^{\Delta}_{H,G}$. Then we have by the decomposition in (1) that

$$A^{\Delta}_{\mathbf{Z}}(H,G) = \bigoplus_{T\in\mathcal{T}} A^{\Delta}_{\mathbf{Z},T}(H,G) \ ,$$

cf. Definition 83(2). Note that all but finitely many summands are equal to $\{0\}$ as $I^{\Delta}_{H,G,T}$ is non-empty only if H and G have subgroups that are isomorphic to T.

Let $T \in \mathcal{T}$. We denote by $A^{\Delta}_{\mathbf{Q},T}(H,G) := \mathbf{Q} \otimes_{\mathbf{Z}} A^{\Delta}_{\mathbf{Z},T}(H,G)$ the $\mathbf{Q}$-span of the subset $I^{\Delta}_{H,G,T}$ of $I^{\Delta}_{H,G}$. Then

$$A^{\Delta}_{\mathbf{Q}}(H,G) = \bigoplus_{T\in\mathcal{T}} A^{\Delta}_{\mathbf{Q},T}(H,G) \ .$$

Lemma 98 (decomposition of $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,H)$)**.** *Write* $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G) := \mathrm{Fix}_{H\times G}(A^{\Delta}_{\mathbf{Z},T}(H,G))$ *and* $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Q},T}(H,G) := \mathrm{Fix}_{H\times G}(A^{\Delta}_{\mathbf{Q},T}(H,G))$.

(1) *We have*

$$\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G) = \bigoplus_{T\in\mathcal{T}} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G) \quad \text{and } \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Q}}(H,G) = \bigoplus_{T\in\mathcal{T}} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Q},T}(H,G) \ .$$

Viewing $\bigoplus_{T\in\mathcal{T}} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$ *also as exterior direct sum, we also write* $(\tilde{b}_T)_{T\in\mathcal{T}} := \sum_{T\in\mathcal{T}} \tilde{b}_T$, *when* $\tilde{b}_T \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$ *for* $T \in \mathcal{T}$.

(2) *We have for* $T_1, T_2 \in \mathcal{T}$ *with* $T_1 \neq T_2$ *that*

$$\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T_1}(H,G) \underset{G}{\cdot} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T_2}(G,P) = 0 \ .$$

So, for $\tilde{b} \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G)$ *and* $\tilde{b}' \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,P)$ *we may uniquely write* $\tilde{b} = \sum_{T\in\mathcal{T}} \tilde{b}_T$ *with* $\tilde{b}_T \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$ *for* $T \in \mathcal{T}$ *and* $\tilde{b}' = \sum_{T\in\mathcal{T}} \tilde{b}'_T$ *with* $\tilde{b}'_T \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(G,P)$ *for* $T \in \mathcal{T}$. *We obtain*

$$\tilde{b} \underset{G}{\cdot} \tilde{b}' = \left(\sum_{T\in\mathcal{T}} \tilde{b}_T\right) \underset{G}{\cdot} \left(\sum_{T\in\mathcal{T}} \tilde{b}'_T\right) = \sum_{T\in\mathcal{T}} \tilde{b}_T \underset{G}{\cdot} \tilde{b}'_T \ .$$

(3) *Recall the* $\mathbf{Z}$*-linear map*

$$\begin{array}{rcl} \mathfrak{m}^{\Delta}_{H,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) & \to & \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G) \\ {[X]} & \mapsto & \sum\limits_{(U,\alpha,V)\in I^{\Delta}_{H,G}} \dfrac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|}{|\,\mathrm{C}_H(U)|}(U,\alpha,V) \ , \end{array}$$

cf. Lemma 86.

For $T \in \mathcal{T}$ *we have*

$$\begin{array}{rcl} \mathfrak{m}^{\Delta}_{H,G,T} := \pi_T \circ \mathfrak{m}^{\Delta}_{H,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) & \to & \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G) \\ {[X]} & \mapsto & \sum\limits_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} \dfrac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}(X)|}{|\,\mathrm{C}_H(U)|}(U,\alpha,V) \ . \end{array}$$

Then

$$\begin{aligned} \mathfrak{m}^{\Delta}_{H,G} = (\mathfrak{m}^{\Delta}_{H,G,T})_{T\in\mathcal{T}} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) &\to \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G) = \bigoplus_{T\in\mathcal{T}} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G) \\ [X] &\mapsto \left(\sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} \frac{|\operatorname{Fix}_{\Delta(U,\alpha,V)}(X)|}{|\operatorname{C}_H(U)|}(U,\alpha,V) \right)_{T\in\mathcal{T}} . \end{aligned}$$

Proof. Ad (1). Using the decomposition from Definition 97(1) and taking (H,G)-fixed points in Definition 97(2) leads to the decompositions.

Ad (2). Recall that

$$\begin{aligned} (-) \underset{G}{\cdot} (=) : A^{\Delta}_{\mathbf{Q}}(H,G) \times A^{\Delta}_{\mathbf{Q}}(G,P) &\to A^{\Delta}_{\mathbf{Q}}(H,P) \\ ((U,\alpha,V),(V',\beta,W)) &\mapsto (U,\alpha,V) \underset{G}{\cdot} (V',\beta,W) := \begin{cases} 0 & \text{if } V \neq V' \\ \dfrac{|\operatorname{C}_G(V)|}{|G|}(U,\alpha\circ\beta,W) & \text{if } V = V' \end{cases} , \end{aligned}$$

cf. Definition 87.

By Lemma 93(3) this map restricts to a bilinear map

$$(-) \underset{G}{\cdot} (=) : \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G) \times \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,P) \to \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,P) .$$

Since $T_1 \neq T_2$ it follows for $(U,\alpha,V) \in I^{\Delta}_{H,G,T_1}$ and $(V',\beta,W) \in I^{\Delta}_{G,P,T_2}$ that we have $V \neq V'$ as $V \cong T_1 \ncong T_2 \cong V'$.

Ad (3). Follows by (1).

□

Definition 99. For $T \in \mathcal{T}$ let $\operatorname{Inj}(T,G)$ denote the set of *injective group morphisms from T to G.*

Note that $\operatorname{Inj}(T,G)$ is empty if T is not isomorphic to a subgroup of G.

We denote $\bar{\lambda} := \lambda|^{\lambda(T)}$ for $\lambda \in \operatorname{Inj}(T,G)$. Note that $\bar{\lambda}$ is an isomorphism from T to $\lambda(T)$.

Remark 100. Note that the set $\operatorname{Inj}(T,G)$ is a $(G,\operatorname{Aut}(T))$-biset via

$$x \cdot \lambda \cdot \omega := \kappa^G_x \circ \lambda \circ \omega$$

for $x \in G$, $\lambda \in \operatorname{Inj}(T,G)$, $\omega \in \operatorname{Aut}(T)$, cf. Remark 15. In particular, $\operatorname{Inj}(T,G)$ is a left G-set.

Definition 101. We denote by $\overline{\operatorname{Inj}}(T,G)$ the set of G-orbits of $\operatorname{Inj}(T,G)$. We denote by $[\lambda]$ the G-orbit of an element $\lambda \in \operatorname{Inj}(T,G)$, cf. Remark 100. So,

$$\overline{\operatorname{Inj}}(T,G) = \{[\lambda] : \lambda \in \operatorname{Inj}(T,G)\} .$$

Remark 102. Note that $\overline{\mathrm{Inj}}(T,G)$ is a right $\mathrm{Out}(T)$-set as $\mathrm{Inn}(T)$ acts trivially on $\overline{\mathrm{Inj}}(T,G)$. In fact, for $t \in T$, $\lambda \in \overline{\mathrm{Inj}}(T)$ we have

$$(\lambda \circ \kappa_t^T)(x) = \lambda({}^t x) = {}^{\lambda(t)}\lambda(x) = \kappa^G_{\lambda(t)}(\lambda(x))$$

and so

$$\lambda \cdot \kappa_t^T = \lambda(t) \cdot \lambda \ .$$

Consequently, $\mathbf{Z}\overline{\mathrm{Inj}}(T,G)$ is a right $\mathbf{Z}\,\mathrm{Out}(T)$-module.

Lemma 103. *Suppose given $T \in \mathcal{T}$ and $\lambda \in \mathrm{Inj}(T,G)$. Then*

$$\mathrm{Stab}_G(\lambda) = \mathrm{C}_G(\lambda(T)) \ .$$

In particular, the G-orbit $[\lambda]$ has $\dfrac{|G|}{|\,\mathrm{C}_G(\lambda(T))|}$ elements.

Proof. We have

$$\begin{array}{rcl} \mathrm{Stab}_G(\lambda) = \{g \in G : g \cdot \lambda = \lambda\} & = & \{g \in G : \kappa_g^G \circ \lambda = \lambda\} \\ & = & \{g \in G : (\kappa_g^G \circ \lambda)(t) = \lambda(t) \text{ for all } t \in T\} \\ & = & \{g \in G : g\lambda(t) = \lambda(t)g \text{ for all } t \in T\} = \mathrm{C}_G(\lambda(T)) \ . \end{array}$$

□

Lemma 104. *Suppose given $T \in \mathcal{T}$ and $\omega \in \mathrm{Aut}(T)$.*

We have the mutually inverse bijections

$$\begin{array}{rcl} \overline{\mathrm{Inj}}(T,G) & \overset{\sim}{\leftrightarrow} & \overline{\mathrm{Inj}}(T,G) \\ [\lambda] & \overset{\varphi}{\mapsto} & [\lambda \circ \omega] \\ [\mu \circ \omega^{-1}] & \overset{\varphi'}{\leftarrow\!\shortmid} & [\mu] \ . \end{array}$$

Proof. Suppose given $[\lambda], [\eta] \in \overline{\mathrm{Inj}}(T,G)$ with $[\lambda] = [\eta]$. Then there exists $g \in G$ such that $\kappa_g^G \circ \lambda = \eta$ and therefore

$$[\eta \circ \omega] = [\kappa_g^G \circ \lambda \circ \omega] = [\lambda \circ \omega] \ .$$

Thus, φ is well-defined.

Analogously, φ' is well-defined.

Moreover, φ and φ' are mutually inverse. □

Lemma 105. *Suppose given $T \in \mathcal{T}$. We have an isomorphism of (H,G)-bisets*

$$\begin{array}{rcl} \hat{f} : \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}} & \to & I^{\Delta}_{H,G,T} \\ \lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} & \mapsto & (\lambda(T), \bar{\lambda} \circ \bar{\mu}^{-1}, \mu(T)) \ . \end{array}$$

Cf. Definition 97(1), Definition 99, Remark 100 and Remark 20.

Proof. We consider the map

$$\begin{array}{rcl} f : \mathrm{Inj}(T,H) \times (\mathrm{Inj}(T,G))^{\mathrm{op}} & \to & I^{\Delta}_{H,G,T} \\ (\lambda, \mu^{\mathrm{op}}) & \mapsto & (\lambda(T), \bar{\lambda} \circ \bar{\mu}^{-1}, \mu(T)) \ . \end{array}$$

By construction, f is well-defined.

We show that f is surjective.

Suppose given isomorphic subgroups $U \xhookrightarrow{\iota_U} H$, $V \xhookrightarrow{\iota_V} G$ and an isomorphism $V \xrightarrow[\sim]{\alpha} U$. There exists exactly one $T \in \mathcal{T}$ such that $T \cong V \cong U$. Choose an isomorphism $T \xrightarrow[\sim]{\varphi} V$. Set $\lambda := \iota_U \circ \alpha \circ \varphi$ and $\mu := \iota_V \circ \varphi$. Then $\mu(T) = V$ and $\lambda(T) = U$. Moreover, $\bar{\lambda} \circ \bar{\mu}^{-1} = \alpha \circ \varphi \circ \varphi^{-1} = \alpha$, cf. Definition 99. So, f is surjective.

We *claim* that for $(h,g) \in H \times G$, $\omega \in \mathrm{Aut}(T)$ and $(\lambda, \mu^{\mathrm{op}}) \in \mathrm{Inj}(T,H) \times (\mathrm{Inj}(T,G)^{\mathrm{op}})$ we have

$$^{(h,g)}f((\lambda, \mu^{\mathrm{op}})) = f((h \cdot \lambda \cdot \omega, \omega^{-1} \cdot \mu^{\mathrm{op}} \cdot g^{-1})) \ .$$

In fact,

$$\begin{array}{rl} & ^{(h,g)}f((\lambda, \mu^{\mathrm{op}})) \\ = & ^{(h,g)}(\lambda(T), \bar{\lambda} \circ \bar{\mu}^{-1}, \mu(T)) \\ = & (^{h}\lambda(T), \kappa_h^{{}^h\lambda(T),\lambda(T)} \circ \bar{\lambda} \circ \bar{\mu}^{-1} \circ \kappa_{g^{-1}}^{\mu(T),{}^g\mu(T)}, {}^g\mu(T)) \\ = & (^{h}\lambda(T), \kappa_h^{{}^h\lambda(T),\lambda(T)} \circ \bar{\lambda} \circ \omega \circ \omega^{-1} \circ \bar{\mu}^{-1} \circ \kappa_{g^{-1}}^{\mu(T),{}^g\mu(T)}, {}^g\mu(T)) \\ = & (^{h}(\lambda \circ \omega)(T), \kappa_h^{{}^h(\lambda\circ\omega)(T),(\lambda\circ\omega)(T)} \circ \bar{\lambda} \circ \omega \circ \omega^{-1} \circ \bar{\mu}^{-1} \circ \kappa_{g^{-1}}^{(\mu\circ\omega)(T),{}^g(\mu\circ\omega)(T)}, {}^g(\mu \circ \omega)(T)) \\ = & ((\kappa_h^H \circ \lambda \circ \omega)(T), \kappa_h^{{}^h(\lambda\circ\omega)(T),(\lambda\circ\omega)(T)} \circ \bar{\lambda} \circ \omega \circ \omega^{-1} \circ \bar{\mu}^{-1} \circ \kappa_{g^{-1}}^{(\mu\circ\omega)(T),{}^g(\mu\circ\omega)(T)}, (\kappa_g^G \circ \mu \circ \omega)(T)) \\ = & f((h \cdot \lambda \cdot \omega, (g \cdot \mu \cdot \omega)^{\mathrm{op}})) \\ = & f((h \cdot \lambda \cdot \omega, \omega^{-1} \cdot \mu^{\mathrm{op}} \cdot g^{-1})) \ . \end{array}$$

This proves the *claim.*

In particular, $f((\lambda, \mu^{\mathrm{op}})) = f((\lambda \cdot \omega, \omega^{-1} \cdot \mu^{\mathrm{op}}))$. So, the map

$$\begin{array}{rcl} \hat{f} : \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}} & \to & I^{\Delta}_{H,G,T} \\ \lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} & \mapsto & \hat{f}(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}}) := f((\lambda, \mu^{\mathrm{op}})) = (\lambda(T), \bar{\lambda} \circ \bar{\mu}^{-1}, \mu(T)) \end{array}$$

is well-defined and surjective, cf. Definition 30.

As for $(h,g) \in H \times G$ we have

$$^{(h,g)}\hat{f}(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}}) = {}^{(h,g)}f((\lambda, \mu^{\mathrm{op}})) = f((h\cdot\lambda, \mu^{\mathrm{op}}\cdot g^{-1})) = \hat{f}((h\cdot\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}}\cdot g^{-1})) \ ,$$

it is an (H,G)-bimap, cf. Remark 31.

We show that $\hat{f}$ is injective.

Suppose given $\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}},\ \lambda' \underset{\mathrm{Aut}(T)}{\times} \mu'^{\mathrm{op}} \in \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}}$ such that

$$(\lambda(T), \bar{\lambda}\circ\bar{\mu}^{-1}, \mu(T)) = (\lambda'(T), \bar{\lambda}'\circ\bar{\mu}'^{-1}, \mu'(T)) \ .$$

Then we have $\lambda(T) = \lambda'(T)$. So, there exists $\omega \in \mathrm{Aut}(T)$ with $\lambda' = \lambda\circ\omega$, namely $\omega := \bar{\lambda}^{-1}\circ\bar{\lambda}'$.

We have $\mu(T) = \mu'(T)$. So, there exists $\tilde{\omega} \in \mathrm{Aut}(T)$ with $\mu' = \mu\circ\tilde{\omega}$, namely $\tilde{\omega} := \bar{\mu}^{-1}\circ\bar{\mu}'$.

Moreover,

$$\bar{\lambda}\circ\bar{\mu}^{-1} = \bar{\lambda}'\circ\bar{\mu}'^{-1} = \bar{\lambda}\circ\omega\circ\tilde{\omega}^{-1}\circ\bar{\mu}^{-1} \ ,$$

i.e. $\tilde{\omega} = \omega$. Thus, $\lambda' \underset{\mathrm{Aut}(T)}{\times} \mu'^{\mathrm{op}} = \lambda\circ\omega \underset{\mathrm{Aut}(T)}{\times} (\mu\circ\omega)^{\mathrm{op}} = \lambda\cdot\omega \underset{\mathrm{Aut}(T)}{\times} \omega^{-1}\cdot\mu^{\mathrm{op}} = \lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}}$.

So, $\hat{f}$ is injective.

Altogether, $\hat{f}$ is bijective. □

Lemma 106. *Let $T \in \mathcal{T}$. Suppose given $\lambda \in \mathrm{Inj}(T,H)$ and $\eta \in \mathrm{Inj}(T,P)$. We have a bijection*

$$\begin{array}{rcl} \mathrm{Inj}(T,G) & \overset{\sim}{\longleftrightarrow} & \Gamma_G(\lambda(T), \bar{\lambda}\circ\bar{\eta}^{-1}, \eta(T)) \\ \mu & \overset{\varphi}{\mapsto} & (\bar{\lambda}\circ\bar{\mu}^{-1}, \mu(T), \bar{\mu}\circ\bar{\eta}^{-1}) \\ \iota_V\circ\alpha^{-1}\circ\bar{\lambda} & \overset{\varphi'}{\hookleftarrow} & (\alpha, V, \beta) \ . \end{array}$$

Cf. Definition 79, Definition 99.

Proof. Suppose given $\mu \in \mathrm{Inj}(T,G)$. Then

$$(\varphi'\circ\varphi)(\mu) = \varphi'((\bar{\lambda}\circ\bar{\mu}^{-1}, \mu(T), \bar{\mu}\circ\bar{\eta}^{-1})) = \iota_{\mu(T)}\circ(\bar{\lambda}\circ\bar{\mu}^{-1})^{-1}\circ\bar{\lambda} = \iota_{\mu(T)}\circ\bar{\mu} = \mu \ .$$

Suppose given $(\alpha, V, \beta) \in \Gamma_G(\lambda(T), \lambda\circ\bar{\eta}^{-1}, \eta(T))$. Note that $\alpha\circ\beta = \bar{\lambda}\circ\bar{\eta}^{-1}$. Then

$$(\varphi\circ\varphi')((\alpha,V,\beta)) = \varphi(\iota_V\circ\alpha^{-1}\circ\bar{\lambda}) = (\bar{\lambda}\circ(\alpha^{-1}\circ\bar{\lambda})^{-1}, (\alpha^{-1}\circ\bar{\lambda})(T), \alpha^{-1}\circ\bar{\lambda}\circ\bar{\eta}^{-1}) = (\alpha, V, \beta) \ ,$$

cf. Definition 79.

So, φ and φ' are mutually inverse. □

Lemma 107. *Concerning $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$, cf. Lemma 98. Concerning $\overline{\mathrm{Inj}}(T,G)$, cf. Definition 101. Recall that $\mathbf{Z}\overline{\mathrm{Inj}}(T,G)$ and $\mathbf{Z}\overline{\mathrm{Inj}}(T,H)$ are right $\mathbf{Z}\,\mathrm{Out}(T)$-modules, cf. Remark 102.*

(1) *For $T \in \mathcal{T}$ we have the $\mathbf{Z}$-linear map*

$$\tau^{\Delta}_{H,G,T} : \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G) \;\to\; \mathrm{Hom}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G), \mathbf{Z}\overline{\mathrm{Inj}}(T,H))\ ,$$

defined by

$$(\tau^{\Delta}_{H,G,T}(a))([\mu]) := \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}[\lambda]$$

for $a = \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} a_{(U,\alpha,V)}(U,\alpha,V) \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$ and $[\mu] \in \overline{\mathrm{Inj}}(T,G)$, cf. Lemma 105. Then $\tau^{\Delta}_{H,G,T}$ is an isomorphism of abelian groups.

We obtain the isomorphism of abelian groups

$$\tau^{\Delta}_{H,G} := \bigoplus_{T\in\mathcal{T}} \tau^{\Delta}_{H,G,T} : \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G) = \bigoplus_{T\in\mathcal{T}} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G) \xrightarrow{\sim} \bigoplus_{T\in\mathcal{T}} \mathrm{Hom}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G), \mathbf{Z}\overline{\mathrm{Inj}}(T,H))\ .$$

Note that $\tau^{\Delta}_{H,G,T} = \pi_T \circ \tau^{\Delta}_{H,G}$.

(2) *Suppose given $T \in \mathcal{T}$. Suppose given*

$$a = \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} a_{(U,\alpha,V)}(U,\alpha,V) \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$$

and

$$b = \sum_{(V',\beta,W)\in I^{\Delta}_{G,P,T}} b_{(V',\beta,W)}(V',\beta,W) \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,P)\ .$$

We have

$$\tau^{\Delta}_{H,P,T}(a \underset{G}{\cdot} b) \;=\; \tau^{\Delta}_{H,G,T}(a) \circ \tau^{\Delta}_{G,P,T}(b)\ .$$

In particular, we have

$$\tau^{\Delta}_{H,P}(\tilde{a} \underset{G}{\cdot} \tilde{b}) \;=\; \tau^{\Delta}_{H,G}(\tilde{a}) \circ \tau^{\Delta}_{G,P}(\tilde{b})$$

for $\tilde{a} \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G)$ and $\tilde{b} \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,P)$, where in the right hand side composition is to be read entrywise.

(3) *Recall that $\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)} = \sum_{S\in\mathcal{L}_G} [S,\mathrm{id}_S,S]^{+}_{G\times G} \in \tilde{\mathrm{B}}_{\mathbf{Z}}(G,G)$, cf. Lemma 93(4).*

Let $\tilde{\mathrm{id}}^{\Delta}_{G,T} := \pi_T(\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}) \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(G,G)$ for $T \in \mathcal{T}$.

We have

$$(\tau^{\Delta}_{G,G,T}(\tilde{\mathrm{id}}^{\Delta}_{G,T}))([\mu]) = [\mu]$$

for $[\mu] \in \overline{\mathrm{Inj}}(T,G)$.

So,

$$\tau^{\Delta}_{G,G}(\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}) = (\mathrm{id}_{\mathbf{Z}\overline{\mathrm{Inj}}(T,G)})_{T\in\mathcal{T}}\ .$$

Proof. *Ad* (1). We show that $\tau^{\Delta}_{H,G,T}$ is well-defined.

We want to show that the element claimed to be $(\tau^{\Delta}_{H,G,T}(a))([\mu])$ is independent of the choice of the representative $\lambda \in \mathrm{Inj}(T,H)$ in each summand and of the choice of the representative $\mu \in \mathrm{Inj}(T,G)$.

Suppose given $[\mu], [\mu'] \in \overline{\mathrm{Inj}}(T,G)$ and $[\lambda], [\lambda'] \in \overline{\mathrm{Inj}}(T,H)$ with $[\mu] = [\mu']$ and $[\lambda] = [\lambda']$. Then there exist $g \in G$ and $h \in H$ such that

$$\mu' = \kappa_g^G \circ \mu \text{ and } \lambda' = \kappa_h^H \circ \lambda \ .$$

Note that $\bar{\mu}' = \kappa_g^{{}^g\mu(T),\mu(T)} \circ \bar{\mu}$ and $\bar{\lambda}' = \kappa_h^{{}^h\lambda(T),\lambda(T)} \circ \bar{\lambda}$.

It suffices to show that

$$a_{(\lambda'(T),\bar{\lambda}'\circ\bar{\mu}'^{-1},\mu'(T))} \stackrel{!}{=} a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))} \ .$$

As $a = \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} a_{(U,\alpha,V)}(U,\alpha,V)$ is an (H,G)-fixed point in $A^{\Delta}_{\mathbf{Z},T}(H,G)$, cf. Definition 97(2), we have for $(h_0,g_0) \in H \times G$ that

$$a_{(U,\alpha,V)} = a_{({}^{h_0}U,\kappa_{h_0}^{{}^{h_0}U,U}\circ\alpha\circ\kappa_{g_0^{-1}}^{V,{}^{g_0}V},{}^{g_0}V)} \ .$$

So,

$$\begin{aligned}
a_{(\lambda'(T),\bar{\lambda}'\circ\bar{\mu}'^{-1},\mu'(T))} &= a_{((\kappa_h^H\circ\lambda)(T),\kappa_h^{{}^h\lambda(T),\lambda(T)}\circ\bar{\lambda}\circ(\kappa_g^{{}^g\mu(T),\mu(T)}\circ\bar{\mu})^{-1},(\kappa_g^G\circ\mu)(T))} \\
&= a_{({}^h\lambda(T),\kappa_h^{{}^h\lambda(T),\lambda(T)}\circ\bar{\lambda}\circ\bar{\mu}^{-1}\circ\kappa_{g^{-1}}^{\mu(T),{}^g\mu(T)},{}^g\mu(T))} \\
&= a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))} \ .
\end{aligned}$$

Now we show $\mathbf{Z}\,\mathrm{Out}(T)$-linearity of the map claimed to be $\tau^{\Delta}_{H,G,T}(a)$.

Suppose given $\omega \in \mathrm{Aut}(T)$ and $[\mu] \in \overline{\mathrm{Inj}}(T,G)$. Note that

$$\overline{(\mu\circ\omega)}^{\,-1} = (\bar{\mu}\circ\omega)^{-1} = \omega^{-1}\circ\bar{\mu}^{-1}$$

and $\omega(T) = T$. So, we need to show that

$$\sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} a_{(\lambda(T),\bar{\lambda}\circ\omega^{-1}\circ\bar{\mu}^{-1},\mu(T))}[\lambda] \stackrel{!}{=} \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}[\lambda\circ\omega] \ .$$

Using the bijection $\overline{\mathrm{Inj}}(T,H) \xrightarrow{\sim} \overline{\mathrm{Inj}}(T,H)$, $[\lambda] \mapsto [\lambda\circ\omega]$, cf. Lemma 104, and substituting $\lambda' = \lambda\circ\omega$, we obtain

$$\begin{aligned}
\sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}[\lambda\circ\omega] &= \sum_{[\lambda']\in\overline{\mathrm{Inj}}(T,H)} a_{((\lambda'\circ\omega^{-1})(T),\bar{\lambda}'\circ\omega^{-1}\circ\bar{\mu}^{-1},\mu(T))}[\lambda'] \\
&= \sum_{[\lambda']\in\overline{\mathrm{Inj}}(T,H)} a_{(\lambda'(T),\bar{\lambda}'\circ\omega^{-1}\circ\bar{\mu}^{-1},\mu(T))}[\lambda'] \ .
\end{aligned}$$

So, the map claimed to be $\tau^{\Delta}_{H,G,T}(a)$ is $\mathbf{Z}\,\mathrm{Out}(T)$-linear.

Hence, the map $\tau^{\Delta}_{H,G,T}$ is a well-defined $\mathbf{Z}$-linear map.

We show that $\tau^{\Delta}_{H,G,T}$ is an isomorphism of abelian groups.

Define the map

$$\tau'^{\Delta}_{H,G,T} : \mathrm{Hom}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G), \mathbf{Z}\overline{\mathrm{Inj}}(T,H)) \;\to\; \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)\ ,$$

as follows.

Let $\psi \in \mathrm{Hom}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G), \mathbf{Z}\overline{\mathrm{Inj}}(T,H))$ be represented by the integral matrix $(b_{[\lambda],[\mu]})_{\substack{[\lambda]\in\overline{\mathrm{Inj}}(T,H),\\ [\mu]\in\overline{\mathrm{Inj}}(T,G)}}$ with respect to the $\mathbf{Z}$-linear bases $\overline{\mathrm{Inj}}(T,H)$ and $\overline{\mathrm{Inj}}(T,G)$.

So,

$$\psi([\mu]) = \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} b_{[\lambda],[\mu]}[\lambda]$$

for $[\mu] \in \overline{\mathrm{Inj}}(T,G)$.

We set

$$\tau'^{\Delta}_{H,G,T}(\psi) := \sum_{\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} \;\in\; \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}}} b_{[\lambda],[\mu]}(\lambda(T), \bar{\lambda}\circ\bar{\mu}^{-1}, \mu(T))\ .$$

We show that $\tau'^{\Delta}_{H,G,T}$ is well-defined.

Recall that by Lemma 105 we have an isomorphism of (H,G)-bisets

$$\begin{aligned} \hat{f} : \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}} \;&\to\; I^{\Delta}_{H,G,T} \\ \lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} \;&\mapsto\; (\lambda(T), \bar{\lambda}\circ\bar{\mu}^{-1}, \mu(T))\ . \end{aligned}$$

First, we show that each summand appearing in the element claimed to be $\tau'^{\Delta}_{H,G,T}(\psi)$ is independent of the choice of $\lambda \in \mathrm{Inj}(T,H)$ and of the choice of $\mu \in \mathrm{Inj}(T,G)$.

Suppose given $\lambda, \lambda' \in \mathrm{Inj}(T,H)$ and $\mu, \mu' \in \mathrm{Inj}(T,G)$ with $\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} = \lambda' \underset{\mathrm{Aut}(T)}{\times} \mu'^{\mathrm{op}}$.

By the bijection $\hat{f}$, recalled above, we have $(\lambda(T), \bar{\lambda}\circ\bar{\mu}^{-1}, \mu(T)) = (\lambda'(T), \bar{\lambda}'\circ\bar{\mu}'^{-1}, \mu'(T))$.

Therefore, it suffices to show that $b_{[\lambda],[\mu]} = b_{[\lambda'],[\mu']}$.

As $\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} = \lambda' \underset{\mathrm{Aut}(T)}{\times} \mu'^{\mathrm{op}}$, there exists $\omega \in \mathrm{Aut}(T)$ such that $\lambda' = \lambda\circ\omega$ and $\mu' = \mu\circ\omega$.

We have, by construction,

$$\psi([\mu\circ\omega]) = \sum_{[\eta]\in\overline{\mathrm{Inj}}(T,H)} b_{[\eta],[\mu\circ\omega]}[\eta]\ .$$

Since the map ψ is $\mathbf{Z}\,\mathrm{Out}(T)$-linear and since we have the bijection

$$\overline{\mathrm{Inj}}(T,H) \xrightarrow{\sim} \overline{\mathrm{Inj}}(T,H),\ \ [\eta] \mapsto [\eta\circ\omega] =: [\eta']$$

from Lemma 104, it follows that

$$\begin{aligned}
\psi([\mu\circ\omega]) &= \psi([\mu]\cdot[\omega]) \\
&= \psi([\mu])\cdot[\omega] \\
&= \sum_{[\eta]\in\overline{\mathrm{Inj}}(T,H)} b_{[\eta],[\mu]}[\eta\circ\omega] \\
&= \sum_{[\eta']\in\overline{\mathrm{Inj}}(T,H)} b_{[\eta'\circ\omega^{-1}],[\mu]}[\eta'] \ ,
\end{aligned}$$

where $[\omega]$ denotes the residue class of ω in $\mathrm{Out}(T)$.

Comparing the coefficients at $[\eta]$ leads to $b_{[\eta],[\mu\circ\omega]} = b_{[\eta\circ\omega^{-1}],[\mu]}$. In particular,

$$b_{[\lambda'],[\mu']} = b_{[\lambda\circ\omega],[\mu\circ\omega]} = b_{[\lambda\circ\omega\circ\omega^{-1}],[\mu]} = b_{[\lambda],[\mu]} \ .$$

Now, we show that the element claimed to be ${\tau'}^{\Delta}_{H,G,T}(\psi)$ lies in $\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$.

Therefore, we need to show that $\sum\limits_{\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} \,\in\, \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}}} b_{[\lambda],[\mu]}(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))$ is an (H,G)-fixed point in $A^{\Delta}_{\mathbf{Z},T}(H,G)$, cf. Definition 97(2).

By Lemma 9, applied to the bijection $\hat{f}$ recalled above, it follows that

$$\sum_{\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} \,\in\, \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}}} b_{[\lambda],[\mu]}(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))$$

is an (H,G)-fixed point if and only if for $(h,g)\in H\times G$ the coefficient $b_{[\lambda],[\mu]}$ at $\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}}$ equals the coefficient at $(h,g)\cdot(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}})$.

But,

$$\begin{aligned}
(h,g)\cdot(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}}) &= h\cdot(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}})\cdot g^{-1} \\
&= h\cdot\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}}\cdot g^{-1} \\
&= h\cdot\lambda \underset{\mathrm{Aut}(T)}{\times} (g\cdot\mu)^{\mathrm{op}} \\
&= \kappa^H_h\circ\lambda \underset{\mathrm{Aut}(T)}{\times} (\kappa^G_g\circ\mu)^{\mathrm{op}}
\end{aligned}$$

has the coefficient $b_{[\kappa^H_h\circ\lambda],[\kappa^G_g\circ\mu]} = b_{[\lambda],[\mu]}$.

Hence, the map ${\tau'}^{\Delta}_{H,G,T}$ is well-defined.

We now show that $\tau^{\Delta}_{H,G,T}$ and ${\tau'}^{\Delta}_{H,G,T}$ are mutually inverse.

Suppose given $a = \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} a_{(U,\alpha,V)}(U,\alpha,V) \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$. Then

$$\begin{aligned}
(\tau'^{\Delta}_{H,G,T} \circ \tau^{\Delta}_{H,G,T})(a) &= \tau'^{\Delta}_{H,G,T}(\ [\mu] \mapsto \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}[\lambda]\ ) \\
&= \sum_{\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} \,\in\, \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}}} a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}(\lambda(T),\lambda\circ\bar{\mu}^{-1},\mu(T)) \\
&\overset{\mathrm{L.105}}{=} \sum_{\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} \,\in\, \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}}} a_{\hat{f}(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}})}\hat{f}(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}}) \\
&= \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} a_{(U,\alpha,V)}(U,\alpha,V) = a\ .
\end{aligned}$$

Suppose given $\psi \in \mathrm{Hom}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G),\mathbf{Z}\overline{\mathrm{Inj}}(T,H))$ represented by $(b_{[\lambda],[\mu]})_{\substack{[\lambda]\in\overline{\mathrm{Inj}}(T,H),\\ [\mu]\in\overline{\mathrm{Inj}}(T,G)}}$ with respect to the $\mathbf{Z}$-linear bases $\overline{\mathrm{Inj}}(T,H)$ and $\overline{\mathrm{Inj}}(T,G)$.

Note that for $a = \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} a_{(U,\alpha,V)}(U,\alpha,V) \in \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G)$ we have

$$\tau^{\Delta}_{H,G,T}(a) : [\mu] \mapsto \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} a_{\hat{f}(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}})}[\lambda]\ ,$$

cf. Lemma 105.

Now,

$$\begin{aligned}
(\tau^{\Delta}_{H,G,T} \circ \tau'^{\Delta}_{H,G,T})(\psi) &= \tau^{\Delta}_{H,G,T}(\sum_{\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} \,\in\, \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}}} b_{[\lambda],[\mu]}(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))\ ) \\
&= \tau^{\Delta}_{H,G,T}(\sum_{\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}} \,\in\, \mathrm{Inj}(T,H) \underset{\mathrm{Aut}(T)}{\times} (\mathrm{Inj}(T,G))^{\mathrm{op}}} b_{[\lambda],[\mu]}\hat{f}(\lambda \underset{\mathrm{Aut}(T)}{\times} \mu^{\mathrm{op}})) \\
&= (\ [\mu] \mapsto \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} b_{[\lambda],[\mu]}[\lambda]\ ) = \psi\ .
\end{aligned}$$

So, $\tau^{\Delta}_{H,G,T}$ and $\tau'^{\Delta}_{H,G,T}$ are mutually inverse.

Ad (2). We have

$$\begin{aligned}
a \underset{G}{\cdot} b &= \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} \sum_{(V',\beta,W)\in I^{\Delta}_{G,P,T}} a_{(U,\alpha,V)}b_{(V',\beta,W)}(U,\alpha,V) \underset{G}{\cdot} (V',\beta,W) \\
&\overset{\mathrm{D.87}}{=} \sum_{\substack{(U,\alpha,V)\in I^{\Delta}_{H,G,T} \\ (V',\beta,W)\in I^{\Delta}_{G,P,T} \\ V=V'}} a_{(U,\alpha,V)}b_{(V,\beta,W)}\frac{|\,\mathrm{C}_G(V)|}{|G|}(U,\alpha\circ\beta,W) \\
&\overset{\mathrm{D.79}}{=} \sum_{(U,\gamma,W)\in I^{\Delta}_{H,P,T}} \left(\sum_{(\alpha,V,\beta)\in\Gamma_G(U,\gamma,W)} a_{(U,\alpha,V)}b_{(V,\beta,W)}\frac{|\,\mathrm{C}_G(V)|}{|G|}\right)(U,\gamma,W) \\
&=: \sum_{(U,\gamma,W)\in I^{\Delta}_{H,P,T}} c_{(U,\gamma,W)}(U,\gamma,W)\ .
\end{aligned}$$

Suppose given $[\mu] \in \overline{\mathrm{Inj}}(T,P)$.

We have

$$(\tau^{\Delta}_{H,P,T}(a \underset{G}{\cdot} b))([\mu]) \overset{(1)}{=} \sum_{[\eta]\in\overline{\mathrm{Inj}}(T,H)} c_{(\eta(T),\bar\eta\circ\bar\mu^{-1},\mu(T))}[\eta] \ .$$

Using the bijection

$$\begin{array}{rcl} \mathrm{Inj}(T,G) & \overset{\sim}{\longleftrightarrow} & \Gamma_G(\eta(T),\bar\eta\circ\bar\mu^{-1},\mu(T)) \\ \lambda & \overset{\varphi}{\mapsto} & (\bar\eta\circ\bar\lambda^{-1},\lambda(T),\bar\lambda\circ\bar\mu^{-1}) \\ \iota_V\circ\alpha^{-1}\circ\bar\eta & \overset{\varphi'}{\leftarrow\!\shortmid} & (\alpha,V,\beta)\ , \end{array}$$

cf. Lemma 106, we obtain for the coefficient of $(\tau^{\Delta}_{H,P,T}(a \underset{G}{\cdot} b))([\mu])$ at $[\eta]$

$$\begin{aligned} c_{(\eta(T),\bar\eta\circ\bar\mu^{-1},\mu(T))} &= \sum_{(\alpha,V,\beta)\in\Gamma_G(\eta(T),\bar\eta\circ\bar\mu^{-1},\mu(T))} a_{(\eta(T),\alpha,V)}b_{(V,\beta,\mu(T))}\frac{|\,\mathrm{C}_G(V)|}{|G|} \\ &= \sum_{\lambda\in\mathrm{Inj}(T,G)} a_{(\eta(T),\bar\eta\circ\bar\lambda^{-1},\lambda(T))}b_{(\lambda(T),\bar\lambda\circ\bar\mu^{-1},\mu(T))}\frac{|\,\mathrm{C}_G(\lambda(T))|}{|G|} \ . \end{aligned}$$

Moreover, we have

$$\begin{aligned} (\tau^{\Delta}_{H,G,T}(a)\circ\tau^{\Delta}_{G,P,T}(b))([\mu]) &\overset{(1)}{=} (\tau^{\Delta}_{H,G,T}(a))\Big(\sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} b_{(\lambda(T),\bar\lambda\circ\bar\mu^{-1},\mu(T))}[\lambda]\Big) \\ &\overset{(1)}{=} \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} b_{(\lambda(T),\bar\lambda\circ\bar\mu^{-1},\mu(T))} \sum_{[\eta]\in\overline{\mathrm{Inj}}(T,H)} a_{(\eta(T),\bar\eta\circ\bar\lambda^{-1},\lambda(T))}[\eta] \\ &= \sum_{[\eta]\in\overline{\mathrm{Inj}}(T,H)}\left(\sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} a_{(\eta(T),\bar\eta\circ\bar\lambda^{-1},\lambda(T))}b_{(\lambda(T),\bar\lambda\circ\bar\mu^{-1},\mu(T))}\right)[\eta] \ . \end{aligned}$$

The coefficient of $(\tau^{\Delta}_{H,G,T}(a)\circ\tau^{\Delta}_{G,P,T}(b))([\mu])$ at $[\eta]$ is given by

$$\begin{aligned} \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} a_{(\eta(T),\bar\eta\circ\bar\lambda^{-1},\lambda(T))}b_{(\lambda(T),\bar\lambda\circ\bar\mu^{-1},\mu(T))} &\overset{\text{L.103}}{=} \sum_{\lambda\in\mathrm{Inj}(T,G)} a_{(\eta(T),\bar\eta\circ\bar\lambda^{-1},\lambda(T))}b_{(\lambda(T),\bar\lambda\circ\bar\mu^{-1},\mu(T))}\frac{|\,\mathrm{C}_G(\lambda(T))|}{|G|} \\ &\overset{\text{see above}}{=} c_{(\eta(T),\bar\eta\circ\bar\mu^{-1},\mu(T))} \ . \end{aligned}$$

Thus, the coefficients at $[\eta]$ are equal and we have $\tau^{\Delta}_{H,P,T}(a \underset{G}{\cdot} b) = \tau^{\Delta}_{H,G,T}(a)\circ\tau^{\Delta}_{G,P,T}(b)$.

So, for $\tilde a\in\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G)$ and $\tilde b\in\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,P)$ we have

$$\begin{aligned} \tau^{\Delta}_{H,P}(\tilde a \underset{G}{\cdot} \tilde b) &= (\tau_{H,P,T}(\pi_T(\tilde a \underset{G}{\cdot} \tilde b)))_{T\in\mathcal{T}} \\ &\overset{\text{L.98(2)}}{=} (\tau^{\Delta}_{H,P,T}(\pi_T(\tilde a) \underset{G}{\cdot} \pi_T(\tilde b)))_{T\in\mathcal{T}} \\ &= (\tau^{\Delta}_{H,G,T}(\pi_T(\tilde a))\circ\tau^{\Delta}_{G,P,T}(\pi_T(\tilde b)))_{T\in\mathcal{T}} \\ &= (\pi_T(\tau^{\Delta}_{H,G}(\tilde a))\circ\pi_T(\tau^{\Delta}_{G,P}(\tilde b)))_{T\in\mathcal{T}} \\ &= \tau^{\Delta}_{H,G}(\tilde a)\circ\tau^{\Delta}_{G,P}(\tilde b)\ , \end{aligned}$$

as the multiplication in the codomain is given by componentwise composition.

Ad (3). Suppose given $[\mu] \in \overline{\mathrm{Inj}}(T, G)$. We have

$$\begin{aligned}
\tilde{\mathrm{id}}^{\Delta}_{G,T} &= \pi_T(\sum_{S\in\mathcal{L}_G} [S, \mathrm{id}_S, S]^+_{G\times G}) \\
&= \pi_T(\sum_{S\in\mathcal{L}_G} \sum_{(U,\alpha,V)\in[S,\mathrm{id}_S,S]_{G\times G}} (U,\alpha,V)) \\
&= \sum_{\substack{S\in\mathcal{L}_G \\ S\cong T}} \sum_{(U,\alpha,V)\in[S,\mathrm{id}_S,S]_{G\times G}} (U,\alpha,V)\ .
\end{aligned}$$

Write $\sum_{\substack{S\in\mathcal{L}_G \\ S\cong T}} \sum_{(U,\alpha,V)\in[S,\mathrm{id}_S,S]_{G\times G}} (U,\alpha,V) =: \sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} a_{(U,\alpha,V)}(U,\alpha,V)\ .$

The coefficient of $\tilde{\mathrm{id}}^{\Delta}_{G,T}$ at (U,α,V) equals 1 if there exists $(g,g') \in G\times G$ and $S \in \mathcal{L}_G$ with $S \cong T$ such that $(U,\alpha,V) = {}^{(g,g')}(S,\mathrm{id}_S,S) = ({}^gS, \kappa^{{}^gS,{}^{g'}S}_{gg'^{-1}}, {}^{g'}S)$ i.e. ${}^gS = U$, ${}^{g'}S = V$ and $\alpha = \kappa^{U,V}_{gg'^{-1}}$.

It equals 0 otherwise.

We have

$$(\tau^{\Delta}_{G,G,T}(\tilde{\mathrm{id}}^{\Delta}_{G,T}))([\mu]) = \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}[\lambda]\ .$$

Now $a_{(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}$ equals 1 if and only if $(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T)) = ({}^gS, \kappa^{{}^gS,{}^{g'}S}_{gg'^{-1}}, {}^{g'}S)$ for some $(g,g') \in G\times G$ and $S \in \mathcal{L}_G$ with $S \cong T$.

If this is the case, then $\bar{\lambda}\circ\bar{\mu}^{-1} = \kappa^{{}^gS,{}^{g'}S}_{gg'^{-1}}$, hence $\lambda = \kappa^G_{gg'^{-1}} \circ \mu$ and therefore λ and μ lie in the same G-orbit and so $[\lambda] = [\mu]$.

Conversely, if $[\lambda] = [\mu]$ we have $a_{(\mu(T),\bar{\mu}\circ\bar{\mu}^{-1},\mu(T))} = a_{(\mu(T),\mathrm{id}_{\mu(T)},\mu(T))} = 1$.

Thus,

$$(\tau^{\Delta}_{G,G,T}(\tilde{\mathrm{id}}^{\Delta}_{G,T}))([\mu]) = [\mu]\ .$$

So, we obtain

$$\begin{aligned}
\tau^{\Delta}_{G,G}(\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}) &= (\tau^{\Delta}_{G,G,T}(\pi_T(\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)})))_{T\in\mathcal{T}} \\
&= (\tau^{\Delta}_{G,G,T}(\tilde{\mathrm{id}}^{\Delta}_{G,T}))_{T\in\mathcal{T}} \\
&= (\mathrm{id}_{\mathbf{Z}\overline{\mathrm{Inj}}(T,G)})_{T\in\mathcal{T}}\ .
\end{aligned}$$

□

2.3.5 The Theorem of Boltje and Danz

Theorem 108. ([1, Theorem 5.5]) *Recall that H, G and P are finite groups. Concerning τ, cf. Lemma 107. Concerning $\mathfrak{m}$, cf. Lemma 94.*

Let

$$\sigma^{\Delta}_{H,G} := \tau^{\Delta}_{H,G} \circ \mathfrak{m}^{\Delta}_{H,G}\ .$$

The following diagram commutes.

$$\begin{tikzcd}[column sep=small] & \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) \arrow[dl, "\mathfrak{m}^{\Delta}_{H,G}"'] \arrow[dr, "\sigma^{\Delta}_{H,G}"] & \\ \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G) = \bigoplus\limits_{T\in\mathcal{T}} \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z},T}(H,G) \arrow[rr, "\tau^{\Delta}_{H,G} = \bigoplus_{T\in\mathcal{T}} \tau^{\Delta}_{H,G,T}"] & & \bigoplus\limits_{T\in\mathcal{T}} \mathrm{Hom}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G), \mathbf{Z}\overline{\mathrm{Inj}}(T,H)) \end{tikzcd}$$

(1) *We have*

$$\begin{array}{rcl} \sigma^{\Delta}_{H,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) & \to & \bigoplus\limits_{T\in\mathcal{T}} \mathrm{Hom}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G), \mathbf{Z}\overline{\mathrm{Inj}}(T,H)) \\ {[X]} & \mapsto & \left([\mu] \mapsto \sum\limits_{[\lambda]\in\overline{\mathrm{Inj}}(T,H)} \dfrac{|\,\mathrm{Fix}_{\Delta(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}(X)|}{|\,\mathrm{C}_H(\lambda(T))|} [\lambda] \right)_{T\in\mathcal{T}} \end{array} .$$

(2) *For $a \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$ and $b \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,P)$ we have*

$$\sigma^{\Delta}_{H,P}(a \underset{G}{\cdot} b) \;=\; \sigma^{\Delta}_{H,G}(a) \circ \sigma^{\Delta}_{G,P}(b) \ ,$$

where in the right hand side composition is to be read entrywise.

(3) *The $\mathbf{Z}$-linear map $\sigma^{\Delta}_{H,G}$ is an injective group morphism with finite cokernel of order* $\prod\limits_{\Delta(U,\alpha,V)\in\mathcal{L}^{\Delta}_{H\times G}} \dfrac{[\mathrm{N}_{H\times G}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\,\mathrm{C}_H(U)|}$, *which divides $|H\times G|^{|\mathcal{L}^{\Delta}_{H\times G}|}$.*

(4) *We obtain a $\mathbf{Q}$-module isomorphism*

$$\mathbf{Q}\otimes\sigma^{\Delta}_{H,G} : \mathrm{B}^{\Delta}_{\mathbf{Q}}(H,G) \to \bigoplus_{T\in\mathcal{T}} \mathrm{Hom}_{\mathbf{Q}\,\mathrm{Out}(T)}(\mathbf{Q}\overline{\mathrm{Inj}}(T,G), \mathbf{Q}\overline{\mathrm{Inj}}(T,H)) \ .$$

(5) *The map*

$$\begin{array}{rcl} \sigma^{\Delta}_{G,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G) & \to & \prod\limits_{T\in\mathcal{T}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) \\ {[X]} & \mapsto & \left([\mu] \mapsto \sum\limits_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} \dfrac{|\,\mathrm{Fix}_{\Delta(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}(X)|}{|\,\mathrm{C}_G(\lambda(T))|} [\lambda] \right)_{T\in\mathcal{T}} \end{array}$$

is an injective ring morphism with image of finite index of order $\prod\limits_{\Delta(U,\alpha,V)\in\mathcal{L}^{\Delta}_{G\times G}} \dfrac{[\mathrm{N}_{G\times G}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\,\mathrm{C}_G(U)|}$, *which divides $|G\times G|^{|\mathcal{L}^{\Delta}_{G\times G}|}$.*
The multiplication in the domain is given by ($\underset{G}{\cdot}$) and the multiplication in the codomain is given by componentwise composition ($\circ$).

(6) *We obtain a $\mathbf{Q}$-algebra isomorphism*

$$\mathbf{Q}\otimes\sigma^{\Delta}_{G,G} : \mathrm{B}^{\Delta}_{\mathbf{Q}}(G,G) \to \prod_{T\in\mathcal{T}} \mathrm{End}_{\mathbf{Q}\,\mathrm{Out}(T)}(\mathbf{Q}\overline{\mathrm{Inj}}(T,G)) \ .$$

Proof. *Ad* (1). We have for $[X] \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$ that

$$\begin{array}{rcl} \sigma^{\Delta}_{H,G}([X]) & = & (\tau^{\Delta}_{H,G} \circ \mathfrak{m}^{\Delta}_{H,G})([X]) \\ & \overset{\text{L.98(3), L.107(1)}}{=} & (\bigoplus_{T\in\mathcal{T}} \tau^{\Delta}_{H,G,T}) \left(\left(\sum_{(U,\alpha,V)\in I^{\Delta}_{H,G,T}} \frac{|\operatorname{Fix}_{\Delta(U,\alpha,V)}(X)|}{|\operatorname{C}_H(U)|}(U,\alpha,V) \right)_{T\in\mathcal{T}} \right) \\ & \overset{\text{L.107(1)}}{=} & \left([\mu] \mapsto \sum_{[\lambda]\in\overline{\operatorname{Inj}}(T,H)} \frac{|\operatorname{Fix}_{\Delta(\lambda(T),\bar{\lambda}\circ\bar{\mu}^{-1},\mu(T))}(X)|}{|\operatorname{C}_H(\lambda(T))|}[\lambda] \right)_{T\in\mathcal{T}} . \end{array}$$

Ad (2). We have for $a \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G)$ and $b \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,P)$

$$\begin{array}{rcl} \sigma^{\Delta}_{H,P}(a \underset{G}{\cdot} b) & = & (\tau^{\Delta}_{H,P} \circ \mathfrak{m}^{\Delta}_{H,P})(a \underset{G}{\cdot} b) \\ & \overset{\text{L.94(1)}}{=} & \tau^{\Delta}_{H,P}(\mathfrak{m}^{\Delta}_{H,G}(a) \underset{G}{\cdot} \mathfrak{m}^{\Delta}_{G,P}(b)) \\ & \overset{\text{L.107(2)}}{=} & \tau^{\Delta}_{H,G}(\mathfrak{m}^{\Delta}_{H,G}(a)) \circ \tau^{\Delta}_{G,P}(\mathfrak{m}^{\Delta}_{G,P}(b)) \\ & = & (\tau^{\Delta}_{H,G} \circ \mathfrak{m}^{\Delta}_{H,G})(a) \circ (\tau^{\Delta}_{G,P} \circ \mathfrak{m}^{\Delta}_{G,P})(b) \\ & = & \sigma^{\Delta}_{H,G}(a) \circ \sigma^{\Delta}_{G,P}(b) . \end{array}$$

Ad (3). As the map $\tau^{\Delta}_{H,G}$ is an isomorphism, cf. Lemma 107(1), and as

$$\tau^{\Delta}_{H,G} \circ \mathfrak{m}^{\Delta}_{H,G} = \sigma^{\Delta}_{H,G} ,$$

the assertion follows as $\mathfrak{m}^{\Delta}_{H,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(H,G) \to \tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(H,G)$ is an injective morphism of abelian groups with finite cokernel of order $\prod_{\Delta(U,\alpha,V)\in\mathcal{L}^{\Delta}_{H\times G}} \frac{[\mathrm{N}_{H\times G}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\operatorname{C}_H(U)|}$, which divides $|H \times G|^{|\mathcal{L}^{\Delta}_{H\times G}|}$, cf. Lemma 94(2).

Ad (4). This follows by (3).

Ad (5). By (3) the map $\sigma^{\Delta}_{G,G}$ is a $\mathbf{Z}$-linear map.

By (2) we have for $a \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ and $b \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ that

$$\sigma^{\Delta}_{G,G}(a \underset{G}{\cdot} b) = \sigma^{\Delta}_{G,G}(a) \circ \sigma^{\Delta}_{G,G}(b) .$$

Moreover, we have

$$\begin{array}{rcl} \sigma^{\Delta}_{G,G}(\mathrm{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)}) & = & (\tau^{\Delta}_{G,G} \circ \mathfrak{m}^{\Delta}_{G,G})(\mathrm{id}_{\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)}) \\ & \overset{\text{L.94(1)}}{=} & \tau^{\Delta}_{G,G}(\mathrm{id}_{\tilde{\mathrm{B}}^{\Delta}_{\mathbf{Z}}(G,G)}) \\ & \overset{\text{L.107(3)}}{=} & (\mathrm{id}_{\mathbf{Z}\overline{\operatorname{Inj}}(T,G)})_{T\in\mathcal{T}} . \end{array}$$

So, $\sigma^{\Delta}_{G,G}$ is a ring morphism.

Ad (6). This follows by (5). $\square$

2.4 The bifree double Burnside ring $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$

Example 109. By Example 75, we know that $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3) \xrightarrow{\delta} \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ is a ring isomorphism. So the ring $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ is well-known.

For sake of illustration of Theorem 108, we nonetheless aim to determine the image of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ under $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3}$.

Recall from Example 75 the subgroups

$$U_0 = 1,\ U_1 = \langle (1,2) \rangle,\ U_2 = \langle (1,3) \rangle,\ U_3 = \langle (2,3) \rangle,\ U_4 = \langle (1,2,3) \rangle \text{ and } \mathrm{S}_3 \text{ of } \mathrm{S}_3$$

as well as the $\mathbf{Z}$-linear basis

$$([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(1)],\ [(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(U_1)],\ [(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(U_4)],\ [(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3)])$$

of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$.

Write $\mathrm{C}_2 := \langle c_2 : c_2^2 \rangle$ and $\mathrm{C}_3 := \langle c_3 : c_3^3 \rangle$.

Note that in this particular case, we may replace $\mathcal{T}$ by

$$\mathcal{T}_{\mathrm{S}_3} := \{1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{S}_3\}$$

in Theorem 108, cf. Notation 96, Definition 99, Definition 101.

Note that $\mathrm{Out}(T)$ is trivial for $T \in \mathcal{T}_{\mathrm{S}_3} \setminus \mathrm{C}_3$ and that $\mathrm{Out}(\mathrm{C}_3) \cong \mathrm{C}_2$.

We have

$$\begin{array}{lclclcl}
\mathrm{Inj}(1, \mathrm{S}_3) & = & \{1 \mapsto \mathrm{id}\} & \rightsquigarrow & \overline{\mathrm{Inj}}(1, \mathrm{S}_3) & = & \{[1 \mapsto \mathrm{id}]\} \\
\mathrm{Inj}(\mathrm{C}_2, \mathrm{S}_3) & = & \{c_2 \mapsto (1,2),\ c_2 \mapsto (1,3),\ c_2 \mapsto (2,3)\} & \rightsquigarrow & \overline{\mathrm{Inj}}(\mathrm{C}_2, \mathrm{S}_3) & = & \{[c_2 \mapsto (1,2)]\} \\
\mathrm{Inj}(\mathrm{C}_3, \mathrm{S}_3) & = & \{c_3 \mapsto (1,2,3),\ c_3 \mapsto (1,3,2)\} & \rightsquigarrow & \overline{\mathrm{Inj}}(\mathrm{C}_3, \mathrm{S}_3) & = & \{[c_3 \mapsto (1,2,3)]\} \\
\mathrm{Inj}(\mathrm{S}_3, \mathrm{S}_3) & = & \{\kappa_x^{\mathrm{S}_3} : x \in \mathrm{S}_3\} & \rightsquigarrow & \overline{\mathrm{Inj}}(\mathrm{S}_3, \mathrm{S}_3) & = & \{[\mathrm{id}_{\mathrm{S}_3}]\}\ .
\end{array}$$

Note that $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3, \mathrm{S}_3)) = {}_{\mathbf{Z}}\langle\, \mathrm{id}_{\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3,\mathrm{S}_3)} \rangle$ and thus

$$\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_3)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3, \mathrm{S}_3)) = \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3, \mathrm{S}_3))\ .$$

Moreover, we may identify

$$\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(1, \mathrm{S}_3)) = \mathbf{Z},\ \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2, \mathrm{S}_3)) = \mathbf{Z},\ \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3, \mathrm{S}_3)) = \mathbf{Z},\ \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_3, \mathrm{S}_3)) = \mathbf{Z}.$$

The map $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3}$ from Theorem 108 decomposes into

$$\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3} = (\pi_T \circ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3})_{T \in \mathcal{T}} =: (\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,T})_{T \in \mathcal{T}}\ .$$

Ad $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,1}$. We consider the map

$$\begin{array}{rcll} \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,1} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) & \to & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(1,\mathrm{S}_3)) & = \mathbf{Z} \\ [X] & \mapsto & \left([1 \mapsto \mathrm{id}] \mapsto \dfrac{|\operatorname{Fix}_{\Delta(1)}(X)|}{|\operatorname{C}_{\mathrm{S}_3}(1)|}[1 \mapsto \mathrm{id}] = \dfrac{|X|}{6}[1 \mapsto \mathrm{id}]\right) & = \dfrac{|X|}{6} \; . \end{array}$$

We obtain

$$\begin{array}{rcl} \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,1}([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(1)]) & = & 6 \; , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,1}([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(U_1)]) & = & 3 \; , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,1}([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(U_4)]) & = & 2 \; , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,1}([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3)]) & = & 1 \; . \end{array}$$

Ad $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_2}$. We consider the map

$$\begin{array}{rcll} \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_2} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) & \to & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2,\mathrm{S}_3)) & = \mathbf{Z} \\ [X] & \mapsto & \left([c_2 \mapsto (1,2)] \mapsto \dfrac{|\operatorname{Fix}_{\Delta(U_1)}(X)|}{\underbrace{|\operatorname{C}_{\mathrm{S}_3}(U_1)|}_{=2}}[c_2 \mapsto (1,2)]\right) & = \dfrac{|\operatorname{Fix}_{\Delta(U_1)}(X)|}{2} \; . \end{array}$$

We will calculate

$$\begin{array}{rcl} \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_2}([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(1)]) & = & 0 \; , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_2}([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(U_1)]) & = & 1 \; , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_2}([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(U_4)]) & = & 0 \; , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_2}([(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3)]) & = & 1 \; . \end{array}$$

In fact, we have

$$\begin{array}{rcl} |\operatorname{Fix}_{\Delta(U_1)}((\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(1))| & \overset{\text{L.12(2)}}{=} & 0, \\ |\operatorname{Fix}_{\Delta(U_1)}((\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(U_1))| & \overset{\text{R.92}}{=} & \dfrac{|\operatorname{N}_{\mathrm{S}_3}(U_1)||\operatorname{C}_{\mathrm{S}_3}(U_1)|}{|U_1|} = 2, \\ |\operatorname{Fix}_{\Delta(U_1)}((\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(U_4))| & \overset{\text{L.12(2)}}{=} & 0 \end{array}$$

and

$$\begin{array}{rl} & |\operatorname{Fix}_{\Delta(U_1)}((\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3))| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3) : (u,v)(g,h)\Delta(\mathrm{S}_3) = (g,h)\Delta(\mathrm{S}_3) \ \forall (u,v) \in \Delta(U_1)\}| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3) : ({}^{g^{-1}}u, {}^{h^{-1}}v) \in \Delta(\mathrm{S}_3) \ \forall (u,v) \in \Delta(U_1)\}| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3) : ({}^{g^{-1}}(1,2), {}^{h^{-1}}(1,2)) \in \Delta(\mathrm{S}_3)\}| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3) : {}^{g^{-1}}(1,2) = {}^{h^{-1}}(1,2)\}| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(\mathrm{S}_3) : hg^{-1} \in \langle (1,2) \rangle\}| \\ = & |\{(\mathrm{id},\mathrm{id})\Delta(\mathrm{S}_3), (\mathrm{id},(1,2))\Delta(\mathrm{S}_3)\}| = 2 \; . \end{array}$$

Ad $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_3}$. We consider the map

$$\begin{array}{rcll}\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_3} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) & \to & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3,\mathrm{S}_3)) & = \quad \mathbf{Z} \\ {}[X] & \mapsto & \left([c_3 \mapsto (1,2,3)] \mapsto \dfrac{|\operatorname{Fix}_{\Delta(U_4)}(X)|}{\underbrace{|\operatorname{C}_{\mathrm{S}_3}(U_4)|}_{=3}} [c_3 \mapsto (1,2,3)] \right) & = \dfrac{|\operatorname{Fix}_{\Delta(U_4)}(X)|}{3} \ . \end{array}$$

We will calculate

$$\begin{array}{rcl}\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_3}([(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(1)]) & = & 0\ , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_3}([(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_1)]) & = & 0\ , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_3}([(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_4)]) & = & 2\ , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{C}_3}([(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3)]) & = & 1\ . \end{array}$$

In fact, we have

$$\begin{array}{rcl}|\operatorname{Fix}_{\Delta(U_4)}((\mathrm{S}_3\times\mathrm{S}_3)/\Delta(1))| & \overset{\text{L.12(2)}}{=} & 0, \\ |\operatorname{Fix}_{\Delta(U_4)}((\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_1))| & \overset{\text{L.12(2)}}{=} & 0, \\ |\operatorname{Fix}_{\Delta(U_4)}((\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_4))| & \overset{\text{R.92}}{=} & \dfrac{|\operatorname{N}_{\mathrm{S}_3}(U_4)||\operatorname{C}_{\mathrm{S}_3}(U_4)|}{|U_4|} = 6 \end{array}$$

and

$$\begin{array}{rl} & |\operatorname{Fix}_{\Delta(U_4)}((\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3))| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3) : (u,v)(g,h)\Delta(\mathrm{S}_3) = (g,h)\Delta(\mathrm{S}_3)\ \forall (u,v)\in\Delta(U_4)\}| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3) : ({}^{g^{-1}}u, {}^{h^{-1}}v) \in \Delta(\mathrm{S}_3)\ \forall (u,v)\in\Delta(U_4)\}| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3) : {}^{g^{-1}}(1,2,3) = {}^{h^{-1}}(1,2,3))\}| \\ = & |\{(g,h)\Delta(\mathrm{S}_3) \in (\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3) : hg^{-1} \in \langle (1,2,3)\rangle\}| \\ = & |\{(\mathrm{id},\mathrm{id})\Delta(\mathrm{S}_3), (\mathrm{id},(1,2,3))\Delta(\mathrm{S}_3), (\mathrm{id},(1,3,2))\Delta(\mathrm{S}_3)\}| = 3. \end{array}$$

Ad $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{S}_3}$. We consider the map

$$\begin{array}{rcll}\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{S}_3} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) & \to & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_3,\mathrm{S}_3)) & = \quad \mathbf{Z} \\ {}[X] & \mapsto & \left([\mathrm{id}_{\mathrm{S}_3}] \mapsto \dfrac{|\operatorname{Fix}_{\Delta(\mathrm{S}_3)}(X)|}{|\operatorname{C}_{\mathrm{S}_3}(\mathrm{S}_3)|}[\mathrm{id}_{\mathrm{S}_3}] = \dfrac{|\operatorname{Fix}_{\Delta(\mathrm{S}_3)}(X)|}{1}[\mathrm{id}_{\mathrm{S}_3}] \right) & = |\operatorname{Fix}_{\Delta(\mathrm{S}_3)}(X)|\ . \end{array}$$

We will calculate

$$\begin{array}{rcl}\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{S}_3}([(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(1)]) & = & 0\ , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{S}_3}([(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_1)]) & = & 0\ , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{S}_3}([(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_4)]) & = & 0\ , \\ \sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3,\mathrm{S}_3}([(\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3)]) & = & 1\ . \end{array}$$

In fact, we have

$$\begin{aligned}
|\,\mathrm{Fix}_{\Delta(\mathrm{S}_3)}((\mathrm{S}_3\times\mathrm{S}_3)/\Delta(1))| &\overset{\text{L.12(2)}}{=} 0\ ,\\
|\,\mathrm{Fix}_{\Delta(\mathrm{S}_3)}((\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_1))| &\overset{\text{L.12(2)}}{=} 0\ ,\\
|\,\mathrm{Fix}_{\Delta(\mathrm{S}_3)}((\mathrm{S}_3\times\mathrm{S}_3)/\Delta(U_4))| &\overset{\text{L.12(2)}}{=} 0\ ,\\
|\,\mathrm{Fix}_{\Delta(\mathrm{S}_3)}((\mathrm{S}_3\times\mathrm{S}_3)/\Delta(\mathrm{S}_3))| &\overset{\text{R.92}}{=} \frac{|\,\mathrm{N}_{\mathrm{S}_3}(\mathrm{S}_3)||\,\mathrm{C}_{\mathrm{S}_3}(\mathrm{S}_3)|}{|\,\mathrm{S}_3\,|} = 1\ .
\end{aligned}$$

After identification, we may write

$$\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3}} \prod_{T\in\mathcal{T}_{\mathrm{S}_3}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) = \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\ .$$

The image of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$ under $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3}$ is given by the span of the columns of

$$A := \begin{pmatrix} 6 & 3 & 2 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 1\end{pmatrix} \in \mathbf{Z}^{4\times 4}\ ,$$

considered as elements of $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$.
For $x\in\mathbf{Z}^{4\times 1}$ we have

$$\begin{aligned}
x^{\mathrm{t}} := (x_1,x_2,x_3,x_4)\in\mathrm{im}(\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3}) \;&\Leftrightarrow\; \text{there exists } y\in\mathbf{Z}^{4\times 1} \text{ with } x = Ay\\
&\Leftrightarrow\; A^{-1}x\in\mathbf{Z}^{4\times 1} \Leftrightarrow \frac{1}{6}\begin{pmatrix} 1 & -3 & -1 & 3\\ 0 & 6 & 0 & -6\\ 0 & 0 & 3 & -3\\ 0 & 0 & 0 & 6\end{pmatrix}\cdot x\in\mathbf{Z}^{4\times 1}\\
&\Leftrightarrow\; \left\{\begin{array}{rl} x_1-3x_2-x_3+3x_4 & \equiv_6\ 0\\ 6x_2-6x_4 & \equiv_6\ 0\\ 3x_3-3x_4 & \equiv_6\ 0\\ 6x_4 & \equiv_6\ 0\end{array}\right.\\
&\Leftrightarrow\; \left\{\begin{array}{rl} x_1-3x_2-x_3+3x_4 & \equiv_2\ 0\\ x_1-3x_2-x_3+3x_4 & \equiv_3\ 0\\ x_3+x_4 & \equiv_2\ 0\end{array}\right.\\
&\Leftrightarrow\; \left\{\begin{array}{rl} x_1+x_2+x_3+x_4 & \equiv_2\ 0\\ x_1-x_3 & \equiv_3\ 0\\ x_3+x_4 & \equiv_2\ 0\end{array}\right.\\
&\Leftrightarrow\; \left\{\begin{array}{lll} x_1 & \equiv_2 & x_2\\ x_3 & \equiv_2 & x_4\\ x_1 & \equiv_3 & x_3\ .\end{array}\right.
\end{aligned}$$

Less formally, we write the isomorphism $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3}|^{\mathrm{im}(\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3})}$ of rings as follows.

$$\begin{array}{rcl} \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) & \xrightarrow{\sim} & \{(x_1,x_2,x_3,x_4)\in \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z} : x_1 \equiv_2 x_2,\ x_3 \equiv_2 x_4,\ x_1 \equiv_3 x_3\} \\ & = & \left(\begin{array}{ccccccc} & & & 3 & & & \\ \mathbf{Z} & -2- & \mathbf{Z} & & \mathbf{Z} & -2- & \mathbf{Z} \end{array} \right) . \end{array}$$

We will illustrate the index formula in Theorem 108(5).

We have chosen $\mathcal{L}^{\Delta}_{\mathrm{S}_3\times\mathrm{S}_3} = \{\Delta(1),\Delta(U_1),\Delta(U_4),\Delta(\mathrm{S}_3)\}$ in Example 75 as set of representatives of conjugacy classes of twisted diagonal subgroups of $\mathrm{S}_3\times\mathrm{S}_3$.

Theorem 108(5) states that the index of $\mathrm{im}(\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3})$ in $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$ is given by

$$\begin{array}{cl} & \displaystyle\prod_{\Delta(U,\alpha,V)\in\mathcal{L}^{\Delta}_{\mathrm{S}_3\times\mathrm{S}_3}} \frac{[\mathrm{N}_{\mathrm{S}_3\times\mathrm{S}_3}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\,\mathrm{C}_{\mathrm{S}_3}(U)|} \\[2ex] = & \displaystyle\frac{[\mathrm{N}_{\mathrm{S}_3\times\mathrm{S}_3}(\Delta(1)) : \Delta(1)]}{|\,\mathrm{C}_{\mathrm{S}_3}(1)|}\cdot\frac{[\mathrm{N}_{\mathrm{S}_3\times\mathrm{S}_3}(\Delta(U_1)) : \Delta(U_1)]}{|\,\mathrm{C}_{\mathrm{S}_3}(U_1)|} \\[2ex] & \displaystyle\cdot\frac{[\mathrm{N}_{\mathrm{S}_3\times\mathrm{S}_3}(\Delta(U_4)) : \Delta(U_4)]}{|\,\mathrm{C}_{\mathrm{S}_3}(U_4)|}\cdot\frac{[\mathrm{N}_{\mathrm{S}_3\times\mathrm{S}_3}(\Delta(\mathrm{S}_3)) : \Delta(\mathrm{S}_3)]}{|\,\mathrm{C}_{\mathrm{S}_3}(\mathrm{S}_3)|} \\[2ex] \overset{\text{R.92}}{=} & \displaystyle\frac{|\,\mathrm{N}_{\mathrm{S}_3}(1)|}{|1|}\cdot\frac{|\,\mathrm{N}_{\mathrm{S}_3}(U_1)|}{|U_1|}\cdot\frac{|\,\mathrm{N}_{\mathrm{S}_3}(U_4)|}{|U_4|}\cdot\frac{|\,\mathrm{N}_{\mathrm{S}_3}(\mathrm{S}_3)|}{|\,\mathrm{S}_3\,|} \\[2ex] = & 6\cdot 1\cdot 2\cdot 1 \\[2ex] = & 12 \end{array}$$

which is the same as $|\det\begin{pmatrix} 6 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}|$.

Chapter 3

The bifree double Burnside ring $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$

Let U, V be subgroups of S_4.

We have the subgroup $\mathrm{Inn}_{\mathrm{S}_4}(V) = \{\kappa_g^V : g \in \mathrm{N}_{\mathrm{S}_4}(V)\} \leq \mathrm{Aut}(V)$, cf. Notation 43, Remark 44.

We have the $(\mathrm{Inn}_{\mathrm{S}_4}(V), \mathrm{Inn}_{\mathrm{S}_4}(U))$-biset

$$\mathrm{Isom}(V, U) = \{V \overset{f}{\underset{\sim}{\leftarrow}} U : f \text{ is an isomorphism of groups}\} ,$$

cf. Remark 45.

Following Boltje and Danz, we have given an embedding of the bifree double Burnside ring $\mathrm{B}^{\Delta}_{\mathbf{Q}}(\mathrm{S}_4, \mathrm{S}_4) = \mathbf{Q} \underset{\mathbf{Z}}{\otimes} \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$ via a direct product of endomorphism rings of permutation modules over outer automorphism groups of subgroups of S_4, cf. Theorem 108. In this chapter, we use this embedding in order to describe an isomorphic copy of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$ as a subring of a product of matrix rings over $\mathbf{Z}$ via congruences of matrix entries.

3.1 Preparations

Using the computer algebra system Magma [6], a system of representatives for the conjugacy classes of subgroups of S_4 is given by

$$\begin{aligned}
U_1 &:= 1 \\
U_2 &:= \langle (1,2) \rangle \\
U_3 &:= \langle (1,2)(3,4) \rangle \\
U_4 &:= \langle (1,2,3) \rangle \\
U_5 &:= \langle (1,2)(3,4), (1,3)(2,4) \rangle \\
U_6 &:= \langle (1,2,3,4) \rangle \\
U_7 &:= \langle (1,2), (3,4) \rangle \\
U_8 &:= \langle (1,2), (1,2,3) \rangle \\
U_9 &:= \langle (1,2,3,4), (1,4)(2,3) \rangle \\
U_{10} &:= \langle (1,2,3), (1,2)(3,4) \rangle \\
U_{11} &:= \langle (1,2), (1,2,3,4) \rangle \ .
\end{aligned}$$

Lemma 110. *We have the following presentations.*

$$
\begin{aligned}
\mathrm{C}_2 := \langle c_2 : c_2^2 \rangle &\xrightarrow{r_2} U_2 \\
c_2 &\mapsto (1,2)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_2 = \langle c_2 : c_2^2 \rangle &\xrightarrow{r_3} U_3 \\
c_2 &\mapsto (1,2)(3,4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_3 := \langle c_3 : c_3^3 \rangle &\xrightarrow{r_4} U_4 \\
c_3 &\mapsto (1,2,3)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_2 \times \mathrm{C}_2 := \langle a : a^2 \rangle \times \langle b : b^2 \rangle &\xrightarrow{r_5} U_5 \\
a &\mapsto (1,2)(3,4) \\
b &\mapsto (1,3)(2,4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_4 := \langle c_4 : c_4^4 \rangle &\xrightarrow{r_6} U_6 \\
c_4 &\mapsto (1,2,3,4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_2 \times \mathrm{C}_2 = \langle a : a^2 \rangle \times \langle b : b^2 \rangle &\xrightarrow{r_7} U_7 \\
a &\mapsto (1,2) \\
b &\mapsto (3,4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{S}_3^{\mathrm{pres}} := \langle r, s : r^3, s^2, (sr)^2 \rangle &\xrightarrow{r_8} U_8 = \mathrm{S}_3 \\
r &\mapsto (1,2,3) \\
s &\mapsto (1,2)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}_8 := \langle d, t : d^4, t^2, (td)^2 \rangle &\xrightarrow{r_9} U_9 \\
d &\mapsto (1,2,3,4) \\
t &\mapsto (1,4)(2,3)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{A}_4^{\mathrm{pres}} := \langle x, y : x^3, y^2, (yx^{-1})^3 \rangle &\xrightarrow{r_{10}} U_{10} = \mathrm{A}_4 \\
x &\mapsto (1,2,3) \\
y &\mapsto (1,2)(3,4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{S}_4^{\mathrm{pres}} := \langle v, w : v^4, w^2, (vw)^3 \rangle &\xrightarrow{r_{11}} U_{11} = \mathrm{S}_4 \\
v &\mapsto (1,2,3,4) \\
w &\mapsto (1,2)
\end{aligned}
$$

Proof. As

$$\begin{aligned} (1,2)^2 &= \mathrm{id}, & (1,2)(3,4)^2 &= \mathrm{id},\\ (1,2,3)^3 &= \mathrm{id}, & (1,3)(2,4)^2 &= \mathrm{id},\\ (1,2,3,4)^4 &= \mathrm{id}, & (3,4)^2 &= \mathrm{id},\\ ((1,2,3)\circ(1,2))^2 = (1,3)^2 &= \mathrm{id}, & (1,4)(2,3)^2 &= \mathrm{id},\\ ((1,4)(2,3)\circ(1,2,3,4))^2 = (1,3)^2 &= \mathrm{id}, & ((1,2)(3,4)\circ(1,2,3)^{-1})^3 = (1,4,3)^3 &= \mathrm{id},\\ ((1,2,3,4)\circ(1,2))^3 = (1,3,4)^3 &= \mathrm{id}, \end{aligned}$$

the maps r_i for $i \in [2,11]$ are surjective group morphisms.

Using the computer algebra system Magma [6] we obtain

$$\begin{aligned} |U_2| = |U_3| &= |\,\mathrm{C}_2\,|,\\ |U_4| &= |\,\mathrm{C}_3\,|,\\ |U_5| = |U_6| = |U_7| &= |\,\mathrm{C}_2\times\mathrm{C}_2\,| = |\,\mathrm{C}_4\,|,\\ |U_8| &= |\langle r,s : r^3, s^2, (sr)^2\rangle|,\\ |U_9| &= |\mathrm{D}_8|,\\ |U_{10}| &= |\langle x,y : x^3, y^2, (yx^{-1})^3\rangle|,\\ |U_{11}| &= |\langle v,w : v^4, w^2, (vw)^3\rangle|\ . \end{aligned}$$

Therefore, r_i is bijective for $i \in [2,11]$. $\square$

Remark 111. Forming isomorphism classes in $\{U_i : i \in [1,11]\}$, we obtain

$$\{U_1\},\ \{U_2,U_3\},\ \{U_4\},\ \{U_5,U_7\},\ \{U_6\},\ \{U_8\},\ \{U_9\},\ \{U_{10}\},\ \{U_{11}\},$$

cf. Lemma 110.

3.2 Z-linear basis of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4)$

We want to use Lemma 85 to construct a $\mathbf{Z}$-linear basis of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4)$, cf. Definition 67.

Lemma 112. *For $i \in [1,11]\setminus\{7,9\}$ we have $\mathrm{Inn}_{\mathrm{S}_4}(U_i) = \mathrm{Aut}(U_i)$. In particular, the $(\mathrm{Inn}_{\mathrm{S}_4}(U_i),\mathrm{Inn}_{\mathrm{S}_4}(U_i))$-biset $\mathrm{Isom}(U_i,U_i) = \mathrm{Aut}(U_i)$ is transitive, hence its only orbit can be represented by id_{U_i}, cf. Notation 43, Remark 45 and Lemma 46.*

Proof. It suffices to show that $\mathrm{Aut}(U_i) \subseteq \mathrm{Inn}_{\mathrm{S}_4}(U_i)$ for $i \in [1,11]\setminus\{7,9\}$, cf. Remark 44.

Ad U_1, U_2, U_3 .

We have $\mathrm{Inn}_{\mathrm{S}_4}(U_1) = 1 = \mathrm{Aut}(U_1)$, $\mathrm{Inn}_{\mathrm{S}_4}(U_2) = 1 = \mathrm{Aut}(U_2)$, $\mathrm{Inn}_{\mathrm{S}_4}(U_3) = 1 = \mathrm{Aut}(U_3)$.

Ad U_4 .

The only nontrivial automorphism of U_4 is given by

$$f : U_4 \to U_4,\ (1,2,3) \mapsto (1,3,2)\ .$$

But, we have $f = \kappa^{U_4}_{(1,2)} \in \mathrm{Inn}_{\mathrm{S}_4}(U_4)$. So, $\mathrm{Inn}_{\mathrm{S}_4}(U_4) = \mathrm{Aut}(U_4)$.

Ad U_5 .

The subgroup U_5 has the presentation $\langle a : a^2\rangle \times \langle b : b^2\rangle$, cf. Lemma 110. Every automorphism of U_5 is of the form

$$\begin{array}{rrcl} \varphi_{k,l,m} : & U_5 & \xrightarrow{\sim} & U_5 \\ & x_2 := (1,2)(3,4) & \mapsto & x_k \\ & x_3 := (1,3)(2,4) & \mapsto & x_l \\ & x_4 := (1,4)(2,3) & \mapsto & x_m \end{array}$$

for $k, l, m \in \{2, 3, 4\}$ such that $\{k, l, m\} = \{2, 3, 4\}$.

Then $\varphi_{k,l,m} = \kappa^{U_5}_{g_{k,l,m}}$ for $g_{k,l,m} : [1, 4] \to [1, 4]$, $1 \mapsto 1$, $2 \mapsto k$, $3 \mapsto l$, $4 \mapsto m$.

Note that $\mathrm{N}_{\mathrm{S}_4}(U_5) = \mathrm{S}_4$. So, we obtain

$$\mathrm{Aut}(U_5) = \{\varphi_{k,l,m} : \{k, l, m\} = \{2, 3, 4\}\} = \{\kappa^{U_5}_{g_{k,l,m}} : \{k, l, m\} = \{2, 3, 4\}\} \subseteq \mathrm{Inn}_{\mathrm{S}_4}(U_5) \ .$$

Ad U_6 .

The only nontrivial automorphism of U_6 is given by

$$f : U_6 \to U_6,\ (1, 2, 3, 4) \mapsto (1, 4, 3, 2) \ .$$

But, we have $f = \kappa^{U_6}_{(1,2)(3,4)} \in \mathrm{Inn}_{\mathrm{S}_4}(U_6)$. So, $\mathrm{Inn}_{\mathrm{S}_4}(U_6) = \mathrm{Aut}(U_6)$.

Ad U_8 .

Note that $\mathrm{Aut}(U_8) = \mathrm{Inn}_{U_8}(U_8)$. As $\mathrm{Inn}_{U_8}(U_8) \subseteq \mathrm{Inn}_{\mathrm{S}_4}(U_8)$ the assertions holds.

Ad U_{10} .

The subgroup U_{10} has the presentation $\langle x, y : x^3, y^2, (yx^{-1})^3\rangle$, cf. Lemma 110. Since automorphisms map generating tuples to generating tuples, this means there are at most $8 \cdot 3 = 24$ automorphisms of U_{10}. Note that $\mathrm{N}_{\mathrm{S}_4}(U_{10}) = \mathrm{S}_4$. Consider the group morphism

$$\begin{array}{rccl} \pi : & \mathrm{S}_4 & \to & \mathrm{Aut}(U_{10}) \\ & g & \mapsto & \kappa^{U_{10}}_g \ , \end{array}$$

cf. Remark 44.

We have $g \in \ker(\pi)$ if and only if $g \in \mathrm{C}_{\mathrm{S}_4}(U_{10})$.

Magma [6] gives $\mathrm{C}_{\mathrm{S}_4}(U_{10}) = \{\mathrm{id}\}$, thus π is injective. So, $\mathrm{S}_4 \cong \mathrm{im}(\pi) = \mathrm{Inn}_{\mathrm{S}_4}(U_{10})$. As $|\mathrm{Aut}(U_{10})| \leq 24$ we obtain that $\mathrm{Inn}_{\mathrm{S}_4}(U_{10}) = \mathrm{Aut}(U_{10}) \cong \mathrm{S}_4$.

Ad U_{11} .

Note that all elements of order 4 in S_4 are conjugate. Note that for a given element $\hat{v}$ of S_4 of order 4 we have $|\{\hat{w} \in \mathrm{S}_4 : |\langle\hat{w}\rangle| = 2 \text{ and } |\langle\hat{v}, \hat{w}\rangle| < 24\}| \geq 3$. Since automorphisms map generating tuples to generating tuples, there are at most $6 \cdot 6 = 36$ automorphisms of U_{11}.

Consider the group morphism

$$\begin{array}{rccl} \varphi : & \mathrm{S}_4 & \to & \mathrm{Aut}(U_{11}) \\ & g & \mapsto & \kappa_g^{U_{11}} \ , \end{array}$$

cf. Remark 44. Since $\mathrm{C}_{\mathrm{S}_4}(U_{11}) = \{\mathrm{id}\}$ the group morphism φ is injective and $\mathrm{S}_4 \cong \mathrm{im}(\varphi) = \mathrm{Inn}_{\mathrm{S}_4}(U_{11})$. Since $24 \leq |\,\mathrm{Aut}(U_{11})| \leq 36$ and 24 divides $|\,\mathrm{Aut}(U_{11})|$, we conclude that $\mathrm{Aut}(U_{11}) = \mathrm{Inn}_{\mathrm{S}_4}(U_{11}) \cong \mathrm{S}_4$. □

Lemma 113. *The* $(\mathrm{Inn}_{\mathrm{S}_4}(U_7), \mathrm{Inn}_{\mathrm{S}_4}(U_7))$*-orbits in* $\mathrm{Isom}(U_7, U_7) = \mathrm{Aut}(U_7)$ *are represented by* id_{U_7} *and*

$$\begin{array}{rccl} \varphi_7 : & U_7 & \to & U_7 \\ & \mathrm{id} & \mapsto & \mathrm{id} \\ & (1,2) & \mapsto & (1,2)(3,4) \\ & (3,4) & \mapsto & (1,2) \\ & (1,2)(3,4) & \mapsto & (3,4) \ . \end{array}$$

Proof. The subgroup U_7 has the presentation $\langle a : a^2 \rangle \times \langle b : b^2 \rangle$, cf. Lemma 110. Since automorphisms map generating tuples to generating tuples, there are at most $3 \cdot 2 = 6$ automorphisms of U_7. We have the automorphisms

$$\begin{array}{rcclrccl} \mathrm{id}_{U_7} : & U_7 & \to & U_7 & \varphi_2 : & U_7 & \to & U_7 \\ & \mathrm{id} & \mapsto & \mathrm{id} & & \mathrm{id} & \mapsto & \mathrm{id} \\ & (1,2) & \mapsto & (1,2) & & (1,2) & \mapsto & (1,2)(3,4) \\ & (3,4) & \mapsto & (3,4) & & (3,4) & \mapsto & (1,2) \\ & (1,2)(3,4) & \mapsto & (1,2)(3,4) \ , & & (1,2)(3,4) & \mapsto & (3,4) \ , \\ \varphi_3 : & U_7 & \to & U_7 & \varphi_4 : & U_7 & \to & U_7 \\ & \mathrm{id} & \mapsto & \mathrm{id} & & \mathrm{id} & \mapsto & \mathrm{id} \\ & (1,2) & \mapsto & (3,4) & & (1,2) & \mapsto & (1,2) \\ & (3,4) & \mapsto & (1,2) & & (3,4) & \mapsto & (1,2)(3,4) \\ & (1,2)(3,4) & \mapsto & (1,2)(3,4) \ , & & (1,2)(3,4) & \mapsto & (3,4) \ , \\ \varphi_5 : & U_7 & \to & U_7 & \varphi_6 : & U_7 & \to & U_7 \\ & \mathrm{id} & \mapsto & \mathrm{id} & & \mathrm{id} & \mapsto & \mathrm{id} \\ & (1,2) & \mapsto & (3,4) & & (1,2) & \mapsto & (1,2)(3,4) \\ & (3,4) & \mapsto & (1,2)(3,4) & & (3,4) & \mapsto & (3,4) \\ & (1,2)(3,4) & \mapsto & (1,2) \ , & & (1,2)(3,4) & \mapsto & (1,2) \ . \end{array}$$

So, $\mathrm{Aut}(U_7) = \{\mathrm{id}_{U_7}\} \cup \{\varphi_i : i \in [2,6]\}$. Note that $(\varphi_2)^3 = \mathrm{id}_{U_7}$, that $(\varphi_3)^2 = \mathrm{id}_{U_7}$ and that $(\varphi_3 \circ \varphi_2)^2 = \mathrm{id}_{U_7}$ and therefore $\mathrm{Aut}(U_7) \cong \mathrm{S}_3$, cf. Lemma 110.

As ${}^{(1,3)(2,4)}(1,2) = (3,4)$ and ${}^{(1,3)(2,4)}(3,4) = (1,2)$, we have $(1,3)(2,4) \in \mathrm{N}_{\mathrm{S}_4}(U_7)$. Moreover,

$$\begin{array}{rcl}\varphi_3 &=& \kappa^{U_7}_{(1,3)(2,4)} \circ \mathrm{id}_{U_7} \\ \varphi_4 &=& \varphi_2 \circ \kappa^{U_7}_{(1,3)(2,4)} \\ \varphi_5 &=& \kappa^{U_7}_{(1,3)(2,4)} \circ \varphi_2 \circ \kappa^{U_7}_{(1,3)(2,4)} \\ \varphi_6 &=& \kappa^{U_7}_{(1,3)(2,4)} \circ \varphi_2 \ .\end{array}$$

As φ_2 is not given by conjugation in S_4, the $(\mathrm{Inn}_{\mathrm{S}_4}(U_7), \mathrm{Inn}_{\mathrm{S}_4}(U_7))$-orbits of the elements id_{U_7} and φ_2 are disjoint. So, the $(\mathrm{Inn}_{\mathrm{S}_4}(U_7), \mathrm{Inn}_{\mathrm{S}_4}(U_7))$-orbits in $\mathrm{Aut}(U_7)$ can be represented by id_{U_7} and $\varphi_7 := \varphi_2$. □

Lemma 114. *The* $(\mathrm{Inn}_{\mathrm{S}_4}(U_9), \mathrm{Inn}_{\mathrm{S}_4}(U_9))$*-orbits in* $\mathrm{Isom}(U_9, U_9) = \mathrm{Aut}(U_9)$ *are represented by* id_{U_9} *and*

$$\begin{array}{rrcl}\varphi_9 : & U_9 & \to & U_9 \\ & (1,2,3,4) & \mapsto & (1,2,3,4) \\ & (1,4)(2,3) & \mapsto & (2,4) \ .\end{array}$$

We have $\mathrm{Inn}_{\mathrm{S}_4}(U_9) = \mathrm{Inn}(U_9) = \{\kappa^{U_9}_{\mathrm{id}}, \kappa^{U_9}_{(1,3)}, \kappa^{U_9}_{(1,2,3,4)}, \kappa^{U_9}_{(1,2)(3,4)}\}$.

We have $\mathrm{Aut}(U_9) = \{\mathrm{id}_{U_9}, \kappa^{U_9}_{(1,3)}, \kappa^{U_9}_{(1,2,3,4)}, \kappa^{U_9}_{(1,2)(3,4)}, \varphi, \kappa^{U_9}_{(1,3)} \circ \varphi, \kappa^{U_9}_{(1,2,3,4)} \circ \varphi, \kappa^{U_9}_{(1,2)(3,4)} \circ \varphi\}$.

Proof. We have $U_9 = \{\mathrm{id}, (1,2,3,4), (1,4,3,2), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,3), (2,4)\}$. Note that the two elements of order 4 in U_9 are conjugate in U_9. Note that given an element $\hat{d}$ of order 4 in U_9, the element $\hat{d}^2$ is of order 2 and $|\langle \hat{d}, \hat{d}^2\rangle| = 4$.

As U_9 can be presented by $\langle d, t : d^4, t^2, (td)^2\rangle$, cf. Lemma 110, and as automorphisms map generating tuples to generating tuples, there are at most $2 \cdot 4 = 8$ automorphisms of U_9. Consider the automorphisms

$$\begin{array}{rrcl}\mathrm{id}_{U_9} : & U_9 & \to & U_9 \\ & (1,2,3,4) & \mapsto & (1,2,3,4) \\ & (1,4)(2,3) & \mapsto & (1,4)(2,3)\end{array} \ , \qquad \begin{array}{rrcl}\varphi : & U_9 & \to & U_9 \\ & (1,2,3,4) & \mapsto & (1,2,3,4) \\ & (1,4)(2,3) & \mapsto & (2,4) \ .\end{array}$$

As φ is not given by conjugation in S_4, the $(\mathrm{Inn}_{\mathrm{S}_4}(U_9), \mathrm{Inn}_{\mathrm{S}_4}(U_9))$-orbits in $\mathrm{Aut}(U_9)$ of the elements id_{U_9} and φ are disjoint.

Note that $\mathrm{N}_{\mathrm{S}_4}(U_9) = U_9$. The group morphism $\psi : U_9 \to \mathrm{Aut}(U_9)$, $g \mapsto \kappa^{U_9}_g$ has the kernel $\mathrm{C}_{U_9}(U_9) = \langle (1,3)(2,4)\rangle$. We have $\kappa^{U_9}_{\mathrm{id}} = \kappa^{U_9}_{(1,3)(2,4)}$, $\kappa^{U_9}_{(1,3)} = \kappa^{U_9}_{(2,4)}$, $\kappa^{U_9}_{(1,2,3,4)} = \kappa^{U_9}_{(1,4,3,2)}$ and $\kappa^{U_9}_{(1,2)(3,4)} = \kappa^{U_9}_{(1,4)(2,3)}$.

Therefore, we have the group of order 4

$$\mathrm{im}(\psi) = \mathrm{Inn}_{\mathrm{S}_4}(U_9) = \{\kappa^{U_9}_g : g \in U_9\} = \{\kappa^{U_9}_{\mathrm{id}}, \kappa^{U_9}_{(1,3)}, \kappa^{U_9}_{(1,2,3,4)}, \kappa^{U_9}_{(1,2)(3,4)}\} \ .$$

In particular, $\mathrm{Inn}_{\mathrm{S}_4}(U_9) = \mathrm{Inn}(U_9)$.

We obtain the eight different automorphisms of U_9:

$$\mathrm{id}_{U_9} \ , \quad \kappa^{U_9}_{(1,3)} \circ \mathrm{id}_{U_9}, \quad \kappa^{U_9}_{(1,2,3,4)} \circ \mathrm{id}_{U_9} \ , \quad \kappa^{U_9}_{(1,2)(3,4)} \circ \mathrm{id}_{U_9} \ ,$$

$$\varphi, \qquad \kappa^{U_9}_{(1,3)} \circ \varphi \ , \qquad \kappa^{U_9}_{(1,2,3,4)} \circ \varphi \ , \qquad \kappa^{U_9}_{(1,2)(3,4)} \circ \varphi \ .$$

Hence, $|\operatorname{Aut}(U_9)| = 8$ and the $(\operatorname{Inn}_{S_4}(U_9), \operatorname{Inn}_{S_4}(U_9))$-orbits in $\operatorname{Aut}(U_9)$ can be represented by id_{U_9} and $\varphi_9 := \varphi$. □

Lemma 115. *The only* $(\operatorname{Inn}_{S_4}(U_3), \operatorname{Inn}_{S_4}(U_2))$*-orbit in* $\operatorname{Isom}(U_3, U_2)$ *is represented by*

$$\begin{array}{rccl} \varphi_{2,3}: & U_2 & \to & U_3 \\ & \mathrm{id} & \mapsto & \mathrm{id} \\ & (1,2) & \mapsto & (1,2)(3,4)\ . \end{array}$$

The only $(\operatorname{Inn}_{S_4}(U_2), \operatorname{Inn}_{S_4}(U_3))$*-orbit in* $\operatorname{Isom}(U_2, U_3)$ *is represented by*

$$\begin{array}{rccl} \varphi_{3,2}: & U_3 & \to & U_2 \\ & \mathrm{id} & \mapsto & \mathrm{id} \\ & (1,2)(3,4) & \mapsto & (1,2)\ . \end{array}$$

Proof. We have $\operatorname{Inn}_{S_4}(U_2) = \operatorname{Aut}(U_2)$, cf. Lemma 112.

By Lemma 46, $\operatorname{Isom}(U_3, U_2)$ is a transitive $(\operatorname{Inn}_{S_4}(U_3), \operatorname{Inn}_{S_4}(U_2))$-biset and $\operatorname{Isom}(U_2, U_3)$ is a transitive $(\operatorname{Inn}_{S_4}(U_2), \operatorname{Inn}_{S_4}(U_3))$-biset. □

Lemma 116. *The* $(\operatorname{Inn}_{S_4}(U_5), \operatorname{Inn}_{S_4}(U_7))$*-orbits in* $\operatorname{Isom}(U_5, U_7)$ *are represented by*

$$\begin{array}{rccl} \varphi_{7,5}: & U_7 & \to & U_5 \\ & \mathrm{id} & \mapsto & \mathrm{id} \\ & (1,2)(3,4) & \mapsto & (1,2)(3,4) \\ & (1,2) & \mapsto & (1,3)(2,4) \\ & (3,4) & \mapsto & (1,4)(2,3)\ . \end{array}$$

The $(\operatorname{Inn}_{S_4}(U_7), \operatorname{Inn}_{S_4}(U_5))$*-orbits in* $\operatorname{Isom}(U_7, U_5)$ *are represented by*

$$\begin{array}{rccl} \varphi_{5,7}: & U_5 & \to & U_7 \\ & \mathrm{id} & \mapsto & \mathrm{id} \\ & (1,2)(3,4) & \mapsto & (1,2)(3,4) \\ & (1,3)(2,4) & \mapsto & (1,2) \\ & (1,4)(2,3) & \mapsto & (3,4)\ . \end{array}$$

Proof. We have $\operatorname{Inn}_{S_4}(U_5) = \operatorname{Aut}(U_5)$, cf. Lemma 112.

By Lemma 46, $\operatorname{Isom}(U_5, U_7)$ is a transitive $(\operatorname{Inn}_{S_4}(U_5), \operatorname{Inn}_{S_4}(U_7))$-biset and $\operatorname{Isom}(U_7, U_5)$ is a transitive $(\operatorname{Inn}_{S_4}(U_7), \operatorname{Inn}_{S_4}(U_5))$-biset. □

Corollary 117. *We obtain the* $\mathbf{Z}$*-linear basis* $\mathcal{B} :=$

$$\left(\begin{array}{lll} [b_1 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_1)], & [b_2 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_2)], & [b_3 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_3)], \\ [b_4 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_4)], & [b_5 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_5)], & [b_6 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_6)], \\ [b_7 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_7)], & [b_7' := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_7, \varphi_7, U_7)], & [b_8 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_8)], \\ [b_9 := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_9)], & [b_9' := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_9, \varphi_9, U_9)], & [b_{10} := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_{10})], \\ [b_{11} := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_{11})], & [b_{3,2} := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_3, \varphi_{2,3}, U_2)], & [b_{2,3} := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_2, \varphi_{3,2}, U_3)] \\ [b_{5,7} := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_5, \varphi_{7,5}, U_7)], & [b_{7,5} := (\mathrm{S}_4 \times \mathrm{S}_4)/\Delta(U_7, \varphi_{5,7}, U_5)] & \end{array}\right)$$

of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$.

Proof. By Lemma 85, Lemma 112, Lemma 113, Lemma 114, Lemma 115 and Lemma 116 we have a system of representatives for the conjugacy classes of twisted diagonal subgroups of $\mathrm{S}_4 \times \mathrm{S}_4$:

$$\begin{array}{llll} \Delta(U_1), & \Delta(U_2), & \Delta(U_3), & \Delta(U_4), \\ \Delta(U_5), & \Delta(U_6), & \Delta(U_7), & \Delta(U_7, \varphi_7, U_7) \\ \Delta(U_8), & \Delta(U_9), & \Delta(U_9, \varphi_9, U_9), & \Delta(U_{10}), \\ \Delta(U_{11}), & \Delta(U_3, \varphi_{2,3}, U_2), & \Delta(U_2, \varphi_{3,2}, U_3) & \Delta(U_5, \varphi_{7,5}, U_7), \\ \Delta(U_7, \varphi_{5,7}, U_5). & & & \end{array}$$

By Definition 67 we obtain the claimed basis. □

3.3 Wedderburn image of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$

Remark 118. Note that we may replace $\mathcal{T}$ in Theorem 108 by

$$\mathcal{T}_{\mathrm{S}_4} = \{1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_3^{\mathrm{pres}}, \mathrm{D}_8, \mathrm{A}_4^{\mathrm{pres}}, \mathrm{S}_4\} ,$$

cf. Notation 96, Definition 99, Definition 101, Lemma 110.

3.3.1 Preparations

Lemma 119. *Concerning* $\overline{\mathrm{Inj}}$*, cf. Definition 99 and Definition 101. Concerning presentations of subgroups in* $\mathcal{T}_{\mathrm{S}_4}$*, cf. Lemma 110.*

(1) *We have* $\overline{\mathrm{Inj}}(1, \mathrm{S}_4) = \{[1 \mapsto \mathrm{id}]\}$.

(2) *We have* $\overline{\mathrm{Inj}}(\mathrm{C}_2, \mathrm{S}_4) = \{[c_2 \mapsto (1,2)], [c_2 \mapsto (1,2)(3,4)]\}$.

(3) *We have* $\overline{\mathrm{Inj}}(\mathrm{C}_3, \mathrm{S}_4) = \{[c_3 \mapsto (1,2,3)]\}$.

(4) *We have* $\overline{\mathrm{Inj}}(\mathrm{C}_4, \mathrm{S}_4) = \{[c_4 \mapsto (1,2,3,4)]\}$.

(5) *We have* $\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4) =$

$$\left\{ \left[\begin{array}{lll} a & \mapsto & (1,2)(3,4) \\ b & \mapsto & (1,3)(2,4) \end{array} \right], \left[\begin{array}{lll} a & \mapsto & (1,2) \\ b & \mapsto & (3,4) \end{array} \right], \left[\begin{array}{lll} a & \mapsto & (1,2)(3,4) \\ b & \mapsto & (1,2) \end{array} \right], \left[\begin{array}{lll} a & \mapsto & (1,2) \\ b & \mapsto & (1,2)(3,4) \end{array} \right] \right\} .$$

(6) *We have* $\overline{\mathrm{Inj}}(\mathrm{S}_3^{\mathrm{pres}}, \mathrm{S}_4) = \left\{ \left[\begin{array}{lll} r & \mapsto & (1,2,3) \\ s & \mapsto & (1,2) \end{array} \right] \right\} .$

(7) *We have* $\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4) = \left\{ \left[\begin{array}{lll} d & \mapsto & (1,2,3,4) \\ t & \mapsto & (1,4)(2,3) \end{array} \right], \left[\begin{array}{lll} d & \mapsto & (1,2,3,4) \\ t & \mapsto & (2,4) \end{array} \right] \right\} .$

(8) *We have* $\overline{\mathrm{Inj}}(\mathrm{A}_4^{\mathrm{pres}}, \mathrm{S}_4) = \left\{ \left[\begin{array}{lll} x & \mapsto & (1,2,3) \\ y & \mapsto & (1,2)(3,4) \end{array} \right] \right\} .$

(9) *We have* $\overline{\mathrm{Inj}}(\mathrm{S}_4, \mathrm{S}_4) = \{[\mathrm{id}_{\mathrm{S}_4}]\}$.

Proof. Ad (1). We have $\mathrm{Inj}(1, \mathrm{S}_4) = \{1 \mapsto \mathrm{id}\}$ and therefore $\overline{\mathrm{Inj}}(1, \mathrm{S}_4) = \{[1 \mapsto \mathrm{id}]\}$.

Ad (2). We have $\mathrm{Inj}(\mathrm{C}_2, \mathrm{S}_4) = \{\mathrm{C}_2 \to \mathrm{S}_4,\ c_2 \mapsto g : g \in \mathrm{S}_4, |\langle g \rangle| = 2\}$. As every transposition in S_4 is conjugate to $(1,2)$ and every double transposition in S_4 is conjugate to $(1,2)(3,4)$ it follows that

$$\overline{\mathrm{Inj}}(\mathrm{C}_2, \mathrm{S}_4) = \{[c_2 \mapsto (1,2)], [c_2 \mapsto (1,2)(3,4)]\} .$$

Ad (3). We have $\mathrm{Inj}(\mathrm{C}_3, \mathrm{S}_4) = \{\mathrm{C}_3 \to \mathrm{S}_4,\ c_3 \mapsto g : g \in \mathrm{S}_4, |\langle g \rangle| = 3\}$. As every 3-cycle in S_4 is conjugate to $(1,2,3)$ it follows that

$$\overline{\mathrm{Inj}}(\mathrm{C}_3, \mathrm{S}_4) = \{[c_3 \mapsto (1,2,3)]\} .$$

Ad (4). We have $\mathrm{Inj}(\mathrm{C}_4, \mathrm{S}_4) = \{\mathrm{C}_4 \to \mathrm{S}_4,\ c_4 \mapsto g : g \in \mathrm{S}_4\, |\langle g \rangle| = 4\}$. As every 4-cycle in S_4 is conjugate to $(1,2,3,4)$ it follows that

$$\overline{\mathrm{Inj}}(\mathrm{C}_4, \mathrm{S}_4) = \{[c_4 \mapsto (1,2,3,4)]\} .$$

Ad (5). Let $\beta : \mathrm{C}_2 \times \mathrm{C}_2 \to \mathrm{S}_4$ be an injective group morphism. As $\mathrm{C}_2 \times \mathrm{C}_2 \cong \mathrm{im}(\beta) \leq \mathrm{S}_4$ it follows that $\mathrm{im}(\beta) \in \{{}^g U_5, {}^g U_7 : g \in \mathrm{S}_4\} = \{U_5, {}^g U_7 : g \in \mathrm{S}_4\}$. As we are interested in the G-orbits of $\mathrm{Inj}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)$ it suffices to consider $\mathrm{im}(\beta) \in \{U_5, U_7\}$.

Now, every group isomorphism $\mathrm{C}_2 \times \mathrm{C}_2 \to \mathrm{im}(\beta)$ is of the form $\omega \circ \beta$ for $\omega \in \mathrm{Aut}(U_5)$ respectively $\omega \in \mathrm{Aut}(U_7)$. By Lemma 112, we have $\mathrm{Aut}(U_5) = \mathrm{Inn}_{\mathrm{S}_4}(U_5)$.

Every automorphism of U_7 is of the form

$$\varphi_{\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right)} : U_7 \to U_7,\ (1,2) \mapsto (1,2)^i (3,4)^j,\ (3,4) \mapsto (1,2)^k (3,4)^l$$

for $\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{F}_2)$. Therefore, we have

$$\mathrm{Aut}(U_7) = \{\varphi_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, \varphi_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)}, \varphi_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}, \varphi_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}, \varphi_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)}, \varphi_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)}\} .$$

Note that conjugation with $(1,3)(2,4)$ interchanges rows on $\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right)$ as

$$ {}^{(1,3)(2,4)}(1,2)^i(3,4)^j = (1,2)^j(3,4)^i \ . $$

So,

$$ \mathrm{Aut}(U_7) = \{\varphi_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, \varphi_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)}, \varphi_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}, \kappa^{\mathrm{S}_4}_{(1,3)(2,4)} \circ \varphi_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, \kappa^{\mathrm{S}_4}_{(1,3)(2,4)} \circ \varphi_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)}, \kappa^{\mathrm{S}_4}_{(1,3)(2,4)} \circ \varphi_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}\} \ . $$

Therefore, we obtain $\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4) =$

$$ \left\{ \left[\begin{array}{lll} a & \mapsto & (1,2)(3,4) \\ b & \mapsto & (1,3)(2,4) \end{array} \right], \left[\begin{array}{lll} a & \mapsto & (1,2) \\ b & \mapsto & (3,4) \end{array} \right], \left[\begin{array}{lll} a & \mapsto & (1,2)(3,4) \\ b & \mapsto & (1,2) \end{array} \right], \left[\begin{array}{lll} a & \mapsto & (1,2) \\ b & \mapsto & (1,2)(3,4) \end{array} \right] \right\} . $$

Ad (6). Let $\gamma : \mathrm{S}_3^{\mathrm{pres}} \to \mathrm{S}_4$ be an injective group morphism. Then it follows that $\mathrm{S}_3 \cong \mathrm{im}(\gamma) \in \{{}^g U_8 : g \in \mathrm{S}_4\}$. It suffices to consider $\mathrm{im}(\gamma) = U_8$.

Now, every group isomorphism $\mathrm{S}_3^{\mathrm{pres}} \to U_8$ is of the form $\omega \circ \gamma$ for $\omega \in \mathrm{Aut}(U_8)$. As $\mathrm{Aut}(U_8) = \mathrm{Inn}_{\mathrm{S}_4}(U_8)$, cf. Lemma 112, we obtain

$$ \overline{\mathrm{Inj}}(\mathrm{S}_3^{\mathrm{pres}}, \mathrm{S}_4) = \left\{ \left[\begin{array}{lll} r & \mapsto & (1,2,3) \\ s & \mapsto & (1,2) \end{array} \right] \right\} . $$

Ad (7). Let $\tau : \mathrm{D}_8 \to \mathrm{S}_4$ be an injective group morphism. Then it follows that $\mathrm{D}_8 \cong \mathrm{im}(\tau) \in \{{}^g U_9 : g \in \mathrm{S}_4\}$. It suffices to consider $\mathrm{im}(\tau) = U_9$.

Now, every group isomorphism $\mathrm{D}_8 \to U_9$ is of the form $\omega \circ \tau$ for $\omega \in \mathrm{Aut}(U_9)$.

By Lemma 114

$$ \mathrm{Aut}(U_9) = \{\mathrm{id}_{U_9}, \kappa^{\mathrm{S}_4}_{(1,3)}, \kappa^{\mathrm{S}_4}_{(1,2,3,4)}, \kappa^{\mathrm{S}_4}_{(1,2)(3,4)}, \varphi, \kappa^{\mathrm{S}_4}_{(1,3)} \circ \varphi, \kappa^{\mathrm{S}_4}_{(1,2,3,4)} \circ \varphi, \kappa^{\mathrm{S}_4}_{(1,2)(3,4)} \circ \varphi\} \ . $$

Therefore, we obtain

$$ \overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4) = \left\{ \left[\begin{array}{lll} d & \mapsto & (1,2,3,4) \\ t & \mapsto & (1,4)(2,3) \end{array} \right], \left[\begin{array}{lll} d & \mapsto & (1,2,3,4) \\ t & \mapsto & (2,4) \end{array} \right] \right\} . $$

Ad (8). Let $\mu : \mathrm{A}_4^{\mathrm{pres}} \to \mathrm{S}_4$ be an injective group morphism. Then it follows that $\mathrm{A}_4 \cong \mathrm{im}(\mu) \in \{{}^g U_{10} : g \in \mathrm{S}_4\} = \{U_{10}\}$. As $\mathrm{Aut}(U_{10}) = \mathrm{Inn}_{\mathrm{S}_4}(U_{10})$, cf. Lemma 112, we obtain

$$ \overline{\mathrm{Inj}}(\mathrm{A}_4, \mathrm{S}_4) = \left\{ \left[\begin{array}{lll} x & \mapsto & (1,2,3) \\ y & \mapsto & (1,2)(3,4) \end{array} \right] \right\} . $$

Ad (9). As $\mathrm{Inn}_{\mathrm{S}_4}(\mathrm{S}_4) = \mathrm{Aut}(\mathrm{S}_4)$, cf. Lemma 112, it follows that $\overline{\mathrm{Inj}}(\mathrm{S}_4, \mathrm{S}_4) = \{[\mathrm{id}_{\mathrm{S}_4}]\}$. $\square$

Corollary 120. *We may identify*

$$\begin{array}{rclcl}
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(1)}(\mathbf{Z}\overline{\mathrm{Inj}}(1,\mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(1,\mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2,\mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2,\mathrm{S}_4)) & = & \mathbf{Z}^{2\times 2}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_3)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3,\mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3,\mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_4)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_4,\mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_4,\mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{S}_3)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_3,\mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_3,\mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{A}_4)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{A}_4,\mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{A}_4,\mathrm{S}_4)) & = & \mathbf{Z}\ , \\
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{S}_4)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_4,\mathrm{S}_4)) & = & \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_4,\mathrm{S}_4)) & = & \mathbf{Z}\ ,
\end{array}$$

cf. Lemma 119.

Lemma 121. *We may identify* $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4)) = \mathbf{Z}^{2\times 2}$. *Then*

$$\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4)) = \{\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right) : u, v \in \mathbf{Z}\} \subseteq \mathbf{Z}^{2\times 2} = \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4))\ .$$

We have the ring isomorphism

$$\begin{array}{rcl}
\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4)) & \xrightarrow{\sim} & \{(a,b) \in \mathbf{Z}\times\mathbf{Z} : a \equiv_2 b\} \subseteq \mathbf{Z}\times\mathbf{Z} \\
\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right) & \mapsto & (u+v, u-v)\ .
\end{array}$$

Note that $\{(a,b) \in \mathbf{Z}\times\mathbf{Z} : a \equiv_2 b\}$ *is a subgroup of index* 2 *in* $\mathbf{Z}\times\mathbf{Z}$.

More symbolically, we write $\{(a,b) \in \mathbf{Z}\times\mathbf{Z} : a \equiv_2 b\} = \Big(\, \mathbf{Z} \overset{1}{\text{———}} 2 \overset{1}{\text{———}} \mathbf{Z} \,\Big)$.

Proof. Recall that

$$\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4) = \left\{ [\rho_1] := \begin{bmatrix} d & \mapsto & (1,2,3,4) \\ t & \mapsto & (1,4)(2,3) \end{bmatrix}, [\rho_2] := \begin{bmatrix} d & \mapsto & (1,2,3,4) \\ t & \mapsto & (2,4) \end{bmatrix} \right\}\ ,$$

cf. Lemma 119.

Hence, we may identify $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4)) = \mathbf{Z}^{2\times 2}$.

So, we have $\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4)) \subseteq \{\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) : u, v, w, x \in \mathbf{Z}\}$.

We have $\mathrm{Out}(\mathrm{D}_8) \cong \mathrm{C}_2$, cf. Lemma 114. More precisely, we define the group morphism

$$\begin{array}{rcl}
\alpha : \mathrm{D}_8 & \to & \mathrm{D}_8 \\
d & \mapsto & d \\
t & \mapsto & td\ .
\end{array}$$

It exists, as $d^4 = 1$, $(td)^2 = 1$ and $((td)d)^2 = td^2td^2 = d^{-2}t^2d^2 = 1$.

Moreover, it is surjective as $\mathrm{D}_8 = \langle t, td\rangle$. We have $\alpha \notin \mathrm{Inn}(\mathrm{D}_8)$ as for $i, j \geq 0$ we have

$${}^{d^i t^j}t = {}^{d^i}t = d^{2i}t \neq d^3t = td\ .$$

Thus, we have $\mathrm{Out}(\mathrm{D}_8) = \{\mathrm{id}\,\mathrm{Inn}(\mathrm{D}_8), \alpha\,\mathrm{Inn}(\mathrm{D}_8)\}$.

Consider the map

$$\begin{array}{rcl} \mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4) & \xrightarrow{\varphi} & \mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4) \\ [\rho_1] & \mapsto & [\rho_1 \circ \alpha] \\ [\rho_2] & \mapsto & [\rho_2 \circ \alpha] \ . \end{array}$$

Note that

$$[\rho_2 \circ \alpha] = \left[\begin{array}{ccccl} d & \mapsto & d & \mapsto & (1,2,3,4) \\ t & \mapsto & td & \mapsto & (2,4) \circ (1,2,3,4) = (14)(23) \end{array} \right] = [\rho_1]$$

and that

$$\begin{array}{rcl} [\rho_1 \circ \alpha] & = & [\kappa^{\mathrm{S}_4}_{(1,2,3,4)} \circ \rho_1 \circ \alpha] \\ & = & \left[\begin{array}{ccccccl} d & \mapsto & d & \mapsto & (1,2,3,4) & \mapsto & (1,2,3,4) \\ t & \mapsto & td & \mapsto & (1,4)(2,3) \circ (1,2,3,4) = (1,3) & \mapsto & (2,4) \end{array} \right] = [\rho_2] \ . \end{array}$$

Thus, we obtain with respect to the standard basis of $\mathbf{Z}^{2\times 2}$ and the basis $([\rho_1], [\rho_2])$ of $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4))$ the representing matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of φ.

Now, an element $\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \in \mathbf{Z}^{2\times 2} = \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(D_8, \mathrm{S}_4))$ is $\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)$-linear if and only if

$$\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \stackrel{!}{=} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \ ,$$

i.e.

$$\left(\begin{smallmatrix} v & u \\ x & w \end{smallmatrix}\right) \stackrel{!}{=} \left(\begin{smallmatrix} w & x \\ u & v \end{smallmatrix}\right) \ ,$$

i.e. $u = x$ and $v = w$.

This proves that $\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4)) = \{\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right) : u, v \in \mathbf{Z}\}$.

Consider the map

$$\begin{array}{rcl} f : \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4)) & \xrightarrow{\sim} & \{(a,b) \in \mathbf{Z} \times \mathbf{Z} : a \equiv_2 b\} \subseteq \mathbf{Z} \times \mathbf{Z} \\ \left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right) & \mapsto & (u+v, u-v) \ . \end{array}$$

The map f is a ring morphism as for $\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right), \left(\begin{smallmatrix} u' & v' \\ v' & u' \end{smallmatrix}\right) \in \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4))$ we have

$$\begin{array}{rcl} f(\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right) + \left(\begin{smallmatrix} u' & v' \\ v' & u' \end{smallmatrix}\right)) & = & (u + u' + v + v', u + u' - v - v') \quad = \quad f(\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right)) + f(\left(\begin{smallmatrix} u' & v' \\ v' & u' \end{smallmatrix}\right)) \\ f(\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} u' & v' \\ v' & u' \end{smallmatrix}\right)) & = & f(\left(\begin{smallmatrix} uu'+vv' & uv'+vu' \\ uv'+vu' & uu'+vv' \end{smallmatrix}\right)) \\ & = & (uu' + vv' + uv' + vu', uu' + vv' - uv' - vu') \\ & = & (u+v, u-v)(u'+v', u'-v') \\ & = & f(\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right)) \cdot f(\left(\begin{smallmatrix} u' & v' \\ v' & u' \end{smallmatrix}\right)) \\ f(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) & = & (1,1) \ . \end{array}$$

It is injective, as for $\left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} u' & v' \\ v' & u' \end{smallmatrix}\right) \in \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4))$ with

$$(u+v, u-v) = (u'+v', u'-v')$$

it follows that $u = u'$ and $v = v'$. Moreover, we have $f(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) = (1,1)$, $f(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)) = (2,0)$ and $f(\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)) = (0,2)$. So, f is bijective.

$\square$

Lemma 122. *We may identify* $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)) = \mathbf{Z}^{4\times 4}$. *Then*

$$\begin{aligned} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)) \quad &= \quad \left\{ \begin{pmatrix} u & v & v & v \\ w & x+y & y & y \\ w & y & x+y & y \\ w & y & y & x+y \end{pmatrix} : u, v, w, x, y \in \mathbf{Z} \right\} \\ &\subseteq \quad \mathbf{Z}^{4\times 4} = \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)) \ . \end{aligned}$$

We have the injective ring morphism

$$\begin{array}{ccc} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)) & \to & \mathbf{Z} \times \mathbf{Z}^{2\times 2} \\ \begin{pmatrix} u & v & v & v \\ w & x+y & y & y \\ w & y & x+y & y \\ w & y & y & x+y \end{pmatrix} & \mapsto & \left(x, \begin{pmatrix} u & 3v \\ w & x+3y \end{pmatrix}\right) \end{array} .$$

Note that its image $\{(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right)) : a, b, c, d, e \in \mathbf{Z},\ a \equiv_3 e,\ c \equiv_3 0\}$ *is a subgroup of index* 9 *in* $\mathbf{Z} \times \mathbf{Z}^{2\times 2}$.

More symbolically, we write $\{(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right)) : a, b, c, d, e \in \mathbf{Z},\ a \equiv_3 e,\ c \equiv_3 0\} \ = \ \begin{array}{cc} & \begin{bmatrix} \mathbf{Z} & (3) \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix} \\ \mathbf{Z} & \end{array}$.

$$\mathbf{Z} \quad \underset{1 \;\; 3 \;\; 1}{\frown} \quad \mathbf{Z}$$

Proof. Recall that $\mathrm{C}_2 \times \mathrm{C}_2 = \langle a : a^2 \rangle \times \langle b : b^2 \rangle$.

Recall that $\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4) =$

$$\left\{ \begin{array}{cc} [\nu_1] := \left[\begin{array}{ccc} a & \mapsto & (1,2)(3,4) \\ b & \mapsto & (1,3)(2,4) \end{array} \right], & [\nu_2] := \left[\begin{array}{ccc} a & \mapsto & (1,2) \\ b & \mapsto & (3,4) \end{array} \right], \\ [\nu_3] := \left[\begin{array}{ccc} a & \mapsto & (1,2)(3,4) \\ b & \mapsto & (1,2) \end{array} \right], & [\nu_4] := \left[\begin{array}{ccc} a & \mapsto & (1,2) \\ b & \mapsto & (1,2)(3,4) \end{array} \right] \end{array} \right\},$$

cf. Lemma 119.

Therefore, we may identify

$$\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)) = \mathbf{Z}^{4\times 4} \ .$$

We have

$$\mathrm{Out}(\mathrm{C}_2 \times \mathrm{C}_2) = \mathrm{Aut}(\mathrm{C}_2 \times \mathrm{C}_2) = \left\langle \alpha := \begin{pmatrix} a & \mapsto & b \\ b & \mapsto & a \end{pmatrix}, \beta := \begin{pmatrix} a & \mapsto & a \\ b & \mapsto & ab \end{pmatrix} \right\rangle .$$

One can identify $\mathrm{Aut}(\mathrm{C}_2 \times \mathrm{C}_2)$ with $\mathrm{GL}_2(\mathbf{F}_2)$ in which case α becomes $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and β becomes $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Consider the map

$$\begin{array}{ccc} \mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4) & \xrightarrow{\varphi_\alpha} & \mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4) \\ [\nu_1] & \mapsto & [\nu_1 \circ \alpha] \\ [\nu_2] & \mapsto & [\nu_2 \circ \alpha] \\ [\nu_3] & \mapsto & [\nu_3 \circ \alpha] \\ [\nu_4] & \mapsto & [\nu_4 \circ \alpha] \ . \end{array}$$

We have

$$[\kappa^{\mathrm{S}_4}_{(2,3)} \circ \nu_1 \circ \alpha] = \begin{bmatrix} a & \mapsto & b & \mapsto & (1,3)(2,4) & \mapsto & (2,3) \circ (1,3)(2,4) \circ (2,3) = (1,2)(3,4) \\ b & \mapsto & a & \mapsto & (1,2)(3,4) & \mapsto & (2,3) \circ (1,2)(3,4) \circ (2,3) = (1,3)(2,4) \end{bmatrix}$$

and therefore $[\nu_1 \circ \alpha] = [\nu_1]$.

We have

$$[\kappa^{\mathrm{S}_4}_{(1,3)(2,4)} \circ \nu_2 \circ \alpha] = \begin{bmatrix} a & \mapsto & b & \mapsto & (3,4) & \mapsto & (1,3)(2,4) \circ (3,4) \circ (1,3)(2,4) = (1,2) \\ b & \mapsto & a & \mapsto & (1,2) & \mapsto & (1,3)(2,4) \circ (1,2) \circ (1,3)(2,4) = (3,4) \end{bmatrix}$$

and therefore $[\nu_2 \circ \alpha] = [\nu_2]$.

We have

$$[\nu_3 \circ \alpha] = \begin{bmatrix} a & \mapsto & b & \mapsto & (1,2) \\ b & \mapsto & a & \mapsto & (1,2)(3,4) \end{bmatrix}$$

and therefore $[\nu_3 \circ \alpha] = [\nu_4]$.

We have

$$[\nu_4 \circ \alpha] = \begin{bmatrix} a & \mapsto & b & \mapsto & (1,2)(3,4) \\ b & \mapsto & a & \mapsto & (1,2) \end{bmatrix}$$

and therefore $[\nu_4 \circ \alpha] = [\nu_3]$.

Thus, we obtain with respect to the standard basis of $\mathbf{Z}^{4\times 4}$ and the basis $([\nu_1], [\nu_2], [\nu_3], [\nu_4])$ of $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4))$ the representing matrix $S := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ of φ_α.

Consider the map

$$\begin{array}{rcl} \mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4) & \xrightarrow{\varphi_\beta} & \mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4) \\ [\nu_1] & \mapsto & [\nu_1 \circ \beta] \\ [\nu_2] & \mapsto & [\nu_2 \circ \beta] \\ [\nu_3] & \mapsto & [\nu_3 \circ \beta] \\ [\nu_4] & \mapsto & [\nu_4 \circ \beta] \,. \end{array}$$

We have

$$[\kappa^{\mathrm{S}_4}_{(3,4)} \circ \nu_1 \circ \beta] =$$

$$\left[\begin{array}{ccccccc} a & \mapsto & a & \mapsto & (1,2)(3,4) & \mapsto & (3,4) \circ (1,2)(3,4) \circ (3,4) = (1,2)(3,4) \\ b & \mapsto & ab & \mapsto & (1,2)(3,4) \circ (1,3)(2,4) & \mapsto & (3,4) \circ (1,4)(2,3) \circ (3,4) = (1,3)(2,4) \end{array}\right]$$

and therefore $[\nu_1 \circ \beta] = [\nu_1]$.

We have

$$[\nu_2 \circ \beta] \quad = \quad \left[\begin{array}{ccccc} a & \mapsto & a & \mapsto & (1,2) \\ b & \mapsto & ab & \mapsto & (1,2)(3,4) \end{array}\right]$$

and therefore $[\nu_2 \circ \beta] = [\nu_4]$.

We have

$$[\kappa^{\mathrm{S}_4}_{(1,3)(2,4)} \circ \nu_3 \circ \beta] =$$

$$\left[\begin{array}{ccccccc} a & \mapsto & a & \mapsto & (1,2)(3,4) & \mapsto & (1,3)(2,4) \circ (1,2)(3,4) \circ (1,3)(2,4) = (1,2)(3,4) \\ b & \mapsto & ab & \mapsto & (3,4) & \mapsto & (1,3)(2,4) \circ (3,4) \circ (1,3)(2,4) = (1,2) \end{array}\right]$$

and therefore $[\nu_3 \circ \beta] = [\nu_3]$.

We have

$$[\nu_4 \circ \beta] \quad = \quad \left[\begin{array}{ccccc} a & \mapsto & a & \mapsto & (1,2) \\ b & \mapsto & ab & \mapsto & (3,4) \end{array}\right]$$

and therefore $[\nu_4 \circ \beta] = [\nu_2]$.

Thus, we obtain with respect to the standard basis of $\mathbf{Z}^{4\times 4}$and the basis $([\nu_1], [\nu_2], [\nu_3], [\nu_4])$ of $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4))$ the representing matrix $T := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ of φ_β.

Now, after identification, an element $\psi := \begin{pmatrix} j & k & l & m \\ n & p & q & r \\ s & t & u & v \\ w & x & y & z \end{pmatrix} \in \mathbf{Z}^{4\times 4} = \mathrm{End}_{\mathbf{Z}}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4))$

is $\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2 \times \mathrm{C}_2)$-linear if and only if $\psi \cdot S = S \cdot \psi$ and $\psi \cdot T = T \cdot \psi$.

The condition $\psi \cdot S = S \cdot \psi$, i.e.

$$\begin{pmatrix} j & k & m & l \\ n & p & r & q \\ s & t & v & u \\ w & x & z & y \end{pmatrix} = \begin{pmatrix} j & k & l & m \\ n & p & q & r \\ s & t & u & v \\ w & x & y & z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\overset{!}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} j & k & l & m \\ n & p & q & r \\ s & t & u & v \\ w & x & y & z \end{pmatrix} = \begin{pmatrix} j & k & l & m \\ n & p & q & r \\ w & x & y & z \\ s & t & u & v \end{pmatrix},$$

means $l = m$, $q = r$, $v = y$, $s = w$, $t = x$ and $u = z$.

The condition $\psi \cdot T = T \cdot \psi$, i.e.

$$\begin{pmatrix} j & m & l & k \\ n & r & q & p \\ s & v & u & t \\ w & z & y & x \end{pmatrix} = \begin{pmatrix} j & k & l & m \\ n & p & q & r \\ s & t & u & v \\ w & x & y & z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\overset{!}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} j & k & l & m \\ n & p & q & r \\ s & t & u & v \\ w & x & y & z \end{pmatrix} = \begin{pmatrix} j & k & l & m \\ w & x & y & z \\ s & t & u & v \\ n & p & q & r \end{pmatrix},$$

means $m = k$, $n = w$, $r = x$, $q = y$, $p = z$ and $v = t$.

So, our condition reads $k = m = l$, $n = s = w$, $p = z = u$ and $q = r = t = v = x = y$.

Hence, we obtain

$$\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2 \times \mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)) = \left\{ \begin{pmatrix} u & v & v & v \\ w & x+y & y & y \\ w & y & x+y & y \\ w & y & y & x+y \end{pmatrix} : u, v, w, x, y \in \mathbf{Z} \right\} \subseteq \mathbf{Z}^{4 \times 4}.$$

We *claim* that $\varepsilon := \dfrac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$ and $1 - \varepsilon = \dfrac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ are orthogonal central idempotents of

$$\mathrm{End}_{\mathbf{C}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{C}\overline{\mathrm{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4)) = \left\{\begin{pmatrix} u & v & v & v \\ w & x+y & y & y \\ w & y & x+y & y \\ w & y & y & x+y \end{pmatrix} : u,v,w,x,y\in\mathbf{C}\right\} \subseteq \mathbf{C}^{4\times 4}\,.$$

The centre of $\mathrm{End}_{\mathbf{C}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{C}\overline{\mathrm{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4))$ is given by

$$\left\{\begin{pmatrix} x+3y & 0 & 0 & 0 \\ 0 & x+y & y & y \\ 0 & y & x+y & y \\ 0 & y & y & x+y \end{pmatrix} : x,y\in\mathbf{C}\right\} ,$$

as the condition

$$\begin{aligned}\begin{pmatrix} u & 0 & 0 & 0 \\ w & 0 & 0 & 0 \\ w & 0 & 0 & 0 \\ w & 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} u & v & v & v \\ w & x+y & y & y \\ w & y & x+y & y \\ w & y & y & x+y \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\overset{!}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u & v & v & v \\ w & x+y & y & y \\ w & y & x+y & y \\ w & y & y & x+y \end{pmatrix} = \begin{pmatrix} u & v & v & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

means $v = w = 0$ and the condition

$$\begin{aligned}\begin{pmatrix} 0 & u & u & u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & x+y & y & y \\ 0 & y & x+y & y \\ 0 & y & y & x+y \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\overset{!}{=} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & x+y & y & y \\ 0 & y & x+y & y \\ 0 & y & y & x+y \end{pmatrix} = \begin{pmatrix} 0 & x+3y & x+3y & x+3y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

means $u = x + 3y$, and as the claimed subset consists of central elements.

Now, we search for a central idempotent.

The condition

$$\left(x\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+y\cdot\begin{pmatrix}3&0&0&0\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{pmatrix}\right)^2 = x^2\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+(2xy+3y^2)\cdot\begin{pmatrix}3&0&0&0\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{pmatrix}$$

$$\stackrel{!}{=} x\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+y\cdot\begin{pmatrix}3&0&0&0\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{pmatrix}$$

means $x \stackrel{!}{\in} \{0,1\}$ and $y \stackrel{!}{=} 2xy+3y^2$.

Suppose that $x := 1$. Then $y = 2y+3y^2$, i.e. $0 = y(1+3y)$. Choosing $y := -\frac{1}{3}$, we obtain the central idempotent $\varepsilon := \dfrac{1}{3}\begin{pmatrix}0&0&0&0\\0&2&-1&-1\\0&-1&2&-1\\0&-1&-1&2\end{pmatrix}$, which proves the *claim*.

We have

$$\varepsilon\,\mathrm{End}_{\mathbf{C}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{C}\overline{\mathrm{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4)) = \left\{\frac{1}{3}\begin{pmatrix}0&0&0&0\\0&2x&-x&-x\\0&-x&2x&-x\\0&-x&-x&2x\end{pmatrix} : x\in\mathbf{C}\right\}.$$

Hence, we have an isomorphism of $\mathbf{C}$-algebras

$$\begin{array}{rcl}\varepsilon\,\mathrm{End}_{\mathbf{C}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{C}\overline{\mathrm{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4)) & \xrightarrow{\sim} & \mathbf{C}\\ x\cdot\varepsilon & \mapsto & x\,.\end{array}$$

We have

$$\begin{aligned}(1-\varepsilon)\,\mathrm{End}_{\mathbf{C}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{C}\overline{\mathrm{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4)) &= \left\{\begin{pmatrix}u&v&v&v\\w&\frac{x}{3}+y&\frac{x}{3}+y&\frac{x}{3}+y\\w&\frac{x}{3}+y&\frac{x}{3}+y&\frac{x}{3}+y\\w&\frac{x}{3}+y&\frac{x}{3}+y&\frac{x}{3}+y\end{pmatrix} =: u,v,w,x,y\in\mathbf{C}\right\}\\ &= \left\{\begin{pmatrix}u&v&v&v\\w&y&y&y\\w&y&y&y\\w&y&y&y\end{pmatrix} : u,v,w,y\in\mathbf{C}\right\}.\end{aligned}$$

We have an isomorphism of $\mathbf{C}$-algebras

$$\begin{array}{rcl}\omega_{(1-\varepsilon)} : (1-\varepsilon)\operatorname{End}_{\mathbf{C}\operatorname{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{C}\overline{\operatorname{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4)) & \xrightarrow{\sim} & \mathbf{C}^{2\times 2} \\ \begin{pmatrix} u & v & v & v \\ w & y & y & y \\ w & y & y & y \\ w & y & y & y \end{pmatrix} & \mapsto & \begin{pmatrix} u & 3v \\ w & 3y \end{pmatrix}\,. \end{array}$$

In fact, $\omega_{(1-\varepsilon)}(1-\varepsilon) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3\cdot\frac{1}{3}\end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ and we always have

$$\begin{aligned} & \omega_{(1-\varepsilon)}\left(\begin{pmatrix} u & v & v & v \\ w & y & y & y \\ w & y & y & y \\ w & y & y & y \end{pmatrix}\cdot\begin{pmatrix} u' & v' & v' & v' \\ w' & y' & y' & y' \\ w' & y' & y' & y' \\ w' & y' & y' & y' \end{pmatrix}\right) \\ = \;& \omega_{(1-\varepsilon)}\left(\begin{pmatrix} uu'+3vw' & uv'+3vy' & uv'+3vy' & uv'+3vy' \\ wu'+3yw' & wv'+3yy' & wv'+3yy' & wv'+3yy' \\ wu'+3yw' & wv'+3yy' & wv'+3yy' & wv'+3yy' \\ wu'+3yw' & wv'+3yy' & wv'+3yy' & wv'+3yy' \end{pmatrix}\right) \\ = \;& \begin{pmatrix} uu'+3vw' & 3(uv'+3vy') \\ wu'+3yw' & 3(uv'+3yy') \end{pmatrix} \\ = \;& \begin{pmatrix} u & 3v \\ w & 3y \end{pmatrix}\cdot\begin{pmatrix} u' & 3v' \\ w' & 3y' \end{pmatrix}\,. \end{aligned}$$

Moreover, note that $\omega_{(1-\varepsilon)}$ is bijective.

So, we obtain the Wedderburn isomorphism

$$\begin{array}{rcl}\operatorname{End}_{\mathbf{C}\operatorname{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{C}\overline{\operatorname{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4)) & \xrightarrow{\sim} & \mathbf{C}\times\mathbf{C}^{2\times 2} \\ \begin{pmatrix} u & v & v & v \\ w & x+y & y & y \\ w & y & x+y & y \\ w & y & y & x+y \end{pmatrix} & \mapsto & \left(x, \left(\begin{smallmatrix} u & 3v \\ w & x+3y \end{smallmatrix}\right)\right)\,, \end{array}$$

which restricts to the embedding

$$\begin{array}{rcl}\operatorname{End}_{\mathbf{Z}\operatorname{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{Z}\overline{\operatorname{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4)) & \to & \mathbf{Z}\times\mathbf{Z}^{2\times 2} \\ \begin{pmatrix} u & v & v & v \\ w & x+y & y & y \\ w & y & x+y & y \\ w & y & y & x+y \end{pmatrix} & \mapsto & \left(x, \left(\begin{smallmatrix} u & 3v \\ w & x+3y \end{smallmatrix}\right)\right)\,. \end{array}$$

□

Remark 123. We have for subgroups U, V of a finite group G that

$$\mathrm{Fix}_V(G/U) = \{gU \in G/U : V^g \subseteq U\} ,$$

cf. Lemma 12.

The following table lists the values of $|\operatorname{Fix}_L(b)|$, where L runs through the twisted diagonal subgroups listed in the left column and where b runs through the $\mathbf{Z}$-linear basis $\mathcal{B}$ from Corollary 117, as listed in the upper row.

For calculation, we have used Magma [6].

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_7'	b_8	b_9	b_9'	b_{10}	b_{11}	$b_{3,2}$	$b_{2,3}$	$b_{5,7}$	$b_{7,5}$
$\Delta(U_1)$	576	288	288	192	144	144	144	144	96	72	72	48	24	288	288	144	144
$\Delta(U_2)$	0	8	0	0	0	0	8	4	8	4	0	0	4	0	0	0	0
$\Delta(U_3)$	0	0	32	0	48	16	16	0	0	24	8	16	8	0	0	16	16
$\Delta(U_4)$	0	0	0	6	0	0	0	0	3	0	0	6	3	0	0	0	0
$\Delta(U_5)$	0	0	0	0	24	0	0	0	0	12	0	8	4	0	0	0	0
$\Delta(U_6)$	0	0	0	0	0	8	0	0	0	4	4	0	4	0	0	0	0
$\Delta(U_7)$	0	0	0	0	0	0	8	0	0	4	0	0	4	0	0	0	0
$\Delta(U_7, \varphi_7, U_7)$	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0
$\Delta(U_8)$	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0
$\Delta(U_9)$	0	0	0	0	0	0	0	0	0	2	0	0	2	0	0	0	0
$\Delta(U_9, \varphi_9, U_9)$	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
$\Delta(U_{10})$	0	0	0	0	0	0	0	0	0	0	0	2	1	0	0	0	0
$\Delta(U_{11})$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\Delta(U_3, \varphi_{2,3}, U_2)$	0	0	0	0	0	0	0	8	0	0	8	0	0	16	0	16	0
$\Delta(U_2, \varphi_{3,2}, U_3)$	0	0	0	0	0	0	0	8	0	0	8	0	0	0	16	0	16
$\Delta(U_5, \varphi_{7,5}, U_7)$	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	8	0
$\Delta(U_7, \varphi_{5,7}, U_5)$	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	8

Remark 124. Recall that $\mathcal{T}_{\mathrm{S}_4} = \{1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_3^{\mathrm{pres}}, \mathrm{D}_8, \mathrm{A}_4^{\mathrm{pres}}, \mathrm{S}_4\}$, cf. Remark 118.

Recall the $\mathbf{Z}$-linear basis $\mathcal{B}$ of $\mathrm{B}^\Delta_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$, cf. Corollary 117.

The injective ring morphism $\sigma^\Delta_{\mathrm{S}_4,\mathrm{S}_4}$ from Theorem 108 decomposes into

$$\sigma^\Delta_{\mathrm{S}_4,\mathrm{S}_4} = (\pi_T \circ \sigma^\Delta_{\mathrm{S}_4,\mathrm{S}_4})_{T \in \mathcal{T}_{\mathrm{S}_4}} =: (\sigma^\Delta_{\mathrm{S}_4,\mathrm{S}_4,T})_{T \in \mathcal{T}_{\mathrm{S}_4}} .$$

Note that $\sigma^\Delta_{\mathrm{S}_4,\mathrm{S}_4,T}$ is a ring morphism for $T \in \mathcal{T}$.

Concerning $\overline{\mathrm{Inj}}(T, \mathrm{S}_4)$ for $T \in \mathcal{T}_{\mathrm{S}_4}$, cf. Lemma 119.

Ad $\sigma^\Delta_{\mathrm{S}_4,\mathrm{S}_4,1}$. Consider the map

$$\begin{array}{rcll} \sigma^\Delta_{\mathrm{S}_4,\mathrm{S}_4,1} : \mathrm{B}^\Delta_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \to & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(1)}(\mathbf{Z}\overline{\mathrm{Inj}}(1,\mathrm{S}_4)) & \overset{\text{C.120}}{=} \mathbf{Z} \\ [X] & \mapsto & \left([1 \mapsto \mathrm{id}] \mapsto \dfrac{|\operatorname{Fix}_{\Delta(U_1)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(1)|}[1 \mapsto \mathrm{id}] = \dfrac{|X|}{24}[1 \mapsto \mathrm{id}]\right) & = \dfrac{|X|}{24} . \end{array}$$

So, $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,1}$ maps

$$\begin{array}{rclrclrcl}
b_1 & \mapsto & 24, & b_2 & \mapsto & 12, & b_3 & \mapsto & 12,\\
b_4 & \mapsto & 8, & b_5 & \mapsto & 6, & b_6 & \mapsto & 6,\\
b_7 & \mapsto & 6, & b_7' & \mapsto & 6, & b_8 & \mapsto & 4,\\
b_9 & \mapsto & 3, & b_9' & \mapsto & 3, & b_{10} & \mapsto & 2,\\
b_{11} & \mapsto & 1, & b_{3,2} & \mapsto & 12, & b_{2,3} & \mapsto & 12,\\
b_{5,7} & \mapsto & 6, & b_{7,5} & \mapsto & 6, & & &
\end{array}$$

cf. Remark 123.

Ad $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2}$. Consider the map

$$\begin{array}{ccccc}
\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2}} & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2,\mathrm{S}_4)) & \overset{\mathrm{C}.120}{=} & \mathbf{Z}^{2\times 2}\\
[X] & \mapsto & \nu_{[X]} & = & \begin{pmatrix} \frac{|\operatorname{Fix}_{\Delta(U_2)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_2,\varphi_{3,2},U_3)}(X)|}{4}\\ \frac{|\operatorname{Fix}_{\Delta(U_3,\varphi_{2,3},U_2)}(X)|}{8} & \frac{|\operatorname{Fix}_{\Delta(U_3)}(X)|}{8}\end{pmatrix}
\end{array}$$

where $\nu_{[X]} :=$

$$\begin{pmatrix}
[c_2\mapsto(1,2)] & \mapsto & \frac{|\operatorname{Fix}_{\Delta(U_2)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_2)|}[c_2\mapsto(1,2)] + \frac{|\operatorname{Fix}_{\Delta(U_3,\varphi_{2,3},U_2)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_3)|}[c_2\mapsto(1,2)(3,4)]\\
[c_2\mapsto(1,2)(3,4)] & \mapsto & \frac{|\operatorname{Fix}_{\Delta(U_2,\varphi_{3,2},U_3)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_2)|}[c_2\mapsto(1,2)] + \frac{|\operatorname{Fix}_{\Delta(U_3)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_3)|}[c_2\mapsto(1,2)(3,4)]
\end{pmatrix}.$$

So, after identification, $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2}$ maps

$$\begin{array}{rclrclrcl}
b_1 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_2 & \mapsto & \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right), & b_3 & \mapsto & \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right),\\
b_4 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_5 & \mapsto & \left(\begin{smallmatrix}0&0\\0&6\end{smallmatrix}\right), & b_6 & \mapsto & \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right),\\
b_7 & \mapsto & \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right), & b_7' & \mapsto & \left(\begin{smallmatrix}1&2\\1&0\end{smallmatrix}\right), & b_8 & \mapsto & \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right),\\
b_9 & \mapsto & \left(\begin{smallmatrix}1&0\\0&3\end{smallmatrix}\right), & b_9' & \mapsto & \left(\begin{smallmatrix}0&2\\1&1\end{smallmatrix}\right), & b_{10} & \mapsto & \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right),\\
b_{11} & \mapsto & \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), & b_{3,2} & \mapsto & \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right), & b_{2,3} & \mapsto & \left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right),\\
b_{5,7} & \mapsto & \left(\begin{smallmatrix}0&0\\2&2\end{smallmatrix}\right), & b_{7,5} & \mapsto & \left(\begin{smallmatrix}0&4\\0&2\end{smallmatrix}\right), & & &
\end{array}$$

cf. Remark 123.

Ad $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_3}$. Consider the map

$$\begin{array}{ccccc}
\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_3}} & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_3)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_3,\mathrm{S}_4)) & \overset{\mathrm{C}.120}{=} & \mathbf{Z}\\
[X] & \mapsto & \left([c_3\mapsto(1,2,3)]\mapsto \frac{|\operatorname{Fix}_{\Delta(U_4)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_4)|}[c_3\mapsto(1,2,3)]\right) & = & \frac{|\operatorname{Fix}_{\Delta(U_4)}(X)|}{3}
\end{array}.$$

So, after identification, $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_3}$ maps

$$\begin{array}{rclrclrcl}
b_1 & \mapsto & 0, & b_2 & \mapsto & 0, & b_3 & \mapsto & 0, \\
b_4 & \mapsto & 2, & b_5 & \mapsto & 0, & b_6 & \mapsto & 0, \\
b_7 & \mapsto & 0, & b'_7 & \mapsto & 0, & b_8 & \mapsto & 1, \\
b_9 & \mapsto & 0, & b'_9 & \mapsto & 0, & b_{10} & \mapsto & 2, \\
b_{11} & \mapsto & 1, & b_{3,2} & \mapsto & 0, & b_{2,3} & \mapsto & 0, \\
b_{5,7} & \mapsto & 0, & b_{7,5} & \mapsto & 0, & & &
\end{array}$$

cf. Remark 123.

Ad $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_4}$. Consider the map

$$\begin{array}{ccccc}
\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_4}} & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_4)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_4,\mathrm{S}_4)) & \stackrel{\mathrm{C.120}}{=} & \mathbf{Z} \\
[X] & \mapsto & \left([c_4 \mapsto (1,2,3,4)] \mapsto \dfrac{|\operatorname{Fix}_{\Delta(U_6)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_6)|}[c_4 \mapsto (1,2,3,4)]\right) & = & \dfrac{|\operatorname{Fix}_{\Delta(U_6)}(X)|}{4}\ .
\end{array}$$

So, after identification, $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_4}$ maps

$$\begin{array}{rclrclrcl}
b_1 & \mapsto & 0, & b_2 & \mapsto & 0, & b_3 & \mapsto & 0, \\
b_4 & \mapsto & 0, & b_5 & \mapsto & 0, & b_6 & \mapsto & 2, \\
b_7 & \mapsto & 0, & b'_7 & \mapsto & 0, & b_8 & \mapsto & 0, \\
b_9 & \mapsto & 1, & b'_9 & \mapsto & 1, & b_{10} & \mapsto & 0, \\
b_{11} & \mapsto & 1, & b_{3,2} & \mapsto & 0, & b_{2,3} & \mapsto & 0, \\
b_{5,7} & \mapsto & 0, & b_{7,5} & \mapsto & 0, & & &
\end{array}$$

cf. Remark 123.

Ad $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2\times\mathrm{C}_2}$. Recall that

$$\overline{\mathrm{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4) = \left\{\begin{array}{l}
[\iota_1] \;:=\; \left[\begin{array}{lcl} a & \mapsto & (1,2)(3,4) \\ b & \mapsto & (1,3)(2,4) \end{array}\right], [\iota_2] \;:=\; \left[\begin{array}{lcl} a & \mapsto & (1,2) \\ b & \mapsto & (3,4) \end{array}\right], \\[4ex]
[\iota_3] \;:=\; \left[\begin{array}{lcl} a & \mapsto & (1,2)(3,4) \\ b & \mapsto & (1,2) \end{array}\right], [\iota_4] \;:=\; \left[\begin{array}{lcl} a & \mapsto & (1,2) \\ b & \mapsto & (1,2)(3,4) \end{array}\right]
\end{array}\right\},$$

cf. Lemma 119.

Consider the map

$$\begin{array}{ccccc}
\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2\times\mathrm{C}_2}} & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2\times\mathrm{C}_2,\mathrm{S}_4)) & \stackrel{\mathrm{L.122}}{\subseteq} & \mathbf{Z}^{4\times 4} \\
[X] & \mapsto & \nu_{[X]} & &
\end{array}$$

where

$$\nu_{[X]} := \left(\begin{array}{lll} [\iota_1] & \mapsto & \frac{|\operatorname{Fix}_{\Delta(U_5)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_5)|}[\iota_1] + \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7}\circ\kappa^{U_5}_{(1,3,2)},U_5)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_2] \\ & & +\frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7},U_5)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_3] + \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7}\circ\kappa^{U_5}_{(2,3)},U_5)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_4] \\ \\ [\iota_2] & \mapsto & \frac{|\operatorname{Fix}_{\Delta(U_5,\kappa^{U_5}_{(1,2,3)}\circ\varphi_{7,5},U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_5)|}[\iota_1] + \frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_2] \\ & & +\frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7,U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_3] + \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7\circ\kappa^{U_7}_{(1,3)(2,4)}),U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_4] \\ \\ [\iota_3] & \mapsto & \frac{|\operatorname{Fix}_{\Delta(U_5,\varphi_{7,5},U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_5)|}[\iota_1] + \frac{|\operatorname{Fix}_{\Delta(U_7,\kappa^{U_7}_{(1,3)(2,4)}\circ\varphi_7\circ\kappa^{U_7}_{(1,3)(2,4)},U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_2] \\ & & +\frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_3] + \frac{|\operatorname{Fix}_{\Delta(U_7,\kappa^{U_7}_{(1,3)(2,4))}\circ\varphi_7,U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_4] \\ \\ [\iota_4] & \mapsto & \frac{|\operatorname{Fix}_{\Delta(U_5,\kappa^{U_5}_{(2,3)}\circ\varphi_{7,5},U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_5)|}[\iota_1] + \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7\circ\kappa^{U_7}_{(1,3)(2,4)},U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_2] \\ & & +\frac{|\operatorname{Fix}_{\Delta(U_7,\kappa^{U_7}_{(1,3)(2,4)}\circ\varphi_7,U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_3] + \frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{|\operatorname{C}_{\mathrm{S}_4}(U_7)|}[\iota_4] \end{array} \right).$$

Let

$$\begin{pmatrix} \frac{|\operatorname{Fix}_{\Delta(U_5)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_5,\kappa^{U_5}_{(1,2,3)}\circ\varphi_{7,5},U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_5,\varphi_{7,5},U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_5,\kappa^{U_5}_{(2,3)}\circ\varphi_{7,5},U_7)}(X)|}{4} \\ \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7}\circ\kappa^{U_5}_{(1,3,2)},U_5)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\kappa^{U_7}_{(1,3)(2,4)}\circ\varphi_7\circ\kappa^{U_7}_{(1,3)(2,4)},U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7\circ\kappa^{U_7}_{(1,3)(2,4)},U_7)}(X)|}{4} \\ \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7},U_5)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7,U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\kappa^{U_7}_{(1,3)(2,4)}\circ\varphi_7,U_7)}(X)|}{4} \\ \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7}\circ\kappa^{U_5}_{(2,3)},U_5)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7\circ\kappa^{U_7}_{(1,3)(2,4)},U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\kappa^{U_7}_{(1,3)(2,4)}\circ\varphi_7,U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{|\operatorname{Fix}_{\Delta(U_5)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_5,\varphi_{7,5},U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_5,\varphi_{7,5},U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_5,\varphi_{7,5},U_7)}(X)|}{4} \\ \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7}}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7,U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7,U_7)}(X)|}{4} \\ \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7},U_5)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7,U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7,U_7)}(X)|}{4} \\ \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_{5,7},U_5)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7,U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7,\varphi_7,U_7)}(X)|}{4} & \frac{|\operatorname{Fix}_{\Delta(U_7)}(X)|}{4} \end{pmatrix}$$

be the representation matrix of $\nu_{[X]}$ with respect to $([\iota_1], [\iota_2], [\iota_3], [\iota_4])$.

Then $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2\times\mathrm{C}_2}$ maps

$$b_1 \mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad b_2 \mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad b_3 \mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$b_4 \mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad b_5 \mapsto \begin{pmatrix}6&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad b_6 \mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$b_7 \mapsto \begin{pmatrix}0&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}, \quad b_7' \mapsto \begin{pmatrix}0&0&0&0\\0&0&1&1\\0&1&0&1\\0&1&1&0\end{pmatrix}, \quad b_8 \mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$b_9 \mapsto \begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \quad b_9' \mapsto \begin{pmatrix}0&1&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\end{pmatrix}, \quad b_{10} \mapsto \begin{pmatrix}2&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$b_{11} \mapsto \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \quad b_{3,2} \mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad b_{2,3} \mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$b_{5,7} \mapsto \begin{pmatrix}0&2&2&2\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad b_{7,5} \mapsto \begin{pmatrix}0&0&0&0\\2&0&0&0\\2&0&0&0\\2&0&0&0\end{pmatrix},$$

cf. Remark 123.

With Lemma 122 we obtain that

$$\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4) \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2\times\mathrm{C}_2}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2\times\mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2\times\mathrm{C}_2, \mathrm{S}_4)) \to \mathbf{Z}\times\mathbf{Z}^{2\times 2}$$

maps

$$b_1 \mapsto \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) \quad b_2 \mapsto \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \quad b_3 \mapsto \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),$$

$$b_4 \mapsto \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), \quad b_5 \mapsto \left(0, \left(\begin{smallmatrix}6&0\\0&0\end{smallmatrix}\right)\right), \quad b_6 \mapsto \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),$$

$$b_7 \mapsto \left(2, \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right)\right), \quad b_7' \mapsto \left(-1, \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right)\right), \quad b_8 \mapsto \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right),$$

$$b_9 \mapsto \left(1, \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right), \quad b'_9 \mapsto \left(0, \left(\begin{smallmatrix} 0 & 3 \\ 1 & 0 \end{smallmatrix}\right)\right), \quad b_{10} \mapsto \left(0, \left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right),$$
$$b_{11} \mapsto \left(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right), \quad b_{3,2} \mapsto \left(0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right), \quad b_{2,3} \mapsto \left(0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right),$$
$$b_{5,7} \mapsto \left(0, \left(\begin{smallmatrix} 0 & 6 \\ 0 & 0 \end{smallmatrix}\right)\right), \quad b_{7,5} \mapsto \left(0, \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)\right).$$

Ad $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{S}_3^{\mathrm{pres}}}$. Consider the map

$$\begin{array}{ccccc} \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{S}_3^{\mathrm{pres}}}} & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{S}_3)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_3^{\mathrm{pres}},\mathrm{S}_4)) & \overset{\mathrm{C.120}}{=} & \mathbf{Z} \\ [X] & \mapsto & \left(\left[\begin{smallmatrix} r & \mapsto & (1,2,3) \\ s & \mapsto & (1,2) \end{smallmatrix}\right] \mapsto \dfrac{|\,\mathrm{Fix}_{\Delta(U_8)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_8)|}\left[\begin{smallmatrix} r & \mapsto & (1,2,3) \\ s & \mapsto & (1,2) \end{smallmatrix}\right]\right) & = & |\,\mathrm{Fix}_{\Delta(U_8)}(X)| \,. \end{array}$$

So, after identification, $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{S}_3^{\mathrm{pres}}}$ maps

$$\begin{array}{rclrclrcl} b_1 & \mapsto & 0, & b_2 & \mapsto & 0, & b_3 & \mapsto & 0, \\ b_4 & \mapsto & 0, & b_5 & \mapsto & 0, & b_6 & \mapsto & 0, \\ b_7 & \mapsto & 0, & b'_7 & \mapsto & 0, & b_8 & \mapsto & 1, \\ b_9 & \mapsto & 0, & b'_9 & \mapsto & 0, & b_{10} & \mapsto & 0, \\ b_{11} & \mapsto & 1, & b_{3,2} & \mapsto & 0, & b_{2,3} & \mapsto & 0, \\ b_{5,7} & \mapsto & 0, & b_{7,5} & \mapsto & 0\,. & & & \end{array}$$

Ad $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{D}_8}$. Recall that

$$\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4) = \left\{ [\rho_1] := \left[\begin{array}{ccc} d & \mapsto & (1,2,3,4) \\ t & \mapsto & (1,4)(2,3) \end{array}\right], [\rho_2] := \left[\begin{array}{ccc} d & \mapsto & (1,2,3,4) \\ t & \mapsto & (2,4) \end{array}\right] \right\},$$

cf. Lemma 119.

Consider the map

$$\begin{array}{ccccc} \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \xrightarrow{\sigma_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{D}_8}} & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8,\mathrm{S}_4)) & \overset{\mathrm{L.121}}{\subseteq} & \mathbf{Z}^{2\times 2} \\ [X] & \mapsto & \nu_{[X]}\ , & & \end{array}$$

where

$$\nu_{[X]} := \left(\begin{array}{ccc} [\rho_1] & \mapsto & \dfrac{|\,\mathrm{Fix}_{\Delta(U_9)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_9)|}[\rho_1] + \dfrac{|\,\mathrm{Fix}_{\Delta(U_9,\varphi_9,U_9)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_9)|}[\rho_2] \\ [\rho_2] & \mapsto & \dfrac{|\,\mathrm{Fix}_{\Delta(U_9,\kappa^{U_9}_{(1,4,3,2)}\circ\varphi_9,U_9)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_9)|}[\rho_1] + \dfrac{|\,\mathrm{Fix}_{\Delta(U_9)}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(U_9)|}[\rho_2] \end{array} \right).$$

Let $A_{[X]} =$

$$\begin{pmatrix} \frac{|\operatorname{Fix}_{\Delta(U_9)}(X)|}{2} & \frac{|\operatorname{Fix}_{\Delta(U_9,\kappa^{U_9}_{(1,4,3,2)}\circ\varphi_9,U_9)}(X)|}{2} \\ \frac{|\operatorname{Fix}_{\Delta(U_9,\varphi_9,U_9)}(X)|}{2} & \frac{|\operatorname{Fix}_{\Delta(U_9)}(X)|}{2} \end{pmatrix} = \begin{pmatrix} \frac{|\operatorname{Fix}_{\Delta(U_9)}(X)|}{2} & \frac{|\operatorname{Fix}_{\Delta(U_9,\varphi_9,U_9)}(X)|}{2} \\ \frac{|\operatorname{Fix}_{\Delta(U_9,\varphi_9,U_9)}(X)|}{2} & \frac{|\operatorname{Fix}_{\Delta(U_9)}(X)|}{2} \end{pmatrix}$$

be the representation matrix of $\nu_{[X]}$ with respect to $([\rho_1],[\rho_2])$.
So, after identification, $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{D}_8}$ maps

$$\begin{array}{lllllllll}
b_1 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_2 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_3 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
b_4 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_5 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_6 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
b_7 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b'_7 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_8 & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
b_9 & \mapsto & \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), & b'_9 & \mapsto & \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), & b_{10} & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
b_{11} & \mapsto & \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), & b_{3,2} & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_{2,3} & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
b_{5,7} & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), & b_{7,5} & \mapsto & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) . & & &
\end{array}$$

With Lemma 121 we obtain that

$$B^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{D}_8}} \operatorname{End}_{\mathbf{Z}\operatorname{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\operatorname{Inj}}(\mathrm{D}_8,\mathrm{S}_4)) \to \mathbf{Z}\times\mathbf{Z}$$

maps

$$\begin{array}{lllllllll}
b_1 & \mapsto & (0,0), & b_2 & \mapsto & (0,0), & b_3 & \mapsto & (0,0), \\
b_4 & \mapsto & (0,0), & b_5 & \mapsto & (0,0), & b_6 & \mapsto & (0,0), \\
b_7 & \mapsto & (0,0), & b'_7 & \mapsto & (0,0), & b_8 & \mapsto & (0,0), \\
b_9 & \mapsto & (1,1), & b'_9 & \mapsto & (1,-1), & b_{10} & \mapsto & (0,0), \\
b_{11} & \mapsto & (1,1), & b_{3,2} & \mapsto & (0,0), & b_{2,3} & \mapsto & (0,0), \\
b_{5,7} & \mapsto & (0,0), & b_{7,5} & \mapsto & (0,0) . & & &
\end{array}$$

Ad $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{A}_4^{\mathrm{pres}}}$. Consider the map

$$\begin{array}{ccccc}
B^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{A}_4^{\mathrm{pres}}}} & \operatorname{End}_{\mathbf{Z}\operatorname{Out}(\mathrm{A}_4^{\mathrm{pres}})}(\mathbf{Z}\overline{\operatorname{Inj}}(\mathrm{A}_4^{\mathrm{pres}},\mathrm{S}_4)) & \stackrel{\mathrm{C.120}}{=} & \mathbf{Z} \\
[X] & \mapsto & \left(\left[\begin{smallmatrix} x & \mapsto & (1,2,3) \\ y & \mapsto & (1,2)(3,4)\end{smallmatrix}\right] \mapsto \frac{|\operatorname{Fix}_{\Delta(U_{10})}(X)|}{|\mathrm{C}_{\mathrm{S}_4}(U_{10})|}\left[\begin{smallmatrix} x & \mapsto & (1,2,3) \\ y & \mapsto & (1,2)(3,4)\end{smallmatrix}\right]\right) & = & |\operatorname{Fix}_{\Delta(U_{10})}(X)| .
\end{array}$$

So, after identification, $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{A}_4^{\mathrm{pres}}}$ maps

$$\begin{array}{rclrclrcl} b_1 & \mapsto & 0, & b_2 & \mapsto & 0, & b_3 & \mapsto & 0, \\ b_4 & \mapsto & 0, & b_5 & \mapsto & 0, & b_6 & \mapsto & 0, \\ b_7 & \mapsto & 0, & b_7' & \mapsto & 0, & b_8 & \mapsto & 0, \\ b_9 & \mapsto & 0, & b_9' & \mapsto & 0, & b_{10} & \mapsto & 2, \\ b_{11} & \mapsto & 1, & b_{3,2} & \mapsto & 0, & b_{2,3} & \mapsto & 0, \\ b_{5,7} & \mapsto & 0, & b_{7,5} & \mapsto & 0\,. & & & \end{array}$$

Ad $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{S}_4}$. Consider the map

$$\begin{array}{ccccc} \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \xrightarrow{\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{S}_4}} & \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{S}_4)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{S}_4,\mathrm{S}_4)) & \overset{\mathrm{C.120}}{=} & \mathbf{Z} \\ [X] & \mapsto & \left([\mathrm{id}_{\mathrm{S}_4}] \mapsto \dfrac{|\,\mathrm{Fix}_{\Delta(U_{11})}(X)|}{|\,\mathrm{C}_{\mathrm{S}_4}(\mathrm{S}_4)|}[\mathrm{id}_{\mathrm{S}_4}]\right) & = & |\,\mathrm{Fix}_{\Delta(U_{11})}(X)|\,. \end{array}$$

So, after identification, $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{S}_4}$ maps

$$\begin{array}{rclrclrcl} b_1 & \mapsto & 0, & b_2 & \mapsto & 0, & b_3 & \mapsto & 0, \\ b_4 & \mapsto & 0, & b_5 & \mapsto & 0, & b_6 & \mapsto & 0, \\ b_7 & \mapsto & 0, & b_7' & \mapsto & 0, & b_8 & \mapsto & 0, \\ b_9 & \mapsto & 0, & b_9' & \mapsto & 0, & b_{10} & \mapsto & 0, \\ b_{11} & \mapsto & 1, & b_{3,2} & \mapsto & 0, & b_{2,3} & \mapsto & 0, \\ b_{5,7} & \mapsto & 0, & b_{7,5} & \mapsto & 0\,. & & & \end{array}$$

3.3.2 Congruences describing an isomorphic copy of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4)$

Write

$$\sigma := \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4} = (\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,1}, \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2}, \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_3}, \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_4}, \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{C}_2\times\mathrm{C}_2}, \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{S}_3}, \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{D}_8}, \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{A}_4}, \sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4,\mathrm{S}_4})\,,$$

cf. Remark 124.

We identify

$$\sigma : \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) \to \prod_{T\in\mathcal{T}_{\mathrm{S}_4}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) \to \Xi := \mathbf{Z}\times\mathbf{Z}^{2\times 2}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}^{2\times 2}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\,.$$

The first factor $\mathbf{Z}$ in Ξ belongs to $T = 1$. The second factor $\mathbf{Z}^{2\times 2}$ in Ξ belongs to $T = \mathrm{C}_2$. The third factor $\mathbf{Z}$ in Ξ belongs to $T = \mathrm{C}_3$. The fourth factor $\mathbf{Z}$ in Ξ belongs to $T = \mathrm{C}_4$. The fifth and the sixth factors $\mathbf{Z}\times\mathbf{Z}^{2\times 2}$ in Ξ belong to $T = \mathrm{C}_2\times\mathrm{C}_2$. The seventh factor $\mathbf{Z}$ in Ξ belongs to $T = \mathrm{S}_3$. The eighth and the ninth factors $\mathbf{Z}\times\mathbf{Z}$ in Ξ belong to $T = \mathrm{D}_8$. The tenth factor $\mathbf{Z}$ in Ξ belongs to $T = \mathrm{A}_4$. The eleventh factor $\mathbf{Z}$ in Ξ belongs to $T = \mathrm{S}_4$.

We have

$$
\begin{array}{rcccccccccccc}
\sigma\ :\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \to & \mathbf{Z} & \times\ \mathbf{Z}^{2\times 2} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z}^{2\times 2} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z} \\
b_1 & \mapsto & (24 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_2 & \mapsto & (12 & ,\ \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_3 & \mapsto & (12 & ,\ \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_4 & \mapsto & (8 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 2 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_5 & \mapsto & (6 & ,\ \left(\begin{smallmatrix}0&0\\0&6\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}6&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_6 & \mapsto & (6 & ,\ \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 2 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_7 & \mapsto & (6 & ,\ \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 2 & ,\ \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_7' & \mapsto & (6 & ,\ \left(\begin{smallmatrix}1&2\\1&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ -1 & ,\ \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_8 & \mapsto & (4 & ,\ \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right) & ,\ 1 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 1 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_9 & \mapsto & (3 & ,\ \left(\begin{smallmatrix}1&0\\0&3\end{smallmatrix}\right) & ,\ 0 & ,\ 1 & ,\ 1 & ,\ \left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right) & ,\ 0 & ,\ 1 & ,\ 1 & ,\ 0 & ,\ 0) \\
b_9' & \mapsto & (3 & ,\ \left(\begin{smallmatrix}0&2\\1&1\end{smallmatrix}\right) & ,\ 0 & ,\ 1 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&3\\1&0\end{smallmatrix}\right) & ,\ 0 & ,\ 1 & ,\ -1 & ,\ 0 & ,\ 0) \\
b_{10} & \mapsto & (2 & ,\ \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right) & ,\ 2 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 2 & ,\ 0) \\
b_{11} & \mapsto & (1 & ,\ \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) & ,\ 1 & ,\ 1 & ,\ 1 & ,\ \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) & ,\ 1 & ,\ 1 & ,\ 1 & ,\ 1 & ,\ 1) \\
b_{3,2} & \mapsto & (12 & ,\ \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_{2,3} & \mapsto & (12 & ,\ \left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_{5,7} & \mapsto & (6 & ,\ \left(\begin{smallmatrix}0&0\\2&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_{7,5} & \mapsto & (6 & ,\ \left(\begin{smallmatrix}0&4\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0)\ .
\end{array}
$$

Let $x := (1\ ,\ \left(\begin{smallmatrix}3&2\\1&1\end{smallmatrix}\right)\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ \left(\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right)\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1)$. We define the ring morphism σ' by $\sigma'(b) = x\cdot\sigma(b)\cdot x^{-1}$ for $b \in \mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4)$

$$
\begin{array}{rcccccccccccc}
\sigma' :\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4) & \to & \mathbf{Z} & \times\ \mathbf{Z}^{2\times 2} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z}^{2\times 2} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z} & \times\ \mathbf{Z} \\
b_1 & \mapsto & (24 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_2 & \mapsto & (12 & ,\ \left(\begin{smallmatrix}6&-12\\2&-4\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_3 & \mapsto & (12 & ,\ \left(\begin{smallmatrix}-8&24\\-4&12\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_4 & \mapsto & (8 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 2 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_5 & \mapsto & (6 & ,\ \left(\begin{smallmatrix}-12&36\\-6&18\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}6&-18\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_6 & \mapsto & (6 & ,\ \left(\begin{smallmatrix}-4&12\\-2&6\end{smallmatrix}\right) & ,\ 0 & ,\ 2 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_7 & \mapsto & (6 & ,\ \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 2 & ,\ \left(\begin{smallmatrix}0&6\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_7' & \mapsto & (6 & ,\ \left(\begin{smallmatrix}-1&8\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ -1 & ,\ \left(\begin{smallmatrix}0&6\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_8 & \mapsto & (4 & ,\ \left(\begin{smallmatrix}6&-12\\2&-4\end{smallmatrix}\right) & ,\ 1 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 1 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_9 & \mapsto & (3 & ,\ \left(\begin{smallmatrix}-3&12\\-2&7\end{smallmatrix}\right) & ,\ 0 & ,\ 1 & ,\ 1 & ,\ \left(\begin{smallmatrix}3&-6\\0&1\end{smallmatrix}\right) & ,\ 0 & ,\ 1 & ,\ 1 & ,\ 0 & ,\ 0) \\
b_9' & \mapsto & (3 & ,\ \left(\begin{smallmatrix}-6&20\\-2&7\end{smallmatrix}\right) & ,\ 0 & ,\ 1 & ,\ 0 & ,\ \left(\begin{smallmatrix}3&-6\\1&-3\end{smallmatrix}\right) & ,\ 0 & ,\ 1 & ,\ -1 & ,\ 0 & ,\ 0) \\
b_{10} & \mapsto & (2 & ,\ \left(\begin{smallmatrix}-4&12\\-2&6\end{smallmatrix}\right) & ,\ 2 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}2&-6\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 2 & ,\ 0) \\
b_{11} & \mapsto & (1 & ,\ \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) & ,\ 1 & ,\ 1 & ,\ 1 & ,\ \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) & ,\ 1 & ,\ 1 & ,\ 1 & ,\ 1 & ,\ 1) \\
b_{3,2} & \mapsto & (12 & ,\ \left(\begin{smallmatrix}4&-8\\2&-4\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_{2,3} & \mapsto & (12 & ,\ \left(\begin{smallmatrix}-12&36\\-4&12\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_{5,7} & \mapsto & (6 & ,\ \left(\begin{smallmatrix}0&4\\0&2\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0) \\
b_{7,5} & \mapsto & (6 & ,\ \left(\begin{smallmatrix}-16&48\\-6&18\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ \left(\begin{smallmatrix}6&-18\\2&-6\end{smallmatrix}\right) & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0 & ,\ 0)\ .
\end{array}
$$

Remark 125. The conjugating element $x := (1, \left(\begin{smallmatrix} 3 & 2 \\ 1 & 1 \end{smallmatrix}\right), 1, 1, 1, \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right), 1, 1, 1, 1, 1) \in \Xi$ was constructed using the computer algebra system Magma [6] for the purpose of simplifying the congruences in $\sigma(\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4))$.

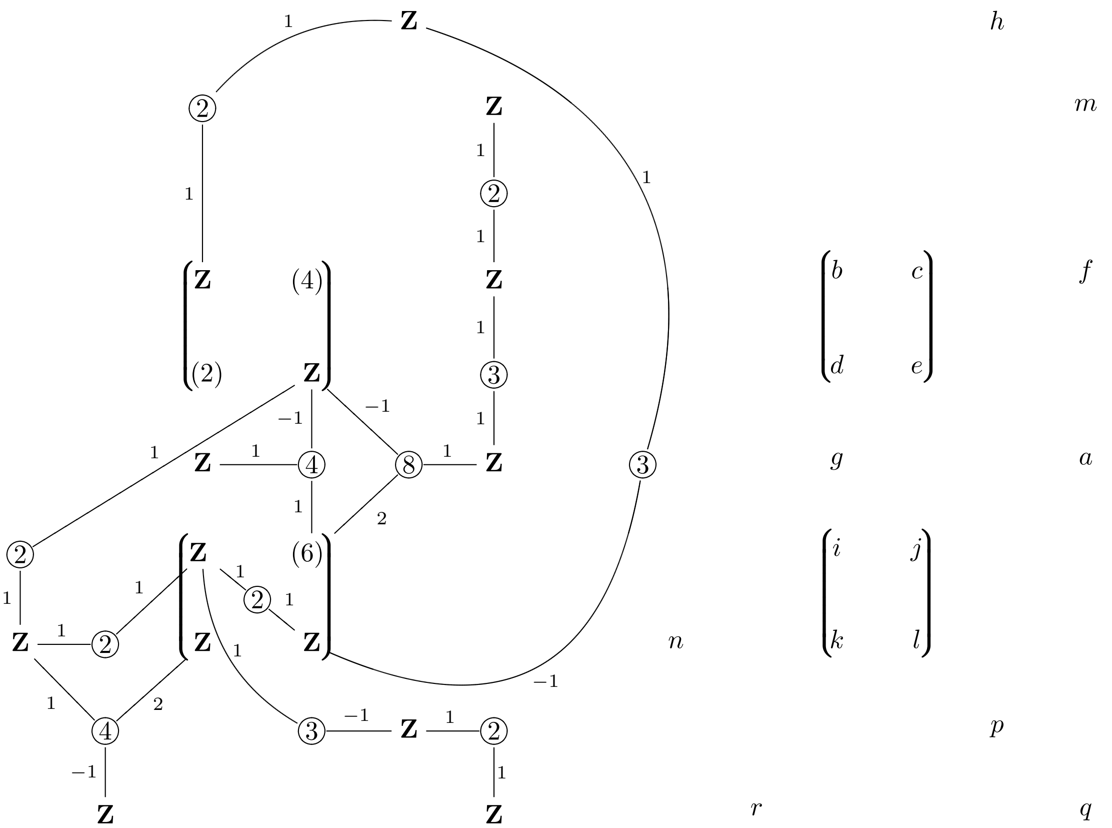
Theorem 126. *Recall* $\Xi = \mathbf{Z} \times \mathbf{Z}^{2\times 2} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$. *Let*

$$\Lambda := \left\{ (a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f, g, h, \left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right), m, n, r, p, q) \in \Xi : \begin{array}{rclclcl} e - a & \equiv_8 & 2j & \equiv_4 & 0 \\ e - g & \equiv_4 & j \\ n - r & \equiv_4 & 2k \\ c & \equiv_4 & 0 \\ d & \equiv_2 & 0 \\ n & \equiv_2 & l & \equiv_2 & e & \equiv_2 & i \\ h & \equiv_2 & b \\ p & \equiv_2 & q \\ m & \equiv_2 & f \\ a & \equiv_3 & f \\ p & \equiv_3 & i \\ h & \equiv_3 & l \\ j & \equiv_3 & 0 \end{array} \right\} .$$

We have a ring isomorphism

$$\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4) \xrightarrow{\sim} \Lambda \subseteq \Xi \; .$$

More symbolically written, we may describe Λ as follows. The letters to the right are the key to this picture.



Herein

$$\begin{tikzcd} \mathbf{Z} \arrow[r, dash, "t"] & \textcircled{s} \arrow[r, dash, "v"] \arrow[d, dash, "u"] & \mathbf{Z} & & x & & z \\ & \mathbf{Z} & & & & y & \end{tikzcd}$$

means $t \cdot x + u \cdot y + v \cdot z \equiv_s 0$, etc.

Proof. Let A be the representation matrix of σ' with respect to the bases $\mathcal{B}$ of $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)$, cf. Corollary 117, and the standard basis of Ξ. We obtain

$$A \;=\; \begin{pmatrix}
24 & 12 & 12 & 8 & 6 & 6 & 6 & 6 & 4 & 3 & 3 & 2 & 1 & 12 & 12 & 6 & 6 \\
0 & 6 & -8 & 0 & -12 & -4 & 2 & -1 & 6 & -3 & -6 & -4 & 1 & 4 & -12 & 0 & -16 \\
0 & -12 & 24 & 0 & 36 & 12 & 0 & 8 & -12 & 12 & 20 & 12 & 0 & -8 & 36 & 4 & 48 \\
0 & 2 & -4 & 0 & -6 & -2 & 0 & 0 & 2 & -2 & -2 & -2 & 0 & 2 & -4 & 0 & -6 \\
0 & -4 & 12 & 0 & 18 & 6 & 2 & 2 & -4 & 7 & 7 & 6 & 1 & -4 & 12 & 2 & 18 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 3 & 3 & 2 & 1 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & -18 & 0 & 6 & 6 & 0 & -6 & -6 & -6 & 0 & 0 & 0 & 6 & -18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & -3 & 0 & 1 & 0 & 0 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$24A^{-1} \;=\; \begin{pmatrix}
1 & 0 & 0 & -24 & -9 & -4 & 0 & 0 & 8 & 2 & 0 & 0 & 12 & 0 & 0 & 4 & -12 \\
0 & 36 & 12 & -72 & -24 & 0 & 0 & -4 & 0 & 0 & -24 & -8 & -24 & 0 & 0 & 0 & 24 \\
0 & -12 & -6 & 36 & 18 & 0 & -6 & -4 & -12 & -2 & 24 & 4 & 0 & 12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & -12 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & -12 & 0 & 0 & -6 & -6 & -4 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 12 & 4 & 0 & -6 & -6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 24 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 & -24 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & -24 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & -12 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & -12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \\
0 & -36 & -12 & 108 & 36 & 0 & 0 & 4 & -12 & -4 & 24 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 6 & -24 & -12 & 0 & 0 & 4 & 0 & 0 & -24 & -4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 4 & -36 & -12 & 0 & -6 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & -6 & 6 & 0 & 0
\end{pmatrix} .$$

Let $\xi := (a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f, g, h, \left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right), m, n, r, p, q) \in \Xi$. We have

$$\xi \in \sigma'(\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4, \mathrm{S}_4)) \quad \Leftrightarrow \quad \exists x \in \mathbf{Z}^{17\times 1} \text{ with } \begin{pmatrix} a \\ b \\ c \\ \vdots \\ q \end{pmatrix} = Ax \Leftrightarrow A^{-1} \cdot \begin{pmatrix} a \\ b \\ c \\ \vdots \\ q \end{pmatrix} \in \mathbf{Z}^{17\times 1}$$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 8 & 0 & 0 & 0 & 10 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 0 & 6 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 10 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 12 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 4 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \\ m \\ n \\ r \\ p \\ q \end{pmatrix} \in 24\mathbf{Z}^{12\times 1}$$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \\ m \\ n \\ r \\ p \\ q \end{pmatrix} \in 8\mathbf{Z}^{12\times 1}$$

$$\text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \\ k \\ l \\ m \\ n \\ r \\ p \\ q \end{pmatrix} \in 3\mathbf{Z}^{4\times 1}$$

$$\Leftrightarrow \left\{ \begin{aligned} a+3e+2j+4l &\equiv_8 0 \\ 2r+2n+4k+4l &\equiv_8 0 \\ 4p+4q &\equiv_8 0 \\ 4m+4f &\equiv_8 0 \\ 2g+2e+2j+4l &\equiv_8 0 \\ 4h+4b &\equiv_8 0 \\ 4n+4l &\equiv_8 0 \\ 2c &\equiv_8 0 \\ 4d &\equiv_8 0 \\ 4e+4l &\equiv_8 0 \\ 4i+4l &\equiv_8 0 \\ 4j &\equiv_8 0 \\ a+2f &\equiv_3 0 \\ p+2i &\equiv_3 0 \\ h+2l &\equiv_3 0 \\ j &\equiv_3 0 \end{aligned} \right. \Leftrightarrow \left\{ \begin{aligned} e-a &\equiv_8 2j \equiv_4 0 \\ e-g &\equiv_4 j \\ n-r &\equiv_4 2k \\ c &\equiv_4 0 \\ d &\equiv_2 0 \\ n &\equiv_2 l \equiv_2 e \equiv_2 i \\ h &\equiv_2 b \\ p &\equiv_2 q \\ m &\equiv_2 f \\ a &\equiv_3 f \\ p &\equiv_3 i \\ h &\equiv_3 l \\ j &\equiv_3 0 \end{aligned} \right.$$

□

Remark 127. Choose

$$\mathcal{L}^{\Delta}_{\mathrm{S}_4\times\mathrm{S}_4} = \left\{ \begin{array}{llll} \Delta(U_1), & \Delta(U_2), & \Delta(U_3), & \Delta(U_4), \\ \Delta(U_5), & \Delta(U_6), & \Delta(U_7), & \Delta(U_7,\varphi_7,U_7), \\ \Delta(U_8), & \Delta(U_9), & \Delta(U_9,\varphi_9,U_9), & \Delta(U_{10}), \\ \Delta(U_{11}), & \Delta(U_3,\varphi_{2,3},U_2), & \Delta(U_2,\varphi_{3,2},U_3) & \Delta(U_5,\varphi_{7,5},U_7), \\ \Delta(U_7,\varphi_{5,7},U_5) & & & \end{array} \right\},$$

cf. Corollary 117, as set of representatives of conjugacy classes of twisted diagonal subgroups of $\mathrm{S}_4\times\mathrm{S}_4$. By Lemma 12(3) we know that for $L\in\Delta_{\mathrm{S}_4\times\mathrm{S}_4}$ we have

$$\mathrm{Fix}_L((\mathrm{S}_4\times\mathrm{S}_4)/L) \;=\; \mathrm{N}_{\mathrm{S}_4\times\mathrm{S}_4}(L)/L\ .$$

Theorem 108(5) states that the index of Λ in $\prod\limits_{T\in\mathcal{T}_{\mathrm{S}_4}}\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G))$ is given by

$$\begin{aligned} & \prod_{\Delta(U,\alpha,V)\in\mathcal{L}^{\Delta}_{\mathrm{S}_4\times\mathrm{S}_4}} \frac{[\mathrm{N}_{\mathrm{S}_4\times\mathrm{S}_4}(\Delta(U,\alpha,V)) : \Delta(U,\alpha,V)]}{|\,\mathrm{C}_{\mathrm{S}_4}(U)|} \\ = & \prod_{\Delta(U,\alpha,V)\in\mathcal{L}^{\Delta}_{\mathrm{S}_4\times\mathrm{S}_4}} \frac{|\,\mathrm{Fix}_{\Delta(U,\alpha,V)}((\mathrm{S}_4\times\mathrm{S}_4)/\Delta(U,\alpha,V))|}{|\,\mathrm{C}_{\mathrm{S}_4}(U)|} \\ \overset{\mathrm{R.123}}{=} & \; 24\cdot 2\cdot 4\cdot 2\cdot 6\cdot 2\cdot 2\cdot 1\cdot 1\cdot 1\cdot 1\cdot 2\cdot 1\cdot 2\cdot 4\cdot 2\cdot 2 \\ = & \; 589824\ . \end{aligned}$$

We will verify this by a direct calculation. We have

$$
\begin{aligned}
[\Xi : \Lambda] \;&=\; \left|\det \begin{pmatrix}
24 & 12 & 12 & 8 & 6 & 6 & 6 & 6 & 4 & 3 & 3 & 2 & 1 & 12 & 12 & 6 & 6 \\
0 & 6 & -8 & 0 & -12 & -4 & 2 & -1 & 6 & -3 & -6 & -4 & 1 & 4 & -12 & 0 & -16 \\
0 & -12 & 24 & 0 & 36 & 12 & 0 & 8 & -12 & 12 & 20 & 12 & 0 & -8 & 36 & 4 & 48 \\
0 & 2 & -4 & 0 & -6 & -2 & 0 & 0 & 2 & -2 & -2 & -2 & 0 & 2 & -4 & 0 & -6 \\
0 & -4 & 12 & 0 & 18 & 6 & 2 & 2 & -4 & 7 & 7 & 6 & 1 & -4 & 12 & 2 & 18 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 3 & 3 & 2 & 1 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & -18 & 0 & 6 & 6 & 0 & -6 & -6 & -6 & 0 & 0 & 0 & 6 & -18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & -3 & 0 & 1 & 0 & 0 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}\right| \\
&=\; 10616832\,.
\end{aligned}
$$

Moreover, we know that

$$\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{D}_8)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{D}_8, \mathrm{S}_4)) = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} : a \equiv_2 b\}$$

is a subgroup of index 2 in $\mathbf{Z} \times \mathbf{Z}$, cf. Lemma 121 and that

$$\mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(\mathrm{C}_2 \times \mathrm{C}_2)}(\mathbf{Z}\overline{\mathrm{Inj}}(\mathrm{C}_2 \times \mathrm{C}_2, \mathrm{S}_4)) = \{(a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right)) : a, b, c, d, e \in \mathbf{Z},\ a \equiv_3 e,\ c \equiv_3 0\}$$

is a subgroup of index 9 in $\mathbf{Z} \times \mathbf{Z}^{2\times 2}$, cf. Lemma 122.

So, $[\prod_{T \in \mathcal{T}_{\mathrm{S}_4}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T, G)) : \Lambda] = \dfrac{[\Xi : \Lambda]}{2 \cdot 9} = \dfrac{10616832}{18} = 589824.$

3.3.3 Localisation at 2: $\mathrm{B}^{\Delta}_{\mathbf{Z}_{(2)}}(\mathrm{S}_4, \mathrm{S}_4)$

Write $R := \mathbf{Z}_{(2)}$.

3.3.3.1 Congruences describing the image of $\mathrm{B}^{\Delta}_{\mathbf{Z}_{(2)}}(\mathrm{S}_4, \mathrm{S}_4)$

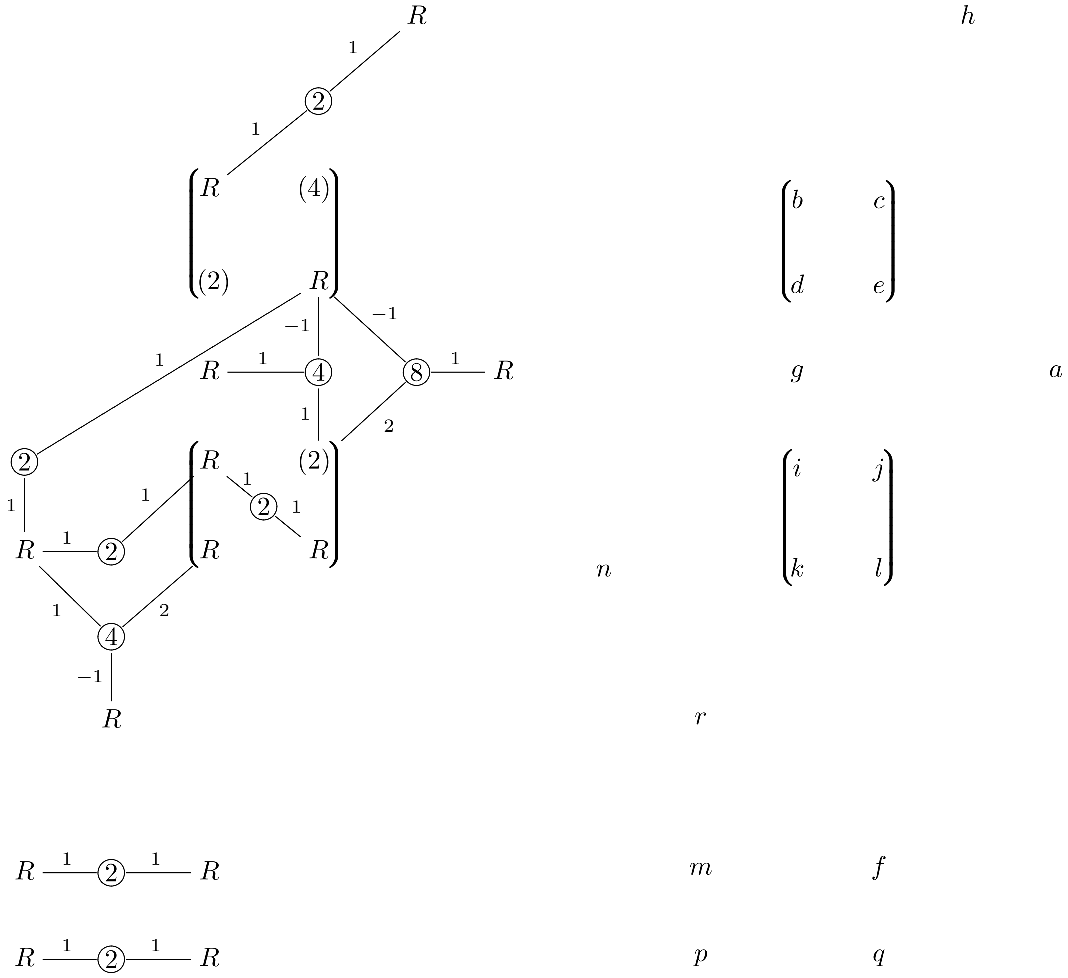
Recall Λ from Theorem 126.

Write $\Xi_{(2)} := R \times R^{2\times 2} \times R \times R \times R \times R^{2\times 2} \times R \times R \times R \times R \times R$.

Corollary 128. *We have*

$$
\Lambda_{(2)} = \left\{ \begin{array}{l} (a, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), f, g, h, \left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right), m, n, r, p, q) \in \Xi_{(2)} : \begin{array}{rclclcl}
e - a & \equiv_8 & 2j & \equiv_4 & 0 & & \\
e - g & \equiv_4 & j & & & & \\
n - r & \equiv_4 & 2k & & & & \\
c & \equiv_4 & 0 & & & & \\
d & \equiv_2 & 0 & & & & \\
n & \equiv_2 & l & \equiv_2 & e & \equiv_2 & i \\
h & \equiv_2 & b & & & & \\
p & \equiv_2 & q & & & & \\
m & \equiv_2 & f & & & &
\end{array} \end{array} \right\}.
$$

More symbolically written, we may describe $\Lambda_{(2)}$ as follows. The letters to the right are the key to this picture.



Remark 129. *We have the orthogonal decomposition* $1_{\Lambda_{(2)}} = \tilde{\mathrm{e}}_1+\tilde{\mathrm{e}}_2+\tilde{\mathrm{e}}_3+\tilde{\mathrm{e}}_4$ *into primitive idempotent elements of* $\Lambda_{(2)}$*, where*

$$
\begin{aligned}
\tilde{\mathrm{e}}_1 &:= \left(0, \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), 0, 0, 1, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0\right)\\
\tilde{\mathrm{e}}_2 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 1, 1\right)\\
\tilde{\mathrm{e}}_3 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 1, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 1, 0, 0, 0, 0\right)\\
\tilde{\mathrm{e}}_4 &:= \left(1, \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), 0, 1, 0, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), 0, 1, 1, 0, 0\right)\ .
\end{aligned}
$$

Proof. As R-algebras, we have

$$
\begin{aligned}
\tilde{\mathrm{e}}_1\Lambda_{(2)}\tilde{\mathrm{e}}_1 &\cong \{(b,h) : b,h \in R,\ b \equiv_2 h\}\\
\tilde{\mathrm{e}}_2\Lambda_{(2)}\tilde{\mathrm{e}}_2 &\cong \{(p,q) : p,q \in R,\ p \equiv_2 q\}\\
\tilde{\mathrm{e}}_3\Lambda_{(2)}\tilde{\mathrm{e}}_3 &\cong \{(f,m) : f,m \in R,\ f \equiv_2 m\}
\end{aligned}
$$

$$\tilde{\mathrm{e}}_4\Lambda_{(2)}\tilde{\mathrm{e}}_4 \;\cong\; \left\{ (a,e,g,\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right),n,r) : a,e,g,i,j,k,l,n,r \in R, \begin{array}{rcl} e-a & \equiv_8 & 2j \equiv_4 0 \\ e-g & \equiv_4 & j \\ n-r & \equiv_4 & 2k \\ n & \equiv_2 & l \equiv_2 e \equiv_2 i \end{array} \right\} =: \Gamma\ .$$

For $s \in \{1,2,3\}$ the ring $\tilde{\mathrm{e}}_s\Lambda_{(2)}\tilde{\mathrm{e}}_s$ is local. Hence, $\tilde{\mathrm{e}}_s$ is primitive for $s \in \{1,2,3\}$.

To show that $\tilde{\mathrm{e}}_4$ is primitive, we show that $\tilde{\mathrm{e}}_4\Lambda_{(2)}\tilde{\mathrm{e}}_4$ is local, i.e. that Γ is local.

We have the R-linear basis of Γ

$$\begin{array}{rcl} 1_\Gamma & := & (1,1,1,\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),1,1) \\ \tilde{\gamma}_1 & := & (4,0,2,\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right),0,0) \\ \tilde{\gamma}_2 & := & (2,2,2,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),0,0) \\ \tilde{\gamma}_3 & := & (8,0,4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),0,0) \\ \tilde{\gamma}_4 & := & (0,8,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),0,0) \\ \tilde{\gamma}_5 & := & (0,0,0,\left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right),0,0) \\ \tilde{\gamma}_6 & := & (0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right),0,0) \\ \tilde{\gamma}_7 & := & (0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),4,0) \\ \tilde{\gamma}_8 & := & (0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right),2,0)\ . \end{array}$$

We claim that the Jacobson radical of Γ is given by

$$J := {}_R\langle 2\cdot 1_\Gamma, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4, \tilde{\gamma}_5, \tilde{\gamma}_6, \tilde{\gamma}_7, \tilde{\gamma}_8\rangle\ .$$

For that, the multiplication table for the basis elements is given by

$(\cdot)$	1_Γ	$\tilde{\gamma}_1$	$\tilde{\gamma}_2$	$\tilde{\gamma}_3$	$\tilde{\gamma}_4$	$\tilde{\gamma}_5$	$\tilde{\gamma}_6$	$\tilde{\gamma}_7$	$\tilde{\gamma}_8$
1_Γ	1_Γ	$\tilde{\gamma}_1$	$\tilde{\gamma}_2$	$\tilde{\gamma}_3$	$\tilde{\gamma}_4$	$\tilde{\gamma}_5$	$\tilde{\gamma}_6$	$\tilde{\gamma}_7$	$\tilde{\gamma}_8$
$\tilde{\gamma}_1$	$\tilde{\gamma}_1$	$-4\tilde{\gamma}_2+3\tilde{\gamma}_3+\tilde{\gamma}_4$	$\tilde{\gamma}_3$	$-8\tilde{\gamma}_2+6\tilde{\gamma}_3+2\tilde{\gamma}_4$	0	0	$2\tilde{\gamma}_1-\tilde{\gamma}_3$	0	$\tilde{\gamma}_5$
$\tilde{\gamma}_2$	$\tilde{\gamma}_2$	$\tilde{\gamma}_3$	$2\tilde{\gamma}_2$	$2\tilde{\gamma}_3$	$2\tilde{\gamma}_4$	0	0	0	0
$\tilde{\gamma}_3$	$\tilde{\gamma}_3$	$-8\tilde{\gamma}_2+6\tilde{\gamma}_3+2\tilde{\gamma}_4$	$2\tilde{\gamma}_3$	$-16\tilde{\gamma}_2+12\tilde{\gamma}_3+4\tilde{\gamma}_4$	0	0	0	0	0
$\tilde{\gamma}_4$	$\tilde{\gamma}_4$	0	$2\tilde{\gamma}_4$	0	$8\tilde{\gamma}_4$	0	0	0	0
$\tilde{\gamma}_5$	$\tilde{\gamma}_5$	$2\tilde{\gamma}_1-\tilde{\gamma}_3$	0	0	0	$2\tilde{\gamma}_5$	0	0	0
$\tilde{\gamma}_6$	$\tilde{\gamma}_6$	0	0	0	0	0	$2\tilde{\gamma}_6$	0	$2\tilde{\gamma}_8-\tilde{\gamma}_7$
$\tilde{\gamma}_7$	$\tilde{\gamma}_7$	0	0	0	0	0	0	$4\tilde{\gamma}_7$	$2\tilde{\gamma}_7$
$\tilde{\gamma}_8$	$\tilde{\gamma}_8$	$\tilde{\gamma}_6$	0	0	0	$2\gamma_8-\gamma_7$	0	$2\tilde{\gamma}_7$	$\tilde{\gamma}_7$

This shows that J is an ideal. We have $J^2 = {}_R\langle 4\cdot 1_\Gamma, 2\tilde{\gamma}_1, 2\tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4, \tilde{\gamma}_5, \tilde{\gamma}_6, \tilde{\gamma}_7, 2\tilde{\gamma}_8\rangle$.

Moreover, J is topologically nilpotent as

$$J^4 \subseteq 2\Gamma\ .$$

Since $\Gamma/J \cong \mathbf{F}_2$, the ideal J indeed is the Jacobson radical of Γ, cf. Corollary 160, and the ring Γ is local. $\square$

3.3.3.2 $\Lambda_{(2)}$ as path algebra modulo relations

Recall that $R = \mathbf{Z}_{(2)}$. We aim to write $\Lambda_{(2)}$, cf. Corollary 128, as path algebra modulo relations.

Write

$$\Lambda'_{(2)} := \{(m,f) \in R \times R : m \equiv_2 f\} ,$$

$$\Lambda''_{(2)} := \{(p,q) \in R \times R : p \equiv_2 q\} ,$$

$$\Lambda'''_{(2)} := \left\{ \left(h, \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right), g, a, n, r, \left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right)\right) \in R \times R^{2\times 2} \times R \times R \times R \times R \times R^{2\times 2} : \begin{array}{r} e - a \equiv_8 2j \equiv_4 0 \\ e - g \equiv_4 j \\ n - r \equiv_4 2k \\ c \equiv_4 0 \\ d \equiv_2 0 \\ n \equiv_2 l \equiv_2 e \equiv_2 i \\ h \equiv_2 b \end{array} \right\} .$$

We identify $\Lambda_{(2)} = \Lambda'_{(2)} \times \Lambda''_{(2)} \times \Lambda'''_{(2)}$.

Ad $\Lambda'_{(2)}$.

We have the R-linear basis $(\tilde{\mathrm{e}}_3 = (1,1),\ \tilde{\nu}_1 = (0,2))$ of $\Lambda'_{(2)}$.

Consider the quiver

$$\Psi' := \left[\, \mathrm{e}_3 \circlearrowleft \nu_1 \,\right] .$$

We have a surjective R-algebra morphism $\varphi' : R\Psi' \to \Lambda'_{(2)}$ by sending

$$\begin{array}{rcl} \mathrm{e}_3 & \mapsto & \tilde{\mathrm{e}}_3 \\ \nu_1 & \mapsto & \tilde{\nu}_1 \ . \end{array}$$

We establish the following multiplication tree, where we underline the elements that are not in a $\mathbf{Q}$-linear relation with previous elements. We double underline the last element in the tree that gets underlined.

$$\underline{\mathrm{e}_3} \xrightarrow[\nu_1]{} \underline{\underline{\nu_1}} \xrightarrow[\nu_1]{} \nu_1^2 = 2\nu_1$$

So, the kernel of φ contains the element

$$\nu_1^2 - 2\nu_1 \ .$$

Let I' be the ideal in $R\Psi'$ generated by $\nu_1^2 - 2\nu_1$. So, $I' \subseteq \mathrm{kern}(\varphi')$. Therefore, φ' induces a surjective R-algebra morphism from $R\Psi'/I'$ to $\Lambda'_{(2)}$.

Note that $R\Psi'/I'$ is R-linearly generated by

$$\mathcal{N}' := \{\mathrm{e}_3 + I, \nu_1 + I\} \ .$$

Note that $|\mathcal{N}'| = 2 = \mathrm{rk}_R(\Lambda'_{(2)})$.

Since we have a surjective R-algebra morphism from $R\Psi'/I'$ to $\Lambda'_{(2)}$, this rank argument shows this morphism to be bijective. In particular, $I' = \mathrm{kern}(\varphi')$.

Ad $\Lambda''_{(2)}$.

We have the R-linear basis $(\tilde{\mathrm{e}}_2 = (1,1),\ \tilde{\nu}_2 = (0,2))$ of $\Lambda'_{(2)}$.

Consider the quiver

$$\Psi'' := \left[{}^{\nu_2} \circlearrowleft \mathrm{e}_2 \right] \ .$$

Analogously to the case of $\Lambda'_{(2)}$ we obtain a surjective R-algebra morphism

$$\begin{array}{rcl} \varphi'' : R\Psi'' & \to & \Lambda''_{(2)} \\ \mathrm{e}_2 & \mapsto & \tilde{\mathrm{e}}_2 \\ \nu_2 & \mapsto & \tilde{\nu}_2 \ , \end{array}$$

which has the kernel $I'' := (\nu_2^2 - 2\nu_2)$.

So, φ'' induces an R-algebra isomorphism from $R\Psi''/I''$ to $\Lambda''_{(2)}$.

Ad $\Lambda'''_{(2)}$.

We have the R-linear basis

$$\begin{array}{rcl}
\tilde{\mathrm{e}}_1 & = & (1, \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), 0,0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\mathrm{e}}_4 & = & (0, \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), 1,1,1,1, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)) \\
\tilde{\alpha}_1 & := & (0, \left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right), 0,0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\alpha}_2 & := & (0, \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right), 0,0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\beta}_1 & := & (0, \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right), 0,0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\gamma}_1 & := & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 2,4,0,0, \left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right)) \\
\tilde{\gamma}_2 & := & (0, \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right), 2,2,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\gamma}_3 & := & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 4,8,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\gamma}_4 & := & (0, \left(\begin{smallmatrix}0&0\\0&8\end{smallmatrix}\right), 0,0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\gamma}_5 & := & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0, \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\gamma}_6 & := & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0, \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right)) \\
\tilde{\gamma}_7 & := & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,4,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\
\tilde{\gamma}_8 & := & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,2,0, \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right))
\end{array}$$

of $\Lambda'''_{(2)}$.

We have $\tilde{\gamma}_3 = \tilde{\gamma}_1\tilde{\gamma}_2,\ \tilde{\gamma}_4 = \tilde{\alpha}_2\tilde{\alpha}_1,\ \tilde{\gamma}_5 = \tilde{\gamma}_1\tilde{\gamma}_8,\ \tilde{\gamma}_6 = \tilde{\gamma}_8\tilde{\gamma}_1,\ \tilde{\gamma}_7 = \tilde{\gamma}_8^2$. Hence, as an R-algebra $\Lambda'''_{(2)}$ is generated by $\tilde{\mathrm{e}}_1, \tilde{\mathrm{e}}_4, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_8$.

Consider the quiver

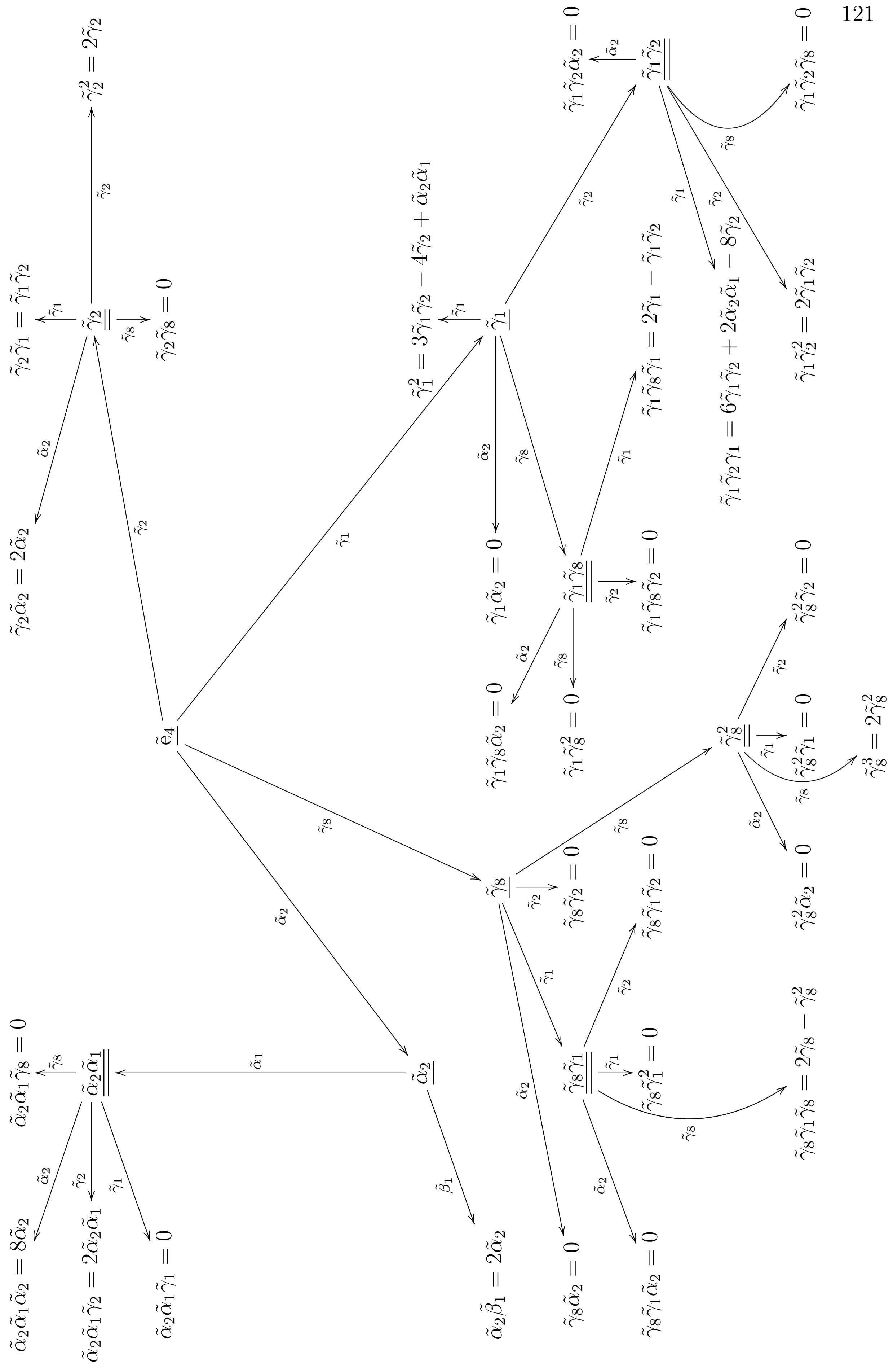
$$\Psi''' := \left[\begin{tikzcd} \mathrm{e}_1 \arrow[loop left, "\beta_1"] \arrow[r, bend left, "\alpha_1"] & \mathrm{e}_4 \arrow[l, bend left, "\alpha_2"] \arrow[loop above, "\gamma_1"] \arrow[loop right, "\gamma_2"] \arrow[loop below, "\gamma_8"] \end{tikzcd} \right] .$$

We have a surjective R-algebra morphism $\varphi''' : R\Psi''' \to \Lambda'''_{(2)}$ by sending

$$\begin{aligned} \mathrm{e}_1 &\mapsto \tilde{\mathrm{e}}_1 \\ \mathrm{e}_4 &\mapsto \tilde{\mathrm{e}}_4 \\ \alpha_1 &\mapsto \tilde{\alpha}_1 \\ \alpha_2 &\mapsto \tilde{\alpha}_2 \\ \beta_1 &\mapsto \tilde{\beta}_1 \\ \gamma_1 &\mapsto \tilde{\gamma}_1 \\ \gamma_2 &\mapsto \tilde{\gamma}_2 \\ \gamma_8 &\mapsto \tilde{\gamma}_8 \,. \end{aligned}$$

We establish the following multiplication trees, where we underline the elements that are not in a **Q**-linear relation with previous elements. We double underline the last elements in the tree that get underlined.

$$\begin{tikzcd} \tilde{\alpha}_1\tilde{\gamma}_8 = 0 & \tilde{\alpha}_1\tilde{\alpha}_2 = 4\tilde{\beta}_1 & \underline{\tilde{\mathrm{e}}_1} \arrow[dl, "\tilde{\alpha}_1"'] \arrow[dr, "\tilde{\beta}_1"] & & \\ \tilde{\alpha}_1\tilde{\gamma}_1 = 0 & \underline{\underline{\tilde{\alpha}_1}} \arrow[ul, "\tilde{\gamma}_8"'] \arrow[u, "\tilde{\alpha}_2"'] \arrow[l, "\tilde{\gamma}_1"'] \arrow[dl, "\tilde{\gamma}_2"] & & \underline{\underline{\tilde{\beta}_1}} \arrow[r, "\tilde{\beta}_1"'] \arrow[d, "\tilde{\alpha}_1"'] & \tilde{\beta}_1^2 = 2\tilde{\beta}_1 \\ \tilde{\alpha}_1\tilde{\gamma}_2 = 2\tilde{\alpha}_1 & & & \tilde{\beta}_1\tilde{\alpha}_1 = 2\tilde{\alpha}_1 & \end{tikzcd}$$



$\tilde{e}_4$
$\tilde{\alpha}_2$
$\tilde{\gamma}_1$
$\tilde{\gamma}_2$
$\tilde{\gamma}_8$
$\tilde{\beta}_1$
$\tilde{\alpha}_1$
$\tilde{\gamma}_2\tilde{\alpha}_2 = 2\tilde{\alpha}_2$
$\tilde{\gamma}_2\tilde{\gamma}_1 = \tilde{\gamma}_1\tilde{\gamma}_2$
$\tilde{\gamma}_2^2 = 2\tilde{\gamma}_2$
$\tilde{\gamma}_2\tilde{\gamma}_8 = 0$
$\tilde{\gamma}_1^2 = 3\tilde{\gamma}_1\tilde{\gamma}_2 - 4\tilde{\gamma}_2 + \tilde{\alpha}_2\tilde{\alpha}_1$
$\tilde{\gamma}_1\tilde{\alpha}_2 = 0$
$\tilde{\gamma}_1\tilde{\gamma}_8$
$\tilde{\gamma}_1\tilde{\gamma}_8\tilde{\alpha}_2 = 0$
$\tilde{\gamma}_1\tilde{\gamma}_8^2 = 0$
$\tilde{\gamma}_1\tilde{\gamma}_8\tilde{\gamma}_2 = 0$
$\tilde{\gamma}_1\tilde{\gamma}_8\tilde{\gamma}_1 = 2\tilde{\gamma}_1 - \tilde{\gamma}_1\tilde{\gamma}_2$
$\tilde{\gamma}_1\tilde{\gamma}_2$
$\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\alpha}_2 = 0$
$\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_1 = 6\tilde{\gamma}_1\tilde{\gamma}_2 + 2\tilde{\alpha}_2\tilde{\alpha}_1 - 8\tilde{\gamma}_2$
$\tilde{\gamma}_1\tilde{\gamma}_2^2 = 2\tilde{\gamma}_1\tilde{\gamma}_2$
$\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_8 = 0$
$\tilde{\gamma}_8^2$
$\tilde{\gamma}_8^2\tilde{\gamma}_2 = 0$
$\tilde{\gamma}_8^2\tilde{\gamma}_1 = 0$
$\tilde{\gamma}_8^3 = 2\tilde{\gamma}_8^2$
$\tilde{\gamma}_8^2\tilde{\alpha}_2 = 0$
$\tilde{\gamma}_8\tilde{\gamma}_2 = 0$
$\tilde{\gamma}_8\tilde{\gamma}_1\tilde{\gamma}_2 = 0$
$\tilde{\gamma}_8\tilde{\gamma}_1$
$\tilde{\gamma}_8\tilde{\gamma}_1^2 = 0$
$\tilde{\gamma}_8\tilde{\gamma}_1\tilde{\gamma}_8 = 2\tilde{\gamma}_8 - \tilde{\gamma}_8^2$
$\tilde{\gamma}_8\tilde{\alpha}_2 = 0$
$\tilde{\gamma}_8\tilde{\gamma}_1\tilde{\alpha}_2 = 0$
$\tilde{\alpha}_2\tilde{\beta}_1 = 2\tilde{\alpha}_2$
$\tilde{\alpha}_2\tilde{\alpha}_1$
$\tilde{\alpha}_2\tilde{\alpha}_1\tilde{\gamma}_8 = 0$
$\tilde{\alpha}_2\tilde{\alpha}_1\tilde{\alpha}_2 = 8\tilde{\alpha}_2$
$\tilde{\alpha}_2\tilde{\alpha}_1\tilde{\gamma}_2 = 2\tilde{\alpha}_2\tilde{\alpha}_1$
$\tilde{\alpha}_2\tilde{\alpha}_1\tilde{\gamma}_1 = 0$

So, the kernel of φ''' contains the elements

$$
\begin{array}{cccc}
\alpha_2\alpha_1\gamma_8 & \gamma_2\alpha_2 - 2\alpha_2 & \gamma_8\alpha_2 & \gamma_1\alpha_2 \\
\alpha_2\alpha_1\alpha_2 - 8\alpha_2 & \gamma_2\gamma_1 - \gamma_1\gamma_2 & \gamma_8\gamma_1\alpha_2 & \gamma_1^2 - 3\gamma_1\gamma_2 + 4\gamma_2 - \alpha_2\alpha_1 \\
\alpha_2\alpha_1\gamma_2 - 2\alpha_2\alpha_1 & \gamma_2^2 - 2\gamma_2 & \gamma_8\gamma_1\gamma_2 & \gamma_1\gamma_2\alpha_2 \\
\alpha_2\alpha_1\gamma_1 & \gamma_2\gamma_8 & \gamma_8\gamma_1^2 & \gamma_1\gamma_8\alpha_2 \\
\alpha_2\beta_1 - 2\alpha_2 & & \gamma_8\gamma_1\gamma_8 - 2\gamma_8 + \gamma_8^2 & \gamma_1\gamma_8^2 \\
 & & \gamma_8\gamma_2 & \gamma_1\gamma_8\gamma_2 \\
 & & \gamma_8^2\gamma_2 & \gamma_1\gamma_8\gamma_1 - 2\gamma_1 + \gamma_1\gamma_2 \\
 & & \gamma_8^2\gamma_1 & \gamma_1\gamma_2\gamma_8 \\
 & & \gamma_8^2\alpha_2 & \gamma_1\gamma_2^2 - 2\gamma_1\gamma_2 \\
 & & \gamma_8^3 - 2\gamma_8^2 & \gamma_1\gamma_2\gamma_1 - 6\gamma_1\gamma_2 - 2\gamma_4 + 8\gamma_2
\end{array}
$$

$$
\begin{array}{c}
\alpha_1\gamma_8 \\
\alpha_1\gamma_1 \\
\alpha_1\gamma_2 - 2\alpha_1 \\
\alpha_1\alpha_2 - 4\beta_1 \\
\beta_1^2 - 2\beta_1 \\
\beta_1\alpha_1 - 2\alpha_1\ .
\end{array}
$$

Let I''' be the ideal in $R\Psi'''$ generated by those elements. So, $I''' \subseteq \mathrm{kern}(\varphi''')$. Therefore, φ''' induces a surjective R-algebra morphism from $R\Psi'''/I'''$ to $\Lambda'''_{(2)}$.

Note that $R\Psi'''/I'''$ is R-linearly generated by

$$
\begin{aligned}
\mathcal{N}''' := \{&\mathrm{e}_1 + I''', \alpha_1 + I''', \beta_1 + I''', \mathrm{e}_4 + I''', \alpha_2 + I''', \alpha_2\tilde{\alpha}_1 + I''', \gamma_1 + I''', \\
&\gamma_2 + I''', \gamma_1\gamma_2 + I''', \gamma_8 + I''', \gamma_1\gamma_8 + I''', \gamma_8\gamma_1 + I''', \gamma_8^2 + I'''\}\ ,
\end{aligned}
$$

since, using the trees above, a product of a double underlined element with k further factors may be written, modulo I''', as an R-linear combination of products of underlined elements with $\leq k-1$ further factors. Moreover, note that $|\mathcal{N}'''| = 13 = \mathrm{rk}_R(\Lambda'''_{(2)})$.

Since we have a surjective R-algebra morphism from $R\Psi'''/I'''$ to $\Lambda'''_{(2)}$, this rank argument shows this morphism to be bijective. In particular, $I''' = \mathrm{kern}(\varphi''')$.

We may reduce the list of generators above to obtain $I''' =$

$$
\left(
\begin{array}{cccc}
\gamma_2\alpha_2 - 2\alpha_2, & \gamma_8\alpha_2, & \gamma_1\alpha_2, & \gamma_2\gamma_1 - \gamma_1\gamma_2, \\
\gamma_1^2 - 3\gamma_1\gamma_2 + 4\gamma_2 - \alpha_2\alpha_1, & \gamma_2^2 - 2\gamma_2, & \gamma_1\gamma_8^2, & \gamma_2\gamma_8, \\
\alpha_2\beta_1 - 2\alpha_2, & \gamma_8\gamma_1\gamma_8 - 2\gamma_8 + \gamma_8^2, & \gamma_8^3 - 2\gamma_8^2, & \gamma_8\gamma_2, \\
\gamma_1\gamma_8\gamma_1 - 2\gamma_1 + \gamma_1\gamma_2, & \alpha_1\gamma_1, & \alpha_1\gamma_2 - 2\alpha_1, & \gamma_1\gamma_2\gamma_1 - 6\gamma_1\gamma_2 - 2\alpha_2\alpha_1 + 8\gamma_2, \\
\alpha_1\gamma_8, & \beta_1\alpha_1 - 2\alpha_1, & \alpha_1\alpha_2 - 4\beta_1, & \beta_1^2 - 2\beta_1
\end{array}
\right)
$$

as

$$
\begin{array}{rcl}
\alpha_2\alpha_1\gamma_8 & = & \alpha_2(\alpha_1\gamma_8) \\
\alpha_2\alpha_1\alpha_2 - 8\alpha_2 & = & \alpha_2(\alpha_1\alpha_2 - 4\beta_1) + 4(\alpha_2\beta_1 - 2\alpha_2) \\
\alpha_2\alpha_1\gamma_2 - 2\alpha_2\alpha_1 & = & \alpha_2(\alpha_1\gamma_2 - 2\alpha_1) \\
\alpha_2\alpha_1\gamma_1 & = & \alpha_2(\alpha_1\gamma_1) \\
\gamma_8\gamma_1\alpha_2 & = & \gamma_8(\gamma_1\alpha_2) \\
\gamma_8\gamma_1\gamma_2 & = & -\gamma_8(\gamma_2\gamma_1 - \gamma_1\gamma_2) + (\gamma_8\gamma_2)\gamma_1 \\
\gamma_8\gamma_1^2 & = & \gamma_8(\gamma_1^2 - 3\gamma_1\gamma_2 + 4\gamma_2 - \alpha_2\alpha_1) + 3\gamma_8\gamma_1\gamma_2 - 4\gamma_8\gamma_2 + (\gamma_8\alpha_2)\alpha_1 \\
\gamma_8^2\gamma_2 & = & \gamma_8(\gamma_8\gamma_2) \\
\gamma_8^2\gamma_1 & = & (\gamma_8\gamma_1\gamma_8 - 2\gamma_8 + \gamma_8^2)\gamma_1 - \gamma_8(\gamma_1\gamma_8\gamma_1 - 2\gamma_1 + \gamma_1\gamma_2) + \gamma_8\gamma_1\gamma_2 \\
\gamma_8^2\alpha_2 & = & \gamma_8(\gamma_8\alpha_2) \\
\gamma_1\gamma_8\alpha_2 & = & \gamma_1(\gamma_8\alpha_2) \\
\gamma_1\gamma_8\gamma_2 & = & \gamma_1(\gamma_8\gamma_2) \\
\gamma_1\gamma_2\gamma_8 & = & \gamma_1(\gamma_2\gamma_8) \\
\gamma_1\gamma_2^2 - 2\gamma_1\gamma_2 & = & \gamma_1(\gamma_2^2 - 2\gamma_2) \\
\gamma_1\gamma_2\alpha_2 & = & \gamma_1(\gamma_2\alpha_2 - 2\alpha_2) + 2\gamma_1\alpha_2\ .
\end{array}
$$

So, we obtain the

Proposition 130. *Recall that $I' = (\nu_1^2 - 2\nu_1)$, $I'' = (\nu_2^2 - 2\nu_2)$ and that $I''' =$*

$$
\left(
\begin{array}{cccc}
\gamma_2\alpha_2 - 2\alpha_2, & \gamma_8\alpha_2, & \gamma_1\alpha_2, & \gamma_2\gamma_1 - \gamma_1\gamma_2, \\
\gamma_1^2 - 3\gamma_1\gamma_2 + 4\gamma_2 - \alpha_2\alpha_1, & \gamma_2^2 - 2\gamma_2, & \gamma_1\gamma_8^2, & \gamma_2\gamma_8, \\
\alpha_2\beta_1 - 2\alpha_2, & \gamma_8\gamma_1\gamma_8 - 2\gamma_8 + \gamma_8^2, & \gamma_8^3 - 2\gamma_8^2, & \gamma_8\gamma_2, \\
\gamma_1\gamma_8\gamma_1 - 2\gamma_1 + \gamma_1\gamma_2, & \alpha_1\gamma_1, & \alpha_1\gamma_2 - 2\alpha_1, & \gamma_1\gamma_2\gamma_1 - 6\gamma_1\gamma_2 - 2\alpha_2\alpha_1 + 8\gamma_2, \\
\alpha_1\gamma_8, & \beta_1\alpha_1 - 2\alpha_1, & \alpha_1\alpha_2 - 4\beta_1, & \beta_1^2 - 2\beta_1
\end{array}
\right) .
$$

Using Theorem 126, we obtain the isomorphisms of R-algebras

$$
\mathrm{B}^{\Delta}_{\mathbf{Z}_{(2)}}(\mathrm{S}_4, \mathrm{S}_4) \xrightarrow[\sim]{\sigma'_{(2)}} \Lambda_{(2)} \ \xrightarrow{\sim}\ R\left[\ {}^{\nu_1}\circlearrowleft \mathrm{e}_3\ \right]/I' \times R\left[\ {}^{\nu_2}\circlearrowleft \mathrm{e}_2\ \right]/I'' \times R\left[
\begin{tikzcd}
\mathrm{e}_1 \arrow[loop left, "\beta_1"] \arrow[r, bend left, "\alpha_1"] & \mathrm{e}_4 \arrow[l, bend left, "\alpha_2"] \arrow[loop above, "\gamma_1"] \arrow[loop right, "\gamma_2"] \arrow[loop below, "\gamma_8"]
\end{tikzcd}
\right]/I'''
$$

$$
\begin{array}{rcl}
\tilde{\mathrm{e}}_3 & \mapsto & (\mathrm{e}_3 + I', 0, 0) \\
\tilde{\nu}_1 & \mapsto & (\nu_1 + I', 0, 0) \\
\tilde{\mathrm{e}}_2 & \mapsto & (0, \mathrm{e}_2 + I'', 0) \\
\tilde{\nu}_2 & \mapsto & (0, \nu_2 + I'', 0) \\
\tilde{\mathrm{e}}_1 & \mapsto & (0, 0, \mathrm{e}_1 + I''') \\
\tilde{\mathrm{e}}_4 & \mapsto & (0, 0, \mathrm{e}_4 + I''') \\
\tilde{\alpha}_j & \mapsto & (0, 0, \alpha_j + I''')\ \textit{for}\ j \in \{1, 2\} \\
\tilde{\beta}_1 & \mapsto & (0, 0, \beta_1 + I''') \\
\tilde{\gamma}_j & \mapsto & (0, 0, \gamma_j + I''')\ \textit{for}\ j \in \{1, 2, 8\}\ .
\end{array}
$$

Corollary 131. *As $\mathbf{F}_2$-algebras, we have*

$$\mathrm{B}^{\Delta}_{\mathbf{F}_2}(\mathrm{S}_4,\mathrm{S}_4)) \cong\ \Lambda/2\Lambda \cong \mathbf{F}_2\ \left[\begin{tikzcd} \mathrm{e}_3 \arrow[loop left, "\nu_1"] \end{tikzcd}\right]/\overline{I'}\times\mathbf{F}_2\left[\begin{tikzcd} \mathrm{e}_2 \arrow[loop left, "\nu_2"] \end{tikzcd}\right]/\overline{I''}\times\mathbf{F}_2\left[\begin{tikzcd} \mathrm{e}_1 \arrow[loop left, "\beta_1"] \arrow[r, bend left, "\alpha_1"] & \mathrm{e}_4 \arrow[l, bend left, "\alpha_2"] \arrow[loop above, "\gamma_1"] \arrow[loop right, "\gamma_2"] \arrow[loop below, "\gamma_8"] \end{tikzcd}\right]/\overline{I'''}$$

where $\overline{I'} = (\nu_1^2)$, $\overline{I''} = (\nu_2^2)$ *and*

$$\overline{I'''} = \left(\begin{array}{cccc} \gamma_2\alpha_2, & \gamma_8\alpha_2, & \gamma_1\alpha_2, & \gamma_2\gamma_1-\gamma_1\gamma_2, \\ \gamma_1^2-\gamma_1\gamma_2-\alpha_2\alpha_1, & \gamma_2^2, & \gamma_1\gamma_8^2, & \gamma_2\gamma_8, \\ \alpha_2\beta_1, & \gamma_8\gamma_1\gamma_8+\gamma_8^2, & \gamma_8^3, & \gamma_8\gamma_2, \\ \gamma_1\gamma_8\gamma_1+\gamma_1\gamma_2, & \alpha_1\gamma_1 & \alpha_1\gamma_2, & \gamma_1\gamma_2\gamma_1, \\ \alpha_1\gamma_8, & \beta_1\alpha_1, & \alpha_1\alpha_2, & \beta_1^2 \end{array}\right),$$

cf. Proposition 130.

3.3.4 Localisation at 3: $\mathrm{B}^{\Delta}_{\mathbf{Z}_{(3)}}(\mathrm{S}_4,\mathrm{S}_4)$

Write $R := \mathbf{Z}_{(3)}$.

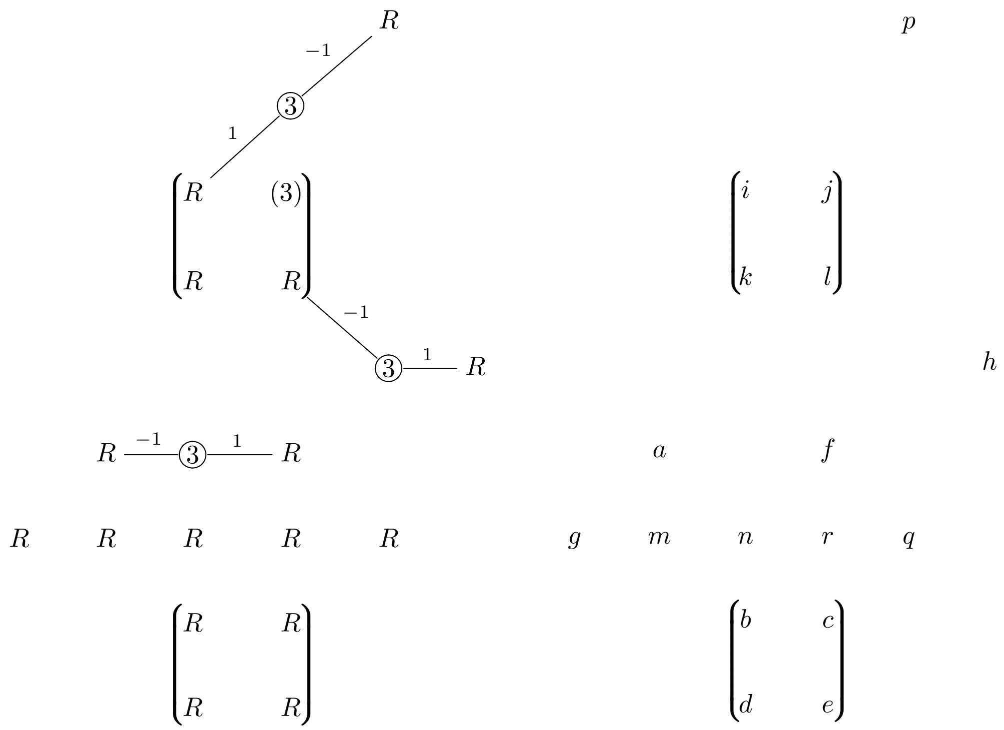
3.3.4.1 Congruences describing the image of $\mathrm{B}^{\Delta}_{\mathbf{Z}_{(3)}}(\mathrm{S}_4,\mathrm{S}_4)$

Recall Λ from Theorem 126. Write $\Xi_{(3)} := R\times R^{2\times 2}\times R\times R\times R\times R^{2\times 2}\times R\times R\times R\times R\times R$.

Corollary 132. *We have*

$$\Lambda_{(3)} = \left\{\begin{array}{rcl} (a,\left(\begin{smallmatrix} b & c\\ d & e\end{smallmatrix}\right),f,g,h,\left(\begin{smallmatrix} i & j\\ k & l\end{smallmatrix}\right),m,n,r,p,q)\in\Xi_{(3)} : a & \equiv_3 & f \\ p & \equiv_3 & i \\ h & \equiv_3 & l \\ j & \equiv_3 & 0 \end{array}\right\}.$$

More symbolically written, we may describe $\Lambda_{(3)}$ as follows. The letters to the right are the key to this picture.



Remark 133. *We have the orthogonal decomposition*

$$1_{\Lambda_{(3)}} = \tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4 + \tilde{e}_5 + \tilde{e}_6 + \tilde{e}_7 + \tilde{e}_8 + \tilde{e}_9 + \tilde{e}_{10}$$

into primitive idempotent elements of $\Lambda_{(3)}$, *where*

$$\begin{aligned}
\tilde{e}_1 &:= \left(1, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 1, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0\right)\\
\tilde{e}_2 &:= \left(0, \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), 0, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0\right)\\
\tilde{e}_3 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), 0, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0\right)\\
\tilde{e}_4 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 1, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0\right)\\
\tilde{e}_5 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 1, \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), 0, 0, 0, 0, 0\right)\\
\tilde{e}_6 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 1, 0\right)\\
\tilde{e}_7 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 1, 0, 0, 0, 0\right)\\
\tilde{e}_8 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 1, 0, 0, 0\right)\\
\tilde{e}_9 &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 1, 0, 0\right)\\
\tilde{e}_{10} &:= \left(0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 1\right).
\end{aligned}$$

Proof. We have $\tilde{e}_s \Lambda_{(3)} \tilde{e}_s \cong R$ for $s \in \{2, 3, 4, 7, 8, 9, 10\}$.

As R-algebras, we have

$$\begin{aligned}
\tilde{e}_1 \Lambda_{(3)} \tilde{e}_1 &\cong \{(a, f) : a, f \in R,\ a \equiv_3 f\}\\
\tilde{e}_5 \Lambda_{(3)} \tilde{e}_5 &\cong \{(h, l) : h, l \in R,\ h \equiv_3 l\}\\
\tilde{e}_6 \Lambda_{(3)} \tilde{e}_6 &\cong \{(i, p) : i, p \in R,\ i \equiv_3 p\}\ .
\end{aligned}$$

Note that for $s \in \{1,5,6\}$ the ring $\tilde{e}_s\Lambda_{(3)}\tilde{e}_s$ is local.

So, $\tilde{e}_s$ is primitive for $s \in [1,10]$. $\square$

3.3.4.2 $\Lambda_{(3)}$ as path algebra modulo relations

Recall that $R = \mathbf{Z}_{(3)}$. We aim to write $\Lambda_{(3)}$, cf. Corollary 128, as path algebra modulo relations.

We have the R-linear basis

$$\begin{array}{rcl}
\tilde{e}_1 & = & (1, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 1,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{e}_2 & = & (0, \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{e}_3 & = & (0, \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{e}_4 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,1,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{e}_5 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,1, \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{e}_6 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), 0,0,0,1,0)\\
\tilde{e}_7 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 1,0,0,0,0)\\
\tilde{e}_8 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,1,0,0,0)\\
\tilde{e}_9 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,1,0,0)\\
\tilde{e}_{10} & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,1)\\
\tilde{\alpha}_1 & = & (0, \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{\alpha}_2 & = & (0, \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{\alpha}_3 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{\beta}_1 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 3,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{\beta}_2 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,3,0)\\
\tilde{\beta}_3 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,3, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0,0,0)\\
\tilde{\beta}_4 & = & (0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0,0,0, \left(\begin{smallmatrix}0&3\\0&0\end{smallmatrix}\right), 0,0,0,0,0)
\end{array}$$

of $\Lambda_{(3)}$.

Note that $\tilde{\beta}_2 = 3\tilde{e}_6 - \tilde{\beta}_4\tilde{\alpha}_3$ and that $\tilde{\beta}_3 = 3\tilde{e}_5 - \tilde{\alpha}_3\tilde{\beta}_4$.

Hence, as an algebra $\Lambda_{(3)}$ is generated by $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{e}_6, \tilde{e}_7, \tilde{e}_8, \tilde{e}_9, \tilde{e}_{10}, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\beta}_1, \tilde{\beta}_4$.

We identify $\tilde{e}_s\Lambda_{(3)}\tilde{e}_s = R$ for $s \in \{4,7,8,9,10\}$. We write $\Lambda_{(3)} \xrightarrow{\pi_s} R$ for the respective projection morphism.

We identify $(\tilde{e}_2 + \tilde{e}_3)\Lambda_{(3)}(\tilde{e}_2 + \tilde{e}_3) = R^{2\times 2}$. We write $\Lambda_{(3)} \xrightarrow{\pi_2} R^{2\times 2}$ for the projection morphism.

Note that $\tilde{e}_1$, $\tilde{e}_2 + \tilde{e}_3$, $\tilde{e}_4$, $\varepsilon := \tilde{e}_5 + \tilde{e}_6$, $\tilde{e}_7$, $\tilde{e}_8$, $\tilde{e}_9$ and $\tilde{e}_{10}$ are central.

Ad $\tilde{e}_1\Lambda_{(3)}$. Consider the quiver

$$\Psi' := \left[\begin{tikzcd} \mathrm{e}_1 \arrow[loop left, "\beta_1"] \end{tikzcd}\right] .$$

We have a surjective R-algebra morphism $\varphi' : R\Psi' \to \tilde{e}_1\Lambda_{(3)}$ by sending

$$\begin{array}{rcl} \mathrm{e}_1 & \mapsto & \tilde{\mathrm{e}}_1 \\ \beta_1 & \mapsto & \tilde{\beta}_1 \ . \end{array}$$

We establish the following multiplication tree, where we underline the elements that are not in a **Q**-linear relation with previous elements. We double underline the last element in the tree that gets underlined.

$$\underline{\tilde{\mathrm{e}}_1} \xrightarrow[\tilde{\beta}_1]{} \underline{\underline{\tilde{\beta}_1}} \xrightarrow[\tilde{\beta}_1]{} \tilde{\beta}_1^2 = 3\tilde{\beta}_1$$

So, the kernel of φ' contains the element $\beta_1^2 - 3\beta_1$. Let I' be the ideal in $R\Psi'$ generated by this element. Therefore, φ' induces a surjective R-algebra morphism from $R\Psi'/I'$ to $\tilde{\mathrm{e}}_1\Lambda_{(3)}$.

Note that $R\Psi'/I'$ is R-linearly generated by

$$\mathcal{N}' := \{\mathrm{e}_1 + I', \beta_1 + I'\} \ .$$

Note that $|\mathcal{N}'| = 2 = \mathrm{rk}_R(\tilde{\mathrm{e}}_1\Lambda_{(3)})$.

Since we have a surjective R-algebra morphism from $R\Psi'/I'$ to $\tilde{\mathrm{e}}_1\Lambda_{(3)}$, this rank argument shows this morphism to be bijective. In particular, $I' = \mathrm{kern}(\varphi')$.

Ad $\varepsilon\Lambda_{(3)}$. Consider the quiver

$$\Psi'' := \left[\begin{tikzcd} \mathrm{e}_5 \arrow[r, bend left, "\alpha_3"] & \mathrm{e}_6 \arrow[l, bend left, "\beta_4"] \end{tikzcd} \right] \ .$$

We have a surjective R-algebra morphism

$$\begin{array}{rcl} \varphi'' : R\Psi'' & \to & \varepsilon\Lambda_{(3)} \\ \mathrm{e}_i & \mapsto & \tilde{\mathrm{e}}_i \text{ for } i \in \{5,6\} \\ \alpha_3 & \mapsto & \tilde{\alpha}_3 \\ \beta_4 & \mapsto & \tilde{\beta}_4 \ . \end{array}$$

We establish the following multiplication trees, where we underline the elements that are not in a **Q**-linear relation with previous elements. We double underline the last elements in the tree that get underlined.

$$\underline{\tilde{\mathrm{e}}_5} \xrightarrow[\tilde{\alpha}_3]{} \underline{\tilde{\alpha}_3} \xrightarrow[\tilde{\beta}_4]{} \underline{\underline{\tilde{\alpha}_3\tilde{\beta}_4}} \xrightarrow[\tilde{\alpha}_3]{} \tilde{\alpha}_3\tilde{\beta}_4\tilde{\alpha}_3 = 3\tilde{\alpha}_3$$

$$\underline{\tilde{\mathrm{e}}_6} \xrightarrow[\tilde{\beta}_4]{} \underline{\tilde{\beta}_4} \xrightarrow[\tilde{\alpha}_3]{} \underline{\underline{\tilde{\beta}_4\tilde{\alpha}_3}} \xrightarrow[\tilde{\beta}_4]{} \tilde{\beta}_4\tilde{\alpha}_3\tilde{\beta}_4 = 3\tilde{\beta}_4$$

The kernel of φ'' contains the elements

$$\alpha_3\beta_4\alpha_3 - 3\alpha_3 \quad , \quad \beta_4\alpha_3\beta_4 - 3\beta_4 \ .$$

Let I'' be the ideal in $R\Psi''$ generated by those elements. So, $I \subseteq \mathrm{kern}(\varphi'')$. Therefore, φ'' induces a surjective R-algebra morphism from $R\Psi''/I''$ to $\varepsilon\Lambda_{(3)}$.

Note that $R\Psi''/I''$ is R-linearly generated by

$$\mathcal{N} := \{\mathrm{e}_5 + I'', \mathrm{e}_6 + I'', \alpha_3 + I'', \beta_4 + I'', \alpha_3\beta_4 + I'', \beta_4\alpha_3 + I''\} \ ,$$

since, using the trees above, a product of a double underlined element with k further factors may be written, modulo I'', as an R-linear combination of products of underlined elements with $\leq k-1$ further factors. Moreover, note that $|\mathcal{N}| = 6 = \mathrm{rk}_R(\varepsilon\Lambda_{(3)})$.

Since we have a surjective R-algebra morphism from $R\Psi''/I''$ to $\varepsilon\Lambda_{(3)}$, this rank argument shows this morphism to be bijective. In particular, $I'' = \mathrm{kern}(\varphi'')$.

So, we obtain the

Proposition 134. *Recall that $I' = (\beta_1^2 - 3\beta_1)$ and that $I'' := (\alpha_3\beta_4\alpha_3 - 3\alpha_3, \beta_4\alpha_3\beta_4 - 3\beta_4)$. Using Theorem 126, we obtain the isomorphism of R-algebras*

$$\mathrm{B}^{\Delta}_{\mathbf{Z}_{(3)}}(\mathrm{S}_4, \mathrm{S}_4) \xrightarrow[\sim]{\sigma'_{(3)}} \Lambda_{(3)} \ \stackrel{\sim}{\to}\ \frac{R\left[\,\beta_1 \circlearrowright \mathrm{e}_1\,\right]}{(\beta_1^2 - 3\beta_1)} \times \frac{R\left[\ \mathrm{e}_5 \underset{\beta_4}{\overset{\alpha_3}{\rightleftarrows}} \mathrm{e}_6\ \right]}{(\alpha_3\beta_4\alpha_3 - 3\alpha_3, \beta_4\alpha_3\beta_4 - 3\beta_4)} \times R^{2\times 2} \times R \times R \times R \times R \times R$$

$$\begin{array}{rcl}
\tilde{\mathrm{e}}_1 & \mapsto & (\mathrm{e}_1 + I', 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0) \\
\tilde{\beta}_1 & \mapsto & (\beta_1 + I', 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0) \\
\tilde{\mathrm{e}}_5 & \mapsto & (0, \mathrm{e}_5 + I'', \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0) \\
\tilde{\mathrm{e}}_6 & \mapsto & (0, \mathrm{e}_6 + I'', \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0) \\
\tilde{\alpha}_3 & \mapsto & (0, \alpha_3 + I'', \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0) \\
\tilde{\beta}_4 & \mapsto & (0, \beta_4 + I'', \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, 0) \\
\tilde{\mathrm{e}}_2 & \mapsto & (0, 0, \pi_2(\tilde{\mathrm{e}}_2), 0, 0, 0, 0, 0) \\
\tilde{\mathrm{e}}_3 & \mapsto & (0, 0, \pi_2(\tilde{\mathrm{e}}_3), 0, 0, 0, 0, 0) \\
\tilde{\alpha}_1 & \mapsto & (0, 0, \pi_2(\tilde{\alpha}_1), 0, 0, 0, 0, 0) \\
\tilde{\alpha}_2 & \mapsto & (0, 0, \pi_2(\tilde{\alpha}_2), 0, 0, 0, 0, 0) \\
\tilde{\mathrm{e}}_4 & \mapsto & (0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \pi_4(\tilde{\mathrm{e}}_4), 0, 0, 0, 0) \\
\tilde{\mathrm{e}}_7 & \mapsto & (0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, \pi_7(\tilde{\mathrm{e}}_7), 0, 0, 0) \\
\tilde{\mathrm{e}}_8 & \mapsto & (0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, \pi_8(\tilde{\mathrm{e}}_8), 0, 0) \\
\tilde{\mathrm{e}}_9 & \mapsto & (0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, \pi_9(\tilde{\mathrm{e}}_9), 0) \\
\tilde{\mathrm{e}}_{10} & \mapsto & (0, 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), 0, 0, 0, 0, \pi_{10}(\tilde{\mathrm{e}}_{10})) \ .
\end{array}$$

Corollary 135. *As $\mathbf{F}_3$-algebras, we have*

$$\mathrm{B}^{\Delta}_{\mathbf{F}_3}(\mathrm{S}_4, \mathrm{S}_4) \cong \Lambda/3\Lambda \ \cong\ \frac{\mathbf{F}_3\left[\,\beta_1 \circlearrowright \mathrm{e}_1\,\right]}{(\beta_1^2)} \times \frac{\mathbf{F}_3\left[\ \mathrm{e}_5 \underset{\beta_4}{\overset{\alpha_3}{\rightleftarrows}} \mathrm{e}_6\ \right]}{(\alpha_3\beta_4\alpha_3, \beta_4\alpha_3\beta_4)} \times \mathbf{F}_3^{2\times 2} \times \mathbf{F}_3 \times \mathbf{F}_3 \times \mathbf{F}_3 \times \mathbf{F}_3 \times \mathbf{F}_3 \ ,$$

cf. Proposition 134.

Chapter 4

The double Burnside ring $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$

Let R be a commutative ring. We often write $\mathrm{B}_R := \mathrm{B}_R(\mathrm{S}_3, \mathrm{S}_3)$.

4.1 Z-linear basis of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$

A **Z**-linear basis of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ consists of elements of the form $[(\mathrm{S}_3 \times \mathrm{S}_3)/U]$, where U runs through representatives for the conjugacy classes of subgroups of $\mathrm{S}_3 \times \mathrm{S}_3$, cf. Lemma 28.

The following calculations were done using the computer algebra system Magma [6].

The group S_3 has the subgroups

$$\begin{array}{lcllcl} V_0 & := & \{\mathrm{id}\}, & V_3 & := & \langle (2,3) \rangle, \\ V_1 & := & \langle (1,2) \rangle, & V_4 & := & \langle (1,2,3) \rangle, \\ V_2 & := & \langle (1,3) \rangle, & V_5 & := & \mathrm{S}_3 \ . \end{array}$$

The set $\{V_0,\, V_1,\, V_4,\, V_5\}$ is a system of representatives for the conjugacy classes of subgroups of S_3. In S_3, we write $a := (1,2)$, $b := (1,2,3)$ and $1 := \mathrm{id}$. So, $V_1 = \langle a \rangle$, $V_4 = \langle b \rangle$ and $V_5 = \langle a, b \rangle$.

A system of representatives for the conjugacy classes of subgroups of $\mathrm{S}_3 \times \mathrm{S}_3$ is given by

$$\begin{array}{lcccl} U_{0,0} & := & V_0 \times V_0 & = & \{(1,1)\}, \\ U_{1,0} & := & V_1 \times V_0 & = & \langle (a,1) \rangle, \\ U_{0,1} & := & V_0 \times V_1 & = & \langle (1,a) \rangle, \\ \Delta(V_1) & = & & & \langle (a,a) \rangle, \\ U_{4,0} & := & V_4 \times V_0 & = & \langle (b,1) \rangle, \\ U_{0,4} & := & V_0 \times V_4 & = & \langle (1,b) \rangle, \\ \Delta(V_4) & = & & & \langle (b,b) \rangle, \\ U_{1,1} & := & V_1 \times V_1 & = & \langle (a,1), (1,a) \rangle, \\ U_{5,0} & := & V_5 \times V_0 & = & \langle (a,1), (b,1) \rangle, \\ U_{0,5} & := & V_0 \times V_5 & = & \langle (1,a), (1,b) \rangle, \\ U_6 & := & & & \langle (a,a), (1,b) \rangle, \end{array}$$

$$\begin{array}{llcll}
U_{4,1} & := & V_4 \times V_1 & = & \langle (b,1),(1,a) \rangle, \\
U_{1,4} & := & V_1 \times V_4 & = & \langle (a,1),(1,b) \rangle, \\
U_7 & := & & & \langle (a,a),(b,1) \rangle, \\
\Delta(V_5) & = & & & \langle (a,a),(b,b) \rangle, \\
U_{4,4} & := & V_4 \times V_4 & = & \langle (b,1),(1,b) \rangle, \\
U_{1,5} & := & V_1 \times V_5 & = & \langle (a,1),(1,a),(1,b) \rangle, \\
U_{5,1} & := & V_5 \times V_1 & = & \langle (a,1),(b,1),(1,a) \rangle, \\
U_{4,5} & := & V_4 \times V_5 & = & \langle (b,1),(1,a),(1,b) \rangle, \\
U_{5,4} & := & V_5 \times V_4 & = & \langle (a,1),(b,1),(1,b) \rangle, \\
U_8 & := & & & \langle (a,a),(b,1),(1,b) \rangle, \\
U_{5,5} & := & V_5 \times V_5 & = & \langle (a,1),(1,a),(b,1),(1,b) \rangle\ .
\end{array}$$

Let

$$\begin{array}{lll}
H_{i,j} & := & [(\mathrm{S}_3 \times \mathrm{S}_3)/U_{i,j}] \quad \text{for } i,j \in \{0,1,4,5\}, \\
H_s & := & [(\mathrm{S}_3 \times \mathrm{S}_3)/U_s] \quad \text{for } s \in [6,8] \text{ and} \\
H_t^{\Delta} & := & [(\mathrm{S}_3 \times \mathrm{S}_3)/\Delta(V_t)] \quad \text{for } t \in \{1,4,5\}\ .
\end{array}$$

So, we obtain the $\mathbf{Z}$-linear basis

$$\mathcal{H} := \begin{array}{l} (H_{0,0}, H_{1,0}, H_{0,1}, H_1^{\Delta}, H_{4,0}, H_{0,4}, H_4^{\Delta}, H_{1,1}, H_{5,0}, H_{0,5}, H_6, \\ H_{4,1}, H_{1,4}, H_7, H_5^{\Delta}, H_{4,4}, H_{1,5}, H_{5,1}, H_{4,5}, H_{5,4}, H_8, H_{5,5}) \end{array}$$

of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$, cf. Lemma 28.

We have the following multiplication table for the basis elements, cf. Lemma 42 and Appendix C.

$(\cdot)$	$H_{0,0}$	$H_{1,0}$	$H_{0,1}$	H_1^Δ	$H_{4,0}$	$H_{0,4}$	H_4^Δ	$H_{1,1}$	$H_{5,0}$	$H_{0,5}$	H_6	$H_{4,1}$	$H_{1,4}$	H_7	H_5^Δ	$H_{4,4}$	$H_{1,5}$	$H_{5,1}$	$H_{4,5}$	$H_{5,4}$	H_8	$H_{5,5}$
$H_{0,0}$	$6H_{0,0}$	$3H_{0,0}$	$6H_{0,1}$	$3H_{0,0}$	$2H_{0,0}$	$6H_{0,4}$	$2H_{0,0}$	$3H_{0,1}$	$H_{0,0}$	$6H_{0,5}$	$3H_{0,4}$	$2H_{0,1}$	$3H_{0,4}$	$H_{0,0}$	$H_{0,0}$	$2H_{0,4}$	$3H_{0,5}$	$H_{0,1}$	$2H_{0,5}$	$H_{0,4}$	$H_{0,4}$	$H_{0,5}$
$H_{1,0}$	$6H_{1,0}$	$3H_{1,0}$	$6H_{1,1}$	$3H_{1,0}$	$2H_{1,0}$	$6H_{1,4}$	$2H_{1,0}$	$3H_{1,1}$	$H_{1,0}$	$6H_{1,5}$	$3H_{1,4}$	$2H_{1,1}$	$3H_{1,4}$	$H_{1,0}$	$H_{1,0}$	$2H_{1,4}$	$3H_{1,5}$	$H_{1,1}$	$2H_{1,5}$	$H_{1,4}$	$H_{1,4}$	$H_{1,5}$
$H_{0,1}$	$3H_{0,0}$	$2H_{0,0}$	$3H_{0,1}$	$H_{0,1}+H_{0,0}$	$H_{0,0}$	$3H_{0,4}$	$H_{0,0}$	$2H_{0,1}$	$H_{0,0}$	$3H_{0,5}$	$H_{0,5}+H_{0,4}$	$H_{0,1}$	$2H_{0,4}$	$H_{0,1}$	$H_{0,1}$	$H_{0,4}$	$2H_{0,5}$	$H_{0,1}$	$H_{0,5}$	$H_{0,4}$	$H_{0,5}$	$H_{0,5}$
H_1^Δ	$3H_{0,0}$	$H_{1,0}+H_{0,0}$	$3H_{0,1}$	$H_1^\Delta+H_{0,0}$	$H_{0,0}$	$3H_{0,4}$	$H_{0,0}$	$H_{1,1}+H_{0,1}$	$H_{1,0}$	$3H_{0,5}$	$H_6+H_{0,4}$	$H_{0,1}$	$H_{1,4}+H_{0,4}$	H_1^Δ	H_1^Δ	$H_{0,4}$	$H_{1,5}+H_{0,5}$	$H_{1,1}$	$H_{0,5}$	$H_{1,4}$	H_6	$H_{1,5}$
$H_{4,0}$	$6H_{4,0}$	$3H_{4,0}$	$6H_{4,1}$	$3H_{4,0}$	$2H_{4,0}$	$6H_{4,4}$	$2H_{4,0}$	$3H_{4,1}$	$H_{4,0}$	$6H_{4,5}$	$3H_{4,4}$	$2H_{4,1}$	$3H_{4,4}$	$H_{4,0}$	$H_{4,0}$	$2H_{4,4}$	$3H_{4,5}$	$H_{4,1}$	$2H_{4,5}$	$H_{4,4}$	$H_{4,4}$	$H_{4,5}$
$H_{0,4}$	$2H_{0,0}$	$H_{0,0}$	$2H_{0,1}$	$H_{0,0}$	$2H_{0,0}$	$2H_{0,4}$	$2H_{0,4}$	$H_{0,1}$	$H_{0,0}$	$2H_{0,5}$	$H_{0,4}$	$2H_{0,1}$	$H_{0,4}$	$H_{0,0}$	$H_{0,4}$	$2H_{0,4}$	$H_{0,5}$	$H_{0,1}$	$2H_{0,5}$	$H_{0,4}$	$H_{0,4}$	$H_{0,5}$
H_4^Δ	$2H_{0,0}$	$H_{0,0}$	$2H_{0,1}$	$H_{0,0}$	$2H_{4,0}$	$2H_{0,4}$	$2H_4^\Delta$	$H_{0,1}$	$H_{4,0}$	$2H_{0,5}$	$H_{0,4}$	$2H_{4,1}$	$H_{0,4}$	$H_{4,0}$	H_4^Δ	$2H_{4,4}$	$H_{0,5}$	$H_{4,1}$	$2H_{4,5}$	$H_{4,4}$	$H_{4,4}$	$H_{4,5}$
$H_{1,1}$	$3H_{1,0}$	$2H_{1,0}$	$3H_{1,1}$	$H_{1,1}+H_{1,0}$	$H_{1,0}$	$3H_{1,4}$	$H_{1,0}$	$2H_{1,1}$	$H_{1,0}$	$3H_{1,5}$	$H_{1,5}+H_{1,4}$	$H_{1,1}$	$2H_{1,4}$	$H_{1,1}$	$H_{1,1}$	$H_{1,4}$	$2H_{1,5}$	$H_{1,1}$	$H_{1,5}$	$H_{1,4}$	$H_{1,5}$	$H_{1,5}$
$H_{5,0}$	$6H_{5,0}$	$3H_{5,0}$	$6H_{5,1}$	$3H_{5,0}$	$2H_{5,0}$	$6H_{5,4}$	$2H_{5,0}$	$3H_{5,1}$	$H_{5,0}$	$6H_{5,5}$	$3H_{5,4}$	$2H_{5,1}$	$3H_{5,4}$	$H_{5,0}$	$H_{5,0}$	$2H_{5,4}$	$3H_{5,5}$	$H_{5,1}$	$2H_{5,5}$	$H_{5,4}$	$H_{5,4}$	$H_{5,5}$
$H_{0,5}$	$H_{0,0}$	$H_{0,0}$	$H_{0,1}$	$H_{0,1}$	$H_{0,0}$	$H_{0,4}$	$H_{0,4}$	$H_{0,1}$	$H_{0,0}$	$H_{0,5}$	$H_{0,5}$	$H_{0,1}$	$H_{0,4}$	$H_{0,1}$	$H_{0,5}$	$H_{0,4}$	$H_{0,5}$	$H_{0,1}$	$H_{0,5}$	$H_{0,4}$	$H_{0,5}$	$H_{0,5}$
H_6	$H_{0,0}$	$H_{1,0}$	$H_{0,1}$	H_1^Δ	$H_{0,0}$	$H_{0,4}$	$H_{0,4}$	$H_{1,1}$	$H_{1,0}$	$H_{0,5}$	H_6	$H_{0,1}$	$H_{1,4}$	H_1^Δ	H_6	$H_{0,4}$	$H_{1,5}$	$H_{1,1}$	$H_{0,5}$	$H_{1,4}$	H_6	$H_{1,5}$
$H_{4,1}$	$3H_{4,0}$	$2H_{4,0}$	$3H_{4,1}$	$H_{4,1}+H_{4,0}$	$H_{4,0}$	$3H_{4,4}$	$H_{4,0}$	$2H_{4,1}$	$H_{4,0}$	$3H_{4,5}$	$H_{4,5}+H_{4,4}$	$H_{4,1}$	$2H_{4,4}$	$H_{4,1}$	$H_{4,1}$	$H_{4,4}$	$H_{4,5}$	$H_{4,1}$	$H_{4,5}$	$H_{4,4}$	$H_{4,5}$	$H_{4,5}$
$H_{1,4}$	$2H_{1,0}$	$H_{1,0}$	$2H_{1,1}$	$H_{1,0}$	$2H_{1,0}$	$2H_{1,4}$	$2H_{1,4}$	$H_{1,1}$	$H_{1,0}$	$2H_{1,5}$	$H_{1,4}$	$2H_{1,1}$	$H_{1,4}$	$H_{1,0}$	$H_{1,4}$	$2H_{1,4}$	$H_{1,5}$	$H_{1,1}$	$2H_{1,5}$	$H_{1,4}$	$H_{1,4}$	$H_{1,5}$
H_7	$3H_{4,0}$	$H_{5,0}+H_{4,0}$	$3H_{4,1}$	$H_7+H_{4,0}$	$H_{4,0}$	$3H_{4,4}$	$H_{4,0}$	$H_{5,1}+H_{4,1}$	$H_{5,0}$	$3H_{4,5}$	$H_8+H_{4,4}$	$H_{4,1}$	$H_{5,4}+H_{4,4}$	H_7	H_7	$H_{4,4}$	$H_{5,5}+H_{4,5}$	$H_{5,1}$	$H_{4,5}$	$H_{5,4}$	H_8	$H_{5,5}$
H_5^Δ	$H_{0,0}$	$H_{1,0}$	$H_{0,1}$	H_1^Δ	$H_{4,0}$	$H_{0,4}$	H_4^Δ	$H_{1,1}$	$H_{5,0}$	$H_{0,5}$	H_6	$H_{4,1}$	$H_{1,4}$	H_7	H_5^Δ	$H_{4,4}$	$H_{1,5}$	$H_{5,1}$	$H_{4,5}$	$H_{5,4}$	H_8	$H_{5,5}$
$H_{4,4}$	$2H_{4,0}$	$H_{4,0}$	$2H_{4,1}$	$H_{4,0}$	$2H_{4,0}$	$2H_{4,4}$	$2H_{4,4}$	$H_{4,1}$	$H_{4,0}$	$2H_{4,5}$	$H_{4,4}$	$2H_{4,1}$	$H_{4,4}$	$H_{4,0}$	$H_{4,4}$	$2H_{4,4}$	$H_{4,5}$	$H_{4,1}$	$2H_{4,5}$	$H_{4,4}$	$H_{4,4}$	$H_{4,5}$
$H_{1,5}$	$H_{1,0}$	$H_{1,0}$	$H_{1,1}$	$H_{1,1}$	$H_{1,0}$	$H_{1,4}$	$H_{1,4}$	$H_{1,1}$	$H_{1,0}$	$H_{1,5}$	$H_{1,5}$	$H_{1,1}$	$H_{1,4}$	$H_{1,1}$	$H_{1,5}$	$H_{1,4}$	$H_{1,5}$	$H_{1,1}$	$H_{1,5}$	$H_{1,4}$	$H_{1,5}$	$H_{1,5}$
$H_{5,1}$	$3H_{5,0}$	$2H_{5,0}$	$3H_{5,1}$	$H_{5,1}+H_{5,0}$	$H_{5,0}$	$3H_{5,4}$	$H_{5,0}$	$2H_{5,1}$	$H_{5,0}$	$3H_{5,5}$	$H_{5,5}+H_{5,4}$	$H_{5,1}$	$2H_{5,4}$	$H_{5,1}$	$H_{5,1}$	$H_{5,4}$	$2H_{5,5}$	$H_{5,1}$	$H_{5,5}$	$H_{5,4}$	$H_{5,5}$	$H_{5,5}$
$H_{4,5}$	$H_{4,0}$	$H_{4,0}$	$H_{4,1}$	$H_{4,1}$	$H_{4,0}$	$H_{4,4}$	$H_{4,4}$	$H_{4,1}$	$H_{4,0}$	$H_{4,5}$	$H_{4,5}$	$H_{4,1}$	$H_{4,4}$	$H_{4,1}$	$H_{4,5}$	$H_{4,4}$	$H_{4,5}$	$H_{4,1}$	$H_{4,5}$	$H_{4,4}$	$H_{4,5}$	$H_{4,5}$
$H_{5,4}$	$2H_{5,0}$	$H_{5,0}$	$2H_{5,1}$	$H_{5,0}$	$2H_{5,0}$	$2H_{5,4}$	$2H_{5,4}$	$H_{5,1}$	$H_{5,0}$	$2H_{5,5}$	$H_{5,4}$	$2H_{5,1}$	$H_{5,4}$	$H_{5,0}$	$H_{5,4}$	$2H_{5,4}$	$H_{5,5}$	$H_{5,1}$	$2H_{5,5}$	$H_{5,4}$	$H_{5,4}$	$H_{5,5}$
H_8	$H_{4,0}$	$H_{5,0}$	$H_{4,1}$	H_7	$H_{4,0}$	$H_{4,4}$	$H_{4,4}$	$H_{5,1}$	$H_{5,0}$	$H_{4,5}$	H_8	$H_{4,1}$	$H_{5,4}$	H_7	H_8	$H_{4,4}$	$H_{5,5}$	$H_{5,1}$	$H_{4,5}$	$H_{5,4}$	H_8	$H_{5,5}$
$H_{5,5}$	$H_{5,0}$	$H_{5,0}$	$H_{5,1}$	$H_{5,1}$	$H_{5,0}$	$H_{5,4}$	$H_{5,4}$	$H_{5,1}$	$H_{5,0}$	$H_{5,5}$	$H_{5,5}$	$H_{5,1}$	$H_{5,4}$	$H_{5,1}$	$H_{5,5}$	$H_{5,4}$	$H_{5,5}$	$H_{5,1}$	$H_{5,5}$	$H_{5,4}$	$H_{5,5}$	$H_{5,5}$

Remark 136. *As a ring,* $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ *is generated by* H_1^Δ, H_4^Δ, $H_{1,1}$ *and* H_8.

Proof. We have

$$\begin{array}{lllll} H_{0,0} & = & H_4^\Delta \cdot H_1^\Delta & & \\ H_{1,0} & = & H_{1,1} \cdot H_4^\Delta & & \\ H_{0,1} & = & H_4^\Delta \cdot H_{1,1} & & \\ H_{4,0} & = & H_8 \cdot H_{0,0} & = & H_8 \cdot H_4^\Delta \cdot H_1^\Delta \\ H_{0,4} & = & H_{0,0} \cdot H_8 & = & H_4^\Delta \cdot H_1^\Delta \cdot H_8 \\ H_{5,0} & = & H_8 \cdot H_{1,0} & = & H_8 \cdot H_{1,1} \cdot H_4^\Delta \\ H_{0,5} & = & H_{0,1} \cdot H_8 & = & H_4^\Delta \cdot H_{1,1} \cdot H_8 \\ H_6 & = & H_1^\Delta \cdot H_8 & & \\ H_{4,1} & = & H_8 \cdot H_{0,1} & = & H_8 \cdot H_4^\Delta \cdot H_{1,1} \end{array} \qquad \begin{array}{lllll} H_{1,4} & = & H_{1,0} \cdot H_8 & = & H_{1,1} \cdot H_4^\Delta \cdot H_8 \\ H_7 & = & H_8 \cdot H_1^\Delta & & \\ H_5 & = & 1_{\mathrm{B}_{\mathbf{Z}}} & & \\ H_{4,4} & = & H_8 \cdot H_4^\Delta & & \\ H_{1,5} & = & H_{1,1} \cdot H_8 & & \\ H_{5,1} & = & H_8 \cdot H_{1,1} & & \\ H_{4,5} & = & H_{4,4} \cdot H_{1,5} & = & H_8 \cdot H_4^\Delta \cdot H_{1,1} \cdot H_8 \\ H_{5,4} & = & H_{5,1} \cdot H_{4,4} & = & H_8 \cdot H_{1,1} \cdot H_8 \cdot H_4^\Delta \\ H_{5,5} & = & H_{5,1} \cdot H_8 & = & H_8 \cdot H_{1,1} \cdot H_8\ . \end{array}$$

□

4.2 $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)/\operatorname{Jac}(\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3))$

Recall that $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3) = \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$. In particular, $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$ is a $\mathbf{Q}$-algebra of dimension 22 with $\mathbf{Q}$-linear basis

$(H_{0,0},\ H_{1,0},\ H_{0,1},\ H_1^\Delta,\ H_{4,0},\ H_{0,4},\ H_4^\Delta,\ H_{1,1},\ H_{5,0},\ H_{0,5},\ H_6,\ H_{4,1},\ H_{1,4},\ H_7,\ H_5^\Delta,\ H_{4,4},\ H_{1,5},\ H_{5,1},\ H_{4,5},\ H_{5,4},\ H_8,\ H_{5,5})$.

Let $\operatorname{Jac}(\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3))$ denote the Jacobson radical of $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$.

We have that

$$\overline{\mathrm{B}}_{\mathbf{Q}} := \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)/\operatorname{Jac}(\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3))$$

is semisimple. For $x \in \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$ we denote by $\overline{x} := x + \operatorname{Jac}(\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3))$.

Magma [6], using `JacobsonRadical`, gives a $\mathbf{Q}$-linear basis of $\operatorname{Jac}(\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3))$.

$$\begin{array}{l} (-\tfrac{1}{4}H_{0,0} + \tfrac{1}{2}H_{1,0} + \tfrac{1}{2}H_{0,1} + \tfrac{1}{4}H_{4,0} + \tfrac{1}{4}H_{0,4} - H_{1,1} - \tfrac{1}{2}H_{4,1} - \tfrac{1}{2}H_{1,4} - \tfrac{1}{4}H_{4,4} + H_{5,5}, \\ \quad \tfrac{1}{2}H_{0,0} - H_1^\Delta - \tfrac{1}{2}H_{4,4} + H_8, \\ \quad \tfrac{1}{2}H_{0,4} - H_{1,4} - \tfrac{1}{2}H_{4,4} + H_{5,4}, \\ \quad \tfrac{1}{2}H_{4,0} - H_{4,1} - \tfrac{1}{2}H_{4,4} + H_{4,5}, \\ \quad \tfrac{1}{2}H_{0,1} - H_{1,1} - \tfrac{1}{2}H_{4,1} + H_{5,1}, \\ \quad \tfrac{1}{2}H_{1,0} - H_{1,1} - \tfrac{1}{2}H_{1,4} + H_{1,5}, \\ \quad \tfrac{1}{2}H_{0,0} - H_1^\Delta - \tfrac{1}{2}H_{4,0} + H_7, \\ \quad \tfrac{1}{2}H_{0,0} - H_1^\Delta - \tfrac{1}{2}H_{0,4} + H_6, \\ \quad \tfrac{1}{2}H_{0,0} - H_{0,1} - \tfrac{1}{2}H_{0,4} + H_{0,5}, \\ \quad \tfrac{1}{2}H_{0,0} - H_{1,0} - \tfrac{1}{2}H_{4,0} + H_{5,0})\ . \end{array}$$

In addition, Magma provides, using `CentralIdempotents`, the following orthogonal decomposition of $1_{\overline{\mathrm{B}}_{\mathbf{Q}}} = \overline{H}_5^{\Delta}$ into primitive central idempotents of $\overline{\mathrm{B}}_{\mathbf{Q}}$. We have

$$1_{\overline{\mathrm{B}}_{\mathbf{Q}}} = \overline{\varepsilon}_1 + \overline{\varepsilon}_2 + \overline{\varepsilon}_3 + \overline{\varepsilon}_4, \text{ where}$$

$$\begin{array}{rcl}
\overline{\varepsilon}_1 & := & \frac{3}{4}\overline{H}_{0,0} - \overline{H}_{1,0} - \overline{H}_{0,1} - \frac{1}{4}\overline{H}_{4,0} - \frac{1}{4}\overline{H}_{0,4} + 2\overline{H}_{1,1} + \frac{3}{4}\overline{H}_{4,4} \\
\overline{\varepsilon}_2 & := & -\overline{H}_{0,0} + \overline{H}_{1,0} + \overline{H}_{0,1} + \overline{H}_1^{\Delta} - 2\overline{H}_{1,1} \\
\overline{\varepsilon}_3 & := & -\frac{1}{4}\overline{H}_{0,0} + \frac{1}{4}\overline{H}_{4,0} + \frac{1}{4}\overline{H}_{0,4} + \frac{1}{2}\overline{H}_4^{\Delta} - \frac{3}{4}\overline{H}_{4,4} \\
\overline{\varepsilon}_4 & := & \frac{1}{2}\overline{H}_{0,0} - \overline{H}_1^{\Delta} - \frac{1}{2}\overline{H}_4^{\Delta} + \overline{H}_5^{\Delta} \ .
\end{array}$$

As

$$1 = \dim_{\mathbf{Q}}(\overline{\varepsilon}_2\overline{\mathrm{B}}_{\mathbf{Q}}) = \dim_{\mathbf{Q}}(\overline{\varepsilon}_3\overline{\mathrm{B}}_{\mathbf{Q}}) = \dim_{\mathbf{Q}}(\overline{\varepsilon}_4\overline{\mathrm{B}}_{\mathbf{Q}})$$

and therefore $\overline{\varepsilon}_2\overline{\mathrm{B}}_{\mathbf{Q}} \cong \mathbf{Q}$, $\overline{\varepsilon}_3\overline{\mathrm{B}}_{\mathbf{Q}} \cong \mathbf{Q}$, $\overline{\varepsilon}_4\overline{\mathrm{B}}_{\mathbf{Q}} \cong \mathbf{Q}$, the idempotents $\overline{\varepsilon}_2$, $\overline{\varepsilon}_3$, $\overline{\varepsilon}_4$ are primitive. Further, $\dim_{\mathbf{Q}}(\overline{\varepsilon}_1\overline{\mathrm{B}}_{\mathbf{Q}}) = 9$. The idempotent $\overline{H}_{5,0}$ lies in $\overline{\varepsilon}_1\overline{\mathrm{B}}_{\mathbf{Q}}$ as

$$\begin{array}{rcl}
\overline{H}_{5,0} \cdot \overline{\varepsilon_1} & = & \frac{3}{4}\overline{H}_{5,0}\overline{H}_{0,0} - \overline{H}_{5,0}\overline{H}_{1,0} - \overline{H}_{5,0}\overline{H}_{0,1} - \frac{1}{4}\overline{H}_{5,0}\overline{H}_{4,0} - \frac{1}{4}\overline{H}_{5,0}\overline{H}_{0,4} + 2\overline{H}_{5,0}\overline{H}_{1,1} + \frac{3}{4}\overline{H}_{5,0}\overline{H}_{4,4} \\
& = & \frac{3}{4} \cdot 6\overline{H}_{5,0} - 3\overline{H}_{5,0} - 6\overline{H}_{5,1} - \frac{1}{4} \cdot 2\overline{H}_{5,0} - \frac{1}{4} \cdot 6\overline{H}_{5,4} + 2 \cdot 3\overline{H}_{5,1} + \frac{3}{4} \cdot 2\overline{H}_{5,4} \\
& = & \overline{H}_{5,0} \ .
\end{array}$$

Write

$$\overline{e} := \overline{H}_{5,0}$$

and $\overline{f} := \overline{\varepsilon}_1 - \overline{e}$.

We have

$$\overline{\varepsilon}_1\overline{\mathrm{B}}_{\mathbf{Q}} = \overline{e}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{e} \oplus \overline{e}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{f} \oplus \overline{f}\,\overline{\mathrm{B}}_{\mathbf{Q}}\overline{e} \oplus \overline{f}\,\overline{\mathrm{B}}_{\mathbf{Q}}\overline{f},$$

where $\dim_{\mathbf{Q}}(\overline{e}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{e}) = 1$ and $\dim_{\mathbf{Q}}(\overline{f}\,\overline{\mathrm{B}}_{\mathbf{Q}}\overline{f}) = 4$.

Magma now offers a basis of $\overline{f}\,\overline{\mathrm{B}}_{\mathbf{Q}}\overline{f}$ which contains an idempotent element. Therefore, we obtain the idempotents $\overline{g}, \overline{h} \in \overline{f}\,\overline{\mathrm{B}}_{\mathbf{Q}}\overline{f}$

$$\begin{array}{rcl}
\overline{g} & := & \frac{4}{3}\overline{H}_{0,0} - 2\overline{H}_{1,0} - \frac{4}{3}\overline{H}_{0,1} - \overline{H}_{4,0} + 2\overline{H}_{1,1} + \overline{H}_{4,1} \\
\\
\overline{h} & := & \overline{f} - \overline{g} = -\frac{1}{12}\overline{H}_{0,0} + \frac{1}{3}\overline{H}_{0,1} + \frac{1}{4}\overline{H}_{4,0} - \frac{1}{4}\overline{H}_{0,4} + \frac{3}{4}\overline{H}_{4,4} - \overline{H}_{4,1} \ .
\end{array}$$

Thus,

$$\overline{\varepsilon}_1\overline{\mathrm{B}}_{\mathbf{Q}} = \overline{e}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{e} \oplus \overline{e}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{g} \oplus \overline{e}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{h} \oplus \overline{g}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{e} \oplus \overline{g}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{g} \oplus \overline{g}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{h} \oplus \overline{h}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{e} \oplus \overline{h}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{g} \oplus \overline{h}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{h},$$

where $\dim_{\mathbf{Q}}(\overline{g}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{g}) = \dim_{\mathbf{Q}}(\overline{h}\overline{\mathrm{B}}_{\mathbf{Q}}\overline{h}) = 1$.

Hence, we have the following orthogonal decomposition of $1_{\overline{\mathrm{B}}_{\mathbf{Q}}}$ into primitive idempotents of $\overline{\mathrm{B}}_{\mathbf{Q}}$.

$$1_{\overline{\mathrm{B}}_{\mathbf{Q}}} = \overline{e} + \overline{g} + \overline{h} + \overline{\varepsilon}_2 + \overline{\varepsilon}_3 + \overline{\varepsilon}_4$$

4.3 $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$

4.3.1 Peirce decomposition

Using the computer algebra system Magma we obtain an orthogonal decomposition of $1_{\mathrm{B}_{\mathbf{Q}}}$ into primitive idempotents of $\mathrm{B}_{\mathbf{Q}}$ by lifting the idempotents of $\overline{\mathrm{B}}_{\mathbf{Q}}$ obtained in section 4.2.

$$\begin{array}{rcr} e & := & -\frac{1}{2}H_{0,0} + H_{1,0} + \frac{1}{2}H_{4,0} \\ g & := & \frac{4}{3}H_{0,0} - 2H_{1,0} - \frac{4}{3}H_{0,1} - H_{4,0} + 2H_{1,1} + H_{4,1} \\ h & := & -\frac{1}{12}H_{0,0} + \frac{1}{3}H_{0,1} + \frac{1}{4}H_{4,0} - \frac{1}{4}H_{0,4} + \frac{3}{4}H_{4,4} - H_{4,1} \\ \varepsilon_2 & := & -H_{0,0} + H_{1,0} + H_{0,1} + H_1^{\Delta} - 2H_{1,1} \\ \varepsilon_3 & := & -\frac{1}{4}H_{0,0} + \frac{1}{4}H_{4,0} + \frac{1}{4}H_{0,4} + \frac{1}{2}H_4^{\Delta} - \frac{3}{4}H_{4,4} \\ \varepsilon_4 & := & \frac{1}{2}H_{0,0} - H_1^{\Delta} - \frac{1}{2}H_4^{\Delta} + H_5^{\Delta} \end{array}$$

Further,

Peirce component	$e\,\mathrm{B}_{\mathbf{Q}}\,e$	$e\,\mathrm{B}_{\mathbf{Q}}\,g$	$e\,\mathrm{B}_{\mathbf{Q}}\,h$	$g\,\mathrm{B}_{\mathbf{Q}}\,e$	$g\,\mathrm{B}_{\mathbf{Q}}\,g$	$g\,\mathrm{B}_{\mathbf{Q}}\,h$	$h\,\mathrm{B}_{\mathbf{Q}}\,e$	$h\,\mathrm{B}_{\mathbf{Q}}\,g$	$h\,\mathrm{B}_{\mathbf{Q}}\,h$
$\dim_{\mathbf{Q}}$	1	1	1	1	1	1	1	1	1

Peirce component	$e\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$e\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3$	$e\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$g\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$g\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3$	$g\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$h\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$h\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3$	$h\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$
$\dim_{\mathbf{Q}}$	0	0	1	0	0	1	0	0	1

Peirce component	$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,e$	$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,g$	$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,h$	$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3$	$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$
$\dim_{\mathbf{Q}}$	0	0	0	1	0	1

Peirce component	$\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,e$	$\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,g$	$\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,h$	$\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3$	$\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$
$\dim_{\mathbf{Q}}$	0	0	0	0	1	0

Peirce component	$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,e$	$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,g$	$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,h$	$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3$	$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$
$\dim_{\mathbf{Q}}$	1	1	1	1	0	3

.

In particular, $e, g, h, \varepsilon_2, \varepsilon_3$ are primitive idempotents, confirming the assertion above.

Pictorially,

$$\mathrm{B}_{\mathbf{Q}} = \begin{bmatrix} e\,\mathrm{B}_{\mathbf{Q}}\,e & e\,\mathrm{B}_{\mathbf{Q}}\,g & e\,\mathrm{B}_{\mathbf{Q}}\,h & 0 & 0 & e\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ g\,\mathrm{B}_{\mathbf{Q}}\,e & g\,\mathrm{B}_{\mathbf{Q}}\,g & g\,\mathrm{B}_{\mathbf{Q}}\,h & 0 & 0 & g\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ h\,\mathrm{B}_{\mathbf{Q}}\,e & h\,\mathrm{B}_{\mathbf{Q}}\,g & h\,\mathrm{B}_{\mathbf{Q}}\,h & 0 & 0 & h\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ 0 & 0 & 0 & \varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 & 0 & \varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ 0 & 0 & 0 & 0 & \varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3 & 0 \\ \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,e & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,g & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,h & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 & 0 & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \end{bmatrix} \quad \begin{array}{l}\text{has Peirce}\\ \text{components}\\ \text{of dimensions}\end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 3 \end{bmatrix}.$$

In a next step, we fix $\mathbf{Q}$-linear bases of the Peirce components of $\mathrm{B}_{\mathbf{Q}} = \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$.

Peirce component	$\mathbf{Q}$-linear basis
$e\,\mathrm{B}_{\mathbf{Q}}\,e$	$e := -\frac{1}{2}H_{0,0} + H_{1,0} + \frac{1}{2}H_{4,0}$
$e\,\mathrm{B}_{\mathbf{Q}}\,g$	$b_{e,g} := \frac{1}{2}H_{0,0} - H_{1,0} - \frac{1}{2}H_{0,1} - \frac{1}{2}H_{4,0} + H_{1,1} + \frac{1}{2}H_{4,1}$
$e\,\mathrm{B}_{\mathbf{Q}}\,h$	$b_{e,h} := -\frac{1}{8}H_{0,0} + \frac{1}{4}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{8}H_{4,0} - \frac{3}{8}H_{0,4} - H_{1,1} - \frac{1}{2}H_{4,1} + \frac{3}{4}H_{1,4} + \frac{3}{8}H_{4,4}$
$g\,\mathrm{B}_{\mathbf{Q}}\,e$	$b_{g,e} := -\frac{4}{3}H_{0,0} + 2H_{1,0} + H_{4,0}$
$g\,\mathrm{B}_{\mathbf{Q}}\,g$	$g := \frac{4}{3}H_{0,0} - 2H_{1,0} - \frac{4}{3}H_{0,1} - H_{4,0} + 2H_{1,1} + H_{4,1}$
$g\,\mathrm{B}_{\mathbf{Q}}\,h$	$b_{g,h} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{4}{3}H_{0,1} + \frac{1}{4}H_{4,0} - H_{0,4} - 2H_{1,1} - H_{4,1} + \frac{3}{2}H_{1,4} + \frac{3}{4}H_{4,4}$
$h\,\mathrm{B}_{\mathbf{Q}}\,e$	$b_{h,e} := -\frac{1}{3}H_{0,0} + H_{4,0}$
$h\,\mathrm{B}_{\mathbf{Q}}\,g$	$b_{h,g} := \frac{1}{3}H_{0,0} - \frac{1}{3}H_{0,1} - H_{4,0} + H_{4,1}$
$h\,\mathrm{B}_{\mathbf{Q}}\,h$	$h := -\frac{1}{12}H_{0,0} + \frac{1}{3}H_{0,1} + \frac{1}{4}H_{4,0} - \frac{1}{4}H_{0,4} + \frac{3}{4}H_{4,4} - H_{4,1}$
$e\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$b_{e,\varepsilon_4} := -\frac{1}{8}H_{0,0} + \frac{1}{4}H_{1,0} + \frac{1}{4}H_{0,1} + \frac{1}{8}H_{4,0} + \frac{1}{8}H_{0,4} - \frac{1}{2}H_{1,1} - \frac{1}{4}H_{0,5} - \frac{1}{4}H_{4,1}$ $- \frac{1}{4}H_{1,4} - \frac{1}{8}H_{4,4} + \frac{1}{2}H_{1,5} + \frac{1}{4}H_{4,5}$
$g\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$b_{g,\varepsilon_4} := -\frac{1}{3}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{2}{3}H_{0,1} + \frac{1}{4}H_{4,0} + \frac{1}{3}H_{0,4} - H_{1,1} - \frac{2}{3}H_{0,5} - \frac{1}{2}H_{4,1}$ $- \frac{1}{2}H_{1,4} - \frac{1}{4}H_{4,4} + H_{1,5} + \frac{1}{2}H_{4,5}$
$h\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$b_{h,\varepsilon_4} := -\frac{1}{12}H_{0,0} + \frac{1}{6}H_{0,1} + \frac{1}{4}H_{4,0} + \frac{1}{12}H_{0,4} - \frac{1}{6}H_{0,5} - \frac{1}{2}H_{4,1} - \frac{1}{4}H_{4,4} + \frac{1}{2}H_{4,5}$
$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$\varepsilon_2 := -H_{0,0} + H_{1,0} + H_{0,1} + H_1^{\Delta} - 2H_{1,1}$
$\varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$b_{\varepsilon_2,\varepsilon_4} := -\frac{1}{2}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_1^{\Delta} + \frac{1}{2}H_{0,4} - H_{1,1} - \frac{1}{2}H_{0,5} - \frac{1}{2}H_6 - \frac{1}{2}H_{1,4}$ $+ H_{1,5}$
$\varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3$	$\varepsilon_3 := -\frac{1}{4}H_{0,0} + \frac{1}{4}H_{4,0} + \frac{1}{4}H_{0,4} + \frac{1}{2}H_4^{\Delta} - \frac{3}{4}H_{4,4}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,e$	$b_{\varepsilon_4,e} := \frac{1}{6}H_{0,0} - \frac{1}{3}H_{1,0} - \frac{1}{6}H_{4,0} + \frac{1}{3}H_{5,0}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,g$	$b_{\varepsilon_4,g} := -\frac{1}{6}H_{0,0} + \frac{1}{3}H_{1,0} + \frac{1}{6}H_{0,1} + \frac{1}{6}H_{4,0} - \frac{1}{3}H_{1,1} - \frac{1}{3}H_{5,0} - \frac{1}{6}H_{4,1} + \frac{1}{3}H_{5,1}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,h$	$b_{\varepsilon_4,h} := \frac{1}{24}H_{0,0} - \frac{1}{12}H_{1,0} - \frac{1}{6}H_{0,1} - \frac{1}{24}H_{4,0} + \frac{1}{8}H_{0,4} + \frac{1}{3}H_{1,1} + \frac{1}{12}H_{5,0}$ $+ \frac{1}{6}H_{4,1} - \frac{1}{4}H_{1,4} - \frac{1}{8}H_{4,4} - \frac{1}{3}H_{5,1} + \frac{1}{4}H_{5,4}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2$	$b_{\varepsilon_4,\varepsilon_2} := -\frac{1}{2}H_{0,0} + \frac{1}{2}H_{1,0} + \frac{1}{2}H_{0,1} + \frac{1}{2}H_1^{\Delta} + \frac{1}{2}H_{4,0} - H_{1,1} - \frac{1}{2}H_{5,0} - \frac{1}{2}H_{4,1}$ $- \frac{1}{2}H_7 + H_{5,1}$
$\varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4$	$(\varepsilon_4 := \frac{1}{2}H_{0,0} - H_1^{\Delta} - \frac{1}{2}H_4^{\Delta} + H_5^{\Delta},$ $b'_{\varepsilon_4,\varepsilon_4} := \frac{1}{24}H_{0,0} - \frac{1}{12}H_{1,0} - \frac{1}{12}H_{0,1} - \frac{1}{24}H_{4,0} - \frac{1}{24}H_{0,4} + \frac{1}{6}H_{1,1} + \frac{1}{12}H_{5,0} + \frac{1}{12}H_{0,5}$ $+ \frac{1}{12}H_{4,1} + \frac{1}{12}H_{1,4} + \frac{1}{24}H_{4,4} - \frac{1}{6}H_{1,5} - \frac{1}{6}H_{5,1} - \frac{1}{12}H_{4,5} - \frac{1}{12}H_{5,4} + \frac{1}{6}H_{5,5}$, $b''_{\varepsilon_4,\varepsilon_4} := \frac{1}{4}H_{0,0} - \frac{3}{4}H_{1,0} - \frac{3}{4}H_{0,1} + \frac{1}{4}H_1^{\Delta} - \frac{1}{4}H_{4,0} - \frac{1}{4}H_{0,4} + \frac{3}{2}H_{1,1} + \frac{3}{4}H_{5,0}$ $+ \frac{3}{4}H_{0,5} - \frac{1}{4}H_6 + \frac{3}{4}H_{4,1} + \frac{3}{4}H_{1,4} - \frac{1}{4}H_7 + \frac{1}{4}H_{4,4} - \frac{3}{2}H_{1,5} - \frac{3}{2}H_{5,1}$ $- \frac{3}{4}H_{4,5} - \frac{3}{4}H_{5,4} + \frac{1}{4}H_8 + \frac{3}{2}H_{5,5})$

Then we have the following multiplication table for the basis elements of $\mathrm{B}_{\mathbf{Q}} = \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$.

$(\cdot)$	e	$b_{e,g}$	$b_{e,h}$	$b_{g,e}$	g	$b_{g,h}$	$b_{h,e}$	$b_{h,g}$	h	b_{e,ε_4}	b_{g,ε_4}	b_{h,ε_4}	ε_2	$b_{\varepsilon_2,\varepsilon_4}$	ε_3	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	$b_{\varepsilon_4,\varepsilon_2}$	ε_4	$b'_{\varepsilon_4,\varepsilon_4}$	$b''_{\varepsilon_4,\varepsilon_4}$
e	e	$b_{e,g}$	$b_{e,h}$	0	0	0	0	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
$b_{e,g}$	0	0	0	e	$b_{e,g}$	$b_{e,h}$	0	0	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0
$b_{e,h}$	0	0	0	0	0	0	e	$b_{e,g}$	$b_{e,h}$	0	0	b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0
$b_{g,e}$	$b_{g,e}$	g	$b_{g,h}$	0	0	0	0	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
g	0	0	0	$b_{g,e}$	g	$b_{g,h}$	0	0	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0
$b_{g,h}$	0	0	0	0	0	0	$b_{g,e}$	g	$b_{g,h}$	0	0	b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0
$b_{h,e}$	$b_{h,e}$	$b_{h,g}$	h	0	0	0	0	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0
$b_{h,g}$	0	0	0	$b_{h,e}$	$b_{h,g}$	h	0	0	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0
h	0	0	0	0	0	0	$b_{h,e}$	$b_{h,g}$	h	0	0	b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0
b_{e,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{e,ε_4}	0	0
b_{g,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{g,ε_4}	0	0
b_{h,ε_4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_{h,ε_4}	0	0
ε_2	0	0	0	0	0	0	0	0	0	0	0	0	ε_2	$b_{\varepsilon_2,\varepsilon_4}$	0	0	0	0	0	0	0	0
$b_{\varepsilon_2,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_2,\varepsilon_4}$	0	0
ε_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ε_3	0	0	0	0	0	0	0
$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,g}$	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,h}$	0	0	0	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0
$b_{\varepsilon_4,\varepsilon_2}$	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_4,\varepsilon_2}$	$b''_{\varepsilon_4,\varepsilon_4} - 12b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0
ε_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b_{\varepsilon_4,e}$	$b_{\varepsilon_4,g}$	$b_{\varepsilon_4,h}$	$b_{\varepsilon_4,\varepsilon_2}$	ε_4	$b'_{\varepsilon_4,\varepsilon_4}$	$b''_{\varepsilon_4,\varepsilon_4}$
$b'_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b'_{\varepsilon_4,\varepsilon_4}$	0	0
$b''_{\varepsilon_4,\varepsilon_4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$b''_{\varepsilon_4,\varepsilon_4}$	0	0

We see that ε_3 is even central, but ε_2 and ε_4 are not.

Lemma 137. *Write* $\mathbf{Q}[\eta, \xi]/(\eta^2, \eta\xi, \xi^2) = \mathbf{Q}[\overline{\eta}, \overline{\xi}]$, *where* $\overline{\xi} := \xi + (\eta^2, \eta\xi, \xi^2)$ *and* $\overline{\eta} := \eta + (\eta^2, \eta\xi, \xi^2)$.

The map

$$\begin{array}{rccc} \mu : & \mathbf{Q}[\overline{\eta}, \overline{\xi}] & \to & \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ & \overline{\eta} & \mapsto & b'_{\varepsilon_4,\varepsilon_4} \\ & \overline{\xi} & \mapsto & b''_{\varepsilon_4,\varepsilon_4} \end{array}$$

is a $\mathbf{Q}$*-algebra isomorphism.*

Proof. Since $\varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 = {}_{\mathbf{Q}}\langle \varepsilon_4, b'_{\varepsilon_4,\varepsilon_4}, b''_{\varepsilon_4,\varepsilon_4} \rangle$ is commutative and

$$\begin{aligned} (b'_{\varepsilon_4,\varepsilon_4})^2 &= 0 \\ (b''_{\varepsilon_4,\varepsilon_4})^2 &= 0 \\ b'_{\varepsilon_4,\varepsilon_4} b''_{\varepsilon_4,\varepsilon_4} &= 0 \, , \end{aligned}$$

the map μ is a well-defined $\mathbf{Q}$-algebra morphism.

As the $\mathbf{Q}$-linear basis $(\varepsilon_4, b'_{\varepsilon_4,\varepsilon_4}, b''_{\varepsilon_4,\varepsilon_4})$ of $\varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4$ lies in the image of μ, the map is surjective. Because of $\dim_{\mathbf{Q}}(\varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4) = 3 = \dim_{\mathbf{Q}}(\mathbf{Q}[\overline{\eta}, \overline{\xi}])$ it is bijective. $\square$

Remark 138. *The ring* $\mathbf{Q}[\overline{\eta}, \overline{\xi}]$ *is local.*

In particular, ε_4 *is a primitive idempotent of* $\mathrm{B}_{\mathbf{Q}}$*, confirming the assertion above.*

Proof. For $u := a + b\overline{\eta} + c\overline{\xi} \in \mathbf{Q}[\overline{\eta}, \overline{\xi}]$ with $a \neq 0$ the inverse is given by $u^{-1} = a^{-1} - a^{-2}b\overline{\eta} - a^{-2}c\overline{\xi}$ as

$$uu^{-1} = aa^{-1} + (-a^{-1}b + a^{-1}b)\overline{\eta} + (-a^{-1}c + a^{-1}c)\overline{\xi} = 1 \ .$$

Therefore, $\mathrm{U}(\mathbf{Q}[\overline{\eta}, \overline{\xi}]) = \mathbf{Q}[\overline{\eta}, \overline{\xi}] \setminus (\overline{\eta}, \overline{\xi})$. Thus, the nonunits of $\mathbf{Q}[\overline{\eta}, \overline{\xi}]$ form an ideal and so $\mathbf{Q}[\overline{\eta}, \overline{\xi}]$ is a local ring. □

Lemma 139. *Write* $\varepsilon_1 := e + g + h$*. We have an isomorphism of* $\mathbf{Q}$*-algebras*

$$\begin{array}{cccc} \nu : & \varepsilon_1 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 & \to & \mathbf{Q}^{3\times 3} \\ & \begin{bmatrix} exe & exg & exh \\ gxe & gxg & gxh \\ hxe & hxg & hxh \end{bmatrix} = \begin{bmatrix} s_{1,1}e & s_{1,2}b_{e,g} & s_{1,3}b_{e,h} \\ s_{2,1}b_{g,e} & s_{2,2}g & s_{2,3}b_{g,h} \\ s_{3,1}b_{h,e} & s_{3,2}b_{h,g} & s_{3,3}h \end{bmatrix} & \mapsto & \begin{pmatrix} s_{1,1} & s_{1,2} & s_{1,3} \\ s_{2,1} & s_{2,2} & s_{2,3} \\ s_{3,1} & s_{3,2} & s_{3,3} \end{pmatrix} \end{array}$$

for $x \in \mathrm{B}_{\mathbf{Q}}$ *and* $s_{i,j} \in \mathbf{Q}$*, for* $i, j \in [1, 3]$*.*

Proof. The $\mathbf{Q}$-linear map ν is bijective.

We have $\nu(\varepsilon_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

We have

$$\begin{aligned}
& \nu\left(\begin{bmatrix} r_{1,1}e & r_{1,2}b_{e,g} & r_{1,3}b_{e,h} \\ r_{2,1}b_{g,e} & r_{2,2}g & r_{2,3}b_{g,h} \\ r_{3,1}b_{h,e} & r_{3,2}b_{h,g} & r_{3,3}h \end{bmatrix} \cdot \begin{bmatrix} \tilde{r}_{1,1}e & \tilde{r}_{1,2}b_{e,g} & \tilde{r}_{1,3}b_{e,h} \\ \tilde{r}_{2,1}b_{g,e} & \tilde{r}_{2,2}g & \tilde{r}_{2,3}b_{g,h} \\ \tilde{r}_{3,1}b_{h,e} & \tilde{r}_{3,2}b_{h,g} & \tilde{r}_{3,3}h \end{bmatrix} \right) \\
= \ & \nu\left(\begin{bmatrix} (r_{1,1}\tilde{r}_{1,1} + r_{1,2}\tilde{r}_{2,1} + r_{1,3}\tilde{r}_{3,1})e & (r_{1,1}\tilde{r}_{1,2} + r_{1,2}\tilde{r}_{2,2} + r_{1,3}\tilde{r}_{3,2})b_{e,g} & (r_{1,1}\tilde{r}_{1,3} + r_{1,2}\tilde{r}_{2,3} + r_{1,3}\tilde{r}_{3,3})b_{e,h} \\ (r_{2,1}\tilde{r}_{1,1} + r_{2,2}\tilde{r}_{2,1} + r_{2,3}\tilde{r}_{3,1})b_{g,e} & (r_{2,1}\tilde{r}_{1,2} + r_{2,2}\tilde{r}_{2,2} + r_{2,3}\tilde{r}_{3,2})g & (r_{2,1}\tilde{r}_{1,3} + r_{2,2}\tilde{r}_{2,3} + r_{2,3}\tilde{r}_{3,3})b_{g,h} \\ (r_{3,1}\tilde{r}_{1,1} + r_{3,2}\tilde{r}_{2,1} + r_{3,3}\tilde{r}_{3,1})b_{h,e} & (r_{3,1}\tilde{r}_{1,2} + r_{3,2}\tilde{r}_{2,2} + r_{3,3}\tilde{r}_{3,2})b_{h,g} & (r_{3,1}\tilde{r}_{1,3} + r_{3,2}\tilde{r}_{2,3} + r_{3,3}\tilde{r}_{3,3})h \end{bmatrix} \right) \\
= \ & \begin{pmatrix} r_{1,1}\tilde{r}_{1,1} + r_{1,2}\tilde{r}_{2,1} + r_{1,3}\tilde{r}_{3,1} & r_{1,1}\tilde{r}_{1,2} + r_{1,2}\tilde{r}_{2,2} + r_{1,3}\tilde{r}_{3,2} & r_{1,1}\tilde{r}_{1,3} + r_{1,2}\tilde{r}_{2,3} + r_{1,3}\tilde{r}_{3,3} \\ r_{2,1}\tilde{r}_{1,1} + r_{2,2}\tilde{r}_{2,1} + r_{2,3}\tilde{r}_{3,1} & r_{2,1}\tilde{r}_{1,2} + r_{2,2}\tilde{r}_{2,2} + r_{2,3}\tilde{r}_{3,2} & r_{2,1}\tilde{r}_{1,3} + r_{2,2}\tilde{r}_{2,3} + r_{2,3}\tilde{r}_{3,3} \\ r_{3,1}\tilde{r}_{1,1} + r_{3,2}\tilde{r}_{2,1} + r_{3,3}\tilde{r}_{3,1} & r_{3,1}\tilde{r}_{1,2} + r_{3,2}\tilde{r}_{2,2} + r_{3,3}\tilde{r}_{3,2} & r_{3,1}\tilde{r}_{1,3} + r_{3,2}\tilde{r}_{2,3} + r_{3,3}\tilde{r}_{3,3} \end{pmatrix} \\
= \ & \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix} \cdot \begin{pmatrix} \tilde{r}_{1,1} & \tilde{r}_{1,2} & \tilde{r}_{1,3} \\ \tilde{r}_{2,1} & \tilde{r}_{2,2} & \tilde{r}_{2,3} \\ \tilde{r}_{3,1} & \tilde{r}_{3,2} & \tilde{r}_{3,3} \end{pmatrix} \\
= \ & \nu\left(\begin{bmatrix} r_{1,1}e & r_{1,2}b_{e,g} & r_{1,3}b_{e,h} \\ r_{2,1}b_{g,e} & r_{2,2}g & r_{2,3}b_{g,h} \\ r_{3,1}b_{h,e} & r_{3,2}b_{h,g} & r_{3,3}h \end{bmatrix} \right) \cdot \nu\left(\begin{bmatrix} \tilde{r}_{1,1}e & \tilde{r}_{1,2}b_{e,g} & \tilde{r}_{1,3}b_{e,h} \\ \tilde{r}_{2,1}b_{g,e} & \tilde{r}_{2,2}g & \tilde{r}_{2,3}b_{g,h} \\ \tilde{r}_{3,1}b_{h,e} & \tilde{r}_{3,2}b_{h,g} & \tilde{r}_{3,3}h \end{bmatrix} \right)
\end{aligned}$$

for $r_{i,j}, \tilde{r}_{i,j} \in \mathbf{Q}$, for $i, j \in [1, 3]$. □

4.3.2 Peirce composition

We aim to construct a $\mathbf{Q}$-algebra $A := \bigoplus_{i,j} A_{i,j}$ such that $A \cong \mathrm{B}_{\mathbf{Q}} = \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$ as a Peirce composite in the sense of Appendix A.

Set $\varepsilon_1 := e + g + h$. Recall that $1_{\mathrm{B}_{\mathbf{Q}}} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, cf. Section 4.3.1.

In a first step to construct such a Peirce-composed $\mathbf{Q}$-algebra with $A \cong \mathrm{B}_{\mathbf{Q}}$ we choose $\mathbf{Q}$-vector spaces $A_{i,j}$ and $\mathbf{Q}$-linear isomorphisms $\gamma_{i,j} : A_{i,j} \xrightarrow{\sim} \varepsilon_i \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_j$ for $i, j \in [1, 4]$.

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix} := \begin{bmatrix} \mathbf{Q}^{3\times 3} & 0 & 0 & \mathbf{Q}^{3\times 1} \\ 0 & \mathbf{Q} & 0 & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} & 0 \\ \mathbf{Q}^{1\times 3} & \mathbf{Q} & 0 & \mathbf{Q}[\overline{\eta}, \overline{\xi}] \end{bmatrix}$$

We let $\gamma_{s,t} := 0$ for $(s,t) \in \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (3,4), (4,3)\}$. Let

$$\begin{array}{rrcl} \gamma_{1,1} := \nu^{-1} : & A_{1,1} & \xrightarrow{\sim} & \varepsilon_1 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 \\ & \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix} & \mapsto & \begin{bmatrix} r_{1,1} e & r_{1,2} b_{e,g} & r_{1,3} b_{e,h} \\ r_{2,1} b_{g,e} & r_{2,2} g & r_{2,3} b_{g,h} \\ r_{3,1} b_{h,e} & r_{3,2} b_{h,g} & r_{3,3} h \end{bmatrix} \end{array} \text{, where } r_{i,j} \in \mathbf{Q} \text{ for } i, j \in [1, 3] \text{, cf. Lemma 139.}$$

$$\begin{array}{rrcl} \gamma_{1,4} : & A_{1,4} & \xrightarrow{\sim} & \varepsilon_1 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ & \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} & \mapsto & \begin{bmatrix} u_1 b_{e,\varepsilon_4} \\ u_2 b_{g,\varepsilon_4} \\ u_3 b_{h,\varepsilon_4} \end{bmatrix} \end{array}$$

$$\begin{array}{rrcl} \gamma_{2,2} : & A_{2,2} & \xrightarrow{\sim} & \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 \\ & u & \mapsto & u\varepsilon_2 \\ \gamma_{2,4} : & A_{2,4} & \xrightarrow{\sim} & \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ & u & \mapsto & u b_{\varepsilon_2,\varepsilon_4} \\ \gamma_{3,3} : & A_{3,3} & \xrightarrow{\sim} & \varepsilon_3 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_3 \\ & u & \mapsto & u\varepsilon_3 \end{array}$$

$$\begin{array}{rrcl} \gamma_{4,1} : & A_{4,1} & \xrightarrow{\sim} & \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 \\ & \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} & \mapsto & \begin{bmatrix} v_1 b_{\varepsilon_4,e} & v_2 b_{\varepsilon_4,g} & v_3 b_{\varepsilon_4,h} \end{bmatrix} \end{array}$$

$$\begin{array}{rrcl} \gamma_{4,2} : & A_{4,2} & \xrightarrow{\sim} & \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 \\ & u & \mapsto & u b_{\varepsilon_4,\varepsilon_2} \end{array}$$

$$\begin{array}{rrcl} \gamma_{4,4} := \mu : & A_{4,4} & \xrightarrow{\sim} & \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ & a + b\overline{\eta} + c\overline{\xi} & \mapsto & a\varepsilon_4 + b b'_{\varepsilon_4,\varepsilon_4} + c b''_{\varepsilon_4,\varepsilon_4} \end{array} \text{, where } a, b, c \in \mathbf{Q} \text{, cf. Lemma 137.}$$

Let $\beta : \mathrm{B}_{\mathbf{Q}} \times \mathrm{B}_{\mathbf{Q}} \to \mathrm{B}_{\mathbf{Q}}$ be the multiplication map on $\mathrm{B}_{\mathbf{Q}}$. Write

$$\beta_{i,j,k} := \beta|^{\varepsilon_i \mathrm{B}_{\mathbf{Q}} \varepsilon_k}_{\varepsilon_i \mathrm{B}_{\mathbf{Q}} \varepsilon_j \times \varepsilon_j \mathrm{B}_{\mathbf{Q}} \varepsilon_k} : \varepsilon_i \mathrm{B}_{\mathbf{Q}} \varepsilon_j \times \varepsilon_j \mathrm{B}_{\mathbf{Q}} \varepsilon_k \to \varepsilon_i \mathrm{B}_{\mathbf{Q}} \varepsilon_k \ .$$

We now construct $\alpha_{i,j,k}$ such that the quadrangle

$$\begin{tikzcd}
A_{i,j} \times A_{j,k} \arrow[r, "\alpha_{i,j,k}"] \arrow[d, "\gamma_{i,j} \times \gamma_{j,k}"'] & A_{i,k} \arrow[d, "\gamma_{i,k}"'] \\
\varepsilon_i \mathrm{B}_{\mathbf{Q}} \varepsilon_j \times \varepsilon_j \mathrm{B}_{\mathbf{Q}} \varepsilon_k \arrow[r, "\beta_{i,j,k}"] & \varepsilon_i \mathrm{B}_{\mathbf{Q}} \varepsilon_k
\end{tikzcd}$$

commutes for $i, j, k \in [1, 4]$.

We let $\alpha_{i,j,k} := 0$ if (i, j), (j, k) or (i, k) is contained in

$$\{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 3)\} \ .$$

Ad $\alpha_{1,1,1}$.

Using Lemma 139 we get

$$\begin{array}{rcl} \alpha_{1,1,1} := \gamma_{1,1}^{-1} \circ \beta_{1,1,1} \circ (\gamma_{1,1} \times \gamma_{1,1}) : A_{1,1} \times A_{1,1} & \to & A_{1,1} \\ (X, Y) & \mapsto & XY \ . \end{array}$$

Ad $\alpha_{1,1,4}$.

We have

$$\begin{array}{rcl} \beta_{1,1,4} : \varepsilon_1 \mathrm{B}_{\mathbf{Q}} \varepsilon_1 \times \varepsilon_1 \mathrm{B}_{\mathbf{Q}} \varepsilon_4 & \to & \varepsilon_1 \mathrm{B}_{\mathbf{Q}} \varepsilon_4 \\ \left(\begin{bmatrix} r_{1,1}e & r_{1,2}b_{e,g} & r_{1,3}b_{e,h} \\ r_{2,1}b_{g,e} & r_{2,2}g & r_{2,3}b_{g,h} \\ r_{3,1}b_{h,e} & r_{3,2}b_{h,g} & r_{3,3}h \end{bmatrix}, \begin{bmatrix} u_1 b_{e,\varepsilon_4} \\ u_2 b_{g,\varepsilon_4} \\ u_3 b_{h,\varepsilon_4} \end{bmatrix} \right) & \mapsto & \begin{bmatrix} (r_{1,1}u_1 + r_{1,2}u_2 + r_{1,3}u_3)b_{e,\varepsilon_4} \\ (r_{2,1}u_1 + r_{2,2}u_2 + r_{2,3}u_3)b_{g,\varepsilon_4} \\ (r_{3,1}u_1 + r_{3,2}u_2 + r_{3,3}u_3)b_{h,\varepsilon_4} \end{bmatrix} \ , \end{array}$$

where $r_{i,j}, u_k \in \mathbf{Q}$ for $i, j, k \in [1, 3]$.

Hence,

$$\begin{array}{rcl} \alpha_{1,1,4} := \gamma_{1,4}^{-1} \circ \beta_{1,1,4} \circ (\gamma_{1,1} \times \gamma_{1,4}) : A_{1,1} \times A_{1,4} & \to & A_{1,4} \\ (X, u) & \mapsto & Xu \ . \end{array}$$

Ad $\alpha_{1,4,1}$.

As $\beta_{1,4,1} : \varepsilon_1 \mathrm{B}_{\mathbf{Q}} \varepsilon_4 \times \varepsilon_4 \mathrm{B}_{\mathbf{Q}} \varepsilon_1 \to \varepsilon_1 \mathrm{B}_{\mathbf{Q}} \varepsilon_1$ is zero, it follows that $\alpha_{1,4,1}$ must be zero.

Ad $\alpha_{1,4,4}$.

We have

$$\begin{array}{rcl} \beta_{1,4,4} : \varepsilon_1 \mathrm{B}_{\mathbf{Q}} \varepsilon_4 \times \varepsilon_4 \mathrm{B}_{\mathbf{Q}} \varepsilon_4 & \to & \varepsilon_1 \mathrm{B}_{\mathbf{Q}} \varepsilon_4 \\ (\begin{bmatrix} u_1 b_{e,\varepsilon_4} \\ u_2 b_{g,\varepsilon_4} \\ u_3 b_{h,\varepsilon_4} \end{bmatrix}, a\varepsilon_4 + bb'_{\varepsilon_4,\varepsilon_4} + cb''_{\varepsilon_4,\varepsilon_4}) & \mapsto & \begin{bmatrix} u_1 a b_{e,\varepsilon_4} \\ u_2 a b_{g,\varepsilon_4} \\ u_3 a b_{h,\varepsilon_4} \end{bmatrix} \ , \end{array}$$

where $u_1, u_2, u_3, a, b, c \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{1,4,4} := \gamma_{1,4}^{-1} \circ \beta_{1,4,4} \circ (\gamma_{1,4} \times \gamma_{4,4}) : A_{1,4} \times A_{4,4} & \to & A_{1,4} \\ (\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, a + b\overline{\eta} + c\overline{\xi}) & \mapsto & \begin{pmatrix} u_1 a \\ u_2 a \\ u_3 a \end{pmatrix} \end{array} .$$

Ad $\alpha_{2,2,2}$.

We have

$$\begin{array}{rcl} \beta_{2,2,2} : \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 \times \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 & \to & \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 \\ (u\varepsilon_2, v\varepsilon_2) & \mapsto & uv\varepsilon_2 \ , \end{array}$$

where $u, v \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{2,2,2} := \gamma_{2,2}^{-1} \circ \beta_{2,2,2} \circ (\gamma_{2,2} \times \gamma_{2,2}) : A_{2,2} \times A_{2,2} & \to & A_{2,2} \\ (u, v) & \mapsto & uv \ . \end{array}$$

Ad $\alpha_{2,2,4}$.

We have

$$\begin{array}{rcl} \beta_{2,2,4} : \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 \times \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 & \to & \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ (u\varepsilon_2, v b_{\varepsilon_2,\varepsilon_4}) & \mapsto & uv b_{\varepsilon_2,\varepsilon_4} \ , \end{array}$$

where $u, v \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{2,2,4} := \gamma_{2,4}^{-1} \circ \beta_{2,2,4} \circ (\gamma_{2,2} \times \gamma_{2,4}) : A_{2,2} \times A_{2,4} & \to & A_{2,4} \\ (u, v) & \mapsto & uv \ . \end{array}$$

Ad $\alpha_{2,4,2}$.

As $\beta_{2,4,2} : \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \times \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2 \to \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_2$ is zero, it follows that $\alpha_{2,4,2}$ must be zero.

Ad $\alpha_{2,4,4}$.

We have

$$\begin{array}{rcl} \beta_{2,4,4} : \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \times \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 & \to & \varepsilon_2 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ (u b_{\varepsilon_2,\varepsilon_4}, a\varepsilon_4 + b b'_{\varepsilon_4,\varepsilon_4} + c b''_{\varepsilon_4,\varepsilon_4}) & \mapsto & uab_{\varepsilon_2,\varepsilon_4} \ , \end{array}$$

where $u, a, b, c \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{2,4,4} := \gamma_{2,4}^{-1} \circ \beta_{2,4,4} \circ (\gamma_{2,4} \times \gamma_{4,4}) : A_{2,4} \times A_{4,4} & \to & A_{2,4} \\ (u, a + b\overline{\eta} + c\overline{\xi}) & \mapsto & ua \ . \end{array}$$

Ad $\alpha_{3,3,3}$.

We have

$$\begin{array}{rcl} \beta_{3,3,3} : \varepsilon_3 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_3 \times \varepsilon_3 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_3 & \to & \varepsilon_3 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_3 \\ (u\varepsilon_3, v\varepsilon_3) & \mapsto & uv\varepsilon_3 \ , \end{array}$$

where $u, v \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{3,3,3} := \gamma_{3,3}^{-1} \circ \beta_{3,3,3} \circ (\gamma_{3,3} \times \gamma_{3,3}) : A_{3,3} \times A_{3,3} & \to & A_{3,3} \\ (u, v) & \mapsto & uv \ . \end{array}$$

Ad $\alpha_{4,1,1}$.

We have

$$\begin{array}{c} \beta_{4,1,1} : \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 \times \varepsilon_1 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 \ \to \ \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 \\ \left(\begin{bmatrix} v_1 b_{\varepsilon_4,e}, v_2 b_{\varepsilon_4,g}, v_3 b_{\varepsilon_4,h} \end{bmatrix}, \begin{bmatrix} r_{1,1}e & r_{1,2}b_{e,g} & r_{1,3}b_{e,h} \\ r_{2,1}b_{g,e} & r_{2,2}g & r_{2,3}b_{g,h} \\ r_{3,1}b_{h,e} & r_{3,2}b_{h,g} & r_{3,3}h \end{bmatrix} \right) \mapsto \\ \begin{bmatrix} (v_1 r_{1,1} + v_2 r_{2,1} + v_3 r_{3,1}) b_{\varepsilon_4,e} & (v_1 r_{1,2} + v_2 r_{2,2} + v_3 r_{3,2}) b_{\varepsilon_4,g} & (v_1 r_{1,3} + v_2 r_{2,3} + v_3 r_{3,3}) b_{\varepsilon_4,h} \end{bmatrix} \ , \end{array}$$

where $r_{i,j}, v_k \in \mathbf{Q}$ for $i, j, k \in [1, 3]$. Therefore

$$\begin{array}{rcl} \alpha_{4,1,1} := \gamma_{4,1}^{-1} \circ \beta_{4,1,1} \circ (\gamma_{4,1} \times \gamma_{1,1}) : A_{4,1} \times A_{1,1} & \to & A_{4,1} \\ (v, X) & \mapsto & vX \ . \end{array}$$

Ad $\alpha_{4,1,4}$.

We have

$$\begin{array}{rcl} \beta_{4,1,4} : \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_1 \times \varepsilon_1 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 & \to & \varepsilon_4 \, \mathrm{B}_{\mathbf{Q}} \, \varepsilon_4 \\ \left(\begin{bmatrix} v_1 b_{\varepsilon_4,e}, v_2 b_{\varepsilon_4,g}, v_3 b_{\varepsilon_4,h} \end{bmatrix}, \begin{bmatrix} u_1 b_{e,\varepsilon_4} \\ u_2 b_{g,\varepsilon_4} \\ u_3 b_{h,\varepsilon_4} \end{bmatrix} \right) & \mapsto & (v_1 u_1 + v_2 u_2 + v_3 u_3) b'_{\varepsilon_4,\varepsilon_4} \ , \end{array}$$

where $v_1, v_2, v_3, u_1, u_2, u_3 \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{4,1,4} := \gamma_{4,4}^{-1} \circ \beta_{4,1,4} \circ (\gamma_{4,1} \times \gamma_{1,4}) : A_{4,1} \times A_{1,4} & \to & A_{4,4} \\ \left(\begin{pmatrix} v_1, v_2, v_3 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \right) & \mapsto & (v_1 u_1 + v_2 u_2 + v_3 u_3) \overline{\eta} \ . \end{array}$$

Ad $\alpha_{4,2,2}$. We have

$$\begin{array}{rcl} \beta_{4,2,2} : \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 \times \varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 & \to & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 \\ (ub_{\varepsilon_4,\varepsilon_2}, v\varepsilon_2) & \mapsto & uvb_{\varepsilon_4,\varepsilon_2}\ , \end{array}$$

where $u, v \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{4,2,2} := \gamma_{4,2}^{-1} \circ \beta_{4,2,2} \circ (\gamma_{4,2} \times \gamma_{2,2}) : A_{4,2} \times A_{2,2} & \to & A_{4,2} \\ (u, v) & \mapsto & uv\ . \end{array}$$

Ad $\alpha_{4,2,4}$. We have

$$\begin{array}{rcl} \beta_{4,2,4} : \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 \times \varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 & \to & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ (ub_{\varepsilon_4,\varepsilon_2}, vb_{\varepsilon_2,\varepsilon_4}) & \mapsto & uv(b''_{\varepsilon_4,\varepsilon_4} - 12b'_{\varepsilon_4,\varepsilon_4})\ , \end{array}$$

where $u, v \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{4,2,4} := \gamma_{4,4}^{-1} \circ \beta_{4,2,4} \circ (\gamma_{4,2} \times \gamma_{2,4}) : A_{4,2} \times A_{2,4} & \to & A_{4,4} \\ (u, v) & \mapsto & uv(\overline{\xi} - 12\overline{\eta})\ . \end{array}$$

Ad $\alpha_{4,4,1}$. We have

$$\begin{array}{rcl} \beta_{4,4,1} : \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \times \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_1 & \to & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_1 \\ (a\varepsilon_4 + bb'_{\varepsilon_4,\varepsilon_4} + cb''_{\varepsilon_4,\varepsilon_4}, \begin{bmatrix} v_1 b_{\varepsilon_4,e} & v_2 b_{\varepsilon_4,g} & v_3 b_{\varepsilon_4,h} \end{bmatrix}) & \mapsto & \begin{bmatrix} av_1 b_{\varepsilon_4,e} & av_2 b_{\varepsilon_4,g} & av_3 b_{\varepsilon_4,h} \end{bmatrix}\ , \end{array}$$

where $a, b, c, v_1, v_2, v_3 \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{4,4,1} := \gamma_{4,1}^{-1} \circ \beta_{4,4,1} \circ\ (\gamma_{4,4} \times \gamma_{4,1}) : A_{4,4} \times A_{4,1} & \to & A_{4,1} \\ (a + b\overline{\eta} + c\overline{\xi}, \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}) & \mapsto & \begin{pmatrix} av_1 & av_2 & av_3 \end{pmatrix}\ . \end{array}$$

Ad $\alpha_{4,4,2}$. We have

$$\begin{array}{rcl} \beta_{4,4,2} : \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \times \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 & \to & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 \\ (a\varepsilon_4 + bb'_{\varepsilon_4,\varepsilon_4} + cb''_{\varepsilon_4,\varepsilon_4}, vb_{\varepsilon_4,\varepsilon_2}) & \mapsto & avb_{\varepsilon_4,\varepsilon_2}\ , \end{array}$$

where $a, b, c, v \in \mathbf{Q}$. Therefore

$$\begin{array}{rcl} \alpha_{4,4,2} := \gamma_{4,2}^{-1} \circ \beta_{4,4,2} \circ (\gamma_{4,4} \times \gamma_{4,2}) : A_{4,4} \times A_{4,2} & \to & A_{4,2} \\ (a + b\overline{\eta} + c\overline{\xi}, v) & \mapsto & av\ . \end{array}$$

Ad $\alpha_{4,4,4}$. Using Lemma 137, we get

$$\begin{array}{rcl} \alpha_{4,4,4} := \gamma_{4,4}^{-1} \circ \beta_{4,4,4} \circ (\gamma_{4,4} \times \gamma_{4,4}) : A_{4,4} \times A_{4,4} & \to & A_{4,4} \\ (a + b\overline{\eta} + c\overline{\xi}, \tilde{a} + \tilde{b}\overline{\eta} + \tilde{c}\overline{\xi}) & \mapsto & a\tilde{a} + (a\tilde{b} + b\tilde{a})\overline{\eta} + (a\tilde{c} + c\tilde{a})\overline{\xi} \\ & & = (a + b\overline{\eta} + c\overline{\xi}) \cdot (\tilde{a} + \tilde{b}\overline{\eta} + \tilde{c}\overline{\xi})\ , \end{array}$$

where $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbf{Q}$.

We now obtain the

Proposition 140. *Peirce composition yields a* $\mathbf{Q}$*-algebra structure on* $A = \bigoplus_{i,j\in[1,4]} A_{i,j}$ *with respect to the multiplication maps* $\alpha_{i,j,k}$ *for* $i, j, k \in [1,4]$*, cf. Lemma 156.*

We obtain a $\mathbf{Q}$*-algebra isomorphism*

$$\begin{array}{rcl} A = \bigoplus\limits_{i,j\in[1,4]} A_{i,j} & \overset{\gamma}{\underset{\sim}{\to}} & \mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3) \\ & & \\ (a_{i,j})_{i,j\in[1,4]} & \mapsto & \sum\limits_{i,j\in[1,4]} \gamma_{i,j}(a_{i,j}) \ . \end{array}$$

Pictorially,

$$\mathrm{B}_{\mathbf{Q}} = \begin{bmatrix} e\,\mathrm{B}_{\mathbf{Q}}\,e & e\,\mathrm{B}_{\mathbf{Q}}\,g & e\,\mathrm{B}_{\mathbf{Q}}\,h & 0 & 0 & e\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ g\,\mathrm{B}_{\mathbf{Q}}\,e & g\,\mathrm{B}_{\mathbf{Q}}\,g & g\,\mathrm{B}_{\mathbf{Q}}\,h & 0 & 0 & g\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ h\,\mathrm{B}_{\mathbf{Q}}\,e & h\,\mathrm{B}_{\mathbf{Q}}\,g & h\,\mathrm{B}_{\mathbf{Q}}\,h & 0 & 0 & h\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ 0 & 0 & 0 & \varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 & 0 & \varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \\ 0 & 0 & 0 & 0 & \varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3 & 0 \\ \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,e & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,g & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,h & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 & 0 & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \end{bmatrix} \overset{\gamma^{-1}}{\underset{\sim}{\longrightarrow}} \begin{bmatrix} \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & 0 & 0 & \mathbf{Q} \\ \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & 0 & 0 & \mathbf{Q} \\ \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & 0 & 0 & \mathbf{Q} \\ 0 & 0 & 0 & \mathbf{Q} & 0 & \mathbf{Q} \\ 0 & 0 & 0 & 0 & \mathbf{Q} & 0 \\ \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & 0 & \mathbf{Q}[\overline{\eta}, \overline{\xi}] \end{bmatrix} = A$$

$$x \;=\; \begin{bmatrix} exe & exg & exh & 0 & 0 & ex\varepsilon_4 \\ gxg & gxg & gxh & 0 & 0 & gx\varepsilon_4 \\ hxh & hxg & hxh & 0 & 0 & hx\varepsilon_4 \\ 0 & 0 & 0 & \varepsilon_2 x\varepsilon_2 & 0 & \varepsilon_2 x\varepsilon_4 \\ 0 & 0 & 0 & 0 & \varepsilon_3 x\varepsilon_3 & 0 \\ \varepsilon_4 xe & \varepsilon_4 xg & \varepsilon_4 xh & \varepsilon_4 x\varepsilon_2 & 0 & \varepsilon_4 x\varepsilon_4 \end{bmatrix}$$

$$||$$

$$\begin{bmatrix} s_{1,1}e & s_{1,2}b_{e,g} & s_{1,3}b_{e,h} & 0 & 0 & t_1 b_{e,\varepsilon_4} \\ s_{2,1}b_{g,e} & s_{2,2}g & s_{2,3}b_{g,h} & 0 & 0 & t_2 b_{g,\varepsilon_4} \\ s_{3,1}b_{h,e} & s_{3,2}b_{h,g} & s_{3,3}h & 0 & 0 & t_3 b_{h,\varepsilon_4} \\ 0 & 0 & 0 & u\varepsilon_2 & 0 & vb_{\varepsilon_2,\varepsilon_4} \\ 0 & 0 & 0 & 0 & w\varepsilon_3 & 0 \\ x_1 b_{\varepsilon_4,e} & x_2 b_{\varepsilon_4,g} & x_3 b_{\varepsilon_4,h} & yb_{\varepsilon_4,\varepsilon_2} & 0 & z_1 + z_2 b'_{\varepsilon_4,\varepsilon_4} + z_3 b''_{\varepsilon_4,\varepsilon_4} \end{bmatrix} \mapsto \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{bmatrix},$$

where $s_{i,j}, t_j, u, v, w, x_j, y, z_j \in \mathbf{Q}$ for $i, j \in [1,3]$.

4.3.3 $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$ as path algebra modulo relations

We aim to write $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$, up to Morita equivalence, as path algebra modulo relations.

We have $\mathrm{B}_{\mathbf{Q}}\,e \overset{\sim}{\longleftrightarrow} \mathrm{B}_{\mathbf{Q}}\,g$ as $\mathrm{B}_{\mathbf{Q}}$-modules, using multiplication with $b_{e,g}$ from the right from $\mathrm{B}_{\mathbf{Q}}\,e$ to $\mathrm{B}_{\mathbf{Q}}\,g$ and multiplication with $b_{g,e}$ from the right from $\mathrm{B}_{\mathbf{Q}}\,g$ to $\mathrm{B}_{\mathbf{Q}}\,e$. Note that $b_{e,g}b_{g,e} = e$ and $b_{g,e}b_{e,g} = g$. Similarly $\mathrm{B}_{\mathbf{Q}}\,e \overset{\sim}{\longleftrightarrow} \mathrm{B}_{\mathbf{Q}}\,h$ as $\mathrm{B}_{\mathbf{Q}}$-modules.

Therefore, $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$ is Morita equivalent to $\mathrm{B}'_{\mathbf{Q}} := (e+\varepsilon_2+\varepsilon_3+\varepsilon_4)\,\mathrm{B}_{\mathbf{Q}}(e+\varepsilon_2+\varepsilon_3+\varepsilon_4)$. Then $\mathrm{B}'_{\mathbf{Q}}$ has the $\mathbf{Q}$-linear basis $(e, b_{e,\varepsilon_4}, \varepsilon_2, b_{\varepsilon_2,\varepsilon_4}, \varepsilon_3, b_{\varepsilon_4,e}, b_{\varepsilon_4,\varepsilon_2}, \varepsilon_4, b'_{\varepsilon_4,\varepsilon_4}, b_{\varepsilon_4,\varepsilon_4})$. Note that $\dim_{\mathbf{Q}}(\mathrm{B}'_{\mathbf{Q}}) = 10$.

Note that $\gamma(e+\varepsilon_2+\varepsilon_3+\varepsilon_4) = \begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}$. Write $A'_{i,j} := A_{i,j}$ for $i, j \in [2,4]$.

Identify $A'_{1,1} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q}&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix} \subseteq A_{1,1}$. Identify $A'_{1,4} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q}\\ 0\\ 0 \end{pmatrix} \subseteq A_{1,4}$. Identify $A'_{4,1} := \mathbf{Q} = \begin{pmatrix} \mathbf{Q}&0&0 \end{pmatrix} \subseteq A_{4,1}$. Identify $A'_{1,j} := 0$ for $j \in [2,3]$. Identify $A'_{j,1} := 0$ for $j \in [2,3]$.

We have the isomorphism $\gamma' := \gamma|_{A'}^{\mathrm{B}'_{\mathbf{Q}}}$

$$\mathrm{B}'_{\mathbf{Q}} = \begin{bmatrix} e\,\mathrm{B}_{\mathbf{Q}}\,e & 0 & 0 & e\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4\\ 0 & \varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 & 0 & \varepsilon_2\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4\\ 0 & 0 & \varepsilon_3\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_3 & 0\\ \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,e & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_2 & 0 & \varepsilon_4\,\mathrm{B}_{\mathbf{Q}}\,\varepsilon_4 \end{bmatrix} \overset{\sim}{\underset{\gamma'}{\longleftarrow}} \begin{bmatrix} \mathbf{Q} & 0 & 0 & \mathbf{Q}\\ 0 & \mathbf{Q} & 0 & \mathbf{Q}\\ 0 & 0 & \mathbf{Q} & 0\\ \mathbf{Q} & \mathbf{Q} & 0 & \mathbf{Q}[\overline{\eta}, \overline{\xi}] \end{bmatrix} = A' \,.$$

The inverse images under γ' of our $\mathbf{Q}$-linear basis elements of $\mathrm{B}'_{\mathbf{Q}}$ are written as follows.

$$e \;\longleftarrow\!\shortmid\; \begin{bmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} =: a_{1,1} \qquad \varepsilon_2 \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} =: a_{2,2}$$

$$\varepsilon_3 \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix} =: a_{3,3} \qquad \varepsilon_4 \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{bmatrix} =: a_{4,4}$$

$$b_{e,\varepsilon_4} \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} =: a_{1,4} \qquad b_{\varepsilon_4,e} \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix} =: a_{4,1}$$

$$b_{\varepsilon_2,\varepsilon_4} \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} =: a_{2,4} \qquad b_{\varepsilon_4,\varepsilon_2} \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix} =: a_{4,2}$$

$$b'_{\varepsilon_4,\varepsilon_4} \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&\overline{\eta} \end{bmatrix} =: a'_{4,4} \qquad b''_{\varepsilon_4,\varepsilon_4} \;\longleftarrow\!\shortmid\; \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&\overline{\xi} \end{bmatrix} =: a''_{4,4}$$

We have the following multiplication table for our basis elements of A'.

$(\cdot)$	$a_{1,1}$	$a_{1,4}$	$a_{2,2}$	$a_{2,4}$	$a_{3,3}$	$a_{4,1}$	$a_{4,2}$	$a_{4,4}$	$a'_{4,4}$	$a''_{4,4}$
$a_{1,1}$	$a_{1,1}$	$a_{1,4}$	0	0	0	0	0	0	0	0
$a_{1,4}$	0	0	0	0	0	0	0	$a_{1,4}$	0	0
$a_{2,2}$	0	0	$a_{2,2}$	$a_{2,4}$	0	0	0	0	0	0
$a_{2,4}$	0	0	0	0	0	0	0	$a_{2,4}$	0	0
$a_{3,3}$	0	0	0	0	$a_{3,3}$	0	0	0	0	0
$a_{4,1}$	$a_{4,1}$	$a'_{4,4}$	0	0	0	0	0	0	0	0
$a_{4,2}$	0	0	$a_{4,2}$	$a''_{4,4} - 12a'_{4,4}$	0	0	0	0	0	0
$a_{4,4}$	0	0	0	0	0	$a_{4,1}$	$a_{4,2}$	$a_{4,4}$	$a'_{4,4}$	$a''_{4,4}$
$a'_{4,4}$	0	0	0	0	0	0	0	$a'_{4,4}$	0	0
$a''_{4,4}$	0	0	0	0	0	0	0	$a''_{4,4}$	0	0

We have $a'_{4,4} = a_{4,1} \cdot a_{1,4}$ and $a''_{4,4} = a_{4,2} \cdot a_{2,4} + 12a_{4,1} \cdot a_{1,4}$. Hence, as a $\mathbf{Q}$-algebra A' is generated by $a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, a_{1,4}, a_{4,1}, a_{2,4}, a_{4,2}$.

Consider the quiver

$$\Psi := \left[\begin{tikzcd} \tilde{a}_{3,3} & \tilde{a}_{2,2} \arrow[r, bend left, "\sigma"] & \tilde{a}_{4,4} \arrow[l, bend left, "\vartheta"] \arrow[r, bend right, "\rho"'] & \tilde{a}_{1,1} \arrow[l, bend right, "\pi"'] \end{tikzcd}\right] .$$

We have a surjective $\mathbf{Q}$-algebra morphism $\varphi : \mathbf{Q}\Psi \to A'$ by sending

$$\begin{array}{rcl} \tilde{a}_{1,1} & \mapsto & a_{1,1} \\ \tilde{a}_{2,2} & \mapsto & a_{2,2} \\ \tilde{a}_{3,3} & \mapsto & a_{3,3} \\ \tilde{a}_{4,4} & \mapsto & a_{4,4} \\ \rho & \mapsto & a_{4,1} \\ \pi & \mapsto & a_{1,4} \\ \vartheta & \mapsto & a_{4,2} \\ \sigma & \mapsto & a_{2,4} \ . \end{array}$$

We establish the following multiplication trees, where we underline the elements that are not in a $\mathbf{Q}$-linear relation with previous elements. We double underline the last elements in the tree that get underlined.

$$\begin{tikzcd} \underline{a_{1,1}} \arrow[r, "a_{1,4}"'] & \underline{\underline{a_{1,4}}} \arrow[r, "a_{4,1}"'] \arrow[d, "a_{4,2}"'] & a_{1,4}a_{4,1} = 0 \\ & a_{1,4}a_{4,2} = 0 & \end{tikzcd}$$

$$\begin{tikzcd} \underline{a_{2,2}} \arrow[r, "a_{2,4}"'] & \underline{\underline{a_{2,4}}} \arrow[r, "a_{4,2}"'] \arrow[d, "a_{4,1}"'] & a_{2,4}a_{4,2} = 0 \\ & a_{2,4}a_{4,1} = 0 & \end{tikzcd}$$

$$
\begin{tikzcd}
 & & & \underline{a_{4,4}} \arrow[dl, "a_{4,2}"'] \arrow[dr, "a_{4,1}"] & & & \\
(a''_{4,4} - 12a'_{4,4})a_{4,1} = 0 & & \underline{a_{4,2}} \arrow[dll, "a_{2,4}"] & & \underline{a_{4,1}} \arrow[r, bend left, "a_{1,4}"] & \underline{\underline{a_{4,1}a_{1,4}}} \arrow[r, "a_{4,1}"'] \arrow[d, "a_{4,2}"'] & a'_{4,4}a_{4,1} = 0 \\
\underline{\underline{a_{4,2}a_{2,4}}} \arrow[u, "a_{4,1}"] \arrow[d, "a_{4,2}"'] & & & & & a'_{4,4}a_{4,2} = 0 & \\
(a''_{4,4} - 12a'_{4,4})a_{4,2} = 0 & & & & & &
\end{tikzcd}
$$

The multiplication tree of the idempotent $a_{3,3}$ consists only of the element $a_{3,3}$.

So, the kernel of φ contains the elements:

$$
\begin{array}{ccccccc}
\pi\rho & , & \sigma\vartheta & , & \rho\pi\rho & , & \vartheta\sigma\rho \ , \\
\pi\vartheta & , & \sigma\rho & , & \rho\pi\vartheta & , & \vartheta\sigma\vartheta \ .
\end{array}
$$

Let I be the ideal in $\mathbf{Q}\Psi$ generated by those elements. So, $I \subseteq \mathrm{kern}(\varphi)$. Therefore, φ induces a surjective $\mathbf{Q}$-algebra morphism from $\mathbf{Q}\Psi/I$ to A'.

We may reduce the list of generators to obtain $I = (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$.

Note that $\mathbf{Q}\Psi/I$ is $\mathbf{Q}$-linearly generated by

$$\mathcal{N} := \{\tilde{a}_{3,3} + I, \tilde{a}_{2,2} + I, \tilde{a}_{4,4} + I, \tilde{a}_{1,1} + I, \sigma + I, \pi + I, \vartheta + I, \rho + I, \vartheta\sigma + I, \rho\pi + I\},$$

since, using the trees above, a product of a double underlined element with further factors is zero. Moreover, note that $|\mathcal{N}| = 10 = \dim_{\mathbf{Q}}(A')$.

Since we have a surjective $\mathbf{Q}$-algebra morphism from $\mathbf{Q}\Psi/I$ to A', this dimension argument shows this morphism to be bijective. In particular, $I = \mathrm{kern}(\varphi)$.

So, we obtain the

Proposition 141. *Recall that $I = (\pi\rho, \sigma\vartheta, \pi\vartheta, \sigma\rho)$. We have the isomorphism of* $\mathbf{Q}$-*algebras*

$$
\mathrm{B}'_{\mathbf{Q}} \xrightarrow[\sim]{\gamma'^{-1}} A' \ \xrightarrow{\sim}\ \mathbf{Q}\left[\begin{tikzcd} \tilde{a}_{3,3} & \tilde{a}_{2,2} \arrow[r, bend left, "\sigma"] & \tilde{a}_{4,4} \arrow[l, bend left, "\vartheta"] \arrow[r, bend right, "\rho"'] & \tilde{a}_{1,1} \arrow[l, bend right, "\pi"'] \end{tikzcd}\right]/I
$$

$$
\begin{array}{rcl}
a_{1,1} & \mapsto & \tilde{a}_{1,1} + I \\
a_{2,2} & \mapsto & \tilde{a}_{2,2} + I \\
a_{3,3} & \mapsto & \tilde{a}_{3,3} + I \\
a_{4,4} & \mapsto & \tilde{a}_{4,4} + I \\
a_{4,1} & \mapsto & \rho + I \\
a_{1,4} & \mapsto & \pi + I \\
a_{4,2} & \mapsto & \vartheta + I \\
a_{2,4} & \mapsto & \sigma + I \ .
\end{array}
$$

In particular, $\mathbf{Q}\Psi/I$ is Morita equivalent to $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3, \mathrm{S}_3)$.

4.4 The image of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$

4.4.1 Congruences describing the image of $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$

Recall that

$$A = \bigoplus_{i,j \in [1,4]} A_{i,j} \overset{\gamma}{\underset{\sim}{\to}} \mathrm{B}_{\mathbf{Q}} \ ,$$

where the Peirce composite A is formed with respect to the multiplication maps $\alpha_{i,j,k}$ for $i, j, k \in [1, 4]$, cf. Proposition 140. Define the subring

$$A_{\mathbf{Z}} := \begin{bmatrix} A_{\mathbf{Z},1,1} & A_{\mathbf{Z},1,2} & A_{\mathbf{Z},1,3} & A_{\mathbf{Z},1,4} \\ A_{\mathbf{Z},2,1} & A_{\mathbf{Z},2,2} & A_{\mathbf{Z},2,3} & A_{\mathbf{Z},2,4} \\ A_{\mathbf{Z},3,1} & A_{\mathbf{Z},3,2} & A_{\mathbf{Z},3,3} & A_{\mathbf{Z},3,4} \\ A_{\mathbf{Z},4,1} & A_{\mathbf{Z},4,2} & A_{\mathbf{Z},4,3} & A_{\mathbf{Z},4,4} \end{bmatrix} := \begin{bmatrix} \mathbf{Z}^{3\times 3} & 0 & 0 & \mathbf{Z}^{3\times 1} \\ 0 & \mathbf{Z} & 0 & \mathbf{Z} \\ 0 & 0 & \mathbf{Z} & 0 \\ \mathbf{Z}^{1\times 3} & \mathbf{Z} & 0 & \mathbf{Z}[\overline{\eta}, \overline{\xi}] \end{bmatrix} \subseteq A \ .$$

In fact, $A_{\mathbf{Z}}$ is a subring of A, as $\alpha_{i,j,k}(A_{\mathbf{Z},i,j} \times A_{\mathbf{Z},j,k}) \subseteq A_{\mathbf{Z},i,k}$ for $i, j, k \in [1, 4]$, cf. Section 4.3.2.

We have

$$\begin{tikzcd} \mathrm{B}_{\mathbf{Q}} \arrow[r, "\gamma^{-1}", "\sim"'] & A \\ \mathrm{B}_{\mathbf{Z}} \arrow[u, hook] \arrow[ur, "\gamma^{-1}|_{\mathrm{B}_{\mathbf{Z}}}"'] & A_{\mathbf{Z}} \arrow[u, hook] \end{tikzcd}$$

Recall that as a ring $\mathrm{B}_{\mathbf{Z}}$ is generated by H_1^{Δ}, H_4^{Δ}, $H_{1,1}$ and H_8, cf. Remark 136.

We have, using the definitions of the $\mathbf{Q}$-linear basis elements in Section 4.3.1,

$$\begin{aligned} H_1^{\Delta} &= 3e - b_{g,e} - \frac{1}{2}b_{h,e} + g - b_{h,g} + \varepsilon_2 \\ H_4^{\Delta} &= 2e - b_{g,e} + b_{h,e} + 2b_{h,g} + 2h + 2\varepsilon_3 \\ H_{1,1} &= 3e + 3b_{e,g} - g - b_{g,e} - \frac{1}{2}b_{h,e} - \frac{1}{2}b_{h,g} \\ H_8 &= e + \frac{4}{3}b_{e,g} + \frac{4}{3}b_{e,h} + \frac{1}{3}g - \frac{2}{3}b_{g,h} + \frac{2}{3}h - \frac{1}{3}b_{h,g} + 3b_{\varepsilon_4,e} + 6b_{\varepsilon_4,g} + \varepsilon_2 \\ &\quad - 2b_{\varepsilon_4,\varepsilon_2} + 2b_{g,\varepsilon_4} - 2b_{h,\varepsilon_4} - 2b_{\varepsilon_2,\varepsilon_4} - 36b'_{\varepsilon_4,\varepsilon_4} + 4b''_{\varepsilon_4,\varepsilon_4} \end{aligned}$$

and therefore

$$\mathrm{B}_{\mathbf{Q}} \xrightarrow{\gamma^{-1}} \begin{bmatrix} \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & 0 & 0 & \mathbf{Q} \\ \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & 0 & 0 & \mathbf{Q} \\ \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & 0 & 0 & \mathbf{Q} \\ 0 & 0 & 0 & \mathbf{Q} & 0 & \mathbf{Q} \\ 0 & 0 & 0 & 0 & \mathbf{Q} & 0 \\ \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & \mathbf{Q} & 0 & \mathbf{Q}[\overline{\eta},\overline{\xi}] \end{bmatrix}$$

$$H_1^{\Delta} \mapsto \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad H_4^{\Delta} \mapsto \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_{1,1} \mapsto \begin{bmatrix} 3 & 3 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad H_8 \mapsto \begin{bmatrix} 1 & \frac{4}{3} & \frac{4}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & 2 \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & -2 & 0 & -36\overline{\eta}+4\overline{\xi} \end{bmatrix}.$$

Consider the following elements of $\mathrm{U}(A)$.

$$x_1 := \begin{bmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ 6 & 6 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_3 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \end{bmatrix},$$

$$x_1^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{2}{3} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 & 0 & 1+6\overline{\eta} \end{bmatrix}.$$

Using this we define the ring morphism $\delta : \mathrm{B}_{\mathbf{Z}} \to A,\ y \mapsto x_3^{-1} \cdot x_2^{-1} \cdot x_1^{-1} \cdot \gamma^{-1}(y) \cdot x_1 \cdot x_2 \cdot x_3$. The conjugating element x_1 was constructed such that the image lies in $A_{\mathbf{Z}}$. The elements $x_2,\ x_3$ serve the purpose of simplifying the congruences of $\delta(\mathrm{B}_{\mathbf{Z}})$.

We obtain

$$\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) \xrightarrow{\delta} \begin{bmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & \mathbf{Z} \\ 0 & 0 & 0 & \mathbf{Z} & 0 & \mathbf{Z} \\ 0 & 0 & 0 & 0 & \mathbf{Z} & 0 \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & \mathbf{Z}[\overline{\eta},\overline{\xi}] \end{bmatrix}$$

$$H_1^{\Delta} \mapsto \begin{bmatrix} -3 & 9 & 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ -6 & 2 & 4 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 36 & -12 & -24 & 0 & 0 & -24\overline{\eta} \end{bmatrix}, \qquad H_4^{\Delta} \mapsto \begin{bmatrix} 8 & -23 & -4 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 12 & -46 & -6 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ -72 & 276 & 36 & 0 & 0 & 36\overline{\eta} \end{bmatrix},$$

$$H_{1,1} \mapsto \begin{bmatrix} 6 & -24 & -4 & 0 & 0 & -4 \\ -9 & 36 & 6 & 0 & 0 & 6 \\ 60 & -240 & -40 & 0 & 0 & -40 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -360 & 1440 & 240 & 0 & 0 & 240\overline{\eta} \end{bmatrix}, \quad H_8 \mapsto \begin{bmatrix} 3 & -5 & -1 & 0 & 0 & -2 \\ -4 & 11 & 2 & 0 & 0 & 2 \\ 26 & -65 & -12 & 0 & 0 & -14 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -120 & 252 & 48 & -2 & 0 & 24\overline{\eta}+4\overline{\xi} \end{bmatrix}.$$

Proposition 142. *The image of δ in $A_{\mathbf{Z}}$ is given by*

$$\Lambda := \left\{ \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1+z_2\overline{\eta}+z_3\overline{\xi} \end{bmatrix} \in A_{\mathbf{Z}} : \begin{array}{c} 2w-2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\ x_1 \equiv_4 0 \\ x_2 \equiv_4 0 \\ x_3 \equiv_4 0 \\ y \equiv_2 0 \\ t_1 \equiv_2 0 \\ t_2 \equiv_2 0 \\ t_3 \equiv_2 0 \\ v \equiv_2 0 \\ \\ x_1 \equiv_3 0 \\ x_2 \equiv_3 0 \\ x_3 \equiv_3 0 \\ z_2 \equiv_3 0 \end{array} \right\}.$$

In particular, we have $\mathrm{B}_{\mathbf{Z}} = \mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3) \cong \Lambda$ as rings.

More symbolically written we have

$$\Lambda = \left(\begin{array}{cccccccc}
\mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\
\mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\
\mathbf{Z} & \mathbf{Z} & \mathbf{Z} & 0 & 0 & (2) & & \\
0 & 0 & 0 & \mathbf{Z} & 0 & (2) & & \\
0 & 0 & 0 & 0 & \mathbf{Z} & 0 & & 8 \\
(12) & (12) & (12) & (2) & 0 & \mathbf{Z} & +(12)\overline{\eta} & +(4)\overline{\xi}
\end{array}\right).$$

Herein

$$\begin{tikzcd} a \arrow[r, dash, "t"] & s \arrow[r, dash, "v"] \arrow[d, dash, "u"] & b \\ & c & \end{tikzcd}$$

means $t \cdot a + u \cdot c + v \cdot b \equiv_s 0$.

Proof. We identify $\mathbf{Z}^{22\times 1}$ and $A_{\mathbf{Z}}$ along the $\mathbf{Z}$-linear isomorphism

$$\begin{pmatrix} s_{1,1} \\ s_{2,1} \\ s_{3,1} \\ s_{1,2} \\ s_{2,2} \\ s_{3,2} \\ s_{1,3} \\ s_{2,3} \\ s_{3,3} \\ x_1 \\ x_2 \\ x_3 \\ u \\ y \\ w \\ t_1 \\ t_2 \\ t_3 \\ v \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{bmatrix}.$$

Let M be the representation matrix of δ, with respect to the bases $\tilde{\mathcal{H}} = (H_{0,0}, H_{0,1}, H_{1,0}, H_1^{\Delta}, H_{0,4}, H_{4,0}, H_4^{\Delta}, H_{1,1}, H_{0,5}, H_{5,0}, H_7, H_{1,4}, H_{4,1}, H_6, H_5^{\Delta}, H_{4,4}, H_{5,1}, H_{1,5}, H_{5,4}, H_{4,5}, H_8, H_{5,5})$ of $\mathrm{B}_{\mathbf{Z}}$ and the standard basis of $A_{\mathbf{Z}}$, cf. Section 4.1. We obtain

$$M = \begin{pmatrix}
0 & 0 & 15 & -3 & 0 & 20 & 8 & 6 & 0 & 25 & 7 & 9 & 8 & -3 & 1 & 12 & 10 & 3 & 15 & 4 & 3 & 5 \\
0 & 0 & -18 & 0 & 0 & -24 & 0 & -9 & 0 & -30 & -12 & -6 & -12 & 0 & 0 & -8 & -15 & -3 & -10 & -4 & -4 & -5 \\
0 & 0 & 126 & -6 & 0 & 168 & 12 & 60 & 0 & 210 & 78 & 48 & 80 & -6 & 0 & 64 & 100 & 21 & 80 & 28 & 26 & 35 \\
-5 & -2 & -60 & 9 & -3 & -55 & -23 & -24 & -1 & -85 & -16 & -36 & -22 & 10 & 0 & -33 & -34 & -12 & -51 & -11 & -5 & -17 \\
6 & 3 & 72 & 3 & 2 & 66 & 2 & 36 & 1 & 102 & 33 & 24 & 33 & 1 & 1 & 22 & 51 & 12 & 34 & 11 & 11 & 17 \\
-42 & -20 & -504 & 2 & -16 & -462 & -46 & -240 & -7 & -714 & -208 & -192 & -220 & 15 & 0 & -176 & -340 & -84 & -272 & -77 & -65 & -119 \\
0 & 0 & -10 & 2 & 0 & -10 & -4 & -4 & 0 & -15 & -3 & -6 & -4 & 2 & 0 & -6 & -6 & -2 & -9 & -2 & -1 & -3 \\
0 & 0 & 12 & 0 & 0 & 12 & 0 & 6 & 0 & 18 & 6 & 4 & 6 & 0 & 0 & 4 & 9 & 2 & 6 & 2 & 2 & 3 \\
0 & 0 & -84 & 4 & 0 & -84 & -6 & -40 & 0 & -126 & -38 & -32 & -40 & 4 & 1 & -32 & -60 & -14 & -48 & -14 & -12 & -21 \\
0 & 0 & -756 & 36 & 0 & -1008 & -72 & -360 & 0 & -1260 & -468 & -288 & -480 & 72 & 0 & -384 & -600 & -108 & -480 & -144 & -120 & -180 \\
252 & 120 & 3024 & -12 & 96 & 2772 & 276 & 1440 & 36 & 4284 & 1248 & 1152 & 1320 & -228 & 0 & 1056 & 2040 & 432 & 1632 & 396 & 252 & 612 \\
0 & 0 & 504 & -24 & 0 & 504 & 36 & 240 & 0 & 756 & 228 & 192 & 240 & -48 & 0 & 192 & 360 & 72 & 288 & 72 & 48 & 108 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -10 & 2 & 0 & -10 & -4 & -4 & 0 & -10 & -4 & -6 & -4 & 2 & 0 & -6 & -4 & -2 & -6 & -2 & -2 & -2 \\
0 & 0 & 12 & 0 & 0 & 12 & 0 & 6 & 0 & 12 & 6 & 4 & 6 & 0 & 0 & 4 & 6 & 2 & 4 & 2 & 2 & 2 \\
0 & 0 & -84 & 4 & 0 & -84 & -6 & -40 & 0 & -84 & -40 & -32 & -40 & 4 & 0 & -32 & -40 & -14 & -32 & -14 & -14 & -14 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 504 & -24 & 0 & 504 & 36 & 240 & 0 & 504 & 240 & 192 & 240 & -48 & 0 & 192 & 240 & 72 & 192 & 72 & 24 & 72 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0
\end{pmatrix}$$

and

$$24M^{-1} = \begin{pmatrix}
-96 & 336 & 72 & 96 & -336 & -72 & -672 & 2352 & 504 & 2 & -2 & 14 & -24 & -12 & -6 & -48 & 168 & 36 & -12 & -42 & 1 & 6 \\
144 & -552 & -120 & -144 & 552 & 120 & 1008 & -3864 & -840 & -4 & 4 & -28 & 24 & 12 & 0 & 72 & -276 & -60 & 12 & 120 & -2 & -18 \\
96 & -336 & -72 & 0 & 0 & 0 & 96 & -336 & -72 & -2 & 0 & -2 & 24 & 12 & 0 & 96 & -336 & -72 & 12 & -48 & -2 & -18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 12 & 0 & 0 & 0 & 0 & 12 & -24 & 0 & 6 \\
72 & -192 & -48 & -72 & 192 & 48 & 504 & -1344 & -336 & -2 & 2 & -14 & 0 & 12 & 6 & 36 & -96 & -24 & 0 & 66 & -1 & -6 \\
-96 & 336 & 72 & 0 & 0 & 0 & -192 & 672 & 144 & 2 & 0 & 4 & 0 & 0 & 6 & 48 & -168 & -36 & 12 & -54 & -1 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & -12 & 0 & 0 \\
-144 & 552 & 120 & 0 & 0 & 0 & -144 & 552 & 120 & 4 & 0 & 4 & -48 & -24 & 0 & -144 & 552 & 120 & -24 & 72 & 4 & 36 \\
0 & 0 & 24 & 0 & 0 & -24 & 0 & 0 & 168 & 4 & -4 & 28 & 0 & -12 & 0 & 0 & 0 & 12 & 0 & -168 & 2 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 96 & -336 & -72 & 0 & 0 & -2 & 0 & 0 & 0 & -96 & 336 & 72 & -12 & 72 & 2 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & -6 \\
-72 & 192 & 48 & 0 & 0 & 0 & -72 & 192 & 48 & 2 & 0 & 2 & 0 & -12 & 0 & -72 & 192 & 48 & 0 & 24 & 2 & 18 \\
144 & -552 & -120 & 0 & 0 & 0 & 288 & -1104 & -240 & -4 & 0 & -8 & 0 & 0 & 0 & -72 & 276 & 60 & -12 & 96 & 2 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 \\
72 & -192 & -48 & 0 & 0 & 0 & 144 & -384 & -96 & -2 & 0 & -4 & 0 & 0 & -18 & -36 & 96 & 24 & 0 & 42 & 1 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & -144 & 552 & 120 & 0 & 0 & 4 & 0 & 0 & 0 & 144 & -552 & -120 & 24 & -120 & -4 & -36 \\
0 & 0 & -24 & 0 & 0 & 0 & 0 & 0 & -24 & -4 & 0 & -4 & 0 & 24 & 0 & 0 & 0 & -24 & 0 & 24 & -4 & -36 \\
0 & 0 & 0 & 0 & 0 & 0 & -72 & 192 & 48 & 0 & 0 & 2 & 0 & 0 & 0 & 72 & -192 & -48 & 0 & -48 & -2 & -18 \\
0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 48 & 4 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & -48 & -2 & -18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 24 & 4 & 36
\end{pmatrix}.$$

Let $\lambda := \begin{bmatrix}
s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\
s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\
s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\
0 & 0 & 0 & u & 0 & v \\
0 & 0 & 0 & 0 & w & 0 \\
x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi}
\end{bmatrix} \in A_{\mathbf{Z}}.$

We have

$$
\begin{aligned}
\lambda \in \Lambda \quad &\Leftrightarrow \quad \exists q \in \mathbf{Z}^{22\times 1} \text{ such that } \lambda = Mq \\
&\Leftrightarrow \quad \exists q \in \mathbf{Z}^{22\times 1} \text{ such that } M^{-1}\cdot\lambda = q \\
&\Leftrightarrow \quad 24M^{-1}\cdot\lambda \in 24\mathbf{Z}^{22\times 1}
\end{aligned}
$$

$$
\Leftrightarrow \quad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&12&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&6&0&0&0&0&18&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&12&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&12&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&12&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&12&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&6
\end{pmatrix}
\begin{pmatrix} s_{1,1}\\ s_{2,1}\\ s_{3,1}\\ \vdots\\ s_{3,3}\\ x_1\\ x_2\\ x_3\\ u\\ y\\ w\\ t_1\\ t_2\\ t_3\\ v\\ z_1\\ z_2\\ z_3 \end{pmatrix} \in 24\mathbf{Z}^{11\times 1}
$$

$$
\Leftrightarrow \quad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&6&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2
\end{pmatrix}
\begin{pmatrix} s_{1,1}\\ s_{2,1}\\ s_{3,1}\\ \vdots\\ s_{3,3}\\ x_1\\ x_2\\ x_3\\ u\\ y\\ w\\ t_1\\ t_2\\ t_3\\ v\\ z_1\\ z_2\\ z_3 \end{pmatrix} \in 8\mathbf{Z}^{11\times 1}
$$

$$
\text{and} \quad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}
\begin{pmatrix} s_{1,1}\\ s_{2,1}\\ s_{3,1}\\ \vdots\\ s_{3,3}\\ x_1\\ x_2\\ x_3\\ u\\ y\\ w\\ t_1\\ t_2\\ t_3\\ v\\ z_1\\ z_2\\ z_3 \end{pmatrix} \in 3\mathbf{Z}^{4\times 1}
$$

$$\Leftrightarrow \left\{ \begin{array}{ll} 2w - 2z_1 & \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\ x_1 & \equiv_4 0 \\ x_2 & \equiv_4 0 \\ x_3 & \equiv_4 0 \\ v & \equiv_2 0 \\ y & \equiv_2 0 \\ t_1 & \equiv_2 0 \\ t_2 & \equiv_2 0 \\ t_3 & \equiv_2 0 \\ & \\ x_1 & \equiv_3 0 \\ x_2 & \equiv_3 0 \\ x_3 & \equiv_3 0 \\ z_2 & \equiv_3 0 \end{array} \right\} .$$

□

Remark 143. *There is no maximal* $\mathbf{Z}$*-order in* A *containing* $A_{\mathbf{Z}}$ *.*
In particular, $A_{\mathbf{Z}}$ *is not a maximal* $\mathbf{Z}$*-order in* A*.*

> So $A_{\mathbf{Z}}$ is not a canonical choice of a $\mathbf{Z}$-order in A, but it nonetheless enables us to describe Λ inside $A_{\mathbf{Z}}$ via congruences.

Proof. Assume that $A^{\mathfrak{m}}_{\mathbf{Z}}$ is a maximal $\mathbf{Z}$-order of $A_{\mathbf{Q}}$ such that

$$A_{\mathbf{Z}} \subseteq A^{\mathfrak{m}}_{\mathbf{Z}} \subset A_{\mathbf{Q}} = A \ .$$

In particular, the primitive idempotents on the main diagonal of A are contained in $A_{\mathbf{Z}}$, hence in $A^{\mathfrak{m}}_{\mathbf{Z}}$.

$$\text{So, } A^{\mathfrak{m}}_{\mathbf{Z}} = \begin{bmatrix} A^{\mathfrak{m}}_{\mathbf{Z},1,1} & A^{\mathfrak{m}}_{\mathbf{Z},1,2} & A^{\mathfrak{m}}_{\mathbf{Z},1,3} & A^{\mathfrak{m}}_{\mathbf{Z},1,4} \\ A^{\mathfrak{m}}_{\mathbf{Z}2,1} & A^{\mathfrak{m}}_{\mathbf{Z},2,2} & A^{\mathfrak{m}}_{\mathbf{Z},2,3} & A^{\mathfrak{m}}_{\mathbf{Z},2,4} \\ A^{\mathfrak{m}}_{\mathbf{Z},3,1} & A^{\mathfrak{m}}_{\mathbf{Z},3,2} & A^{\mathfrak{m}}_{\mathbf{Z},3,3} & A^{\mathfrak{m}}_{\mathbf{Z},3,4} \\ A^{\mathfrak{m}}_{\mathbf{Z},4,1} & A^{\mathfrak{m}}_{\mathbf{Z},4,2} & A^{\mathfrak{m}}_{\mathbf{Z},4,3} & A^{\mathfrak{m}}_{\mathbf{Z},4,4} \end{bmatrix} \subset \begin{bmatrix} \mathbf{Q}^{3\times 3} & 0 & 0 & \mathbf{Q}^{3\times 1} \\ 0 & \mathbf{Q} & 0 & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} & 0 \\ \mathbf{Q}^{1\times 3} & \mathbf{Q} & 0 & \mathbf{Q}[\overline{\eta}, \overline{\xi}] \end{bmatrix} .$$

We now consider

$$\mathbf{Z}[\overline{\eta}, \overline{\xi}] = A_{\mathbf{Z},4,4} \subseteq A^{\mathfrak{m}}_{\mathbf{Z},4,4} \subseteq A_{4,4} = \mathbf{Q}[\overline{\eta}, \overline{\xi}] \ .$$

Let

$$(b_1, b_2, b_3) := (u_1 + v_1\overline{\eta} + w_1\overline{\xi}, u_2 + v_2\overline{\eta} + w_2\overline{\xi}, u_3 + v_3\overline{\eta} + w_3\overline{\xi})$$

be a $\mathbf{Z}$-linear basis of $A^{\mathfrak{m}}_{\mathbf{Z},4,4}$, where $u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbf{Q}$.

Let $T := \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \in \mathbf{Q}^{3\times 3}$.

We *claim* that $u_1, u_2, u_3 \in \mathbf{Z}$. We have for $k \in \mathbf{Z}_{\geq 1}$ and $u, v, w \in \mathbf{Q}$ that

$$(u + v\overline{\eta} + w\overline{\xi})^k = u^k + ku^{k-1}v\overline{\eta} + ku^{k-1}w\overline{\xi} \ .$$

Consider the $\mathbf{Q}$-algebra morphism $\pi : A_{4,4} \to \mathbf{Q}$, $u + v\overline{\eta} + w\overline{\xi} \mapsto u$.

As $A^{\mathtt{m}}_{\mathbf{Z},4,4}$ is a finitely generated $\mathbf{Z}$-module, so is $\pi(A^{\mathtt{m}}_{\mathbf{Z},4,4})$. *Assume* that $u \in \mathbf{Q} \setminus \mathbf{Z}$. Then the submodule ${}_{\mathbf{Z}}\langle u^k : k \geq 1\rangle \subseteq \pi(A^{\mathtt{m}}_{\mathbf{Z},4,4}) \subseteq \mathbf{Q}$ is not a finitely generated $\mathbf{Z}$-module. This is a contradiction and therefore proves the *claim*.

Since $1 \in A_{\mathbf{Z},4,4} \subseteq A^{\mathtt{m}}_{\mathbf{Z},4,4}$, there exist $s_1, s_2, s_3 \in \mathbf{Z}$ such that $s_1u_1 + s_2u_2 + s_3u_3 = 1$. Thus, $\gcd(s_1, s_2, s_3) = 1$.

By the elementary divisor theorem there exist matrices $M \in \mathrm{GL}_1(\mathbf{Z}) = \{-1, 1\}$ and $N \in \mathrm{GL}_3(\mathbf{Z})$ such that

$$M \begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix} N = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

see e.g. [4, III. Theorem 7.8.]. We may assume $M = 1$.

Hence the matrix $S := N^{-1}$ is of the form $\begin{pmatrix} s_1 & s_2 & s_3 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} \in \mathrm{GL}_3(\mathbf{Z})$.

Write $ST =: \begin{pmatrix} 1 & \check{v}_1 & \check{w}_1 \\ \check{u}_2 & \check{v}_2 & \check{w}_2 \\ \check{u}_3 & \check{v}_3 & \check{w}_3 \end{pmatrix} \in \mathbf{Q}^{3\times 3}$, where $\check{u}_2, \check{u}_3 \in \mathbf{Z}$.

Let $P := \begin{pmatrix} 1 & 0 & 0 \\ -\check{u}_2 & 1 & 0 \\ -\check{u}_3 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbf{Z})$ and write $PST =: \begin{pmatrix} 1 & \tilde{v}_1 & \tilde{w}_1 \\ 0 & \tilde{v}_2 & \tilde{w}_2 \\ 0 & \tilde{v}_3 & \tilde{w}_3 \end{pmatrix} \in \mathbf{Q}^{3\times 3}$.

Then

$$(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) := (1 + \tilde{v}_1\overline{\eta} + \tilde{w}_1\overline{\xi}, \tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi}, \tilde{v}_3\overline{\eta} + \tilde{w}_3\overline{\xi})$$

is a $\mathbf{Z}$-linear basis of $A^{\mathtt{m}}_{\mathbf{Z},4,4}$.

Choose $t \in \mathbf{Z}_{\geq 2}$ and set

$$A^{\mathtt{m}'}_{\mathbf{Z}} := \begin{bmatrix} A^{\mathtt{m}}_{\mathbf{Z},1,1} & A^{\mathtt{m}}_{\mathbf{Z},1,2} & A^{\mathtt{m}}_{\mathbf{Z},1,3} & A^{\mathtt{m}}_{\mathbf{Z},1,4} \\ A^{\mathtt{m}}_{\mathbf{Z},2,1} & A^{\mathtt{m}}_{\mathbf{Z},2,2} & A^{\mathtt{m}}_{\mathbf{Z},2,3} & A^{\mathtt{m}}_{\mathbf{Z},2,4} \\ A^{\mathtt{m}}_{\mathbf{Z},3,1} & A^{\mathtt{m}}_{\mathbf{Z},3,2} & A^{\mathtt{m}}_{\mathbf{Z},3,3} & A^{\mathtt{m}}_{\mathbf{Z},3,4} \\ A^{\mathtt{m}}_{\mathbf{Z},4,1} & A^{\mathtt{m}}_{\mathbf{Z},4,2} & A^{\mathtt{m}}_{\mathbf{Z},4,3} & A^{\mathtt{m}}_{\mathbf{Z},4,4} + {}_{\mathbf{Z}}\langle \frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi})\rangle \end{bmatrix} .$$

We remark that $\pi(A^{\mathtt{m}}_{\mathbf{Z},4,4}) = \pi(A^{\mathtt{m}}_{\mathbf{Z},4,4} + {}_{\mathbf{Z}}\langle \frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi})\rangle)$.

We *claim* that $A_{\mathbf{Z}}^{\mathfrak{m}'} \subseteq A$ is a subring. As $A_{\mathbf{Z},4,4}^{\mathfrak{m}} \subseteq A$ is a subring and as we have from Section 4.3.2 that

$$\begin{array}{lcl}\alpha_{1,4,4} : A_{1,4} \times A_{4,4} & \to & A_{1,4} \\ (\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, a + b\overline{\eta} + c\overline{\xi}) & \mapsto & \begin{pmatrix} u_1 a \\ u_2 a \\ u_3 a \end{pmatrix} ,\end{array}$$

$$\begin{array}{lcl}\alpha_{2,4,4} : A_{2,4} \times A_{4,4} & \to & A_{2,4} \\ (u, a + b\overline{\eta} + c\overline{\xi}) & \mapsto & ua \ .\end{array}$$

$$\begin{array}{lcl}\alpha_{4,4,1} : A_{4,4} \times A_{4,1} & \to & A_{4,1} \\ (a + b\overline{\eta} + c\overline{\xi}, (v_1 \ v_2 \ v_3)) & \mapsto & (av_1 \ av_2 \ av_3) \ ,\end{array}$$

$$\begin{array}{lcl}\alpha_{4,4,2} : A_{4,4} \times A_{4,2} & \to & A_{4,2} \\ (a + b\overline{\eta} + c\overline{\xi}, v) & \mapsto & av \ ,\end{array}$$

$$\begin{array}{lcl}\alpha_{4,4,4} : A_{4,4} \times A_{4,4} & \to & A_{4,4} \\ (a + b\overline{\eta} + c\overline{\xi}, \tilde{a} + \tilde{b}\overline{\eta} + \tilde{c}\overline{\xi}) & \mapsto & a\tilde{a} + (a\tilde{b} + b\tilde{a})\overline{\eta} + (c\tilde{a} + a\tilde{c})\overline{\xi} \ ,\end{array}$$

it suffices to show that $A_{\mathbf{Z},4,4}^{\mathfrak{m}} + {}_{\mathbf{Z}}\langle \frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi})\rangle = {}_{\mathbf{Z}}\langle 1 + \tilde{v}_1\overline{\eta} + \tilde{w}_1\overline{\xi}, \frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi}), \tilde{v}_3\overline{\eta} + \tilde{w}_3\overline{\xi}\rangle$ is a subring of $\mathbf{Q}[\overline{\eta}, \overline{\xi}]$. Since $A_{\mathbf{Z},4,4}^{\mathfrak{m}} \subseteq A_{4,4}$ is a subring, we have $\tilde{b}_1^2 \in A_{\mathbf{Z},4,4}^{\mathfrak{m}}$. Moreover, we obtain

$$\begin{array}{rcccc}(\frac{1}{t}\tilde{b}_2)^2 & = & (\frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi}))^2 & = & 0 \\ \tilde{b}_1\frac{1}{t}\tilde{b}_2 & = & (1 + \tilde{v}_1\overline{\eta} + \tilde{w}_1\overline{\xi})\frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi}) & = & \frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi}) \\ \tilde{b}_1\tilde{b}_3 & = & (1 + \tilde{v}_1\overline{\eta} + \tilde{w}_1\overline{\xi})(\tilde{v}_3\overline{\eta} + \tilde{w}_3\overline{\xi}) & = & \tilde{v}_3\overline{\eta} + \tilde{w}_3\overline{\xi} \\ (\frac{1}{t}\tilde{b}_2)\tilde{b}_3 & = & \frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi})(\tilde{v}_3\overline{\eta} + \tilde{w}_3\overline{\xi}) & = & 0 \\ \tilde{b}_3\tilde{b}_3 & = & (\tilde{v}_3\overline{\eta} + \tilde{w}_3\overline{\xi})(\tilde{v}_3\overline{\eta} + \tilde{w}_3\overline{\xi}) & = & 0 \ .\end{array}$$

Hence, $A_{\mathbf{Z},4,4}^{\mathfrak{m}} + {}_{\mathbf{Z}}\langle \frac{1}{t}(\tilde{v}_2\overline{\eta} + \tilde{w}_2\overline{\xi})\rangle$ is closed under multiplication.

So, $A_{\mathbf{Z}}^{\mathfrak{m}'}$ is a $\mathbf{Z}$-order containing $A_{\mathbf{Z}}^{\mathfrak{m}}$. Therefore, we have

$$A_{\mathbf{Z}} \subseteq A_{\mathbf{Z}}^{\mathfrak{m}} \subset A_{\mathbf{Z}}^{m'} \subset A$$

which contradicts the assumption that $A_{\mathbf{Z}}^{\mathfrak{m}}$ is a maximal order. $\square$

4.4.2 Localisation at 2: $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3, \mathrm{S}_3)$

Write $R := \mathbf{Z}_{(2)}$. Write $R[\overline{\eta}, \overline{\xi}] := R[\eta, \xi]/(\eta^2, \eta\xi, \xi^2)$.

4.4.2.1 Congruences describing the image of $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3, \mathrm{S}_3)$

Corollary 144. *We have*

$$\Lambda_{(2)} = \left\{ \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{bmatrix} \in \begin{bmatrix} R & R & R & 0 & 0 & R \\ R & R & R & 0 & 0 & R \\ R & R & R & 0 & 0 & R \\ 0 & 0 & 0 & R & 0 & R \\ 0 & 0 & 0 & 0 & R & 0 \\ R & R & R & R & 0 & R[\overline{\eta}, \overline{\xi}] \end{bmatrix} : \begin{array}{r} 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \\ t_1 \equiv_2 0 \\ t_2 \equiv_2 0 \\ t_3 \equiv_2 0 \\ v \equiv_2 0 \\ x_1 \equiv_4 0 \\ x_2 \equiv_4 0 \\ x_3 \equiv_4 0 \\ y \equiv_2 0 \end{array} \right\} \subseteq A_R,$$

cf. Proposition 142.

In particular, we have $\mathrm{B}_R = \mathrm{B}_R(\mathrm{S}_3, \mathrm{S}_3) \cong \Lambda_{(2)}$ *as R-algebras.*

More symbolically written we have

$$\Lambda_{(2)} = \begin{pmatrix} R & R & R & 0 & 0 & (2) & & \\ R & R & R & 0 & 0 & (2) & & \\ R & R & R & 0 & 0 & (2) & & \\ 0 & 0 & 0 & R & 0 & (2) & & \\ 0 & 0 & 0 & 0 & R & 0 & & 8 \\ (4) & (4) & (4) & (2) & 0 & R & +(4)\overline{\eta} & +(4)\overline{\xi} \end{pmatrix}.$$

Remark 145. *We claim that* $1_{\Lambda_{(2)}} = \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5$ *is an orthogonal decomposition into primitive idempotents, where*

$$\mathrm{e}_1 := \begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_2 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_3 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},$$

$$\mathrm{e}_4 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_5 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}.$$

Proof. We have $\mathrm{e}_1 \Lambda_{(2)} \mathrm{e}_1 \cong R$, $\mathrm{e}_2 \Lambda_{(2)} \mathrm{e}_2 \cong R$, $\mathrm{e}_3 \Lambda_{(2)} \mathrm{e}_3 \cong R$ and $\mathrm{e}_4 \Lambda_{(2)} \mathrm{e}_4 \cong R$. So, it follows that $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4$ are primitive.

As R-algebras, we have

$$\begin{aligned} \mathrm{e}_5 \Lambda_{(2)} \mathrm{e}_5 &\cong \Big\{ \begin{pmatrix} w, & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{pmatrix} \in R \times R[\overline{\eta}, \overline{\xi}] : 2w - 2z_1 \equiv_8 z_2 \equiv_4 z_3 \equiv_4 0 \Big\} =: \Gamma \\ &\subseteq R \times R[\overline{\eta}, \overline{\xi}] \,. \end{aligned}$$

To show that e_5 is primitive, we show that the ring $\mathrm{e}_5 \Lambda_{(2)} \mathrm{e}_5$ is local, i.e. we show that Γ is local.

We have the R-linear basis (b_1, b_2, b_3, b_4) of Γ, where

$$\begin{aligned} b_1 &= \begin{pmatrix} 1, & 1 \end{pmatrix}, \quad & b_2 &= \begin{pmatrix} 0, & 2 + 4\overline{\eta} \end{pmatrix}, \\ b_3 &= \begin{pmatrix} 0, & 8\overline{\eta} \end{pmatrix}, \quad & b_4 &= \begin{pmatrix} 0, & 4\overline{\xi} \end{pmatrix}. \end{aligned}$$

We *claim* that the Jacobson radical of Γ is given by $J :=_R \langle 2b_1, b_2, b_3, b_4 \rangle$, that $\Gamma/J \cong \mathbf{F}_2$ and that Γ is local.

In fact, the multiplication table for the basis elements is given by

$(\cdot)$	b_1	b_2	b_3	b_4
b_1	b_1	b_2	b_3	b_4
b_2	b_2	$2b_2 + b_3$	$2b_3$	$2b_4$
b_3	b_3	$2b_3$	0	0
b_4	b_4	$2b_4$	0	0

.

This shows that J is an ideal. Moreover, J is topologically nilpotent as

$$J^3 = {}_R\langle 8b_1, 4b_2, 2b_3, 4b_4\rangle \subseteq 2\Gamma\ .$$

Since $\Gamma/J \cong \mathbf{F}_2$, the *claim* follows by Corollary 160. □

4.4.2.2 $\Lambda_{(2)}$ as path algebra modulo relations

Recall that $R = \mathbf{Z}_{(2)}$. We aim to write $\Lambda_{(2)}$, up to Morita equivalence, as path algebra modulo relations.

The R-algebra $\Lambda_{(2)} = (\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5)\Lambda_{(2)}(\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5)$ is Morita equivalent to $\Lambda'_{(2)} := (\mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5)\Lambda_{(2)}(\mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5)$ since $\Lambda_{(2)}\,\mathrm{e}_1 \cong \Lambda_{(2)}\,\mathrm{e}_2 \cong \Lambda_{(2)}\,\mathrm{e}_3$ using multiplication with elements of $\Lambda_{(2)}$ with a single nonzero entry 1 in the upper (3×3)-corner.

We have the R-linear basis of $\Lambda'_{(2)}$

$$\mathrm{e}_3 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_4 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_5 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},$$

$$\tau_1 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&4&0&0&0 \end{bmatrix}, \quad \tau_2 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&2\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \tau_3 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&2&0&0 \end{bmatrix},$$

$$\tau_4 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&2\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \tau_5 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&8\overline{\eta} \end{bmatrix}, \quad \tau_6 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&4\overline{\xi} \end{bmatrix},$$

$$\tau_7 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&2+4\overline{\eta} \end{bmatrix}.$$

We have $\tau_5 = \tau_1\tau_2$ and $\tau_6 = \tau_3\tau_4 + 6\tau_1\tau_2$. Hence, as an R-algebra $\Lambda'_{(2)}$ is generated by $\mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5, \tau_1, \tau_2, \tau_3, \tau_4, \tau_7$.

Consider the quiver

$$\Psi := \left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[rr, bend left, "\tilde{\tau}_2"] && \tilde{\mathrm{e}}_5 \arrow[ll, bend left, "\tilde{\tau}_1"] \arrow[loop right, "\tilde{\tau}_7"] \arrow[rr, bend right, "\tilde{\tau}_3"'] && \tilde{\mathrm{e}}_4 \arrow[ll, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right] .$$

We have a surjective R-algebra morphism $\varphi : R\Psi \to \Lambda'_{(2)}$ by sending

$$\begin{array}{lllllllll}
\tilde{\mathrm{e}}_3 & \mapsto & \mathrm{e}_3 \ , & \tilde{\tau}_1 & \mapsto & \tau_1 \ , & \tilde{\tau}_4 & \mapsto & \tau_4 \ , \\
\tilde{\mathrm{e}}_4 & \mapsto & \mathrm{e}_4 \ , & \tilde{\tau}_2 & \mapsto & \tau_2 \ , & \tilde{\tau}_7 & \mapsto & \tau_7 \ . \\
\tilde{\mathrm{e}}_5 & \mapsto & \mathrm{e}_5 \ , & \tilde{\tau}_3 & \mapsto & \tau_3 \ , & & &
\end{array}$$

We establish the following multiplication trees, where we underline the elements that are not in a R-linear relation with previous elements. We double underline the last elements in the tree that get underlined.

$$\begin{tikzcd}
\underline{\mathrm{e}_3} \arrow[rr, "\tau_2"'] && \underline{\underline{\tau_2}} \arrow[rr, "\tau_1"'] \arrow[dl, "\tau_3"'] \arrow[dr, "\tau_7"] && \tau_2\tau_1 = 0 \\
& \tau_2\tau_3 = 0 && \tau_2\tau_7 = 2\tau_2 &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{e}_4} \arrow[rr, "\tau_4"'] && \underline{\underline{\tau_4}} \arrow[rr, "\tau_1"'] \arrow[dl, "\tau_3"'] \arrow[dr, "\tau_7"] && \tau_4\tau_1 = 0 \\
& \tau_4\tau_3 = 0 && \tau_4\tau_7 = 2\tau_4 &
\end{tikzcd}$$

$$\begin{tikzcd}
& & \tau_7^2 = 2\tau_7 + \tau_1\tau_2 & & & \\
& \tau_7\tau_1 = 2\tau_1 & \underline{\underline{\tau_7}} \arrow[l, "\tau_1"'] \arrow[r, "\tau_3"'] \arrow[u, "\tau_7"'] & \tau_7\tau_3 = 2\tau_3 & & \\
& & \underline{\mathrm{e}_5} \arrow[u, "\tau_7"'] \arrow[dll, "\tau_3"'] \arrow[dr, "\tau_1"] & & & \\
\underline{\tau_3} \arrow[d, "\tau_4"'] & & & \underline{\tau_1} \arrow[r, "\tau_2"'] & \underline{\underline{\tau_1\tau_2}} \arrow[r, "\tau_3"'] \arrow[d, "\tau_7"'] \arrow[dr, "\tau_1"] & \tau_1\tau_2\tau_3 = 0 \\
\underline{\underline{\tau_3\tau_4}} \arrow[r, "\tau_3"'] \arrow[dr, "\tau_1"] \arrow[dd, "\tau_7"'] & \tau_3\tau_4\tau_3 = 0 & & & \tau_1\tau_2\tau_7 = 2\tau_1\tau_2 & \tau_1\tau_2\tau_1 = 0 \\
& \tau_3\tau_4\tau_1 = 0 & & & & \\
\tau_3\tau_4\tau_7 = 2\tau_3\tau_4 & & & & &
\end{tikzcd}$$

So, the kernel of φ contains the elements:

$$\begin{array}{ccccccccc}
\tilde\tau_2\tilde\tau_1 & , & \tilde\tau_1\tilde\tau_2\tilde\tau_1 & , & \tilde\tau_4\tilde\tau_1 & , & \tilde\tau_3\tilde\tau_4\tilde\tau_1 & , & \tilde\tau_7\tilde\tau_1 - 2\tilde\tau_1, \\
\tilde\tau_2\tilde\tau_3 & , & \tilde\tau_1\tilde\tau_2\tilde\tau_3 & , & \tilde\tau_4\tilde\tau_3 & , & \tilde\tau_3\tilde\tau_4\tilde\tau_3 & , & \tilde\tau_7\tilde\tau_3 - 2\tilde\tau_3, \\
\tilde\tau_2\tilde\tau_7 - 2\tilde\tau_2 & , & \tilde\tau_1\tilde\tau_2\tilde\tau_7 - 2\tilde\tau_1\tilde\tau_2 & , & \tilde\tau_4\tilde\tau_7 - 2\tilde\tau_4 & , & \tilde\tau_3\tilde\tau_4\tilde\tau_7 - 2\tilde\tau_3\tilde\tau_4 & , & \tilde\tau_7^2 - 2\tilde\tau_7 - \tilde\tau_1\tilde\tau_2 \ .
\end{array}$$

Let I be the ideal generated by these elements. So $I \subseteq \mathrm{kern}(\varphi)$. Therefore, φ induces a surjective R-algebra morphism from $R\Psi/I$ to $\Lambda'_{(2)}$. We may reduce the list of generators to obtain

$$I = (\tilde\tau_2\tilde\tau_1, \tilde\tau_4\tilde\tau_1, \tilde\tau_7\tilde\tau_1 - 2\tilde\tau_1, \tilde\tau_2\tilde\tau_3, \tilde\tau_4\tilde\tau_3, \tilde\tau_7\tilde\tau_3 - 2\tilde\tau_3, \tilde\tau_2\tilde\tau_7 - 2\tilde\tau_2, \tilde\tau_4\tilde\tau_7 - 2\tilde\tau_4, \tilde\tau_7^2 - 2\tilde\tau_7 - \tilde\tau_1\tilde\tau_2) \ .$$

Note that $R\Psi/I$ is R-linearly generated by

$$\mathcal{N} := \{\tilde{\mathrm{e}}_4 + I, \tilde{\mathrm{e}}_3 + I, \tilde{\mathrm{e}}_5 + I, \tilde\tau_1 + I, \tilde\tau_2 + I, \tilde\tau_3 + I, \tilde\tau_4 + I, \tilde\tau_7 + I, \tilde\tau_3\tilde\tau_4 + I, \tilde\tau_1\tilde\tau_2 + I\} \ ,$$

since, using the trees above, a product of a double underlined element with k further factors may be written, modulo I, as an R-linear combination of products of underlined elements with $\leq k-1$ further factors. Moreover, note that $|\mathcal{N}| = 10 = \mathrm{rk}_R(\Lambda'_{(2)})$.

Since we have a surjective R-algebra morphism from $R\Psi/I$ to $\Lambda'_{(2)}$, this rank argument shows this morphism to be bijective. In particular, $I = \mathrm{kern}(\varphi)$.

So, we obtain the

Proposition 146. *Recall that* $I = \left(\begin{array}{ccccc} \tilde\tau_2\tilde\tau_1 & , & \tilde\tau_2\tilde\tau_3 & , & \tilde\tau_2\tilde\tau_7 - 2\tilde\tau_2, \\ \tilde\tau_4\tilde\tau_1 & , & \tilde\tau_4\tilde\tau_3 & , & \tilde\tau_4\tilde\tau_7 - 2\tilde\tau_4, \\ \tilde\tau_7\tilde\tau_1 - 2\tilde\tau_1 & , & \tilde\tau_7\tilde\tau_3 - 2\tilde\tau_3 & , & \tilde\tau_7^2 - 2\tilde\tau_7 - \tilde\tau_1\tilde\tau_2 \end{array}\right)$.

We have the isomorphism of $\mathbf{Z}_{(2)}$*-algebras*

$$\Lambda'_{(2)} \ \overset{\sim}{\to} \ R\left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde\tau_2"] & \tilde{\mathrm{e}}_5 \arrow[l, bend left, "\tilde\tau_1"] \arrow[loop right, "\tilde\tau_7"] \arrow[r, bend right, "\tilde\tau_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde\tau_4"'] \end{tikzcd}\right]/I$$

$$\begin{array}{rcl}
\mathrm{e}_i & \mapsto & \tilde{\mathrm{e}}_i + I \text{ for } i \in [3,5] \\
\tau_j & \mapsto & \tilde\tau_j + I \text{ for } j \in [1,7] \setminus \{5,6\} \ .
\end{array}$$

Recall that $\mathrm{B}_{\mathbf{Z}_{(2)}}(\mathrm{S}_3, \mathrm{S}_3)$ *is Morita equivalent to* $\Lambda'_{(2)}$.

Corollary 147. *As* $\mathbf{F}_2$*-algebras, we have*

$$\Lambda'_{(2)}/2\Lambda'_{(2)} \ \cong \ \mathbf{F}_2 \left[\begin{tikzcd} \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde\tau_2"] & \tilde{\mathrm{e}}_5 \arrow[l, bend left, "\tilde\tau_1"] \arrow[loop right, "\tilde\tau_7"] \arrow[r, bend right, "\tilde\tau_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde\tau_4"'] \end{tikzcd}\right] / \left(\begin{array}{ccccc} \tilde\tau_2\tilde\tau_1 & , & \tilde\tau_2\tilde\tau_3 & , & \tilde\tau_2\tilde\tau_7, \\ \tilde\tau_4\tilde\tau_1 & , & \tilde\tau_4\tilde\tau_3 & , & \tilde\tau_4\tilde\tau_7, \\ \tilde\tau_7\tilde\tau_1 & , & \tilde\tau_7\tilde\tau_3 & , & \tilde\tau_7^2 - \tilde\tau_1\tilde\tau_2 \end{array}\right) \ .$$

Recall that $\mathrm{B}_{\mathbf{F}_2}(\mathrm{S}_3, \mathrm{S}_3)$ *is Morita equivalent to* $\Lambda'_{(2)}/2\Lambda'_{(2)}$.

4.4.3 Localisation at 3: $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3, \mathrm{S}_3)$

Write $R := \mathbf{Z}_{(3)}$. Write $R[\overline{\eta}, \overline{\xi}] := R[\eta, \xi]/(\eta^2, \eta\xi, \xi^2)$.

4.4.3.1 Congruences describing the image of $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3, \mathrm{S}_3)$

Corollary 148. *We have*

$$\Lambda_{(3)} := \left\{ \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 & t_1 \\ s_{2,1} & s_{2,2} & s_{2,3} & 0 & 0 & t_2 \\ s_{3,1} & s_{3,2} & s_{3,3} & 0 & 0 & t_3 \\ 0 & 0 & 0 & u & 0 & v \\ 0 & 0 & 0 & 0 & w & 0 \\ x_1 & x_2 & x_3 & y & 0 & z_1 + z_2\overline{\eta} + z_3\overline{\xi} \end{bmatrix} \in \begin{bmatrix} R & R & R & 0 & 0 & R \\ R & R & R & 0 & 0 & R \\ R & R & R & 0 & 0 & R \\ 0 & 0 & 0 & R & 0 & R \\ 0 & 0 & 0 & 0 & R & 0 \\ R & R & R & R & 0 & R[\overline{\eta}, \overline{\xi}] \end{bmatrix} : \begin{array}{c} x_1 \equiv_3 0 \\ x_2 \equiv_3 0 \\ x_3 \equiv_3 0 \\ z_2 \equiv_3 0 \end{array} \right\} \subseteq A_R ,$$

cf. Proposition 142.

In particular, we have $\mathrm{B}_R = \mathrm{B}_R(\mathrm{S}_3, \mathrm{S}_3) \cong \Lambda_{(3)}$ *as* R*-algebras.*

More symbolically written we have

$$\Lambda_{(3)} = \begin{pmatrix} R & R & R & 0 & 0 & R & & \\ R & R & R & 0 & 0 & R & & \\ R & R & R & 0 & 0 & R & & \\ 0 & 0 & 0 & R & 0 & R & & \\ 0 & 0 & 0 & 0 & R & 0 & & \\ (3) & (3) & (3) & R & 0 & R & +(3)\overline{\eta} & +R\overline{\xi} \end{pmatrix} .$$

Remark 149. *We claim that* $1_{\Lambda_{(3)}} = \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 + \mathrm{e}_6$ *is an orthogonal decomposition into primitive idempotents, where*

$$\mathrm{e}_1 := \begin{bmatrix} 1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_2 := \begin{bmatrix} 0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_3 := \begin{bmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{bmatrix},$$

$$\mathrm{e}_4 := \begin{bmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_5 := \begin{bmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_6 := \begin{bmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1 \end{bmatrix}.$$

Proof. We have $\mathrm{e}_s\,\Lambda_{(3)}\,\mathrm{e}_s \cong R$ for $s \in [1,5]$. Therefore it follows that $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5$ are primitive.

We *claim* that the ring $\mathrm{e}_6\,\Lambda_{(3)}\,\mathrm{e}_6 \cong R[\overline{\eta}, \overline{\xi}]$ is local. In particular, e_6 then is a primitive idempotent.

We have

$$\mathrm{U}(R[\overline{\eta}, \overline{\xi}]) = R[\overline{\eta}, \overline{\xi}] \setminus (3, \overline{\eta}, \overline{\xi}) \ .$$

In fact, for $u := a + b\overline{\eta} + c\overline{\xi}$ with $a \in R \setminus (3)$ and $b, c \in R$, the inverse is given by $u^{-1} = a^{-1} - a^{-2}b\overline{\eta} - a^{-2}c\overline{\xi}$ as

$$uu^{-1} = aa^{-1} + (-a^{-1}b + a^{-1}b)\overline{\eta} + (-a^{-1}c + a^{-1}c)\overline{\xi} = 1 \ .$$

Thus the nonunits of $R[\overline{\eta}, \overline{\xi}]$ form an ideal and so $R[\eta, \xi]/(\eta^2, \eta\xi, \xi^2)$ is a local ring. This proves the *claim*. □

4.4.3.2 $\Lambda_{(3)}$ as path algebra modulo relations

Recall that $R = \mathbf{Z}_{(3)}$.

We aim to write $\Lambda_{(3)}$, up to Morita equivalence, as path algebra modulo relations. The R-algebra $\Lambda_{(3)} = (\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 + \mathrm{e}_6)\Lambda_{(3)}(\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 + \mathrm{e}_6)$ is Morita equivalent to $\Lambda'_{(3)} := (\mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 + \mathrm{e}_6)\Lambda_{(3)}(\mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 + \mathrm{e}_6)$ since $\Lambda_{(3)}\,\mathrm{e}_1 \cong \Lambda_{(3)}\,\mathrm{e}_2 \cong \Lambda_{(3)}\,\mathrm{e}_3$ using multiplication with elements of $\Lambda_{(3)}$ with a single nonzero entry 1 in the upper (3×3)-corner.

We have the R-linear basis of $\Lambda'_{(3)}$

$$\mathrm{e}_3 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_4 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \mathrm{e}_5 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{bmatrix},$$

$$\mathrm{e}_6 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix}, \quad \tau_1 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&3&0&0&0 \end{bmatrix}, \quad \tau_2 := \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},$$

$$\tau_3 \;:=\; \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0 \end{bmatrix}, \quad \tau_4 \;:=\; \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad \tau_5 \;:=\; \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&3\overline{\eta} \end{bmatrix},$$

$$\tau_6 \;:=\; \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\overline{\xi} \end{bmatrix}.$$

We have $\tau_5 = \tau_1\tau_2$ and $\tau_6 = \tau_3\tau_4 + 4\tau_1\tau_2$. Hence, as an algebra $\Lambda'_{(3)}$ is generated by $\mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5, \mathrm{e}_6, \tau_1, \tau_2, \tau_3, \tau_4$.

Consider the quiver

$$\Psi := \left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_6 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right].$$

We have a surjective R-algebra morphism $\varphi : R\Psi \to \Lambda'_{(3)}$ by sending

$$\begin{aligned} \tilde{\mathrm{e}}_3 &\mapsto \mathrm{e}_3 , & \tilde{\tau}_1 &\mapsto \tau_1 , \\ \tilde{\mathrm{e}}_4 &\mapsto \mathrm{e}_4 , & \tilde{\tau}_2 &\mapsto \tau_2 , \\ \tilde{\mathrm{e}}_5 &\mapsto \mathrm{e}_5 , & \tilde{\tau}_3 &\mapsto \tau_3 , \\ \tilde{\mathrm{e}}_6 &\mapsto \mathrm{e}_6 , & \tilde{\tau}_4 &\mapsto \tau_4 . \end{aligned}$$

We establish the following multiplication trees, where we underline the elements that are not in an R-linear relation with previous elements. We double underline the last elements in the tree that get underlined.

$$\begin{tikzcd} \underline{\mathrm{e}_4} \arrow[r, "\tau_4"'] & \underline{\underline{\tau_4}} \arrow[r, "\tau_3"'] \arrow[d, "\tau_1"'] & \tau_4\tau_3 = 0 \\ & \tau_4\tau_1 = 0 & \end{tikzcd}$$

$$\begin{tikzcd} \underline{\mathrm{e}_3} \arrow[r, "\tau_2"'] & \underline{\underline{\tau_2}} \arrow[r, "\tau_1"'] \arrow[d, "\tau_3"'] & \tau_2\tau_1 = 0 \\ & \tau_2\tau_3 = 0 & \end{tikzcd}$$

$$
\begin{tikzcd}
 & & \underline{\mathrm{e}_6} \arrow[dl, "\tau_3"'] \arrow[dr, "\tau_1"] & & & \\
\tau_3\tau_4\tau_3 = 0 & \underline{\tau_3} \arrow[dl, "\tau_4"'] & & \underline{\tau_1} \arrow[r, bend left, "\tau_2"] & \underline{\underline{\tau_1\tau_2}} \arrow[r, "\tau_1"] \arrow[d, "\tau_3"'] & \tau_1\tau_2\tau_1 = 0 \\
\underline{\underline{\tau_3\tau_4}} \arrow[u, "\tau_3"] \arrow[d, "\tau_1"'] & & & & \tau_1\tau_2\tau_3 = 0 & \\
\tau_3\tau_4\tau_1 = 0 & & & & &
\end{tikzcd}
$$

The multiplication tree of the idempotent e_5 consists only of the element e_5.

So, the kernel of φ contains the elements:

$$
\begin{array}{ccccl}
\tilde{\tau}_4\tilde{\tau}_3 & , & \tilde{\tau}_2\tilde{\tau}_1 & , & \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_3 \ , \\
\tilde{\tau}_4\tilde{\tau}_1 & , & \tilde{\tau}_2\tilde{\tau}_3 & , & \tilde{\tau}_3\tilde{\tau}_4\tilde{\tau}_1 \ , \\
 & & & & \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3 \ , \\
 & & & & \tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_1 \ .
\end{array}
$$

Let I be the ideal generated by those elements. So, $I \subseteq \mathrm{kern}(\varphi)$. Therefore, φ induces a surjective R-algebra morphism from $R\Psi/I$ to $\Lambda'_{(3)}$. We may reduce the list of generators to obtain $I = (\tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_2\tilde{\tau}_1,\ \tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_3)$.

Note that $R\Psi/I$ is R-linearly generated by

$$
\mathcal{N} := \{\tilde{\mathrm{e}}_3 + I, \tilde{\mathrm{e}}_4 + I, \tilde{\mathrm{e}}_5 + I, \tilde{\mathrm{e}}_6 + I, \tilde{\tau}_1 + I, \tilde{\tau}_2 + I, \tilde{\tau}_3 + I, \tilde{\tau}_4 + I, \tilde{\tau}_3\tilde{\tau}_4 + I, \tilde{\tau}_1\tilde{\tau}_2 + I\} \ ,
$$

since, using the trees above, a product of a double underlined element with further factors is zero. Moreover, note that $|\mathcal{N}| = 10 = \mathrm{rk}_R(\Lambda'_{(3)})$.

Since we have an surjective algebra morphism from $R\Psi/I$ to $\Lambda'_{(3)}$, this rank argument shows this morphism to be bijective. In particular, $I = \mathrm{kern}(\varphi)$.

So, we obtain the

Proposition 150. *Recall that* $I = (\tilde{\tau}_4\tilde{\tau}_3,\ \tilde{\tau}_2\tilde{\tau}_1,\ \tilde{\tau}_4\tilde{\tau}_1,\ \tilde{\tau}_2\tilde{\tau}_3)$.

We have the isomorphism of R-algebras

$$
\begin{array}{rcl}
\Lambda'_{(3)} & \stackrel{\sim}{\to} & R\left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde{\tau}_2"] & \tilde{\mathrm{e}}_6 \arrow[l, bend left, "\tilde{\tau}_1"] \arrow[r, bend right, "\tilde{\tau}_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde{\tau}_4"'] \end{tikzcd}\right]/I \\
\mathrm{e}_i & \mapsto & \tilde{\mathrm{e}}_i + I \textit{ for } i \in [3,6] \\
\tau_i & \mapsto & \tilde{\tau}_i + I \textit{ for } i \in [1,4]
\end{array}
$$

Recall that $\mathrm{B}_{\mathbf{Z}_{(3)}}(\mathrm{S}_3, \mathrm{S}_3)$ *is Morita equivalent to* $\Lambda'_{(3)}$.

Corollary 151. *As* $\mathbf{F}_3$*-algebras, we have*

$$\Lambda'_{(3)}/3\Lambda'_{(3)} \;\cong\; \mathbf{F}_3 \left[\begin{tikzcd} \tilde{\mathrm{e}}_5 & \tilde{\mathrm{e}}_3 \arrow[r, bend left, "\tilde\tau_2"] & \tilde{\mathrm{e}}_6 \arrow[l, bend left, "\tilde\tau_1"] \arrow[r, bend right, "\tilde\tau_3"'] & \tilde{\mathrm{e}}_4 \arrow[l, bend right, "\tilde\tau_4"'] \end{tikzcd} \right] / (\tilde\tau_4\tilde\tau_3,\ \tilde\tau_2\tilde\tau_1,\ \tilde\tau_4\tilde\tau_1,\ \tilde\tau_2\tilde\tau_3)\ .$$

Recall that $\mathrm{B}_{\mathbf{F}_3}(\mathrm{S}_3, \mathrm{S}_3)$ *is Morita equivalent to* $\Lambda'_{(3)}/3\Lambda'_{(3)}$.

Appendix A

Peirce composition

Let R be a commutative ring.

Definition 152. Suppose given $r \in \mathbf{Z}_{\geq 0}$. A *Peirce composition* $((A_{i,j})_{i,j\in[1,r]}, (\alpha_{i,j,k})_{i,j,k\in[1,r]})$ of size r consists of a tuple of R-modules

$$(A_{i,j})_{i,j\in[1,r]}$$

and a tuple of R-bilinear maps

$$(\alpha_{i,j,k} : A_{i,j} \times A_{j,k} \to A_{i,k})_{i,j,k\in[1,r]}$$

such that the following conditions hold.

(1) We have

$$\alpha_{i,k,l}(\alpha_{i,j,k}(a_{i,j}, a'_{j,k}), a''_{k,l}) = \alpha_{i,j,l}(a_{i,j}, \alpha_{j,k,l}(a'_{j,k}, a''_{k,l})) \ .$$

for $i, j, k, l \in [1, r]$ and $a_{i,j} \in A_{i,j}$, $a'_{j,k} \in A_{j,k}$, $a''_{k,l} \in A_{k,l}$.

(2) For $j \in [1, r]$, there exists $e_j \in A_{j,j}$ fulfilling

$$\begin{aligned} \alpha_{j,j,k}(e_j, a_{j,k}) &= a_{j,k} \\ \text{and } \alpha_{i,j,j}(a_{i,j}, e_j) &= a_{i,j} \end{aligned}$$

for $i, k \in [1, r]$, $a_{i,j} \in A_{i,j}$, $a_{j,k} \in A_{j,k}$.

Remark 153. A Peirce composition as above can be seen as an R-linear preadditive category $\mathcal{X}$ with objects $[1, k]$, morphisms ${}_{\mathcal{X}}(i, j) = A_{i,j}$ for $i, j \in [1, k]$ and composition given by the maps $\alpha_{i,j,k}$.

Remark 154. *The element $e_j \in A_{j,j}$ in Definition 152(2) is unique.*

Proof. Suppose that $\tilde{e}_j \in A_{j,j}$ is another element fulfilling (2). Then

$$e_j = \alpha_{j,j,j}(e_j, \tilde{e}_j) = \tilde{e}_j \ .$$

□

Notation 155. Suppose given a Peirce composition $\big((A_{i,j})_{i,j\in[1,r]},(\alpha_{i,j,k})_{i,j,k\in[1,r]}\big)$. Set $A := \bigoplus_{i,j\in[1,r]} A_{i,j}$. For $i,j\in[1,r]$ and $x\in A_{i,j}$, consider the element

$$(\xi_{k,l})_{k,l\in[1,r]}\in A \text{ defined by } \xi_{k,l} := \begin{cases} x & \text{if } (k,l)=(i,j) \\ 0 & \text{otherwise} \end{cases} .$$

By abuse of notation, we often write $x := (\xi_{k,l})_{k,l\in[1,r]}$. In particular, for $a=(a_{i,j})_{i,j\in[1,r]}$ we may write $a=(a_{i,j})_{i,j\in[1,r]} = \sum_{i,j\in[1,r]} a_{i,j}$.

Lemma 156. *Suppose given a Peirce composition* $\big((A_{i,j})_{i,j\in[1,r]},(\alpha_{i,j,k})_{i,j,k\in[1,r]}\big)$. *Set*

$$A := \bigoplus_{i,j\in[1,r]} A_{i,j}$$

For $a=(a_{i,j})_{i,j\in[1,r]}\in A$ *and* $b=(b_{j,k})_{j,k\in[1,r]}\in A$ *we let*

$$a\cdot b = a\underset{A}{\cdot} b := \Big(\sum_{j\in[1,r]} \alpha_{i,j,k}(a_{i,j},b_{j,k})\Big)_{i,k\in[1,r]}\in A\ .$$

Let

$$1_A = \sum_{i\in[1,r]} e_i \in A\ .$$

So 1_A *has entry* e_i *at position* (i,i) *for* $i\in[1,r]$, *and it has entry* 0 *at position* (i,j) *for* $i,j\in[1,r]$ *with* $i\neq j$.

The element $1_A\in A$ *is neutral with respect to* $(\,\cdot\,)$. *Let*

$$\begin{array}{rcl} R & \xrightarrow{\varphi} & A \\ 1_R & \mapsto & 1_A\ . \end{array}$$

Then

$$A = (A,+,\cdot,\varphi)$$

is an R*-algebra, called the* Peirce composite *of* $\big((A_{i,j})_{i,j\in[1,r]},(\alpha_{i,j,k})_{i,j,k\in[1,r]}\big)$.

We have the orthogonal decomposition

$$1_A = e_1+\cdots+e_r\ ,$$

into idempotents, where $e_iAe_j = A_{i,j}$ *for* $i,j\in[1,r]$.

Proof. Suppose given $i,j,k,l\in[1,r]$, $a_{i,j}\in A_{i,j}$ and $b_{l,k}\in A_{l,k}$. Then

$$a_{i,j}\underset{A}{\cdot} b_{l,k} = 0 \text{ if } j\neq l\ ,$$

and

$$a_{i,j}\underset{A}{\cdot} b_{l,k} = \alpha_{i,j,k}(a_{i,j},b_{j,k}) \text{ if } j=l\ .$$

Note that for $s \in R$ and $a = (a_{i,j})_{i,j\in[1,r]} \in A$ we get

$$(s1_A) \underset{A}{\cdot} a = (\alpha_{i,i,j}(se_i, a_{i,j}))_{i,j\in[1,r]} = s(\alpha_{i,i,j}(e_i, a_{i,j}))_{i,j\in[1,r]} = s(a_{i,j})_{i,j\in[1,r]} = sa$$

$$a \underset{A}{\cdot} (s1_A) = (\alpha_{i,j,j}(a_{i,j}, se_j))_{i,j\in[1,r]} = s(\alpha_{i,j,j}(a_{i,j}, e_j))_{i,j\in[1,r]} = s(a_{i,j})_{i,j\in[1,r]} = sa \ .$$

In particular, the element $1_A \in A$ is neutral with respect to $(\,\cdot\,)$.

Moreover, we have for $a = (a_{i,j})_{i,j\in[1,r]},\ b = (b_{k,l})_{k,l\in[1,r]}$ and $c = (c_{m,n})_{m,n\in[1,r]} \in A$ that

$$\begin{aligned}
\Big(a \underset{A}{\cdot} b\Big) \underset{A}{\cdot} c &= \Big((a_{i,j})_{i,j\in[1,r]} \underset{A}{\cdot} (b_{k,l})_{k,l\in[1,r]}\Big) \underset{A}{\cdot} (c_{m,n})_{m,n\in[1,r]} \\
&= \Big(\sum_{j\in[1,r]} \alpha_{i,j,k}(a_{i,j}, b_{j,k})\Big)_{i,k\in[1,r]} \underset{A}{\cdot} (c_{m,n})_{m,n\in[1,r]} \\
&= \Big(\sum_{j,k\in[1,r]} \alpha_{i,k,l}(\alpha_{i,j,k}(a_{i,j}, b_{j,k}), c_{k,l})\Big)_{i,l\in[1,r]} \\
&= \Big(\sum_{j,k\in[1,r]} \alpha_{i,j,l}(a_{i,j}, \alpha_{j,k,l}(b_{j,k}, c_{k,l}))\Big)_{i,l\in[1,r]} \\
&= (a_{i,j})_{i,j\in[1,r]} \underset{A}{\cdot} \Big(\sum_{k\in[1,r]} \alpha_{j,k,l}(b_{j,k}, c_{k,l})\Big)_{j,l\in[1,r]} \qquad = \quad a \underset{A}{\cdot} \Big(b \underset{A}{\cdot} c\Big) \ .
\end{aligned}$$

Thus, $(\,\underset{A}{\cdot}\,)$ is associative.

Furthermore, we have for $a = (a_{i,j})_{i,j\in[1,r]},\ b = (b_{k,l})_{k,l\in[1,r]}$ and $c = (c_{m,n})_{m,n\in[1,r]} \in A$ that

$$\begin{aligned}
(a + b) \underset{A}{\cdot} c &= (a_{i,j} + b_{i,j})_{i,j\in[1,r]} \underset{A}{\cdot} (c_{m,n})_{m,n\in[1,r]} \\
&= \Big(\sum_{j\in[1,r]} \alpha_{i,j,n}(a_{i,j} + b_{i,j}, c_{j,n})\Big)_{i,n\in[1,r]} \\
&= \Big(\sum_{j\in[1,r]} \alpha_{i,j,n}(a_{i,j}, c_{j,n}) + \alpha_{i,j,n}(b_{i,j}, c_{j,n})\Big)_{i,n\in[1,r]} \\
&= \Big(\sum_{j\in[1,r]} \alpha_{i,j,n}(a_{i,j}, c_{j,n})\Big)_{i,n\in[1,r]} + \Big(\sum_{j\in[1,r]} \alpha_{i,j,n}(b_{i,j}, c_{j,n})\Big)_{i,n\in[1,r]} \quad = \quad a \underset{A}{\cdot} c + b \underset{A}{\cdot} c
\end{aligned}$$

and similarly $a \underset{A}{\cdot} (b + c) = a \underset{A}{\cdot} b + a \underset{A}{\cdot} c$.

The R-linear map

$$\begin{array}{rcl} R & \stackrel{\varphi}{\to} & A \\ 1_R & \mapsto & 1_A \ , \end{array}$$

is a ring morphism as for $s, t \in R$ we have

$$\varphi(s) \cdot \varphi(t) = (s1_A) \underset{A}{\cdot} (t1_A) \stackrel{\text{see above}}{=} st1_A = \varphi(st) \ .$$

Because of $s1_A \in \mathrm{Z}(A)$, for $s \in R$, it follows that $A = (A, +, \cdot, \varphi)$ is an R-algebra. Note that $\varphi(s) \cdot x = s \cdot x$, the latter using the R-module structure on A given by construction in Notation 155.

For $i, j \in [1, r]$ we have

$$e_i \cdot_A e_j = \begin{cases} \alpha_{i,i,i}(e_i, e_i) = e_i & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \ . \end{cases}$$

For $a = (a_{k,l})_{k,l \in [1,r]} = \sum_{k,l \in [1,r]} a_{k,l} \in A$ it follows that

$$e_i \underset{A}{\cdot} a \underset{A}{\cdot} e_j = \alpha_{i,j,j}(\alpha_{i,i,j}(e_i, a_{i,j}), e_j) = \alpha_{i,j,j}(a_{i,j}, e_j) = a_{i,j} \in A_{i,j} \ .$$

So, we have the orthogonal decomposition

$$1_A = e_1 + \cdots + e_r \ ,$$

into idempotents, where $e_i A e_j = A_{i,j}$ for $i, j \in [1, r]$. □

Remark 157. Note that in general, the ring $e_i A e_i$ is not local and the idempotent e_i is not necessarily primitive.

Lemma 158 (Transport of structure)**.** *Suppose given a tuple of R-modules $(A_{i,j})_{i,j \in [1,r]}$. Let $B = (B, \underset{B}{\cdot})$ be an R-algebra and let $\beta : B \times B \to B$ be the multiplication map on B. Suppose given an orthogonal decomposition*

$$1_B = \varepsilon_1 + \cdots + \varepsilon_r$$

into idempotents and R-linear isomorphisms

$$A_{i,j} \xrightarrow{\gamma_{i,j}} \varepsilon_i B \varepsilon_j$$

for $i, j \in [1, r]$. Write

$$\beta_{i,j,k} := \beta|^{\varepsilon_i B \varepsilon_k}_{\varepsilon_i B \varepsilon_j \times \varepsilon_j B \varepsilon_k} : \varepsilon_i B \varepsilon_j \times \varepsilon_j B \varepsilon_k \to \varepsilon_i B \varepsilon_k$$

and set

$$\alpha_{i,j,k} := \gamma_{i,k}^{-1} \circ \beta_{i,j,k} \circ (\gamma_{i,j} \times \gamma_{j,k})$$

for $i, j, k \in [1, r]$.

Then

$$\big((A_{i,j})_{i,j \in [1,r]}, (\alpha_{i,j,k})_{i,j,k \in [1,r]}\big)$$

is a Peirce composition, where $e_i := \gamma_{i,i}^{-1}(\varepsilon_i)$ for $i \in [1, r]$.

Moreover,

$$\begin{array}{rccc} \gamma : A = \bigoplus\limits_{i,j \in [1,r]} A_{i,j} & \xrightarrow{\sim} & \bigoplus\limits_{i,j \in [1,r]} \varepsilon_i B \varepsilon_j = B \\ (a_{i,j})_{i,j \in [1,r]} & \mapsto & (\gamma_{i,j}(a_{i,j}))_{i,j \in [1,r]} \end{array}$$

is an isomorphism of R-algebras.

Proof. We have to show that the conditions in Definition 152 hold.

Ad (1). As $\gamma_{i,l}$ is in particular injective, it suffices to show that

$$\gamma_{i,l}(\alpha_{i,k,l}(\alpha_{i,j,k}(a_{i,j}, a_{j,k}), a_{k,l})) \overset{!}{=} \gamma_{i,l}(\alpha_{i,j,l}(a_{i,j}, \alpha_{j,k,l}(a_{j,k}, a_{k,l}))) \ .$$

for $i, j, k, l \in [1, r]$, $a_{i,j} \in A_{i,j}$, $a_{j,k} \in A_{j,k}$ and $a_{k,l} \in A_{k,l}$. In fact,

$$\begin{aligned} \gamma_{i,l}(\alpha_{i,k,l}(\alpha_{i,j,k}(a_{i,j}, a_{j,k}), a_{k,l})) &= \gamma_{i,k}(\alpha_{i,j,k}(a_{i,j}, a_{j,k})) \underset{B}{\cdot} \gamma_{k,l}(a_{k,l}) \\ &= (\gamma_{i,j}(a_{i,j}) \underset{B}{\cdot} \gamma_{j,k}(a_{j,k})) \underset{B}{\cdot} \gamma_{k,l}(a_{k,l}) \\ &= \gamma_{i,j}(a_{i,j}) \underset{B}{\cdot} (\gamma_{j,k}(a_{j,k}) \underset{B}{\cdot} \gamma_{k,l}(a_{k,l})) \\ &= \gamma_{i,j}(a_{i,j}) \underset{B}{\cdot} \gamma_{j,l}(\alpha_{j,k,l}(a_{j,k}, a_{k,l})) \\ &= \gamma_{i,l}(\alpha_{i,j,l}(a_{i,j}, \alpha_{j,k,l}(a_{j,k}, a_{k,l}))) \ . \end{aligned}$$

Ad (2). We have

$$\alpha_{j,j,k}(e_j, a_{j,k}) = (\gamma_{j,k}^{-1} \circ \beta_{j,j,k} \circ (\gamma_{j,j} \times \gamma_{j,k}))(e_j, a_{j,k}) = \gamma_{j,k}^{-1}(\varepsilon_j \underset{B}{\cdot} \gamma_{j,k}(a_{j,k})) = \gamma_{j,k}^{-1}(\gamma_{j,k}(a_{j,k})) = a_{j,k}$$

$$\alpha_{i,j,j}(a_{i,j}, e_j) = (\gamma_{i,j}^{-1} \circ \beta_{i,j,j} \circ (\gamma_{i,j} \times \gamma_{j,j}))(a_{i,j}, e_j) = \gamma_{i,j}^{-1}(\gamma_{i,j}(a_{i,j}) \underset{B}{\cdot} \varepsilon_j) = \gamma_{i,j}^{-1}(\gamma_{i,j}(a_{i,j})) = a_{i,j}$$

for $i, j, k \in [1, r]$.

We show that $\gamma : A \to B$ is an R-algebra isomorphism. By construction, γ is a bijective R-linear map. We obtain for $a = (a_{i,j})_{i,j \in [1,r]} \in A$ and $\tilde{a} = (\tilde{a}_{j,k})_{j,k \in [1,r]} \in A$

$$\begin{aligned} \gamma(a) \underset{B}{\cdot} \gamma(\tilde{a}) &= \gamma(\sum_{i,j \in [1,r]} a_{i,j}) \underset{B}{\cdot} \gamma(\sum_{j,k \in [1,r]} \tilde{a}_{j,k}) \\ &= (\sum_{i,j \in [1,r]} \gamma_{i,j} a_{i,j}) \underset{B}{\cdot} (\sum_{j,k \in [1,r]} \gamma_{j,k} \tilde{a}_{j,k}) \\ &= \sum_{i,j,k \in [1,r]} \beta_{i,j,k} \circ (\gamma_{i,j} \times \gamma_{j,k})(a_{i,j}, \tilde{a}_{j,k}) \\ &= \sum_{i,j,k \in [1,r]} \gamma_{i,k} \circ \gamma_{i,k}^{-1} \circ \beta_{i,j,k} \circ (\gamma_{i,j} \times \gamma_{j,k})(a_{i,j}, \tilde{a}_{j,k}) \\ &= \sum_{i,j,k \in [1,r]} \gamma_{i,k} \circ \alpha_{i,j,k}(a_{i,j}, \tilde{a}_{j,k}) \\ &= \gamma(\sum_{i,j,k \in [1,r]} \alpha_{i,j,k}(a_{i,j}, \tilde{a}_{j,k})) \\ &= \gamma((\sum_{j \in [1,r]} a_{i,j} \cdot \tilde{a}_{j,k})_{i,k \in [1,r]}) \\ &= \gamma(a \underset{A}{\cdot} \tilde{a}) \ . \end{aligned}$$

Moreover, $\gamma(1_A) = \gamma(\sum_{i \in [1,r]} e_i) = \sum_{i \in [1,r]} \gamma_{i,i}(e_i) = \sum_{i \in [1,r]} \gamma_{i,i}(\gamma_{i,i}^{-1}(\varepsilon_i)) = \sum_{i \in [1,r]} \varepsilon_i = 1_B$. $\square$

Appendix B

Jacobson radical

Let R be a discrete valuation ring with maximal ideal π. Let Λ be an R-order. Let $\operatorname{Jac}(\Lambda)$ be the Jacobson radical of Λ, i.e. $\operatorname{Jac}(\Lambda) = \bigcap_{T \text{ is a simple right } \Lambda\text{-module}} \operatorname{Ann}_\Lambda T$.

The ring Λ is called local, if its set of non-units $\Lambda \setminus \mathrm{U}(\Lambda)$ is an ideal in Λ. This is equivalent to $\Lambda / \operatorname{Jac}(\Lambda)$ being a skewfield, cf. [5, Remark 192].

An ideal $N \subseteq \Lambda$ is called a topologically nilpotent ideal of Λ, if there exists $m \in \mathbf{Z}_{\geq 0}$ such that $N^m \subseteq \pi\Lambda$.

Note that $\operatorname{Jac}(\Lambda)$ is the terminal topologically nilpotent ideal of Λ, cf. [5, Lemma 213].

Proposition 159. *Let $N \subseteq \Lambda$ be a topologically nilpotent ideal of Λ. Let $S_1, \ldots, S_k$ be simple right Λ-modules such that*

$$\bigcap_{i \in [1,k]} \operatorname{Ann}_\Lambda S_i \subseteq N \ .$$

Then $N = \operatorname{Jac}(\Lambda)$.

Proof. As $\operatorname{Jac}(\Lambda)$ is the terminal topologically nilpotent ideal of Λ, it follows that $N \subseteq \operatorname{Jac}(\Lambda)$. We have that $\operatorname{Jac}(\Lambda) = \bigcap_{T \text{ is a simple right } \Lambda\text{-module}} \operatorname{Ann}_\Lambda T$. Thus,

$$\operatorname{Jac}(\Lambda) \subseteq \bigcap_{i \in [1,k]} \operatorname{Ann}_\Lambda S_i \subseteq N \subseteq \operatorname{Jac}(\Lambda) \ .$$

So $\operatorname{Jac}(\Lambda) = N$. □

Corollary 160. *Let $N \subseteq \Lambda$ be a topologically nilpotent ideal of Λ such that Λ/N is a skew field. Then*

$$N = \operatorname{Jac}(\Lambda)$$

and Λ is local.

Proof. We claim that Λ/N is a simple right Λ-module. Let $x + N \in (\Lambda/N)^\times$. As Λ/N is a skew field, we have $1_\Lambda + N \in \langle x + N \rangle_\Lambda$. So $\langle x + N \rangle_\Lambda = \Lambda/N$ for every $x + N \in \Lambda/N$. Thus, Λ/N is a simple Λ-module.

For $x \in \operatorname{Ann}_\Lambda(\Lambda/N) = \{\lambda \in \Lambda : \mu\lambda + N = 0 + N, \text{ for } \mu \in \Lambda\}$ we in particular have

$$1_\Lambda \cdot x + N = x + N = 0 + N \Leftrightarrow x \in N \ .$$

So $\operatorname{Ann}_\Lambda(\Lambda/N) \subseteq N$ and $N = \operatorname{Jac}(\Lambda)$ follows with Proposition 159. □

Appendix C

Multiplication in $\mathrm{B}_{\mathbf{Z}}(\mathrm{S}_3, \mathrm{S}_3)$ via Magma

First, we define $\mathrm{S}_3 \times \mathrm{S}_3$, together with the tuple of inclusion morphisms `p` and the tuple of projection morphisms `q`. Here `p[1]` is the inclusion of the first factor, `p[2]` is the inclusion of the second factor, `q[1]` is the projection on the first factor and `q[2]` is the projection on the second factor.

```
S:=SymmetricGroup(3);
D,p,q:=DirectProduct(S,S);
```

Then we list representatives for the conjugacy classes of subgroups of $\mathrm{S}_3 \times \mathrm{S}_3$.

```
 SL := [X‘subgroup : X in Subgroups(D)];
```

In a next step, we implement the star relation from Lemma 41. For subgroups M, L of $\mathrm{S}_3 \times \mathrm{S}_3$ the list `A` equals the subset $M * L$ of $\mathrm{S}_3 \times \mathrm{S}_3$, the group `N` equals $M * L$ as group. Afterwards, we determine the number of the representative in `SL` that is conjugate to `N` in $\mathrm{S}_3 \times \mathrm{S}_3$.

```
Relation:=function(M,L)
 A:=[];
 for x in D do
     K:=[k : k in S| p[1](q[1](x))*p[2](k) in M and p[1](k)*p[2](q[2](x)) in L];
     if #K ge 1 then
            A:=Include(A,x);
     end if;
 end for;
 N:=PermutationGroup<6| A>;
 for i in [1..#SL] do
      if IsConjugate(D,N,SL[i]) eq true then
         io:=i;
      end if;
 end for;
 return io;
end function;
```

Now, we implement the Mackey formula from Lemma 42. First we let Magma choose a system of double coset representatives `H` of `q[2](M)\S/q[1](L)`. For each $h \in$ `H` we conjugate the subgroup L with $(h, 1)$ and use `Relation(M,L)` to obtain the number of the representative that is conjugate to $M *^{(h,1)} L$. We list those numbers in `E`. Then `E_red` is the list of distinct elements in `E`. In `E_coeff` we list the elements of `E_red`, preceded by the respective multiplicity in `E`.

```
Multiplication:=function(M,L)
 E:=[];
 R:=[];
 H:=[h: h in DoubleCosetRepresentatives(S,q[2](M), q[1](L))];
 for h in H do
   a:=p[1](h);
   L:=sub<D|[a*l*a^-1: l in L]>;
   E:=Append(E, Relation(M,L));
 end for;
 E_red := SetToSequence(SequenceToSet(E));
 E_coeff := [<#[1 : s in E | s eq e],e> : e in E_red];
 return E_coeff;
end function;
```

As a result, to calculate the product of $[(\mathrm{S}_3 \times \mathrm{S}_3)/\mathtt{M}]$ and $[(\mathrm{S}_3 \times \mathrm{S}_3)/\mathtt{L}]$, we may let `P:=Multiplication(M,L)` and obtain

$$[(\mathrm{S}_3 \times \mathrm{S}_3)/\mathtt{M}] \underset{\mathrm{S}_3}{\cdot} [(\mathrm{S}_3 \times \mathrm{S}_3)/\mathtt{L}] = \sum_{\mathtt{x} \in \mathtt{P}} \mathtt{x}[1] \, [(\mathrm{S}_3 \times \mathrm{S}_3)/\mathtt{SL}[\mathtt{x}[2]]] \ .$$

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Zusammenfassung

Seien G, H und P endliche Gruppen.

Eine (H,G)-Bimenge M besteht aus einer endlichen Menge M und aus einer H-Linksoperation sowie einer G-Rechtsoperation, die miteinander vertauschen, d.h.

$$(h \cdot m) \cdot g = h \cdot (m \cdot g)$$

für $h \in H$, $g \in G$ und $m \in M$.

Sei M eine (H,G)-Bimenge und N eine (G,P)-Bimenge. Das kartesische Produkt $M \times N$ wird eine (H,P)-Bimenge durch $h(m,n)p = (hm, np)$ für $h \in H$, $(m,n) \in M \times N$ und $p \in P$. Es ist auch eine Links-G-Menge durch $g(m,n) = (mg^{-1}, gn)$ für $g \in G$ und $(m,n) \in M \times N$. Wir nennen die Menge der G-Bahnen auf $M \times N$ das *Tensorprodukt* $M \underset{G}{\times} N$ von M und N. Dies ist eine (H,P)-Bimenge.

Nun definieren wir den *doppelten Burnsidering* $\mathrm{B}_{\mathbf{Z}}(G,G)$; dies ist die freie abelsche Gruppe, die von den Isomorphieklassen der endlichen (H,G)-Bimengen erzeugt wird, modulo der Relation

$$[M] + [N] = [M \sqcup N]$$

für endliche (G,G)-Bimengen M, N. Die Multiplikation wird definiert durch

$$[M] \underset{G}{\cdot} [N] = [M \underset{G}{\times} N] \text{ für } (G,G)\text{-Bimengen } M, N.$$

In $\mathrm{B}_{\mathbf{Z}}(G,G)$ ist $\mathrm{id}_{\mathrm{B}_{\mathbf{Z}}(G,G)} = [G]$ das neutrale Element der Multiplikation.

Der Ring $\mathrm{B}_{\mathbf{Z}}(G,G)$ hat die $\mathbf{Z}$-lineare Basis $([(G \times G)/U] : U \in \mathcal{L}_{G\times G})$, wobei $\mathcal{L}_{G\times G}$ ein Vertretersystem der Konjugationsklassen von Untergruppen von $G \times G$ ist.

Wir folgen Boltje und Danz, die einen Wedderburnisomorphismus für einen Teilring $\mathrm{B}^{\Delta}_{\mathbf{Q}}(G,G) \subseteq \mathrm{B}_{\mathbf{Q}}(G,G) = \mathbf{Q} \underset{\mathbf{Z}}{\otimes} \mathrm{B}_{\mathbf{Z}}(G,G)$ konstruiert haben. Dieser schränkt auf $\mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G)$ zu einem injektiven Ringmorphismus ein

$$\begin{array}{rcl} \sigma^{\Delta}_{G,G} : \mathrm{B}^{\Delta}_{\mathbf{Z}}(G,G) & \to & \prod\limits_{T \in \mathcal{T}} \mathrm{End}_{\mathbf{Z}\,\mathrm{Out}(T)}(\mathbf{Z}\overline{\mathrm{Inj}}(T,G)) \\ {[X]} & \mapsto & \left([\mu] \mapsto \sum\limits_{[\lambda] \in \overline{\mathrm{Inj}}(T,G)} \dfrac{|\,\mathrm{Fix}_{\Delta(\lambda(T), \bar{\lambda}\circ\bar{\mu}^{-1}, \mu(T))}(X)|}{|\,\mathrm{C}_G(\lambda(T))|} [\lambda] \right)_{T \in \mathcal{T}} . \end{array}$$

Im Fall von $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_3,\mathrm{S}_3)$ bzw. $\mathrm{B}^{\Delta}_{\mathbf{Z}}(\mathrm{S}_4,\mathrm{S}_4)$ geben wir eine explizite Beschreibung einer isomorphen Kopie des Bildes dieser Einschränkung $\sigma^{\Delta}_{\mathrm{S}_3,\mathrm{S}_3}$ bzw. $\sigma^{\Delta}_{\mathrm{S}_4,\mathrm{S}_4}$ in einem Produkt von Matrizenringen über $\mathbf{Z}$ an. Die Beschreibung erfolgt durch Kongruenzen von Tupeleinträgen. Wir beschreiben $\mathrm{B}^{\Delta}_{R}(\mathrm{S}_4,\mathrm{S}_4)$ für $R \in \{\mathbf{Z}_{(2)}, \mathbf{F}_2, \mathbf{Z}_{(3)}, \mathbf{F}_3\}$ als Pfadalgebren über R.

Im Gegensatz zu $\mathrm{B}^{\Delta}_{\mathbf{Q}}(G,G)$ ist $\mathrm{B}_{\mathbf{Q}}(G,G)$ nicht halbeinfach und daher nicht isomorph zu einem direkten Produkt von Matrizenringen über $\mathbf{Q}$.

Wir arbeiten direkt mit der Definition von $\mathrm{B}_{\mathbf{Q}}(\mathrm{S}_3,\mathrm{S}_3)$ und erhalten Beschreibungen für $\mathrm{B}_R(\mathrm{S}_3,\mathrm{S}_3)$ für $R \in \{\mathbf{Q}, \mathbf{Z}_{(2)}, \mathbf{F}_2, \mathbf{Z}_{(3)}, \mathbf{F}_3\}$ als Pfadalgebren über R.

Hiermit versichere ich,

1. dass ich meine Arbeit selbstständig verfasst habe,
2. dass ich keine anderen als die angegebenen Quellen benutzt und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,
3. dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und
4. dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, im Dezember 2017

Nora Krauß