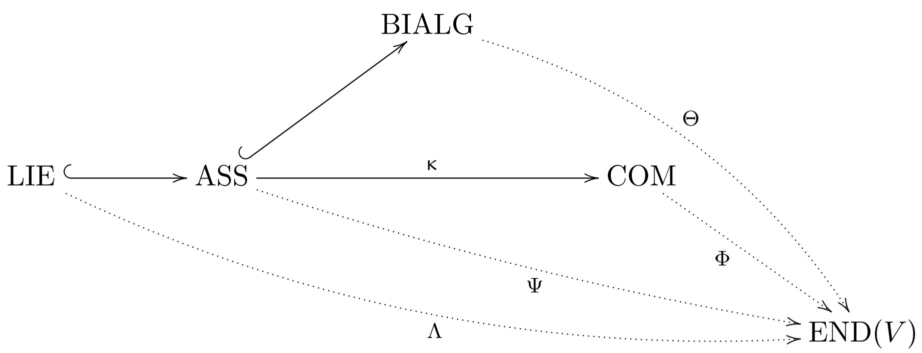
Operads

in the sense of Mac Lane



University of Stuttgart
Germany

Master's Thesis

Vanda Eggert

February 2017

Contents

0 Introduction

0.1 Preoperads and operads

Operads determine algebra structures. For instance, there is an operad ASS such that algebras over ASS are associative algebras; cf. §0.3.1. There is an operad COM such that algebras over COM are commutative algebras; cf. §0.3.3 and the diagram in §0.3.5. Here the principle of forming an algebra over an operad is the same as the principle of forming a module over a ring.

The theory of operads we develop differs from the classical notion of an operad. In fact, the linear operads we define in §6 closely resemble the PROPs, as defined by S. Mac Lane. Concerning the notion of an operad in the classical sense and the connection to Mac Lane's PROPs, cf. also §0.4 below.

With all definitions and constructions we will always handle two different cases. On the one hand, we will define set-preoperads and set-operads, where all involved maps are mere maps between sets. On the other hand, we will define linear preoperads and linear operads over a commutative ring R where all involved sets are R-modules and all involved maps are R-linear. This may lead to some repetition, yet the distinction is usually necessary.

0.1.1 Preoperads

In §2 we treat the basic theory of set-preoperads and linear preoperads as well as basic examples.

A *set-preoperad* is essentially a strict monoidal category (cf. **[10,** VII.1**]**) with $\mathbb{Z}_{\geqslant 0}$ as the set of objects, where the monoid structure is given by addition in $\mathbb{Z}_{\geqslant 0}$. More precisely, a set-preoperad $(\mathcal{P}_0, \boxtimes, \cdot)$ consists of a biindexed set $\mathcal{P}_0 = (\mathcal{P}_0(m,n))_{m,n\geqslant 0}$ with distinguished identity elements $\mathrm{id}_m \in \mathcal{P}_0(m,m)$ for $m \in \mathbb{Z}_{\geqslant 0}$, an associative multiplication, given by multiplication maps

$$(\boxtimes) : \mathcal{P}_0(m,n) \times \mathcal{P}_0(m',n') \longrightarrow \mathcal{P}_0(m+m', n+n'),$$

and an associative composition, given by composition maps

$$(\cdot) : \mathcal{P}_0(m,n) \times \mathcal{P}_0(n,k) \longrightarrow \mathcal{P}_0(m,k),$$

such that certain compatibility conditions are satisfied; cf. Definition 2.6.

A *linear preoperad* $(\mathcal{P}, \boxtimes, \cdot)$ over a commutative ring R is defined similarly, with the additional properties that we ask $\mathcal{P}(m,n)$ to be an R-module and that we ask for the multiplication maps

$$(\boxtimes) : \mathcal{P}(m,n) \otimes \mathcal{P}(m',n') \longrightarrow \mathcal{P}(m+m', n+n')$$

and the composition maps

$$(\cdot) : \mathcal{P}(m,n) \otimes \mathcal{P}(n,k) \longrightarrow \mathcal{P}(m,k)$$

to be R-linear maps.

A *morphism* of (set- or linear) preoperads is a biindexed map $\varphi = (\varphi(m,n))_{m,n\geqslant 0} : \mathcal{P} \longrightarrow \mathcal{Q}$ that is compatible with the structure of the preoperad, that is, compatible with multiplication, composition and identities.

A *subpreoperad* of a (set- or linear) preoperad is a biindexed subset $\mathcal{Q} \subseteq \mathcal{P}$ that contains all identity elements $\mathrm{id}_{\mathcal{P},m}$ and that is closed under multiplication and composition of $\mathcal{P}$. In the case of linear preoperads we additionally ask for $\mathcal{Q}(m,n) \subseteq \mathcal{P}(m,n)$ to be a submodule.

For a (set- or linear) preoperad $\mathcal{P}$ and a biindexed subset $X \subseteq \mathcal{P}$, the subpreoperad *generated by* X is the smallest subpreoperad of $\mathcal{P}$ containing X, i.e.

$$_{\text{preop}}\langle X \rangle = \bigcap \{\mathcal{R} \subseteq \mathcal{P} : \mathcal{R} \text{ is a subpreoperad with } X \subseteq \mathcal{R}\}.$$

A basic example is the set-preoperad Map_0, where $\mathrm{Map}_0(m,n)$ consists of maps $f : [1,m] \longrightarrow [1,n]$ with the usual composition of maps and with multiplication defined by stacking and renumbering, i.e. for $f \in \mathrm{Map}_0(m,n)$, $f' \in \mathrm{Map}_0(m',n')$ we have $f \boxtimes f' \in \mathrm{Map}_0(m+m', n+n')$ given by

$$i(f \boxtimes f') := \begin{cases} if & \text{if } i \in [1,m] \\ (i-m)f' + n & \text{if } i \in [m+1, m+m']. \end{cases}$$

The set-subpreoperad of Map_0 consisting only of monotone maps is called Ass_0. Similarly, the set-subpreoperad of Map_0 consisting only of bijective maps is called Sym_0. By linearly extending to $\mathrm{Map}(m,n) := R\,\mathrm{Map}_0(m,n)$ and $(\boxtimes_{\mathrm{Map}}) := R(\boxtimes_{\mathrm{Map}_0})$ and $(\cdot_{\mathrm{Map}}) := R(\cdot_{\mathrm{Map}_0})$, we obtain the linear preoperad $\mathrm{Map} = R\,\mathrm{Map}_0$. Similarly, the linear subpreoperads $\mathrm{Ass} \subseteq \mathrm{Map}$ and $\mathrm{Sym} \subseteq \mathrm{Map}$ are defined.

For a set X, the set-preoperad $\mathrm{End}_0(X)$ has $\mathrm{End}_0(X)(m,n)$ consisting of all maps $f : X^{\times m} \longrightarrow X^{\times n}$ with the usual composition of maps and with multiplication defined by joining tuples.

Similarly, for an R-module V, the linear preoperad $\mathrm{End}(V)$ has $\mathrm{End}(V)(m,n)$ consisting of R-linear maps $f : V^{\otimes m} \longrightarrow V^{\otimes n}$ with the usual composition of maps and with the tensor product of maps as multiplication.

In §3 we establish the connection between linear preoperads and operads in the classical sense, as defined by J.P. May **[13,** Definition 1.1**]**, which for our purposes we call "absolute operads".

In §4 we construct the free set-preoperad $\mathrm{Free}_0(X)$ for a biindexed set X and the linear preoperad $\mathrm{Free}(X) = R\,\mathrm{Free}_0(X)$. Furthermore, we define *presentations* of preoperads, so that we may write a set-preoperad as $\mathcal{P}_0 \cong {}_{\text{spo}}\langle X \mid Y \rangle$ and a linear preoperad as $\mathcal{P} \cong {}_{\text{lpo}}\langle X \mid Y \rangle$ using a biindexed set X of *generators* and a biindexed set Y of *relations*.

More precisely, the free set-preoperad $\mathrm{Free}_0(X)$ over a biindexed set X has $\mathrm{Free}_0(X)(m,n)$ consisting of equivalence classes of certain words which are built from letters being elements of X that are formally multiplied by identities on both sides.

Theorem (cf. Theorem 4.32, Theorem 4.33) We have

$$\begin{aligned} \mathrm{Ass}_0 &\xleftarrow{\sim} {}_{\text{spo}}\langle \hat{\varepsilon}, \hat{\mu} \mid \big((\hat{\mu} \boxtimes \mathrm{id}_1)\hat{\mu}\,,\, (\mathrm{id}_1 \boxtimes \hat{\mu})\hat{\mu}\big), \big((\mathrm{id}_1 \boxtimes \hat{\varepsilon})\hat{\mu}\,,\, \mathrm{id}_1\big), \big((\hat{\varepsilon} \boxtimes \mathrm{id}_1)\hat{\mu}\,,\, \mathrm{id}_1\big) \rangle \\ \mathrm{Ass} &\xleftarrow{\sim} {}_{\text{lpo}}\langle \hat{\varepsilon}, \hat{\mu} \mid \big((\hat{\mu} \boxtimes \mathrm{id}_1)\hat{\mu} - (\mathrm{id}_1 \boxtimes \hat{\mu})\hat{\mu}\big), \big((\mathrm{id}_1 \boxtimes \hat{\varepsilon})\hat{\mu} - \mathrm{id}_1\big), \big((\hat{\varepsilon} \boxtimes \mathrm{id}_1)\hat{\mu} - \mathrm{id}_1\big) \rangle, \end{aligned}$$

where $\hat{\mu}$ maps to μ, the unique element in $\mathrm{Ass}_0(2,1)$ and where $\hat{\varepsilon}$ maps to ε, the unique element in $\mathrm{Ass}_0(0,1)$.

In §5 we define algebras over preoperads and take a closer look at Ass_0-algebras and Ass-algebras.

For a set-preoperad $\mathcal{P}_0$, a $\mathcal{P}_0$-*algebra* (X, ϱ_0) is a set X together with an *action morphism* of set-preoperads $\varrho_0 : \mathcal{P}_0 \longrightarrow \mathrm{End}_0(X)$. Similarly, for a linear preoperad $\mathcal{P}$, a $\mathcal{P}$-*algebra* (V, ϱ) is an R-module V together with an *action morphism* of linear preoperads $\varrho : \mathcal{P} \longrightarrow \mathrm{End}(V)$.

The set-preoperad Ass_0 and the linear preoperad Ass have the property that every Ass_0-algebra (X, ψ_0) is an associative monoid and every Ass-algebra (V, ψ) is an associative algebra over R.

Conversely, we show that every associative monoid can be turned into an Ass_0-algebra and that every associative algebra over R can be turned into an Ass-algebra. We show this with two different approaches. One uses the usual convention of dropping all brackets when associativity is known and leads to an explicit formula for the action morphism, the other involves the presentations for Ass_0 and Ass, yet merely shows the existence of an action morphism.

It turns out that a morphism ϱ_0 from Map_0 to $\mathrm{End}_0(X)$ does not, however, have to yield a commutative monoid, since the image of the transposition $(1,2) \in \mathrm{Map}_0(2,2)$ under ϱ_0 is in $\mathrm{End}_0(X)(2,2)$ but does not necessarily have to be the map

$$\begin{aligned} \tau_X : \; X^{\times 2} &\longrightarrow X^{\times 2} \\ (x,y) &\longmapsto (y,x). \end{aligned}$$

If we could prescribe $(1,2)\varrho_0 = \tau_X$, then $\tau_X \mu_X = (1,2)\varrho_0 \cdot \mu\varrho_0 = ((1,2)\cdot\mu)\varrho_0 = \mu\varrho_0 = \mu_X$ would force the resulting monoid to be commutative.

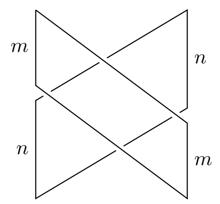
So in order to obtain similar results for commutative monoids, commutative algebras or for Lie algebras, we need to impose some extra structure and study set-operads and linear operads.

0.1.2 Operads

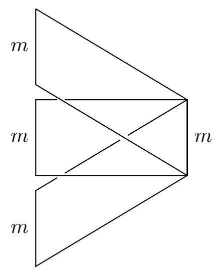
In §6 we treat the basic theory of set-operads and linear operads as well as algebras over set-operads and linear operads.

For the definition of set-operads and linear operads we need certain elements of Map_0.

For $m, n \in \mathbb{Z}_{\geqslant 0}$ we have the blockwise transposition $s_{m,n} \in \mathrm{Sym}_0(m+n, m+n) \subseteq \mathrm{Map}_0(m+n, m+n)$, which we can illustrate as follows.



Furthermore, for $k, m \in \mathbb{Z}_{\geqslant 0}$ the map $h_{k,m} \in \mathrm{Map}_0(km, m)$ maps k blocks of size m to one such block. For the case $k = 3$ this can be illustrated as follows.



A *set-operad* $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ is a set-preoperad $\mathcal{P}_0^{\mathrm{pre}}$ together with a morphism of set-preoperads

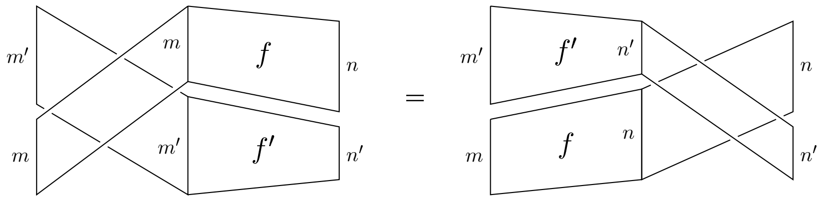
$$\mathfrak{p}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathcal{P}_0^{\mathrm{pre}}$$

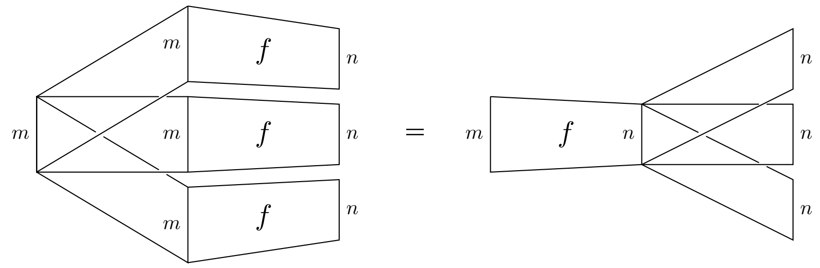
satisfying the following two conditions.

(so1) We always have $(s_{m,m'}^{\mathrm{op}}\mathfrak{p}_0) \cdot (f \boxtimes f') = (f' \boxtimes f) \cdot (s_{n,n'}^{\mathrm{op}}\mathfrak{p}_0)$ for $f \in \mathcal{P}_0^{\mathrm{pre}}(m,n)$, $f' \in \mathcal{P}_0^{\mathrm{pre}}(m',n')$.

(so2) We always have $(h_{k,m}^{\mathrm{op}}\mathfrak{p}_0) \cdot f^{\boxtimes k} = f \cdot (h_{k,n}^{\mathrm{op}}\mathfrak{p}_0)$ for $f \in \mathcal{P}_0^{\mathrm{pre}}(m,n)$.

We can illustrate the conditions (so1) and, e.g. for the case $k = 3$, (so2) as follows.





Equivalently, the morphism $\mathfrak{p}_0$ has to satisfy the following assertion $\mathrm{B}_0(a)$ for all $a \in \mathrm{Map}_0(m, n)$.

$\mathrm{B}_0(a)$: For $l = (l_i)_{i\in[1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$ and $r = (r_i)_{i\in[1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$ and for $f_i \in \mathcal{P}_0^{\mathrm{pre}}(l_i, r_i)$ we have

$$\Big(\underset{i\in[1,n]}{\boxtimes} f_i\Big) \cdot (a_{[r]}^{\mathrm{op}}\mathfrak{p}_0) = (a_{[l]}^{\mathrm{op}}\mathfrak{p}_0) \cdot \Big(\underset{j\in[1,m]}{\boxtimes} f_{ja}\Big),$$

where $a_{[r]} : \big[1, \sum_{j\in[1,m]} r_{ja}\big] \longrightarrow \big[1, \sum_{i\in[1,n]} r_i\big]$ and $a_{[l]} : \big[1, \sum_{j\in[1,m]} l_{ja}\big] \longrightarrow \big[1, \sum_{i\in[1,n]} l_i\big]$ are block versions of the map $a : [1, m] \longrightarrow [1, n]$; cf. Definition 6.8.

Whenever necessary, the multiplication on $\mathcal{P}_0^{\mathrm{pre}}$ is also written $(\boxtimes_{\mathcal{P}_0^{\mathrm{pre}}})$ or simply $(\boxtimes_{\mathcal{P}_0})$. Likewise, the composition is also written $(\cdot_{\mathcal{P}_0^{\mathrm{pre}}})$ or simply $(\cdot_{\mathcal{P}_0})$.

Given set-operads $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ and $\mathcal{Q}_0 = (\mathcal{Q}_0^{\mathrm{pre}}, \mathfrak{q}_0)$, then a *morphism of set-operads* φ_0 is given by a morphism of set-preoperads $\varphi_0^{\mathrm{pre}} : \mathcal{P}_0^{\mathrm{pre}} \longrightarrow \mathcal{Q}_0^{\mathrm{pre}}$ that satisfies $\mathfrak{p}_0\varphi_0^{\mathrm{pre}} = \mathfrak{q}_0$.

$$\begin{array}{ccc} \mathcal{P}_0^{\mathrm{pre}} & \xrightarrow{\varphi_0^{\mathrm{pre}}} & \mathcal{Q}_0^{\mathrm{pre}} \\ & {\scriptstyle \mathfrak{p}_0}\nwarrow \quad \nearrow{\scriptstyle \mathfrak{q}_0} & \\ & \mathrm{Map}_0^{\mathrm{op}} & \end{array}$$

So if we consider the morphism of set-operads φ_0 as a mere morphism of set-preoperads, we often write it φ_0^{pre}.

A *set-suboperad* of a set-operad $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ is given by a set-subpreoperad $\mathcal{Q}_0 \subseteq \mathcal{P}_0^{\mathrm{pre}}$ such that $\mathrm{Im}(\mathfrak{p}_0) \subseteq \mathcal{Q}_0$, together with the restriction $\mathfrak{p}_0\big|^{\mathcal{Q}_0}$.

For example, defining the morphism $\mathfrak{e}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{End}_0(X)$ for a set X to send $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m, n)$ to the map

$$\begin{aligned} X^{\times m} &\longrightarrow X^{\times n} \\ (x_1, \ldots, x_m) &\longmapsto (x_{1f}, \ldots, x_{nf}) \end{aligned}$$

in $\mathrm{End}_0(X)(m, n)$, we obtain the set-operad $\mathrm{END}_0(X) := (\mathrm{End}_0(X), \mathfrak{e}_0)$. So in particular, we have $\mathrm{END}_0(X)^{\mathrm{pre}} = \mathrm{End}_0(X)$.

For a set-operad $\mathcal{P}_0$, a $\mathcal{P}_0$-*algebra* (X, ϱ_0) consists of a set X and an *action morphism* of set-operads $\varrho_0 : \mathcal{P}_0 \longrightarrow \mathrm{END}_0(X)$.

Unlike with preoperads, linear operads can not be defined in complete analogy to set-operads, since there seems to be no sensible definition of a morphism of linear preoperads from $\mathrm{Map}^{\mathrm{op}}$ to $\mathrm{End}(V)$ for an R-module V. Moreover, we have to drop condition (so2) when passing to the linear case.

A *linear operad* $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ is a linear preoperad $\mathcal{P}^{\mathrm{pre}}$ together with a morphism of linear preoperads

$$\mathfrak{p} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathcal{P}^{\mathrm{pre}}$$

satisfying the following condition.

(lo) We always have $(s^{\mathrm{op}}_{m,m'}\mathfrak{p}) \cdot (f \boxtimes f') = (f' \boxtimes f) \cdot (s^{\mathrm{op}}_{n,n'}\mathfrak{p})$ for $f \in \mathcal{P}^{\mathrm{pre}}(m,n)$, $f' \in \mathcal{P}^{\mathrm{pre}}(m',n')$.

Equivalently, the morphism $\mathfrak{p}$ has to satisfy the assertion $\mathrm{B}(a)$ for all $a \in \mathrm{Sym}_0(m,n)$.

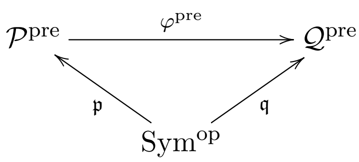
$\mathrm{B}(a)$: For $l = (l_i)_{i\in[1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$ and $r = (r_i)_{i\in[1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$ and for $f_i \in \mathcal{P}^{\mathrm{pre}}(l_i, r_i)$ we have

$$\Big(\underset{i\in[1,n]}{\boxtimes} f_i\Big) \cdot \big(a^{\mathrm{op}}_{[r]}\mathfrak{p}\big) = \big(a^{\mathrm{op}}_{[l]}\mathfrak{p}\big) \cdot \Big(\underset{j\in[1,m]}{\boxtimes} f_{ja}\Big);$$

cf. Definition 6.8.

Whenever necessary, the multiplication in $\mathcal{P}^{\mathrm{pre}}$ is also written $(\boxtimes_{\mathcal{P}^{\mathrm{pre}}})$ or simply $(\boxtimes_{\mathcal{P}})$. Likewise, the composition is also written $(\cdot_{\mathcal{P}^{\mathrm{pre}}})$ or simply $(\cdot_{\mathcal{P}})$.

Given linear operads $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ and $\mathcal{Q} = (\mathcal{Q}^{\mathrm{pre}}, \mathfrak{q})$, then a *morphism of linear operads* φ is given by a morphism of linear preoperads $\varphi^{\mathrm{pre}} : \mathcal{P}^{\mathrm{pre}} \longrightarrow \mathcal{Q}^{\mathrm{pre}}$ that satisfies $\mathfrak{p}\varphi^{\mathrm{pre}} = \mathfrak{q}$.



A *linear suboperad* of a linear operad $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ is given by a linear subpreoperad $\mathcal{Q} \subseteq \mathcal{P}^{\mathrm{pre}}$ such that $\mathrm{Im}(\mathfrak{p}) \subseteq \mathcal{Q}$, together with the restriction $\mathfrak{p}\big|^{\mathcal{Q}}$.

For a linear operad $\mathcal{P}$ and a biindexed subset $X \subseteq \mathcal{P}^{\mathrm{pre}}$, the linear suboperad *generated by* X is the smallest linear suboperad of $\mathcal{P}$ containing X, i.e.

$${}_{\mathrm{op}}\langle\, X \,\rangle = \bigcap \big\{\mathcal{R} \subseteq \mathcal{P} : \mathcal{R} \text{ is a linear suboperad with } X \subseteq \mathcal{R}^{\mathrm{pre}}\big\}.$$

For example, defining the morphism $\mathfrak{e} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{End}(V)$ for an R-module V to send an element $f^{\mathrm{op}} \in \mathrm{Sym}^{\mathrm{op}}(m,m)$ to the map

$$\begin{aligned} V^{\otimes m} &\longrightarrow V^{\otimes m} \\ v_1 \otimes \ldots \otimes v_m &\longmapsto v_{1f} \otimes \ldots \otimes v_{mf} \end{aligned}$$

in $\mathrm{End}(V)(m,m)$, we obtain the linear operad $\mathrm{END}(V) := (\mathrm{End}(V), \mathfrak{e})$. So $\mathrm{END}(V)^{\mathrm{pre}} = \mathrm{End}(V)$.

For a linear operad $\mathcal{P}$, a $\mathcal{P}$-*algebra* (V, ϱ) consists of an R-module V and an *action morphism* of linear operads $\varrho : \mathcal{P} \longrightarrow \mathrm{END}(V)$.

0.2 Particular set-operads

In §7 we discuss the set-operad ASS_0, in §9 we discuss the set-operad COM_0.

0.2.1 The set-operad ASS_0

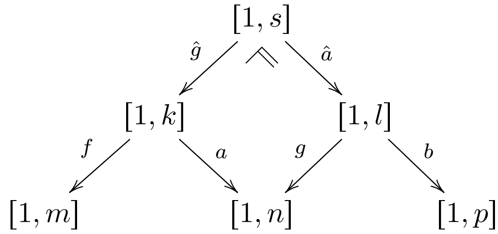
The set-operad ASS_0 is defined using fractions of elements of Ass_0 and Map_0.

For $m, n \in \mathbb{Z}_{\geqslant 0}$ the set $\mathrm{ASS}_0^{\mathrm{pre}}(m,n)$ consists of fractions $f \setminus a$, where $f \in \mathrm{Map}_0(k,m)$ and where $a \in \mathrm{Ass}_0(k,n)$ for some $k \in \mathbb{Z}_{\geqslant 0}$. We have $\mathrm{id}_{\mathrm{ASS}_0,m} = \mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Ass}_0,m}$.

The product of fractions $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$ and $f' \setminus a' \in \mathrm{ASS}_0^{\mathrm{pre}}(m',n')$ is defined using multiplication in Map_0 and Ass_0, i.e.

$$(f \setminus a) \boxtimes_{\mathrm{ASS}_0} (f' \setminus a') := (f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Ass}_0} a').$$

The composite of fractions $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$ and $g \setminus b \in \mathrm{ASS}_0^{\mathrm{pre}}(n,p)$ is defined using what we call a *sorted pullback*. A sorted pullback $([1,s], \hat{g}, \hat{a})$ of $a \in \mathrm{Ass}_0(k,n)$ and $g \in \mathrm{Map}_0(l,n)$ is a pullback of a and g with the additional property that $\hat{a}$ is monotone and that $\hat{g}\big|_{\hat{a}^{-1}(i)}^{a^{-1}(ig)}$ is monotone for $i \in [1,l]$. This yields a diagram



and allows us to define $(f \setminus a) \cdot_{\mathrm{ASS}_0} (g \setminus b) := (\hat{g} f) \setminus (\hat{a} b)$.

Then $\mathrm{ASS}_0 := (\mathrm{ASS}_0^{\mathrm{pre}}, \mathfrak{a}_0)$ is a set-operad, where $\mathfrak{a}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}$ has $\mathfrak{a}_0(m,n)$ mapping $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$ to $f \setminus \mathrm{id}_{\mathrm{Ass}_0,n} \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$.

We have the morphism of set-preoperads $\alpha_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}$ defined by

$$\begin{aligned} \alpha_0(m,n) : \mathrm{Ass}_0(m,n) &\longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}(m,n) \\ a &\longmapsto \mathrm{id}_{\mathrm{Map}_0,m} \setminus a. \end{aligned}$$

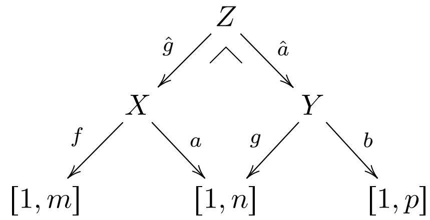
0.2.2 The set-operad COM_0

The set-preoperad $\mathrm{COM}_0^{\mathrm{pre}}$ has $\mathrm{COM}_0^{\mathrm{pre}}(m,n)$ consisting of equivalence classes of tuples (f,a) where $f : X \longrightarrow [1,m]$ and $a : X \longrightarrow [1,n]$ are maps and X is a finite set. Two tuples (f,a) and $(\tilde{f}, \tilde{a})$ with $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,m] \xleftarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{a}} [1,n]$, where X and $\tilde{X}$ are finite sets, are considered equivalent if there exists a bijective map $u : X \longrightarrow \tilde{X}$ such that $uf = \tilde{f}$ and $ua = \tilde{a}$. We denote the equivalence class of a tuple (f,a) by $f \setminus a$.

So, similar to the elements of $\mathrm{ASS}_0^{\mathrm{pre}}$, the elements of $\mathrm{COM}_0^{\mathrm{pre}}$ are fractions of maps, but, in contrast to the case $\mathrm{ASS}_0^{\mathrm{pre}}$, in $\mathrm{COM}_0^{\mathrm{pre}}$ we allow expansion by bijective maps. By abuse of notation, we use the same fraction notation in both cases.

The identity elements of $\mathrm{COM}_0^{\mathrm{pre}}$ are $\mathrm{id}_{\mathrm{COM}_0,m} := \mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Map}_0,m}$. The multiplication in $\mathrm{COM}_0^{\mathrm{pre}}$ is defined using the disjoint union of the maps and renumbering in the image; cf. Definition 9.3.

The composite of fractions $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$ and $g \setminus b \in \mathrm{COM}_0^{\mathrm{pre}}(n,p)$ is defined using a pullback of a and g. So we have the following diagram.



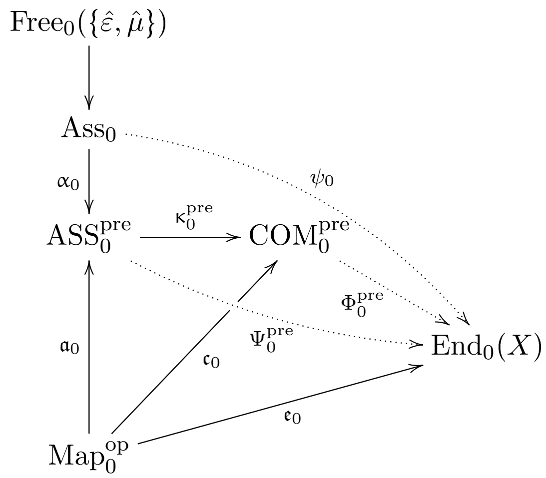
This allows us to define $(f \setminus a) \cdot_{\mathrm{COM}_0} (g \setminus b) := (\hat{g} f) \setminus (\hat{a} b)$.

Then $\mathrm{COM}_0 := (\mathrm{COM}_0^{\mathrm{pre}}, \mathfrak{c}_0)$ is a set-operad, where $\mathfrak{c}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{COM}_0^{\mathrm{pre}}$ has $\mathfrak{c}_0(m,n)$ mapping $f^{\mathrm{op}} \in \mathrm{Map}_0(m,n)$ to $f \setminus \mathrm{id}_{\mathrm{Map}_0,n} \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$.

We have the morphism of set-operads $\kappa_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{COM}_0$ that has $\kappa_0^{\mathrm{pre}}(m,n)$ mapping a fraction $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$ to the fraction $(f \setminus a)\kappa_0^{\mathrm{pre}} := f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$. The maps $\kappa_0^{\mathrm{pre}}(m,n)$ are surjective, but in general not injective.

0.2.3 Overview of the discussed set-preoperads and set-operads

The following diagram illustrates the set-preoperads and morphisms of such under consideration. Here X is a set.



In §5 we show that for a given morphism of set-preoperads $\psi_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{End}_0(X)$, i.e. for a given Ass_0-algebra (X, ψ_0), we get a monoid $(X, \mu_X, \varepsilon_X)$ with multiplication $\mu_X = \mu\psi_0$ and unit $\varepsilon_X = \varepsilon\psi_0$.

Conversely, given a monoid $(X, \mu_X, \varepsilon_X)$, then there exists a morphism of set-preoperads $\psi_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{End}_0(X)$ such that $\mu_X = \mu\psi_0$ and $\varepsilon_X = \varepsilon\psi_0$, which turns (X, ψ_0) into an Ass_0-algebra.

Given a morphism of set-operads $\Psi_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{END}_0(X)$, i.e. given an ASS_0-algebra (X, Ψ_0), then $(X, \alpha_0 \Psi_0^{\mathrm{pre}}) = (X, \psi_0)$ is an Ass_0-algebra. Hence $(X, \mu_X, \varepsilon_X)$ is a monoid with multiplication $\mu_X = \mu\psi_0 = \mu\alpha_0\Psi_0^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}}$ and unit $\varepsilon_X = \varepsilon\psi_0 = \varepsilon\alpha_0\Psi_0^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\Psi_0^{\mathrm{pre}}$.

Conversely, given a monoid $(X, \mu_X, \varepsilon_X)$, we can define the morphism of set-preoperads $\psi_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{End}_0(X)$ as explained above. Then a universal property of the diagram

$$\mathrm{Ass}_0 \xrightarrow{\alpha_0} \mathrm{ASS}_0^{\mathrm{pre}} \xleftarrow{\mathfrak{a}_0} \mathrm{Map}_0^{\mathrm{op}}$$

of set-preoperads induces, when compared with the diagram

$$\mathrm{Ass}_0 \xrightarrow{\psi_0} \mathrm{End}_0(X) \xleftarrow{\mathfrak{e}_0} \mathrm{Map}_0^{\mathrm{op}}$$

of set-preoperads, a uniquely determined morphism of set-preoperads $\Psi_0^{\mathrm{pre}} : \mathrm{ASS}_0^{\mathrm{pre}} \longrightarrow \mathrm{End}_0(X)$ satisfying $\alpha_0\Psi_0^{\mathrm{pre}} = \psi_0$ and $\mathfrak{a}_0\Psi_0^{\mathrm{pre}} = \mathfrak{e}$. In particular, $\Psi_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{END}_0(X)$ is a morphism of set-operads.

So (X, Ψ_0) is an ASS_0-algebra and we have $\mu_X = \mu\psi_0 = (\mu\alpha_0)\Psi_0^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}}$ as well as $\varepsilon_X = \varepsilon\psi_0 = (\varepsilon\alpha_0)\Psi_0^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\Psi_0^{\mathrm{pre}}$.

Theorem (cf. Propositions 5.3, 5.4, 7.16 and 7.18)**.** A monoid corresponds to an Ass_0-algebra, which in turn corresponds to an ASS_0-algebra, using the correspondences just described.

Given a morphism of set-operads $\Phi_0 : \mathrm{COM}_0 \longrightarrow \mathrm{END}_0(X)$, i.e. given a COM_0-algebra (X, Φ_0), then $(X, \mu_X, \varepsilon_X)$ is a commutative monoid with the multiplication $\mu_X = (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}}$ and the unit $\varepsilon_X = (\mathrm{id}_0 \setminus \varepsilon)\Phi_0^{\mathrm{pre}}$.

Given a morphism of set-operads $\Psi_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{END}_0(X)$, the morphism of set-operads κ_0 has a universal property that ensures that, under certain circumstances, there exists a morphism of set-operads $\Phi_0 : \mathrm{COM}_0 \longrightarrow \mathrm{END}_0(X)$ satisfying $\kappa_0\Phi_0 = \Psi_0$.

More precisely, if $\Psi_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{END}_0(X)$ satisfies $(\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}} = ((1,2) \setminus \mu)\Psi_0^{\mathrm{pre}}$, where μ is the unique element in $\mathrm{Ass}_0(2,1)$ and where $(1,2) \in \mathrm{Sym}_0(2,2)$ is the transposition, then there exists a uniquely determined morphism of set-operads $\Phi_0 : \mathrm{COM}_0 \longrightarrow \mathrm{END}_0(X)$ satisfying $\kappa_0\Phi_0 = \Psi_0$.

So conversely, given a commutative monoid $(X, \mu_X, \varepsilon_X)$, then it is in particular an associative monoid, hence there exists a uniquely determined morphism of set-operads $\Psi_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{END}_0(X)$ with $\mu_X = (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}}$ and $\varepsilon_X = (\mathrm{id}_0 \setminus \varepsilon)\Psi_0^{\mathrm{pre}}$. Since $(X, \mu_X, \varepsilon_X)$ is a commutative monoid, the morphism of set-operads Ψ_0 satisfies the condition of the universal property, hence there exists a unique morphism of set-operads $\Phi_0 : \mathrm{COM}_0 \longrightarrow \mathrm{END}_0(X)$ satisfying $\kappa_0\Phi_0 = \Psi_0$.

So (X, Φ_0) is a COM_0-algebra with $\mu_X = (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\kappa_0^{\mathrm{pre}}\Phi_0^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}}$ and $\varepsilon_X = (\mathrm{id}_0 \setminus \varepsilon)\Psi_0^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\kappa_0^{\mathrm{pre}}\Phi_0^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\Phi_0^{\mathrm{pre}}$.

Theorem (cf. Propositions 9.11 and 9.15)**.** Commutative monoids correspond to COM_0-algebras, using the correspondence just described.

Omitting the set-preoperads $\mathrm{Free}_0(\{\hat{\varepsilon}, \hat{\mu}\})$ and Ass_0, the above diagram of set-preoperads can be written as the following diagram of set-operads.

$$\begin{array}{ccc}
\mathrm{ASS}_0 & \xrightarrow{\kappa_0} & \mathrm{COM}_0 \\
 & \overset{\Psi_0}{\searrow} & \downarrow{\scriptstyle \Phi_0} \\
 & & \mathrm{END}_0(X)
\end{array}$$

We can also interpret this diagram as follows. The fact that every commutative monoid is in particular an associative monoid translates to the fact that, by composition of the action morphism with κ_0, every COM_0-algebra can be turned into an ASS_0-algebra.

0.3 Particular linear operads

In §7 we discuss the linear operad ASS, in §8 we discuss the linear operad BIALG, in §9 we discuss the linear operad COM and in §10 we discuss the linear operad LIE.

0.3.1 **The linear operad** ASS

We define the linear preoperad $\mathrm{ASS}^{\mathrm{pre}} = R\,\mathrm{ASS}_0^{\mathrm{pre,bij}}$, where $\mathrm{ASS}_0^{\mathrm{pre,bij}}$ is the set-subpreoperad of $\mathrm{ASS}_0^{\mathrm{pre}}$ defined by

$$\mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n) := \{f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n) : f \text{ is bijective}\}.$$

Let $\mathfrak{a} = R\Big(\mathfrak{a}_0\big|_{\mathrm{Sym}_0^{\mathrm{op}}}^{\mathrm{ASS}_0^{\mathrm{pre,bij}}}\Big) : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{ASS}^{\mathrm{pre}}$. Then $\mathrm{ASS} := (\mathrm{ASS}^{\mathrm{pre}}, \mathfrak{a})$ is a linear operad.

We have the morphism of linear preoperads $\alpha : \mathrm{Ass} \longrightarrow \mathrm{ASS}^{\mathrm{pre}}$ defined by

$$\begin{aligned} \alpha(m,n) : \quad \mathrm{Ass}(m,n) &\longrightarrow \mathrm{ASS}^{\mathrm{pre}}(m,n) \\ a &\longmapsto \mathrm{id}_{\mathrm{Map}_0,m} \setminus a \qquad \text{for } a \in \mathrm{Ass}_0(m,n). \end{aligned}$$

0.3.2 **The linear operad** BIALG

The linear operad $\mathrm{BIALG} := (\mathrm{BIALG}^{\mathrm{pre}}, \mathfrak{b})$ is defined as follows. We let $\mathrm{BIALG}^{\mathrm{pre}} := R\,\mathrm{ASS}_0^{\mathrm{pre}}$ and $\mathfrak{b} := R\big(\mathfrak{a}_0\big|_{\mathrm{Sym}_0^{\mathrm{op}}}\big) : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{BIALG}^{\mathrm{pre}}$.

So ASS is a set-suboperad of BIALG, consisting of formal linear combinations of fractions with bijective denominators. Viewed from their origin, both ASS and BIALG arise from ASS_0. For BIALG we use all denominators, whereas for ASS we only use bijective denominators.

0.3.3 **The linear operad** COM

We define the linear preoperad $\mathrm{COM}^{\mathrm{pre}} = R\,\mathrm{COM}_0^{\mathrm{pre,bij}}$, where $\mathrm{COM}_0^{\mathrm{pre,bij}}$ is the set-subpreoperad of $\mathrm{COM}_0^{\mathrm{pre}}$ defined by

$$\mathrm{COM}_0^{\mathrm{pre,bij}}(m,n) := \{f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n) : f \text{ is bijective}\}.$$

Let $\mathfrak{c} = R\Big(\mathfrak{c}_0\big|_{\mathrm{Sym}_0^{\mathrm{op}}}^{\mathrm{COM}_0^{\mathrm{pre,bij}}}\Big) : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{COM}^{\mathrm{pre}}$. Then $\mathrm{COM} := (\mathrm{COM}^{\mathrm{pre}}, \mathfrak{c})$ is a linear operad.

We have the morphism of linear operads $\kappa : \mathrm{ASS} \longrightarrow \mathrm{COM}$ which has $\kappa(m,n)$ mapping a fraction $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)$ to the fraction $(f \setminus a)\kappa := f \setminus a \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)$ for $m,n \in \mathbb{Z}_{\geqslant 0}$.

0.3.4 **The linear operad** LIE

The linear operad LIE) is the linear suboperad of ASS generated by the element

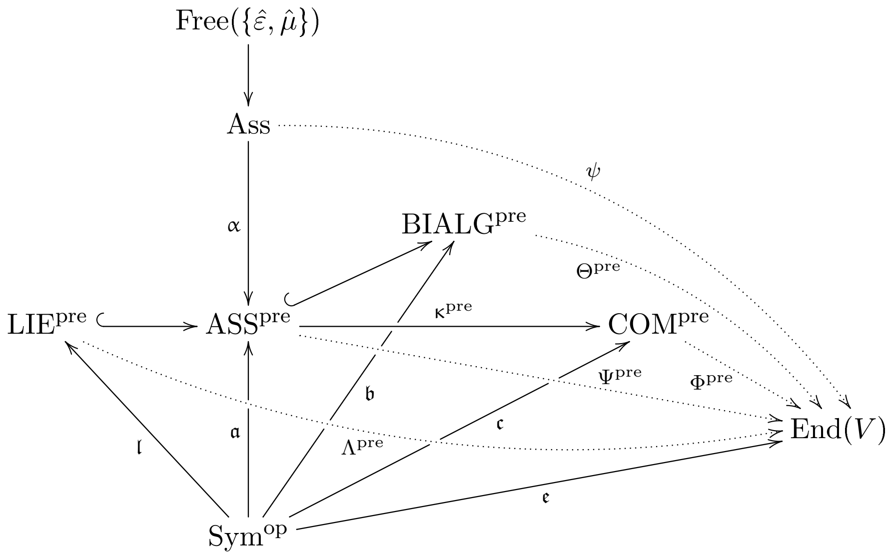
$$\lambda := (\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu) \in R(\mathrm{ASS}_0^{\mathrm{pre,bij}}(2,1)) = \mathrm{ASS}^{\mathrm{pre}}(2,1).$$

So we have $\mathrm{LIE} = {}_{\mathrm{op}}\langle\, \lambda \,\rangle$, that is, $\mathrm{LIE} = (\mathrm{LIE}^{\mathrm{pre}}, \mathfrak{l})$, where $\mathrm{LIE}^{\mathrm{pre}} = {}_{\mathrm{preop}}\langle \{\lambda\} \cup \mathrm{Im}(\mathfrak{a}) \,\rangle \subseteq \mathrm{ASS}^{\mathrm{pre}}$ and where $\mathfrak{l} := \mathfrak{a}\big|^{\mathrm{LIE}^{\mathrm{pre}}} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{LIE}^{\mathrm{pre}}$.

In the context of absolute operads, Aguiar and Livernet have defined the operad *Lie* as a suboperad of *Ass* generated by a commutator element; cf. **[1,** §5.3**]**.

0.3.5 Overview of the discussed linear preoperads and linear operads

The following diagram illustrates the linear preoperads and morphisms of such under consideration. Here V is an R-module.



In §5 we show that for a given morphism of linear preoperads $\psi : \mathrm{Ass} \longrightarrow \mathrm{End}(V)$, i.e. for a given Ass-algebra (V, ψ), we get an R-algebra $(V, \mu_V, \varepsilon_V)$ with multiplication $\mu_V = \mu\psi$ and unit $\varepsilon_V = \varepsilon\psi$.

Conversely, given an associative algebra $(V, \mu_V, \varepsilon_V)$, then there exists a morphism of linear preoperads $\psi : \mathrm{Ass} \longrightarrow \mathrm{End}(V)$ such that $\mu_V = \mu\psi$ and $\varepsilon_V = \varepsilon\psi$, which turns (V, ψ) into an Ass-algebra.

Given a morphism of linear operads $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$, i.e. given an ASS-algebra (V, Ψ), then $(V, \alpha\Psi^{\mathrm{pre}}) = (V, \psi)$ is an Ass-algebra. Hence $(V, \mu_V, \varepsilon_V)$ is an associative algebra with multiplication $\mu_V = \mu\psi = \mu\alpha\Psi^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}}$ and unit $\varepsilon_V = \varepsilon\psi = \varepsilon\alpha\Psi^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}}$.

Conversely, given an associative algebra $(V, \mu_V, \varepsilon_V)$, we define a morphism of linear preoperads $\psi : \mathrm{Ass} \longrightarrow \mathrm{End}(V)$ as explained above. Then a universal property of the diagram

$$\mathrm{Ass} \xrightarrow{\alpha} \mathrm{ASS}^{\mathrm{pre}} \xleftarrow{\mathfrak{a}} \mathrm{Sym}^{\mathrm{op}}$$

of linear preoperads induces, when compared with the diagram

$$\mathrm{Ass} \xrightarrow{\psi} \mathrm{End}(V) \xleftarrow{\mathfrak{e}} \mathrm{Sym}^{\mathrm{op}}$$

of linear preoperads, a uniquely determined morphism of linear preoperads $\Psi^{\mathrm{pre}} : \mathrm{ASS}^{\mathrm{pre}} \longrightarrow \mathrm{End}(V)$ satisfying $\alpha\Psi^{\mathrm{pre}} = \psi$ and $\mathfrak{a}\Psi^{\mathrm{pre}} = \mathfrak{e}$. In particular, $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ is a morphism of linear operads.

So (V, Ψ) is an ASS-algebra that satisfies $\mu_V = \mu\psi = (\mu\alpha)\Psi^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}}$ as well as $\varepsilon_V = \varepsilon\psi = (\varepsilon\alpha)\Psi^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}}$.

Theorem (cf. Propositions 5.6, 5.7, 7.22 and 7.24)**.** An algebra corresponds to an Ass-algebra, which in turn corresponds to an ASS-algebra, using the correspondences just described.

Proposition (cf. Proposition 8.3)**.** Given a morphism of linear operads $\Theta : \mathrm{BIALG} \longrightarrow \mathrm{END}(V)$, i.e. given a BIALG-algebra (V, Θ), then $(V, \mu_V, \varepsilon_V, \Delta_V, \eta_V)$ is a bialgebra with

$$\begin{array}{llcl}
\text{multiplication} & \mu_V & = & (\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}} \in \mathrm{End}(V)(2,1) \\
\text{unit} & \varepsilon_V & = & (\mathrm{id}_0 \setminus \varepsilon)\Theta^{\mathrm{pre}} \in \mathrm{End}(V)(0,1) \\
\text{comultiplication} & \Delta_V & = & (\mu \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} \in \mathrm{End}(V)(1,2) \\
\text{counit} & \eta_V & = & (\varepsilon \setminus \mathrm{id}_0)\Theta^{\mathrm{pre}} \in \mathrm{End}(V)(1,0).
\end{array}$$

Given a morphism of linear operads $\Phi : \mathrm{COM} \longrightarrow \mathrm{END}(V)$, i.e. given a COM-algebra (V, Φ), then $(V, \mu_V, \varepsilon_V)$ is a commutative algebra with the multiplication $\mu_V = (\mathrm{id}_2 \setminus \mu)\Phi^{\mathrm{pre}}$ and the unit $\varepsilon_V = (\mathrm{id}_0 \setminus \varepsilon)\Phi^{\mathrm{pre}}$.

Given a morphism of linear operads $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$, the morphism of linear operads κ has a universal property that ensures that, under certain circumstances, there exists a morphism of linear operads $\Phi : \mathrm{COM} \longrightarrow \mathrm{END}(V)$ satisfying $\kappa\Phi = \Psi$.

More precisely, if $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ satisfies $(\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} = ((1,2) \setminus \mu)\Psi^{\mathrm{pre}}$, where μ is the unique element in $\mathrm{Ass}_0(2,1)$ and where $(1,2) \in \mathrm{Sym}_0(2,2)$ is the transposition, then there exists a uniquely determined morphism of linear operads $\Phi : \mathrm{COM} \longrightarrow \mathrm{END}(V)$ satisfying $\kappa\Phi = \Psi$.

So conversely, given a commutative algebra $(V, \mu_V, \varepsilon_V)$, then it is in particular an associative algebra, hence there exists a uniquely determined morphism of linear operads $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ with $\mu_V = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}}$ and $\varepsilon_V = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}}$. Since $(V, \mu_V, \varepsilon_V)$ is a commutative algebra, the morphism of linear operads Ψ satisfies the condition of the universal property, hence there exists a unique morphism of linear operads $\Phi : \mathrm{COM} \longrightarrow \mathrm{END}(V)$ satisfying $\kappa\Phi = \Psi$.

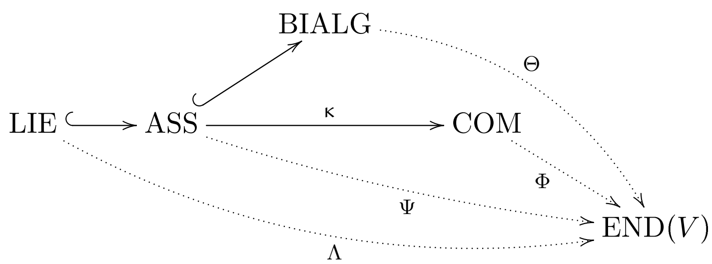
So (V, Φ) is a COM-algebra and we have $\mu_V = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\kappa^{\mathrm{pre}}\Phi^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\Phi^{\mathrm{pre}}$ and $\varepsilon_V = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\kappa^{\mathrm{pre}}\Phi^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\Phi^{\mathrm{pre}}$.

Theorem (cf. Propositions 9.19 and 9.21)**.** Commutative algebras correspond to COM-algebras, using the correspondence just described.

Proposition (cf. Proposition 10.2)**.** Suppose $2 \in \mathcal{U}(R)$. Let V be an R-module. Given a morphism of linear operads $\Lambda : \mathrm{LIE} \longrightarrow \mathrm{END}(V)$, i.e. given a LIE-algebra (V, Λ), then $(V, [-,=])$ is a Lie algebra with Lie bracket $[v, w] := (v \otimes w)\lambda_V$ for $v, w \in V$, where

$$\lambda_V := \lambda\Lambda^{\mathrm{pre}} = ((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu))\Lambda^{\mathrm{pre}}.$$

Omitting the linear preoperads $\mathrm{Free}(\{\hat{\varepsilon}, \hat{\mu}\})$ and Ass , the above diagram of linear preoperads can be written as the following diagram of linear operads.



The fact that every commutative algebra is in particular an associative algebra translates to the fact that, by composition of the action morphism with κ, every COM-algebra can be turned into an ASS-algebra.

The fact that a bialgebra has an underlying associative algebra, obtained by forgetting comultiplication and counit, translates to the fact that we may restrict the action morphism $\mathrm{BIALG} \longrightarrow \mathrm{END}(V)$ from BIALG to ASS.

Given an associative algebra V, we have the ASS-algebra (V, Ψ). So by restricting Ψ to LIE we obtain the LIE-algebra $(V, \Psi\big|_{\mathrm{LIE}}) = (V, \Lambda)$. So in this case V is a Lie algebra with Lie bracket

$$\begin{aligned}[v, w] &= (v \otimes w)((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu))\Lambda^{\mathrm{pre}} \\ &= (v \otimes w)((\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} - ((1,2) \setminus \mu)\Psi^{\mathrm{pre}}) \\ &= (v \otimes w)\mu_V - (w \otimes v)\mu_V\end{aligned}$$

for $v, w \in V$. That is, $(V, [-,=])$ is the commutator Lie algebra for the associative algebra $(V, \mu_V, \varepsilon_V)$.

In other words, the fact that each associative R-algebra has a commutator Lie algebra translates to the fact that we may restrict the action morphism $\mathrm{ASS} \longrightarrow \mathrm{END}(V)$ from ASS to LIE.

However, starting with a morphism of linear operads $\Lambda : \mathrm{LIE} \longrightarrow \mathrm{END}(V)$, we do not know whether there exists a morphism of linear operads $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ with $\Psi\big|_{\mathrm{LIE}} = \Lambda$.

0.4 Historical Context

In his 1963 thesis, F. W. Lawvere defined algebraic theories, which are, except for one additional property, a first version of what would later be known as PROPs; cf. **[7,** p.869, l.14**]**.

Then, also in 1963, S. Mac Lane, one of the interlocutors of Lawvere, defined a PROP (short for product and permutation category) as follows; cf. **[8,** §6**]** and **[9,** §V.24**]**, written partially in collaboration with J. F. Adams. Using his notation, he takes a category $\mathcal{H}$ with the natural numbers as objects such that for $n \in \mathbb{Z}_{\geqslant 0}$, the symmetric group $S(n)$ is a subgroup of the group of all invertible elements of $\mathcal{H}\binom{n}{n} := \hom_{\mathcal{H}}(n,n)$, together with a functor $\otimes : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ with object function $m \otimes m' = m + m'$ satisfying (1)–(3) below.

(1) $\otimes$ is associative, so we have $f \otimes (f' \otimes f'') = (f \otimes f') \otimes f''$.

(2) For $\sigma \in S(n)$ and $\sigma' \in S(n')$ we have that $\sigma \otimes \sigma' \in \mathcal{H}\binom{n+n'}{n+n'}$ is the permutation that acts on the first n letters as σ does and on the remaining n' letters as σ' does.

(3) The blockwise transposition $\tau_{(n,n')} \in \mathcal{H}\binom{n+n'}{n+n'}$ that interchanges the first block of n letters with the second block of n' letters satisfies the following. For $f \in \mathcal{H}\binom{m}{n}$ and $f' \in \mathcal{H}\binom{m'}{n'}$ we have

$$\tau_{(n,n')} \circ (f \otimes f') = (f' \otimes f) \circ \tau_{(m,m')} .$$

Note that $(f \circ g) \otimes (f' \circ g') = (f \otimes f') \circ (g \otimes g')$, whenever defined, since $\otimes$ is a functor.

So essentially, a PROP, enriched in R-modules, is the same as a linear operad, we merely use different notation. For instance, in linear operads, we obtain symmetric group elements as images of $\mathrm{Sym}^{\mathrm{op}}$-elements under the action morphism. Moreover, property (3) is equivalent to condition (lo) for a linear operad.

Our concept of set-operads, however, does not entirely fit into the concept of a PROP. In set-operads we consider the image of every map $[1,m] \longrightarrow [1,n]$ under the action morphism, not only of bijective maps, and thus have the additional condition (so2).

From the theory of PROPs and their complex version, called PACTs, later the theory of operads was developed as a somewhat reduced version. Operads in the classical sense can be defined in any symmetric monoidal category. The first definition was by J. P. May in 1972 over compactly generated Hausdorff spaces; cf. **[13,** Definition 1.1**]**.

An operad in the category of R-modules is given by R-modules $\mathcal{P}(m)$ for $m \in \mathbb{Z}_{\geqslant 0}$, each carrying a symmetric group action. There are structure morphisms

$$\gamma_{n;m_1,\ldots,m_n} : \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_n) \longrightarrow \mathcal{P}(m_1 + \ldots + m_n)$$

for $n, m_1, \ldots, m_n \in \mathbb{Z}_{\geqslant 0}$ and a unit morphism $\eta : R \longrightarrow \mathcal{P}(1)$ satisfying certain associativity, unity and equivariance axioms; cf. **[13,** Definition 1.1**]** or **[12,** Definition II.1.4.**]**.

There is an equivalent definition of an operads using not the composition morphisms $\gamma_{n;m_1,\ldots,m_n}$ but partial composition products

$$\left(\underset{i}{\circ}\right) : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$$

for $i \in [1, m]$ with suitable associativity, unity and equivariance axioms; cf. **[1,** §1.1**]** or under the name "pseudo-operad" in **[12,** Definition II.1.16**]**. The equivalence of these definitions was shown e.g. in **[12,** II.1.7.1**]**.

Forgetting the symmetric groups actions on the R-modules $\mathcal{P}(m)$, we obtain a nonsymmetric operad; cf. **[1,** §1.1**]** or **[12,** Definition II.1.18 and §II.1.7.1**]** ("nonsymmetric pseudo-operads with unit"). These nonsymmetric operads are equivalent to the absolute operads we consider in §3. An absolute operad can be obtained from a linear preoperad $\mathcal{P}$ by forgetting those $\mathcal{P}(m, n)$ with $m \neq 1$, defining $\mathcal{P}(m) := \mathcal{P}(m, 1)$ and by defining the partial composition products $\underset{i}{\circ}$ by

$$f \underset{i}{\circ} g := (\mathrm{id}_{i-1} \boxtimes g \boxtimes \mathrm{id}_{m-i}) \cdot f \in \mathcal{P}(m+n-1)$$

for $f \in \mathcal{P}(m)$ and $g \in \mathcal{P}(n)$.

0.5 Open Questions

Recall that we show that associative algebras correspond to Ass-algebras and to ASS-algebras.

However, in §8, we only show that every BIALG-algebra is a bialgebra. Moreover, in §10, we only show that for certain commutative rings R, every LIE-algebra is a Lie algebra. In both situations we do not show the converse statement, i.e. we do not show that every bialgebra can be turned into a BIALG-algebra or that every Lie algebra can be turned into a LIE-algebra.

One way to show this would be to define the free linear operad FREE(X) for a biindexed set X and to find generators and relations, i.e. presentations for BIALG and LIE.

More generally, we may ask for a left adjoint to the forgetful functor from linear operads to linear preoperads – which should then map Free(X) to FREE(X), as well as Ass to ASS.

0.6 Conventions

1. Let R be a commutative (unital) ring. We denote by 1_R the unit of R. The ring R will play the role of the ground ring, if applicable. A linear map between R-modules is an R-linear map.

2. We write $\mathbb{Z}$ for the set of integers. We write $\mathbb{Z}_{\geqslant m} := \{n \in \mathbb{Z} : n \geqslant m\}$ for $m \in \mathbb{Z}$.

3. For $m, n \in \mathbb{Z}_{\geqslant 0}$ we write $[m, n] := \{i \in \mathbb{Z}_{\geqslant 0} : m \leqslant i \leqslant n\}$. Note that if $m > n$, then we have $[m, n] = \varnothing$.

4. For a finite set X we denote by $|X|$ the cardinality of X.

5. We will often write $m \geqslant 0$ instead of $m \in \mathbb{Z}_{\geqslant 0}$ for brevity, so the abbreviation $m \geqslant 0$ will always imply that m is an integer.

6. We write maps on the right, so for sets X and Y and a map $f : X \longrightarrow Y$ we write xf for the image of $x \in X$ under f. However, we write the inverse image on the left, so for a subset $S \subseteq Y$ we let
$$f^{-1}(S) = \{x \in X : xf \in S\}.$$
Furthermore, for $y \in Y$ we abbreviate $f^{-1}(y) := f^{-1}(\{y\}) = \{x \in X : xf = y\}$.

7. Composition of morphisms in a category is also written on the right, i.e.
$$\Big(X \xrightarrow{\ a\ } Y \xrightarrow{\ b\ } Z \Big) = \Big(X \xrightarrow{\ ab\ } Z \Big).$$

8. The identity morphism on an object X of a category is written id_X.

9. Suppose given sets X and Y and a map $f : X \longrightarrow Y$. For a subset $S \subseteq X$ we write $f\big|_S$ for the restricted map
$$\begin{aligned} f\big|_S : S &\longrightarrow Y \\ s &\longmapsto s(f\big|_S) := sf. \end{aligned}$$
Moreover, suppose given a subset $T \subseteq Y$ such that $xf \in T$ for $x \in X$, i.e. $\mathrm{Im}(f) \subseteq T$. Then we write $f\big|^T$ for the restricted map
$$\begin{aligned} f\big|^T : X &\longrightarrow T \\ x &\longmapsto x(f\big|^T) := xf. \end{aligned}$$
We will also use a combined version of this. Suppose given a subset $S \subseteq X$ and a subset $T \subseteq Y$ such that $sf \in T$ for $s \in S$. Then we have the restricted map
$$\begin{aligned} f\big|_S^T = (f\big|_S)\big|^T : S &\longrightarrow T \\ s &\longmapsto s(f\big|_S^T) := sf. \end{aligned}$$

10. Suppose given finite and linearly ordered sets X and Y. Suppose given a map $f : X \longrightarrow Y$. Then we say that f is *isotone*, if it is bijective and monotone.

    Note that the composite of isotone maps is again isotone. Moreover, given the isotone maps $f, g : X \longrightarrow Y$, then we have $f = g$, so there exists at most one isotone map between the finite and linearly ordered sets X and Y.

11. In §2 we will introduce the set-preoperad $\mathrm{End}_0(X)$ for some set X and the linear preoperad $\mathrm{End}(V)$ for some R-module V. We will not use $\mathrm{End}(V)$ to denote the endomorphism ring of V.

12. Let $n \in \mathbb{Z}_{\geqslant 0}$ and let X be a set. We abbreviate $X^{\times n} := X \times \ldots \times X$ for the n-fold cartesian product. For $n = 0$ this means $X^{\times 0} = \{()\}$, where $()$ is the empty tuple. Furthermore, for $m, n \in \mathbb{Z}_{\geqslant 0}$ we can define a bijective map $X^{\times m} \times X^{\times n} \longrightarrow X^{\times (m+n)}$ by joining the tuples, i.e. we have

$$\begin{aligned} X^{\times m} \times X^{\times n} &\longrightarrow X^{\times (m+n)} \\ \big((x_1, \ldots, x_m), (x_{m+1}, \ldots, x_{m+n})\big) &\longmapsto (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}), \end{aligned}$$

thus identifying $X^{\times m} \times X^{\times n} = X^{\times (m+n)}$. We will write

$$(x_1, \ldots, x_m) \times (x_{m+1}, \ldots, x_{m+n}) := (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n})$$

for $x_1, \ldots, x_{m+n} \in X$.

Moreover, we identify $X^{\times 1} = X$ via the bijective map $X^{\times 1} \longrightarrow X, (x) \longmapsto x$.

Note that joining tuples is associative, i.e. for $m, m', m'' \in \mathbb{Z}_{\geqslant 0}$ and $x_1, \ldots, x_{m+m'+m''} \in X$ we have

$$\begin{aligned} &\big((x_1, \ldots, x_m) \times (x_{m+1}, \ldots, x_{m+m'})\big) \times (x_{m+m'+1}, \ldots, x_{m+m'+m''}) \\ &= (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+m'}, x_{m+m'+1}, \ldots, x_{m+m'+m''}) \\ &= (x_1, \ldots, x_m) \times \big((x_{m+1}, \ldots, x_{m+m'}) \times (x_{m+m'+1}, \ldots, x_{m+m'+m''})\big). \end{aligned}$$

13. Suppose given finite sets X_1, X_2 and Y_1, Y_2 and maps $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$. The cartesian product of f_1 and f_2 is the map

$$\begin{aligned} f_1 \times f_2 : X_1 \times X_2 &\longrightarrow Y_1 \times Y_2 \\ (x_1, x_2) &\longmapsto x_1 f_1 \times x_2 f_2 . \end{aligned}$$

14. A *monoid* $(X, \mu_X, \varepsilon_X)$ is a set X together with a *multiplication* map $\mu_X : X \times X \longrightarrow X$ and a *unit* map $\varepsilon_X : \{()\} = X^{\times 0} \longrightarrow X$ such that μ_X is associative, i.e.

$$(\mu_X \times \mathrm{id}_X)\mu_X = (\mathrm{id}_X \times \mu_X)\mu_X : X^{\times 3} \longrightarrow X,$$

and such that

$$(\varepsilon_X \times \mathrm{id}_X)\mu_X = \mathrm{id}_X = (\mathrm{id}_X \times \varepsilon_X)\mu_X .$$

Note that this is equivalent to the definition of a monoid $(X, \bullet, 1_X)$ with associative multiplication $(\bullet)$ and neutral element 1_X if we define $x \bullet y := (x, y)\mu_X \in X$ for $x, y \in X$ and $1_X := ()\varepsilon_X \in X$.

We define $\tau_X : X^{\times 2} \longrightarrow X^{\times 2}$ to be the map that maps $(x, y) \in X^{\times 2}$ to $(x, y)\tau_X = (y, x)$. The monoid $(X, \mu_X, \varepsilon_X)$ is said to be *commutative*, if $\tau_X \mu_X = \mu_X$.

15. For $n \in \mathbb{Z}_{\geqslant 0}$ and for an R-module V we write $V^{\otimes n} := V \otimes \ldots \otimes V$ for the n-fold tensor product. Furthermore, we identify $X^{\otimes 0} = R$, $V^{\otimes 1} = V$ and $R^{\otimes n} = R$ for $n \in \mathbb{Z}_{\geqslant 0}$.

16. An *R-algebra* $(V, \mu_V, \varepsilon_V)$ is an R-module V together with an R-linear *multiplication* map $\mu_V : V \otimes V \longrightarrow V$ and an R-linear *unit* map $\varepsilon_V : R = V^{\otimes 0} \longrightarrow V$ such that μ_V is associative, i.e.

$$(\mu_V \otimes \mathrm{id}_V)\mu_V = (\mathrm{id}_V \otimes \mu_V)\mu_V : V^{\otimes 3} \longrightarrow V,$$

and such that

$$(\varepsilon_V \otimes \mathrm{id}_V)\mu_V = \mathrm{id}_V = (\mathrm{id}_V \otimes \varepsilon_V)\mu_V .$$

Note that this is equivalent to the definition of an R-algebra $(V, \bullet, 1_V)$ with associative multiplication $(\bullet)$ and neutral element 1_V if we define $v \bullet w := (v, w)\mu_V \in V$ for $v, w \in V$ and $1_V := 1_R \varepsilon_V \in V$.

We define $\tau_V : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ to be the map that maps $v \otimes w \in V^{\otimes 2}$ to $(v \otimes w)\tau_V = w \otimes v$ for $v, w \in V$. The R-algebra $(V, \mu_V, \varepsilon_V)$ is said to be *commutative*, if $\tau_V \mu_V = \mu_V$.

17. A *bialgebra* $(V, \mu_V, \varepsilon_V, \Delta_V, \eta_V)$ is an R-module V together with an R-linear *multiplication* map $\mu_V : V^{\otimes 2} \longrightarrow V$, an R-linear *unit* map $\varepsilon_V : R = V^{\otimes 0} \longrightarrow V$, an R-linear *comultiplication* map $\Delta_V : V \longrightarrow V^{\otimes 2}$ and an R-linear *counit* map $\eta_V : V \longrightarrow V^{\otimes 0} = R$ such that the following hold.

    - $(V, \mu_V, \varepsilon_V)$ is an R-algebra, that is, we have $(\mu_V \otimes \mathrm{id}_V)\mu_V = (\mathrm{id}_V \otimes \mu_V)\mu_V$ and $(\varepsilon_V \otimes \mathrm{id}_V)\mu_V = \mathrm{id}_V = (\mathrm{id}_V \otimes \varepsilon_V)\mu_V$.
    - (V, Δ_V, η_V) is an R-coalgebra, that is, we have $\Delta_V(\Delta_V \otimes \mathrm{id}_V) = \Delta_V(\mathrm{id}_V \otimes \Delta_V)$ and $\Delta_V(\eta_V \otimes \mathrm{id}_V) = \mathrm{id}_V = \Delta_V(\mathrm{id}_V \otimes \eta_V)$.
    - The following compatibility conditions are satisfied.
        - We have $\mu_V \Delta_V = (\Delta_V \otimes \Delta_V)(\mathrm{id}_V \otimes \tau_V \otimes \mathrm{id}_V)(\mu_V \otimes \mu_V)$, where τ_V is the R-linear map $\tau_V : V \otimes V \longrightarrow V \otimes V$, $v \otimes w \longmapsto w \otimes v$ as above.
        - We have $\mu_V \eta_V = \eta_V \otimes \eta_V$.
        - We have $\varepsilon_V \Delta_V = \varepsilon_V \otimes \varepsilon_V$.
        - We have $\varepsilon_V \eta_V = \mathrm{id}_R$.

    Cf. **[4,** Definition 4.1.3**]**.

18. We denote by $\mathcal{U}(R)$ the set of units of R. That is, $\mathcal{U}(R)$ consists of all invertible elements of R.

1 Preliminaries

1.1 Tensor products

We will give a brief definition and some properties of the tensor product of R-modules. For all proofs and for further properties we refer to **[14,** §1.3**]**.

Definition 1.1. Let $m \in \mathbb{Z}_{\geqslant 0}$. Let $V_1, \ldots, V_m$ be R-modules. We denote by $\bigotimes_{i \in [1,m]} V_i = V_1 \otimes \ldots \otimes V_m$ the tensor product of $V_1, \ldots, V_m$ over R; cf. e.g. **[14,** Definition 13**]**.

Note that the tensor product $V_1 \otimes \ldots \otimes V_m$ has the R-linear generating set

$$\{v_1 \otimes \ldots \otimes v_m : v_i \in V_i \text{ for } i \in [1, m]\}\,;$$

cf. **[14,** Lemma 14**]**.

Let $\mu_{V_1,\ldots,V_m}$ be the map

$$\begin{aligned} \mu_{V_1,\ldots,V_m} : V_1 \times \ldots \times V_m &\longrightarrow V_1 \otimes \ldots \otimes V_m \\ (v_1, \ldots, v_m) &\longmapsto v_1 \otimes \ldots \otimes v_m\,. \end{aligned}$$

In the case $m = 0$ we identify $\bigotimes_{i \in [1,0]} V_i = R$.

Furthermore, for an R-module V we also write $V^{\otimes m} := \bigotimes_{i \in [1,m]} V$. In particular, $V^{\otimes 0} = R$.

Lemma 1.2 (Universal property of the tensor product)**.** *Let $m \in \mathbb{Z}_{\geqslant 0}$ and let $V_1, \ldots, V_m$ and M be R-modules. Let $f : V_1 \times \ldots \times V_m \longrightarrow M$ be an R-multilinear map; cf.* **[14,** Definition 7**]**.

There exists a unique R-linear map $\bar{f} : V_1 \otimes \ldots \otimes V_m \longrightarrow M$ such that $\mu_{V_1,\ldots,V_m} \bar{f} = f$.

Proof. For the proof see **[14,** Lemma 16**]**. □

Lemma 1.3. *Let $m \in \mathbb{Z}_{\geqslant 0}$ and let $V_1, \ldots, V_m$ be R-modules. Let $j \in [2, m]$. There exists the unique R-linear isomorphism*

$$\begin{aligned} \psi : \quad \Big(\bigotimes_{i \in [1,j-1]} V_i\Big) \otimes \Big(\bigotimes_{i \in [j,m]} V_i\Big) &\longrightarrow \bigotimes_{i \in [1,m]} V_i \\ (v_1 \otimes \ldots \otimes v_{j-1}) \otimes (v_j \otimes \ldots \otimes v_m) &\longmapsto v_1 \otimes \ldots \otimes v_m\,. \end{aligned}$$

Proof. For the proof see **[14,** Lemma 19**]**. □

Remark 1.4. We will use the isomorphism ψ to identify $\Big(\bigotimes_{i \in [1,j-1]} V_i\Big) \otimes \Big(\bigotimes_{i \in [j,m]} V_i\Big) = \bigotimes_{i \in [1,m]} V_i$ for $m \in \mathbb{Z}_{\geqslant 0}$ and R-modules $V_1, \ldots, V_m$.

In particular, given an R-module V, then we identify $V^{\otimes m} \otimes V^{\otimes n} = V^{\otimes (m+n)}$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ using ψ.

Remark 1.5. Furthermore, we identify $R \otimes V = V = V \otimes R$ and $V^{\otimes 1} = V$ for any R-module V.

Remark 1.6. Note that by identification via ψ we have

$$(\xi \otimes \xi') \otimes \xi'' = \xi \otimes \xi' \otimes \xi'' = \xi \otimes (\xi' \otimes \xi'')$$

for $m, m', m'' \in \mathbb{Z}_{\geqslant 0}$ and $\xi \in V^{\otimes m}$, $\xi' \in V^{\otimes m'}$ and $\xi'' \in V^{\otimes m''}$, as can be seen on elementary tensors.

Definition 1.7. Let $m \in \mathbb{Z}_{\geqslant 0}$. Let $V_1, \ldots, V_m$ and $W_1, \ldots, W_m$ be R-modules and let $f_i : V_i \longrightarrow W_i$ be an R-linear map for $i \in [1, m]$. We define the tensor product of $f_1, \ldots, f_m$ as follows. Let

$$\begin{aligned} f_1 \otimes \ldots \otimes f_m : \qquad V_1 \otimes \ldots \otimes V_m &\longrightarrow W_1 \otimes \ldots \otimes W_m \\ \sum_{j \in [1,m]} r_j (v_{1,j} \otimes \ldots \otimes v_{m,j}) &\longmapsto \sum_{j \in [1,m]} r_j (v_{1,j} f_1 \otimes \ldots \otimes v_{m,j} f_m). \end{aligned}$$

This is a well-defined R-linear map, as proven in **[14,** Definition/Lemma 20**]**.

1.2 The free R-module on a set X

Definition 1.8. Let X be a set. Then we can define the free R-module with basis X by taking all formal R-linear combinations of elements in X, i.e.

$$RX := \Big\{ \sum_{x \in X} r_x x : r_x \in R,\ \{x \in X : r_x \neq 0\} \text{ finite} \Big\}.$$

Referring to an element of RX by $\sum_{x \in X} r_x x$, we suppose that $r_x \in R$ for $x \in X$ and that the set $\{x \in X : r_x \neq 0\}$ is finite without further comment.

Identifying along the injective map

$$\begin{aligned} X &\longrightarrow RX \\ y &\longmapsto \sum_{x \in X} \delta_{x,y} x, \end{aligned}$$

where $\delta_{x,y} = \left\{ \begin{array}{ll} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{array} \right\}$, we obtain $X \subseteq RX$.

Given sets X, Y and a map $u : X \longrightarrow Y$ then the map

$$\begin{aligned} Ru : \qquad RX &\longrightarrow RY \\ \sum_{x \in X} r_x x &\longmapsto \sum_{x \in X} r_x\, (xu) \end{aligned}$$

is R-linear.

We have $R(uv) = (Ru)(Rv)$ for sets X, Y and Z and maps $u : X \longrightarrow Y$, $v : Y \longrightarrow Z$. Furthermore, we have $R\,\mathrm{id}_X = \mathrm{id}_{RX}$ for a set X and the identity map $\mathrm{id}_X : X \longrightarrow X$. So $X \mapsto RX$ and $u \mapsto Ru$ defines a functor from the category of sets to the category of R-modules.

Remark 1.9. Let X be a set. Let M be an R-module and let $\varphi : X \longrightarrow M$ be a map. Then the map

$$\begin{aligned} \bar{\varphi} : \qquad RX &\longrightarrow M \\ \sum_{x \in X} r_x x &\longmapsto \sum_{x \in X} r_x (x\varphi) \end{aligned}$$

is the uniquely determined R-linear map $RX \longrightarrow M$ such that $x\bar{\varphi} = x\varphi$ for $x \in X$.

Remark 1.10. We have the following isomorphism.

$$\begin{aligned} RX \otimes RY &\longrightarrow R\,(X \times Y) \\ \Big(\sum_{x \in X} r_x x \Big) \otimes \Big(\sum_{y \in Y} s_y y \Big) &\longmapsto \sum_{(x,y) \in X \times Y} r_x s_y (x, y) \\ \sum_{(x,y) \in X \times Y} t_{(x,y)} (x \otimes y) &\longleftarrow\!\shortmid \sum_{(x,y) \in X \times Y} t_{(x,y)} (x, y) \end{aligned}$$

We use this isomorphism to identify $R(X \times Y) = RX \otimes RY$.

1.3 Equivalence relations

Lemma 1.11 (Generated equivalence relation). *Let X be a set and $(\sim) \subseteq X \times X$ be a relation on X. We define a relation $(\approx)$ on X as follows. For $x, x' \in X$ let $x \approx x'$ if there exist $n \geqslant 1$ and $c_1, \ldots, c_n \in X$ such that $x = c_1$, $x' = c_n$ and such that $(c_i, c_{i+1}) \in (\sim)$ or $(c_{i+1}, c_i) \in (\sim)$ for $i \in [1, n-1]$.*

Then $(\approx)$ is an equivalence relation on X.

We say that $(\approx)$ is the equivalence on X generated by $(\sim)$.

Proof. *Reflexivity.* Suppose given $x \in X$. Then, setting $n := 1$ and $c_1 := x$, we have $x \approx x$.

Symmetry. Suppose given $x, x' \in X$ such that $x \approx x'$. So there exist $n \geqslant 1$ and $c_1, \ldots c_n \in X$ such that $x = c_1$, $x' = c_n$ and such that $(c_i, c_{i+1}) \in (\sim)$ or $(c_{i+1}, c_i) \in (\sim)$ for $i \in [1, n-1]$.

Define $c'_i := c_{n-i+1} \in X$ for $i \in [1, n]$.

Then we have $c'_1 = c_{n-1+1} = c_n = x'$ and $c'_n = c_{n-n+1} = c_1 = x$ and for $i \in [1, n-1]$ we have $(c'_i, c'_{i+1}) = (c_{n-i+1}, c_{n-i}) \in (\sim)$ or $(c'_{i+1}, c'_i) = (c_{n-i}, c_{n-i+1}) \in (\sim)$, hence $x' \approx x$.

Transitivity. Suppose given $x, x', x'' \in X$ such that $x \approx x'$ and $x' \approx x''$. So there exist $n, n' \geqslant 1$ and $c_1, \ldots, c_n, c'_1, \ldots c'_{n'} \in X$ such that $x = c_1$, $x' = c_n$ and $(c_i, c_{i+1}) \in (\sim)$ or $(c_{i+1}, c_i) \in (\sim)$ for $i \in [1, n-1]$ and such that $x' = c'_1$, $x'' = c'_{n'}$ and $(c'_j, c'_{j+1}) \in (\sim)$ or $(c'_{j+1}, c'_j) \in (\sim)$ for $j \in [1, n'-1]$.

Define $n'' := n + n'$. For $i \in [1, n'']$ define $c''_i := \left\{ \begin{array}{ll} c_i & \text{if } i \in [1, n] \\ c'_{i-n} & \text{if } i \in [n+1, n+n'] \end{array} \right\}$.

Then we have $x = c''_1$, $x'' = c''_{n''}$ and since $c''_n = c_n = x' = c'_1 = c''_{n+1}$ we have $(c''_i, c''_{i+1}) \in (\sim)$ or $(c''_{i+1}, c''_i) \in (\sim)$ for $i \in [1, n+n']$. Hence $x \approx x''$. □

Lemma 1.12. *Let X, Y be sets. Let $(\sim) \subseteq X \times X$ be a relation on X and let $(\approx)$ be the equivalence relation on X generated by $(\sim)$; cf. Lemma* 1.11*. Let $f : X \longrightarrow Y$ be a map such that for $x, x' \in X$ with $x \sim x'$ we have $xf = x'f$.*

Then for $x, x' \in X$ with $x \approx x'$ we have $xf = x'f$.

So there exists a unique map $\bar{f} : \frac{X}{(\approx)} \longrightarrow Y$ that maps the equivalence class of $x \in X$ to xf.

Proof. Suppose given $x, x' \in X$ with $x \approx x'$. So there exist $n \geqslant 0$ and $c_1, \ldots, c_n \in X$ such that $x = c_1$, $x' = c_n$ and such that $c_i \sim c_{i+1}$ or $c_{i+1} \sim c_i$ for $i \in [1, n-1]$. So by assumption we have $xf = c_1 f$, $c_n f = x'f$ and $c_i f = c_{i+1} f$ for $i \in [1, n-1]$. So we have $xf = c_1 f = c_2 f = \ldots = c_n f = x'f$. □

1.4 Disjoint unions

Definition 1.13. Let $n \in \mathbb{Z}_{\geqslant 0}$ and let X_i be a set for $i \in [1, n]$. Then the (exterior) disjoint union of $X_1, \ldots, X_n$ is defined as follows.

$$\bigsqcup_{i \in [1,n]} X_i := X_1 \sqcup X_2 \sqcup \ldots \sqcup X_n := \{(i, x_i) : i \in [1, n], x_i \in X_i\}$$

Definition 1.14. Let $n \in \mathbb{Z}_{\geqslant 0}$ and let X_i and Y_i be sets for $i \in [1, n]$. Let $f_i : X_i \longrightarrow Y_i$ be a map for $i \in [1, n]$. The disjoint union of $f_1, \ldots f_n$ is defined as follows.

$$\begin{array}{rcl} \bigsqcup_{i \in [1,n]} f_i := f_1 \sqcup \ldots \sqcup f_n : \bigsqcup_{i \in [1,n]} X_i & \longrightarrow & \bigsqcup_{i \in [1,n]} Y_i \\ (i, x_i) & \longmapsto & (i, x_i f_i) \end{array}$$

Remark 1.15.

(1) Suppose given a set X and $n \in \mathbb{Z}_{\geqslant 0}$. Then we write $X^{\sqcup n} := \bigsqcup_{i \in [1,n]} X = X \sqcup \ldots \sqcup X$.

Note that in the case $n = 0$ this means that $X^{\sqcup 0} = \bigsqcup_{i \in [1,0]} X = \varnothing$. Furthermore, we identify $X^{\sqcup 1} = X$.

(2) Suppose given sets X and Y and a map $f : X \longrightarrow Y$ and $n \in \mathbb{Z}_{\geqslant 0}$. Then we write $f^{\sqcup n} := \bigsqcup_{i \in [1,n]} f = f \sqcup \ldots \sqcup f : X^{\sqcup n} \longrightarrow Y^{\sqcup n}$.

Note that in the case $n = 0$ this means that $f^{\sqcup 0} : X^{\sqcup 0} = \varnothing \longrightarrow Y^{\sqcup 0} = \varnothing$, so $f^{\sqcup 0} = \mathrm{id}_{\varnothing}$. Furthermore, we identify $f^{\sqcup 1} = f$.

Definition 1.16. Let X, X', X'' be sets. Define the maps $\gamma_{(X,X'),X''} : X \sqcup X' \sqcup X'' \longrightarrow (X \sqcup X') \sqcup X''$ and $\gamma_{X,(X',X'')} : X \sqcup X' \sqcup X'' \longrightarrow X \sqcup (X' \sqcup X'')$ as follows.

$$\begin{array}{rrll} \gamma_{(X,X'),X''} : & X \sqcup X' \sqcup X'' & \longrightarrow & (X \sqcup X') \sqcup X'' \\ & (1,x) & \longmapsto & (1,(1,x)) \\ & (2,x') & \longmapsto & (1,(2,x')) \\ & (3,x'') & \longmapsto & (2,x'') \end{array}$$

$$\begin{array}{rrll} \gamma_{X,(X',X'')} : & X \sqcup X' \sqcup X'' & \longrightarrow & X \sqcup (X' \sqcup X'') \\ & (1,x) & \longmapsto & (1,x) \\ & (2,x') & \longmapsto & (2,(1,x')) \\ & (3,x'') & \longmapsto & (2,(2,x'')) \end{array}$$

These are bijective maps.

Lemma 1.17. *Let X, X', X'', Y, Y', Y'' be sets and $f : X \longrightarrow Y$, $f' : X' \longrightarrow Y'$ and $f'' : X'' \longrightarrow Y''$ be maps. Then we have the following commutative diagrams* (i) *and* (ii)*.*

$$\text{(i)} \qquad \begin{CD} X \sqcup X' \sqcup X'' @>{f \sqcup f' \sqcup f''}>> Y \sqcup Y' \sqcup Y'' \\ @V{\gamma_{(X,X'),X''}}VV @VV{\gamma_{(Y,Y'),Y''}}V \\ (X \sqcup X') \sqcup X'' @>>{(f \sqcup f') \sqcup f''}> (Y \sqcup Y') \sqcup Y'' \end{CD}$$

$$\text{(ii)} \qquad \begin{CD} X \sqcup X' \sqcup X'' @>{f \sqcup f' \sqcup f''}>> Y \sqcup Y' \sqcup Y'' \\ @V{\gamma_{X,(X',X'')}}VV @VV{\gamma_{Y,(Y',Y'')}}V \\ X \sqcup (X' \sqcup X'') @>>{f \sqcup (f' \sqcup f'')}> Y \sqcup (Y' \sqcup Y'') \end{CD}$$

Proof. We will show that (i) is a commutative diagram. Suppose given $z \in X \sqcup X' \sqcup X''$. We have to show that

$$z(f \sqcup f' \sqcup f'')\gamma_{(Y,Y'),Y''} \stackrel{!}{=} z\gamma_{(X,X'),X''}((f \sqcup f') \sqcup f'').$$

Case 1: $z = (1, x)$ for some $x \in X$. Then we have

$$\begin{aligned} (1,x)(f \sqcup f' \sqcup f'')\gamma_{(Y,Y'),Y''} &= (1, xf)\gamma_{(Y,Y'),Y''} \\ &= (1,(1,xf)) \end{aligned}$$

and on the other hand we have

$$\begin{aligned}(1,x)\gamma_{(X,X'),X''}((f\sqcup f')\sqcup f'') &= (1,(1,x))((f\sqcup f')\sqcup f'')\\ &= (1,(1,x)(f\sqcup f'))\\ &= (1,(1,xf)).\end{aligned}$$

Case 2: $z=(2,x')$ for some $x'\in X'$. Then we have

$$\begin{aligned}(2,x')(f\sqcup f'\sqcup f'')\gamma_{(Y,Y'),Y''} &= (2,x'f')\gamma_{(Y,Y'),Y''}\\ &= (1,(2,x'f'))\end{aligned}$$

and on the other hand we have

$$\begin{aligned}(2,x')\gamma_{(X,X'),X''}((f\sqcup f')\sqcup f'') &= (1,(2,x'))((f\sqcup f')\sqcup f'')\\ &= (1,(2,x')(f\sqcup f'))\\ &= (1,(2,x'f')).\end{aligned}$$

Case 3: $z=(3,x'')$ for some $x''\in X''$. Then we have

$$\begin{aligned}(3,x'')(f\sqcup f'\sqcup f'')\gamma_{(Y,Y'),Y''} &= (3,x''f'')\gamma_{(Y,Y'),Y''}\\ &= (2,x''f'')\end{aligned}$$

and on the other hand we have

$$\begin{aligned}(3,x'')\gamma_{(X,X'),X''}((f\sqcup f')\sqcup f'') &= (2,x'')((f\sqcup f')\sqcup f'')\\ &= (2,x''f'').\end{aligned}$$

So in all three cases we have $z(f\sqcup f'\sqcup f'')\gamma_{(Y,Y'),Y''} = z\gamma_{(X,X'),X''}((f\sqcup f')\sqcup f'')$. □

Definition 1.18. Let $n\in\mathbb{Z}_{\geqslant 0}$ and $k=(k_i)_{i\in[1,n]}$, where $k_i\in\mathbb{Z}_{\geqslant 0}$ for $i\in[1,n]$. We have the bijective map

$$\begin{aligned}\varphi_k : \Big[1, \sum_{i\in[1,n]} k_i\Big] &\longrightarrow \bigsqcup_{i\in[1,n]} [1,k_i]\\ t &\longmapsto \Big(t\chi_k\,,\, t-\sum_{s\in[1,t\chi_k-1]} k_s\Big),\end{aligned}$$

where χ_k is the map

$$\begin{aligned}\chi_k : \Big[1, \sum_{i\in[1,n]} k_i\Big] &\longrightarrow [1,n]\\ t &\longmapsto \min\Big\{u\in[1,n] : \sum_{s\in[1,u]} k_s \geqslant t\Big\}.\end{aligned}$$

Its inverse map is

$$\begin{aligned}\varphi_k^{-1} : \bigsqcup_{i\in[1,n]} [1,k_i] &\longrightarrow \Big[1, \sum_{i\in[1,n]} k_i\Big]\\ (i,x) &\longmapsto \Big(\sum_{s\in[1,i-1]} k_s\Big) + x.\end{aligned}$$

Example 1.19. Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $k = (k_i)_{i \in [1,n]}$, where $k_i = m$ for $i \in [1,n]$. Then for $s \in [1,n]$ we have $\sum\limits_{i \in [1,s]} k_i = sm$.

For $t \in \big[1, \sum\limits_{i \in [1,n]} k_i\big] = [1, nm]$ we can write $t = \underline{t}m + \bar{t}$ in a unique way with $\underline{t} \in [0, n-1]$ and $\bar{t} \in [1, m]$. Then we have

$$\begin{aligned} t\chi_k &= \min\big\{u \in [1,n] : \sum_{s \in [1,u]} k_s \geqslant t\big\} \\ &= \min\big\{u \in [1,n] : um \geqslant \underline{t}m + \bar{t}\,\big\} \\ &= \underline{t} + 1. \end{aligned}$$

Hence we have

$$\begin{aligned} t\varphi_k &= \Big(t\chi_k\,,\, t - \sum_{s \in [1, t\chi_k - 1]} k_s\Big) \\ &= (\underline{t} + 1\,,\, t - \underline{t}m) \\ &= (\underline{t} + 1\,,\, \bar{t}\,) \end{aligned}$$

for $t \in [1, nm]$.

Moreover, for $(i,j) \in [1,m]^{\sqcup n}$ we have

$$(i,j)\varphi_k^{-1} = (i-1)m + j.$$

Lemma 1.20. *Let $m, m', m'' \in \mathbb{Z}_{\geqslant 0}$. We have the following commutative diagrams* (i) *and* (ii).

$$\text{(i)} \qquad \begin{array}{ccc} [1,m] \sqcup [1,m'] \sqcup [1,m''] & \xrightarrow{\;\varphi^{-1}_{(m,m',m'')}\;} & [1, m+m'+m''] \\ \Big\downarrow{\scriptstyle \gamma_{([1,m],[1,m']),[1,m'']}} & & \Big\uparrow{\scriptstyle \varphi^{-1}_{(m+m',m'')}} \\ ([1,m] \sqcup [1,m']) \sqcup [1,m''] & \xrightarrow[\;\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']}\;]{} & [1, m+m'] \sqcup [1,m''] \end{array}$$

$$\text{(ii)} \qquad \begin{array}{ccc} [1,m] \sqcup [1,m'] \sqcup [1,m''] & \xrightarrow{\;\varphi^{-1}_{(m,m',m'')}\;} & [1, m+m'+m''] \\ \Big\downarrow{\scriptstyle \gamma_{[1,m],([1,m'],[1,m''])}} & & \Big\uparrow{\scriptstyle \varphi^{-1}_{(m,m'+m'')}} \\ [1,m] \sqcup ([1,m'] \sqcup [1,m'']) & \xrightarrow[\;\mathrm{id}_{[1,m]} \sqcup \varphi^{-1}_{(m',m'')}\;]{} & [1,m] \sqcup [1, m'+m''] \end{array}$$

Proof. We will show that (i) is a commutative diagram. So let $z \in [1,m] \sqcup [1,m'] \sqcup [1,m'']$. We have to show that

$$z\varphi^{-1}_{(m,m',m'')} \overset{!}{=} z\gamma_{([1,m],[1,m']),[1,m'']}(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')}\,.$$

Case 1: $z = (1,i)$ for some $i \in [1,m]$. Then we have

$$(1,i)\varphi^{-1}_{(m,m',m'')} = i$$

and on the other hand we have

$$\begin{aligned}
(1,i)\gamma_{([1,m],[1,m']),[1,m'']}(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} &= (1,(1,i))(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} \\
&= (1,(1,i)\varphi^{-1}_{(m,m')})\varphi^{-1}_{(m+m',m'')} \\
&= (1,i)\varphi^{-1}_{(m+m',m'')} \\
&= i.
\end{aligned}$$

Case 2: $z = (2, i')$ for some $i' \in [1, m']$. Then we have

$$(2,i')\varphi^{-1}_{(m,m',m'')} = m + i'$$

and on the other hand we have

$$\begin{aligned}
(2,i')\gamma_{([1,m],[1,m']),[1,m'']}(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} &= (1,(2,i'))(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} \\
&= (1,(2,i')\varphi^{-1}_{(m,m')})\varphi^{-1}_{(m+m',m'')} \\
&= (1,m+i')\varphi^{-1}_{(m+m',m'')} \\
&= m+i'.
\end{aligned}$$

Case 3: $z = (3, i'')$ for some $i'' \in [1, m'']$. Then we have

$$(3,i'')\varphi^{-1}_{(m,m',m'')} = m + m' + i''$$

and on the other hand we have

$$\begin{aligned}
(3,i'')\gamma_{([1,m],[1,m']),[1,m'']}(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} &= (2,i'')(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} \\
&= (2,i''\,\mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} \\
&= (2,i'')\varphi^{-1}_{(m+m',m'')} \\
&= (m+m')+i''.
\end{aligned}$$

□

Definition 1.21. Let $k \in \mathbb{Z}_{\geqslant 1}$. Let X be a set. Define the bijective map

$$\begin{aligned}
\gamma_{k,X} : X^{\sqcup k} &\longrightarrow X^{\sqcup (k-1)} \sqcup X \\
(i,x) &\longmapsto \begin{cases} (1,(i,x)) & \text{if } i \in [1,k-1] \\ (2,x) & \text{if } i = k. \end{cases}
\end{aligned}$$

Lemma 1.22. *Let $k \in \mathbb{Z}_{\geqslant 1}$. Let X, Y be sets and let $f : X \longrightarrow Y$ be a map. Then we have*

$$\gamma_{k,X}(f^{\sqcup(k-1)} \sqcup f) = f^{\sqcup k}\gamma_{k,Y}\,.$$

So we have the following commutative diagram.

$$\begin{array}{ccc}
X^{\sqcup k} & \xrightarrow{\quad f^{\sqcup k} \quad} & Y^{\sqcup k} \\
\Big\downarrow{\scriptstyle \gamma_{k,X}} & & \Big\downarrow{\scriptstyle \gamma_{k,Y}} \\
X^{\sqcup(k-1)} \sqcup X & \xrightarrow[f^{\sqcup(k-1)} \sqcup f]{} & Y^{\sqcup(k-1)} \sqcup Y
\end{array}$$

Proof. Suppose given $(i, x) \in X^{\sqcup k}$, that is, $i \in [1, k]$ and $x \in X$. Since $\gamma_{k,Y}$ is bijective, we have to show that

$$(i, x)\gamma_{k,X}(f^{\sqcup(k-1)} \sqcup f)\gamma_{k,Y}^{-1} \overset{!}{=} (i, x)f^{\sqcup k}\,.$$

We have

$$\begin{aligned}
(i, x)\gamma_{k,X}(f^{\sqcup(k-1)} \sqcup f)\gamma_{k,Y}^{-1} &= \begin{cases} (1, (i, x))(f^{\sqcup(k-1)} \sqcup f)\gamma_{k,Y}^{-1} & \text{if } i \in [1, k-1] \\ (2, x)(f^{\sqcup(k-1)} \sqcup f)\gamma_{k,Y}^{-1} & \text{if } i = k \end{cases} \\
&= \begin{cases} (1, (i, x)f^{\sqcup(k-1)})\gamma_{k,Y}^{-1} & \text{if } i \in [1, k-1] \\ (2, xf)\gamma_{k,Y}^{-1} & \text{if } i = k \end{cases} \\
&= \begin{cases} (1, (i, xf))\gamma_{k,Y}^{-1} & \text{if } i \in [1, k-1] \\ (2, xf)\gamma_{k,Y}^{-1} & \text{if } i = k \end{cases} \\
&= \begin{cases} (i, xf) & \text{if } i \in [1, k-1] \\ (k, xf) & \text{if } i = k \end{cases} \\
&= (i, xf) \\
&= (i, x)f^{\sqcup k}\,.
\end{aligned}$$

□

Lemma 1.23. *Let $m \in \mathbb{Z}_{\geqslant 0}$, $k \in \mathbb{Z}_{\geqslant 1}$. Let $l_i := m$ for $i \in [1, k]$ and let $l := (l_i)_{i\in[1,k]} \in (\mathbb{Z}_{\geqslant 0})^{\times k}$ and $\hat{l} := (l_i)_{i\in[1,k-1]} \in (\mathbb{Z}_{\geqslant 0})^{\times(k-1)}$. Then we have*

$$\varphi_l^{-1} = \gamma_{k,[1,m]}(\varphi_{\hat{l}}^{-1} \sqcup \mathrm{id}_{[1,m]})\varphi_{((k-1)m,m)}^{-1}\,.$$

So we have the following commutative diagram.

$$\begin{array}{ccc}
[1, m]^{\sqcup k} & \xrightarrow{\quad\varphi_l^{-1}\quad} & [1, km] \\
\Big\downarrow{\scriptstyle \gamma_{k,[1,m]}} & & \Big\uparrow{\scriptstyle \varphi_{((k-1)m,m)}^{-1}} \\
[1, m]^{\sqcup(k-1)} \sqcup [1, m] & \xrightarrow[\varphi_{\hat{l}}^{-1} \sqcup \mathrm{id}_{[1,m]}]{} & [1, (k-1)m] \sqcup [1, m]
\end{array}$$

Proof. Suppose given $(i, j) \in [1, m]^{\sqcup k}$. We have

$$\begin{aligned}
(i, j)\gamma_{k,[1,m]}(\varphi_{\hat{l}}^{-1} \sqcup \mathrm{id}_{[1,m]})\varphi_{((k-1)m,m)}^{-1} &= \begin{cases} (1, (i, j))(\varphi_{\hat{l}}^{-1} \sqcup \mathrm{id}_{[1,m]})\varphi_{((k-1)m,m)}^{-1} & \text{if } i \in [1, k-1] \\ (2, j)(\varphi_{\hat{l}}^{-1} \sqcup \mathrm{id}_{[1,m]})\varphi_{((k-1)m,m)}^{-1} & \text{if } i = k \end{cases} \\
&= \begin{cases} (1, (i, j)\varphi_{\hat{l}}^{-1})\varphi_{((k-1)m,m)}^{-1} & \text{if } i \in [1, k-1] \\ (2, j\,\mathrm{id}_{[1,m]})\varphi_{((k-1)m,m)}^{-1} & \text{if } i = k \end{cases} \\
&= \begin{cases} (1, (i-1)m + j)\varphi_{((k-1)m,m)}^{-1} & \text{if } i \in [1, k-1] \\ (2, j)\varphi_{((k-1)m,m)}^{-1} & \text{if } i = k \end{cases} \\
&= \begin{cases} (i-1)m + j & \text{if } i \in [1, k-1] \\ (k-1)m + j & \text{if } i = k \end{cases} \\
&= (i-1)m + j \\
&= (i, j)\varphi_l^{-1}\,.
\end{aligned}$$

□

Lemma 1.24. *Let X, Y, Z, X', Y', Z' be sets and let $f : X \longrightarrow Y$, $f' : X' \longrightarrow Y'$, $g : Y \longrightarrow Z$ and $g' : Y' \longrightarrow Z'$ be maps. Furthermore, let $\mathrm{id}_{[1,0]} : [1,0] \longrightarrow [1,0]$ be the unique map from the empty set to itself. For some set T define the following bijective maps.*

$$\begin{array}{rrl} u_T : & T \sqcup [1,0] & \longrightarrow T \\ & (1,t) & \longmapsto t \end{array} \qquad\qquad \begin{array}{rrl} v_T : & [1,0] \sqcup T & \longrightarrow T \\ & (2,t) & \longmapsto t \end{array}$$

Then (i), (ii) *and* (iii) *hold.*

(i) *We have* $(f \sqcup f')(g \sqcup g') = (fg) \sqcup (f'g')$.

(ii) *We have* $(f \sqcup \mathrm{id}_{[1,0]})u_Y = u_X f$.

(iii) *We have* $(\mathrm{id}_{[1,0]} \sqcup f)v_Y = v_X f$.

Proof. Ad (i). Note that for $x \in X$ we have

$$(1,x)(f \sqcup f')(g \sqcup g') = (1, xf)(g \sqcup g') = (1, xfg) = (1,x)((fg) \sqcup (f'g'))$$

and for $x' \in X'$ we have

$$(2,x')(f \sqcup f')(g \sqcup g') = (2, x'f')(g \sqcup g') = (2, x'f'g') = (2,x')((fg) \sqcup (f'g')).$$

So we have $(f \sqcup f')(g \sqcup g') = (fg) \sqcup (f'g')$.

Ad (ii). Let $\xi \in X \sqcup [1,0]$. Then $\xi = (1,x)$ for some $x \in X$. Then we have

$$(1,x)(f \sqcup \mathrm{id}_{[1,0]})u_Y = (1, xf)u_Y = xf = (1,x)u_X f.$$

Ad (iii). Let $\xi \in [1,0] \sqcup X$. Then $\xi = (2,x)$ for some $x \in X$. Then we have

$$(2,x)(\mathrm{id}_{[1,0]} \sqcup f)v_Y = (2, xf)v_Y = xf = (2,x)v_X f.$$

□

Remark 1.25. Let $n \in \mathbb{Z}_{\geqslant 0}$. Note that

$$\begin{array}{rrl} u_{[1,n]} : & [1,n] \sqcup [1,0] & \longrightarrow [1,n] = [1, n+0] \\ & (1,i) & \longmapsto i \end{array} \qquad \begin{array}{rrl} v_{[1,n]} : & [1,0] \sqcup [1,n] & \longrightarrow [1,n] = [1, 0+n] \\ & (2,i) & \longmapsto i = 0 + i. \end{array}$$

So $u_{[1,n]} = \varphi_{(n,0)}^{-1}$ and $v_{[1,n]} = \varphi_{(0,n)}^{-1}$.

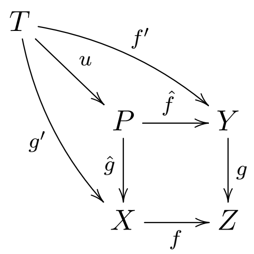
1.5 Pullbacks

Definition 1.26. Let X, Y, Z be sets and $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ be maps. Furthermore, let P be a set and $\hat{f} : P \longrightarrow Y$ and $\hat{g} : P \longrightarrow X$ be maps. We say that the tuple $(P, \hat{g}, \hat{f})$ is a *pullback* of f and g if (P1) and (P2) hold.

(P1) We have $\hat{f}g = \hat{g}f$, that is, the following diagram commutes.

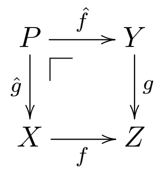
$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & Y \\ {\scriptstyle \hat{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

(P2) Given a set T and maps $f' : T \longrightarrow X$ and $g' : T \longrightarrow Y$ such that $f'g = g'f$, then there exists a unique map $u : T \longrightarrow P$ such that $u\hat{f} = f'$ and $u\hat{g} = g'$.



In the above situation we will also often say that the quadrangle (P, X, Y, Z) is a pullback, drawing the attention closer to the involved sets than the involved maps, which then need to be known from context.

A quadrangle being a pullback is often expressed graphically as follows.



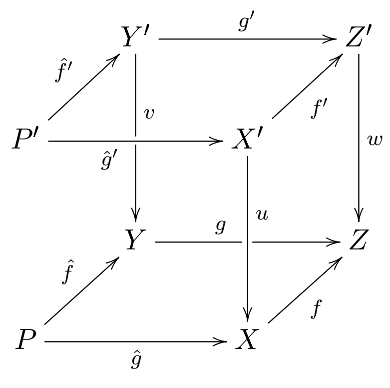
Remark 1.27. *Let X, Y, Z be sets and $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be maps. Suppose given two pullbacks $(P, \hat{g}, \hat{f})$ and $(P', \hat{g}', \hat{f}')$ of f and g. Then there exists a uniquely determined bijective map $u : P' \longrightarrow P$ such that $u\hat{g} = \hat{g}'$ and $u\hat{f} = \hat{f}'$.*

Proof. By (P2) there exist uniquely determined maps $u : P' \longrightarrow P$ and $v : P \longrightarrow P'$ satisfying $u\hat{g} = \hat{g}'$, $u\hat{f} = \hat{f}'$, $v\hat{g}' = \hat{g}$ and $v\hat{f}' = \hat{f}$.

So we get a map $vu : P \longrightarrow P$ satisfying $(vu)\hat{g} = v\hat{g}' = \hat{g}$ and $(vu)\hat{f} = v\hat{f}' = \hat{f}$. But the identity map $\mathrm{id}_P : P \longrightarrow P$ also satisfies $\mathrm{id}_P\, \hat{g} = \hat{g}$ and $\mathrm{id}_P\, \hat{f} = \hat{f}$. Since $(P, \hat{g}, \hat{f})$ is a pullback of f and g we have $vu = \mathrm{id}_P$. Similarly, we get $uv = \mathrm{id}_{P'}$. Thus u is bijective. □

We generalize Remark 1.27 somewhat.

Lemma 1.28. *Let X, Y, Z, X', Y', Z' be sets and let $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$, $f' : X' \longrightarrow Z'$ and $g' : Y' \longrightarrow Z'$ be maps. Let $(P, \hat{g}, \hat{f})$ be a pullback of f and g and let $(P', \hat{g}', \hat{f}')$ be a pullback of f' and g'. Let $u : X' \longrightarrow X$, $v : Y' \longrightarrow Y$ and $w : Z' \longrightarrow Z$ be maps such that $uf = f'w$ and $vg = g'w$. That is, we have the following commutative diagram.*



Then (1) *and* (2) *hold.*

(1) *There exists a unique map* $s : P' \longrightarrow P$ *such that* $s\hat{f} = \hat{f}'v$ *and* $s\hat{g} = \hat{g}'u$.

(2) *If* u, v, w *are bijective, then so is* s.

Proof. Ad (1). Since $(P', \hat{g}', \hat{f}')$ is a pullback of f' and g' we have $\hat{f}'g' = \hat{g}'f'$. Moreover, since $uf = f'w$ and $vg = g'w$ we have

$$\hat{f}'vg = \hat{f}'g'w = \hat{g}'f'w = \hat{g}'uf.$$

So we have the following commutative diagram.

$$\begin{array}{ccc} P' & \xrightarrow{\hat{f}'v} & Y \\ {\scriptstyle \hat{g}'u}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z, \end{array}$$

Since $(P, \hat{g}, \hat{f})$ is a pullback of f and g, by (P2) there exists a unique map $s : P' \longrightarrow P$ such that $s\hat{f} = \hat{f}'v$ and $s\hat{g} = \hat{g}'u$.

Ad (2). If u, v, w are bijective, we can apply (1) to (u^{-1}, v^{-1}, w^{-1}) and obtain a map $\tilde{s} : P \longrightarrow P'$ satisfying $\tilde{s}\hat{f}' = \hat{f}v^{-1}$ and $\tilde{s}\hat{g}' = \hat{g}u^{-1}$. This implies

$$\begin{aligned} \mathrm{id}_P\, \hat{f} &= \hat{f} = \hat{f}v^{-1}v = \tilde{s}\hat{f}'v = \tilde{s}s\hat{f} \\ \mathrm{id}_P\, \hat{g} &= \hat{g} = \hat{g}u^{-1}u = \tilde{s}\hat{g}'u = \tilde{s}s\hat{g}. \end{aligned}$$

By (P2) for $(P, \hat{g}, \hat{f})$, we have $\mathrm{id}_P = \tilde{s}s$.

In the same way we obtain $s\tilde{s} = \mathrm{id}_{P'}$. Hence s is bijective. □

Lemma 1.29. *Let* X, Y, Z *be sets and* $f : X \longrightarrow Z$ *and* $f : Y \longrightarrow Z$ *be maps. Define*

$$S := \{(x, y) \in X \times Y : xf = yg\}$$

and the maps

$$\begin{array}{rrcl} \check{g} : & S & \longrightarrow & X \\ & (x, y) & \longmapsto & x \end{array} \qquad \begin{array}{rrcl} \check{f} : & S & \longrightarrow & Y \\ & (x, y) & \longmapsto & y. \end{array}$$

Then $(S, \check{g}, \check{f})$ *is a pullback of* f *and* g*. We often refer to it as the* standard pullback *of* f *and* g.

Proof. Ad (P1). Suppose given $(x, y) \in S$, that is, $xf = yg$. Then we have

$$(x, y)\check{f}g = yg = xf = (x, y)\check{g}f.$$

Hence $\check{f}g = \check{g}f$.

Ad (P2). Suppose given a set T and maps $g' : T \longrightarrow X$ and $f' : T \longrightarrow Y$ such that $f'g = g'f$. We have to show that there exists a uniquely determined map $u : T \longrightarrow S$ such that $u\check{f} = f'$ and $u\check{g} = g'$.

Existence. Define u by $tu := (tg', tf')$ for $t \in T$. We have $(tg')f = t(g'f) = t(f'g) = (tf')g$, hence $tu \in S$ for $t \in T$. Moreover, we have

$$\begin{aligned} t(u\check{f}) &= (tg', tf')\check{f} = tf' \\ t(u\check{g}) &= (tg', tf')\check{g} = tg' \end{aligned}$$

for $t \in T$. Hence we have $u\check{f} = f'$ and $u\check{g} = g'$.

Uniqueness. Let $v : T \longrightarrow S$ be a map satisfying $v\check{f} = f'$ and $v\check{g} = g'$. Then define $(x_t, y_t) := tv$ for $t \in T$ with $x_t \in X$ and $y_t \in Y$ such that $x_t f = y_t g$. Then we have

$$\begin{aligned} tf' &= t(v\check{f}) = (x_t, y_t)\check{f} = y_t \\ tg' &= t(v\check{g}) = (x_t, y_t)\check{g} = x_t\,, \end{aligned}$$

so $tu = (tf', tg') = (x_t, y_t) = tv$ for $t \in T$. Hence we have $u = v$. □

Remark 1.30. Let X, Y, Z be sets and let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be maps. Let $(P, \hat{g}, \hat{f})$ be a pullback of f and g. Then $(P, \hat{f}, \hat{g})$ is a pullback of g and f, as we take from (P1) and (P2).

Remark 1.31. Consider the following commutative diagram.

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

Then for $y \in Y$ and $\tilde{p} \in \tilde{f}^{-1}(y)$ we have $\tilde{p}\tilde{f} = y$ and

$$\tilde{p}(\tilde{g}f) = \tilde{p}(\tilde{f}g) = yg,$$

hence $\tilde{p}\tilde{g} \in f^{-1}(yg)$. So we have the restricted map $\tilde{g}|_{\tilde{f}^{-1}(y)}^{f^{-1}(yg)}$ for $y \in Y$.

The next property is a criterion to decide whether a tuple $(P, \hat{g}, \hat{f})$ is a pullback.

Lemma 1.32. *Let X, Y, Z be sets and $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be maps. Let P be a set and $\hat{g} : P \longrightarrow X$ and $\hat{f} : P \longrightarrow Y$ be maps.*

Then $(P, \hat{g}, \hat{f})$ is a pullback of f and g if and only if (P1) *holds and the following condition* (P) *is satisfied.*

(P) *The map $\hat{g}|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)}$ is bijective for $y \in Y$.*

Proof. First assume that $(P, \hat{g}, \hat{f})$ satisfies the conditions (P1) and (P). We have to show that it also satisfies condition (P2).

Let T be a set and $g' : T \longrightarrow X$ and $f' : T \longrightarrow Y$ be maps such that $f'g = g'f$. We have to show that there exists a uniquely determined map $u : T \longrightarrow P$ such that $u\hat{f} = f'$ and $u\hat{g} = g'$.

Existence. Let $t \in T$. Then $tf' \in Y$ and $tg' \in f^{-1}(tf'g)$ since $(tg')f = tf'g$. Since $\hat{g}|_{\hat{f}^{-1}(tf')}^{f^{-1}(tf'g)}$ is bijective we can define $u : T \longrightarrow P$ as follows. Let

$$tu := tg' \left(\hat{g}|_{\hat{f}^{-1}(tf')}^{f^{-1}(tf'g)}\right)^{-1} \in \hat{f}^{-1}(tf') \subseteq P$$

for $t \in T$.

Then u satisfies $tu\hat{f} = tf'$ and

$$tu\hat{g} = tu \left(\hat{g}|_{\hat{f}^{-1}(tf')}^{f^{-1}(tf'g)}\right) = tg' \left(\hat{g}|_{\hat{f}^{-1}(tf')}^{f^{-1}(tf'g)}\right)^{-1} \left(\hat{g}|_{\hat{f}^{-1}(tf')}^{f^{-1}(tf'g)}\right) = tg'$$

for $t \in T$. Hence $u\hat{f} = f'$ and $u\hat{g} = g'$.

Uniqueness. Let $v : T \longrightarrow P$ be a map satisfying $v\hat{g} = g'$ and $v\hat{f} = f'$. Then for $t \in T$ we have $tv \in \hat{f}^{-1}(tf')$. We have

$$tv \left(\hat{g}\big|_{\hat{f}^{-1}(tf')}^{f^{-1}(tf'g)} \right) = tv\hat{g} = tg',$$

hence

$$tv = tg' \left(\hat{g}\big|_{\hat{f}^{-1}(tf')}^{f^{-1}(tf'g)} \right)^{-1} = tu$$

for $t \in T$. So $v = u$.

Now let $(P, \hat{g}, \hat{f})$ be a pullback of f and g. We have to show that $(P, \hat{g}, \hat{f})$ satisfies condition (P), that is, that $\hat{g}\big|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)}$ is bijective for $y \in Y$. We will first show that this is true for the standard pullback $(S, \check{g}, \check{f})$; cf. Lemma 1.29.

So suppose given $y \in Y$. Note that we have $\check{f}^{-1}(y) = \{(x, y_0) \in S : y = y_0\}$.

Injectivity. Suppose given $(x, y), (x', y) \in \check{f}^{-1}(y)$ with $(x, y)\check{g}\big|_{\check{f}^{-1}(y)}^{f^{-1}(yg)} = (x', y)\check{g}\big|_{\check{f}^{-1}(y)}^{f^{-1}(yg)}$. Then we have

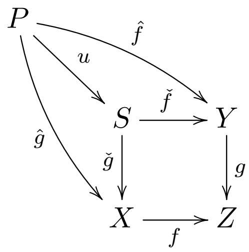
$$x = (x, y)\check{g} = (x, y)\check{g}\big|_{\check{f}^{-1}(y)}^{f^{-1}(yg)} = (x', y)\check{g}\big|_{\check{f}^{-1}(y)}^{f^{-1}(yg)} = (x', y)\check{g} = x'.$$

Surjectivity. Suppose given $x \in f^{-1}(yg)$. This means that $xf = yg$, hence $(x, y) \in S$. Moreover, we have $(x, y) \in \check{f}^{-1}(y)$. Hence we have

$$(x, y)\check{g}\big|_{\check{f}^{-1}(y)}^{f^{-1}(yg)} = (x, y)\check{g} = x.$$

This shows that the standard pullback $(S, \check{g}, \check{f})$ satisfies (P).

Now let $(P, \hat{g}, \hat{f})$ be a pullback of f and g. Then there exists a unique bijective map $u : P \longrightarrow S$ such that $u\check{f} = \hat{f}$ and $u\check{g} = \hat{g}$; cf. Remark 1.27.



Suppose given $y \in Y$. We already know that $\check{g}\big|_{\check{f}^{-1}(y)}^{f^{-1}(yg)}$ is bijective.
Note that for $t \in \hat{f}^{-1}(y)$ we have $tu\check{f} = t\hat{f} = y$, hence $tu \in \check{f}^{-1}(y)$. So the restriction $u\big|_{\hat{f}^{-1}(y)}^{\check{f}^{-1}(y)}$ is defined and injective.

Conversely, given $r \in \check{f}^{-1}(y)$, we have $ru^{-1}\hat{f} = ru^{-1}u\check{f} = r\check{f} = y$, hence $ru^{-1} \in \hat{f}^{-1}(y)$. Moreover, $(ru^{-1})u\big|_{\hat{f}^{-1}(y)}^{\check{f}^{-1}(y)} = ru^{-1}u = r$. Hence $u\big|_{\hat{f}^{-1}(y)}^{\check{f}^{-1}(y)}$ is bijective. We have

$$\hat{g}\big|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)} = (u\check{g})\big|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)} = \left(u\big|_{\hat{f}^{-1}(y)}^{\check{f}^{-1}(y)} \right) \left(\check{g}\big|_{\check{f}^{-1}(y)}^{f^{-1}(yg)} \right),$$

so $\hat{g}\big|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)}$ is bijective as the composite of bijective maps. □

Note that with Remark 1.30 we can also interchange the roles of $\hat{g}$ and $\hat{f}$ in Lemma 1.32 and ask for $\hat{f}\big|_{\hat{g}^{-1}(x)}^{g^{-1}(xf)}$ to be bijective for $x \in X$.

Lemma 1.33. *Let X, Y, Z, T be sets and let $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$, $f' : T \longrightarrow Y$ and $g' : T \longrightarrow X$ be maps such that $f'g = g'f$ and such that g and g' are bijective maps. Then (T, g', f') is a pullback of f and g.*

$$\begin{array}{ccc} T & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow\wr & \ulcorner & \wr\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

Proof. By Lemma 1.32 and Remark 1.30, we have to show that $f'|_{g'^{-1}(x)}^{g^{-1}(xf)}$ is bijective for $x \in X$.

But since g and g' are bijective we have $|g'^{-1}(x)| = |g^{-1}(xf)| = 1$ for $x \in X$. So $f'|_{g'^{-1}(x)}^{g^{-1}(xf)}$ is bijective for $x \in X$. □

Corollary 1.34. *Let X be a set, $n \in \mathbb{Z}_{\geqslant 0}$ and let $f : X \longrightarrow [1, n]$ be a map. Then (X, id_X, f) is a pullback of f and $\mathrm{id}_{[1,n]}$.*

$$\begin{array}{ccc} X & \xrightarrow{f} & [1,n] \\ {\scriptstyle \mathrm{id}_X}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{id}_{[1,n]}} \\ X & \xrightarrow[f]{} & [1,n] \end{array}$$

Lemma 1.35. *Consider the following pulback.*

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & Y \\ {\scriptstyle \hat{g}}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

(i) *If f is injective, then so is $\hat{f}$.*

(ii) *If f is surjective, then so is $\hat{f}$.*

(iii) *If f is bijective, then so is $\hat{f}$.*

Proof. Ad (i). Suppose given $p, \tilde{p} \in P$ with $p\hat{f} = \tilde{p}\hat{f}$.

We have $p\hat{g}f = p\hat{f}g = \tilde{p}\hat{f}g = \tilde{p}\hat{g}f$. Since f is injective, this implies that $p\hat{g} = \tilde{p}\hat{g}$.

Now since $p\hat{f} = \tilde{p}\hat{f}$, we have $p, \tilde{p} \in \hat{f}^{-1}(p\hat{f})$. By Lemma 1.32 we know that $\hat{g}|_{\hat{f}^{-1}(p\hat{f})}^{f^{-1}(p\hat{f}g)}$ is injective, so

$$p\hat{g}|_{\hat{f}^{-1}(p\hat{f})}^{f^{-1}(p\hat{f}g)} = p\hat{g} = \tilde{p}\hat{g} = \tilde{p}\hat{g}|_{\hat{f}^{-1}(p\hat{f})}^{f^{-1}(p\hat{f}g)}$$

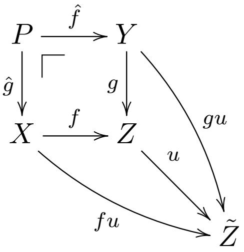
implies $p = \tilde{p}$.

Ad (ii). Suppose given $y \in Y$. Since f is surjective, there exists $x \in X$ such that $xf = yg$, so $y \in g^{-1}(xf)$. Since by Lemma 1.32 and Remark 1.30 the map $\hat{f}|_{\hat{g}^{-1}(x)}^{g^{-1}(xf)}$ is surjective, there exists $p \in \hat{g}^{-1}(x)$ such that

$$p\hat{f} = p\hat{f}|_{\hat{g}^{-1}(x)}^{g^{-1}(xf)} = y.$$

Ad (iii). Since f is bijective, the map $\hat{f}$ is injective by (i) and surjective by (ii). □

Lemma 1.36. *Let $X, Y, Z, \tilde{Z}$ be sets and let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be maps. Let $(P, \hat{g}, \hat{f})$ be a pullback of f and g and let $u : Z \longrightarrow \tilde{Z}$ be an injective map. Then $(P, \hat{g}, \hat{f})$ is also a pullback of fu and gu.*



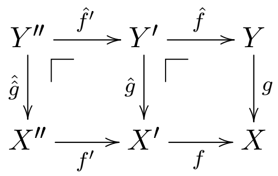
Proof. First note that we have $\hat{f}gu = \hat{g}fu$.

Suppose given $y \in Y$. We have to show that $\hat{g}|_{\hat{f}^{-1}(y)}^{(fu)^{-1}(ygu)}$ is bijective.

Suppose given $x \in X$. Since u is injective, we have $x \in (fu)^{-1}(ygu)$ if and only if $xfu = ygu$, i.e. $xf = yg$, i.e. $x \in f^{-1}(yg)$. This implies that $(fu)^{-1}(ygu) = f^{-1}(yg)$.

Now since $(P, \hat{g}, \hat{f})$ is a pullback of f and g, we know that $\hat{g}|_{\hat{f}^{-1}(y)}^{(fu)^{-1}(ygu)} = \hat{g}|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)}$ is bijective; cf. Lemma 1.32. □

Lemma 1.37. *Let X, Y, X', Y', X'', Y'' be sets and let $f : X' \longrightarrow X$, $g : Y \longrightarrow X$, $f' : X'' \longrightarrow X'$, $\hat{g} : Y' \longrightarrow X'$, $\hat{f} : Y' \longrightarrow Y$, $\hat{\hat{g}} : Y'' \longrightarrow X''$ and $\hat{f}' : Y'' \longrightarrow Y'$ be maps such that $(Y', \hat{g}, \hat{f})$ is a pullback of f and g and $(Y'', \hat{\hat{g}}, \hat{f}')$ is a pullback of f' and $\hat{g}$.*



Then $(Y'', \hat{\hat{g}}, (\hat{f}'\hat{f}))$ is a pullback of $(f'f)$ and g.

Proof. Since $\hat{\hat{g}}(f'f) = (\hat{\hat{g}}f')f = (\hat{f}'\hat{g})f = \hat{f}'(\hat{g}f) = \hat{f}'(\hat{f}g) = (\hat{f}'\hat{f})g$, by Lemma 1.32 and Remark 1.30 it suffices to show that $(\hat{f}'\hat{f})|_{\hat{\hat{g}}^{-1}(x'')}^{g^{-1}(x''f'f)}$ is bijective for $x'' \in X''$.

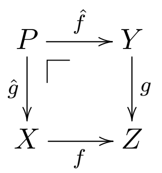
So suppose given $x'' \in X''$. We have

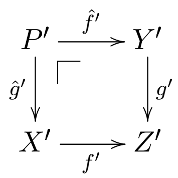
$$(\hat{f}'\hat{f})|_{\hat{\hat{g}}^{-1}(x'')}^{g^{-1}(x''f'f)} = \left(\hat{f}'|_{\hat{\hat{g}}^{-1}(x'')}^{\hat{g}^{-1}(x''f')}\right)\left(\hat{f}|_{\hat{g}^{-1}(x''f')}^{g^{-1}(x''f'f)}\right).$$

Since $(Y', \hat{g}, \hat{f})$ is a pullback of f and g, we know that $\hat{f}|_{\hat{g}^{-1}(x''f')}^{g^{-1}(x''f'f)}$ is bijective. Moreover, since $(Y'', \hat{\hat{g}}, \hat{f}')$ is a pullback of f' and $\hat{g}$, we know that $\hat{f}'|_{\hat{\hat{g}}^{-1}(x'')}^{\hat{g}^{-1}(x''f')}$ is bijective. So $(\hat{f}'\hat{f})|_{\hat{\hat{g}}^{-1}(x'')}^{g^{-1}(x''f'f)}$ is bijective as the composite of bijective maps. □

Finally, we are going to need compatibility of disjoint unions and pullbacks.

Lemma 1.38. *Let X, Y, Z, X', Y', Z' be sets and let $f : X \longrightarrow Z$, $g : Y :\longrightarrow Z$, $f' : X' \longrightarrow Z'$ and $g' : Y' \longrightarrow Z'$ be sets. Let $(P, \hat{g}, \hat{f})$ be a pullback of f and g and $(P', \hat{g}', \hat{f}')$ be a pullback of f' and g'.*





Then $(P \sqcup P', \hat{g} \sqcup \hat{g}', \hat{f} \sqcup \hat{f}')$ *is a pullback of* $f \sqcup f'$ *and* $g \sqcup g'$*; cf. Definition* 1.13 *and Definition* 1.14*.*

$$\begin{array}{ccc} P \sqcup P' & \xrightarrow{\hat{f} \sqcup \hat{f}'} & Y \sqcup Y' \\ {\scriptstyle \hat{g} \sqcup \hat{g}'} \downarrow & & \downarrow {\scriptstyle g \sqcup g'} \\ X \sqcup X' & \xrightarrow[f \sqcup f']{} & Z \sqcup Z' \end{array}$$

Proof. Ad (P1). By Lemma 1.24 (i) and since $(P, \hat{g}, \hat{f})$ is a pullback of f and g and $(P', \hat{g}', \hat{f}')$ is a pullback of f' and g' we have

$$(\hat{f} \sqcup \hat{f}')(g \sqcup g') = \hat{f}g \sqcup \hat{f}'g' = \hat{g}f \sqcup \hat{g}'f' = (\hat{g} \sqcup \hat{g}')(f \sqcup f').$$

Ad (P). Suppose given $\xi \in Y \sqcup Y'$. We have to show that $(\hat{g} \sqcup \hat{g}')\big|_{(\hat{f} \sqcup \hat{f}')^{-1}(\xi)}^{(f \sqcup f')^{-1}(\xi(g \sqcup g'))}$ is bijective.

Case 1: $\xi = (1, y)$ for some $y \in Y$. Then $\xi(g \sqcup g') = (1, y)(g \sqcup g') = (1, yg) \in Z \sqcup Z'$. We have $(\hat{f} \sqcup \hat{f}')^{-1}(1, y) = \{(1, p) : p \in \hat{f}^{-1}(y)\}$ and $(f \sqcup f')^{-1}(1, yg) = \{(1, x) : x \in f^{-1}(yg)\}$.

Injectivity. Suppose given $(1, p), (1, \tilde{p}) \in (\hat{f} \sqcup \hat{f}')^{-1}(1, y)$, that is, $p, \tilde{p} \in \hat{f}^{-1}(y)$, and suppose that

$$(1, p)(\hat{g} \sqcup \hat{g}') = (1, p)(\hat{g} \sqcup \hat{g}')\big|_{(\hat{f} \sqcup \hat{f}')^{-1}(1,y)}^{(f \sqcup f')^{-1}(1,yg)} = (1, \tilde{p})(\hat{g} \sqcup \hat{g}')\big|_{(\hat{f} \sqcup \hat{f}')^{-1}(1,y)}^{(f \sqcup f')^{-1}(1,yg)} = (1, \tilde{p})(\hat{g} \sqcup \hat{g}').$$

But this means that

$$(1, p\hat{g}) = (1, p)(\hat{g} \sqcup \hat{g}') = (1, \tilde{p})(\hat{g} \sqcup \hat{g}') = (1, \tilde{p}\hat{g}),$$

hence $p\hat{g} = \tilde{p}\hat{g}$.

Since $p, \tilde{p} \in \hat{f}^{-1}(y)$ and since $\hat{g}\big|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)}$ is injective, this implies that $p = \tilde{p}$.

Surjectivity. Suppose given $(1, x) \in (f \sqcup f')^{-1}(1, yg)$, that is, $x \in f^{-1}(yg)$. Since $\hat{g}\big|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)}$ is surjective there exists $p \in \hat{f}^{-1}(y)$ such that $p\hat{g} = p\hat{g}\big|_{\hat{f}^{-1}(y)}^{f^{-1}(yg)} = x$. So we have $(1, p) \in (\hat{f} \sqcup \hat{f}')^{-1}(1, y)$ and

$$(1, p)(\hat{g} \sqcup \hat{g}')\big|_{(\hat{f} \sqcup \hat{f}')^{-1}(1,y)}^{(f \sqcup f')^{-1}(1,yg)} = (1, p)(\hat{g} \sqcup \hat{g}') = (1, p\hat{g}) = (1, x).$$

Case 2: $\xi = (2, y')$ for some $y' \in Y'$. Then $\xi(g \sqcup g') = (2, y')(g \sqcup g') = (2, y'g') \in Z \sqcup Z'$. We have $(\hat{f} \sqcup \hat{f}')^{-1}(2, y') = \{(2, p') : p' \in \hat{f}'^{-1}(y')\}$ and $(f \sqcup f')^{-1}(2, y'g') = \{(2, x') : x' \in f'^{-1}(y'g')\}$.

Injectivity. Suppose given $(2, p'), (2, \tilde{p}') \in (\hat{f} \sqcup \hat{f}')^{-1}(2, y')$, that is $p', \tilde{p}' \in \hat{f}'^{-1}(y')$ and suppose that

$$(2, p')(\hat{g} \sqcup \hat{g}') = (2, p')(\hat{g} \sqcup \hat{g}')\big|_{(\hat{f} \sqcup \hat{f}')^{-1}(2,y')}^{(f \sqcup f')^{-1}(2,y'g')} = (2, \tilde{p}')(\hat{g} \sqcup \hat{g}')\big|_{(\hat{f} \sqcup \hat{f}')^{-1}(2,y')}^{(f \sqcup f')^{-1}(2,y'g')} = (2, \tilde{p}')(\hat{g} \sqcup \hat{g}').$$

But this means that

$$(2, p'\hat{g}') = (2, p')(\hat{g} \sqcup \hat{g}') = (2, \tilde{p}')(\hat{g} \sqcup \hat{g}') = (2, \tilde{p}'\hat{g}'),$$

hence $p'\hat{g}' = \tilde{p}'\hat{g}'$.

Since $p', \tilde{p}' \in \hat{f}'^{-1}(y')$ and since $\hat{g}'\big|_{\hat{f}'^{-1}(y')}^{f'^{-1}(y'g')}$ is injective, this implies that $p' = \tilde{p}'$.

Surjectivity. Suppose given $(2, x') \in (f \sqcup f')^{-1}(2, y'g')$, that is, $x' \in f'^{-1}(y'g')$. Since $\hat{g}'\big|_{\hat{f}'^{-1}(y')}^{f'^{-1}(y'g')}$ is surjective, there exists $p' \in \hat{f}'^{-1}(y)$ such that $p'\hat{g}' = p'\hat{g}'\big|_{\hat{f}'^{-1}(y')}^{f'^{-1}(y'g')} = x'$. This means that we have $(2, p') \in (\hat{f} \sqcup \hat{f}')^{-1}(2, y')$ and

$$(2, p')(\hat{g} \sqcup \hat{g}')\big|_{(\hat{f} \sqcup \hat{f}')^{-1}(2,y')}^{(f \sqcup f')^{-1}(2,y'g')} = (2, p')(\hat{g} \sqcup \hat{g}') = (2, p'\hat{g}') = (2, x').$$

□

2 Preoperads

2.1 Biindexed sets

Definition 2.1. A *biindexed set* (X, s, t) consists of a set X and of maps $s : X \longrightarrow \mathbb{Z}_{\geqslant 0}$ and $t : X \longrightarrow \mathbb{Z}_{\geqslant 0}$.

Suppose given $m, n, \in \mathbb{Z}_{\geqslant 0}$. Let $X(m,n) := \{x \in X : xs = m \text{ and } xt = n\}$. We will often also write $X = (X(m,n))_{m,n\geqslant 0}$ for the biindexed set.

Note that for $x \in X$ we have $x \in X(xs, xt)$.

Definition 2.2. Let (X, s_X, t_X) and (Y, s_Y, t_Y) be biindexed sets and let $\varphi : X \longrightarrow Y$ be a map. Then φ is called a *biindexed map*, if for $x \in X$ we have

$$(x\varphi)s_Y = xs_X \qquad \text{and} \qquad (x\varphi)t_Y = xt_X \,.$$

This means that given $m, n \in \mathbb{Z}_{\geqslant 0}$, then the restrictions $\varphi|_{X(m,n)}^{Y(m,n)} : X(m,n) \longrightarrow Y(m,n)$ are maps. Define $\varphi(m,n) := \varphi|_{X(m,n)}^{Y(m,n)}$. We will often also write $\varphi = (\varphi(m,n))_{m,n\geqslant 0}$ for a biindexed map.

Remark 2.3. Let $(X, s_X, t_X), (Y, s_Y, t_Y)$ and (Z, s_Z, t_Z) be biindexed sets and let $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ be biindexed maps. The composite $\varphi\psi : X \longrightarrow Z$ is a biindexed map, since for $x \in X$ we have $(x(\varphi\psi))s_Z = ((x\varphi)\psi)s_Z = (x\varphi)s_Y = xs_X$ and $(x(\varphi\psi))t_Z = ((x\varphi)\psi)t_Z = (x\varphi)t_Y = xt_X$.

So for $m, n \in \mathbb{Z}_{\geqslant 0}$ we have $(\varphi\psi)(m,n) = \varphi(m,n)\psi(m,n)$.

Definition 2.4. Let (X, s, t) be a biindexed set. A *biindexed subset of* X is given by $(Y, s|_Y, t|_Y)$ for some subset $Y \subseteq X$.

Note that this means $Y(m,n) \subseteq X(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$. We will write $Y \subseteq X$ to indicate that Y is a biindexed subset of X.

Definition 2.5. Let (X, s_X, t_X) and (Y, s_Y, t_Y) be biindexed sets. We define

$$X \times Y := \big\{(x,y) \in X \times Y : xs_X = ys_Y \text{ and } xt_X = yt_Y\big\}.$$

Furthermore, we define maps $s_{x \times Y}$ and $t_{X \times Y}$ follows.

$$s_{X\times Y} : \begin{array}{ccc} X \times Y & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ (x,y) & \longmapsto & xs_X = ys_Y \end{array} \qquad\qquad t_{X\times Y} : \begin{array}{ccc} X \times Y & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ (x,y) & \longmapsto & xt_X = yt_Y \end{array}$$

Then we have the biindexed set $(X \times Y, s_{X\times Y}, t_{X\times Y})$.

Note that for $m, n \in \mathbb{Z}_{\geqslant 0}$ we have $(X \times Y)(m,n) = X(m,n) \times Y(m,n)$.

2.2 Set-preoperads and linear preoperads

Definition 2.6. A *set-preoperad* $\mathcal{P}_0 = (\mathcal{P}_0, \boxtimes, \cdot)$ consists of

- a biindexed set $(\mathcal{P}_0(m,n))_{m,n\geqslant 0}$,
- *identity elements* $\mathrm{id}_m := \mathrm{id}_{\mathcal{P}_0,m} \in \mathcal{P}_0(m,m)$ for $m \in \mathbb{Z}_{\geqslant 0}$,
- *multiplication* maps

  $$\begin{aligned} (\boxtimes) := (\boxtimes_{\mathcal{P}_0}) : \mathcal{P}_0(m,n) \times \mathcal{P}_0(m',n') &\longrightarrow \mathcal{P}_0(m+m', n+n') \\ (f, f') &\longmapsto f \boxtimes_{\mathcal{P}_0} f' =: f \boxtimes f' \end{aligned}$$

  for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$,

- *composition* maps

$$(\cdot) := (\cdot_{\mathcal{P}_0}) : \mathcal{P}_0(m,n) \times \mathcal{P}_0(n,k) \longrightarrow \mathcal{P}_0(m,k)$$
$$(f,g) \longmapsto f \cdot_{\mathcal{P}_0} g =: f \cdot g =: fg$$

  for $m, n, k \in \mathbb{Z}_{\geqslant 0}$

such that the following axioms hold.

- concerning multiplication

  (m1) Associativity: We have $(f \boxtimes f') \boxtimes f'' = f \boxtimes (f' \boxtimes f'')$ for $m, n, m', n', m'', n'' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$, $f' \in \mathcal{P}_0(m',n')$ and $f'' \in \mathcal{P}_0(m'',n'')$.

  (m2) We have $\mathrm{id}_0 \boxtimes f = f \boxtimes \mathrm{id}_0 = f$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$.

- concerning composition

  (c1) Associativity: We have $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for $m, n, k, l \in \mathbb{Z}_{\geqslant 0}$, and $f \in \mathcal{P}_0(m,n)$, $g \in \mathcal{P}_0(n,k)$, and $h \in \mathcal{P}_0(k,l)$.

  (c2) We have $\mathrm{id}_m \cdot f = f$ and $f \cdot \mathrm{id}_n = f$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$.

- concerning multiplication and composition

  (mc1) We have $(f \boxtimes f') \cdot (g \boxtimes g') = (f \cdot g) \boxtimes (f' \cdot g')$ for $m, n, k, m', n', k' \in \mathbb{Z}_{\geqslant 0}$, $f \in \mathcal{P}_0(m,n)$, $f' \in \mathcal{P}_0(m',n')$, $g \in \mathcal{P}_0(n,k)$ and $g' \in \mathcal{P}_0(n',k')$.

  (mc2) The identity elements satisfy $\mathrm{id}_m = \mathrm{id}_1^{\boxtimes m}$ for $m \in \mathbb{Z}_{\geqslant 0}$, where for $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$ the product $f^{\boxtimes k} \in \mathcal{P}_0(km, kn)$ is defined as follows. We let $f^{\boxtimes 0} := \mathrm{id}_0$ and for $k \geqslant 1$ we let $f^{\boxtimes k} := f^{\boxtimes (k-1)} \boxtimes f$.

Remark 2.7. Note that (mc2) implies that $\mathrm{id}_m \boxtimes \mathrm{id}_n = \mathrm{id}_{m+n}$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Remark 2.8. One could summarize this definition by saying that a set-preoperad is a *strict monoidal category* with $\mathbb{Z}_{\geqslant 0}$ as set of objects; cf. **[10,** VII.1**]**.

A linear-preoperad over the ring R can be defined similarly.

Definition 2.9. A *linear preoperad* $\mathcal{P} = (\mathcal{P}, \boxtimes, \cdot)$ over R consists of

- a biindexed set $(\mathcal{P}(m,n))_{m,n \geqslant 0}$, where $\mathcal{P}(m,n)$ is an R-module for $m, n \in \mathbb{Z}_{\geqslant 0}$,
- *identity elements* $\mathrm{id}_m := \mathrm{id}_{\mathcal{P},m} \in \mathcal{P}(m,m)$ for $m \in \mathbb{Z}_{\geqslant 0}$,
- R-linear *multiplication* maps

$$(\boxtimes) := (\boxtimes_{\mathcal{P}}) : \mathcal{P}(m,n) \otimes \mathcal{P}(m',n') \longrightarrow \mathcal{P}(m+m', n+n')$$
$$f \otimes f' \longmapsto f \boxtimes_{\mathcal{P}} f' =: f \boxtimes f'$$

  for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$,

- R-linear *composition* maps

$$(\cdot) := (\cdot_{\mathcal{P}}) : \mathcal{P}(m,n) \otimes \mathcal{P}(n,k) \longrightarrow \mathcal{P}(m,k)$$
$$f \otimes g \longmapsto f \cdot_{\mathcal{P}} g =: f \cdot g =: fg$$

  for $m, n, k \in \mathbb{Z}_{\geqslant 0}$

such that the following axioms hold.

- concerning multiplication

(M1) Associativity: We have $(f \boxtimes f') \boxtimes f'' = f \boxtimes (f' \boxtimes f'')$ for $m, n, m', n', m'', n'' \in \mathbb{Z}_{\geqslant 0}$, and $f \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m',n')$ and $f'' \in \mathcal{P}(m'',n'')$.

(M2) We have $\mathrm{id}_0 \boxtimes f = f \boxtimes \mathrm{id}_0 = f$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$.

- concerning composition

(C1) Associativity: We have $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for $m, n, k, l \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$, $g \in \mathcal{P}(n,k)$ and $h \in \mathcal{P}(k,l)$.

(C2) We have $\mathrm{id}_m \cdot f = f$ and $f \cdot \mathrm{id}_n = f$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$.

- concerning multiplication and composition

(MC1) We have $(f \boxtimes f') \cdot (g \boxtimes g') = (f \cdot g) \boxtimes (f' \cdot g')$ for $m, n, k, m', n', k' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m',n')$, $g \in \mathcal{P}(n,k)$ and $g' \in \mathcal{P}(n',k')$.

(MC2) The identity elements satisfy $\mathrm{id}_m = \mathrm{id}_1^{\boxtimes m}$ for $m \in \mathbb{Z}_{\geqslant 0}$, where for $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$ the product $f^{\boxtimes k} \in \mathcal{P}(km, kn)$ is defined as follows. We let $f^{\boxtimes 0} := \mathrm{id}_0$ and for $k \geqslant 1$ we let $f^{\boxtimes k} := f^{\boxtimes (k-1)} \boxtimes f$.

So basically the definitions of a set-preoperad and of a linear preoperad over R are the same except in the R-linear case we ask for the multiplication and composition maps to be R-linear maps. We will often write an index 0 as in $\mathcal{P}_0$ to indicate that we are dealing with set-preoperads and not linear preoperads.

Furthermore, we will often drop the additional "over R". In these cases, all occuring linearities are R-linearities.

Remark 2.10. We can always view a linear preoperad $(\mathcal{P}, \boxtimes, \cdot)$ as a set-properad by forgetting that $\mathcal{P}(m,n)$ is an R-module for $m, n \in \mathbb{Z}_{\geqslant 0}$ and instead viewing it as a set and by using the multiplication maps

$$\begin{aligned} \mathcal{P}(m,n) \times \mathcal{P}(m',n') &\longrightarrow \mathcal{P}(m+m', n+n') \\ (f, f') &\longmapsto f \boxtimes f' \end{aligned}$$

for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and the composition maps

$$\begin{aligned} \mathcal{P}(m,n) \times \mathcal{P}(n,k) &\longrightarrow \mathcal{P}(m,k) \\ (f, g) &\longmapsto f \cdot g \end{aligned}$$

for $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and by forgetting that they are R-bilinear.

Whenever we do not want to make the distinction between linear preoperads and set-preoperads (e.g. if the statement is true for both) we will simply write preoperad.

Lemma 2.11. *Let $(\mathcal{P}, \boxtimes, \cdot)$ be a preoperad. We have the preoperad $(\mathcal{P} \times \mathcal{P}, \boxtimes_{\mathcal{P} \times \mathcal{P}}, \cdot_{\mathcal{P} \times \mathcal{P}})$ with*

- *identity elements* $\mathrm{id}_{\mathcal{P} \times \mathcal{P}, m} := (\mathrm{id}_{\mathcal{P},m}, \mathrm{id}_{\mathcal{P},m})$ *for* $m \in \mathbb{Z}_{\geqslant 0}$ *,*

- *multiplication defined by* $(f, \tilde{f}) \boxtimes_{\mathcal{P} \times \mathcal{P}} (f', \tilde{f}') := (f \boxtimes f', \tilde{f} \boxtimes \tilde{f}')$ *for* $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ *and* $f, \tilde{f} \in \mathcal{P}(m,n)$*,* $f', \tilde{f}' \in \mathcal{P}(m',n')$*,*

- *composition defined by* $(f,\tilde{f}) \cdot_{\mathcal{P}\times\mathcal{P}} (g,\tilde{g}) := (f\cdot g, \tilde{f}\cdot\tilde{g})$ *for* $m,n,k \in \mathbb{Z}_{\geqslant 0}$ *and* $f,\tilde{f} \in \mathcal{P}(m,n)$, $g,\tilde{g} \in \mathcal{P}(n,k)$.

Proof. We have $(\mathcal{P}\times\mathcal{P})(m,n) = \mathcal{P}(m,n)\times\mathcal{P}(m,n)$ for $m,n\in\mathbb{Z}_{\geqslant 0}$; cf. Definition 2.5.

Note that in case $\mathcal{P}$ is a linear preoperad, these cartesian products are again R-modules. Moreover, since in that case $(\boxtimes)$ and $(\cdot)$ are R-linear maps, so are $(\boxtimes_{\mathcal{P}\times\mathcal{P}})$ and $(\cdot_{\mathcal{P}\times\mathcal{P}})$.

The properties (m1) – (mc2) follow from the respective properties of $\mathcal{P}$ since $(\boxtimes_{\mathcal{P}\times\mathcal{P}})$ and $(\cdot_{\mathcal{P}\times\mathcal{P}})$ are defined entry-wise. □

Recall that given sets X, Y and a map $u : X \longrightarrow Y$ then the map $Ru : RX \longrightarrow RY$ is R-linear. Furthermore, recall that we can identify $R(X\times Y) = RX\otimes RY$ and that by identifying along some injective map $X \longrightarrow RX$ we may write $X \subseteq RX$; cf. Definition 1.8.

Remark 2.12. *Let* $\mathcal{P}_0$ *be a set-preoperad. Then* $R\mathcal{P}_0 = (R\mathcal{P}_0(m,n))_{m,n\geqslant 0}$ *is a linear preoperad over* R *with* R*-linear multiplication maps*

$$R\,(\boxtimes_{\mathcal{P}_0}) : R\mathcal{P}_0(m,n)\otimes R\mathcal{P}_0(m',n') = R\big(\mathcal{P}_0(m,n)\times\mathcal{P}_0(m',n)\big) \longrightarrow R\mathcal{P}_0(m+m',n+n')$$

for $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$*,* R*-linear composition maps*

$$R\,(\cdot_{\mathcal{P}_0}) : R\mathcal{P}_0(m,n)\otimes R\mathcal{P}_0(n,k) = R\big(\mathcal{P}_0(m,n)\times\mathcal{P}_0(n,k)\big) \longrightarrow R\mathcal{P}_0(m,k)$$

for $m,n,k\in\mathbb{Z}_{\geqslant 0}$ *and identity elements* $\mathrm{id}_{R\mathcal{P}_0,m} := \mathrm{id}_{\mathcal{P}_0,m}$ *for* $m\in\mathbb{Z}_{\geqslant 0}$.

Proof. Note that

$$\Big(\sum_{f\in\mathcal{P}_0(m,n)} r_f\, f\Big) \boxtimes_{R\mathcal{P}_0} \Big(\sum_{f'\in\mathcal{P}_0(m',n')} r'_{f'}\, f'\Big) = \sum_{\substack{f\in\mathcal{P}_0(m,n)\\ f'\in\mathcal{P}_0(m',n')}} r_f\, r'_{f'} (f \boxtimes_{\mathcal{P}_0} f')$$

$$\Big(\sum_{f\in\mathcal{P}_0(m,n)} r_f\, f\Big) \cdot_{R\mathcal{P}_0} \Big(\sum_{g\in\mathcal{P}_0(n,k)} r'_{g}\, g\Big) = \sum_{\substack{f\in\mathcal{P}_0(m,n)\\ g\in\mathcal{P}_0(n,k)}} r_f\, r'_{g}\, (f \cdot_{\mathcal{P}_0} g)$$

for $m,n,k,m',n'\in\mathbb{Z}_{\geqslant 0}$. So the required properties for $R\mathcal{P}_0$ follow from the properties of $\mathcal{P}_0$. □

Definition 2.13. Let $(\mathcal{P},\boxtimes,\cdot)$ be a preoperad. Then the *opposite preoperad* $\mathcal{P}^{\mathrm{op}}$ is defined as follows.

- Let $\mathcal{P}^{\mathrm{op}}(m,n) := \{f^{\mathrm{op}} : f\in\mathcal{P}(n,m)\}$ for $m,n\in\mathbb{Z}_{\geqslant 0}$.
- Let $\mathrm{id}_{\mathcal{P}^{\mathrm{op}},m} := (\mathrm{id}_{\mathcal{P},m})^{\mathrm{op}}$ for $m\in\mathbb{Z}_{\geqslant 0}$.
- Multiplication is given by

$$\begin{aligned}(\boxtimes_{\mathrm{op}}) := (\boxtimes_{\mathcal{P}^{\mathrm{op}}}) : \mathcal{P}^{\mathrm{op}}(m,n)\times\mathcal{P}^{\mathrm{op}}(m',n') &\longrightarrow \mathcal{P}^{\mathrm{op}}(m+m',n+n')\\ (f^{\mathrm{op}}, f'^{\,\mathrm{op}}) &\longmapsto f^{\mathrm{op}}\boxtimes_{\mathrm{op}} f'^{\,\mathrm{op}} := (f\boxtimes f')^{\mathrm{op}}\end{aligned}$$

  for $m,n,m',n'\in\mathbb{Z}_{\geqslant 0}$.

- Composition is given by

$$\begin{aligned}(\cdot_{\mathrm{op}}) := (\cdot_{\mathcal{P}^{\mathrm{op}}}) : \mathcal{P}^{\mathrm{op}}(m,n)\times\mathcal{P}^{\mathrm{op}}(n,k) &\longrightarrow \mathcal{P}^{\mathrm{op}}(m,k)\\ (f^{\mathrm{op}}, g^{\mathrm{op}}) &\longmapsto f^{\mathrm{op}}\cdot_{\mathrm{op}} g^{\mathrm{op}} := (g\cdot f)^{\mathrm{op}}\end{aligned}$$

  for $m,n,k\in\mathbb{Z}_{\geqslant 0}$.

2.3 Morphisms of preoperads

Now we will be interested in studying maps between (both set- and linear) preoperads that respect the basic structure.

Definition 2.14. Let $\mathcal{P}_0$, $\mathcal{Q}_0$ be set-preoperads. A *morphism* $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ *of set-preoperads* is a biindexed map $\varphi_0 = (\varphi_0(m,n))_{m,n \geqslant 0}$ such that (1) and (2) hold.

(1) We have $\mathrm{id}_{\mathcal{P}_0,m}\, \varphi_0(m,m) = \mathrm{id}_{\mathcal{Q}_0,m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

(2) Suppose given $m, n, m', n', k \in \mathbb{Z}_{\geqslant 0}$. The following two diagrams both commute.

$$\begin{array}{ccc} \mathcal{P}_0(m,n) \times \mathcal{P}_0(m',n') & \xrightarrow{\varphi_0(m,n)\times\varphi_0(m',n')} & \mathcal{Q}_0(m,n) \times \mathcal{Q}_0(m',n') \\ {\scriptstyle (\boxtimes_{\mathcal{P}_0})}\big\downarrow & & \big\downarrow{\scriptstyle (\boxtimes_{\mathcal{Q}_0})} \\ \mathcal{P}_0(m+m',n+n') & \xrightarrow{\varphi_0(m+m',n+n')} & \mathcal{Q}_0(m+m',n+n') \end{array}$$

$$\begin{array}{ccc} \mathcal{P}_0(m,n) \times \mathcal{P}_0(n,k) & \xrightarrow{\varphi_0(m,n)\times\varphi_0(n,k)} & \mathcal{Q}_0(m,n) \times \mathcal{Q}_0(n,k) \\ {\scriptstyle (\cdot_{\mathcal{P}_0})}\big\downarrow & & \big\downarrow{\scriptstyle (\cdot_{\mathcal{Q}_0})} \\ \mathcal{P}_0(m,k) & \xrightarrow{\varphi_0(m,k)} & \mathcal{Q}_0(m,k) \end{array}$$

That is, for $f \in \mathcal{P}_0(m,n)$, $f' \in \mathcal{P}_0(m',n')$, $g \in \mathcal{P}_0(n,k)$ we have

$$(f\varphi_0(m,n)) \boxtimes_{\mathcal{Q}_0} \left(f'\varphi_0(m',n')\right) = \left(f \boxtimes_{\mathcal{P}_0} f'\right) \varphi_0(m+m',n+n')$$
$$(f\varphi_0(m,n)) \cdot_{\mathcal{Q}_0} (g\varphi_0(n,k)) = (f \cdot_{\mathcal{P}_0} g)\, \varphi_0(m,k).$$

Remark 2.15. Since $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ is a map, cf. Definition 2.2, we can write the equations in Definition 2.14 (2) as

$$f\varphi_0 \boxtimes_{\mathcal{Q}_0} f'\varphi_0 = \left(f \boxtimes_{\mathcal{P}_0} f'\right) \varphi_0$$
$$f\varphi_0 \cdot_{\mathcal{Q}_0} g\varphi_0 = (f \cdot_{\mathcal{P}_0} g)\, \varphi_0$$

for $f \in \mathcal{P}_0(m,n), f' \in \mathcal{P}_0(m',n'), g \in \mathcal{P}_0(n,k)$ and $m,n,m',n',k \in \mathbb{Z}_{\geqslant 0}$.

Again we can define morphisms of linear preoperads similarly.

Definition 2.16. Let $\mathcal{P}$, $\mathcal{Q}$ be linear preoperads. A *morphism* $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ *of linear preoperads* is a biindexed map $\varphi = (\varphi(m,n))_{m,n \geqslant 0}$ such that (0), (1) and (2) hold.

(0) The maps $\varphi(m,n)$ are linear for $m,n \in \mathbb{Z}_{\geqslant 0}$.

(1) We have $\mathrm{id}_{\mathcal{P},m}\, \varphi(m,m) = \mathrm{id}_{\mathcal{Q},m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

(2) Suppose given $m, n, m', n', k \in \mathbb{Z}_{\geqslant 0}$. The following two diagrams both commute.

$$\begin{array}{ccc} \mathcal{P}(m,n) \otimes \mathcal{P}(m',n') & \xrightarrow{\varphi(m,n)\otimes\varphi(m',n')} & \mathcal{Q}(m,n) \otimes \mathcal{Q}(m',n') \\ {\scriptstyle (\boxtimes_{\mathcal{P}})}\big\downarrow & & \big\downarrow{\scriptstyle (\boxtimes_{\mathcal{Q}})} \\ \mathcal{P}(m+m',n+n') & \xrightarrow{\varphi(m+m',n+n')} & \mathcal{Q}(m+m',n+n') \end{array}$$

$$
\begin{array}{ccc}
\mathcal{P}(m,n) \otimes \mathcal{P}(n,k) & \xrightarrow{\varphi(m,n)\otimes\varphi(n,k)} & \mathcal{Q}(m,n) \otimes \mathcal{Q}(n,k) \\
{\scriptstyle (\cdot_{\mathcal{P}})}\big\downarrow & & \big\downarrow{\scriptstyle (\cdot_{\mathcal{Q}})} \\
\mathcal{P}(m,k) & \xrightarrow{\varphi(m,k)} & \mathcal{Q}(m,k)
\end{array}
$$

That is, for $f \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m',n')$, $g \in \mathcal{P}(n,k)$ we have

$$
\begin{aligned}
(f\varphi(m,n)) \boxtimes_{\mathcal{Q}} \left(f'\varphi(m',n')\right) &= \left(f \boxtimes_{\mathcal{P}} f'\right) \varphi(m+m', n+n') \\
(f\varphi(m,n)) \cdot_{\mathcal{Q}} (g\varphi(n,k)) &= (f \cdot_{\mathcal{P}} g)\, \varphi(m,k).
\end{aligned}
$$

Note that, as with morphisms of set-preoperads, we may abbreviate $f\varphi = f\varphi(m,n)$ for $m,n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$ when it is clear which linear map $\varphi(m,n)$ is needed.

Remark 2.17. Recall that given linear preoperads $\mathcal{P}$ and $\mathcal{Q}$ then by Remark 2.10 we can view them as set-preoperads. Furthermore, given a morphism of set-preoperads $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ we can then view this as a morphism of set-preoperads by forgetting that $\varphi(m,n)$ is linear for $m,n \in \mathbb{Z}_{\geqslant 0}$.

The following Definition 2.18, Definition 2.19 and Lemma 2.20 pertain to set-preoperads and to linear preoperads.

Definition 2.18.

(1) Let $\mathcal{P}$ be a preoperad. The *identity morphism* $\mathrm{id}_{\mathcal{P}} = (\mathrm{id}_{\mathcal{P}}(m,n))_{m,n\geqslant 0} = (\mathrm{id}_{\mathcal{P}(m,n)})_{m,n\geqslant 0}$ is given by $f\,\mathrm{id}_{\mathcal{P}(m,n)} = f$ for $m,n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$.

(2) Given morphisms of preoperads $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ and $\psi : \mathcal{Q} \longrightarrow \mathcal{R}$, then the composite $\varphi\psi$ is given by $(\varphi\psi)(m,n) = \varphi(m,n)\psi(m,n)$ for $m,n \in \mathbb{Z}_{\geqslant 0}$.

This is again a morphism of preoperads since for $m, m', n, n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$ and $f' \in \mathcal{P}(m',n')$ we have

$$
\begin{aligned}
\left(f(\varphi\psi)\right) \boxtimes_{\mathcal{R}} \left(f'(\varphi\psi)\right) &= \left((f\varphi)\psi\right) \boxtimes_{\mathcal{R}} \left((f'\varphi)\psi\right) \\
&= \left((f\varphi) \boxtimes_{\mathcal{Q}} (f'\varphi)\right)\psi \\
&= \left((f \boxtimes_{\mathcal{P}} f')\varphi\right)\psi \\
&= (f \boxtimes_{\mathcal{P}} f')(\varphi\psi)
\end{aligned}
$$

and since for $m,n,k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$ and $g \in \mathcal{P}(n,k)$ we have

$$
\begin{aligned}
\left(f\varphi\psi)\right) \cdot_{\mathcal{R}} \left(g(\varphi\psi)\right) &= \left((f\varphi)\psi\right) \cdot_{\mathcal{R}} \left((f'\varphi)\psi\right) \\
&= \left((f\varphi) \cdot_{\mathcal{Q}} (g\varphi)\right)\psi \\
&= \left((f \cdot_{\mathcal{P}} g)\varphi\right)\psi \\
&= (f \cdot_{\mathcal{P}} g)(\varphi\psi).
\end{aligned}
$$

Furthermore, note that if φ and ψ are morphisms of linear preoperads over R, then the composite $\varphi(m,n)\psi(m,n)$ for $m,n \in \mathbb{Z}_{\geqslant 0}$ is again an R-linear map.

Definition 2.19. Let $\mathcal{P}$ and $\mathcal{Q}$ be preoperads and let $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of preoperads. Then φ is called an *isomorphism* if there exists a morphism of preoperads $\psi : \mathcal{Q} \longrightarrow \mathcal{P}$ such that $\varphi\psi = \mathrm{id}_{\mathcal{P}}$ and $\psi\varphi = \mathrm{id}_{\mathcal{Q}}$.

We then say that $\mathcal{P}$ and $\mathcal{Q}$ are *isomorphic* and write $\mathcal{P} \cong \mathcal{Q}$.

Lemma 2.20. *Let $\mathcal{P}$ and $\mathcal{Q}$ be preoperads and let $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of preoperads. Then φ is an isomorphism if and only if $\varphi(m,n)$ is bijective for $m,n \in \mathbb{Z}_{\geqslant 0}$.*

Proof. The morphism φ is an isomorphism if and only if there exists a morphism of preoperads $\psi : \mathcal{Q} \longrightarrow \mathcal{P}$ such that $\varphi\psi = \mathrm{id}_{\mathcal{P}}$ and $\psi\varphi = \mathrm{id}_{\mathcal{Q}}$, i.e. we have

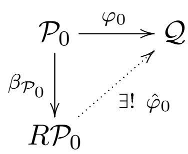
$$\begin{aligned}\varphi(m,n) \cdot \psi(m,n) &= (\varphi \cdot \psi)(m,n) = \mathrm{id}_{\mathcal{P}}(m,n) = \mathrm{id}_{\mathcal{P}(m,n)}\\ \psi(m,n) \cdot \varphi(m,n) &= (\psi \cdot \varphi)(m,n) = \mathrm{id}_{\mathcal{Q}}(m,n) = \mathrm{id}_{\mathcal{Q}(m,n)}\,,\end{aligned}$$

i.e. $\varphi(m,n)$ is bijective. □

Remark 2.21. Note that the given preoperads $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ and isomorphisms of preoperads $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ and $\psi : \mathcal{Q} \longrightarrow \mathcal{R}$, then the composite $\varphi\psi : \mathcal{P} \longrightarrow \mathcal{R}$ is also an isomorphism of preoperads.

Definition 2.22. Let $\mathcal{P}_0$ be a set-preoperad. Since $\mathcal{P}_0(m,n) \subseteq (R\mathcal{P}_0)(m,n) = R(\mathcal{P}_0(m,n))$ for $m,n \in \mathbb{Z}_{\geqslant 0}$; cf. Definition 1.8 and Remark 2.12, we can define a morphism $\beta_{\mathcal{P}_0} : \mathcal{P}_0 \longrightarrow R\mathcal{P}_0$ by $f\beta_{\mathcal{P}_0} = f$ for $f \in \mathcal{P}_0(m,n)$ and $m,n \in \mathbb{Z}_{\geqslant 0}$.

Lemma 2.23. *Let $\mathcal{P}_0$ be a set-preoperad and $\mathcal{Q}$ be a linear preoperad. Recall that we can also view $\mathcal{Q}$ as a set-preoperad. Let $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}$ be a morphism of set-preoperads. Then there exists a unique morphism of linear preoperads $\hat{\varphi} : R\mathcal{P}_0 \longrightarrow \mathcal{Q}$ such that $\beta_{\mathcal{P}_0}\hat{\varphi}_0 = \varphi_0$; cf.* Definition 2.22.



Proof. By Remark 1.9, for $m,n \in \mathbb{Z}_{\geqslant 0}$ there exists a uniquely determined linear map $\hat{\varphi}_0(m,n) : (R\mathcal{P}_0)(m,n) \longrightarrow \mathcal{Q}(m,n)$ such that $f\hat{\varphi} = f\hat{\varphi}_0(m,n) = f\varphi_0(m,n) = f\varphi$ for $f \in \mathcal{P}_0(m,n)$. So it remains to show that $\hat{\varphi}_0 := (\hat{\varphi}_0(m,n))_{m,n\geqslant 0} : R\mathcal{P}_0 \longrightarrow \mathcal{Q}$ is a morphism of linear preoperads.

First note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have $\mathrm{id}_{R\mathcal{P}_0,m}\,\hat{\varphi}_0 = \mathrm{id}_{\mathcal{P}_0,m}\,\varphi_0 = \mathrm{id}_{\mathcal{Q}}$.

Suppose given $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $\sum\limits_{f\in\mathcal{P}_0(m,n)} r_f\, f \in (R\mathcal{P}_0)(m,n)$ and $\sum\limits_{f'\in\mathcal{P}_0(m',n')} s_{f'}\, f' \in (R\mathcal{P}_0)(m',n')$.

Then we have

$$\begin{aligned}\Bigg(\Bigg(\sum_{f\in\mathcal{P}_0(m,n)} r_f\, f\Bigg) \boxtimes_{R\mathcal{P}_0} \Bigg(\sum_{f'\in\mathcal{P}_0(m',n')} s_{f'}\, f'\Bigg)\Bigg)\hat{\varphi}_0 &= \Bigg(\sum_{\substack{f\in\mathcal{P}_0(m,n)\\ f'\in\mathcal{P}_0(m',n')}} r_f\, s_{f'}\, \big(f \boxtimes_{\mathcal{P}_0} f'\big)\Bigg)\hat{\varphi}_0\\ &= \sum_{\substack{f\in\mathcal{P}_0(m,n)\\ f'\in\mathcal{P}_0(m',n')}} r_f\, s_{f'} \big(\big(f \boxtimes_{\mathcal{P}_0} f'\big)\varphi_0\big)\\ &= \sum_{\substack{f\in\mathcal{P}_0(m,n)\\ f'\in\mathcal{P}_0(m',n')}} r_f\, s_{f'} \big(f\varphi_0 \boxtimes_{\mathcal{Q}} f'\varphi_0\big)\\ &= \Bigg(\sum_{f\in\mathcal{P}_0(m,n)} r_f\, (f\varphi_0)\Bigg) \boxtimes_{\mathcal{Q}} \Bigg(\sum_{f'\in\mathcal{P}_0(m',n')} s_{f'}\, (f'\varphi_0)\Bigg)\\ &= \Bigg(\sum_{f\in\mathcal{P}_0(m,n)} r_f\, f\Bigg)\hat{\varphi}_0 \boxtimes_{\mathcal{Q}} \Bigg(\sum_{f'\in\mathcal{P}_0(m',n')} s_{f'}\, f'\Bigg)\hat{\varphi}_0\,.\end{aligned}$$

In the same way we can see that given $m,n,k \in \mathbb{Z}_{\geqslant 0}$ and $\sum\limits_{f\in\mathcal{P}_0(m,n)} r_f\, f \in (R\mathcal{P}_0)(m,n)$ and $\sum\limits_{g\in\mathcal{P}_0(n,k)} s_g\, g \in (R\mathcal{P}_0)(n,k)$, then we have

$$\Bigg(\Bigg(\sum_{f\in\mathcal{P}_0(m,n)} r_f\, f\Bigg) \cdot_{R\mathcal{P}_0} \Bigg(\sum_{g\in\mathcal{P}_0(n,k)} s_g\, g\Bigg)\Bigg)\hat{\varphi}_0 = \Bigg(\sum_{f\in\mathcal{P}_0(m,n)} r_f\, f\Bigg)\hat{\varphi}_0 \cdot_{\mathcal{Q}} \Bigg(\sum_{g\in\mathcal{P}_0(n,k)} s_g\, g\Bigg)\hat{\varphi}_0\,.$$

□

Lemma 2.24. *Let $\mathcal{P}_0$ and $\mathcal{Q}_0$ be set-preoperads and let $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ be a morphism of set-preoperads. Then $R\varphi_0 := \big(R(\varphi_0(m,n))\big)_{m,n \geqslant 0} : R\mathcal{P}_0 \longrightarrow R\mathcal{Q}_0$ is the unique morphism of linear preoperads such that $\beta_{\mathcal{P}_0}(R\varphi_0) = \varphi_0 \beta_{\mathcal{Q}_0}$.*

$$\begin{array}{ccc} \mathcal{P}_0 & \xrightarrow{\varphi_0} & \mathcal{Q}_0 \\ {\scriptstyle \beta_{\mathcal{P}_0}}\big\downarrow & & \big\downarrow{\scriptstyle \beta_{\mathcal{Q}_0}} \\ R\mathcal{P}_0 & \xrightarrow[R\varphi_0]{} & R\mathcal{Q}_0 \end{array}$$

Proof. Note that $R\mathcal{Q}_0$ is a linear preoperad over R that we can view as a set-preoperad and that $\psi_0 := \varphi_0\beta_{\mathcal{Q}_0} : \mathcal{P}_0 \longrightarrow R\mathcal{Q}_0$ is a morphism of set-preoperads.

So by Lemma 2.23, there exists a uniquely determined morphism $\hat{\psi}_0 : R\mathcal{P}_0 \longrightarrow R\mathcal{Q}_0$ of linear preoperads such that $\beta_{\mathcal{P}_0}\hat{\psi}_0 = \psi_0 = \varphi_0\beta_{\mathcal{Q}_0}$.

Since for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$ we have

$$f\beta_{\mathcal{P}_0}(R\varphi_0) = f(R\varphi_0) = f\varphi_0 = (f\varphi_0)\beta_{\mathcal{Q}_0},$$

we have $R\varphi_0 = \hat{\psi}_0$. □

Remark 2.25.

(1) Let $\mathcal{P}_0$, $\mathcal{Q}_0$ and $\mathcal{R}_0$ be set-preoperads and let $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ and $\psi_0 : \mathcal{Q}_0 \longrightarrow \mathcal{R}_0$ be morphisms of set-preoperads. Then we have $(R\varphi_0)(R\psi_0) = R(\varphi_0\psi_0)$.

(2) Let $\mathcal{P}_0$ be a set-preoperad. Then we have $R\,\mathrm{id}_{\mathcal{P}_0} = \mathrm{id}_{R\mathcal{P}_0}$.

(3) Let $\mathcal{P}_0$ and $\mathcal{Q}_0$ be set-preoperads and let $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ be an isomorphism of set-preoperads. Then $R\varphi_0 : R\mathcal{P}_0 \longrightarrow R\mathcal{Q}_0$ is an isomorphism of linear preoperads.

2.4 Subpreoperads of set-preoperads and linear preoperads

Definition 2.26. Let $\mathcal{P}_0, \mathcal{Q}_0$ be set-preoperads. Then $\mathcal{Q}_0$ is said to be a *set-subpreoperad* of $\mathcal{P}_0$ if (1),(2) and (3) hold.

(1) We have $\mathcal{Q}_0(m,n) \subseteq \mathcal{P}_0(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$, so $\mathcal{Q}_0 \subseteq \mathcal{P}_0$ is a biindexed subset.

(2) We have $\mathrm{id}_{\mathcal{P}_0,m} \in \mathcal{Q}_0(m,m)$ for $m \in \mathbb{Z}_{\geqslant 0}$.

(3) The composition maps as well as the multiplication maps of $\mathcal{P}_0$ restrict to the respective maps of $\mathcal{Q}_0$, that is, for $m, n, m', n', k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{Q}_0(m,n)$, $f' \in \mathcal{Q}_0(m',n')$ and $g \in \mathcal{Q}_0(n,k)$ we have $f \boxtimes_{\mathcal{P}_0} f' = f \boxtimes_{\mathcal{Q}_0} f'$ and $f \cdot_{\mathcal{P}_0} g = f \cdot_{\mathcal{Q}_0} g$.

Lemma 2.27. *Let $(\mathcal{P}_0, \boxtimes_{\mathcal{P}_0}, \cdot_{\mathcal{P}_0})$ be a set-preoperad and $\mathcal{Q}_0 = (\mathcal{Q}_0(m,n))_{m,n \geqslant 0}$ a biindexed set satisfying* (s1), (s2) *and* (s3).

(s1) *We have $\mathcal{Q}_0(m,n) \subseteq \mathcal{P}_0(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.*

(s2) *We have $\mathrm{id}_{\mathcal{P}_0,m} \in \mathcal{Q}_0(m,m)$ for $m \in \mathbb{Z}_{\geqslant 0}$.*

(s3) *We have $\mathcal{Q}_0(m,n) \boxtimes_{\mathcal{P}_0} \mathcal{Q}_0(m',n') \subseteq \mathcal{Q}_0(m+m', n+n')$ and $\mathcal{Q}_0(m,n) \cdot_{\mathcal{P}_0} \mathcal{Q}_0(n,k) \subseteq \mathcal{Q}_0(m,k)$ for $m, n, k, m', n' \in \mathbb{Z}_{\geqslant 0}$, that is, $\mathcal{Q}_0$ is closed under multiplication and composition.*

Define $\mathrm{id}_{\mathcal{Q}_0,m} := \mathrm{id}_{\mathcal{P}_0,m}$ *for* $m \in \mathbb{Z}_{\geqslant 0}$ *and define* $(\boxtimes_{\mathcal{Q}_0})$ *by* $f \boxtimes_{\mathcal{Q}_0} f' := f \boxtimes_{\mathcal{P}_0} f'$ *for* $m, m', n, n' \in \mathbb{Z}_{\geqslant 0}$ *and* $f \in \mathcal{Q}_0(m,n)$, $f' \in \mathcal{Q}_0(m',n')$ *and define* $(\cdot_{\mathcal{Q}_0})$ *by* $f \cdot_{\mathcal{Q}_0} g := f \cdot_{\mathcal{P}_0} g$ *for* $m, n, k \in \mathbb{Z}_{\geqslant 0}$ *and* $f \in \mathcal{Q}_0(m,n)$, $g \in \mathcal{Q}_0(n,k)$.

Then $(\mathcal{Q}_0, \boxtimes_{\mathcal{Q}_0}, \cdot_{\mathcal{Q}_0})$ *is a set-subpreoperad of* $\mathcal{P}_0$.

Proof. Let $\mathcal{Q}_0 = (\mathcal{Q}_0(m,n))_{m,n\geqslant 0}$ be a biindexed set satisfying (s1), (s2) and (s3). Since the multiplication and composition maps of $\mathcal{P}_0$ restrict to those of $\mathcal{Q}_0$ by definition and by (s2) we have $\mathrm{id}_{\mathcal{P}_0,m} \in \mathcal{Q}_0$ for $m \in \mathbb{Z}_{\geqslant 0}$, all that needs to be shown is that $\mathcal{Q}_0$ is in fact a set-preoperad.

Since (m2),(c2) and (mc2) are true for $\mathcal{P}_0$ they hold for $\mathcal{Q}_0$ since $\mathrm{id}_{\mathcal{Q}_0,m} = \mathrm{id}_{\mathcal{P}_0,m}$ for $m \in \mathbb{Z}_{\geqslant 0}$ and since $\mathcal{Q}_0(m,n) \subseteq \mathcal{P}_0(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.

The properties (m1), (c1) and (mc1) for $\mathcal{Q}_0$ are inherited from $\mathcal{P}_0$. □

Definition 2.28. Let $\mathcal{P}, \mathcal{Q}$ be linear preoperads over R. Then $\mathcal{Q}$ is said to be a *linear subpreoperad* of $\mathcal{P}$ if (1),(2) and (3) hold.

(1) The R-module $\mathcal{Q}(m,n)$ is a submodule of $\mathcal{P}(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.

(2) We have $\mathrm{id}_{\mathcal{P},m} \in \mathcal{Q}(m,m)$ for all $m \in \mathbb{Z}_{\geqslant 0}$.

(3) The composition maps as well as the multiplication maps of $\mathcal{P}$ restrict to the respective maps of $\mathcal{Q}$, that is for $m, n, m', n', k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{Q}(m,n)$, $f' \in \mathcal{Q}(m',n')$ and $g \in \mathcal{Q}(n,k)$ we have $f \boxtimes_{\mathcal{P}} f' = f \boxtimes_{\mathcal{Q}} f'$ and $f \cdot_{\mathcal{P}} g = f \cdot_{\mathcal{Q}} g$.

Lemma 2.29. *Let* $(\mathcal{P}, \boxtimes_{\mathcal{P}}, \cdot_{\mathcal{P}})$ *be a linear preoperad over* R *and* $\mathcal{Q} = (\mathcal{Q}(m,n))_{m,n\geqslant 0}$ *a biindexed set satisfying* (S1), (S2) *and* (S3).

(S1) $\mathcal{Q}(m,n)$ *is a submodule of* $\mathcal{P}(m,n)$ *for* $m, n \in \mathbb{Z}_{\geqslant 0}$.

(S2) *We have* $\mathrm{id}_{\mathcal{P},m} \in \mathcal{Q}(m,m)$ *for* $m \in \mathbb{Z}_{\geqslant 0}$.

(S3) *We have* $\mathcal{Q}(m,n) \boxtimes_{\mathcal{P}} \mathcal{Q}(m',n') \subseteq \mathcal{Q}(m+m', n+n')$ *and* $\mathcal{Q}(m,n) \cdot_{\mathcal{P}} \mathcal{Q}(n,k) \subseteq \mathcal{Q}(m,k)$ *for* $m, m', n, n', k \in \mathbb{Z}_{\geqslant 0}$, *that is,* $\mathcal{Q}$ *is closed under multiplication and composition.*

Define $\mathrm{id}_{\mathcal{Q},m} := \mathrm{id}_{\mathcal{P},m}$ *for* $m \in \mathbb{Z}_{\geqslant 0}$ *and define* $(\boxtimes_{\mathcal{Q}})$ *by* $f \boxtimes_{\mathcal{Q}} f' := f \boxtimes_{\mathcal{P}} f'$ *for* $m, m', n, n' \in \mathbb{Z}_{\geqslant 0}$ *and* $f \in \mathcal{Q}(m,n)$, $f' \in \mathcal{Q}(m',n')$ *and define* $(\cdot_{\mathcal{Q}})$ *by* $f \cdot_{\mathcal{Q}} g := f \cdot_{\mathcal{P}} g$ *for* $m, n, k \in \mathbb{Z}_{\geqslant 0}$ *and* $f \in \mathcal{Q}(m,n)$, $g \in \mathcal{Q}(n,k)$.

Then $(\mathcal{Q}, \boxtimes_{\mathcal{Q}}, \cdot_{\mathcal{Q}})$ *is a linear subpreoperad of* $\mathcal{P}$.

Proof. Let $\mathcal{Q} = (\mathcal{Q}(m,n))_{m,n\geqslant 0}$ be a biindexed set satisfying (S1), (S2) and (S3). Note that this implies that $(\boxtimes_{\mathcal{Q}})$ and $(\cdot_{\mathcal{Q}})$ define R-linear maps.

Again we only need to show that $\mathcal{Q}$ is a linear preoperad over R.

Since (M2), (C2) and (MC2) are true for $\mathcal{P}$, they also hold for $\mathcal{Q}$, since $\mathrm{id}_{\mathcal{Q},m} = \mathrm{id}_{\mathcal{P},m}$ for $m \in \mathbb{Z}_{\geqslant 0}$ and since $\mathcal{Q}(m,n) \subseteq \mathcal{P}(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.

The properties (M1), (C1) and (MC1) are inherited from $\mathcal{P}$. □

Since the definitions for set-subreoperads and linear subpreoperads are the same except for the fact that in the R-linear case we are dealing with submodules instead of mere subsets, we are often going to write subpreoperad instead of set-subpreoperad or linear subpreoperad. In these cases, when it comes to verifying that a biindexed subset is a subpreoperad, we are often going to write (s1), (s2) and (s3) for the properties from Lemma 2.27 and 2.29.

The following Example 2.30, Lemma 2.31 and Definition 2.32 pertain to set-preoperads and to linear preoperads.

Example 2.30.

(1) Let $\mathcal{P}$ and $\mathcal{Q}$ be preoperads. Let $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of preoperads. The image $\mathrm{Im}(\varphi) = (\mathrm{Im}(\varphi(m,n)))_{m,n\geqslant 0}$ is a subpreoperad of $\mathcal{Q}$.

(2) Let $\mathcal{P}$ be a preoperad and let $\mathcal{Q} \subseteq \mathcal{P}$ be a subpreoperad of $\mathcal{P}$. Then the inclusion

$$\begin{aligned} \iota_{\mathcal{Q},\mathcal{P}} : \mathcal{Q} &\longrightarrow \mathcal{P} \\ f &\longmapsto f \end{aligned}$$

is a morphism of preoperads.

Lemma 2.31. *Let I be a set. Let $\mathcal{P}$ be a preoperad. Let $\mathcal{Q}_i \subseteq \mathcal{P}$ be a subpreoperad for $i \in I$. Then the intersection $\bigcap_{i\in I} \mathcal{Q}_i = \left(\bigcap_{i\in I} \mathcal{Q}_i(m,n)\right)_{m,n\geqslant 0}$ is a subpreoperad of $\mathcal{P}$.*

Proof. *Ad* (s1). Since $\mathcal{Q}_i(m,n) \subseteq \mathcal{P}(m,n)$ for $i \in I$, we have $\bigcap_{i\in I} \mathcal{Q}_i(m,n) \subseteq \mathcal{P}(m,n)$. In the case of linear preoperads the intersection is again an R-module.

Ad (s2). We have $\mathrm{id}_{\mathcal{P},m} \in \mathcal{Q}_i(m,m)$ for $i \in I$, hence $\mathrm{id}_{\mathcal{P},m} \in \bigcap_{i\in I} \mathcal{Q}_i(m,m)$.

Ad (s3). Suppose given $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \bigcap_{i\in I} \mathcal{Q}_i(m,n)$, $f' \in \bigcap_{i\in I} \mathcal{Q}_i(m',n')$, hence $f \in \mathcal{Q}_i(m,n)$ and $f' \in \mathcal{Q}_i(m',n')$ for $i \in I$. By (s3) for $\mathcal{Q}_i$ we have $f \boxtimes_{\mathcal{P}} f' \in \mathcal{Q}_i(m+m',n+n')$ for $i \in I$, hence $f \boxtimes_{\mathcal{P}} f' \in \bigcap_{i\in I} \mathcal{Q}_i(m+m',n+n')$.

Now suppose given $m,n,k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \bigcap_{i\in I} \mathcal{Q}_i(m,n)$, $g \in \bigcap_{i\in I} \mathcal{Q}_i(n,k)$, so $f \in \mathcal{Q}_i(m,n)$ and $g \in \mathcal{Q}_i(n,k)$ for $i \in I$. By (s3) for $\mathcal{Q}_i$ we have $f \cdot_{\mathcal{P}} g \in \mathcal{Q}_i(m,k)$ for $i \in I$, hence $fg \in \bigcap_{i\in I} \mathcal{Q}_i(m,k)$. □

Definition 2.32. Let $\mathcal{P}$ be a preoperad and let $X \subseteq \mathcal{P}$ be a biindexed subset. We define the subpreoperad of $\mathcal{P}$ *generated by* X by

$$_{\text{preop}}\langle\, X \,\rangle := \bigcap \{\mathcal{Q} \subseteq \mathcal{P} : \mathcal{Q} \text{ is a subpreoperad with } X \subseteq \mathcal{Q}\}\,.$$

The biindexed subset $_{\text{preop}}\langle\, X \,\rangle$ is a subpreoperad of $\mathcal{P}$ by Lemma 2.31. It is the smallest subpreoperad of $\mathcal{P}$ that contains X, i.e. given a subpreoperad $\mathcal{Q} \subseteq \mathcal{P}$ with $X \subseteq \mathcal{Q}$, then we have $_{\text{preop}}\langle\, X \,\rangle \subseteq \mathcal{Q}$.

2.5 Congruences on set-preoperads

Definition 2.33. Let $(\mathcal{P}_0, \boxtimes, \cdot)$ be a set-preoperad. A *congruence* on $\mathcal{P}_0$ is a biindexed subset $(\equiv) \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ such that (1), (2) and (3) hold. For $(f,\tilde{f}) \in (\equiv)(m,n)$ with $m,n \in \mathbb{Z}_{\geqslant 0}$ we also write $f \equiv \tilde{f}$.

(1) For $m,n \in \mathbb{Z}_{\geqslant 0}$ the subset $(\equiv)(m,n) \subseteq \mathcal{P}_0(m,n) \times \mathcal{P}_0(m,n)$ is an equivalence relation on $\mathcal{P}_0(m,n)$.

(2) Suppose given $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $(f,\tilde{f}) \in (\equiv)(m,n)$ and $(f',\tilde{f}') \in (\equiv)(m',n')$. Then we have $(f \boxtimes f', \tilde{f} \boxtimes \tilde{f}') \in (\equiv)(m+m',n+n')$.

So if we have $f \equiv \tilde{f}$ and $f' \equiv \tilde{f}'$, then we have $(f \boxtimes f') \equiv (\tilde{f} \boxtimes \tilde{f}')$.

(3) Suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $(f, \tilde{f}) \in (\equiv)(m, n)$ and $(g, \tilde{g}) \in (\equiv)(n, k)$, then we have $(f \cdot g, \tilde{f} \cdot \tilde{g}) \in (\equiv)(m, k)$.

So if $f \equiv \tilde{f}$ and $g \equiv \tilde{g}$, then we have $(f \cdot g) \equiv (\tilde{f} \cdot \tilde{g})$.

We will denote the congruence classes of $\mathcal{P}_0$ with respect to $(\equiv)$ by $[f]_\equiv$ for $f \in \mathcal{P}_0(m, n)$ and $m, n \in \mathbb{Z}_{\geqslant 0}$. So for $m, n \in \mathbb{Z}_{\geqslant 0}$ we have $[f]_\equiv = [\tilde{f}]_\equiv$ for $f, \tilde{f} \in \mathcal{P}_0(m, n)$ if and only if $f \equiv \tilde{f}$, i.e. $(f, \tilde{f}) \in (\equiv)(m, n)$.

Lemma 2.34. *Let I be a set. Let $\mathcal{P}_0$ be a set-preoperad. Let $(\equiv_i) \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ be a congruence on $\mathcal{P}_0$ for $i \in I$. Then the intersection $\bigcap_{i \in I} (\equiv_i) = \left(\bigcap_{i \in I} (\equiv_i)(m, n) \right)_{m,n \geqslant 0}$ is a congruence on $\mathcal{P}_0$.*

Proof. First note that the intersection $\bigcap_{i \in I} (\equiv_i)(m, n)$ again defines an equivalence relation on $\mathcal{P}_0$.

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $(f, \tilde{f}) \in \bigcap_{i \in I} (\equiv_i)(m, n)$, $(f', \tilde{f}') \in \bigcap_{i \in I} (\equiv_i)(m', n')$. So we have $(f, \tilde{f}) \in (\equiv_i)(m, n)$ and $(f' \tilde{f}') \in (\equiv_i)(m', n')$ for $i \in I$. Since $(\equiv_i)$ is a congruence for $i \in I$, we have $((f \boxtimes f'), (\tilde{f} \boxtimes \tilde{f}')) \in (\equiv_i)(m + m', n + n')$ for $i \in I$, hence $(f \boxtimes f', \tilde{f} \boxtimes \tilde{f}') \in \bigcap_{i \in I} (\equiv_i)(m + m', n + n')$.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $(f, \tilde{f}) \in \bigcap_{i \in I} (\equiv_i)(m, n)$, $(g, \tilde{g}) \in \bigcap_{i \in I} (\equiv_i)(n, k)$ for $i \in I$. So we have $(f, \tilde{f}) \in (\equiv_i)(m, n)$ and $(g, \tilde{g}) \in (\equiv_i)(n, k)$ for $i \in I$. Since $(\equiv_i)$ is a congruence for $i \in I$, we have $(fg, \tilde{f}\tilde{g}) \in (\equiv_i)(m, k)$ for $i \in I$, hence $(fg, \tilde{f}\tilde{g}) \in \bigcap_{i \in I} (\equiv_i)(m, k)$. □

Definition 2.35. Let $\mathcal{P}_0$ be a set-preoperad and $X \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ be a biindexed subset. The congruence *generated by* X is defined by

$$(\equiv_X) := \bigcap \{ C \subseteq \mathcal{P}_0 \times \mathcal{P}_0 : C \text{ is a congruence with } X \subseteq C \}.$$

We call X the *generating set for* $(\equiv_X)$.

The congruence $(\equiv_X)$ is the smallest congruence on $\mathcal{P}_0$ containing X, i.e. given a congruence $(\equiv) \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ with $X \subseteq (\equiv)$, then we have $(\equiv_X) \subseteq (\equiv)$.

For the congruence class of $f \in \mathcal{P}_0(m, n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ we will often write $[f]_X := [f]_{\equiv_X}$.

Remark 2.36. Note that given a congruence $(\equiv) \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ for a set-preoperad $\mathcal{P}_0$, then the congruence generated by $(\equiv)$ is $(\equiv)$ itself. So every congruence has a generating set.

In the following Lemma we will introduce a certain kind of congruence on a set-preoperad.

Lemma 2.37. *Let $\mathcal{P}_0, \mathcal{T}_0$ be set-preoperads and let $\tau_0 : \mathcal{P}_0 \longrightarrow \mathcal{T}_0$ be a morphism of set-preoperads. Define the biindexed subset $(\equiv^{\tau_0}) \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ by*

$$(\equiv^{\tau_0})(m, n) := \{ (f, \tilde{f}) \in \mathcal{P}_0(m, n) \times \mathcal{P}_0(m, n) : f\tau_0 = \tilde{f}\tau_0 \}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$. Then $(\equiv^{\tau_0})$ is a congruence on $\mathcal{P}_0$.

Proof. First note that $(\equiv^{\tau_0})(m, n)$ defines an equivalence relation on $\mathcal{P}_0(m, n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f, \tilde{f} \in \mathcal{P}_0(m, n)$ with $(f, \tilde{f}) \in (\equiv^{\tau_0})(m, n)$ and $f', \tilde{f}' \in \mathcal{P}_0(m, n)$ with $(f', \tilde{f}') \in (\equiv^{\tau_0})(m', n')$. That is, we have $f\tau_0 = \tilde{f}\tau_0$ and $f'\tau_0 = \tilde{f}'\tau_0$. Then we have

$$(f \boxtimes_{\mathcal{P}_0} f')\tau_0 = (f\tau_0) \boxtimes_{\mathcal{T}_0} (f'\tau_0) = (\tilde{f}\tau_0) \boxtimes_{\mathcal{T}_0} (\tilde{f}'\tau_0) = (\tilde{f} \boxtimes_{\mathcal{P}_0} \tilde{f}')\tau_0,$$

hence $\big((f \boxtimes_{\mathcal{P}_0} f'), (\tilde{f} \boxtimes_{\mathcal{P}_0} \tilde{f}')\big) \in (\equiv^{\tau_0})(m+m', n+n')$.

Now suppose given $m, n, k' \in \mathbb{Z}_{\geqslant 0}$ and $f, \tilde{f} \in \mathcal{P}_0(m,n)$ with $(f, \tilde{f}) \in (\equiv^{\tau_0})(m,n)$ and $g, \tilde{g} \in \mathcal{P}_0(n,k)$ with $(g, \tilde{g}) \in (\equiv^{\tau_0})(n,k)$. That is, we have $f\tau_0 = \tilde{f}\tau_0$ and $g\tau_0 = \tilde{g}\tau_0$. Then we have

$$(f \cdot_{\mathcal{P}_0} g)\tau_0 = (f\tau_0) \cdot_{\mathcal{T}_0} (g\tau_0) = (\tilde{f}\tau_0) \cdot_{\mathcal{T}_0} (\tilde{g}\tau_0) = (\tilde{f} \cdot_{\mathcal{P}_0} \tilde{g})\tau_0 \,,$$

hence $\big((f \cdot_{\mathcal{P}_0} g), (\tilde{f} \cdot_{\mathcal{P}_0} \tilde{g})\big) \in (\equiv^{\tau_0})(m,k)$. □

Definition 2.38 (factor set-preoperad)**.** Let $\mathcal{P}_0$ be a set-preoperad and let $(\equiv) \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ be a congruence on $\mathcal{P}_0$. We define the factor set-preoperad $\frac{\mathcal{P}_0}{(\equiv)}$ as follows.

- Let $\left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m,n) := \big\{[f]_\equiv : f \in \mathcal{P}_0(m,n)\big\}$.
- Let $\mathrm{id}_{\left(\frac{\mathcal{P}_0}{(\equiv)}\right),m} := [\mathrm{id}_{\mathcal{P}_0,m}]_\equiv$ for $m \in \mathbb{Z}_{\geqslant 0}$.
- The multiplication is given by

$$\begin{aligned}(\boxtimes) := \big(\boxtimes_{\frac{\mathcal{P}_0}{(\equiv)}}\big) : \Big(\frac{\mathcal{P}_0}{(\equiv)}\Big)(m,n) \times \Big(\frac{\mathcal{P}_0}{(\equiv)}\Big)(m',n') &\longrightarrow \Big(\frac{\mathcal{P}_0}{(\equiv)}\Big)(m+m',n+n') \\ \big([f]_\equiv, [f']_\equiv\big) &\longmapsto [f]_\equiv \boxtimes [f']_\equiv := [f \boxtimes_{\mathcal{P}_0} f']_\equiv\end{aligned}$$

  for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$.
- The composition is given by

$$\begin{aligned}(\cdot) := \big(\cdot_{\frac{\mathcal{P}_0}{(\equiv)}}\big) : \Big(\frac{\mathcal{P}_0}{(\equiv)}\Big)(m,n) \times \Big(\frac{\mathcal{P}_0}{(\equiv)}\Big)(n,k) &\longrightarrow \Big(\frac{\mathcal{P}_0}{(\equiv)}\Big)(m,k) \\ \big([f]_\equiv, [f']_\equiv\big) &\longmapsto [f]_\equiv \cdot [f']_\equiv := [f \cdot_{\mathcal{P}_0} f']_\equiv\end{aligned}$$

  for $m, n, k \in \mathbb{Z}_{\geqslant 0}$.

We will show now that this is a set-preoperad. Denote the multiplication and composition on $\frac{\mathcal{P}_0}{(\equiv)}$ by $(\boxtimes)$ and $(\cdot)$ and on $\mathcal{P}_0$ by $(\boxtimes_{\mathcal{P}_0})$ and $(\cdot_{\mathcal{P}_0})$.

First note that given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f, \tilde{f} \in \mathcal{P}_0(m,n)$ with $f \equiv \tilde{f}$ and $f', \tilde{f}' \in \mathcal{P}_0(m',n')$ with $f' \equiv \tilde{f}'$, we have $f \boxtimes_{\mathcal{P}_0} f' \equiv \tilde{f} \boxtimes_{\mathcal{P}_0} \tilde{f}'$, since $(\equiv)$ is a congruence on $\mathcal{P}_0$. Hence we have $[f \boxtimes_{\mathcal{P}_0} f']_\equiv = [\tilde{f} \boxtimes_{\mathcal{P}_0} \tilde{f}']_\equiv$. So the multiplication $(\boxtimes)$ is well-defined.

Moreover, given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f, \tilde{f} \in \mathcal{P}_0(m,n)$ with $f \equiv \tilde{f}$ and $g, \tilde{g} \in \mathcal{P}_0(n,k)$ with $g \equiv \tilde{g}$, we have $f \cdot_{\mathcal{P}_0} g \equiv \tilde{f} \cdot_{\mathcal{P}_0} \tilde{g}$, since $(\equiv)$ is a congruence on $\mathcal{P}_0$. Hence $[f \cdot_{\mathcal{P}_0} g]_\equiv = [\tilde{f} \cdot_{\mathcal{P}_0} \tilde{g}]_\equiv$. So the composition $(\cdot)$ is well-defined.

Ad (m1). Suppose given $m, n, m', n', m'', n'' \in \mathbb{Z}_{\geqslant 0}$ and $[f]_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m,n)$, $[f']_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m',n')$ and $[f'']_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m'',n'')$. We have

$$\begin{aligned}([f]_\equiv \boxtimes [f']_\equiv) \boxtimes [f'']_\equiv &= [f \boxtimes_{\mathcal{P}_0} f']_\equiv \boxtimes [f'']_\equiv \\ &= [(f \boxtimes_{\mathcal{P}_0} f') \boxtimes_{\mathcal{P}_0} f'']_\equiv \\ &= [f \boxtimes_{\mathcal{P}_0} (f' \boxtimes_{\mathcal{P}_0} f'')]_\equiv \\ &= [f]_\equiv \boxtimes [f' \boxtimes_{\mathcal{P}_0} f'']_\equiv \\ &= [f]_\equiv \boxtimes ([f']_\equiv \boxtimes [f'']_\equiv).\end{aligned}$$

Ad (m2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $[f]_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m,n)$. Then we have

$$[f]_\equiv \boxtimes [\mathrm{id}_{\mathcal{P}_0,0}]_\equiv = [f \boxtimes_{\mathcal{P}_0} \mathrm{id}_{\mathcal{P}_0,0}]_\equiv = [f]_\equiv = [\mathrm{id}_{\mathcal{P}_0,0} \boxtimes_{\mathcal{P}_0} f]_\equiv = [\mathrm{id}_{\mathcal{P}_0,0}]_\equiv \boxtimes [f]_\equiv \,.$$

Ad (c1). Suppose given $m, n, k, l \in \mathbb{Z}_{\geqslant 0}$ and $[f]_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m,n)$, $[g]_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(n,k)$ and $[h]_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(k,l)$. We have

$$\begin{aligned}([f]_\equiv \cdot [g]_\equiv) \cdot [h]_\equiv &= [f \cdot_{\mathcal{P}_0} g]_\equiv \cdot [h]_\equiv \\ &= [(f \cdot_{\mathcal{P}_0} g) \cdot_{\mathcal{P}_0} h]_\equiv \\ &= [f \cdot_{\mathcal{P}_0} (g \cdot_{\mathcal{P}_0} h)]_\equiv \\ &= [f]_\equiv \cdot [g \cdot_{\mathcal{P}_0} h]_\equiv \\ &= [f]_\equiv \cdot ([g]_\equiv \cdot [h]_\equiv).\end{aligned}$$

Ad (c2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $[f]_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m,n)$. Then we have

$$[f]_\equiv \cdot [\mathrm{id}_{\mathcal{P}_0,n}]_\equiv = [f \cdot_{\mathcal{P}_0} \mathrm{id}_{\mathcal{P}_0,n}]_\equiv = [f]_\equiv = [\mathrm{id}_{\mathcal{P}_0,n} \cdot_{\mathcal{P}_0} f]_\equiv = [\mathrm{id}_{\mathcal{P}_0,n}]_\equiv \cdot [f]_\equiv\,.$$

Ad (mc1). Suppose given $m, n, k, m', n', k' \in \mathbb{Z}_{\geqslant 0}$ and $[f]_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m,n)$, $[f']_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m',n')$, $[g]_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(n,k)$ and $[g']_\equiv \in \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(n',k')$. We have

$$\begin{aligned}([f]_\equiv \boxtimes [f']_\equiv) \cdot ([g]_\equiv \boxtimes [g']_\equiv) &= [f \boxtimes_{\mathcal{P}_0} f']_\equiv \cdot [g \boxtimes_{\mathcal{P}_0} g']_\equiv \\ &= [(f \boxtimes_{\mathcal{P}_0} f') \cdot_{\mathcal{P}_0} (g \boxtimes_{\mathcal{P}_0} g')]_\equiv \\ &= [(f \cdot_{\mathcal{P}_0} g) \boxtimes_{\mathcal{P}_0} (f' \cdot_{\mathcal{P}_0} g')]_\equiv \\ &= [f \cdot_{\mathcal{P}_0} g]_\equiv \boxtimes [f' \cdot_{\mathcal{P}_0} g']_\equiv \\ &= ([f]_\equiv \cdot [g]_\equiv) \boxtimes ([f']_\equiv \cdot [g']_\equiv).\end{aligned}$$

Ad (mc2). Suppose given $m \in \mathbb{Z}_{\geqslant 0}$. We have

$$[\mathrm{id}_{\mathcal{P}_0,m}]_\equiv = \left[\mathrm{id}_{\mathcal{P}_0,1}^{\boxtimes_{\mathcal{P}_0} m}\right]_\equiv = [\mathrm{id}_{\mathcal{P}_0,1}]_\equiv^{\boxtimes m}.$$

Hence $\frac{\mathcal{P}_0}{(\equiv)}$ is a set-preoperad.

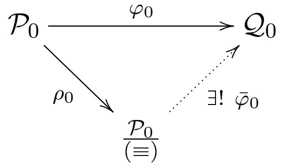
Definition 2.39. Let $\mathcal{P}_0$ be a set-preoperad and $(\equiv)$ be a congruence on $\mathcal{P}_0$. The congruence class morphism $\rho_0 := \rho_{0,(\equiv)} : \mathcal{P}_0 \longrightarrow \frac{\mathcal{P}_0}{(\equiv)}$ is defined as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ we let

$$\begin{aligned}\rho_0(m,n) : \mathcal{P}_0(m,n) &\longrightarrow \left(\frac{\mathcal{P}_0}{(\equiv)}\right)(m,n) \\ f &\longmapsto [f]_\equiv\,.\end{aligned}$$

This defines a morphism since $\rho_0(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ maps identity elements to identity elements and the composition and multiplication on $\frac{\mathcal{P}_0}{(\equiv)}$ are defined using the composition and multiplication of representatives.

Lemma 2.40 (Universal property of the factor set-preoperad)**.** *Let $\mathcal{P}_0$ and $\mathcal{Q}_0$ be set-preoperads. Let $(\equiv) \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ be a congruence on $\mathcal{P}_0$ with generating set $X \subseteq \mathcal{P}_0 \times \mathcal{P}_0$, i.e. $(\equiv) = (\equiv_X)$.*

Let $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ be a morphism of set-preoperads such that $f\varphi_0(m,n) = \tilde{f}\varphi_0(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $(f, \tilde{f}) \in X(m,n)$. Then there exists a uniquely determined morphism of set-preoperads $\bar{\varphi}_0 : \frac{\mathcal{P}_0}{(\equiv)} \longrightarrow \mathcal{Q}_0$ such that $\rho_0 \bar{\varphi}_0 = \varphi_0$.



Proof. Existence. Define $\bar{\varphi}_0 = (\bar{\varphi}_0(m,n))_{m,n\geqslant 0}$ by

$$\begin{aligned}\bar{\varphi}_0(m,n) : \Big(\frac{\mathcal{P}_0}{(\equiv)}\Big)(m,n) &\longrightarrow \mathcal{Q}_0(m,n) \\ [f]_\equiv &\longmapsto f\varphi_0(m,n)\end{aligned}$$

for $m,n \in \mathbb{Z}_{\geqslant 0}$.

For $m,n \in \mathbb{Z}_{\geqslant 0}$ and $(f,\tilde{f}) \in X(m,n)$ we have $f\varphi_0 = \tilde{f}\varphi_0$, hence $(f,\tilde{f}) \in (\equiv^{\varphi_0})(m,n)$; cf. Lemma 2.37. So $(\equiv^{\varphi_0})(m,n)$ is a congruence on $\mathcal{P}_0$ with $X \subseteq (\equiv^{\varphi_0})$. So by the definition of the generated congruence we have $(\equiv) = (\equiv_X) \subseteq (\equiv^{\varphi_0})$. So for $m,n \in \mathbb{Z}_{\geqslant 0}$ and for $f,\tilde{f} \in \mathcal{P}_0(m,n)$ with $[f]_\equiv = [\tilde{f}]_\equiv$ we have $f\varphi_0 = \tilde{f}\varphi_0$, so the map $\bar{\varphi}_0$ is well-defined.

Now we have to show that $\bar{\varphi}_0$ is a morphism of set-preoperads.

First note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have $[\mathrm{id}_{\mathcal{P}_0,m}]_\equiv\bar{\varphi}_0 = \mathrm{id}_{\mathcal{P}_0,m}\,\varphi_0 = \mathrm{id}_{\mathcal{Q}_0,m}$.

Moreover, for $m,n,k,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$, $f' \in \mathcal{P}_0(m',n')$, $g \in \mathcal{P}(n,k)$ we have

$$\begin{aligned}([f]_\equiv \boxtimes [f']_\equiv)\bar{\varphi}_0 &= [f \boxtimes_{\mathcal{P}_0} f']_\equiv\bar{\varphi}_0 \\ &= (f \boxtimes_{\mathcal{P}_0} f')\varphi_0 \\ &= f\varphi_0 \boxtimes_{\mathcal{Q}_0} f'\varphi_0 \\ &= [f]_\equiv\bar{\varphi}_0 \boxtimes_{\mathcal{Q}_0} [f']_\equiv\bar{\varphi}_0\end{aligned}$$

and

$$\begin{aligned}([f]_\equiv \cdot [g]_\equiv)\bar{\varphi}_0 &= [f \cdot_{\mathcal{P}_0} g]_\equiv\bar{\varphi}_0 \\ &= (f \cdot_{\mathcal{P}_0} g)\varphi_0 \\ &= f\varphi_0 \cdot_{\mathcal{Q}_0} g\varphi_0 \\ &= [f]_\equiv\bar{\varphi}_0 \cdot_{\mathcal{Q}_0} [g]_\equiv\bar{\varphi}_0.\end{aligned}$$

So $\bar{\varphi}_0$ is a morphism of set-preoperads. Moreover, for $m,n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$ we have $f(\rho_0\bar{\varphi}_0) = [f]_\equiv\bar{\varphi}_0 = f\varphi_0$. Hence $\rho_0\bar{\varphi}_0 = \varphi_0$.

Uniqueness. Let $\chi_0 : \frac{\mathcal{P}_0}{(\equiv)} \longrightarrow \mathcal{Q}_0$ be a morphism of set-preoperads such that $\rho_0\chi_0 = \varphi_0$. Then for $m,n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$ we have $[f]_\equiv\bar{\varphi}_0 = f\varphi_0 = f(\rho_0\chi_0) = [f]_\equiv\chi_0$, hence $\bar{\varphi}_0 = \chi_0$. □

2.6 Ideals of linear preoperads

Much like in the study of rings we will now introduce ideals of linear preoperads. This will lead to linear factor preoperads similar to the construction for set-preoperads.

Definition 2.41. Let $\mathcal{P}$ be a linear preoperad over R and let $\mathcal{I} = (\mathcal{I}(m,n))_{m,n\geqslant 0}$ be a biindexed set. We say that $\mathcal{I}$ is an *ideal* of $\mathcal{P}$ if (I1) and (I2) hold.

(I1) $\mathcal{I}(m,n) \subseteq \mathcal{P}(m,n)$ is a submodule for $m,n \in \mathbb{Z}_{\geqslant 0}$.

(I2) For $m,n,k,m',n' \in \mathbb{Z}_{\geqslant 0}$ we have

- (mr) $\mathcal{I}(m,n) \boxtimes \mathcal{P}(m',n') \subseteq \mathcal{I}(m+m',n+n')$
- (ml) $\mathcal{P}(m,n) \boxtimes \mathcal{I}(m',n') \subseteq \mathcal{I}(m+m',n+n')$
- (cr) $\mathcal{I}(m,n) \cdot \mathcal{P}(n,k) \subseteq \mathcal{I}(m,k)$
- (cl) $\mathcal{P}(m,n) \cdot \mathcal{I}(n,k) \subseteq \mathcal{I}(m,k)$.

Example 2.42. Let $\mathcal{P}, \mathcal{Q}$ be linear preoperads over R and let $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of linear preoperads. The *kernel of* φ given by $(\ker \varphi)(m,n) := \ker(\varphi(m,n))$ is an ideal of $\mathcal{P}$.

Ad (I1). Since the maps $\varphi(m,n) : \mathcal{P}(m,n) \longrightarrow \mathcal{Q}(m,n)$ are linear maps, $\ker(\varphi(m,n)) \subseteq \mathcal{P}(m,n)$ is a submodule for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Ad (I2). Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m',n')$. Then we have

$$\left(f \boxtimes_{\mathcal{P}} f'\right)\varphi = (f\varphi) \boxtimes_{\mathcal{P}} \left(f'\varphi\right).$$

If we choose $f \in \ker(\varphi(m,n))$, then $f\varphi(m,n) = 0$. Since $(\boxtimes_{\mathcal{P}})$ is R-linear, we have $(f \boxtimes_{\mathcal{P}} f')\varphi = 0$, which shows (mr). If we choose $f' \in \ker(\varphi(m',n'))$, then $f'\varphi(m',n') = 0$. Hence $(f \boxtimes_{\mathcal{P}} f')\varphi = 0$, showing (ml).

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$, $g \in \mathcal{P}(n,k)$. Then we have

$$(f \cdot_{\mathcal{P}} g)\varphi = (f\varphi) \cdot_{\mathcal{P}} (g\varphi).$$

If we choose $f \in \ker(\varphi(m,n))$ then $f\varphi(m,n) = 0$. Since $(\cdot_{\mathcal{P}})$ is R-linear, we have $(f \cdot_{\mathcal{P}} g)\varphi = 0$, which shows (cr). If we choose $g \in \ker(\varphi(n,k))$ then $g\varphi(n,k) = 0$. Hence $(f \cdot_{\mathcal{P}} g)\varphi = 0$, showing (cl).

Lemma 2.43. *Let J be a set. Let $\mathcal{P}$ be a linear preoperad. Let $\mathcal{I}_j \subseteq \mathcal{P}$ be an ideal of $\mathcal{P}$ for $j \in J$. Then the intersection $\bigcap_{j \in J} \mathcal{I}_j = \left(\bigcap_{j \in J} \mathcal{I}_j(m,n)\right)_{m,n \geqslant 0}$ is also an ideal of $\mathcal{P}$.*

Proof. Ad (I1). The intersection $\bigcap_{j \in J} \mathcal{I}_j(m,n) \subseteq \mathcal{P}(m,n)$ is a submodule since $\mathcal{I}_j(m,n) \subseteq \mathcal{P}(m,n)$ is a submodule for $j \in J$.

Ad (I2). Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \bigcap_{j \in J} \mathcal{I}_j(m,n)$ and $f' \in \mathcal{P}(m',n')$. That is, we have $f \in \mathcal{I}_j(m,n)$ for $j \in J$.

Then by (I2) for $\mathcal{I}_j$ we have $f \boxtimes_{\mathcal{P}} f' \in \mathcal{I}_j(m+m', n+n')$ for $j \in J$, hence $f \boxtimes_{\mathcal{P}} f' \in \bigcap_{j \in J} \mathcal{I}_j(m+m', n+n')$, which shows (mr). Analogously, (ml) holds.

Now suppose given $m, n, k, l \in \mathbb{Z}_{\geqslant 0}$ and $f \in \bigcap_{j \in J} \mathcal{I}_j(m,n)$, $g \in \mathcal{P}(n,k)$ and $h \in \bigcap_{j \in J} \mathcal{I}_j(k,l)$, so $f \in \mathcal{I}_j(m,n)$ and $h \in \mathcal{I}_j(k,l)$ for $j \in J$. By (I) for $\mathcal{I}_j$ we have $f \cdot_{\mathcal{P}} g \in \mathcal{I}_i(m,k)$ and $g \cdot_{\mathcal{P}} h \in \mathcal{I}_j(n,l)$ for $j \in J$, so $f \cdot_{\mathcal{P}} g \in \bigcap_{j \in J} \mathcal{I}_j(m,k)$ and $g \cdot_{\mathcal{P}} h \in \bigcap_{j \in J} \mathcal{I}_j(n,l)$, which shows (cr) and (cl). □

Definition 2.44. Let $\mathcal{P}$ be a linear preoperad and let $X \subseteq \mathcal{P}$ be a biindexed subset. The ideal *generated by* X is defined by

$${}_{\text{ideal}}\langle X \rangle := \bigcap \left\{\mathcal{I} \subseteq \mathcal{P} : \mathcal{I} \text{ is an ideal with } X \subseteq \mathcal{I}\right\}.$$

From Lemma 2.43 we know that ${}_{\text{ideal}}\langle X \rangle$ is an ideal of $\mathcal{P}$. It is the smallest ideal of $\mathcal{P}$ containing X, i.e. given an ideal $\mathcal{I} \subseteq \mathcal{P}$ with $X \subseteq \mathcal{I}$, then we have ${}_{\text{ideal}}\langle X \rangle \subseteq \mathcal{I}$.

We say that X is a *generating set* for the ideal ${}_{\text{ideal}}\langle X \rangle$.

Remark 2.45. Let $\mathcal{I} \subseteq \mathcal{P}$ be an ideal of the linear preoperad $\mathcal{P}$. Then the ideal generated by $\mathcal{I}$ is $\mathcal{I}$ itself. So every ideal has a generating set.

Similar to factor set-preoperads we are now able to define linear factor preoperads.

Definition 2.46 (linear factor preoperad)**.** Let $\mathcal{P}$ be a linear preoperad over R. Let $\mathcal{I}$ be an ideal of $\mathcal{P}$. We define the *linear factor preoperad* $\frac{\mathcal{P}}{\mathcal{I}}$ as follows.

- Let $\left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,n) := \frac{\mathcal{P}(m,n)}{\mathcal{I}(m,n)} = \{f + \mathcal{I}(m,n) : f \in \mathcal{P}(m,n)\}$ for $m,n \in \mathbb{Z}_{\geqslant 0}$.
  We often write $[f]_{\mathcal{I}} := f + \mathcal{I}(m,n)$ for $f \in \mathcal{P}(m,n)$ and $m,n \in \mathbb{Z}_{\geqslant 0}$.
- Let $\mathrm{id}_{\left(\frac{\mathcal{P}}{\mathcal{I}}\right),m} := \mathrm{id}_{\mathcal{P},m} + \mathcal{I}(m,m) = [\mathrm{id}_{\mathcal{P},m}]_{\mathcal{I}}$ for $m \in \mathbb{Z}_{\geqslant 0}$.
- The multiplication is given by

$$
\begin{aligned}
(\boxtimes) := \left(\boxtimes_{\frac{\mathcal{P}}{\mathcal{I}}}\right) : \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,n) \otimes \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m',n') &\longrightarrow \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m+m',n+n') \\
[f]_{\mathcal{I}} \otimes [f']_{\mathcal{I}} &\longmapsto [f]_{\mathcal{I}} \boxtimes [f']_{\mathcal{I}} := \left[f \boxtimes_{\mathcal{P}} f'\right]_{\mathcal{I}}
\end{aligned}
$$

  for $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$.
- The composition is given by

$$
\begin{aligned}
(\cdot) := \left(\cdot_{\frac{\mathcal{P}}{\mathcal{I}}}\right) : \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,n) \otimes \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(n,k) &\longrightarrow \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,k) \\
[f]_{\mathcal{I}} \otimes [g]_{\mathcal{I}} &\longmapsto [f]_{\mathcal{I}} \cdot [g]_{\mathcal{I}} := [f \cdot_{\mathcal{P}} g]_{\mathcal{I}}
\end{aligned}
$$

  for $m,n,k \in \mathbb{Z}_{\geqslant 0}$.

First we will show that the multiplication and composition maps are well-defined. Note that by Lemma 1.2 it suffices to show that the corresponding maps

$$
\begin{aligned}
\left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,n) \times \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m',n') &\longrightarrow \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m+m',n+n') \\
\left([f]_{\mathcal{I}}, [f']_{\mathcal{I}}\right) &\longmapsto [f \boxtimes_{\mathcal{P}} f']_{\mathcal{I}}
\end{aligned}
$$

for $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and

$$
\begin{aligned}
\left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,n) \times \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(n,k) &\longrightarrow \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,k) \\
\left([f]_{\mathcal{I}}, [g]_{\mathcal{I}}\right) &\longmapsto [f \cdot_{\mathcal{P}} g]_{\mathcal{I}}
\end{aligned}
$$

for $m,n,k \in \mathbb{Z}_{\geqslant 0}$ are well-defined and bilinear.

Suppose given $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $f, \tilde{f} \in \mathcal{P}(m,n)$ such that $[f]_{\mathcal{I}} = [\tilde{f}]_{\mathcal{I}}$ and $f', \tilde{f}' \in \mathcal{P}(m',n')$ such that $[f']_{\mathcal{I}} = [\tilde{f}']_{\mathcal{I}}$, that is $f - \tilde{f} \in \mathcal{I}(m,n)$ and $f' - \tilde{f}' \in \mathcal{I}(m',n')$. Since $(\boxtimes_{\mathcal{P}})$ is linear we have

$$
\begin{aligned}
f \boxtimes_{\mathcal{P}} f' - \tilde{f} \boxtimes_{\mathcal{P}} \tilde{f}' &= f \boxtimes_{\mathcal{P}} f' - \tilde{f} \boxtimes_{\mathcal{P}} f' + \tilde{f} \boxtimes_{\mathcal{P}} f' - \tilde{f} \boxtimes_{\mathcal{P}} \tilde{f}' \\
&= (f - \tilde{f}) \boxtimes_{\mathcal{P}} f' - \tilde{f} \boxtimes_{\mathcal{P}} (f' - \tilde{f}').
\end{aligned}
$$

Now since $f - \tilde{f} \in \mathcal{I}(m,n)$, by (mr) we have $(f - \tilde{f}) \boxtimes_{\mathcal{P}} f' \in \mathcal{I}(m+m',n+n')$. Moreover, since $f' - \tilde{f}' \in \mathcal{I}(m',n')$, by (ml) we have $\tilde{f} \boxtimes_{\mathcal{P}} (f' - \tilde{f}') \in \mathcal{I}(m+m',n+n')$. Hence we have $f \boxtimes_{\mathcal{P}} f' - \tilde{f} \boxtimes_{\mathcal{P}} \tilde{f}' \in \mathcal{I}(m+m',n+n')$, so $[f \boxtimes_{\mathcal{P}} f']_{\mathcal{I}} = [\tilde{f} \boxtimes_{\mathcal{P}} \tilde{f}']_{\mathcal{I}}$. This shows that the map is well-defined.

It is bilinear since for $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $f_1, f_2 \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m',n')$ and $r \in R$ we have $(([f_1]_{\mathcal{I}} + r[f_2]_{\mathcal{I}}), [f']_{\mathcal{I}}) = ([f_1 + rf_2]_{\mathcal{I}}, [f']_{\mathcal{I}})$ and

$$
\left[(f_1 + rf_2) \boxtimes_{\mathcal{P}} f'\right]_{\mathcal{I}} = \left[(f_1 \boxtimes_{\mathcal{P}} f') + r(f_2 \boxtimes_{\mathcal{P}} f')\right]_{\mathcal{I}} = \left[f_1 \boxtimes_{\mathcal{P}} f'\right]_{\mathcal{I}} + r\left[f_2 \boxtimes_{\mathcal{P}} f'\right]_{\mathcal{I}}
$$

and analogously for the second argument.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f, \tilde{f} \in \mathcal{P}(m,n)$ such that $[f]_\mathcal{I} = [\tilde{f}]_\mathcal{I}$ and $g, \tilde{g} \in \mathcal{P}(n,k)$ such that $[g]_\mathcal{I} = [\tilde{g}]_\mathcal{I}$, that is, $f - \tilde{f} \in \mathcal{I}(m,n)$ and $g - \tilde{g} \in \mathcal{I}(n,k)$. Since $(\cdot_\mathcal{P})$ is linear we have

$$\begin{aligned} fg - \tilde{f}\tilde{g} &= fg - \tilde{f}g + \tilde{f}g - \tilde{f}\tilde{g} \\ &= (f - \tilde{f})g + f(g - \tilde{g}). \end{aligned}$$

Now since $f - \tilde{f} \in \mathcal{I}(m,n)$, by (cr) we have $(f - \tilde{f})g \in \mathcal{I}(m,k)$. Moreover, since $g - \tilde{g} \in \mathcal{I}(n,k)$, by (cl) we have $\tilde{f}(g - \tilde{g}) \in \mathcal{I}(m,k)$. Hence we have $f \cdot_\mathcal{P} g - \tilde{f} \cdot_\mathcal{P} \tilde{g} \in \mathcal{I}(m,k)$, so $[f \cdot_\mathcal{P} g]_\mathcal{I} = [\tilde{f} \cdot_\mathcal{P} \tilde{g}]_\mathcal{I}$. This shows that the map is well-defined. Using the definition of the residue classes $[f]_\mathcal{I}$ and the linearity of $(\cdot_\mathcal{P})$ we see that it is a bilinear map.

This shows that $(\boxtimes)$ and $(\cdot)$ are well-defined.

Ad (M1). Suppose given $m, n, m', n', m'', n'' \in \mathbb{Z}_{\geqslant 0}$, as well as $f \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m',n')$ and $f'' \in \mathcal{P}(m'',n'')$. Then we have

$$\begin{aligned} [f]_\mathcal{I} \boxtimes ([f']_\mathcal{I} \boxtimes [f'']_\mathcal{I}) &= [f]_\mathcal{I} \boxtimes [f' \boxtimes_\mathcal{P} f'']_\mathcal{I} \\ &= [f \boxtimes_\mathcal{P} (f' \boxtimes_\mathcal{P} f'')]_\mathcal{I} \\ &= [(f \boxtimes_\mathcal{P} f') \boxtimes_\mathcal{P} f'']_\mathcal{I} \\ &= [f \boxtimes_\mathcal{P} f']_\mathcal{I} \boxtimes [f'']_\mathcal{I} \\ &= ([f]_\mathcal{I} \boxtimes [f']_\mathcal{I}) \boxtimes [f'']_\mathcal{I}. \end{aligned}$$

Ad (M2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$. Then we have

$$[f]_\mathcal{I} \boxtimes [\mathrm{id}_{\mathcal{P},0}]_\mathcal{I} = [f \boxtimes_\mathcal{P} \mathrm{id}_{\mathcal{P},0}]_\mathcal{I} = [f]_\mathcal{I} = [\mathrm{id}_{\mathcal{P},0} \boxtimes_\mathcal{P} f]_\mathcal{I} = [\mathrm{id}_{\mathcal{P},0}]_\mathcal{I} \boxtimes [f]_\mathcal{I}.$$

Ad (C1). Suppose given $m, n, k, l \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$, $g \in \mathcal{P}(n,k)$ and $h \in \mathcal{P}(k,l)$. Then we have

$$\begin{aligned} [f]_\mathcal{I} \cdot ([g]_\mathcal{I} \cdot [h]_\mathcal{I}) &= [f]_\mathcal{I} \cdot [g \cdot_\mathcal{P} h]_\mathcal{I} \\ &= [f \cdot_\mathcal{P} (g \cdot_\mathcal{P} h)]_\mathcal{I} \\ &= [(f \cdot_\mathcal{P} g) \cdot_\mathcal{P} h]_\mathcal{I} \\ &= [f \cdot_\mathcal{P} g]_\mathcal{I} \cdot [h]_\mathcal{I} \\ &= ([f]_\mathcal{I} \cdot [g]_\mathcal{I}) \cdot [h]_\mathcal{I}. \end{aligned}$$

Ad (C2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$. Then we have

$$[f]_\mathcal{I} \cdot [\mathrm{id}_{\mathcal{P},n}]_\mathcal{I} = [f \cdot_\mathcal{P} \mathrm{id}_{\mathcal{P},n}]_\mathcal{I} = [f]_\mathcal{I} = [\mathrm{id}_{\mathcal{P},m} \cdot_\mathcal{P} f]_\mathcal{I} = [\mathrm{id}_{\mathcal{P},m}]_\mathcal{I} \cdot [f]_\mathcal{I}.$$

Ad (MC1). Suppose given $m, n, k, m', n', k' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m',n')$, $g \in \mathcal{P}(n,k)$ and $g' \in \mathcal{P}(n',k')$. Then we have

$$\begin{aligned} ([f]_\mathcal{I} \boxtimes [f']_\mathcal{I}) \cdot ([g]_\mathcal{I} \boxtimes [g']_\mathcal{I}) &= [f \boxtimes_\mathcal{P} f']_\mathcal{I} \cdot [g \boxtimes_\mathcal{P} g']_\mathcal{I} \\ &= \big[(f \boxtimes_\mathcal{P} f') \cdot_\mathcal{P} (g \boxtimes_\mathcal{P} g')\big]_\mathcal{I} \\ &= \big[(f \cdot_\mathcal{P} g) \boxtimes_\mathcal{P} (f' \cdot_\mathcal{P} g')\big]_\mathcal{I} \\ &= [f \cdot_\mathcal{P} g]_\mathcal{I} \boxtimes [f' \cdot_\mathcal{P} g']_\mathcal{I} \\ &= ([f]_\mathcal{I} \cdot [g]_\mathcal{I}) \boxtimes ([f']_\mathcal{I} \cdot [g']_\mathcal{I}). \end{aligned}$$

Ad (MC2). For $m \in \mathbb{Z}_{\geqslant 0}$ we have $[\mathrm{id}_{\mathcal{P},m}]_\mathcal{I} = \big[\,\mathrm{id}_{\mathcal{P},1}^{\boxtimes_\mathcal{P} m}\,\big]_\mathcal{I} = [\mathrm{id}_{\mathcal{P},1}]_\mathcal{I}^{\boxtimes m}$.

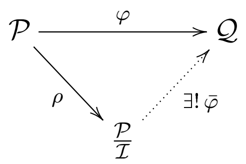
Definition 2.47. The residue class morphism $\rho := \rho_{\mathcal{I}} : \mathcal{P} \longrightarrow \frac{\mathcal{P}}{\mathcal{I}}$ is defined as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ we let

$$\begin{aligned} \rho(m,n) : \mathcal{P}(m,n) &\longrightarrow \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,n) \\ f &\longmapsto f + \mathcal{I}(m,n) = [f]_{\mathcal{I}} \,. \end{aligned}$$

This defines a morphism since the $\rho(m,n)$ are R- linear, map identity elements to identity elements and since the composition and multiplication on $\frac{\mathcal{P}}{\mathcal{I}}$ are defined using the composition and multiplication of representatives.

Lemma 2.48 (Universal property of the linear factor preoperad)**.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be linear preoperads over R. Let $\mathcal{I}$ be an ideal of $\mathcal{P}$ with generating set $X \subseteq \mathcal{P}$, i.e. $\mathcal{I} = {}_{\text{ideal}}\langle\, X \,\rangle$.*

Let $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of linear preoperads such that $x\varphi(m,n) = 0$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $x \in X(m,n)$. Then there exists a uniquely determined morphism of linear preoperads $\bar{\varphi} : \frac{\mathcal{P}}{\mathcal{I}} \longrightarrow \mathcal{Q}$ such that $\rho\bar{\varphi} = \varphi$.



Proof. Existence. Define $\bar{\varphi} = (\bar{\varphi}(m,n))_{m,n \geqslant 0}$ by

$$\begin{aligned} \bar{\varphi}(m,n) : \left(\frac{\mathcal{P}}{\mathcal{I}}\right)(m,n) &\longrightarrow \mathcal{Q}(m,n) \\ [f]_{\mathcal{I}} &\longmapsto f\varphi(m,n) \end{aligned}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$.

For $m, n \in \mathbb{Z}_{\geqslant 0}$ and $x \in X(m,n)$ we have $x\varphi(m,n) = 0$, hence $x \in \ker(\varphi)(m,n)$; cf. Example 2.42. So $\ker(\varphi)$ is an ideal in $\mathcal{P}$ with $X \subseteq \ker(\varphi)$. Hence we have ${}_{\text{ideal}}\langle\, X \,\rangle = \mathcal{I} \subseteq \ker(\varphi)$. So for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{I}(m,n)$ we have $f\varphi(m,n) = 0$. This shows that the biindexed map $\bar{\varphi}$ is well-defined.

Now we have to show that $\bar{\varphi}$ is a morphism of linear preoperads.

First note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have $[\mathrm{id}_{\mathcal{P},m}]_{\mathcal{I}}\bar{\varphi} = \mathrm{id}_{\mathcal{P},m}\,\varphi = \mathrm{id}_{\mathcal{Q},m}$.

Moreover, for $m, n, m', n', k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m',n')$ and $g \in \mathcal{P}(n,k)$ we have

$$\begin{aligned} \left([f]_{\mathcal{I}} \boxtimes [f']_{\mathcal{I}}\right)\bar{\varphi} &= [f \boxtimes_{\mathcal{P}} f']_{\mathcal{I}}\bar{\varphi} \\ &= \left(f \boxtimes_{\mathcal{P}} f'\right)\varphi \\ &= f\varphi \boxtimes_{\mathcal{Q}} f'\varphi \\ &= [f]_{\mathcal{I}}\bar{\varphi} \boxtimes_{\mathcal{Q}} [f']_{\mathcal{I}}\bar{\varphi} \end{aligned}$$

and

$$\begin{aligned} ([f]_{\mathcal{I}} \cdot [g]_{\mathcal{I}})\,\bar{\varphi} &= [f \cdot_{\mathcal{P}} g]_{\mathcal{I}}\bar{\varphi} \\ &= (f \cdot_{\mathcal{P}} g)\,\varphi \\ &= f\varphi \cdot_{\mathcal{Q}} g\varphi \\ &= [f]_{\mathcal{I}}\bar{\varphi} \cdot_{\mathcal{Q}} [g]_{\mathcal{I}}\bar{\varphi}. \end{aligned}$$

So $\bar{\varphi}$ is a morphism of linear preoperads. Furthermore, for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$ we have $f\,(\rho\bar{\varphi}) = [f]_{\mathcal{I}}\bar{\varphi} = f\varphi$. Hence $\rho\bar{\varphi} = \varphi$.

Uniqueness. Let $\chi : \frac{\mathcal{P}}{\mathcal{I}} \longrightarrow \mathcal{Q}$ be a morphism of linear preoperads such that $\rho\chi = \varphi$. Then for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$ we have $[f]_{\mathcal{I}}\bar{\varphi} = f\varphi = f\,(\rho\chi) = [f]_{\mathcal{I}}\chi$, hence $\bar{\varphi} = \chi$. □

Lemma 2.49. *Let $\mathcal{P}$ and $\mathcal{Q}$ be linear preoperads over R and let $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of linear preoperads. Recall that the kernel* $\ker(\varphi)$ *of φ is an ideal in $\mathcal{P}$; cf. Example* 2.42 *and that the image* $\mathrm{Im}(\varphi)$ *of φ is a linear-subpreoperad of $\mathcal{Q}$; cf. Example* 2.30 (1).

We have the isomorphism $\psi : \frac{\mathcal{P}}{\ker(\varphi)} \longrightarrow \mathrm{Im}(\varphi)$ defined by

$$\begin{aligned}\psi(m,n) : \Big(\frac{\mathcal{P}}{\ker\varphi}\Big)(m,n) &\longrightarrow \mathrm{Im}(\varphi)(m,n)\\ [f]_{\ker(\varphi)} &\longmapsto f\varphi\end{aligned}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Proof. Since $f\varphi(m,n) = 0$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in (\ker(\varphi))(m,n)$, by Lemma 2.48 there exists a uniquely determined morphism of linear preoperads $\bar{\varphi}$ such that $\rho\bar{\varphi} = \varphi$. It is defined by

$$\begin{aligned}\bar{\varphi}(m,n) : \Big(\frac{\mathcal{P}}{\ker(\varphi)}\Big)(m,n) &\longrightarrow \mathcal{Q}(m,n)\\ [f]_{\ker(\varphi)} &\longmapsto f\varphi(m,n).\end{aligned}$$

So given $[f]_{\ker(\varphi)} \in \Big(\frac{\mathcal{P}}{\ker\varphi}\Big)(m,n)$, we have $[f]_{\ker(\varphi)}\bar{\varphi}(m,n) \in \mathrm{Im}(\varphi)(m,n)$. So define $\psi := \bar{\varphi}\big|^{\mathrm{Im}(\varphi)}$. We know that $\psi : \frac{\mathcal{P}}{\ker(\varphi)} \longrightarrow \mathrm{Im}(\varphi)$ is a morphism of linear preoperads. We have to show that it is an isomorphism. By Lemma 2.20 it suffices to show that $\psi(m,n)$ is bijective for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Injectivity. Suppose given $[f]_{\ker(\varphi)}, [\tilde{f}]_{\ker(\varphi)} \in \frac{\mathcal{P}}{\ker(\varphi)}(m,n)$ with $[f]_{\ker(\varphi)}\psi(m,n) = [\tilde{f}]_{\ker(\varphi)}\psi(m,n)$. But that implies

$$f\varphi(m,n) = [f]_{\ker(\varphi)}\bar{\varphi}(m,n) = [f]_{\ker(\varphi)}\psi(m,n) = [\tilde{f}]\psi(m,n) = [\tilde{f}]_{\ker(\varphi)}\bar{\varphi}(m,n) = \tilde{f}\varphi(m,n),$$

hence $0 = f\varphi(m,n) - \tilde{f}\varphi(m,n) = (f - \tilde{f})\varphi(m,n)$. So we have $f - \tilde{f} \in \ker(\varphi)(m,n)$, hence $[f]_{\ker(\varphi)} = [\tilde{f}]_{\ker(\varphi)}$. This shows that $\psi(m,n)$ is injective.

Surjectivity. Suppose given $g \in \mathrm{Im}(\varphi)(m,n)$. Then there exists $f \in \mathcal{P}(m,n)$ with $g = f\varphi(m,n)$, hence $[f]_{\ker(\varphi)}\psi(m,n) = f\varphi(m,n) = g$. This shows that $\psi(m,n)$ is surjective. □

2.7 A comparison Lemma for congruences and ideals

Definition 2.50. Let $\mathcal{P}_0$ be a set-preoperad and $Y \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ be a biindexed subset. Define the biindexed subset $D_Y := (D_Y(m,n))_{m,n\geqslant 0} \subseteq R\mathcal{P}_0$ as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ let

$$D_Y(m,n) := \big\{f - \tilde{f} : (f, \tilde{f}) \in Y(m,n)\big\} \subseteq R\mathcal{P}_0\,.$$

Then we can define an ideal of $R\mathcal{P}_0$ by

$$\mathcal{I}_Y := {}_{\mathrm{ideal}}\langle\, D_Y \,\rangle \subseteq R\mathcal{P}_0\,.$$

Note that for biindexed subsets $Y, Z \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ with $Y \subseteq Z$ we have $D_Y \subseteq D_Z$, hence $\mathcal{I}_Y \subseteq \mathcal{I}_Z$.

In particular, given a biindexed subset $Y \subseteq \mathcal{P}_0 \times \mathcal{P}_0$, then $D_Y \subseteq D_{(\equiv_Y)}$, hence $\mathcal{I}_Y \subseteq \mathcal{I}_{(\equiv_Y)}$.

Definition 2.51. Let $\mathcal{P}_0$ be a set-preoperad and let $\mathcal{I} \subseteq R\mathcal{P}_0$ be an ideal. We define the biindexed subset $C_{\mathcal{I}} = (C_{\mathcal{I}}(m,n))_{m,n\geqslant 0} \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ let

$$C_{\mathcal{I}}(m,n) := \big\{(f, \tilde{f}) \in (\mathcal{P}_0 \times \mathcal{P}_0)(m,n) : f - \tilde{f} \in \mathcal{I}(m,n)\big\} \subseteq \mathcal{P}_0 \times \mathcal{P}_0.$$

Lemma 2.52. *The biindexed subset $C_{\mathcal{I}} \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ is a congruence on $\mathcal{P}_0$.*

Proof. Let $m, n \in \mathbb{Z}_{\geqslant 0}$. We show that $C_{\mathcal{I}}(m,n)$ is an equivalence relation on $\mathcal{P}_0(m,n)$.

For $f \in \mathcal{P}_0(m,n)$ we have $f - f = 0 \in \mathcal{I}(m,n)$, so $(f,f) \in C_{\mathcal{I}}(m,n)$. Moreover, for $f, \tilde{f} \in \mathcal{P}_0(m,n)$ with $(f,\tilde{f}) \in C_{\mathcal{I}}(m,n)$ we have $f - \tilde{f} \in \mathcal{I}(m,n)$, so we have $\tilde{f} - f = -(f - \tilde{f}) \in \mathcal{I}(m,n)$, hence $(\tilde{f}, f) \in C_{\mathcal{I}}(m,n)$. Furthermore, suppose given $f, \tilde{f}, \tilde{\tilde{f}} \in \mathcal{P}_0(m,n)$ with $(f,\tilde{f}), (\tilde{f},\tilde{\tilde{f}}) \in C_{\mathcal{I}}(m,n)$. This means that $f - \tilde{f}, \tilde{f} - \tilde{\tilde{f}} \in \mathcal{I}(m,n)$. So we have $f - \tilde{\tilde{f}} = (f - \tilde{f}) + (\tilde{f} - \tilde{\tilde{f}}) \in \mathcal{I}(m,n)$, hence $(f, \tilde{\tilde{f}}) \in C_{\mathcal{I}}(m,n)$.

Now we show that $C_{\mathcal{I}}$ is a congruence on $\mathcal{P}_0$.

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $(f,\tilde{f}) \in C_{\mathcal{I}}(m,n)$, $(f',\tilde{f}') \in C_{\mathcal{I}}(m',n')$. That is, we have $f - \tilde{f} \in \mathcal{I}(m,n)$ and $f' - \tilde{f}' \in \mathcal{I}(m',n')$. Then we have

$$(f \boxtimes f') - (\tilde{f} \boxtimes \tilde{f}') = (f \boxtimes f') - (\tilde{f} \boxtimes f') + (\tilde{f} \boxtimes f') - (\tilde{f} \boxtimes \tilde{f}') = \big((f - \tilde{f}) \boxtimes f'\big) + \big(\tilde{f} \boxtimes (f' - \tilde{f}')\big) \in \mathcal{I}(m+m', n+n'),$$

since $\mathcal{I}$ is an ideal in $R\mathcal{P}_0$. So we have $(f \boxtimes f', \tilde{f} \boxtimes \tilde{f}') \in C_{\mathcal{I}}(m+m', n+n')$.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $(f,\tilde{f}) \in C_{\mathcal{I}}(m,n)$, $(g,\tilde{g}) \in C_{\mathcal{I}}(n,k)$. So we have $f - \tilde{f} \in \mathcal{I}(m,n)$ and $g - \tilde{g} \in \mathcal{I}(n,k)$. Then we have

$$(f \cdot g) - (\tilde{f} \cdot \tilde{g}) = (f \cdot g) - (\tilde{f} \cdot g) + (\tilde{f} \cdot g) - (\tilde{f} \cdot \tilde{g}) = \big((f - \tilde{f}) \cdot g\big) + \big(\tilde{f} \cdot (g - \tilde{g})\big) \in \mathcal{I}(m,k),$$

since $\mathcal{I}$ is an ideal in $R\mathcal{P}_0$. So we have $(f \cdot g, \tilde{f} \cdot \tilde{g}) \in C_{\mathcal{I}}(m,k)$.

This shows that $C_{\mathcal{I}}$ is a congruence on $\mathcal{P}_0$. □

Lemma 2.53. *Let $\mathcal{P}_0$ be a set-preoperad and let $Y \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ be a biindexed subset. Then we have $\mathcal{I}_Y = \mathcal{I}_{(\equiv_Y)}$; cf. Definition* 2.50.

Proof. Since we already know that $\mathcal{I}_Y \subseteq \mathcal{I}_{(\equiv_Y)}$, we have to show that $\mathcal{I}_{(\equiv_Y)} \overset{!}{\subseteq} \mathcal{I}_Y$.

Moreover, since $\mathcal{I}_Y = {}_{\text{ideal}}\langle D_Y \rangle$ and $\mathcal{I}_{(\equiv_Y)} = {}_{\text{ideal}}\langle D_{(\equiv_Y)} \rangle$, it suffices to show that $D_{(\equiv_Y)} \overset{!}{\subseteq} \mathcal{I}_Y$, since then we have that $\mathcal{I}_Y$ is an ideal of $R\mathcal{P}_0$ containing $D_{(\equiv_Y)}$, so by the definition of the generated ideal we have $\mathcal{I}_{(\equiv_Y)} \subseteq \mathcal{I}_Y$; cf. Definition 2.44.

Let $m, n \in \mathbb{Z}_{\geqslant 0}$. First note that for $(f,\tilde{f}) \in Y(m,n)$ we have $f - \tilde{f} \in D_Y(m,n)$, so in particular $f - \tilde{f} \in \mathcal{I}_Y(m,n)$. So we have $(f,\tilde{f}) \in C_{\mathcal{I}_Y}(m,n)$. Hence we have $Y \subseteq C_{\mathcal{I}_Y}$. Since $(\equiv_Y)$ is the congruence on $\mathcal{P}_0$ generated by Y, we have $(\equiv_Y) \subseteq C_{\mathcal{I}_Y}$, since $C_{\mathcal{I}_Y}$ is also a congruence on $\mathcal{P}_0$ containing Y; cf. Lemma 2.52.

Now let $d \in D_{(\equiv_Y)}(m,n)$. So there exist $f, \tilde{f} \in \mathcal{P}_0(m,n)$ such that $d = f - \tilde{f}$ and $(f,\tilde{f}) \in (\equiv_Y)(m,n)$. Since $(\equiv_Y) \subseteq C_{\mathcal{I}_Y}$, we have $(f,\tilde{f}) \in C_{\mathcal{I}_Y}(m,n)$, hence $d = f - \tilde{f} \in \mathcal{I}_Y(m,n)$.

This shows that $D_{(\equiv_Y)} \subseteq \mathcal{I}_Y$ is a biindexed subset and completes the proof. □

Lemma 2.54 (Comparison Lemma)**.** *Let $\mathcal{P}_0$ be a set-preoperad and let $Y \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ be a biindexed subset. Recall that we have defined the biindexed subset $D_Y = (D_Y(m,n))_{m,n \geqslant 0} \subseteq R\mathcal{P}_0$ by*

$$D_Y(m,n) = \big\{ f - \tilde{f} : (f,\tilde{f}) \in Y(m,n) \big\}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$ and the ideal $\mathcal{I}_Y := {}_{\text{ideal}}\langle D_Y \rangle$ in $R\mathcal{P}_0$ in Definition 2.50.

Let $(\equiv_Y) \subseteq \mathcal{P}_0 \times \mathcal{P}_0$ be the congruence on $\mathcal{P}_0$ generated by Y.

Then there exists the isomorphism of linear preoperads

$$\begin{aligned} \chi_{\mathcal{P}_0,Y} : \qquad R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big) &\longrightarrow \frac{R\mathcal{P}_0}{\mathcal{I}_Y} \\ \sum_{f \in \mathcal{P}_0(m,n)} r_f\,[f]_{\equiv_Y} &\longmapsto \Big[\sum_{f \in \mathcal{P}_0(m,n)} r_f\, f\Big]_{\mathcal{I}_Y} = \sum_{f \in \mathcal{P}_0(m,n)} r_f\,[f]_{\mathcal{I}_Y} \end{aligned}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Proof. Note that by Lemma 2.53 we have $\mathcal{I}_Y = \mathcal{I}_{(\equiv_Y)}$. We abbreviate $\mathcal{I} := \mathcal{I}_Y = \mathcal{I}_{(\equiv_Y)}$. Furthermore, we denote by $[f]_Y := [f]_{\equiv_Y}$ the congruence class of $f \in \mathcal{P}_0(m,n)$ for $m,n \in \mathbb{Z}_{\geqslant 0}$ with respect to $(\equiv_Y)$.

Let $\varphi_0 : \mathcal{P}_0 \longrightarrow \frac{R\mathcal{P}_0}{\mathcal{I}}$ be the composed morphism of set-preoperads

$$\mathcal{P}_0 \xrightarrow{\beta_{\mathcal{P}_0}} R\mathcal{P}_0 \xrightarrow{\rho} \frac{R\mathcal{P}_0}{\mathcal{I}}\,; \qquad \varphi_0 = \beta_{\mathcal{P}_0}\rho$$

cf. Definition 2.22 and Definition 2.47.

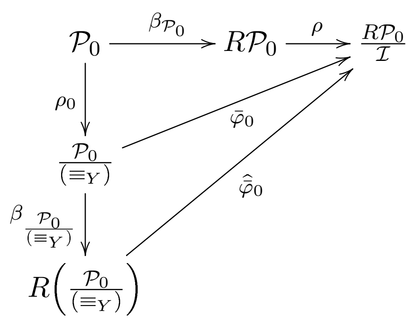
Suppose given $m,n \in \mathbb{Z}_{\geqslant 0}$ and $(f,\tilde{f}) \in Y(m,n)$. Then we have $f - \tilde{f} \in D_Y(m,n)$, hence in particular $f - \tilde{f} \in \mathcal{I}(m,n)$, i.e. $[f]_{\mathcal{I}} = [\tilde{f}]_{\mathcal{I}}$. So we have

$$f\varphi_0 = f\beta_{\mathcal{P}_0}\rho = f\rho = [f]_{\mathcal{I}} = [\tilde{f}]_{\mathcal{I}} = \tilde{f}\rho = \tilde{f}\beta_{\mathcal{P}_0}\rho = \tilde{f}\varphi_0\,.$$

So by the universal property of the factor set-preoperad there exists a uniquely determined morphism $\bar{\varphi}_0 : \frac{\mathcal{P}_0}{(\equiv_Y)} \longrightarrow \frac{R\mathcal{P}_0}{\mathcal{I}}$ of set-preoperads such that $\rho_0\bar{\varphi}_0 = \varphi_0$; cf. Lemma 2.40.

Since $\frac{R\mathcal{P}_0}{\mathcal{I}}$ is a linear preoperad and since $\bar{\varphi}_0 : \frac{\mathcal{P}_0}{(\equiv_Y)} \longrightarrow \frac{R\mathcal{P}_0}{\mathcal{I}}$ is a morphism of set-preoperads, by Lemma 2.23 there exists a uniquely determined morphism $\hat{\bar{\varphi}}_0 : R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big) \longrightarrow \frac{R\mathcal{P}_0}{\mathcal{I}}$ of linear preoperads such that $\beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}}\hat{\bar{\varphi}}_0 = \bar{\varphi}_0$.

So we have the following commutative diagram.



Note that for $m,n \in \mathbb{Z}_{\geqslant 0}$ and $\sum\limits_{f \in \mathcal{P}_0(m,n)} r_f\,[f]_Y \in \Big(R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big)\Big)(m,n)$ we have

$$\begin{aligned}
\Big(\sum_{f \in \mathcal{P}_0(m,n)} r_f\,[f]_Y\Big)\hat{\bar{\varphi}}_0 &\overset{2.23}{=} \sum_{f \in \mathcal{P}_0(m,n)} r_f\,([f]_Y\bar{\varphi}_0) \\
&\overset{2.40}{=} \sum_{f \in \mathcal{P}_0(m,n)} r_f\,(f\varphi_0) \\
&= \sum_{f \in \mathcal{P}_0(m,n)} r_f\,(f\beta_{\mathcal{P}_0}\rho) \\
&= \sum_{f \in \mathcal{P}_0(m,n)} r_f\,(f\rho) \\
&= \sum_{f \in \mathcal{P}_0(m,n)} r_f\,[f]_{\mathcal{I}} \\
&= \Big[\sum_{f \in \mathcal{P}_0(m,n)} r_f\,f\Big]_{\mathcal{I}},
\end{aligned}$$

so $\chi_{\mathcal{P}_0,Y} = \hat{\bar{\varphi}}_0$.

On the other hand, let $\psi_0 : \mathcal{P}_0 \longrightarrow R(\frac{\mathcal{P}_0}{(\equiv_Y)})$ be the composed morphism of set-preoperads

$$\begin{array}{ccccc} \mathcal{P}_0 & \xrightarrow{\ \rho_0\ } & \frac{\mathcal{P}_0}{(\equiv_Y)} & \xrightarrow{\ \beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}}\ } & R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big). \\ & & \underbrace{\hspace{8em}}_{\psi_0} & & \end{array}$$

Since $R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big)$ is a linear preoperad and since $\psi_0 : \mathcal{P}_0 \longrightarrow R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big)$ is a morphism of set-preoperads, by Lemma 2.23 there exists a uniquely determined morphism $\hat{\psi}_0 : R\mathcal{P}_0 \longrightarrow R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big)$ of linear preoperads such that $\beta_{\mathcal{P}_0}\hat{\psi}_0 = \psi_0$.

Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $x \in D_Y(m,n)$. This means that $x = f - \tilde{f}$ for some $(f, \tilde{f}) \in Y(m,n)$. In particular, we have $[f]_Y = [\tilde{f}]_Y$. Then we have

$$\begin{aligned} x\hat{\psi}_0 &= (f - \tilde{f})\hat{\psi}_0 \\ &= f\psi_0 - \tilde{f}\psi_0 \\ &= f(\rho_0\,\beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}}) - \tilde{f}(\rho_0\,\beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}}) \\ &= [f]_Y\beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}} - [\tilde{f}]_Y\beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}} \\ &= [f]_Y - [\tilde{f}]_Y \\ &= 0. \end{aligned}$$

So we have $D_Y \subseteq \ker(\psi_0)$. Since $\mathcal{I} = {}_{\text{ideal}}\langle\, D_Y \,\rangle$, we can apply Lemma 2.48.

Hence there exists a uniquely determined morphism of linear preoperads $\bar{\hat{\psi}}_0 : \frac{R\mathcal{P}_0}{\mathcal{I}} \longrightarrow R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big)$ such that $\rho\bar{\hat{\psi}}_0 = \hat{\psi}_0$.

So we have the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{P}_0 & \xrightarrow{\ \rho_0\ } & \frac{\mathcal{P}_0}{(\equiv_Y)} & \xrightarrow{\ \beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}}\ } & R\Big(\frac{\mathcal{P}_0}{(\equiv_Y)}\Big) \\ \downarrow{\scriptstyle \beta_{\mathcal{P}}} & & & \nearrow{\scriptstyle \hat{\psi}_0} & \\ R\mathcal{P}_0 & & & & \\ \downarrow{\scriptstyle \rho} & & \nearrow{\scriptstyle \bar{\hat{\psi}}_0} & & \\ \frac{R\mathcal{P}_0}{\mathcal{I}} & & & & \end{array}$$

Note that for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $\Big[\sum_{f \in \mathcal{P}_0(m,n)} r_f\, f\Big]_{\mathcal{I}} \in \Big(\frac{R\mathcal{P}_0}{\mathcal{I}}\Big)(m,n)$ we have

$$\begin{aligned} \Big[\sum_{f \in \mathcal{P}_0(m,n)} r_f\, f\Big]_{\mathcal{I}}\bar{\hat{\psi}}_0 &\overset{2.48}{=} \Big(\sum_{f \in \mathcal{P}_0(m,n)} r_f\, f\Big)\hat{\psi}_0 \\ &\overset{2.23}{=} \sum_{f \in \mathcal{P}_0(m,n)} r_f\,(f\psi_0) \\ &= \sum_{f \in \mathcal{P}_0(m,n)} r_f\big(f\rho_0\,\beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}}\big) \\ &= \sum_{f \in \mathcal{P}_0(m,n)} r_f\big([f]_Y\beta_{\frac{\mathcal{P}_0}{(\equiv_Y)}}\big) \\ &= \sum_{f \in \mathcal{P}_0(m,n)} r_f\,[f]_Y\,. \end{aligned}$$

So for $m, n \in \mathbb{Z}_{\geqslant 0}$ we have

$$\Big(\sum_{f\in\mathcal{P}_0(m,n)} r_f\,[f]_Y\Big)\widehat{\bar{\varphi}}_0\bar{\bar{\psi}}_0 = \Big[\sum_{f\in\mathcal{P}_0(m,n)} r_f\,f\Big]_{\mathcal{I}}\bar{\bar{\psi}}_0 = \sum_{f\in\mathcal{P}_0(m,n)} r_f\,[f]_Y$$

$$\Big[\sum_{f\in\mathcal{P}_0(m,n)} r_f\,f\Big]_{\mathcal{I}}\bar{\bar{\psi}}_0\widehat{\bar{\varphi}}_0 = \Big(\sum_{f\in\mathcal{P}_0(m,n)} r_f\,[f]_Y\Big)\widehat{\bar{\varphi}}_0 = \Big[\sum_{f\in\mathcal{P}_0(m,n)} r_f\,f\Big]_{\mathcal{I}},$$

hence $\widehat{\bar{\varphi}}_0\bar{\bar{\psi}}_0 = \mathrm{id}_{R\left(\frac{\mathcal{P}_0}{(\equiv_Y)}\right)}$ and $\bar{\bar{\psi}}_0\widehat{\bar{\varphi}}_0 = \mathrm{id}_{\frac{R\mathcal{P}_0}{\mathcal{I}}}$.

So $\chi_{\mathcal{P}_0,Y} = \widehat{\bar{\varphi}}_0 : R\left(\frac{\mathcal{P}_0}{(\equiv_Y)}\right) \longrightarrow \frac{R\mathcal{P}_0}{\mathcal{I}}$ is an isomorphism of linear preoperads. $\square$

2.8 Some basic examples

2.8.1 Some preoperads

We will now give basic examples of both set-preoperads and linear preoperads over R that will be needed later.

Definition 2.55. Let X be a set. Define the set-preoperad $\mathrm{End}_0(X)$ as follows.

- Let $\mathrm{End}_0(X)(m,n) := \big\{f : X^{\times m} \xrightarrow{f} X^{\times n} \text{ is a map}\big\}$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.
- Let $\mathrm{id}_m := \mathrm{id}_{\mathrm{End}_0,m} := \mathrm{id}_{V^{\times m}}$ for $m \in \mathbb{Z}_{\geqslant 0}$.
- Let the multiplication $(\boxtimes) := (\boxtimes_{\mathrm{End}_0})$ be the cartesian product of maps, that is, given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}_0(X)(m,n)$, $f' \in \mathrm{End}_0(X)(m',n')$ then for $x_1, \dots, x_{m+m'} \in X$ we have
$$(x_1, \dots x_{m+m'})(f \boxtimes f') := (x_1, \dots, x_m)f \times (x_{m+1}, \dots, x_{m+m'})f',$$
where $(x_1, \dots, x_m)f \times (x_{m+1}, \dots, x_{m+m'})f'$ is defined by joining the tuples, that is, given $y_1, \dots, y_{n+n'} \in X$, then $(y_1, \dots, y_n) \times (y_{n+1}, \dots, y_{n+n'}) := (y_1, \dots, y_n, y_{n+1}, \dots, y_{n+n'})$.
- Let the composition $(\cdot) := (\cdot_{\mathrm{End}_0})$ be given by the composition of maps, so given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}_0(X)(m,n)$, $g \in \mathrm{End}_0(X)(n,k)$, then for $x_1, \dots x_m \in X$ we have
$$(x_1, \dots, x_m)(f \cdot_{\mathrm{End}_0} g) = (x_1, \dots, x_m)(fg) = \big((x_1, \dots, x_m)f\big)g.$$

Now we will show that this is actually a set-preoperad.

Ad (m1). First note that joining tuples is associative. Suppose given $m, n, k, m', n', k' \in \mathbb{Z}_{\geqslant 0}$, $f \in \mathrm{End}_0(m,n)$, $f' \in \mathrm{End}_0(X)(m',n')$ and $f'' \in \mathrm{End}_0(X)(m'',n'')$. Then for $x_1, \dots, x_{m+m'+m''} \in X$ we have

$$\begin{aligned}
&(x_1, \dots, x_{m+m'+m''})\big((f \boxtimes f') \boxtimes f''\big)\\
&\quad= (x_1, \dots, x_{m+m'})(f \boxtimes f') \times (x_{m+m'+1}, \dots, x_{m+m'+m''})f''\\
&\quad= \big((x_1, \dots, x_m)f \times (x_{m+1}, \dots, x_{m+m'})f'\big) \times (x_{m+m'+1}, \dots, x_{m+m'+m''})f''\\
&\quad= (x_1, \dots, x_m)f \times \big((x_{m+1}, \dots, x_{m+m'})f' \times (x_{m+m'+1}, \dots, x_{m+m'+m''})f''\big)\\
&\quad= (x_1, \dots, x_m)f \times (x_{m+1}, \dots, x_{m+m'+m''})(f' \boxtimes f'')\\
&\quad= (x_1, \dots, x_{m+m'+m''})\big(f \boxtimes (f' \boxtimes f'')\big).
\end{aligned}$$

Ad (m2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}_0(X)(m,n)$. Then for $x_1, \ldots, x_m \in X$ we have

$$\begin{aligned}(x_1, \ldots, x_m)(f \boxtimes \mathrm{id}_0) &= \big((x_1, \ldots, x_m) \times (\,)\big)(f \boxtimes \mathrm{id}_0) \\ &= (x_1, \ldots, x_m)f \times (\,)\,\mathrm{id}_0 \\ &= (x_1, \ldots, x_m)f \times (\,) \\ &= (x_1, \ldots, x_m)f,\end{aligned}$$

where $(\,)$ is the empty tuple, the only element of $X^{\times 0}$.

In the same way we see that $(x_1, \ldots, x_m)(\mathrm{id}_0 \boxtimes f) = (x_1, \ldots, x_m)f$ for $x_1, \ldots, x_m \in X$. So we have $f \boxtimes \mathrm{id}_0 = f = \mathrm{id}_0 \boxtimes f$.

Ad (c1). The composition of maps is known to be associative.

Ad (c2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}_0(X)(m,n)$. Then for $x_1, \ldots, x_m \in X$ we have

$$\begin{aligned}(x_1, \ldots, x_m)(f \cdot \mathrm{id}_n) &= \big((x_1, \ldots, x_m)f\big)\,\mathrm{id}_n \\ &= (x_1, \ldots, x_m)f \\ &= \big((x_1, \ldots, x_m)\,\mathrm{id}_m\big)f \\ &= (x_1, \ldots, x_m)(\mathrm{id}_m \cdot f).\end{aligned}$$

Ad (mc1). Suppose given $m, n, k, m', n', k' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}_0(X)(m,n)$, $f' \in \mathrm{End}_0(X)(m',n')$, $g \in \mathrm{End}_0(X)(n,k)$ and $g' \in \mathrm{End}_0(n',k')$. For $x_1, \ldots, x_{m+m'} \in X$ we have

$$\begin{aligned}(x_1, \ldots, x_{m+m'})(f \boxtimes f') \cdot (g \boxtimes g') &= \big((x_1, \ldots x_m)f \times (x_{m+1}, \ldots, x_{m+m'})f'\big)(g \boxtimes g') \\ &= \big((x_1, \ldots, x_m)f\big)g \times \big((x_{m+1}, \ldots, x_{m+m'})f'\big)g' \\ &= (x_1, \ldots, x_m)(f \cdot g) \times (x_{m+1}, \ldots, x_{m+m'})(f' \cdot g') \\ &= (x_1, \ldots, x_{m+m'})\big((f \cdot g) \boxtimes (f' \cdot g')\big).\end{aligned}$$

Ad (mc2). Suppose given $m \in \mathbb{Z}_{\geqslant 0}$ and $x_1, \ldots, x_m \in X$. We have

$$(x_1, \ldots, x_m)\,\mathrm{id}_m = (x_1, \ldots, x_m) = ((x_1\,\mathrm{id}_1), \ldots, (x_m\,\mathrm{id}_1)) = (x_1, \ldots, x_m)(\mathrm{id}_1^{\boxtimes m}).$$

This shows that $\mathrm{End}_0(X)$ is in fact a set-preoperad.

We will also need a similar linear preoperad over R.

Definition 2.56. Let V be an R-module. Define the linear preoperad $\mathrm{End}(V)$ as follows.

- Let $\mathrm{End}(V)(m,n) := \mathrm{Hom}_R\,(V^{\otimes m}, V^{\otimes n})$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.
- Let $\mathrm{id}_m := \mathrm{id}_{\mathrm{End},m} := \mathrm{id}_{V^{\otimes m}}$ for $m \in \mathbb{Z}_{\geqslant 0}$.
- The multiplication $(\boxtimes_{\mathrm{End}}) := (\otimes)$ is given by the tensor product of R-linear maps, that is, given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}(V)(m,n)$, $f' \in \mathrm{End}(V)(m',n')$, we have

  $$(v_1 \otimes \ldots \otimes v_m \otimes v_{m+1} \otimes \ldots \otimes v_{m+m'})\,\big(f \otimes f'\big) = (v_1 \otimes \ldots \otimes v_m)\,f \otimes (v_{m+1} \otimes \ldots \otimes v_{m+m'})\,f'$$

  for $v_1, \ldots, v_{m+m'} \in V$. This defines an R-linear map.
- The composition $(\cdot_{\mathrm{End}}) := (\cdot)$ is given by the composition of maps, so given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}(V)(m,n)$, $g \in \mathrm{End}(V)(n,k)$, we have

  $$(v_1 \otimes \ldots \otimes v_m)(f \cdot_{\mathrm{End}} g) = (v_1 \otimes \ldots \otimes v_m)(fg) = ((v_1 \otimes \ldots \otimes v_m)f)g$$

  for $v_1, \ldots, v_m \in V$. This also defines an R-linear map.

We now have to show that this in fact is a linear preoperad over R.

Ad (M1). The tensor product $(\otimes)$ of R-linear maps is known to be associative.

Ad (M2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \operatorname{End}(V)(m,n)$. Since $V^{\otimes 0} = R$ we have

$$(v_1 \otimes \ldots \otimes v_m)\,(\mathrm{id}_{V^{\otimes 0}} \otimes f) = (v_1 \otimes \ldots \otimes v_m)\, f$$

for $v_1, \ldots, v_m \in V$, hence $\mathrm{id}_0 \otimes f = f$. In the same way we obtain $f \otimes \mathrm{id}_0 = f$.

Ad (C1). The composition of maps is known to be associative.

Ad (C2). Suppose given $m \in \mathbb{Z}_{\geqslant 0}$ and $f \in \operatorname{End}(V)(m,n)$. Then for $v_1, \ldots, v_m \in V$ we have

$$\begin{aligned}(v_1 \otimes \ldots \otimes v_m)\,(\mathrm{id}_m \cdot f) &= (v_1 \otimes \ldots \otimes v_m)\, f \\ &= \big((v_1 \otimes \ldots v_m) f\big)\, \mathrm{id}_n \\ &= (v_1 \otimes \ldots \otimes v_m)(f \cdot \mathrm{id}_n).\end{aligned}$$

So we have $\mathrm{id}_m \cdot f = f = f \cdot \mathrm{id}_n$.

Ad (MC1). Suppose given $m, n, k, m', n', k' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \operatorname{End}(V)(m,n)$, $f' \in \operatorname{End}(V)(m',n')$, $g \in \operatorname{End}(V)(n,k)$ and $g' \in \operatorname{End}(V)(n',k')$. Then for $v_1, \ldots, v_{m+m'} \in V$ we have

$$\begin{aligned}&(v_1 \otimes \ldots \otimes v_m \otimes v_{m+1} \otimes \ldots \otimes v_{m+m'})\big((f \otimes f') \cdot (g \otimes g')\big) \\ &\quad= \big((v_1 \otimes \ldots \otimes v_m \otimes v_{m+1} \otimes \ldots \otimes v_{m+m'})(f \otimes f')\big)(g \otimes g') \\ &\quad= \big((v_1 \otimes \ldots \otimes v_m) f \otimes (v_{m+1} \otimes \ldots \otimes v_{m+m'}) f'\big)(g \otimes g') \\ &\quad= \big((v_1 \otimes \ldots \otimes v_m) f\big) g \otimes \big((v_{m+1} \otimes \ldots \otimes v_{m+m'}) f'\big) g' \\ &\quad= (v_1 \otimes \ldots \otimes v_m)(f \cdot g) \otimes (v_{m+1} \otimes \ldots \otimes v_{m+m'})(f' \cdot g') \\ &\quad= (v_1 \otimes \ldots \otimes v_m \otimes v_{m+1} \otimes \ldots \otimes v_{m+m'})\big((f \cdot g) \otimes (f' \cdot g')\big).\end{aligned}$$

So $(f \otimes f') \cdot (g \otimes g') = (f \cdot g) \otimes (f' \cdot g')$.

Ad (MC2). For $m \in \mathbb{Z}_{\geqslant 0}$ and $v_1, \ldots, v_m$ we have

$$(v_1 \otimes \ldots \otimes v_m)\, \mathrm{id}_m = v_1 \otimes \ldots \otimes v_m = (v_1\, \mathrm{id}_1) \otimes \ldots \otimes (v_m\, \mathrm{id}_1) = (v_1 \otimes \ldots \otimes v_m)\, \mathrm{id}_1^{\otimes m} .$$

So $\mathrm{id}_m = \mathrm{id}_1^{\otimes m}$.

This shows that $\operatorname{End}(V)$ is in fact a linear preoperad.

We will also often use the following set-preoperad.

Definition 2.57. The set-preoperad Map_0 is defined as follows.

- Let $\mathrm{Map}_0(m,n) := \big\{ f : [1,m] \xrightarrow{f} [1,n] \text{ is a map} \big\}$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.
- Let $\mathrm{id}_m := \mathrm{id}_{\mathrm{Map}_0, m} := \mathrm{id}_{[1,m]}$, the identity map on $[1,m]$ for $m \in \mathbb{Z}_{\geqslant 0}$.
- The multiplication is given by

  $$\begin{aligned}(\boxtimes) := (\boxtimes_{\mathrm{Map}}) : \mathrm{Map}_0(m,n) \times \mathrm{Map}_0(m',n') &\longrightarrow \mathrm{Map}_0(m+m', n+n') \\ (f, f') &\longmapsto f \boxtimes f'\end{aligned}$$

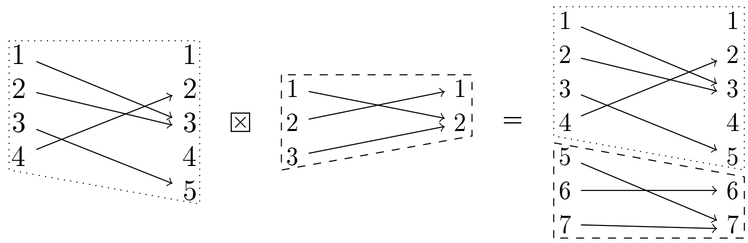
  for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$, where for $i \in [1, m+m']$ we have

  $$i\,\big(f \boxtimes f'\big) := \begin{cases} if & \text{if } i \in [1,m] \\ (i-m)f + n & \text{if } i \in [m+1, m+m']. \end{cases}$$

- The composition $(\cdot) := (\cdot_{\mathrm{Map}_0})$ is the composition of maps, so given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Map}_0(m,n)$, $g \in \mathrm{Map}_0(n,k)$ then for $i \in [1,m]$ we let

$$i(f \cdot_{\mathrm{Map}_0} g) = i(fg) = (if)g.$$

Informally, multiplication is given by stacking the maps and renumbering. Pictorially, we have e.g.



Now we will show that this defines a set-preoperad.

Note that for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$, $f \in \mathrm{Map}_0(m,n)$ and $f' \in \mathrm{Map}_0(m',n')$ and $i \in [1, m+m']$ we have

$$i(f \boxtimes f') \in [1,n] \Leftrightarrow i \in [1,m].$$

Ad (m1). Suppose given $m, n, m', n', m''n'' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Map}_0(m,n)$, $f' \in \mathrm{Map}_0(m',n')$ and $f'' \in \mathrm{Map}_0(m'',n'')$. Then for $i \in [1, m+m'+m'']$ we have

$$\begin{aligned}
i\big(f \boxtimes (f' \boxtimes f'')\big) &= \begin{cases} if & \text{if } i \in [1,m] \\ (i-m)(f' \boxtimes f'') + n & \text{if } i \in [m+1, m+m'+m''] \end{cases} \\
&= \begin{cases} if & \text{if } i \in [1,m] \\ (i-m)f' + n & \text{if } i-m \in [1,m'] \\ ((i-m)-m')f'' + n + n' & \text{if } i-m \in [m'+1, m'+m''] \end{cases} \\
&= \begin{cases} if & \text{if } i \in [1,m] \\ (i-m)f' + n & \text{if } i \in [m+1, m+m'] \\ (i-(m+m'))f'' + (n+n') & \text{if } i \in [m+m'+1, m+m'+m''] \end{cases} \\
&= \begin{cases} i(f \boxtimes f') & \text{if } i \in [1, m+m'] \\ (i-(m+m'))f'' + (n+n') & \text{if } i \in [m+m'+1, m+m'+m''] \end{cases} \\
&= i\big((f \boxtimes f') \boxtimes f''\big).
\end{aligned}$$

Ad (m2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Map}_0(m,n)$. Then for $i \in [1, m+0]$ we have

$$\begin{aligned}
i\,(\mathrm{id}_0 \boxtimes f) &= \begin{cases} i\,\mathrm{id}_0 & \text{if } i \in [1,0] = \varnothing \\ (i-0)f + 0 & \text{if } i \in [0+1, 0+m] \end{cases} \\
&= if \\
&= \begin{cases} if & \text{if } i \in [1,m] \\ (i-m)\,\mathrm{id}_0 + n & \text{if } i \in [m+1, m+0] = \varnothing \end{cases} \\
&= i\,(f \boxtimes \mathrm{id}_0)\,.
\end{aligned}$$

Ad (c1). The composition of maps is known to be associative.

Ad (c2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Map}_0(m,n)$ then for $i \in [1,m]$ we have

$$i(f \cdot \mathrm{id}_n) = (if)\,\mathrm{id}_n = if = (i\,\mathrm{id}_m)f = i(\mathrm{id}_m \cdot f).$$

Ad (mc1). Suppose given $m, n, k, m', n', k' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \operatorname{Map}_0(m, n)$, $f' \in \operatorname{Map}_0(m', n')$, $g \in \operatorname{Map}_0(n, k)$ and $g' \in \operatorname{Map}_0(n', k')$. Then for $i \in [1, m + m']$ we have

$$\begin{aligned}
i\left(f \boxtimes f'\right)\left(g \boxtimes g'\right) &= \begin{cases} i\,(f \boxtimes f')\,g & \text{if } i(f \boxtimes f') \in [1, n] \\ (i(f \boxtimes f') - n)g' + k & \text{if } i(f \boxtimes f') \in [n+1, n+n'] \end{cases} \\
&= \begin{cases} i\,(f \boxtimes f')\,g & \text{if } i \in [1, m] \\ (i(f \boxtimes f') - n)g' + k & \text{if } i \in [m+1, m+m'] \end{cases} \\
&= \begin{cases} (if)g & \text{if } i \in [1, m] \\ \big(((i-m)f' + n) - n\big)g' + k & \text{if } i \in [m+1, m+m'] \end{cases} \\
&= \begin{cases} i(fg) & \text{if } i \in [1, m] \\ (i-m)(f'g') + k & \text{if } i \in [m+1, m+m'] \end{cases} \\
&= i\big((fg) \boxtimes (f'g')\big).
\end{aligned}$$

Ad (mc2). We prove this via induction on $m \geqslant 0$. For $m = 0$ this is the definition. Now let $m \geqslant 1$ and assume that we already know $\mathrm{id}_{m-1} = \mathrm{id}_1^{\boxtimes(m-1)} \in \operatorname{Map}_0(m-1, m-1)$. Then for $i \in [1, m]$ we have

$$\begin{aligned}
i\,\mathrm{id}_1^{\boxtimes m} &= i\big(\,\mathrm{id}_1^{\boxtimes(m-1)} \boxtimes \mathrm{id}_1\,\big) \\
&= \begin{cases} i\,\mathrm{id}_1^{\boxtimes(m-1)} & \text{if } i \in [1, m-1] \\ (i - (m-1))\,\mathrm{id}_1 + (m-1) & \text{if } i = m \end{cases} \\
&= \begin{cases} i\,\mathrm{id}_{m-1} & \text{if } i \in [1, m-1] \\ (i - (m-1)) + (m-1) & \text{if } i = m \end{cases} \\
&= i = i\,\mathrm{id}_m\ .
\end{aligned}$$

This proves that Map_0 is a set-preoperad.

The next two examples are certain set-subpreoperads of the set-preoperad Map_0.

Definition 2.58. Define the set-subpreoperad Ass_0 of Map_0 by

$$\operatorname{Ass}_0(m, n) := \big\{f \in \operatorname{Map}_0(m, n) : f \text{ is monotone}\big\} \subseteq \operatorname{Map}_0(m, n)$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$.

We have to verify the axioms (s1)–(s3) from Lemma 2.27 to show that Ass_0 is a set-subpreoperad of Map_0 and hence a set-preoperad.

Ad (s1). We know that $\operatorname{Ass}_0(m, n) \subseteq \operatorname{Map}_0(m, n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Ad (s2). Suppose given $m \in \mathbb{Z}_{\geqslant 0}$. The identity map $\mathrm{id}_m = \mathrm{id}_{\operatorname{Map}_0, m} : [1, m] \longrightarrow [1, m]$ is monotone, hence $\mathrm{id}_m \in \operatorname{Ass}_0$.

Ad (s3). Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$, $f \in \operatorname{Ass}_0(m, n)$ and $f' \in \operatorname{Ass}_0(m', n')$. Then for $i, j \in [1, m + m']$ with $i < j$ we have

$$i\left(f \boxtimes f'\right) = \begin{cases} if & \text{if } i \in [1, m] \\ (i-m)f' + n & \text{if } i \in [m+1, m+m'] \end{cases}$$

$$j\left(f \boxtimes f'\right) = \begin{cases} jf & \text{if } j \in [1, m] \\ (j-m)f' + n & \text{if } j \in [m+1, m+m']. \end{cases}$$

Case 1: $j \in [1, m]$. Then, since $i < j$, we have $i \in [1, m]$. Now since f is monotone, we have

$$i(f \boxtimes f') = if \leqslant jf = j(f \boxtimes f').$$

Case 2: $j \in [m+1, m+m']$. There are two possibilities: either also $i \in [m+1, m+m']$ or $i \in [1, m]$. If $i \in [1, m]$, then

$$i(f \boxtimes f') = if \leqslant n \leqslant (j-m)f' + n = j(f \boxtimes f').$$

If $i, j \in [m+1, m+m']$, then $i - m < j - m$ and since f' is monotone we have

$$i(f \boxtimes f') = (i-m)f' + n \leqslant (j-m)f' + n = j(f \boxtimes f').$$

This shows that $i(f \boxtimes f') \leqslant j(f \boxtimes f')$ for $i \in [1, m+m']$. So $f \boxtimes f'$ is a monotone map, hence $f \boxtimes f' \in \mathrm{Ass}_0(m+m', n+n')$.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$, $f \in \mathrm{Ass}_0(m, n)$ and $g \in \mathrm{Ass}_0(n, k)$. Then $f \cdot_{\mathrm{Map}_0} g$ is also a monotone map, since for $i, j \in [1, m]$ with $i < j$ we have

$$i(fg) = (if)g \leqslant (jf)g = j(fg).$$

This shows (s3).

So Ass_0 is a set-subpreoperad of Map_0. Hence Ass_0 is a set-preoperad.

Definition 2.59. Define the set-subpreoperad Sym_0 of Map_0 by

$$\mathrm{Sym}_0(m, n) := \{f \in \mathrm{Map}_0(m, n) : f \text{ is bijective}\} \subseteq \mathrm{Map}_0(m, n)$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Again, to show that this is a set-preoperad we are going to verify the axioms (s1)–(s3) from Lemma 2.27 to show that Sym_0 is a set-subpreoperad of Map_0.

Ad (s1). We already know that $\mathrm{Sym}_0(m, n) \subseteq \mathrm{Map}_0(m, n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Ad (s2). Suppose given $m \in \mathbb{Z}_{\geqslant 0}$. The identity map $\mathrm{id}_m = \mathrm{id}_{\mathrm{Map}_0, m} : [1, m] \longrightarrow [1, m]$ is bijective, hence $\mathrm{id}_m \in \mathrm{Sym}_0(m, m)$.

Ad (s3). Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Sym}_0(m, n)$, $f' \in Sym_0(m', n')$. Since f and f' are bijective, there exist inverse maps $f^{-1} \in \mathrm{Map}_0(n, m)$ and $f'^{-1} \in \mathrm{Map}_0(n', m')$. That is, we have $ff^{-1} = \mathrm{id}_m$, $f^{-1}f = \mathrm{id}_n$, $f'f'^{-1} = \mathrm{id}_{m'}$ and $f'^{-1}f' = \mathrm{id}_{n'}$.

So by (mc2) for Map_0 we have

$$\begin{aligned}
(f \boxtimes f')(f^{-1} \boxtimes f'^{-1}) &= (ff^{-1}) \boxtimes (f'f'^{-1}) \\
&= \mathrm{id}_m \boxtimes \mathrm{id}_{m'} \\
&= \mathrm{id}_{m+m'} \\
(f^{-1} \boxtimes f'^{-1})(f \boxtimes f') &= (f^{-1}f) \boxtimes (f'^{-1}f') \\
&= \mathrm{id}_n \boxtimes \mathrm{id}_{n'} \\
&= \mathrm{id}_{n+n'},
\end{aligned}$$

so $f \boxtimes f'$ is bijective. Hence $f \boxtimes f' \in \mathrm{Sym}_0(m+m', n+n')$.

Now let $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Sym}_0(m, n)$, $g \in \mathrm{Sym}_0(n, k)$. The composite of bijective maps is known to be bijective. Hence $fg \in \mathrm{Sym}_0(m, k)$.

This shows that Sym_0 is a set-subpreoperad of Map_0 and hence a set-preoperad.

Definition 2.60. We define the following linear preoperads; cf. Remark 2.12.

- $\mathrm{Map} := R\,\mathrm{Map}_0$ with multiplication $(\boxtimes_{\mathrm{Map}}) := R(\boxtimes_{\mathrm{Map}_0})$ and composition $(\cdot_{\mathrm{Map}}) := R(\cdot_{\mathrm{Map}_0})$. Recall that this means that for $m, n, k, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $\sum\limits_{f \in \mathrm{Map}_0(m,n)} r_f\, f \in \mathrm{Map}(m, n)$, $\sum\limits_{f' \in \mathrm{Map}_0(m',n')} r'_{f'}\, f' \in \mathrm{Map}(m', n')$ and $\sum\limits_{g \in \mathrm{Map}_0(n,k)} s_g\, g \in \mathrm{Map}(n, k)$ we have

$$\Big(\sum_{f \in \mathrm{Map}_0(m,n)} r_f\, f\Big) \boxtimes_{\mathrm{Map}} \Big(\sum_{f' \in \mathrm{Map}_0(m',n')} r'_{f'}\, f'\Big) = \sum_{\substack{f \in \mathrm{Map}_0(m,n) \\ f' \in \mathrm{Map}_0(m',n')}} r_f\, r'_{f'} \big(f \boxtimes_{\mathrm{Map}_0} f'\big)$$
$$\Big(\sum_{f \in \mathrm{Map}_0(m,n)} r_f\, f\Big) \cdot_{\mathrm{Map}} \Big(\sum_{g \in \mathrm{Map}_0(n,k)} s_g\, g\Big) = \sum_{\substack{f \in \mathrm{Map}_0(m,n) \\ g \in \mathrm{Map}_0(n,k)}} r_f\, s_g \big(f \cdot_{\mathrm{Map}_0} g\big).$$

- $\mathrm{Ass} := R\,\mathrm{Ass}_0$ with multiplication $(\boxtimes_{\mathrm{Ass}}) := R(\boxtimes_{\mathrm{Ass}_0})$ and composition $(\cdot_{\mathrm{Ass}}) := R(\cdot_{\mathrm{Ass}_0})$.
- $\mathrm{Sym} := R\,\mathrm{Sym}_0$ with multiplication $(\boxtimes_{\mathrm{Sym}}) := R(\boxtimes_{\mathrm{Sym}_0})$ and composition $(\cdot_{\mathrm{Sym}}) := R(\cdot_{\mathrm{Sym}_0})$.

The linear preoperads Ass and Sym are linear subpreoperads of the linear preoperad Map.

2.8.2 Some morphisms of preoperads

Definition 2.61. Let X be a set. Define the biindexed map

$$\mathfrak{e}_0 = (\mathfrak{e}_0(m, n))_{m,n \geqslant 0} : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{End}_0(X)$$

as follows. Given $f^{\mathrm{op}} \in \mathrm{Map}_0(m, n)$ (corresponding to a map $f : [1, n] \longrightarrow [1, m]$), we define

$$\begin{aligned} f^{\mathrm{op}}\mathfrak{e}_0(m, n) : \qquad X^{\times m} &\longrightarrow X^{\times n} \\ (x_1, \ldots, x_m) &\longmapsto (x_{1f}, \ldots, x_{nf}). \end{aligned}$$

Lemma 2.62. *The biindexed map* $\mathfrak{e}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{End}_0(X)$ *is a morphism of set-preoperads.*

Proof. First note that for $m \in \mathbb{Z}_{\geqslant 0}$ and $x_1, \ldots, x_m \in X$ we have

$$\begin{aligned} (x_1, \ldots, x_m)(\mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}},m}\, \mathfrak{e}_0) &= (x_1, \ldots, x_m)\big((\mathrm{id}_{\mathrm{Map}_0,m})^{\mathrm{op}}\mathfrak{e}_0\big) \\ &= (x_{1\,\mathrm{id}_{\mathrm{Map}_0,m}}, \ldots, x_{m\,\mathrm{id}_{\mathrm{Map}_0,m}}) \\ &= (x_1, \ldots, x_m). \end{aligned}$$

Hence $\mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}\, \mathfrak{e}_0 = \mathrm{id}_{X^{\times m}} = \mathrm{id}_{\mathrm{End}_0,m}$.

Now suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m, n)$, $f'^{\,\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m', n')$. Then for $x_1, \ldots, x_{m+m'} \in X$ by defining $y_i := x_{i+m}$ for $i \in [1, m']$ we have

$$\begin{aligned} (x_1, \ldots, x_{m+m'})(f^{\mathrm{op}} \boxtimes_{\mathrm{Map}_0^{\mathrm{op}}} f'^{\,\mathrm{op}})\mathfrak{e}_o &= (x_1, \ldots, x_{m+m'})(f \boxtimes_{\mathrm{Map}_0} f')^{\mathrm{op}}\mathfrak{e}_0 \\ &= (x_{1(f \boxtimes_{\mathrm{Map}_0} f')}, \ldots, x_{(n+n')(f \boxtimes_{\mathrm{Map}_0} f')}) \\ &= (x_{1f}, \ldots, x_{nf}, x_{(n+1-n)f'+m}, \ldots, x_{(n+n'-n)f'+m}) \\ &= (x_{1f}, \ldots, x_{nf}, x_{1f'+m}, \ldots, x_{n'f'+m}) \\ &= (x_{1f}, \ldots, x_{nf}, y_{1f'}, \ldots, y_{n'f'}) \\ &= (x_1, \ldots, x_m)(f^{\mathrm{op}}\mathfrak{e}_0) \times (y_1, \ldots, y_{m'})(f'^{\,\mathrm{op}}\mathfrak{e}_0) \\ &= (x_1, \ldots, x_m, y_1, \ldots, y_{m'})(f^{\mathrm{op}}\mathfrak{e}_0 \boxtimes_{\mathrm{End}_0} f'^{\,\mathrm{op}}\mathfrak{e}_0) \\ &= (x_1, \ldots, x_{m+m'})(f^{\mathrm{op}}\mathfrak{e}_0 \boxtimes_{\mathrm{End}_0} f'^{\,\mathrm{op}}\mathfrak{e}_0). \end{aligned}$$

So we have $(f^{\mathrm{op}} \boxtimes_{\mathrm{Map}_0^{\mathrm{op}}} f'^{\,\mathrm{op}})\mathfrak{e}_0 = (f^{\mathrm{op}}\mathfrak{e}_0) \boxtimes_{\mathrm{End}_0} (f'^{\,\mathrm{op}}\mathfrak{e}_0)$.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m, n)$, $g^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(n, k)$.

Then for $x_1, \ldots, x_m \in S$ by defining $y_i := x_{if}$ for $i \in [1, n]$ we have

$$\begin{aligned}
(x_1, \ldots, x_m)(f^{\mathrm{op}}\mathfrak{e}_0) \cdot_{\mathrm{End}_0} (g^{\mathrm{op}}\mathfrak{e}_0) &= (x_{1f}, \ldots, x_{nf})(g^{\mathrm{op}}\mathfrak{e}_0) \\
&= (y_1, \ldots, y_n)(g^{\mathrm{op}}\mathfrak{e}_0) \\
&= (y_{1g}, \ldots, y_{kg}) \\
&= (x_{(1g)f}, \ldots, x_{(kg)f}) \\
&= (x_{1(gf)}, \ldots, x_{k(gf)}) \\
&= (x_1, \ldots, x_m)((g \cdot_{\mathrm{Map}_0} f)^{\mathrm{op}}\mathfrak{e}_0) \\
&= (x_1, \ldots, x_m)\big((f^{\mathrm{op}} \cdot_{\mathrm{Map}_0^{\mathrm{op}}} g^{\mathrm{op}})\mathfrak{e}_0\big).
\end{aligned}$$

So we have $(f^{\mathrm{op}}\mathfrak{e}_0) \cdot_{\mathrm{End}_0} (g^{\mathrm{op}}\mathfrak{e}_0) = (f^{\mathrm{op}} \cdot_{\mathrm{Map}_0^{\mathrm{op}}} g^{\mathrm{op}})\mathfrak{e}_0$.

This shows that $\mathfrak{e}_0$ is a morphism of set-preoperads. □

Example 2.63. Suppose given a set X. Let $(1,2) \in \mathrm{Sym}_0(2,2)$ be the transposition. Then $(1,2)^{\mathrm{op}}\mathfrak{e}_0 \in \mathrm{End}_0(X)(2,2)$ is the map

$$\begin{aligned}
(1,2)^{\mathrm{op}}\mathfrak{e}_0 : \quad X^{\times 2} &\longrightarrow X^{\times 2} \\
(x_1, x_2) &\longmapsto (x_{1(1,2)}, x_{2(1,2)}) = (x_2, x_1).
\end{aligned}$$

Definition 2.64. Let V be an R-module. Define the biindexed map

$$\mathfrak{e} = (\mathfrak{e}(m,n))_{m,n \geqslant 0} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{End}(V)$$

as follows. Using linear extension, it suffices to define $\mathfrak{e}$ by the images of elements $f^{\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)$ for $m \in \mathbb{Z}_{\geqslant 0}$ since $\mathrm{Sym}^{\mathrm{op}} = R\,\mathrm{Sym}_0^{\mathrm{op}}$.

So given $m \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)$ we define

$$\begin{aligned}
f^{\mathrm{op}}\mathfrak{e}(m,m) : \qquad V^{\otimes m} &\longrightarrow V^{\otimes m} \\
v_1 \otimes \ldots \otimes v_m &\longmapsto v_{1f} \otimes \ldots \otimes v_{mf} .
\end{aligned}$$

We need to verify that this map is well-defined. In order to do this, consider the corresponding map

$$\begin{aligned}
\widehat{f^{\mathrm{op}}\mathfrak{e}}(m,m) : \qquad V^{\times m} &\longrightarrow V^{\otimes m} \\
(v_1, \ldots, v_m) &\longmapsto v_{1f} \otimes \ldots \otimes v_{mf}
\end{aligned}$$

for $m \in \mathbb{Z}_{\geqslant 0}$. By Lemma 1.2 it suffices to show that this map is R-multilinear.

Let $i \in [1,m]$ and $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m \in V$. Let $k \in \mathbb{Z}_{\geqslant 0}$ and $v_{i,j} \in V$ and $r_j \in R$ for $j \in [1,k]$. We have to show that we have

$$\Big(v_1, \ldots, v_{i-1}, \sum_{j \in [1,k]} r_j v_{i,j}, v_{i+1}, \ldots, v_m\Big)\big(\widehat{f^{\mathrm{op}}\mathfrak{e}}\big) \stackrel{!}{=} \sum_{j \in [1,k]} r_j\,(v_1, \ldots, v_{i-1}, v_{i,j}, v_{i+1}, \ldots, v_m)\big(\widehat{f^{\mathrm{op}}\mathfrak{e}}\big).$$

Define $v_i := \sum_{j \in [1,k]} r_j v_{i,j}$. Since f is bijective, there exists a unique $l \in [1,m]$ such that $lf = i$, in particular, $uf \in [1,m]\backslash\{i\}$ for $u \in [1,m]\backslash\{l\}$.

So we have

$$\begin{aligned}
&\Big(v_1, \ldots, v_{i-1}, \sum_{j\in[1,k]} r_j v_{i,j}, v_{i+1}, \ldots, v_m\Big)\big(\widehat{f^{\mathrm{op}}\mathfrak{e}}\big)\\
&\qquad = (v_1, \ldots, v_m)\big(\widehat{f^{\mathrm{op}}\mathfrak{e}}\big)\\
&\qquad = v_{1f} \otimes \ldots \otimes v_{(l-1)f} \otimes v_{lf} \otimes v_{(l+1)f} \otimes \ldots \otimes v_{mf}\\
&\qquad = v_{1f} \otimes \ldots \otimes v_{(l-1)f} \otimes v_i \otimes v_{(l+1)f} \otimes \ldots \otimes v_{mf}\\
&\qquad = v_{1f} \otimes \ldots \otimes v_{(l-1)f} \otimes \Big(\sum_{j\in[1,k]} r_j v_{i,j}\Big) \otimes v_{(l+1)f} \otimes \ldots \otimes v_{mf}\\
&\qquad = \sum_{j\in[1,k]} r_j \left(v_{1f} \otimes \ldots \otimes v_{(l-1)f} \otimes v_{i,j} \otimes v_{(l+1)f} \otimes \ldots \otimes v_{mf}\right)\\
&\qquad = \sum_{j\in[1,k]} r_j \left(v_{1f} \otimes \ldots \otimes v_{(l-1)f} \otimes v_{lf,j} \otimes v_{(l+1)f} \otimes \ldots \otimes v_{mf}\right)\\
&\qquad = \sum_{j\in[1,k]} r_j \left(v_1, \ldots, v_{i-1}, v_{i,j}, v_{i+1}, \ldots, v_m\right)\big(\widehat{f^{\mathrm{op}}\mathfrak{e}}\big).
\end{aligned}$$

This shows that $(\widehat{f^{\mathrm{op}}\mathfrak{e}})(m,m)$ is a multilinear map for $m \in \mathbb{Z}_{\geqslant 0}$, hence $(f^{\mathrm{op}}\mathfrak{e})(m,m)$ is a well-defined linear map for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Lemma 2.65. *The biindexed map* $\mathfrak{e} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{End}(V)$ *is a morphism of linear preoperads.*

Proof. Recall that by Lemma 2.23 it suffices to show that the restricted map $\mathrm{Sym}_0^{\mathrm{op}} \longrightarrow \mathrm{End}(V)$ is a morphism of set-preoperads.

First note that for $m \in \mathbb{Z}_{\geqslant 0}$ and $v_1, \ldots, v_m \in V$ we have

$$\begin{aligned}
(v_1 \otimes \ldots \otimes v_m)(\mathrm{id}_{\mathrm{Sym}_0^{\mathrm{op}},m}\,\mathfrak{e}) &= (v_1 \otimes \ldots \otimes v_m)\big((\mathrm{id}_{\mathrm{Sym}_0,m})^{\mathrm{op}}\mathfrak{e}\big)\\
&= v_{1\,\mathrm{id}_{\mathrm{Sym}_0,m}} \otimes \ldots \otimes v_{m\,\mathrm{id}_{\mathrm{Sym}_0,m}}\\
&= v_1 \otimes \ldots \otimes v_m,
\end{aligned}$$

hence $\mathrm{id}_{\mathrm{Sym}_0^{\mathrm{op}},m}\,\mathfrak{e} = \mathrm{id}_{V^{\otimes m}} = \mathrm{id}_{\mathrm{End}(V),m}$.

Now suppose given $m, m' \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)$, $f'^{\,\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m',m')$.

Then for $v_1, \ldots, v_{m+m'} \in V$ by defining $w_i := v_{i+m}$ for $i \in [1, m']$ we have

$$\begin{aligned}
(v_1 \otimes \ldots \otimes v_{m+m'})(f^{\mathrm{op}} \boxtimes_{\mathrm{Sym}_0^{\mathrm{op}}} f'^{\,\mathrm{op}})\mathfrak{e} &= (v_1 \otimes \ldots \otimes v_{m+m'})(f \boxtimes_{\mathrm{Sym}_0} f')^{\mathrm{op}}\mathfrak{e}\\
&= v_{1(f \boxtimes_{\mathrm{Sym}_0} f')} \otimes \ldots \otimes v_{(m+m')(f \boxtimes_{\mathrm{Sym}_0} f')}\\
&= v_{1f} \otimes \ldots \otimes v_{nf} \otimes v_{(m+1-m)f'+m} \otimes \ldots \otimes v_{(m+m'-m)f'+m}\\
&= v_{1f} \otimes \ldots \otimes v_{mf} \otimes v_{1f'+m} \otimes \ldots \otimes v_{m'f'+m}\\
&= v_{1f} \otimes \ldots \otimes v_{mf} \otimes w_{1f'} \otimes \ldots \otimes w_{m'f'}\\
&= (v_1 \otimes \ldots \otimes v_m)f^{\mathrm{op}}\mathfrak{e} \otimes (w_1 \otimes \ldots \otimes w_{m'})f'^{\,\mathrm{op}}\mathfrak{e}\\
&= (v_1 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots \otimes w_{m'})(f^{\mathrm{op}}\mathfrak{e} \otimes f'^{\,\mathrm{op}}\mathfrak{e})\\
&= (v_1 \otimes \ldots \otimes v_{m+m'})(f^{\mathrm{op}}\mathfrak{e} \boxtimes_{\mathrm{End}} f'^{\,\mathrm{op}}\mathfrak{e}).
\end{aligned}$$

So $(f^{\mathrm{op}} \boxtimes_{\mathrm{Sym}_0^{\mathrm{op}}} f'^{\,\mathrm{op}})\mathfrak{e} = f^{\mathrm{op}}\mathfrak{e} \otimes f'^{\,\mathrm{op}}\mathfrak{e} = f^{\mathrm{op}}\mathfrak{e} \boxtimes_{\mathrm{End}} f'^{\,\mathrm{op}}\mathfrak{e}$.

Now suppose given $m \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}}, g^{\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)$. Then for $v_1, \ldots, v_m \in V$ by defining $w_i := v_{if}$ for $i \in [1,m]$ we have

$$\begin{aligned}
(v_1 \otimes \ldots \otimes v_m)(f^{\mathrm{op}}\mathfrak{e}) \cdot_{\mathrm{End}} (g^{\mathrm{op}}\mathfrak{e}) &= (v_{1f} \otimes \ldots \otimes v_{mf})(g^{\mathrm{op}}\mathfrak{e}) \\
&= (w_1 \otimes \ldots \otimes w_m)(g^{\mathrm{op}}\mathfrak{e}) \\
&= w_{1g} \otimes \ldots \otimes w_{mg} \\
&= v_{(1g)f} \otimes \ldots \otimes v_{(mg)f} \\
&= v_{1(gf)} \otimes \ldots \otimes v_{m(gf)} \\
&= (v_1 \otimes \ldots \otimes v_m)(g \cdot_{\mathrm{Sym}_0} f)^{\mathrm{op}}\mathfrak{e} \\
&= (v_1 \otimes \ldots \otimes v_m)(f^{\mathrm{op}} \cdot_{\mathrm{Sym}_0^{\mathrm{op}}} g^{\mathrm{op}})\mathfrak{e}.
\end{aligned}$$

So $(f^{\mathrm{op}} \cdot_{\mathrm{Sym}_0^{\mathrm{op}}} g^{\mathrm{op}})\mathfrak{e} = f^{\mathrm{op}}\mathfrak{e} \cdot_{\mathrm{End}} g^{\mathrm{op}}\mathfrak{e}$.

Hence by Lemma 2.23 the biindexed map $\mathfrak{e} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{End}(V)$ is a morphism of linear preoperads. $\square$

Example 2.66. Suppose given an R-module V. Let $(1,2) \in \mathrm{Sym}_0(2,2)$ be the transposition. Then $(1,2)^{\mathrm{op}}\mathfrak{e} \in \mathrm{End}(V)(2,2)$ is the R-linear map defined by

$$\begin{aligned}
(1,2)^{\mathrm{op}}\mathfrak{e} : \quad V^{\otimes 2} &\longrightarrow V^{\otimes 2} \\
v_1 \otimes v_2 &\longmapsto v_{1(1,2)} \otimes v_{2(1,2)} = v_2 \otimes v_1
\end{aligned}$$

for $v_1, v_2 \in V$.

3 Absolute operads

The original concept of an operad is not the same as the preoperads described above. In this section we will give the usual definition of a (non-symmetric) operad and show how an operad (which we will call an absolute operad) can be obtained from a preoperad.

Definition 3.1. An absolute operad $\mathcal{P}^{\mathrm{abs}}$ in the category of R-modules consists of

- R-modules $\mathcal{P}^{\mathrm{abs}}(m)$ for $m \in \mathbb{Z}_{\geqslant 0}$,
- an element $I \in \mathcal{P}^{\mathrm{abs}}(1)$ called *unit*,
- R-linear composition maps

$$\left(\underset{i}{\circ}\right) : \mathcal{P}^{\mathrm{abs}}(m) \otimes \mathcal{P}^{\mathrm{abs}}(n) \longrightarrow \mathcal{P}^{\mathrm{abs}}(m+n-1)$$
$$f \otimes g \longmapsto f \underset{i}{\circ} g$$

  for $i \in [1, m]$ and $m, n \in \mathbb{Z}_{\geqslant 0}$,

such that for $m, n, k \in \mathbb{Z}_{\geqslant 0}$, $f \in \mathcal{P}^{\mathrm{abs}}(m)$, $g \in \mathcal{P}^{\mathrm{abs}}(n)$ and $h \in \mathcal{P}^{\mathrm{abs}}(k)$ the following axioms hold.

(A1) We have $\left(f \underset{i}{\circ} g\right) \underset{j+n-1}{\circ} h = \left(f \underset{j}{\circ} h\right) \underset{i}{\circ} g$ for $i, j \in [1, m]$ with $i < j$.

(A2) We have $\left(f \underset{i}{\circ} g\right) \underset{j+i-1}{\circ} h = f \underset{i}{\circ} \left(g \underset{j}{\circ} h\right)$ for $i \in [1, m]$ and $j \in [1, n]$.

(U) We have $I \underset{1}{\circ} f = f = f \underset{i}{\circ} I$ for $i \in [1, m]$;

cf. **[1,** 1.1**]** or **[12,** Definition II.1.16 **]** ("pseudo-operad").

Lemma 3.2. *Let* $(\mathcal{P}, \boxtimes, \cdot)$ *be a linear preoperad over* R. *Define* $\mathcal{P}^{\mathrm{abs}}(m) := \mathcal{P}(m, 1)$ *for* $m \in \mathbb{Z}_{\geqslant 0}$. *Then* $\mathcal{P}^{\mathrm{abs}}$ *together with* $I := \mathrm{id}_{\mathcal{P},1}$ *and the composition maps*

$$\left(\underset{i}{\circ}\right) : \mathcal{P}^{\mathrm{abs}}(m) \otimes \mathcal{P}^{\mathrm{abs}}(n) \longrightarrow \mathcal{P}^{\mathrm{abs}}(m+n-1)$$
$$f \otimes g \longmapsto f \underset{i}{\circ} g := (\mathrm{id}_{\mathcal{P},i-1} \boxtimes g \boxtimes \mathrm{id}_{\mathcal{P},m-i}) \cdot f$$

is an absolute operad as in Definition 3.1.

Proof. Ad (A1). Suppose given $i, j \in [1, m]$ with $i < j$. Then we have

$$\begin{aligned}
\left(f \underset{i}{\circ} g\right) \underset{j+n-1}{\circ} h &= (\mathrm{id}_{j+n-2} \boxtimes h \boxtimes \mathrm{id}_{m+n-1-(j+n-1)}) \cdot (f \underset{i}{\circ} g) \\
&= (\mathrm{id}_{j+n-2} \boxtimes h \boxtimes \mathrm{id}_{m-j}) \cdot (f \underset{i}{\circ} g) \\
&= (\mathrm{id}_{j+n-2} \boxtimes h \boxtimes \mathrm{id}_{m-j}) \cdot (\mathrm{id}_{i-1} \boxtimes g \boxtimes \mathrm{id}_{m-i}) \cdot f \\
&\overset{\text{(MC2)}}{=} (\mathrm{id}_{i-1} \boxtimes \mathrm{id}_n \boxtimes \mathrm{id}_{j-i-1} \boxtimes h \boxtimes \mathrm{id}_{m-j}) \cdot (\mathrm{id}_{i-1} \boxtimes g \boxtimes \mathrm{id}_{j-i-1} \boxtimes \mathrm{id}_1 \boxtimes \mathrm{id}_{m-j}) \cdot f \\
&\overset{\text{(MC1)}}{=} (\mathrm{id}_{i-1} \boxtimes (\mathrm{id}_n \cdot g) \boxtimes \mathrm{id}_{j-i-1} \boxtimes (h \cdot \mathrm{id}_1) \boxtimes \mathrm{id}_{m-j}) \cdot f \\
&\overset{\text{(C2)}}{=} (\mathrm{id}_{i-1} \boxtimes (g \cdot \mathrm{id}_1) \boxtimes \mathrm{id}_{j-i-1} \boxtimes (\mathrm{id}_k \cdot h) \boxtimes \mathrm{id}_{m-j}) \cdot f \\
&\overset{\text{(MC1)}}{=} (\mathrm{id}_{i-1} \boxtimes g \boxtimes \mathrm{id}_{j-i-1} \boxtimes \mathrm{id}_k \boxtimes \mathrm{id}_{m-j}) \cdot (\mathrm{id}_{i-1} \boxtimes \mathrm{id}_1 \boxtimes \mathrm{id}_{j-i-1} \boxtimes h \boxtimes \mathrm{id}_{m-j}) \cdot f \\
&\overset{\text{(MC2)}}{=} (\mathrm{id}_{i-1} \boxtimes g \boxtimes \mathrm{id}_{m+k-1-i}) \cdot (\mathrm{id}_{j-1} \boxtimes h \boxtimes \mathrm{id}_{m-j}) \cdot f \\
&= (\mathrm{id}_{i-1} \boxtimes g \boxtimes \mathrm{id}_{m+k-1-i}) \cdot (f \underset{j}{\circ} h) \\
&= \left(f \underset{j}{\circ} h\right) \underset{i}{\circ} g.
\end{aligned}$$

Ad (A2). Let $i \in [1,m]$ and $j \in [1,n]$. Then we have

$$\begin{aligned}
\Big(f \underset{i}{\circ} g\Big) \underset{j+i-1}{\circ} h &= (\mathrm{id}_{j+i-2} \boxtimes h \boxtimes \mathrm{id}_{m+n-1-(j+i-1)}) \cdot (f \underset{i}{\circ} g) \\
&= (\mathrm{id}_{j+i-2} \boxtimes h \boxtimes \mathrm{id}_{m+n-i-j}) \cdot (f \underset{i}{\circ} g) \\
&= (\mathrm{id}_{j+i-2} \boxtimes h \boxtimes \mathrm{id}_{m+n-i-j}) \cdot (\mathrm{id}_{i-1} \boxtimes g \boxtimes \mathrm{id}_{m-i}) \cdot f \\
&\overset{\text{(MC2)}}{=} (\mathrm{id}_{i-1} \boxtimes \mathrm{id}_{j-1} \boxtimes h \boxtimes \mathrm{id}_{n-j} \boxtimes \mathrm{id}_{m-i}) \cdot (\mathrm{id}_{i-1} \boxtimes g \boxtimes \mathrm{id}_{m-i}) \cdot f \\
&\overset{\text{(MC1)}}{=} (\mathrm{id}_{i-1} \boxtimes ((\mathrm{id}_{j-1} \boxtimes h \boxtimes \mathrm{id}_{n-j}) \cdot g) \boxtimes \mathrm{id}_{m-i}) \cdot f \\
&= (\mathrm{id}_{i-1} \boxtimes (g \underset{j}{\circ} h) \boxtimes \mathrm{id}_{m-i}) \cdot f \\
&= f \underset{i}{\circ} \Big(g \underset{j}{\circ} h\Big).
\end{aligned}$$

Ad (U). We have

$$I \underset{1}{\circ} f = f \cdot I = f \cdot \mathrm{id}_1 \overset{\text{(C2)}}{=} f$$

and for $i \in [1,m]$ we have

$$f \underset{i}{\circ} I = (\mathrm{id}_{i-1} \boxtimes \mathrm{id}_1 \boxtimes \mathrm{id}_{m-i}) \cdot f \overset{\text{(MC2)}}{=} \mathrm{id}_n \cdot f \overset{\text{(C2)}}{=} f.$$

□

Definition 3.3. Let $\mathcal{P}^{\mathrm{abs}}$ and $\mathcal{Q}^{\mathrm{abs}}$ be absolute operads. A *morphism* $\varphi^{\mathrm{abs}} : \mathcal{P}^{\mathrm{abs}} \longrightarrow \mathcal{Q}^{\mathrm{abs}}$ *of absolute operads* consists of R-linear maps $\varphi^{\mathrm{abs}}(m) : \mathcal{P}^{\mathrm{abs}}(m) \longrightarrow \mathcal{Q}^{\mathrm{abs}}(m)$ such that (1) and (2) hold.

(1) We have $I_{\mathcal{P}^{\mathrm{abs}}}\varphi(1) = I_{\mathcal{Q}^{abs}}$.

(2) We have $\Big(f \underset{i}{\circ} g\Big)\varphi^{\mathrm{abs}}(m+n-1) = (f\varphi^{\mathrm{abs}}(m)) \underset{i}{\circ} (g\varphi^{\mathrm{abs}}(n))$ for $m,n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}^{\mathrm{abs}}(m)$, $g \in \mathcal{P}^{\mathrm{abs}}(n)$ and $i \in [1,m]$;

cf. **[3,** §4**]** or **[11,** Definition 6**]**, called "homomorphism" in the latter.

We write $\varphi^{\mathrm{abs}} = (\varphi^{\mathrm{abs}}(m))_{m \geqslant 0} : \mathcal{P}^{\mathrm{abs}} \longrightarrow \mathcal{Q}^{\mathrm{abs}}$ for the morphism.

For $m \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}^{\mathrm{abs}}(m)$ we will also write $f\varphi^{\mathrm{abs}} := f\varphi^{\mathrm{abs}}(m)$ if it is clear which map $\varphi^{\mathrm{abs}}(m)$ is involved.

Lemma 3.4. *Let $\mathcal{P}, \mathcal{Q}$ be linear preoperads over R and let $\varphi = (\varphi(m,n))_{m,n \geqslant 0} : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of linear preoperads. Define $\varphi^{\mathrm{abs}}(m) := \varphi(m,1)$ for $m \in \mathbb{Z}_{\geqslant 0}$.*

Then $\varphi^{\mathrm{abs}} = (\varphi^{\mathrm{abs}}(m))_{m \geqslant 0} : \mathcal{P}^{\mathrm{abs}} \longrightarrow \mathcal{Q}^{\mathrm{abs}}$ is a morphism of absolute operads as in Definition 3.3.

Proof. Since φ is a morphism of linear preoperads we have

$$I_{\mathcal{P}^{\mathrm{abs}}}\varphi^{\mathrm{abs}}(1) = \mathrm{id}_{\mathcal{P},1}\,\varphi(1,1) = \mathrm{id}_{\mathcal{Q},1} = I_{\mathcal{Q}^{\mathrm{abs}}}.$$

Suppose given $m,n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}^{\mathrm{abs}}(m) = \mathcal{P}(m,1)$, $g \in \mathcal{P}^{\mathrm{abs}}(n) = \mathcal{P}(n,1)$ and $i \in [1,m]$. Then we have

$$\begin{aligned}
\Big(f \underset{i}{\circ} g\Big)\varphi^{\mathrm{abs}} &= (f \underset{i}{\circ} g)\varphi \\
&= \Big((\mathrm{id}_{\mathcal{P},i-1} \boxtimes g \boxtimes \mathrm{id}_{\mathcal{P}.m-i})f\Big)\varphi \\
&= \Big((\mathrm{id}_{\mathcal{P}.i-1} \boxtimes g \boxtimes \mathrm{id}_{\mathcal{P},m-i})\varphi\Big) \cdot (f\varphi) \\
&= \Big((\mathrm{id}_{\mathcal{P},i-1}\,\varphi) \boxtimes (g\varphi) \boxtimes (\mathrm{id}_{\mathcal{P},m-i}\,\varphi)\Big) \cdot (f\varphi) \\
&= (\mathrm{id}_{\mathcal{Q},i-1} \boxtimes (g\varphi^{\mathrm{abs}}) \boxtimes \mathrm{id}_{\mathcal{Q},m-i}) \cdot (f\varphi^{\mathrm{abs}}) \\
&= (f\varphi^{\mathrm{abs}}) \underset{i}{\circ} (g\varphi^{\mathrm{abs}}).
\end{aligned}$$

Note that during this calculation φ had to be applied to terms that do not appear in the absolute operad. $\square$

Remark 3.5.

(1) Let $\mathcal{P}$ be a linear preoperad over R and $\mathrm{id}_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P}$ be the identity morphism. Then $\mathrm{id}_{\mathcal{P}}^{\mathrm{abs}} : \mathcal{P}^{\mathrm{abs}} \longrightarrow \mathcal{P}^{\mathrm{abs}}$ is the identity morphism of absolute operads.

(2) Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be linear preoperads over R. Let $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ and $\psi : \mathcal{Q} \longrightarrow \mathcal{R}$ be morphisms of linear preoperads. Then we have $(\varphi\psi)^{\mathrm{abs}} = \varphi^{\mathrm{abs}}\psi^{\mathrm{abs}}$.

This means that

$$\begin{aligned} (-)^{\mathrm{abs}} : (\text{linear preoperads}) &\longrightarrow (\text{absolute operads}) \\ \mathcal{P} &\longrightarrow \mathcal{P}^{\mathrm{abs}} \\ \varphi &\longrightarrow \varphi^{\mathrm{abs}} \end{aligned}$$

is a functor from the category of linear preoperads to the category of absolute operads.

4 The free preoperad

Our aim in this chapter will be to define the free set-preoperad $\mathrm{Free}_0(X)$ on a biindexed set X. By construction, the set-preoperad $\mathrm{Free}_0(X)$ will contain equivalence classes of words which are built from letters being elements of X, formally multiplied with identities on both sides.

4.1 Operations on words

For all of §4.1, let (X, s, t) be a biindexed set.

Recall that this means that X is a set and that $s : X \longrightarrow \mathbb{Z}_{\geqslant 0}$ and $t : X \longrightarrow \mathbb{Z}_{\geqslant 0}$ are maps. Recall that we write $X(m,n) = \{x \in X : xs = m \text{ and } xt = n\}$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $X = (X(m,n))_{m,n \geqslant 0}$. Note that for $x \in X$ we have $x \in X(xs, xt)$.

Furthermore, for $x \in X$ we write $xd := xt - xs \in \mathbb{Z}$.

Definition 4.1. We define

$$\begin{aligned}\mathrm{Words}(X) := \{q_0(l_1,x_1,r_1)q_1(l_2,x_2,r_2)q_2\cdots q_{k-1}(l_k,x_k,r_k)q_k : {} & k \in \mathbb{Z}_{\geqslant 0}\,,\\ & l_i, r_i \in \mathbb{Z}_{\geqslant 0}\,,\ x_i \in X \text{ for } i \in [1,k],\\ & q_{i-1} = l_i + x_i s + r_i \text{ and } q_i = l_i + x_i t + r_i \text{ for } i \in [1,k]\}.\end{aligned}$$

Note that for $w = q_0(l_1,x_1,r_1)q_1(l_2,x_2,r_2)q_2\cdots q_{k-1}(l_k,x_k,r_k)q_k \in \mathrm{Words}(X)$ we have

$$l_i + x_i t + r_i = q_i = l_{i+1} + x_{i+1}s + r_{i+1}$$

for $i \in [1, k-1]$.

Given $w = q_0(l_1,x_1,r_1)q_1(l_2,x_2,r_2)q_2\cdots q_{k-1}(l_k,x_k,r_k)q_k \in \mathrm{Words}(X)$, we say that w has *length* k. Note that words of length 0 are of the form $w = q_0$ for $q_0 \in \mathbb{Z}_{\geqslant 0}$.

Furthermore, we extend the maps $s : X \longrightarrow \mathbb{Z}_{\geqslant 0}$ and $t : X \longrightarrow \mathbb{Z}_{\geqslant 0}$ on $\mathrm{Words}(X)$ as follows. For $w = q_0(l_1,x_1,r_1)q_1(l_2,x_2,r_2)q_2\cdots q_{k-1}(l_k,x_k,r_k)q_k \in \mathrm{Words}(X)$ we let $ws = q_0$ and $wt = q_k$. So if $k \geqslant 1$ we have $ws = q_0 = l_1 + x_1 s + r_1$ and $wt = q_k = l_k + x_k t + r_k$. We have the biindexed set $(\mathrm{Words}(X), s, t)$.

Remark 4.2. We have the biindexed map

$$\begin{aligned} X &\longrightarrow \mathrm{Words}(X)\\ x &\longmapsto xs(0,x,0)xt. \end{aligned}$$

We can define a composition on the biindexed set $\mathrm{Words}(X)$ as follows:

$$\begin{aligned}(\cdot) : \mathrm{Words}(X)(m,n) \times \mathrm{Words}(X)(n,p) &\longrightarrow \mathrm{Words}(X)(m,p)\\ (w,v) &\longmapsto wv\end{aligned}$$

for $m, n, p \in \mathbb{Z}_{\geqslant 0}$, where for $w = q_0(l_1,x_1,r_1)q_1\cdots q_{k-1}(l_k,x_k,r_k)q_k \in \mathrm{Words}(X)$ such that $q_0 = m$ and $q_k = n$ and for $v = p_0(\lambda_1,y_1,\rho_1)p_1\cdots p_{\kappa-1}(\lambda_\kappa,y_\kappa,\rho_\kappa)p_\kappa \in \mathrm{Words}(X)$ such that $p_0 = n$ and $p_\kappa = p$ we have

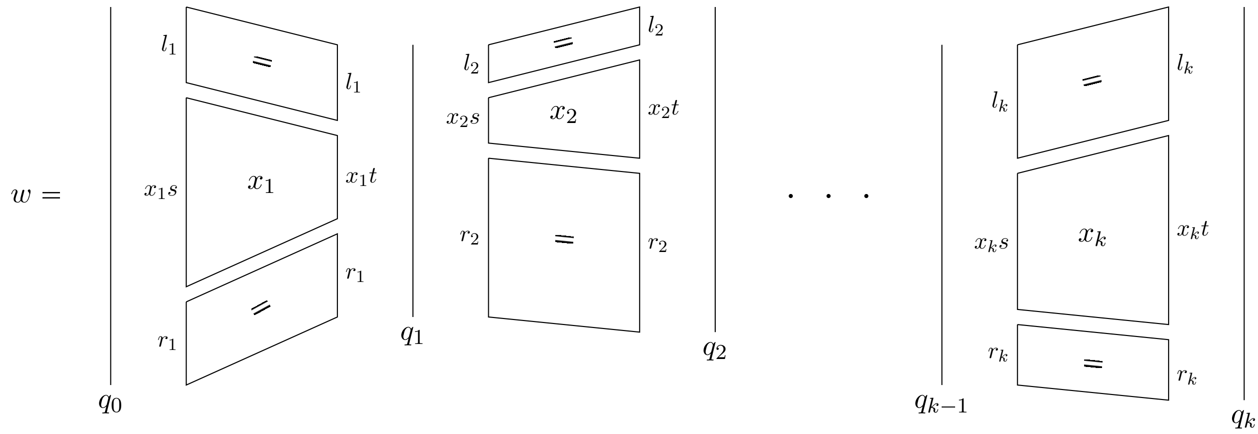
$$wv = q_0(l_1,x_1,r_1)q_1\cdots q_{k-1}(l_k,x_k,r_k)q_k(\lambda_1,y_1,\rho_1)p_1\cdots p_{\kappa-1}(\lambda_\kappa,y_\kappa,\rho_\kappa)p_\kappa\,.$$

Note that $q_k = n = p_0$.

Remark 4.3. Note that this composition is associative, i.e. given $w, v, u \in \text{Words}(X)$ such that $wt = vs$ and $vt = us$, then we have

$$(wv)u = w(vu).$$

We will often illustrate such words as follows.



Remark 4.4. Suppose given a word $w = q_0(l_1, x_1, r_1)q_1 \cdots q_{k-1}(l_k, x_k, r_k)q_k \in \text{Words}(X)$. Then for $i \in [1, k]$ we have

$$\begin{aligned} q_i - q_{i-1} &= (l_i + x_i t + r_i) - (l_i + x_i s + r_i) \\ &= x_t - x_s \\ &= x_i d \end{aligned}$$

and for $i, j \in [0, k]$ with $i < j$ we have

$$q_j - q_i = \sum_{u \in [i+1, j]} (q_u - q_{u-1}) = \sum_{u \in [i+1, j]} x_u d.$$

Now consider the following relation on $\text{Words}(X)$.

Definition 4.5 (Elementary equivalence of words)**.** Let

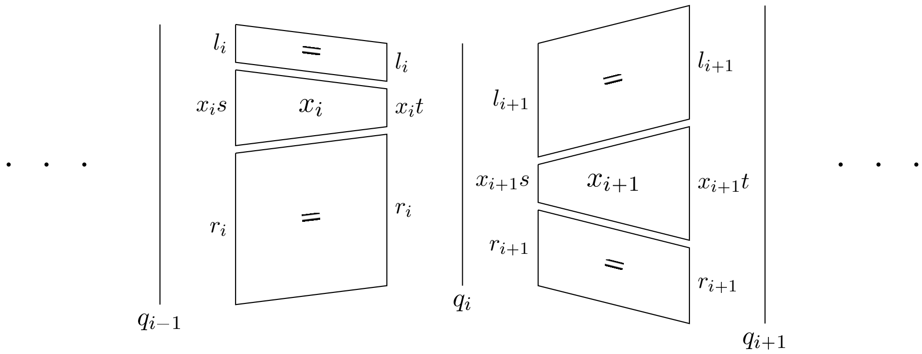
$$\begin{aligned} w &= q_0\,(l_1, x_1, r_1)\,q_1 \cdots q_{k-1}\,(l_k, x_k, r_k)\,q_k \in \text{Words}(X) \\ \tilde{w} &= \tilde{q}_0(\tilde{l}_1, \tilde{x}_1, \tilde{r}_1)\tilde{q}_1 \ldots \tilde{q}_{\tilde{k}-1}(\tilde{l}_{\tilde{k}}, \tilde{x}_{\tilde{k}}, \tilde{r}_{\tilde{k}})\tilde{q}_{\tilde{k}} \in \text{Words}(X) \end{aligned}$$

be two words. We say that w is *elementarily equivalent* to $\tilde{w}$, written $w \sim \tilde{w}$, if $k = \tilde{k}$ and if there exists $i \in [1, k-1]$ such that

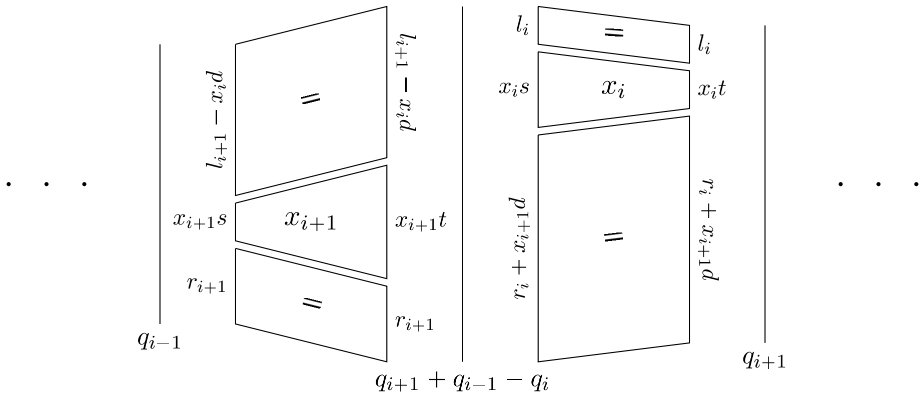
- $l_i + x_i t \leqslant l_{i+1}$
- $l_j = \tilde{l}_j$, $r_j = \tilde{r}_j$ and $x_j = \tilde{x}_j$ for $j \in [1, k] \setminus \{i, i+1\}$
- $q_j = \tilde{q}_j$ for $j \in [0, k] \setminus \{i\}$
- $\tilde{l}_i = l_{i+1} - x_i d$, $\tilde{x}_i = x_{i+1}$ and $\tilde{r}_i = r_{i+1}$
- $\tilde{l}_{i+1} = l_i$, $\tilde{x}_{i+1} = x_i$ and $\tilde{r}_{i+1} = r_i + x_{i+1} d$
- $\tilde{q}_i = q_{i+1} + q_{i-1} - q_i$.

Note that if $k \in \{0,1\}$, then $w \nsim \tilde{w}$.

Pictorially, we have that the word w described by



is elementarily equivalent to the word $\tilde{w}$ described by



where the dotted regions of w and $\tilde{w}$ coincide.

Remark 4.6. *Let* $w = q_0\,(l_1, x_1, r_1)\,q_1 \cdots q_{k-1}\,(l_k, x_k, r_k)\,q_k \in \mathrm{Words}(X)$. *Suppose that there exists* $i \in [1, k-1]$ *such that* $l_i + x_i t \leqslant l_{i+1}$. *Define*

- $\tilde{l}_j := l_j$, $\tilde{r}_j := r_j$ *and* $\tilde{x}_j := x_j$ *for* $j \in [1,k] \setminus \{i, i+1\}$
- $\tilde{q}_j := q_j$ *for* $j \in [0,k] \setminus \{i\}$
- $\tilde{l}_i := l_{i+1} - x_i d$, $\tilde{x}_i := x_{i+1}$ *and* $\tilde{r}_i := r_{i+1}$
- $\tilde{l}_{i+1} := l_i$, $\tilde{x}_{i+1} := x_i$ *and* $\tilde{r}_{i+1} := r_i + x_{i+1}d$
- $\tilde{q}_i := q_{i+1} + q_{i-1} - q_i$.

Then $\tilde{w} := \tilde{q}_0(\tilde{l}_1, \tilde{x}_1, \tilde{r}_1)\tilde{q}_1 \ldots \tilde{q}_{k-1}(\tilde{l}_k, \tilde{x}_k, \tilde{r}_k)\tilde{q}_k \in \mathrm{Words}(X)$ *and* $w \sim \tilde{w}$.

Proof. We have to show that $\tilde{w} \in \mathrm{Words}(X)$; cf. Definition 4.1.

First note that $\tilde{l}_i = l_{i+1} - x_i d = l_{i+1} - x_i t + x_i s \geqslant l_i + x_i s \geqslant 0$ and that

$$\begin{aligned}
\tilde{r}_{i+1} &= r_i + x_{i+1}d \\
&= r_i + x_{i+1}t - x_{i+1}s \\
&= (q_i - l_i - x_i t) + x_{i+1}t - x_{i+1}s \\
&= r_{i+1} + x_{i+1}s + l_{i+1} - l_i - x_i t + x_{i+1}t - x_{i+1}s \\
&= r_{i+1} + (l_{i+1} - l_i - x_i t) + x_{i+1}t \\
&\geqslant r_{i+1} + x_{i+1}t \geqslant 0.
\end{aligned}$$

Moreover, for $j \in [1,k] \setminus \{i, i+1\}$ we have

$$\tilde{l}_j + \tilde{x}_j t + \tilde{r}_j = l_j + x_j t + r_j = q_j = \tilde{q}_j \,.$$

Finally, we have

$$\begin{aligned}
\tilde{l}_i + \tilde{x}_i s + \tilde{r}_i &= l_{i+1} - x_i d + x_{i+1} s + r_{i+1} \\
&= q_i - x_i d \\
&= q_i - (q_i - q_{i-1}) \\
&= q_{i-1} \\
&= \tilde{q}_{i-1} \\
\tilde{l}_i + \tilde{x}_i t + \tilde{r}_i &= l_{i+1} - x_i d + x_{i+1} t + r_{i+1} \\
&= l_{i+1} + x_{i+1} s + r_{i+1} - x_{i+1} s - x_i d + x_{i+1} t \\
&= q_i + x_{i+1} d - x_i d \\
&= q_i + (q_{i+1} - q_i) - (q_i - q_{i-1}) \\
&= q_{i+1} + q_{i-1} - q_i \\
&= \tilde{q}_i \\
\tilde{l}_{i+1} + \tilde{x}_{i+1} s + \tilde{r}_{i+1} &= l_i + x_i s + r_i + x_{i+1} d \\
&= l_i + x_i t + r_i - x_i t + x_i s + x_{i+1} d \\
&= q_i - x_i d + x_{i+1} d \\
&= q_i - (q_i - q_{i-1}) + (q_{i+1} - q_i) \\
&= q_{i+1} + q_{i-1} - q_i \\
&= \tilde{q}_i \\
\tilde{l}_{i+1} + \tilde{x}_{i+1} t + \tilde{r}_{i+1} &= l_i + x_i t + r_i + x_{i+1} d \\
&= q_i + x_{i+1} d \\
&= q_i + (q_{i+1} - q_i) \\
&= q_{i+1} \\
&= \tilde{q}_{i+1} \,.
\end{aligned}$$

Thus, $\tilde{w} \in \mathrm{Words}(X)$. By construction, we have $w \sim \tilde{w}$. □

When dealing with elementary equivalence of words we will often omit those parts of the words that coincide, writing

$$\begin{aligned}
&\big(\cdots q_{i-1}(l_i, x_i, r_i) q_i (l_{i+1}, x_{i+1}, r_{i+1}) q_{i+1} \cdots \big) \\
\sim &\big(\cdots q_{i-1}(l_{i+1} - x_i d, x_{i+1}, r_{i+1})(q_{i+1} + q_{i-1} - q_i)(l_i, x_i, r_i + x_{i+1} d) q_{i+1} \cdots \big).
\end{aligned}$$

Remark 4.7. *Let $w = q_0\,(l_1, x_1, r_1)\,q_1 \cdots q_{k-1}\,(l_k, x_k, r_k)\,q_k \in \mathrm{Words}(X)$. Suppose thet there exists $i \in [1, k-1]$ such that $l_i \geqslant l_{i+1} + x_{i+1} s$. Define*

- $\tilde{l}_j := l_j$, $\tilde{x}_j := x_j$ *and* $\tilde{r}_j := r_j$ *for* $j \in [1, k-1] \setminus \{i, i+1\}$
- $\tilde{q}_j := q_j$ *for* $j \in [0, k] \setminus \{i\}$
- $\tilde{l}_i := l_{i+1}$, $\tilde{x}_i := x_{i+1}$ *and* $\tilde{r}_i := r_{i+1} - x_i d$
- $\tilde{l}_{i+1} := l_i + x_{i+1} d$, $\tilde{x}_{i+1} := x_i$ *and* $\tilde{r}_{i+1} := r_i$
- $q'_i := q_{i-1} + q_{i+1} - q_i$.

Then $\tilde{w} := \tilde{q}_0(\tilde{l}_1, \tilde{x}_1, \tilde{r}_1)\tilde{q}_1 \cdots \tilde{q}_{k-1}(\tilde{l}_k, \tilde{x}_k, \tilde{r}_k)\tilde{q}_k \in \mathrm{Words}(X)$ *and* $\tilde{w} \sim w$.

Proof. We have to show that $\tilde{w} \in \mathrm{Words}(X)$; cf. Definition 4.1.

First note that $\tilde{l}_{i+1} = l_i + x_{i+1}d \geqslant l_{i+1} + x_{i+1}s + x_{i+1}d = l_{i+1} + x_{i+1}t = \tilde{l}_i + \tilde{x}_{i+1}t \geqslant 0$ and that

$$\begin{aligned}
\tilde{r}_i &= r_{i+1} - x_i d \\
&= r_{i+1} - x_i t + x_i s \\
&= (q_i - l_{i+1} - x_{i+1}s) - x_i t + x_i s \\
&= l_i + x_i t + r_i - l_{i+1} - x_{i+1}s - x_i t + x_i s \\
&= (l_i - l_{i+1} - x_{i+1}s) + r_i + x_i s \\
&\geqslant r_i + x_i s \\
&\geqslant 0.
\end{aligned}$$

Moreover, for $j \in [1,k] \setminus \{i, i+1\}$ we have

$$\tilde{l}_j + \tilde{x}_j t + \tilde{r}_j = l_j + x_j t + r_j = q_j = \tilde{q}_j\,.$$

Finally, we have

$$\begin{aligned}
\tilde{l}_i + \tilde{x}_i s + \tilde{r}_i &= (l_{i+1} + x_{i+1}s + r_{i+1}) - x_i d \\
&= (l_i + x_i t + r_i) - (x_i t - x_i s) \\
&= l_i + x_i s + r_i \\
&= q_{i-1} \\
&= \tilde{q}_{i-1} \\
\tilde{l}_i + \tilde{x}_i t + \tilde{r}_i &= l_{i+1} + x_{i+1}t + r_{i+1} - x_i d \\
&= l_{i+1} + x_{i+1}s + r_{i+1} + x_{i+1}d - x_i d \\
&= q_i + (q_{i+1} - q_i) - (q_i - q_{i-1}) \\
&= q_{i+1} + q_{i-1} - q_i \\
&= \tilde{q}_i \\
\tilde{l}_{i+1} + \tilde{x}_{i+1}s + \tilde{r}_{i+1} &= l_i + x_{i+1}d + x_i s + r_i \\
&= l_i + x_i t + r_i - x_i d + x_{i+1}d \\
&= q_i - (q_i - q_{i-1}) + (q_{i+1} - q_i) \\
&= q_{i+1} + q_{i-1} - q_i \\
&= \tilde{q}_i \\
\tilde{l}_{i+1} + \tilde{x}_{i+1}t + \tilde{r}_{i+1} &= l_i + x_{i+1}d + x_i t + r_i \\
&= (l_i + x_i t + r_i) + (x_{i+1}t - x_{i+1}s) \\
&= (l_{i+1} + x_{i+1}s + r_{i+1}) + (x_{i+1}t - x_{i+1}s) \\
&= l_{i+1} + x_{i+1}t + r_{i+1} \\
&= q_{i+1} \\
&= \tilde{q}_{i+1}\,.
\end{aligned}$$

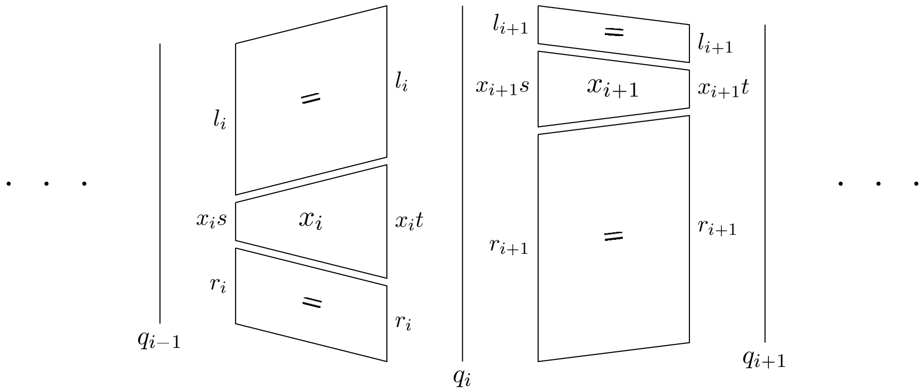
So $\tilde{w} \in \mathrm{Words}(X)$.

Recall that $l_i \geqslant l_{i+1} + x_{i+1}s$ implies $\tilde{l}_{i+1} \geqslant \tilde{l}_i + \tilde{x}_i t$. Therefore we have

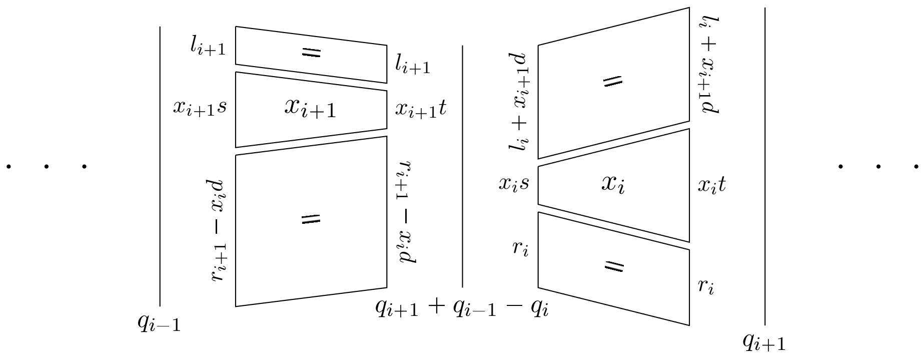
$$\begin{aligned}
w' &= \big(\cdots \tilde{q}_{i-1}(\tilde{l}_i, \tilde{x}_i, \tilde{r}_i)\tilde{q}_i(\tilde{l}_{i+1}, \tilde{x}_{i+1}, \tilde{r}_{i+1})\tilde{q}_{i+1}\cdots\big) \\
&\sim \big(\cdots \tilde{q}_{i-1}(\tilde{l}_{i+1} - \tilde{x}_i d, \tilde{x}_{i+1}, \tilde{r}_{i+1})(\tilde{q}_{i+1} + \tilde{q}_{i-1} - \tilde{q}_i)(\tilde{l}_i, \tilde{x}_i, \tilde{r}_i + \tilde{x}_{i+1}d)\tilde{q}_{i+1}\cdots\big) \\
&= \big(\cdots q_{i-1}(l_i + x_{i+1}d - x_{i+1}d, x_i, r_i)q_i(l_{i+1}, x_{i+1}, r_{i+1} - x_i d + x_i d)q_{i+1}\cdots\big) \\
&= \big(\cdots q_{i-1}(l_i, x_i, r_i)q_i(l_{i+1}, x_{i+1}, r_{i+1})q_{i+1}\cdots\big) \\
&= w.
\end{aligned}$$

Hence $\tilde{w} \sim w$. □

Pictorially, we have that w is described by



and that $\tilde{w}$ is described by



Definition 4.8 (The equivalence relation $(\approx)$)**.** Now we define $(\approx)$ to be the equivalence relation on $\mathrm{Words}(X)$ generated by $(\sim)$.

Recall that this means $w \approx \tilde{w}$ if and only if there exist words $v_1, \ldots, v_n$ for some $n \geqslant 1$ such that $w = v_1$, $\tilde{w} = v_n$ and such that $v_i \sim v_{i+1}$ or $v_{i+1} \sim v_i$ for $i \in [1, n-1]$; cf. Definition 1.11.

Write $[w]$ for the equivalence class of the word w with respect to $(\approx)$.

Note that given $w, \tilde{w} \in \mathrm{Words}(X)$ such that $w \approx \tilde{w}$, then we have $ws = \tilde{w}s$, $wt = \tilde{w}t$ and the two words have the same length.

Definition 4.9. Let $w = q_0\,(l_1, x_1, r_1)\,q_1 \cdots q_{k-1}\,(l_k, x_k, r_k)\,q_k \in \mathrm{Words}(X)(q_0, q_k)$ and let $q \in \mathbb{Z}_{\geqslant 0}$. Define

$$\begin{aligned}
w \rhd q &:= (q_0 + q)\,(l_1, x_1, r_1 + q)\,(q_1 + q) \cdots (q_{k-1} + q)\,(l_k, x_k, r_k + q)\,(q_k + q) \\
q \lhd w &:= (q + q_0)\,(q + l_1, x_1, r_1)\,(q + q_1) \cdots (q + q_{k-1})\,(q + l_k, x_k, r_k)\,(q + q_k).
\end{aligned}$$

Note that $w \rhd q, q \lhd w \in \mathrm{Words}(X)(q_0 + q, q_k + q)$.

Lemma 4.10 (Properties of $(\rhd)$ and $(\lhd)$)**.** *Let $m, n, h \in \mathbb{Z}_{\geqslant 0}$ and let $w, \tilde{w} \in \mathrm{Words}(X)(m, n)$ with $[w] = [\tilde{w}]$ and $v \in \mathrm{Words}(X)(n, h)$. Let $p, q \in \mathbb{Z}_{\geqslant 0}$. Then we have*

(i) $(w \rhd q) \rhd p = (w \rhd p) \rhd q = w \rhd (p + q)$ *and* $p \lhd (q \lhd w) = q \lhd (p \lhd w) = (p + q) \lhd w$

(ii) $p \lhd (w \rhd q) = (p \lhd w) \rhd q$

(iii) $(wv) \rhd q = (w \rhd q)\,(v \rhd q)$ *and* $q \lhd (wv) = (q \lhd w)\,(q \lhd v)$

(iv) $[w \rhd q] = [\tilde{w} \rhd q]$ *and* $[q \lhd w] = [q \lhd \tilde{w}]$.

Proof. Write

$$\begin{aligned} w &= q_0(l_1, x_1, r_1)q_1 \cdots q_{k-1}(l_k, x_k, r_k)q_k \\ \tilde{w} &= \tilde{q}_0(\tilde{l}_1, \tilde{x}_1, \tilde{r}_1)\tilde{q}_1 \cdots \tilde{q}_{k-1}(\tilde{l}_k, \tilde{x}_k, \tilde{r}_k)\tilde{q}_k \\ v &= p_0(\lambda_1, y_1, \rho_1)p_1 \cdots p_{\kappa-1}(\lambda_\kappa, y_\kappa, \rho_\kappa)p_\kappa\,. \end{aligned}$$

So $q_0 = m = \tilde{q}_0$, $q_k = n = \tilde{q}_k = p_0$ and $p_\kappa = h$.

Ad (i). We will show the property for $(\rhd)$. We have

$$\begin{aligned} &(w \rhd q) \rhd p \\ &\quad = \big((q_0 + q)(l_1, x_1, r_1 + q)(q_1 + q) \cdots (q_{k-1} + q)(l_k, x_k, r_k + q)(q_k + q)\big) \rhd p \\ &\quad = (q_0 + q + p)(l_1, x_1, r_1 + q + p)(q_1 + q + p) \cdots (q_{k-1} + q + p)(l_k, x_k, r_k + q + p)(q_k + q + p) \\ &\quad = w \rhd (q + p) \\ &\quad = (w \rhd p) \rhd q. \end{aligned}$$

Ad (ii). We have

$$\begin{aligned} &p \lhd (w \rhd q) \\ &\quad = p \lhd \big((q_0 + q)(l_1, x_1, r_1 + q)(q_1 + q) \cdots (q_{k-1} + q)(l_k, x_k, r_k + q)(q_k + q)\big) \\ &\quad = (p + q_0 + q)(p + l_1, x_1, r_1 + q)(p + q_1 + q) \cdots (p + q_{k-1} + q)(p + l_k, x_k, r_k + q)(p + q_k + q) \\ &\quad = \big((p + q_0)(p + l_1, x_1, r_1)(p + q_1) \cdots (p + q_{k-1})(p + l_k, x_k, r_k)(p + q_k)\big) \rhd q \\ &\quad = (p \lhd w) \rhd q. \end{aligned}$$

Ad (iii). We will show the property for $(\rhd)$. We have

$$\begin{aligned} &(wv) \rhd q \\ &\quad = \big(q_0(l_1, x_1, r_1)q_1 \cdots q_{k-1}(l_k, x_k, r_k)p_0(\lambda_1, y_1, \rho_1)p_1 \cdots p_{\kappa-1}(\lambda_\kappa, y_\kappa, \rho_\kappa)p_\kappa\big) \rhd q \\ &\quad = (q_0 + q)(l_1, x_1, r_1 + q)(q_1 + q) \cdots (q_{k-1} + q)(l_k, x_k, r_k + q)(p_0 + q)(\lambda_1, y_1, \rho_1 + q)(p_1 + q) \cdots \\ &\qquad\qquad \cdots (p_{\kappa-1} + q)(\lambda_\kappa, y_\kappa, \rho_\kappa + q)(p_\kappa + q) \\ &\quad = (w \rhd q)(v \rhd q). \end{aligned}$$

Ad (iv): Consider the map

$$\begin{aligned} f : \mathrm{Words}(X) &\longrightarrow \frac{\mathrm{Words}(X)}{(\approx)} \\ w &\longmapsto [w \rhd q]. \end{aligned}$$

Recall that we have $w, \tilde{w} \in \mathrm{Words}(X)(m, n)$ with $[w] = [\tilde{w}]$ and that we have to show that $wf = \tilde{w}f$. By Lemma 1.12, we can assume $w \sim \tilde{w}$.

So there exists $i \in [1, k-1]$ such that $l_i + x_i t \leqslant l_{i+1}$ and we have $\tilde{l}_j = l_j$, $\tilde{r}_j = r_j$ and $\tilde{x}_j = x_j$ for $j \in [1, k-1] \setminus \{i, i+1\}$ and $\tilde{q}_j = q_j$ for $j \in [0, k] \setminus \{i\}$ as well as $\tilde{l}_i = l_{i+1} - x_i d$, $\tilde{x}_i = x_{i+1}$, $\tilde{r}_i = r_{i+1}$, $\tilde{q}_i = q_{i+1} + q_{i-1} - q_i$, $\tilde{l}_{i+1} = l_i$, $\tilde{x}_{i+1} = x_i$ and $\tilde{r}_{i+1} = r_i + x_{i+1} d$; cf. Definition 4.5.

Then

$$\begin{aligned} w \rhd q &= (q_0 + q) \cdots (q_{i-1} + q)(l_i, x_i, r_i + q)(q_i + q)(l_{i+1}, x_{i+1}, r_{i+1} + q)(q_{i+1} + q) \cdots (q_k + q) \\ \tilde{w} \rhd q &= (q_0 + q) \cdots (q_{i-1} + q)(\tilde{l}_i, \tilde{x}_i, \tilde{r}_i + q)(\tilde{q}_i + q)(\tilde{l}_{i+1}, \tilde{x}_{i+1}, \tilde{r}_{i+1} + q)(q_{i+1} + q) \cdots (q_k + q). \end{aligned}$$

We have $l_i + x_i t \leqslant l_{i+1}$ and $\tilde{l}_j = l_j$, $\tilde{r}_j + q = r_j + q$ and $\tilde{x}_j = x_j$ for $j \in [1, k-1] \setminus \{i, i+1\}$ and $\tilde{q}_j = q_j$ for $j \in [0, k] \setminus \{i\}$. Moreover, we have $\tilde{l}_i = l_{i+1} - x_i d$, $\tilde{x}_i = x_{i+1}$, $\tilde{l}_{i+1} = l_i$ and $\tilde{x}_{i+1} = x_i$, as well as $\tilde{r}_i + q = r_{i+1} + q$, $\tilde{r}_{i+1} + q = r_i + q + x_{i+1} d$ and $\tilde{q}_i + q = q_{i+1} + q_{i-1} - q_i + q = (q_{i+1} + q) + (q_{i-1} + q) - (q_i + q)$.

So we have $w \rhd q \sim \tilde{w} \rhd q$. So in particular $wf = [w \rhd q] = [\tilde{w} \rhd q] = \tilde{w} f$.

The corresponding property for $(\lhd)$ follows analogously, except for the inequality, which is satisfied since $(l_i + q) + x_i t \leqslant l_{i+1} + q$. $\square$

Definition 4.11. Let $w \in \mathrm{Words}(X)$. Let $q \in \mathbb{Z}_{\geqslant 0}$. Define $[w] \rhd q := [w \rhd q]$ and $q \lhd [w] := [q \lhd w]$.

Lemma 4.12. *Let $w, v \in \mathrm{Words}(X)$ be words of length 1, i.e. $w = ws(l, x, r)wt$ and $v = vs(\lambda, y, \rho)vt$ for some $l, r, \lambda, \rho \in \mathbb{Z}_{\geqslant 0}$ and $x, y \in X$. Then we have*

$$(v \rhd ws) \cdot (vt \lhd w) \sim (vs \lhd w) \cdot (v \rhd wt).$$

Proof. We have $\lambda + yt = vt - \rho \leqslant vt + l$, so by the definition of elementary equivalence in Definition 4.5 we have

$$\begin{aligned} (v \rhd ws) \cdot (vt \lhd w) &= (vs + ws)(\lambda, y, \rho + ws)(vt + ws)(l + vt, x, r)(wt + vt) \\ &\sim (vs + ws)(\tilde{l}, x, r)\tilde{q}(\lambda, y, \tilde{\rho})(wt + vt), \end{aligned}$$

where

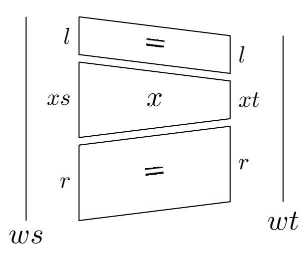
$$\begin{aligned} \tilde{l} &:= l + vt - yd = l + (\lambda + yt + \rho) - (yt - ys) = l + (\lambda + ys + \rho) = l + vs \\ \tilde{q} &:= (wt + vt) + (ws + vs) - (vt + ws) = wt + vt + ws + vs - vt - ws = wt + vs \\ \tilde{\rho} &:= \rho + ws + xd = \rho + (l + xs + r) + (xt - xs) = \rho + (l + xt + r) = \rho + wt. \end{aligned}$$

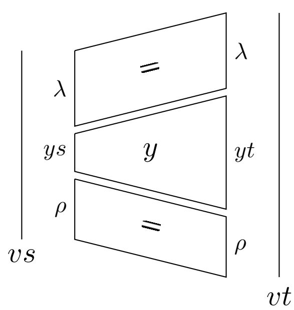
So we have

$$\begin{aligned} (v \rhd ws) \cdot (vt \lhd w) &= (vs + ws)(\lambda, y, \rho + ws)(vt + ws)(l + vt, x, r)(wt + vt) \\ &\sim (vs + ws)(l + vs, x, r)(wt + vs)(\lambda, y, \rho + wt)(wt + vt) \\ &= (vs \lhd w) \cdot (v \rhd wt). \end{aligned}$$

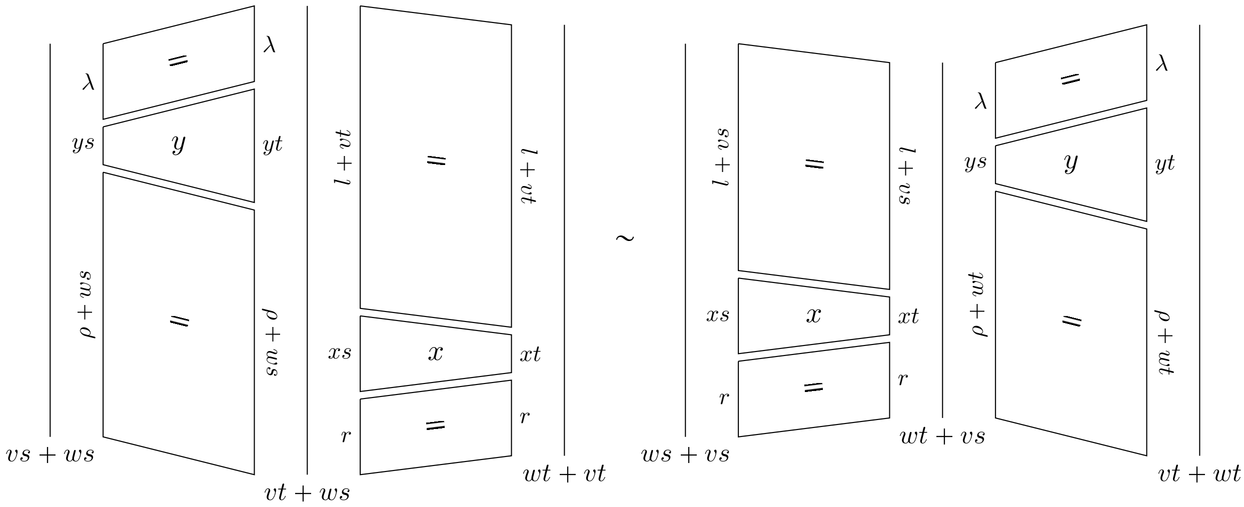
$\square$

Pictorially, we have that, given

$w =$ 

$v =$ 

then



Lemma 4.13. *Suppose given $w, v \in \mathrm{Words}(X)$ such that v is of length 1. Then we have*

$$(v \rhd ws) \cdot (vt \lhd w) \approx (vs \lhd w) \cdot (v \rhd wt).$$

Proof. Write $v = vs(\lambda, y, \rho)vt$, where $\lambda, \rho \in \mathbb{Z}_{\geqslant 0}$ and $y \in X$ and write

$$w = q_0(l_1, x_1, r_1)q_1 \cdots q_{k-1}(l_k, x_k, r_k)q_k\,.$$

Note that $q_0 = ws$ and $q_k = wt$.

If $k = 0$ the assertion holds because then $w = q_0 \in \mathbb{Z}_{\geqslant 0}$ and we have

$$(v \rhd ws) \cdot (vt \lhd w) = (v \rhd q_0) \cdot (q_0 + vt) = v \rhd q_0 = (vs + q_0) \cdot (v \rhd q_0) = (vs \lhd w) \cdot (v \rhd wt).$$

So we may assume that $k \geqslant 1$.

For $i \in [1, k]$ define $w_i := q_{i-1}(l_i, x_i, r_i)q_i \in \mathrm{Words}(X)$, which is of length 1. We have $w_i s = q_{i-1}$ and $w_i t = q_i$ for $i \in [1, k]$, so $w_i t = w_{i+1} s$ for $i \in [1, k-1]$. Moreover, we have $w = w_1 w_2 \cdots w_k$. By Lemma 4.10 (iii), we have

$$(q \lhd w) = (q \lhd (w_1 w_2 \cdots w_k)) = (q \lhd w_1)(q \lhd w_2) \cdots (q \lhd w_k)$$

for $q \in \mathbb{Z}_{\geqslant 0}$. So using Lemma 4.12 iteratively, we get

$$\begin{aligned}
(v \rhd ws)(vt \lhd w) &= (v \rhd w_1 s)(vt \lhd (w_1 \cdots w_k)) \\
&\overset{4.10\text{ (iii)}}{=} (v \rhd w_1 s)(vt \lhd w_1) \cdots (vt \lhd w_k) \\
&\overset{4.12}{\sim} (vs \lhd w_1)(v \rhd w_1 t)(vt \lhd w_2) \cdots (vt \lhd w_k) \\
&= (vs \lhd w_1)(v \rhd w_2 s)(vt \lhd w_2) \cdots (vt \lhd w_k) \\
&\overset{4.12}{\sim} (vs \lhd w_1)(vs \lhd w_2)(v \rhd w_2 t) \cdots (vt \lhd w_k) \\
&\sim \cdots \\
&\sim (vs \lhd w_1)(vs \lhd w_2) \cdots (vs \lhd w_k)(v \rhd w_k t) \\
&\overset{4.10\text{ (iii)}}{=} (vs \lhd (w_1 \cdots w_k))(v \rhd w_k t) \\
&= (vs \lhd w)(v \rhd wt).
\end{aligned}$$

So we have $(v \rhd ws)(vt \lhd w) \approx (vs \lhd w)(v \rhd wt)$. $\square$

Lemma 4.14. *Let $v, w \in \mathrm{Words}(X)$. We have*

$$(v \rhd ws) \cdot (vt \lhd w) \approx (vs \lhd w) \cdot (v \rhd wt).$$

Proof. Note that if v has length 0, then $v = p_0 \in \mathbb{Z}_{\geqslant 0}$ and we have

$$(v \rhd ws) \cdot (vt \lhd w) = (p_0 + ws) \cdot (p_0 \lhd w) = p_0 \lhd w = (p_0 \lhd w) \cdot (p_0 + wt) = (vs \lhd w) \cdot (v \rhd wt).$$

So in this case the assertion holds and we may assume that the length of v is at least 1.

As in the proof of Lemma 4.13, we can write

$$v = v_1 v_2 \ldots v_\kappa\,,$$

where $\kappa \geqslant 1$ and where $v_j \in \mathrm{Words}(X)$ of length 1 for $j \in [1, \kappa]$ such that $v_1 s = vs$, $v_\kappa t = vt$ and $v_j t = v_{j+1} s$ for $j \in [1, \kappa - 1]$.

Again we can use Lemma 4.10 (iii). By using Lemma 4.13 iteratively, we get

$$\begin{array}{rcl}
(v \rhd ws)(vt \lhd w) & = & ((v_1 v_2 \cdots v_\kappa) \rhd ws)(vt \lhd w) \\
& \overset{4.10\ \text{(iii)}}{=} & (v_1 \rhd ws)(v_2 \rhd ws) \cdots (v_\kappa \rhd ws)(v_\kappa t \lhd w) \\
& \overset{4.13}{\approx} & (v_1 \rhd ws) \cdots (v_{\kappa-1} \rhd ws)(v_\kappa s \lhd w)(v_\kappa \rhd wt) \\
& = & (v_1 \rhd ws) \cdots (v_{\kappa-1} \rhd ws)(v_{\kappa-1} t \lhd w)(v_\kappa \rhd wt) \\
& \overset{4.13}{\approx} & (v_1 \rhd ws) \cdots (v_{\kappa-1} s \lhd w)(v_{\kappa-1} \rhd wt)(v_\kappa \rhd wt) \\
& \approx & \cdots \\
& \approx & (v_1 s \lhd w)(v_1 \rhd wt)(v_2 \rhd wt) \cdots (v_\kappa \rhd wt) \\
& \overset{4.10\ \text{(iii)}}{=} & (v_1 s \lhd w)((v_1 v_2 \cdots v_\kappa) \rhd wt) \\
& = & (vs \lhd w)(v \rhd wt).
\end{array}$$

□

4.2 Definition of the free set-preoperad

Definition 4.15. Let (X, s, t) be a biindexed set. We define the free set-preoperad $\mathrm{Free}_0(X)$ on X as follows.

As a biindexed set, define $\mathrm{Free}_0(X) := \frac{\mathrm{Words}(X)}{(\approx)}$ together with the maps

$$\begin{array}{rrcl}
s: & \mathrm{Free}_0(X) & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\
& [w] & \longmapsto & ws
\end{array}
\qquad
\begin{array}{rrcl}
t: & \mathrm{Free}_0(X) & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\
& [w] & \longmapsto & wt.
\end{array}$$

So for

$$w = q_0(l_1, x_1, r_1)q_1 \cdots q_{k-1}(l_k, x_k, r_k)q_k \in \mathrm{Words}(X)$$

we have $[w]s = ws = q_0$ and $[w]t = wt = q_k$; cf. Remark 4.2.

Hence for $m, n \in \mathbb{Z}_{\geqslant 0}$ the set $\mathrm{Free}_0(X)(m, n)$ consists of equivalence classes $[w]$ with respect to the equivalence relation $(\approx)$ of words of the form

$$w = q_0\,(l_1, x_1, r_1)\,q_1\,(l_2, x_2, r_2)\,q_2 \cdots q_{k-1}\,(l_k, x_k, r_k)\,q_k \in \mathrm{Words}(X),$$

where $m = q_0$ and $n = q_k$.

The composition in $\mathrm{Free}_0(X)$ is given by

$$\begin{aligned}(\cdot) : \mathrm{Free}_0(X)(m,n) \times \mathrm{Free}_0(X)(n,p) &\longrightarrow \mathrm{Free}_0(X)(m,p)\\ ([w],[v]) &\longmapsto [w]\cdot[v] := [wv]\end{aligned}$$

for $m,n,p \in \mathbb{Z}_{\geqslant 0}$.

The multiplication is given by

$$\begin{aligned}(\boxtimes) : \mathrm{Free}_0(X)(m,n) \times \mathrm{Free}_0(X)(m',n') &\longrightarrow \mathrm{Free}_0(X)(m+m',n+n')\\ \big([w],[w']\big) &\longmapsto [w]\boxtimes[w'] := ([w] \rhd w's)\cdot(wt \lhd [w'])\end{aligned}$$

for $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$.

For $m \in \mathbb{Z}_{\geqslant 0}$, the identity element in $\mathrm{Free}_0(X)(m,m)$ is $\mathrm{id}_m := \mathrm{id}_{\mathrm{Free}_0(X),m} := [m]$, the empty word, which has length $k = 0$ and $[m]s = [m]t = m$.

Now we have to show that this in fact defines a set-preoperad.

First we show that the composition is well-defined. It suffices to show that for $w, \tilde{w} \in \mathrm{Words}(X)(m,n)$ with $w \sim \tilde{w}$ and $v, \tilde{v} \in \mathrm{Words}(X)(n,p)$ with $v \sim \tilde{v}$ we have $wv \approx \tilde{w}v$ and $wv \approx w\tilde{v}$; cf. Lemma 1.12.

So suppose given $w, \tilde{w} \in \mathrm{Words}(X)(m,n)$ with $w \sim \tilde{w}$ and $v \in \mathrm{Words}(X)(n,p)$. Write

$$\begin{aligned}w &= q_0(l_1,x_1,r_1)q_1\cdots q_{k-1}(l_k,x_k,r_k)q_k\\ \tilde{w} &= \tilde{q}_0(\tilde{l}_1,\tilde{x}_1,\tilde{r}_1)\tilde{q}_1\cdots \tilde{q}_{k-1}(\tilde{l}_k,\tilde{x}_k,\tilde{r}_k)\tilde{q}_k\\ v &= p_0(\lambda_1,y_1,\rho_1)p_1\cdots p_{\kappa-1}(\lambda_\kappa,y_\kappa,\rho_\kappa)p_\kappa\,,\end{aligned}$$

where $m = q_0 = \tilde{q}_0$, $n = q_k = \tilde{q}_k = p_0$, $p = p_\kappa$ and where for some $i \in [1,k]$ we have $l_i + x_i t \leqslant l_{i+1}$ and we have $\tilde{l}_j = l_j$, $\tilde{r}_j = r_j$ and $\tilde{x}_j = x_j$ for $j \in [1,k]\setminus\{i,i+1\}$ and $\tilde{q}_j = q_j$ for $j \in [0,k]\setminus\{i\}$, as well as $\tilde{l}_i = l_{i+1} - x_i d$, $\tilde{x}_i = x_{i+1}$, $\tilde{r}_i = r_{i+1}$, $\tilde{l}_{i+1} = l_i$, $\tilde{x}_{i+1} = x_i$, $\tilde{r}_{i+1} = r_i + x_{i+1}d$ and $\tilde{q}_i = q_{i+1} + q_{i-1} - q_i$; cf. Definition 4.5.

Then we have

$$\begin{aligned}wv &= q_0(l_1,x_1,r_1)q_1\cdots q_{k-1}(l_k,x_k,r_k)q_k(\lambda_1,y_1,\rho_1)p_1\cdots p_{\kappa-1}(\lambda_\kappa,y_\kappa,\rho_\kappa)p_\kappa\\ \tilde{w}v &= \tilde{q}_0(\tilde{l}_1,\tilde{x}_1,\tilde{r}_1)\tilde{q}_1\cdots \tilde{q}_{k-1}(\tilde{l}_k,\tilde{x}_k,\tilde{r}_k)\tilde{q}_k(\lambda_1,y_1,\rho_1)p_1\cdots p_{\kappa-1}(\lambda_\kappa,y_\kappa,\rho_\kappa)p_\kappa\,.\end{aligned}$$

So the conditions of Definition 4.5 are still satisfied for wv and $\tilde{w}v$. Hence $wv \sim \tilde{w}v$, so in particular $wv \approx \tilde{w}v$.

Analogously we can also conclude that given $w \in \mathrm{Words}(X)(m,n)$ and $v, \tilde{v} \in \mathrm{Words}(X)(n,p)$ with $v \sim \tilde{v}$, then we have $wv \approx w\tilde{v}$.

This shows that the composition is well-defined.

Now we show that the multiplication is well-defined. But we already know that $[w] \rhd w's$ and $ws \lhd [w']$ are well-defined for $v \in \mathrm{Words}(X)(m,n)$ and $w' \in \mathrm{Words}(X)(m',n')$; cf. Lemma 4.10 (iv) and Definition 4.11. So since the composition is well-defined, the multiplication is also well-defined.

Ad (c1). Suppose given $m,n,p,q \in \mathbb{Z}_{\geqslant 0}$ and words $w \in \mathrm{Words}(X)(m,n)$, $v \in \mathrm{Words}(X)(n,p)$ and $u \in \mathrm{Words}(X)(p,q)$. That is, we have $w,v,u \in \mathrm{Words}(X)$ with $wt = n = vs$ and $vt = p = us$. Then we have

$$\begin{aligned}([w]\cdot[v])\cdot[u] &= [wv]\cdot[u]\\ &= [(wv)u]\\ &\overset{4.3}{=} [w(vu)]\\ &= [w]\cdot[vu]\\ &= [w]\cdot([v]\cdot[u]).\end{aligned}$$

Ad (c2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $w \in \mathrm{Words}(X)(m,n)$, i.e. $ws = m$ and $wt = n$. Then we have

$$[w] \cdot \mathrm{id}_n = [w] \cdot \mathrm{id}_{wt} = [w] \cdot [wt] = [w \cdot wt] = [w] = [ws \cdot w] = [ws] \cdot [w] = \mathrm{id}_{ws} \cdot [w] = \mathrm{id}_m \cdot [w].$$

Ad (m1). Suppose given $w, w', w'' \in \mathrm{Words}(X)$. Note that we have $(wt \lhd [w'])t = wt + w't$ and $([w'] \rhd w''s)s = w's + w''s$. Then we have

$$\begin{aligned}
([w] \boxtimes [w']) \boxtimes [w''] &= \big(([w] \rhd w's) \cdot (wt \lhd [w'])\big) \boxtimes [w''] \\
&= \Big(\big(([w] \rhd w's) \cdot (wt \lhd [w'])\big) \rhd w''s\Big) \cdot \big((wt + w't) \lhd [w'']\big) \\
&= \big[\big((w \rhd w's)(wt \lhd w')\big) \rhd w''s\big] \cdot [(wt + w't) \lhd w''] \\
&\overset{4.10\ \text{(iii)}}{=} \big[\big((w \rhd w's) \rhd w''s\big)\big((wt \lhd w') \rhd w''s\big)\big] \cdot [(wt + w't) \lhd w''] \\
&\overset{4.10\ \text{(i)}}{=} \big[\big(w \rhd (w's + w''s)\big)\big((wt \lhd w') \rhd w''s\big)\big] \cdot [wt \lhd (w't \lhd w'')] \\
&\overset{4.10\ \text{(ii)}}{=} \big[\big(w \rhd (w's + w''s)\big)\big(wt \lhd (w' \rhd w''s)\big)\big] \cdot [wt \lhd (w't \lhd w'')] \\
&= [w \rhd (w's + w''s)] \cdot \big[\big(wt \lhd (w' \rhd w''s)\big)\big(wt \lhd (w't \lhd w'')\big)\big] \\
&\overset{4.10\ \text{(iii)}}{=} [w \rhd (w's + w''s)] \cdot \big[wt \lhd \big((w' \rhd w''s)(w't \lhd w'')\big)\big] \\
&= \big([w] \rhd (w's + w''s)\big) \cdot \Big(wt \lhd \big(([w'] \rhd w''s) \cdot (w't \lhd [w''])\big)\Big) \\
&= [w] \boxtimes \big(([w'] \rhd w''s) \cdot (w't \lhd [w''])\big) \\
&= [w] \boxtimes ([w'] \boxtimes [w'']).
\end{aligned}$$

Ad (m2). Suppose given $w \in \mathrm{Words}(X)$. We have

$$\begin{aligned}
[w] \boxtimes \mathrm{id}_0 &= ([w] \rhd \mathrm{id}_0\, s) \cdot (wt \lhd \mathrm{id}_0) \\
&= ([w] \rhd [0]s) \cdot (wt \lhd [0]) \\
&= [w] \cdot [wt] \\
&= [w] \\
&= [ws] \cdot [w] \\
&= ([0] \rhd ws) \cdot ([0]t \lhd [w]) \\
&= (\mathrm{id}_0 \rhd ws) \cdot (\mathrm{id}_0\, t \lhd [w]) \\
&= \mathrm{id}_0 \boxtimes [w].
\end{aligned}$$

Ad (mc1). Suppose given $w, w', v, v' \in \mathrm{Words}(X)$ such that $wt = vs$ and $w't = v's$. We have

$$\begin{aligned}
([w] \cdot [v]) \boxtimes ([w'] \cdot [v']) &= [wv] \boxtimes [w'v'] \\
&= ([wv] \rhd w's) \cdot (vt \lhd [w'v']) \\
&= [(wv) \rhd w's] \cdot [vt \lhd (w'v')] \\
&\overset{4.10\ \text{(iii)}}{=} [(w \rhd w's)(v \rhd w's)] \cdot [(vt \lhd w')(vt \lhd v')] \\
&= [w \rhd w's] \cdot [(v \rhd w's)(vt \lhd w')] \cdot [vt \lhd v'] \\
&\overset{4.14}{=} [w \rhd w's] \cdot [(vs \lhd w')(v \rhd w't)] \cdot [vt \lhd v'] \\
&= [(w \rhd w's)(wt \lhd w')] \cdot [(v \rhd v's)(vt \lhd v')] \\
&= \big(([w] \rhd w's) \cdot (wt \lhd [w'])\big) \cdot \big(([v] \rhd v's) \cdot \big(vt \lhd [v']\big)\big) \\
&= ([w] \boxtimes [w']) \cdot ([v] \boxtimes [v']).
\end{aligned}$$

Ad (mc2). We have to show that for $m \geqslant 0$ we have $\mathrm{id}_m \overset{!}{=} \mathrm{id}_1^{\boxtimes m}$. We show this via induction on $m \geqslant 0$.

For $m = 0$ this is the definition. So let $m \geqslant 1$ and assume that the statement is true for $m - 1$. Then we have

$$\begin{aligned} \mathrm{id}_1^{\boxtimes m} &= \mathrm{id}_1^{\boxtimes (m-1)} \boxtimes \mathrm{id}_1 \\ &\overset{\text{ind.}}{=} \mathrm{id}_{m-1} \boxtimes \mathrm{id}_1 \\ &= [m-1] \boxtimes [1] \\ &= ([m-1] \rhd [1]s) \cdot ([m-1]t \lhd [1]) \\ &= [(m-1)+1] \cdot [(m-1)+1] \\ &= [m] \cdot [m] \\ &= [m] \\ &= \mathrm{id}_m \ . \end{aligned}$$

This shows that $\mathrm{Free}_0(X)$ is in fact a set-preoperad.

Remark 4.16. Given a finite biindexed set $X = \{x_1, \ldots, x_n\}$, then we often also abbreviate $\mathrm{Free}_0(x_1, \ldots, x_n) := \mathrm{Free}_0(\{x_1, \ldots, x_n\})$.

Remark 4.17. *Let $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and let $[w] \in \mathrm{Free}_0(X)(m, n)$. Then we have $[w] \rhd p = [w] \boxtimes \mathrm{id}_p$ and $p \lhd [w] = \mathrm{id}_p \boxtimes [w]$.*

Proof. We have

$$\begin{aligned} [w] \boxtimes \mathrm{id}_p &= ([w] \rhd \mathrm{id}_p\, s) \cdot (wt \lhd \mathrm{id}_p) \\ &= ([w] \rhd p) \cdot (wt \lhd [p]) \\ &= ([w] \rhd p) \cdot (wt + p) \\ &= [w] \rhd p \\ \mathrm{id}_p \boxtimes [w] &= (\mathrm{id}_p \rhd ws) \cdot (\mathrm{id}_p\, t \lhd [w]) \\ &= ([p] \rhd ws) \cdot (p \lhd [w]) \\ &= (p + ws) \cdot (p \lhd [w]) \\ &= p \lhd [w]. \end{aligned}$$

□

4.3 The universal property of the free set-preoperad

Recall that set-preoperads have underlying biindexed sets and that morphisms of set-preoperads are biindexed maps that are compatible with composition, multiplication and identities.

Definition 4.18. Let (X, s, t) be a biindexed set and let $\mathrm{Free}_0(X)$ be the free set-preoperad on X; cf. Definition 4.15.

We define the biindexed map $\iota_0 := \iota_{0,X} : X \longrightarrow \mathrm{Free}_0(X)$ as follows.

$$\begin{aligned} \iota_{0,X} : X &\longrightarrow \mathrm{Free}_0(X) \\ x &\longmapsto [xs(0, x, 0)xt] \end{aligned}$$

Theorem 4.19 (Universal property of the free set-preoperad)**.** *Let (X, s, t) be a biindexed set. Let $\mathcal{T}_0$ be a set-preoperad and $\varphi_0 : X \longrightarrow \mathcal{T}_0$ be a biindexed map.*

Then there exists a uniquely determined morphism of set-preoperads $\phi_0 : \mathrm{Free}_0(X) \longrightarrow \mathcal{T}_0$ such that $\iota_0 \phi_0 = \varphi_0$.

$$\begin{array}{ccc} X & \xrightarrow{\varphi_0} & \mathcal{T}_0 \\ {\scriptstyle \iota_0}\downarrow & \nearrow{\scriptstyle \exists!\,\phi_0} & \\ \mathrm{Free}_0(X) & & \end{array}$$

Proof. Uniqueness. First note that given $w = q_0(l_1, x_1, r_1)q_1 \cdots q_{k-1}(l_k, x_k, r_k)q_k \in \mathrm{Words}(X)$ we can write

$$w = w_1 w_2 \cdots w_k\,,$$

where $w_i := q_{i-1}(l_i, x_i, r_i)q_i$ for $i \in [1, k]$, provided $k \geqslant 1$. If $k = 0$, then $w = q_0 = q_k$.

For $i \in [1, k]$ we have

$$\begin{aligned} \mathrm{id}_{l_i} \boxtimes [x_i s\,(0, x_i, 0)\,x_i t] \boxtimes \mathrm{id}_{r_i} &\overset{4.17}{=} l_i \lhd ([x_i s\,(0, x_i, 0)\,x_i t] \rhd r_i) \\ &= [(l_i + x_i s + r_i)\,(l_i, x_i, r_i)\,(l_i + x_i t + r_i)] \\ &= [q_{i-1}\,(l_i, x_i, r_i)\,q_i] \\ &= [w_i]\,. \end{aligned}$$

Hence we have

$$\begin{aligned} [w] &= \mathrm{id}_{q_0} \cdot [w_1 w_2 \cdots w_k] \cdot \mathrm{id}_{q_k} \\ &= \mathrm{id}_{q_0} \cdot [w_1] \cdot [w_2] \cdots [w_k] \cdot \mathrm{id}_{q_k} \\ &= \mathrm{id}_{q_0} \cdot (\mathrm{id}_{l_1} \boxtimes [x_1 s\,(0, x_1, 0)\,x_1 t] \boxtimes \mathrm{id}_{r_1}) \cdots (\mathrm{id}_{l_k} \boxtimes [x_k s\,(0, x_k, 0)\,x_k t] \boxtimes \mathrm{id}_{r_k}) \cdot \mathrm{id}_{q_k} \\ &= \mathrm{id}_{q_0} \cdot (\mathrm{id}_{l_1} \boxtimes x_1\iota_0 \boxtimes \mathrm{id}_{r_1}) \cdots (\mathrm{id}_{l_k} \boxtimes x_k\iota_0 \boxtimes \mathrm{id}_{r_k}) \cdot \mathrm{id}_{q_k}\;. \end{aligned}$$

Now suppose given a morphism of set-preoperads $\chi_0 : \mathrm{Free}_0(X) \longrightarrow \mathcal{T}_0$ satisfying $\iota_0\chi_0 = \varphi_0$. Then we have

$$\begin{aligned} [w]\chi_0 &= \big(\mathrm{id}_{q_0} \cdot (\mathrm{id}_{l_1} \boxtimes x_1\iota_0 \boxtimes \mathrm{id}_{r_1}) \cdots (\mathrm{id}_{l_k} \boxtimes x_k\iota_0 \boxtimes \mathrm{id}_{r_k}) \cdot \mathrm{id}_{q_k}\big)\chi_0 \\ &= (\mathrm{id}_{q_0}\,\chi_0) \cdot (\mathrm{id}_{l_1} \boxtimes x_1\iota_0 \boxtimes \mathrm{id}_{r_1})\chi_0 \cdots (\mathrm{id}_{l_k} \boxtimes x_k\iota_0 \boxtimes \mathrm{id}_{r_k})\chi_0 \cdot (\mathrm{id}_{q_k}\,\chi_0) \\ &= (\mathrm{id}_{q_0}\,\chi_0) \cdot (\mathrm{id}_{l_1}\,\chi_0 \boxtimes (x_1\iota_0)\chi_0 \boxtimes \mathrm{id}_{r_1}\,\chi_0) \cdots (\mathrm{id}_{l_k}\,\chi_0 \boxtimes (x_k\iota_0)\chi_0 \boxtimes \mathrm{id}_{r_k}\,\chi_0) \cdot (\mathrm{id}_{q_k}\,\chi_0) \\ &= \mathrm{id}_{\mathcal{T}_0, q_0} \cdot (\mathrm{id}_{\mathcal{T}_0, l_1} \boxtimes x_1\varphi_0 \boxtimes \mathrm{id}_{\mathcal{T}_0, r_1}) \cdots (\mathrm{id}_{\mathcal{T}_0, l_k} \boxtimes x_k\varphi_0 \boxtimes \mathrm{id}_{\mathcal{T}_0, r_k}) \cdot \mathrm{id}_{\mathcal{T}_0, q_k}\,, \end{aligned}$$

so such a morphism of set-operads χ_0 is uniquely determined by φ_0.

Existence. First we define a biindexed map $\hat{\phi}_0 : \mathrm{Words}(X) \longrightarrow \mathcal{T}_0$ as follows.

For $w = q_0(l_1, x_1, r_1)q_1 \cdots q_{k-1}(l_k, x_k, r_k)q_k \in \mathrm{Words}(X)$ define

$$w\hat{\phi}_0 := \mathrm{id}_{\mathcal{T}_0, q_0}\,(\mathrm{id}_{\mathcal{T}_0, l_1} \boxtimes x_1\varphi_0 \boxtimes \mathrm{id}_{\mathcal{T}_0, r_1}) \cdots (\mathrm{id}_{\mathcal{T}_0, l_k} \boxtimes x_k\varphi_0 \boxtimes \mathrm{id}_{\mathcal{T}_0, r_k})\,\mathrm{id}_{\mathcal{T}_0, q_k}\;.$$

Furthermore, define $\phi_0 : \mathrm{Free}_0(X) \longrightarrow \mathcal{T}_0$ by

$$[w]\phi_0 := w\hat{\phi}_0 = \mathrm{id}_{\mathcal{T}_0, q_0}\,(\mathrm{id}_{\mathcal{T}_0, l_1} \boxtimes x_1\varphi_0 \boxtimes \mathrm{id}_{\mathcal{T}_0, r_1}) \cdots (\mathrm{id}_{\mathcal{T}_0, l_k} \boxtimes x_k\varphi_0 \boxtimes \mathrm{id}_{\mathcal{T}_0, r_k})\,\mathrm{id}_{\mathcal{T}_0, q_k}\;.$$

From now on, during this proof we will often write id_m instead of $\mathrm{id}_{\mathcal{T}_0, m}$.

By Lemma 1.12, in order to show that ϕ_0 is well-defined, it suffices to show that for $w, \tilde{w} \in \mathrm{Words}(X)$ with $w \sim \tilde{w}$ we have $w\hat{\phi}_0 = \tilde{w}\hat{\phi}_0$.

So suppose given $w, \tilde{w} \in \mathrm{Words}(X)$ with $w \sim \tilde{w}$. We write

$$\begin{aligned} w &= q_0(l_1,x_1,r_1)q_1 \cdots q_{i-1}(l_i,x_i,r_i)q_i(l_{i+1},x_{i+1},r_{i+1})q_{i+1} \cdots q_{k-1}(l_k,x_k,r_k)q_k \\ \tilde{w} &= q_0(l_1,x_1,r_1)q_1 \cdots q_{i-1}(\tilde{l}_i,\tilde{x}_i,\tilde{r}_i)\tilde{q}_i(\tilde{l}_{i+1},\tilde{x}_{i+1},\tilde{r}_{i+1})q_{i+1} \cdots q_{k-1}(l_k,x_k,r_k)q_k\,, \end{aligned}$$

where $l_i + x_i t \leqslant l_{i+1}$ and $\tilde{l}_i = l_{i+1} - x_i d$, $\tilde{x}_i = x_{i+1}$, $\tilde{r}_i = r_{i+1}$, $\tilde{l}_{i+1} = l_i$, $\tilde{x}_{i+1} = x_i$, $\tilde{r}_{i+1} = r_i + x_{i+1} d$ and $\tilde{q}_i = q_{i+1} + q_{i-1} - q_i$.

Note that we have

$$w\hat{\phi}_0 = (q_0(l_1,x_1,r_1)q_1)\,\hat{\phi}_0 \cdot (q_1(l_2,x_2,r_2)q_2)\,\hat{\phi}_0 \cdots (q_{k-1}(l_k,x_k,r_k)q_k)\,\hat{\phi}_0\,.$$

So it suffices to show that

$$(q_{i-1}(l_i,x_i,r_i)q_i)\hat{\phi}_0 \cdot (q_i(l_{i+1},x_{i+1},r_{i+1})q_{i+1})\hat{\phi}_0 \overset{!}{=} (q_{i-1}(\tilde{l}_i,\tilde{x}_i,\tilde{r}_i)\tilde{q}_i)\hat{\phi}_0 \cdot (\tilde{q}_i(\tilde{l}_{i+1},\tilde{x}_{i+1},\tilde{r}_{i+1})q_{i+1})\hat{\phi}_0\,.$$

Note that $l_i + x_i t + r_i = q_i = l_{i+1} + x_{i+1}s + r_{i+1}$, hence

$$\begin{aligned} r_i + x_{i+1}d - (l_{i+1} - l_i - x_i t) &= r_i + x_{i+1}t - x_{i+1}s - l_{i+1} + l_i + x_i t \\ &= r_i + x_i t + l_i - (l_{i+1} + x_{i+1}s) + x_{i+1}t \\ &= q_i - (q_i - r_{i+1}) + x_{i+1}t \\ &= x_{i+1}t + r_{i+1}. \end{aligned}$$

This means that $l_{i+1} - l_i - x_i t + x_{i+1}t + r_{i+1} = r_i + x_{i+1}d$ and $l_{i+1} - l_i - x_i t + x_{i+1}s + r_{i+1} = r_i$. We have

$$\begin{aligned} &(q_{i-1}(\tilde{l}_i,\tilde{x}_i,\tilde{r}_i)\tilde{q}_i)\hat{\phi}_0 \cdot (\tilde{q}_i(\tilde{l}_{i+1},\tilde{x}_{i+1},\tilde{r}_{i+1})q_{i+1})\hat{\phi}_0 \\ &\quad= \big(q_{i-1}(l_{i+1} - x_i d, x_{i+1}, r_{i+1})(q_{i+1} + q_{i-1} - q_i)\big)\hat{\phi}_0 \cdot \big((q_{i+1} + q_{i-1} - q_i)(l_i, x_i, r_i + x_{i+1}d)q_{i+1}\big)\hat{\phi}_0 \\ &\quad= \mathrm{id}_{q_{i-1}}(\mathrm{id}_{l_{i+1}-x_i d} \boxtimes x_{i+1}\varphi_0 \boxtimes \mathrm{id}_{r_{i+1}})\,\mathrm{id}_{q_{i+1}+q_{i-1}-q_i}(\mathrm{id}_{l_i} \boxtimes x_i\varphi_0 \boxtimes \mathrm{id}_{r_i+x_{i+1}d})\,\mathrm{id}_{q_{i+1}} \\ &\quad= (\mathrm{id}_{l_{i+1}-x_i d} \boxtimes x_{i+1}\varphi_0 \boxtimes \mathrm{id}_{r_{i+1}}) \cdot (\mathrm{id}_{l_i} \boxtimes x_i\varphi_0 \boxtimes \mathrm{id}_{r_i+x_{i+1}d}) \\ &\quad= (\mathrm{id}_{l_i} \boxtimes \mathrm{id}_{x_i s} \boxtimes \mathrm{id}_{l_{i+1}-l_i-x_i t} \boxtimes x_{i+1}\varphi_0 \boxtimes \mathrm{id}_{r_{i+1}}) \cdot (\mathrm{id}_{l_i} \boxtimes x_i\varphi_0 \boxtimes \mathrm{id}_{l_{i+1}-l_i-x_i t} \boxtimes \mathrm{id}_{x_{i+1}t} \boxtimes \mathrm{id}_{r_{i+1}}) \\ &\quad= (\mathrm{id}_{l_i} \boxtimes \mathrm{id}_{x_i s} \cdot x_i\varphi_0 \boxtimes \mathrm{id}_{l_{i+1}-l_i-x_i t} \boxtimes x_{i+1}\varphi_0 \cdot \mathrm{id}_{x_{i+1}t} \boxtimes \mathrm{id}_{r_{i+1}}) \\ &\quad= (\mathrm{id}_{l_i} \boxtimes x_i\varphi_0 \cdot \mathrm{id}_{x_i t} \boxtimes \mathrm{id}_{l_{i+1}-l_i-x_i t} \boxtimes \mathrm{id}_{x_{i+1}s} \cdot x_{i+1}\varphi_0 \boxtimes \mathrm{id}_{r_{i+1}}) \\ &\quad= (\mathrm{id}_{l_i} \boxtimes x_i\varphi_0 \boxtimes \mathrm{id}_{l_{i+1}-l_i-x_i t} \boxtimes \mathrm{id}_{x_{i+1}s} \boxtimes \mathrm{id}_{r_{i+1}}) \cdot (\mathrm{id}_{l_i} \boxtimes \mathrm{id}_{x_i t} \boxtimes \mathrm{id}_{l_{i+1}-l_i-x_i t} \boxtimes x_{i+1}\varphi_0 \boxtimes \mathrm{id}_{r_{i+1}}) \\ &\quad= (\mathrm{id}_{l_i} \boxtimes x_i\varphi_0 \boxtimes \mathrm{id}_{r_i}) \cdot (\mathrm{id}_{l_{i+1}} \boxtimes x_{i+1}\varphi_0 \boxtimes \mathrm{id}_{r_{i+1}}) \\ &\quad= \mathrm{id}_{q_{i-1}}(\mathrm{id}_{l_i} \boxtimes x_i\varphi_0 \boxtimes \mathrm{id}_{r_i})\,\mathrm{id}_{q_i}(\mathrm{id}_{l_{i+1}} \boxtimes x_{i+1}\varphi_0 \boxtimes \mathrm{id}_{r_{i+1}})\,\mathrm{id}_{q_{i+1}} \\ &\quad= (q_{i-1}(l_i,x_i,r_i)q_i)\hat{\phi}_0 \cdot (q_i(l_{i+1},x_{i+1},r_{i+1})q_{i+1})\hat{\phi}_0\,. \end{aligned}$$

This completes the proof that $\phi_0 : \mathrm{Free}_0(X) \longrightarrow \mathcal{T}_0$ is a well-defined biindexed map. It remains to show that it is a morphism of set-preoperads.

First note that we have $\mathrm{id}_{\mathrm{Free}_0(X),m}\,\phi_0 = [m]\phi_0 = \mathrm{id}_{\mathcal{T}_0,m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

For $w, v \in \mathrm{Words}(X)$ with $wt = vs$ we have

$$([w] \cdot [v])\phi_0 = [wv]\,\phi_0 = (wv)\hat{\phi}_0 = (w\hat{\phi}_0) \cdot (v\hat{\phi}_0) = ([w]\phi_0) \cdot ([v]\phi_0).$$

Note that for $w = q_0(l_1,x_1,r_1)q_1 \cdots q_{k-1}(l_k,x_k,r_k)q_k \in \mathrm{Words}(X)$, where $k \geqslant 1$, and for $p \in \mathbb{Z}_{\geqslant 0}$ we have

$$\begin{aligned} ([w] \vartriangleright p)\phi_0 &= \big[(q_0+p)(l_1,x_1,r_1+p)(q_1+p) \cdots (q_{k-1}+p)(l_k,x_k,r_k+p)(q_k+p)\big]\phi_0 \\ &= \mathrm{id}_{q_0+p}(\mathrm{id}_{l_1} \boxtimes x_1\varphi_0 \boxtimes \mathrm{id}_{r_1+p}) \cdots (\mathrm{id}_{l_k} \boxtimes x_k\varphi_0 \boxtimes \mathrm{id}_{r_k+p})\,\mathrm{id}_{q_k+p} \\ &= (\mathrm{id}_{l_1} \boxtimes x_1\varphi_0 \boxtimes \mathrm{id}_{r_1} \boxtimes \mathrm{id}_p) \cdots (\mathrm{id}_{l_k} \boxtimes x_k\varphi_0 \boxtimes \mathrm{id}_{r_k} \boxtimes \mathrm{id}_p) \\ &\overset{\text{(mc1)}}{=} \big((\mathrm{id}_{l_1} \boxtimes x_1\varphi_0 \boxtimes \mathrm{id}_{r_1}) \cdots (\mathrm{id}_{l_k} \boxtimes x_k\varphi_0 \boxtimes \mathrm{id}_{r_k})\big) \boxtimes (\mathrm{id}_p \cdots \mathrm{id}_p) \\ &= [w]\phi_0 \boxtimes \mathrm{id}_p\,. \end{aligned}$$

This also holds for $w = m \in \text{Words}(X)$ for $m \in \mathbb{Z}_{\geqslant 0}$ since $[w] = [m] = \text{id}_{\text{Free}_0(X),m}$, hence

$$([w] \rhd p)\,\phi_0 = [m+p]\phi_0 = \text{id}_{m+p} = \text{id}_m \boxtimes \text{id}_p = \text{id}_m\,\phi_0 \boxtimes \text{id}_p = [w]\phi_0 \boxtimes \text{id}_p\ .$$

In the same way we obtain $(p \lhd [w])\phi_0 = \text{id}_p \boxtimes [w]\phi_0$ for $w \in \text{Words}(X)$ and $p \in \mathbb{Z}_{\geqslant 0}$.

Now suppose given $w, w' \in \text{Words}(X)$. We have

$$\begin{aligned} ([w] \boxtimes [w'])\phi_0 &= \big(([w] \rhd w's) \cdot (wt \lhd [w'])\big)\phi_0 \\ &= ([w] \rhd w's)\phi_0 \cdot (wt \lhd [w'])\phi_0 \\ &= ([w]\phi_0 \boxtimes \text{id}_{w's}) \cdot (\text{id}_{wt} \boxtimes [w']\phi_0) \\ &\overset{\text{(mc2)}}{=} ([w]\phi_0 \cdot \text{id}_{wt}) \boxtimes (\text{id}_{w's} \cdot [w']\phi_0) \\ &= [w]\phi_0 \boxtimes [w']\phi_0\ . \end{aligned}$$

□

4.4 The universal property of the free linear preoperad

Definition 4.20. Let $X = (X(m,n))_{m,n \geqslant 0}$ be a biindexed set. We define the linear preoperad $\text{Free}(X)$ on X by $\text{Free}(X) := R\,\text{Free}_0(X)$; cf. Definition 4.15 and Remark 2.12. Recall that this means the following.

- We have $\text{Free}(X)(m,n) = R\,\text{Free}_0(X)(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.
- We have $\text{id}_{\text{Free}(X),m} = \text{id}_{\text{Free}_0(X),m} = [m]$ for $m \in \mathbb{Z}_{\geqslant 0}$.
- Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$.
  Then for $\sum_{f \in \text{Words}(X)(m,n)} r_f\,[f] \in \text{Free}(X)(m,n)$ and $\sum_{f' \in \text{Words}(X)(m',n')} r_{f'}\,[f'] \in \text{Free}(X)\,(m',n')$ we have
  $$\Big(\sum_{f \in \text{Words}(X)(m,n)} r_f\,[f]\Big) \boxtimes_{\text{Free}} \Big(\sum_{f' \in \text{Words}(X)(m',n')} r_{f'}\,[f']\Big) = \sum_{\substack{f \in \text{Words}(X)(m,n) \\ f' \in \text{Words}(X)(m',n')}} r_f\, r_{f'} \big([f] \boxtimes_{\text{Free}_0} [f']\big).$$
- Suppose given $m, n, p \in \mathbb{Z}_{\geqslant 0}$.
  Then for $\sum_{f \in \text{Words}(X)(m,n)} r_f\,[f] \in \text{Free}(X)(m,n)$ and $\sum_{g \in \text{Words}(X)(n,p)} r_g\,[g] \in \text{Free}(X)(n,p)$ we have
  $$\Big(\sum_{f \in \text{Words}(X)(m,n)} r_f\,[f]\Big) \cdot_{\text{Free}} \Big(\sum_{g \in \text{Words}(X)(n,p)} r_g\,[g]\Big) = \sum_{\substack{f \in \text{Words}(X)(m,n) \\ g \in \text{Words}(X)(n,p)}} r_f\, r_g \big([f] \cdot_{\text{Free}_0} [g]\big).$$

As in the non-linear case, we will often abbreviate $\text{Free}(x_1, \ldots, x_n) := \text{Free}(\{x_1, \ldots, x_n\})$ for a finite biindexed set $\{x_1, \ldots, x_n\}$.

Definition 4.21. Let (X, s, t) be a biindexed set and let $\text{Free}(X)$ be the free linear preoperad on X; cf. Definition 4.20.

We define the biindexed map $\iota := \iota_X : X \longrightarrow \text{Free}(X)$ as follows.

$$\begin{aligned} \iota_X : X &\longrightarrow \text{Free}(X) \\ x &\longmapsto [xs(0,x,0)xt] \end{aligned}$$

Theorem 4.22 (Universal property of the free linear preoperad)**.** *Let (X, s, t) be a biindexed set. Let $\mathcal{T}$ be a linear preoperad and $\varphi : X \longrightarrow \mathcal{T}$ be a biindexed map.*

Then there exists a uniquely determined morphism of linear preoperads $\phi : \mathrm{Free}(X) \longrightarrow \mathcal{T}$ such that $\iota\phi = \varphi$.

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & \mathcal{T} \\ {\scriptstyle \iota}\downarrow & \nearrow_{\exists!\,\phi} & \\ \mathrm{Free}(X) & & \end{array}$$

Proof. Viewing $\mathcal{T}$ as a set-preoperad, by Theorem 4.19 we know that there exists a uniquely determined morphism of set-preoperads $\phi_0 : \mathrm{Free}_0(X) \longrightarrow \mathcal{T}$ such that $\iota_0\phi_0 = \varphi$.

Moreover, since $\mathrm{Free}(X) = R\,\mathrm{Free}_0(X)$, by Lemma 2.23, there exists a uniquely determined morphism of linear preoperads $\phi : \mathrm{Free}(X) \longrightarrow \mathcal{T}$ such that $[f]\phi = [f]\phi_0$ for $[f] \in \mathrm{Free}_0(X)(m, n)$ and $m, n \in \mathbb{Z}_{\geqslant 0}$.

So we have a uniquely determined morphism of linear preoperads $\phi : \mathrm{Free}(X) \longrightarrow \mathcal{T}$ such that

$$x\iota\phi = [xs(0, x, 0)xt]\phi = [xs(0, x, 0)xt]\phi_0 = x\iota_0\phi_0 = x\varphi$$

for $x \in X$, hence $\iota\phi = \varphi$. □

4.5 Presentations of preoperads

Definition 4.23. Let X be a biindexed set. Let $Y \subseteq \mathrm{Free}_0(X) \times \mathrm{Free}_0(X)$ be a biindexed subset and let $(\equiv_Y) \subseteq \mathrm{Free}_0(X) \times \mathrm{Free}_0(X)$ be the congruence on $\mathrm{Free}_0(X)$ generated by Y; cf. Definition 2.35.

Then we define the set-preoperad ${}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle := \frac{\mathrm{Free}_0(X)}{(\equiv_Y)}$.

Let $\mathcal{P}_0$ be a set-preoperad. If $\mathcal{P}_0 \cong {}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle$ for a biindexed set X and a biindexed subset $Y \subseteq \mathrm{Free}_0(X) \times \mathrm{Free}_0(X)$, then we say that ${}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle$ is a *presentation of* $\mathcal{P}_0$.

We call X a *set of generators* and Y a *set of relators* for $\mathcal{P}_0$.

Given finite biindexed sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\} \subseteq \mathrm{Free}_0(X) \times \mathrm{Free}_0(X)$, we also often write

$${}_{\mathrm{spo}}\langle\, x_1, \ldots, x_n \mid y_1, \ldots, y_m \,\rangle := {}_{\mathrm{spo}}\langle \{x_1, \ldots, x_n\} \mid \{y_1, \ldots, y_m\} \rangle .$$

Lemma 4.24 (Universal property)**.** *Let X be a biindexed set. Let $Y \subseteq \mathrm{Free}_0(X) \times \mathrm{Free}_0(X)$ be a biindexed subset and let $(\equiv_Y) \subseteq \mathrm{Free}_0(X) \times \mathrm{Free}_0(X)$ be the congruence generated by Y. Let $\mathcal{T}_0$ be a set-preoperad and let $\varphi_0 : X \longrightarrow \mathcal{T}_0$ be a biindexed map such that $[f]\phi_0 = [\tilde{f}]\phi_0$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $([f], [\tilde{f}]) \in Y(m, n)$.*

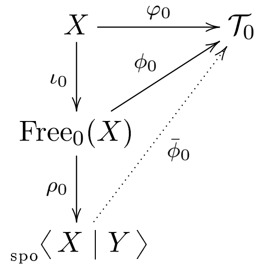
Recall that we have a uniquely determined morphism of set-preoperads $\phi_0 : \mathrm{Free}_0(X) \longrightarrow \mathcal{T}_0$ such that $\iota_0\phi_0 = \varphi_0$; cf. Theorem 4.19.

Recall the congruence class morphism

$$\begin{aligned} \rho_0 := \rho_{0,(\equiv_Y)} : \mathrm{Free}_0(X) &\longrightarrow \frac{\mathrm{Free}_0(X)}{(\equiv_Y)} = {}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle \\ [f] &\longmapsto [[f]]_Y \, ; \end{aligned}$$

cf. Definition 2.39.

Then there exists a uniquely determined morphism of set-preoperads $\bar{\phi}_0 : {}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle \longrightarrow \mathcal{T}_0$ *such that* $(\iota_0\rho_0)\bar{\phi}_0 = \varphi_0$ *, i.e. such that* $\big[[xs(0,x,0)xt]\big]_Y \bar{\phi}_0 = x\varphi_0$ *for* $x \in X$*.*



Proof. By Theorem 4.19 the morphism of set-preoperads $\phi_0 : \mathrm{Free}_0(X) \longrightarrow \mathcal{T}_0$ exists uniquely.

Moreover, since $[f]\phi_0 = [\tilde{f}]\phi_0$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $([f], [\tilde{f}]) \in Y(m,n)$, by Lemma 2.40 there exists a uniquely determined morphism of set-preoperads $\bar{\phi}_0 : \frac{\mathrm{Free}_0(X)}{(\equiv_Y)} \longrightarrow \mathcal{T}_0$ such that $\rho_0\bar{\phi}_0 = \phi_0$.

So we have a unique morphism of set-preoperads $\bar{\phi}_0 : {}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle \longrightarrow \mathcal{T}_0$ such that

$$\iota_0\rho_0\bar{\phi}_0 = \iota_0\phi_0 = \varphi_0 \,.$$

□

Definition 4.25. Let X be a biindexed set. Let $Y \subseteq \mathrm{Free}(X)$ be a biindexed subset and let $\mathcal{I} := {}_{\mathrm{ideal}}\langle\, Y \,\rangle \subseteq \mathrm{Free}(X)$ be the ideal generated by Y; cf. Definition 2.44.

Then we define the linear preoperad ${}_{\mathrm{lpo}}\langle\, X \mid Y \,\rangle := \frac{\mathrm{Free}(X)}{\mathcal{I}}$.

Let $\mathcal{P}$ be a linear preoperad. If $\mathcal{P} \cong {}_{\mathrm{lpo}}\langle\, X \mid Y \,\rangle$ for a biindexed set X and a biindexed subset $Y \subseteq \mathrm{Free}(X) \times \mathrm{Free}(X)$, then we say that ${}_{\mathrm{lpo}}\langle\, X \mid Y \,\rangle$ is a *presentation of* $\mathcal{P}$.

We call X a *set of generators* and Y a *set of relations* for $\mathcal{P}$.

Given finite biindexed sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\} \subseteq \mathrm{Free}(X)$, we also often write

$${}_{\mathrm{lpo}}\langle\, x_1, \ldots, x_n \mid y_1, \ldots, y_m \,\rangle := {}_{\mathrm{lpo}}\langle \{x_1, \ldots, x_n\} \mid \{y_1, \ldots, y_m\} \,\rangle \,.$$

Lemma 4.26. *Let* X *be a biindexed set. Let* $Y \subseteq \mathrm{Free}(X)$ *be a biindexed subset and let* $\mathcal{I} := {}_{\mathrm{ideal}}\langle\, Y \,\rangle \subseteq \mathrm{Free}(X)$ *be the ideal generated by* Y*. Let* $\mathcal{T}$ *be a linear preoperad and let* $\varphi : X \longrightarrow \mathcal{T}$ *be a biindexed map such that* $f\phi = 0$ *for* $m, n \in \mathbb{Z}_{\geqslant 0}$ *and* $f \in Y(m,n)$*.*

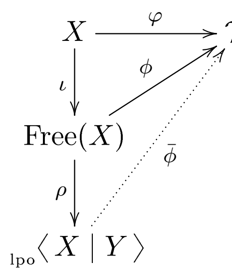
Recall that we have a uniquely determined morphism of linear preoperads $\phi : \mathrm{Free}(X) \longrightarrow \mathcal{T}$ *such that* $\iota\phi = \varphi$ *; cf. Theorem* 4.22*.*

Recall the residue class morphism

$$\begin{aligned} \rho := \rho_{\mathcal{I}} : \mathrm{Free}(X) &\longrightarrow \frac{\mathrm{Free}(X)}{\mathcal{I}} = {}_{\mathrm{lpo}}\langle\, X \mid Y \,\rangle \\ [f] &\longrightarrow \big[[f]\big]_{\mathcal{I}} \,; \end{aligned}$$

cf. Definition 2.47*.*

Then there exists a uniquely determined morphism of linear preoperads $\bar{\phi} : {}_{\mathrm{lpo}}\langle\, X \mid Y \,\rangle \longrightarrow \mathcal{T}$ *such that* $(\iota\rho)\bar{\phi} = \varphi$ *, i.e. such that* $\big[[xs(0,x,0)xt]\big]_{\mathcal{I}} \bar{\phi} = x\varphi$ *for* $x \in X$*.*



Proof. By Lemma 4.22 the morphism of linear preoperads $\phi : \mathrm{Free}(X) \longrightarrow \mathcal{T}$ exists uniquely.

Moreover, since $f\phi = 0$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in Y(m,n)$, by Lemma 2.48 there exists a uniquely determined morphism of linear preoperads $\bar{\phi} : \frac{\mathrm{Free}(X)}{\mathcal{I}} \longrightarrow \mathcal{T}$ such that $\rho\bar{\phi} = \phi$.

So we have a unique morphism of linear preoperads $\bar{\phi} : {}_{\mathrm{lpo}}\langle\, X \mid Y \,\rangle \longrightarrow \mathcal{T}$ such that

$$\iota\rho\bar{\phi} = \iota\phi = \varphi.$$

□

Lemma 4.27. *Let X be a biindexed set and let $Y \subseteq \mathrm{Free}_0(X) \times \mathrm{Free}_0(X)$ be a biindexed subset. Let $(\equiv_Y) \subseteq \mathrm{Free}_0(X) \times \mathrm{Free}_0(X)$ be the congruence on $\mathrm{Free}_0(X)$ generated by Y.*

Suppose given a set-preoperad $\mathcal{P}_0$ and an isomorphism of set-preoperads $\sigma_0 : {}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle \longrightarrow \mathcal{P}_0$.

Let $D_Y(m,n) := \big\{ f - \tilde{f} : (f, \tilde{f}) \in Y(m,n) \big\}$ for $m, n \in \mathbb{Z}_{\geqslant 0}$; cf. Definition 2.50.

Then we have the isomorphism of linear preoperads $\sigma : {}_{\mathrm{lpo}}\langle\, X \mid D_Y \,\rangle \longrightarrow R\mathcal{P}_0$ such that we have the following commutative diagram.

$$\begin{array}{ccccc}
{}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle & \xrightarrow{\beta_{\frac{\mathrm{Free}_0(X)}{(\equiv_Y)}}} & R\Big(\frac{\mathrm{Free}_0}{(\equiv_Y)}\Big) & \xrightarrow{\chi} & {}_{\mathrm{lpo}}\langle\, X \mid D_Y \,\rangle \\
{\scriptstyle \sigma_0}\downarrow\wr & & & & \wr\downarrow{\scriptstyle \sigma} \\
\mathcal{P}_0 & & \xrightarrow[\beta_{\mathcal{P}_0}]{} & & R\mathcal{P}_0
\end{array}$$

Here we abbreviate $\chi := \chi_{\mathrm{Free}_0(X),Y}$; cf. Lemma 2.54.

So in particular, ${}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle \cong \mathcal{P}_0$ implies ${}_{\mathrm{lpo}}\langle\, X \mid D_Y \,\rangle \cong R\mathcal{P}_0$.

Proof. Let $\mathcal{I} := \mathcal{I}_Y = {}_{\mathrm{ideal}}\langle\, D_Y \,\rangle$. By Lemma 2.54 there exists the isomorphism of linear preoperads

$$\begin{aligned}
\chi = \chi_{\mathrm{Free}_0(X),Y} : \quad R\Big(\frac{\mathrm{Free}_0(X)}{(\equiv_Y)}\Big) &\longrightarrow \frac{\mathrm{Free}(X)}{\mathcal{I}} = {}_{\mathrm{lpo}}\langle\, X \mid D_Y \,\rangle \\
\sum_{f \in \mathcal{P}_0(m,n)} r_f\, [f]_{\equiv_Y} &\longmapsto \Big[\sum_{f \in \mathcal{P}_0(m,n)} r_f\, f \Big]_{\mathcal{I}_Y} = \sum_{f \in \mathcal{P}_0(m,n)} r_f\, [f]_{\mathcal{I}_Y}
\end{aligned}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$. Moreover, by Remark 2.25 (3), the linear extension $R\sigma_0 : R\Big(\frac{\mathrm{Free}_0(X)}{(\equiv_Y)}\Big) \longrightarrow R\mathcal{P}_0$ is an isomorphism of linear preoperads; cf. Lemma 2.23. Hence by Remark 2.21, the composite $\sigma := \chi^{-1}\big(R\sigma_0\big) : {}_{\mathrm{lpo}}\langle\, X \mid D_Y \,\rangle \longrightarrow R\mathcal{P}_0$ is an isomorphism of linear preoperads. □

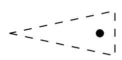
4.6 A presentation for Ass_0

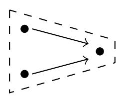
Definition 4.28. Recall that the set-preoperad Ass_0 has $\mathrm{Ass}_0(m,n)$ consisting of monotone maps $[1,m] \longrightarrow [1,n]$ for $m, n \in \mathbb{Z}_{\geqslant 0}$; cf. Definition 2.58. Now define

$$\begin{aligned}
\varepsilon &: [1,0] \longrightarrow [1,1] \\
\mu &: [1,2] \longrightarrow [1,1]
\end{aligned}$$

to be the unique elements in $\mathrm{Ass}_0(0,1)$ and $\mathrm{Ass}_0(2,1)$, respectively.

Pictorially, we have

$\varepsilon =$ 

$\mu =$ 

Note that in Ass_0 we have $(\mu \boxtimes \mathrm{id}_1)\mu = (\mathrm{id}_1 \boxtimes \mu)\mu$ and $(\mathrm{id}_1 \boxtimes \varepsilon)\mu = (\varepsilon \boxtimes \mathrm{id}_1)\mu = \mathrm{id}_1$.

Definition 4.29. We can more generally define $\mu_m \in \mathrm{Ass}_0(m,1)$ to be the unique element in $\mathrm{Ass}_0(m,1)$ for $m \in \mathbb{Z}_{\geqslant 0}$. Note that we have $\mu_0 = \varepsilon \in \mathrm{Ass}_0(0,1)$, $\mu_1 = \mathrm{id}_1 \in \mathrm{Ass}_0(1,1)$ and $\mu_2 = \mu \in \mathrm{Ass}_0(2,1)$.

Furthermore, note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have $(\mu_m \boxtimes \mathrm{id}_1)\mu \in \mathrm{Ass}_0(m+1,1)$, so $(\mu_m \boxtimes \mathrm{id}_1)\mu = \mu_{m+1}$. Hence every μ_m can be written as product and composite of ε, id_1 and μ.

Remark 4.30. *Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Ass}_0(m,n)$. Then we have*

$$a = \mu_{i_1} \boxtimes \mu_{i_2} \boxtimes \ldots \boxtimes \mu_{i_n} ,$$

where $i_j = |a^{-1}(j)|$ for $j \in [1,n]$.

Proof. Define $\tilde{a} := \mu_{i_1} \boxtimes \mu_{i_2} \boxtimes \ldots \boxtimes \mu_{i_n} \in \mathrm{Ass}_0(m,n)$. Then by the definition of $(\boxtimes) = (\boxtimes_{\mathrm{Ass}_0})$ we have

$$\begin{aligned} h\tilde{a} &= \Big(h - \sum_{k \in [1,ha-1]} i_k\Big)\mu_{i_{ha}} + \sum_{k \in [1,ha-1]} 1 \\ &= 1 + ha - 1 = ha \end{aligned}$$

for $h \in [1,n]$. So $a = \tilde{a} = \mu_{i_1} \boxtimes \mu_{i_2} \boxtimes \ldots \boxtimes \mu_{i_n}$. □

Definition 4.31. Define the biindexed set $X := \{\hat{\varepsilon}, \hat{\mu}\}$ with $\hat{\varepsilon} \in X(0,1)$ and $\hat{\mu} \in X(2,1)$.

Recall the injective biindexed maps

$$\begin{aligned} X &\longrightarrow \mathrm{Words}(X) \\ x &\longmapsto xs(0,x,0)xt \end{aligned} \qquad\qquad \begin{aligned} X &\longrightarrow \mathrm{Free}_0(X) \\ x &\longmapsto [xs(0,x,0)xt]. \end{aligned}$$

Abusing notation, we will refer to both $0(0,\hat{\varepsilon},0)1 \in \mathrm{Words}(X)(0,1)$ and $[0(0,\hat{\varepsilon},0)1] \in \mathrm{Free}_0(X)(0,1)$ also by $\hat{\varepsilon}$ and to both $2(0,\hat{\mu},0)1 \in \mathrm{Words}(X)(2,1)$ and $[2(0,\hat{\mu},0)1] \in \mathrm{Free}_0(X)(2,1)$ also by $\hat{\mu}$.

Furthermore, define

$$\begin{aligned} \gamma &:= \big((\hat{\mu} \boxtimes_{\mathrm{Free}_0} \mathrm{id}_{\mathrm{Free}_0,1}) \cdot_{\mathrm{Free}_0} \hat{\mu} \,,\, (\mathrm{id}_{\mathrm{Free}_0,1} \boxtimes_{\mathrm{Free}_0} \hat{\mu}) \cdot_{\mathrm{Free}_0} \hat{\mu}\big) && \in \mathrm{Free}_0(X)(3,1) \times \mathrm{Free}_0(X)(3,1) \\ \nu_{\mathrm{l}} &:= \big((\hat{\varepsilon} \boxtimes_{\mathrm{Free}_0} \mathrm{id}_{\mathrm{Free}_0,1}) \cdot_{\mathrm{Free}_0} \hat{\mu} \,,\, \mathrm{id}_{\mathrm{Free}_0,1}\big) && \in \mathrm{Free}_0(X)(1,1) \times \mathrm{Free}_0(X)(1,1) \\ \nu_{\mathrm{r}} &:= \big((\mathrm{id}_{\mathrm{Free}_0,1} \boxtimes_{\mathrm{Free}_0} \hat{\varepsilon}) \cdot_{\mathrm{Free}_0} \hat{\mu} \,,\, \mathrm{id}_{\mathrm{Free}_0,1}\big) && \in \mathrm{Free}_0(X)(1,1) \times \mathrm{Free}_0(X)(1,1) \end{aligned}$$

and finally $Y := \{\gamma, \nu_{\mathrm{l}}, \nu_{\mathrm{r}}\}$.

Then we have the biindexed map $\varphi_0 : X \longrightarrow \mathrm{Ass}_0$ mapping $\hat{\varepsilon}$ to ε and $\hat{\mu}$ to μ.

Recall the congruence class morphism $\rho_0 : \mathrm{Free}_0(X) \longrightarrow \frac{\mathrm{Free}_0(X)}{(\equiv_Y)}$ defined by

$$\begin{aligned} \rho_0(m,n) : \mathrm{Free}_0(X)(m,n) &\longrightarrow \Big(\frac{\mathrm{Free}_0(X)}{(\equiv_Y)}\Big)(m,n) \\ f &\longmapsto [f]_Y \end{aligned}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$; cf. Definition 2.39, where $[f]_Y := [f]_{\equiv_Y}$ is the congruence class of $f \in \mathrm{Free}_0(X)(m,n)$ with respect to $(\equiv_Y)$.

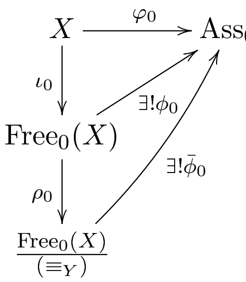
By Theorem 4.19, there exists a uniquely determined morphism of set-preoperads

$$\phi_0 : \mathrm{Free}_0(X) \longrightarrow \mathrm{Ass}_0$$

such that $\iota_0\phi_0 = \varphi_0$, i.e. such that $[xs(0,x,0)xt]\phi_0 = x\varphi_0$ for $x \in X$. We have

$$\begin{aligned}
\big((\hat{\mu} \boxtimes_{\mathrm{Free}_0} \mathrm{id}_{\mathrm{Free}_0,1}) \cdot_{\mathrm{Free}_0} \hat{\mu}\big)\phi_0 &= (\mu \boxtimes \mathrm{id}_1)\cdot \mu \\
&= (\mathrm{id}_1 \boxtimes \mu)\cdot \mu \\
&= \big((\mathrm{id}_{\mathrm{Free}_0,1} \boxtimes_{\mathrm{Free}_0} \hat{\mu}) \cdot_{\mathrm{Free}_0} \hat{\mu}\big)\phi_0 \\
\big((\hat{\varepsilon} \boxtimes_{\mathrm{Free}_0} \mathrm{id}_{\mathrm{Free}_0,1}) \cdot_{\mathrm{Free}_0} \hat{\mu}\big)\phi_0 &= (\varepsilon \boxtimes \mathrm{id}_1)\cdot \mu \\
&= \mathrm{id}_1 \\
&= \mathrm{id}_{\mathrm{Free}_0,1}\,\phi_0 \\
\big((\mathrm{id}_{\mathrm{Free}_0,1} \boxtimes_{\mathrm{Free}_0} \hat{\varepsilon}) \cdot_{\mathrm{Free}_0} \hat{\mu}\big)\phi_0 &= (\mathrm{id}_1 \boxtimes \varepsilon)\cdot \mu \\
&= \mathrm{id}_1 \\
&= \mathrm{id}_{\mathrm{Free}_0,1}\,\phi_0\,.
\end{aligned}$$

So $f\phi_0 = \tilde{f}\phi_0$ for $m,n \in \mathbb{Z}_{\geqslant 0}$ and $(f,\tilde{f}) \in Y(m,n)$. By Lemma 4.24 there exists a uniquely determined morphism of set-preoperads $\bar{\phi}_0 : {}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle = \frac{\mathrm{Free}_0(X)}{(\equiv_Y)} \longrightarrow \mathrm{Ass}_0$ such that $\iota_0\rho_0\bar{\phi}_0 = \varphi_0$.



So define $\mathrm{Ass}_{0,\mathrm{P}} := {}_{\mathrm{spo}}\langle\, X \mid Y \,\rangle = \frac{\mathrm{Free}_0(X)}{(\equiv_Y)}$ and consider the morphism of set-preoperads

$$\bar{\phi}_0 : \mathrm{Ass}_{0,\mathrm{P}} = \frac{\mathrm{Free}_0(X)}{(\equiv_Y)} \longrightarrow \mathrm{Ass}_0$$

that is uniquely determined and given by $[f]_Y\bar{\phi}_0 = f\phi_0$ for $f \in \mathrm{Free}_0(X)(m,n)$ for $m,n \in \mathbb{Z}_{\geqslant 0}$.

So in particular, we have $\big[[xs(0,x,0)xt]\big]_Y\bar{\phi}_0 = [xs(0,x,0)xt]\phi_0 = x\iota_0\phi_0 = x\varphi_0$ for $m,n \in \mathbb{Z}_{\geqslant 0}$ and $x \in X(m,n)$, which implies $[\hat{\varepsilon}]_Y\bar{\phi}_0 = \varepsilon$ and $[\hat{\mu}]_Y\bar{\phi}_0 = \mu$.

Theorem 4.32. *The morphism $\bar{\phi}_0 : \mathrm{Ass}_{0,P} \longrightarrow \mathrm{Ass}_0$ of set-preoperads is an isomorphism.*

So in particular, abbreviating $(\boxtimes) := (\boxtimes_{\mathrm{Free}_0})$, $(\cdot) := (\cdot_{\mathrm{Free}_0})$ and $\mathrm{id}_1 := \mathrm{id}_{\mathrm{Free}_0,1}$ we have

$$\mathrm{Ass}_0 \overset{\sim}{\longleftarrow} {}_{\mathrm{spo}}\langle\, \hat{\mu}, \hat{\varepsilon} \mid \big((\hat{\mu} \boxtimes \mathrm{id}_1)\cdot \hat{\mu}\,,\,(\mathrm{id}_1 \boxtimes \hat{\mu})\cdot \hat{\mu}\big), \big((\mathrm{id}_1 \boxtimes \hat{\varepsilon})\cdot \hat{\mu}\,,\,\mathrm{id}_1\,\big), \big((\hat{\varepsilon} \boxtimes \mathrm{id}_1)\cdot \hat{\mu}\,,\,\mathrm{id}_1\,\big)\,\rangle\,.$$

Proof. We have to show that $\bar{\phi}_0(m,n)$ is bijective for $m,n \in \mathbb{Z}_{\geqslant 0}$; cf. Lemma 2.20.

Surjectivity. Suppose given $a \in \mathrm{Ass}_0(m,n)$. By Remark 4.30 we can write $a = \mu_{i_1} \boxtimes \mu_{i_2} \boxtimes \ldots \boxtimes \mu_{i_n}$, where $i_j = |a^{-1}(j)|$ for $j \in [1,n]$ and where μ_i is the unique element in $\mathrm{Ass}_0(i,1)$ for $i \in \mathbb{Z}_{\geqslant 0}$; cf. Definition 4.29.

Now define recursively

$$\hat{\mu}_i := \begin{cases} \hat{\varepsilon} & \text{if } i = 0 \\ \mathrm{id}_{\mathrm{Free}_0,1} & \text{if } i = 1 \\ (\hat{\mu}_{i-1} \boxtimes_{\mathrm{Free}_0} \mathrm{id}_{\mathrm{Free}_0,1}) \cdot_{\mathrm{Free}_0} \hat{\mu} & \text{if } i \geqslant 2 \end{cases}$$

for $i \in \mathbb{Z}_{\geqslant 0}$. Note that $\hat{\mu}_i \in \mathrm{Free}_0(X)(i,1)$ for $i \in \mathbb{Z}_{\geqslant 0}$, hence $[\hat{\mu}_i]_Y \bar{\phi}_0 = \hat{\mu}_i \phi_0 = \mu_i$ for $i \in \mathbb{Z}_{\geqslant 0}$. So we have

$$\begin{aligned}\big[\hat{\mu}_{i_1} \boxtimes_{\mathrm{Free}_0} \hat{\mu}_{i_2} \boxtimes_{\mathrm{Free}_0} \dots \boxtimes_{\mathrm{Free}_0} \hat{\mu}_{i_n}\big]_Y \bar{\phi}_0 &= \big([\hat{\mu}_{i_1}]_Y \boxtimes_{\mathrm{Ass}_{0,\mathrm{P}}} [\hat{\mu}_{i_2}]_Y \boxtimes_{\mathrm{Ass}_{0,\mathrm{P}}} \dots \boxtimes_{\mathrm{Ass}_{0,\mathrm{P}}} [\hat{\mu}_{i_n}]_Y\big)\bar{\phi}_0 \\ &= [\hat{\mu}_{i_1}]_Y \bar{\phi}_0 \boxtimes [\hat{\mu}_{i_2}]_Y \bar{\phi}_0 \boxtimes \dots \boxtimes [\hat{\mu}_{i_n}]_Y \bar{\phi}_0 \\ &= \mu_{i_1} \boxtimes \mu_{i_2} \boxtimes \dots \boxtimes \mu_{i_n} \\ &= a.\end{aligned}$$

Hence $\bar{\phi}_0(m,n)$ is surjective for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Injectivity. In order to show that $\bar{\phi}_0(m,n)$ is injective for $m, n \in \mathbb{Z}_{\geqslant 0}$ we will need a couple of steps.

Step 1: *Finding a standard form for elements in* $\mathrm{Free}_0(X)$.

Claim 1.1. Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and $[w] \in \mathrm{Free}_0(X)(m,n)$ with length $u \geqslant 2$, that is,

$$w = \tilde{q}_0(\tilde{l}_1, x_1, \tilde{r}_1)\tilde{q}_1 \cdots \tilde{q}_{u-1}(\tilde{l}_u, x_u, \tilde{r}_u)\tilde{q}_u \in \mathrm{Words}(X)(m,n),$$

where $\tilde{l}_i, \tilde{r}_i \in \mathbb{Z}_{\geqslant 0}$ and $x_i \in \{\hat{\varepsilon}, \hat{\mu}\}$ for $i \in [1,u]$ and $m = \tilde{q}_0 = \tilde{l}_1 + x_1 s + \tilde{r}_1$, $n = \tilde{q}_u = \tilde{l}_u + x_u t + \tilde{r}_u$ and $\tilde{l}_i + x_i t + \tilde{r}_i = \tilde{q}_i = \tilde{l}_{i+1} + x_{i+1}s + \tilde{r}_{i+1}$ for $i \in [1, u-1]$.

Then there exists

$$v = q_0(l_1, \mu, r_1)q_1(l_2, \mu, r_2)q_2 \cdots q_{k-1}(l_k, \mu, r_k)q_k(\lambda_\kappa, \varepsilon, \rho_\kappa)p_{\kappa-1} \cdots p_2(\lambda_1, \varepsilon, \rho_1)p_1 \in \mathrm{Words}(X)(m,n)$$

with $l_i \leqslant l_{i+1}$ for $i \in [1, k-1]$ and $\lambda_i \leqslant \lambda_{i+1}$ for $i \in [1, \kappa - 1]$ such that $[w] \equiv_Y [v]$.

We say that v is in *standard form*.

Note that words of length 0 or 1 are already in standard form.

Proof of Claim 1.1. For the proof of this Claim we denote by $(\boxtimes)$ and $(\cdot)$ the multiplication and composition in Free_0 and for $m \in \mathbb{Z}_{\geqslant 0}$ we denote by id_m the identity element in $\mathrm{Free}_0(m,m)$.

For a word $w = \tilde{q}_0(\tilde{l}_1, x_1, \tilde{r}_1)\tilde{q}_1 \cdots \tilde{q}_{u-1}(\tilde{l}_u, x_u, \tilde{r}_u)\tilde{q}_u \in \mathrm{Words}(X)(m,n)$ we define the measure $\sigma_w = \Big(u, \sum\limits_{i \in [1,u]} \tilde{l}_i\Big) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$. Let Σ_w be the set of all measures for the different representatives of the element $\big[[w]\big]_Y$, that is, $\Sigma_w = \{\sigma_z : [w] \equiv_Y [z], z \in \mathrm{Words}(X)(m,n)\}$.

Now we endow Σ_w with the lexicographic order, that is, for $(i,j), (i',j') \in \Sigma_w$ we have $(i,j) < (i',j')$ if and only if $i < i'$ or ($i = i'$ and $j < j'$).

Since for $w \in \mathrm{Words}(X)(m,n)$ we have $\Sigma_w \subseteq \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ there has to be a representative $z \in \mathrm{Words}(X)(m,n)$ with $[z] \equiv_Y [w]$ that has minimal measure $\sigma_z = \min\{\Sigma_w\} \geqslant (0,0)$.

Assume now that this representative z is not in standard form.

Write $z = \check{q}_0(\check{l}_1, \check{x}_1, \check{r}_1)\check{q}_1 \cdots \check{q}_{\check{u}-1}(\check{l}_{\check{u}}, \check{x}_{\check{u}}, \check{r}_{\check{u}})\check{q}_{\check{u}}$. Since z is not in standard form, at least one of the following occur in z.

(1) There exists $i \in [1, \check{u} - 1]$ such that $\check{x}_i = \hat{\varepsilon}$ and $\check{x}_{i+1} = \hat{\mu}$.

(2) There exists $i \in [1, \check{u} - 1]$ such that $\check{x}_i = \hat{\mu} = \check{x}_{i+1}$ but $\check{l}_i > \check{l}_{i+1}$.

(3) There exists $i \in [1, \check{u} - 1]$ such that $\check{x}_i = \hat{\varepsilon} = \check{x}_{i+1}$ but $\check{l}_i < \check{l}_{i+1}$.

To arrive at a contradiction, we search for an element $z' \in \mathrm{Words}(X)(m,n)$ with $[z'] \equiv_Y [z]$ and $\sigma_{z'} < \sigma_z$.

For the following calculation note that $\hat{\mu}d = \hat{\mu}t - \hat{\mu}s = 1 - 2 = -1$ and $\hat{\varepsilon}d = \hat{\varepsilon}t - \hat{\varepsilon}s = 1 - 0 = 1$.

Case (1). There are two different ways how to obtain z': using the congruence on $\mathrm{Free}_0(X)$ or using the equivalence relation valid in $\mathrm{Free}_0(X)$. We have the following four cases.

(1.a.i) *Case* $\check{l}_i = \check{l}_{i+1}$. Since $\check{l}_i + 1 + \check{r}_i = \check{q}_i = \check{l}_{i+1} + 2 + \check{r}_{i+1}$ that implies $\check{r}_i = \check{r}_{i+1} + 1$. Furthermore, we have $\check{q}_{i-1} = \check{l}_i + \check{r}_i = \check{l}_{i+1} + \check{r}_{i+1} + 1 = \check{q}_{i+1}$. So we have

$$\begin{aligned}
\big[\check{q}_{i-1}(\check{l}_i, \hat{\varepsilon}, \check{r}_i)\check{q}_i(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1})\check{q}_{i+1}\big] &= (\mathrm{id}_{\check{l}_i} \boxtimes \hat{\varepsilon} \boxtimes \mathrm{id}_{\check{r}_i}) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\
&= (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\varepsilon} \boxtimes \mathrm{id}_{\check{r}_{i+1}+1}) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\
&= \big(\mathrm{id}_{\check{l}_{i+1}} \boxtimes (\hat{\varepsilon} \boxtimes \mathrm{id}_1) \boxtimes \mathrm{id}_{\check{r}_{i+1}}\big) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\
&= (\mathrm{id}_{\check{l}_{i+1}} \boxtimes ((\hat{\varepsilon} \boxtimes \mathrm{id}_1) \cdot \hat{\mu}) \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\
&\equiv_Y (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \mathrm{id}_1 \boxtimes \mathrm{id}_{\check{r}_{i+1}}) && \text{(using } \nu_{\mathrm{l}}) \\
&= \mathrm{id}_{\check{l}_{i+1}+1+\check{r}_{i+1}} \\
&= \mathrm{id}_{\check{q}_{i+1}} = [\check{q}_{i+1}].
\end{aligned}$$

So by defining

$$z' = \check{q}_0(\check{l}_1, \check{x}_1, \check{r}_1)\check{q}_1 \cdots \check{q}_{i-2}(\check{l}_{i-1}, \check{x}_{i-1}, \check{r}_{i-1})\check{q}_{i+1}(\check{l}_{i+1}, \check{x}_{i+1}, \check{r}_{i+1})\check{q}_{i+2} \cdots \check{q}_{\check{u}-1}(\check{l}_{\check{u}}, \check{x}_{\check{u}}, \check{r}_{\check{u}})\check{q}_{\check{u}},$$

we obtain a word $z' \in \mathrm{Words}(X)(m, n)$ with $[z'] \equiv_Y [z] \equiv_Y [w]$ and

$$\sigma_{z'} = \Big(\check{u} - 2, \Big(\sum_{j \in [1, \check{u}]} \check{l}_j\Big) - \check{l}_i - \check{l}_{i+1}\Big) < \sigma_z,$$

a *contradiction.*

(1.a.ii) *Case* $\check{l}_i = \check{l}_{i+1} + 1$. This implies $\check{r}_i = \check{r}_{i+1}$ and $\check{q}_{i-1} = \check{l}_i + \check{r}_i = \check{l}_{i+1} + 1 + \check{r}_{i+1} = \check{q}_{i+1}$. Then we have

$$\begin{aligned}
\big[\check{q}_{i-1}(\check{l}_i, \hat{\varepsilon}, \check{r}_i)\check{q}_i(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1})\check{q}_{i+1}\big] &= (\mathrm{id}_{\check{l}_i} \boxtimes \hat{\varepsilon} \boxtimes \mathrm{id}_{\check{r}_i}) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\
&= (\mathrm{id}_{\check{l}_{i+1}+1} \boxtimes \hat{\varepsilon} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\
&= \big(\mathrm{id}_{\check{l}_{i+1}} \boxtimes (\mathrm{id}_1 \boxtimes \hat{\varepsilon}) \boxtimes \mathrm{id}_{\check{r}_{i+1}}\big) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\
&= (\mathrm{id}_{\check{l}_{i+1}} \boxtimes ((\mathrm{id}_1 \boxtimes \hat{\varepsilon}) \cdot \hat{\mu}) \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\
&\equiv_Y (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \mathrm{id}_1 \boxtimes \mathrm{id}_{\check{r}_{i+1}}) && \text{(using } \nu_{\mathrm{r}}) \\
&= \mathrm{id}_{\check{l}_{i+1}+1+\check{r}_{i+1}} \\
&= \mathrm{id}_{\check{q}_{i+1}} = [\check{q}_{i+1}].
\end{aligned}$$

So again by defining

$$z' = \check{q}_0(\check{l}_1, \check{x}_1, \check{r}_1)\check{q}_1 \cdots \check{q}_{i-2}(\check{l}_{i-1}, \check{x}_{i-1}, \check{r}_{i-1})\check{q}_{i+1}(\check{l}_{i+1}, \check{x}_{i+1}, \check{r}_{i+1})\check{q}_{i+2} \cdots \check{q}_{\check{u}-1}(\check{l}_{\check{u}}, \check{x}_{\check{u}}, \check{r}_{\check{u}})\check{q}_{\check{u}},$$

we get a word $z' \in \mathrm{Words}(X)(m, n)$ with $[z'] \equiv_Y [z] \equiv_Y [w]$ and

$$\sigma_{z'} = \Big(\check{u} - 2, \Big(\sum_{j \in [1, \check{u}]} \check{l}_j) - \check{l}_i - \check{l}_{i+1}\Big) < \sigma_z,$$

a *contradiction.*

(1.b.i) *Case* $\check{l}_i < \check{l}_{i+1}$. This implies that $\check{l}_{i+1} \geqslant \check{l}_i + 1 = \check{l}_i + \hat{\varepsilon}t$, so by the definition of the equivalence relation $(\approx)$ on $\mathrm{Words}(X)$ we have

$$\begin{aligned}
\check{q}_{i-1}(\check{l}_i, \hat{\varepsilon}, \check{r}_i)\check{q}_i(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1})\check{q}_{i+1} &\approx \check{q}_{i-1}(\check{l}_{i+1} - \hat{\varepsilon}d, \hat{\mu}, \check{r}_{i+1})(\check{q}_{i+1} + \check{q}_{i-1} - \check{q}_i)(\check{l}_i, \hat{\varepsilon}, \check{r}_i + \hat{\mu}d)\check{q}_{i+1} \\
&= \check{q}_{i-1}(\check{l}_{i+1} - 1, \hat{\mu}, \check{r}_{i+1})(\check{q}_{i+1} + \check{q}_{i-1} - \check{q}_i)(\check{l}_i, \hat{\varepsilon}, \check{r}_i + (-1))\check{q}_{i+1}.
\end{aligned}$$

So define

$$z' = \check{q}'_0(\check{l}'_1, \check{x}'_1, \check{r}'_1)\check{q}'_1 \cdots \check{q}'_{\check{u}-1}(\check{l}'_{\check{u}}, \check{x}'_{\check{u}}, \check{r}'_{\check{u}})\check{q}'_{\check{u}}$$

with

* $\check{l}'_i = \check{l}_{i+1} - 1$, $\check{r}'_i = \check{r}_{i+1}$ and $\check{x}'_i = \check{x}_{i+1}$
* $\check{l}'_{i+1} = \check{l}_i$, $\check{r}'_{i+1} = \check{r}_i - 1$ and $\check{x}'_{i+1} = \check{x}_i$
* $\check{l}'_j = \check{l}_j$, $\check{r}'_j = \check{r}_j$ and $\check{x}'_j = \check{x}_j$ for $j \in [1, \check{u}] \setminus \{i, i+1\}$.

Then we have $[z'] = [z]$ and

$$\sigma_{z'} = \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}'_j\Big) = \Big(\check{u}\,, \Big(\sum_{j\in[1,\check{u}]} \check{l}_j\Big) - 1\Big) < \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}_j\Big) = \sigma_z\,,$$

a *contradiction*.

(1.b.ii) *Case* $\check{l}_i > \check{l}_{i+1} + 1$. This implies $\check{l}_i \geqslant \check{l}_{i+1} + 2 = \check{l}_{i+1} + \hat{\mu}s$, so using the equivalence relation $(\approx)$ on Words(X) we have

$$\begin{aligned}\check{q}_{i-1}(\check{l}_i, \hat{\varepsilon}, \check{r}_i)\check{q}_i(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1})\check{q}_{i+1} &\approx \check{q}_{i-1}(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1} - \hat{\varepsilon}d)(\check{q}_{i+1} + \check{q}_{i-1} - \check{q}_i)(\check{l}_i + \hat{\mu}d, \hat{\varepsilon}, \check{r}_i)\check{q}_{i+1} \\ &= \check{q}_{i-1}(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1} - 1)(\check{q}_{i+1} + \check{q}_{i-1} - \check{q}_i)(\check{l}_i + (-1), \hat{\varepsilon}, \check{r}_i)\check{q}_{i+1}\end{aligned}$$

So define

$$z' = \check{q}'_0(\check{l}'_1, \check{x}'_1, \check{r}'_1)\check{q}'_1 \cdots \check{q}'_{\check{u}-1}(\check{l}'_{\check{u}}, \check{x}'_{\check{u}}, \check{r}'_{\check{u}})\check{q}'_{\check{u}}$$

with

* $\check{l}'_i = \check{l}_{i+1}$, $\check{r}'_i = \check{r}_{i+1} - 1$ and $\check{x}'_i = \check{x}_{i+1}$
* $\check{l}'_{i+1} = \check{l}_i - 1$, $\check{r}'_{i+1} = \check{r}_i$ and $\check{x}'_{i+1} = \check{x}_i$
* $\check{l}'_j = \check{l}_j$, $\check{r}'_j = \check{r}_j$ and $\check{x}'_j = \check{x}_j$ for $j \in [1, \check{u}] \setminus \{i, i+1\}$.

Then we have $[z'] = [z]$ and

$$\sigma_{z'} = \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}'_j\Big) = \Big(\check{u}\,, \Big(\sum_{j\in[1,\check{u}]} \check{l}_j\Big) - 1\Big) < \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}_j\Big) = \sigma_z\,,$$

a *contradiction*.

Case (2). Again we can either use the congruence $(\equiv_Y)$ or the equivalence relation $(\approx)$ on Words(X). We have the following two cases.

(2.a) *Case* $\check{l}_i = \check{l}_{i+1} + 1$. Since $\check{l}_i + 1 + \check{r}_i = \check{q}_i = \check{l}_{i+1} + 2 + \check{r}_{i+1}$ this implies $\check{r}_i = \check{r}_{i+1}$. So we have

$$\begin{aligned}\big[\check{q}_{i-1}(\check{l}_i, \hat{\mu}, \check{r}_i)\check{q}_i(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1})\check{q}_{i+1}\big] &= (\mathrm{id}_{\check{l}_i} \boxtimes \check{\mu} \boxtimes \mathrm{id}_{\check{r}_i}) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\ &= (\mathrm{id}_{\check{l}_{i+1}+1} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\ &= \big(\mathrm{id}_{\check{l}_{i+1}} \boxtimes (\mathrm{id}_1 \boxtimes \hat{\mu}) \boxtimes \mathrm{id}_{\check{r}_{i+1}}\big) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\ &= (\mathrm{id}_{\check{l}_{i+1}} \boxtimes ((\mathrm{id}_1 \boxtimes \hat{\mu})\hat{\mu}) \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\ &\equiv_Y (\mathrm{id}_{\check{l}_{i+1}} \boxtimes ((\hat{\mu} \boxtimes \mathrm{id}_1)\hat{\mu}) \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \qquad \text{(using } \gamma \text{)} \\ &= \big(\mathrm{id}_{\check{l}_{i+1}} \boxtimes (\hat{\mu} \boxtimes \mathrm{id}_1) \boxtimes \mathrm{id}_{\check{r}_{i+1}}\big) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\ &= (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}+1}) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\ &= (\mathrm{id}_{\check{l}_i-1} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_i+1}) \cdot (\mathrm{id}_{\check{l}_{i+1}} \boxtimes \hat{\mu} \boxtimes \mathrm{id}_{\check{r}_{i+1}}) \\ &= \check{q}_{i-1}(\check{l}_i - 1, \hat{\mu}, \check{r}_i + 1)\check{q}_i(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1})\check{q}_{i+1}\,.\end{aligned}$$

So in this case define

$$z' = \check{q}'_0(\check{l}'_1, \check{x}'_1, \check{r}'_1)\check{q}'_1 \cdots \check{q}'_{\check{u}-1}(\check{l}'_{\check{u}}, \check{x}'_{\check{u}}, \check{r}'_{\check{u}})\check{q}'_{\check{u}} \in \mathrm{Words}(X)(m, n)$$

with

* $\check{l}'_i = \check{l}_i - 1$, $\check{r}'_i = \check{r}_i + 1$ and $\check{x}'_i = \check{x}_{i+1}$
* $\check{l}'_{i+1} = \check{l}_{i+1}$, $\check{r}'_{i+1} = \check{r}_{i+1}$ and $\check{x}'_{i+1} = \check{x}_i$
* $\check{l}'_j = \check{l}_j$, $\check{r}'_j = \check{r}_j$ and $\check{x}'_j = \check{x}_j$ for $j \in [1, \check{u}] \setminus \{i, i+1\}$.

Then we have $[z'] = [z]$ and

$$\sigma_{z'} = \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}'_j\Big) = \Big(\check{u}\,, \Big(\sum_{j\in[1,\check{u}]} \check{l}_j\Big) - 1\Big) < \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}_j\Big) = \sigma_z\,,$$

a *contradiction.*

(2.b) *Case* $\check{l}_i > \check{l}_{i+1} + 1$. This means that $\check{l}_i \geqslant \check{l}_{i+1} + 2 = \check{l}_{i+1} + \hat{\mu}s$, so using the equivalence relation $(\approx)$ on Words(X) we have

$$\begin{aligned}\check{q}_{i-1}(\check{l}_i, \hat{\mu}, \check{r}_i)\check{q}_i(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1})\check{q}_{i+1} &\approx \check{q}_{i-1}(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1} - \hat{\mu}d)(\check{q}_{i+1} + \check{q}_{i-1} - \check{q}_i)(\check{l}_i + \hat{\mu}d, \hat{\mu}, \check{r}_i)\check{q}_{i+1}\\ &= \check{q}_{i-1}(\check{l}_{i+1}, \hat{\mu}, \check{r}_{i+1} - (-1))(\check{q}_{i+1} + \check{q}_{i-1} - \check{q}_i)(\check{l}_i + (-1), \hat{\mu}, \check{r}_i)\check{q}_{i+1}\end{aligned}$$

Now define

$$z' = \check{q}'_0(\check{l}'_1, \check{x}'_1, \check{r}'_1)\check{q}'_1 \cdots \check{q}'_{\check{u}-1}(\check{l}'_{\check{u}}, \check{x}'_{\check{u}}, \check{r}'_{\check{u}})\check{q}'_{\check{u}} \in \text{Words}(X)(m, n)$$

with

* $\check{l}'_i = \check{l}_{i+1}$, $\check{r}'_i = \check{r}_{i+1} + 1$ and $\check{x}'_i = \check{x}_{i+1}$
* $\check{l}'_{i+1} = \check{l}_i - 1$, $\check{r}'_{i+1} = \check{r}_i$ and $\check{x}'_{i+1} = \check{x}_i$
* $\check{l}'_j = \check{l}_j$, $\check{r}'_j = \check{r}_j$ and $\check{x}'_j = \check{x}_j$ for $j \in [1, \check{u}] \setminus \{i, i+1\}$.

Then we have $[z'] = [z]$ and

$$\sigma_{z'} = \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}'_j\Big) = \Big(\check{u}\,, \Big(\sum_{j\in[1,\check{u}]} \check{l}_j\Big) - 1\Big) < \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}_j\Big) = \sigma_z\,,$$

a *contradiction.*

Case (3). We have $\check{l}_i < \check{l}_{i+1}$. This means that $\check{l}_i + \hat{\varepsilon}t = \check{l}_i + 1 \leqslant \check{l}_{i+1}$, so using the equivalence relation $(\approx)$ on Words(X) we obtain

$$\begin{aligned}\check{q}_{i-1}(\check{l}_i, \hat{\varepsilon}, \check{r}_i)\check{q}_i(\check{l}_{i+1}, \hat{\varepsilon}, \check{r}_{i+1})\check{q}_{i+1} &\approx \check{q}_{i-1}(\check{l}_{i+1} - \hat{\varepsilon}d, \hat{\varepsilon}, \check{r}_{i+1})(\check{q}_{i-1} + \check{q}_{i+1} - \check{q}_i)(\check{l}_i, \hat{\varepsilon}, \check{r}_i + \hat{\varepsilon}d)\check{q}_{i+1}\\ &= \check{q}_{i-1}(\check{l}_{i+1} - 1, \hat{\varepsilon}, \check{r}_{i+1})(\check{q}_{i-1} + \check{q}_{i+1} - \check{q}_i)(\check{l}_i, \hat{\varepsilon}, \check{r}_i + 1)\check{q}_{i+1}\,.\end{aligned}$$

Now define

$$z' = \check{q}'_0(\check{l}'_1, \check{x}'_1, \check{r}'_1)\check{q}'_1 \cdots \check{q}'_{\check{u}-1}(\check{l}'_{\check{u}}, \check{x}'_{\check{u}}, \check{r}'_{\check{u}})\check{q}'_{\check{u}} \in \text{Words}(X)(m, n)$$

with

* $\check{l}'_i = \check{l}_{i+1} - 1$, $\check{r}'_i = \check{r}_{i+1}$ and $\check{x}'_i = \check{x}_{i+1}$
* $\check{l}'_{i+1} = \check{l}_i$, $\check{r}'_{i+1} = \check{r}_i + 1$ and $\check{x}'_{i+1} = \check{x}_i$
* $\check{l}'_j = \check{l}_j$, $\check{r}'_j = \check{r}_j$ and $\check{x}'_j = \check{x}_j$ for $j \in [1, u] \setminus \{i, i+1\}$.

Then we have $[z'] = [z]$ and

$$\sigma_{z'} = \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}'_j\Big) = \Big(\check{u}\,, \Big(\sum_{j\in[1,\check{u}]} \check{l}_j\Big) - 1\Big) < \Big(\check{u}\,, \sum_{j\in[1,\check{u}]} \check{l}_j\Big) = \sigma_z\,,$$

a *contradiction.*

This proves *Claim 1.1.*

Step 2: *Injectivity of the map restricted to words in standard form.*

Now we will show that different standard forms are being sent to different elements of Ass_0 by $\bar{\phi}_0$.

Let $m \in \mathbb{Z}_{\geqslant 1}$. Let

$$v = q_0(l_1, \hat{\mu}, r_1)q_1(l_2, \hat{\mu}, r_2)q_2 \cdots q_{k-1}(l_k, \hat{\mu}, r_k)q_k(\lambda_\kappa, \hat{\varepsilon}, \rho_\kappa)p_{\kappa-1} \cdots p_1(\lambda_1, \hat{\varepsilon}, \rho_1)p_0 \in \mathrm{Words}(X)(m, n)$$

be in standard form, that is, $l_i \leqslant l_{i+1}$ for $i \in [1, k-1]$ and $\lambda_i \leqslant \lambda_{i+1}$ for $i \in [1, \kappa - 1]$.

Note that $n \geqslant 1$. Furthermore, note that $k \in [0, m-1]$ and $\kappa \in [1, n-1]$.

Then we have

$$\big[[v]\big]_Y \bar{\phi}_0 = (\mathrm{id}_{l_1} \boxtimes \mu \boxtimes \mathrm{id}_{r_1}) \cdots (\mathrm{id}_{l_k} \boxtimes \mu \boxtimes \mathrm{id}_{r_k}) \cdot (\mathrm{id}_{\lambda_\kappa} \boxtimes \varepsilon \boxtimes \mathrm{id}_{\rho_\kappa}) \cdots (\mathrm{id}_{\lambda_1} \boxtimes \varepsilon \boxtimes \mathrm{id}_{\rho_1}).$$

By abuse of notation we will also write

$$\big[[v]\big]_Y \bar{\phi}_0 = (l_1, \mu, r_1) \cdots (l_k, \mu, r_k) \cdot (\lambda_\kappa, \varepsilon, \rho_\kappa) \cdots (\lambda_1, \varepsilon, \rho_1) \in \mathrm{Ass}_0(m, n),$$

the same notation we know from $\mathrm{Free}_0(X)$. In particular, if $k = 0$ and $\kappa = 0$, then $m = q_0 = p_0 = n$ and the empty composite is $\mathrm{id}_m = \mathrm{id}_n$.

We know that $\big[[v]\big]_Y \bar{\phi}_0 \in \mathrm{Ass}_0(m, n)$, so it is a monotone map from $[1, m]$ to $[1, n]$.

Claim 2.1. Any monotone map $f \in \mathrm{Ass}_0(m, n)$ can be written in a unique way as the composite of a surjective monotone map $f_{\mathrm{sur}} \in \mathrm{Ass}_0(m, p)$ and an injective monotone map $f_{\mathrm{in}} \in \mathrm{Ass}_0(p, n)$ for some $p \in \mathbb{Z}_{\geqslant 0}$, so $f = f_{\mathrm{sur}} \cdot f_{\mathrm{in}}$.

Proof of Claim 2.1. This is the factorisation over the image. In particular $p = |\operatorname{Im}(f)|$. This proves *Claim 2.1.*

Now define

$$a := (l_1, \mu, r_1) \cdot (l_2, \mu, r_2) \cdots (l_k, \mu, r_k) \in \mathrm{Ass}_0(m, m-k)$$
$$b := (\lambda_\kappa, \varepsilon, \rho_\kappa) \cdot (\lambda_{\kappa-1}, \varepsilon, \rho_{\kappa-1}) \cdots (\lambda_1, \varepsilon, \rho_1) \in \mathrm{Ass}_0(n-\kappa, n).$$

We have $m - k = \big|\operatorname{Im}\big(\big[[v]\big]_Y \bar{\phi}_0\big)\big| = n - \kappa$ and a is surjective and b is injective. So Claim 2.1 implies that $\big[[v]\big]_Y \bar{\phi}_0 = ab$ is the unique decomposition.

We will show in Claim 2.2 and Claim 2.3 that the indices l_i and λ_j for $i \in [1, k]$, $j \in [1, \kappa]$ can be obtained from the sizes of the fibres of the maps a and b.

Claim 2.2. Consider $a = (l_1, \mu, r_1) \cdot (l_2, \mu, r_2) \cdots (l_k, \mu, r_k) \in \mathrm{Ass}_0(m, m-k)$. Recall that $k \in [0, m]$ and that $l_i \leqslant l_{i+1}$ for $i \in [1, k-1]$. Define $l_0 := 0$. For $i \in [1, m-k]$ define $f_i := |a^{-1}(i)|$. Then (i) and (ii) hold.

(i) We have $l_s + 1 = \min\{j \in [1, m-k] : \sum_{i \in [1,j]} (f_i - 1) \geqslant s\}$ for $s \in [1, k]$.

(ii) We have $f_i - 1 = 0$ for $i \in [l_k + 2, m-k]$.

Proof of Claim 2.2. We will show this via induction on the number of factors k.

Case $k = 0$. Then $a = \mathrm{id}_m$. So we have $f_i = 1$ for all $i \in [1, m]$, hence (ii) is true. In particular $f_1 - 1 = 0$, hence $\min\{j \in [1, m-k] : \sum_{i \in [1,j]} (f_i - 1) \geqslant 0\} = 1 = l_0 + 1$, so (i) holds.

Case $k > 0$. We have $a = (l_1, \mu, r_1) \cdots (l_{k-1}, \mu, r_{k-1}) \cdot (l_k, \mu, r_k) \in \mathrm{Ass}_0(m, m-k)$.

By induction, (i) and (ii) hold for $a' = (l_1, \mu, r_1) \cdots (l_{k-1}, \mu, r_{k-1}) \in \mathrm{Ass}_0(m, m-k+1)$. So denoting the fibre sizes of a' by $f'_i := |a^{-1}(i)|$ for $i \in [1, m-k+1]$, then (i′) and (ii′) hold.

(i′) We have $\sum_{i\in[1,l_s]}(f'_i-1)<s$ and $\sum_{i\in[1,l_s+1]}(f'_i-1)\geqslant s$ for $s\in[1,k-1]$.

(ii′) We have $f'_i-1=0$ for $i\in[l_{k-1}+2,m-k+1]$.

Note that for $i\in[1,m-k]$ the fibre sizes f_i of a satisfy

$$f_i-1=\begin{cases} f'_i-1 & \text{if } i\in[1,l_k] \\ f'_i+f'_{i+1}-1 & \text{if } i=l_k+1 \\ f'_{i+1}-1 & \text{if } i\in[l_k+2,m-k]. \end{cases}$$

Ad (ii). Let $i\in[l_k+2,m-k]$. Since $l_k\geqslant l_{k-1}$ we have $i+1\in[l_{k-1}+2,m-k+1]$ and $f_i-1=f'_{i+1}-1=0$ by (ii′).

Ad (i). We have to show that for $s\in[1,k]$ we have $\sum_{i\in[1,l_s]}(f_i-1)\overset{!}{>}s$ and $\sum_{i\in[1,l_s+1]}(f_i-1)\overset{!}{\geqslant}s$.

Case 1: $s\in[1,k-1]$. We have $s\leqslant k-1\leqslant k$, so $l_s\leqslant l_{k-1}\leqslant l_k$. This implies that $f_i-1=f'_i-1$ for $i\in[1,l_s]$ and that $f_{l_s+1}\geqslant f'_{l_s+1}$.

So we have

$$\sum_{i\in[1,l_s]}(f_i-1)=\sum_{i\in[1,l_s]}(f'_i-1)<s$$

by (i′) and

$$\begin{aligned}\sum_{i\in[1,l_s+1]}(f_i-1)&=\Big(\sum_{i\in[1,l_s]}(f_i-1)\Big)+(f_{l_s+1}-1)\\ &=\Big(\sum_{i\in[1,l_s]}(f'_i-1)\Big)+(f_{l_s+1}-1)\\ &\geqslant\Big(\sum_{i\in[1,l_s]}(f'_i-1)\Big)+(f'_{l_s+1}-1)\\ &=\sum_{i\in[1,l_s+1]}(f'_i-1)\geqslant s\end{aligned}$$

by (i′).

Case 2: $s=k$. Note that $\sum_{i\in[1,m-k]}f_i=m$. We also know that $f_i=1$ for $i\in[l_k+2,m-k]$ by (ii). This implies that

$$\begin{aligned}\sum_{i\in[1,l_k+1]}(f_i-1)&=\Big(\sum_{i\in[1,l_k+1]}f_i\Big)-(l_k+1)\\ &=\Big(\sum_{i\in[1,m-k]}f_i\Big)-\Big(\sum_{i\in[l_k+2,m-k]}f_i\Big)-(l_k+1)\\ &=m-(m-k-(l_k+2)+1)\cdot 1-(l_k+1)\\ &=k.\end{aligned}$$

With the same argument one can see that $\sum_{i\in[1,l_{k-1}+1]}(f'_i-1)=k-1$, and since $l_{k-1}\leqslant l_k<l_k+1$

we have

$$\begin{aligned}
\sum_{i\in[1,l_k]}(f_i-1) &= \sum_{i\in[1,l_k]}(f'_i-1)\\
&= \Big(\sum_{i\in[1,l_{k-1}]}(f'_i-1)\Big)+\Big(\sum_{i\in[l_{k-1}+1,l_k]}(f'_i-1)\Big)\\
&\overset{\text{(ii}')}{=} \begin{cases}\sum_{i\in[1,l_{k-1}]}(f'_i-1) & \text{if } l_{k-1}=l_k\\ \Big(\sum_{i\in[1,l_{k-1}]}(f'_i-1)\Big)+(f'_{l_{k-1}+1}-1) & \text{if } l_{k-1}<l_k\end{cases}\\
&= \begin{cases}\sum_{i\in[1,l_{k-1}]}(f'_i-1)<k-1 & \text{if } l_{k-1}=l_k\\ \sum_{i\in[1,l_{k-1}+1]}(f'_i-1)=k-1 & \text{if } l_{k-1}<l_k\end{cases}\\
&< k,
\end{aligned}$$

which shows (i).

This proves *Claim 2.2.*

Claim 2.3. Consider $b=(\lambda_\kappa,\varepsilon,\rho_\kappa)\cdot(\lambda_{\kappa-1},\varepsilon,\rho_{\kappa-1})\cdots(\lambda_1,\varepsilon,\rho_1)\in \mathrm{Ass}_0(n-\kappa,n)$. Recall that $\kappa\in[0,n-1]$ and that $\lambda_i\leqslant\lambda_{i+1}$ for $i\in[1,\kappa-1]$. Define $\lambda_0:=0$. For $i\in[1,n-\kappa]$ define $f_i:=|b^{-1}(i)|$. Then (i) and (ii) hold.

(i) We have $\lambda_s+s=\min\{j\in[1,n-\kappa]:\sum_{i\in[1,j]}(1-f_i)=s\}$ for $s\in[1,\kappa]$.

(ii) We have $1-f_i=0$ for $i\in[\lambda_\kappa+\kappa+1,n]$.

Proof of Claim 2.3. Again we will show this via induction on the number of factors κ.

Case $\kappa=0$. This means that $b=\mathrm{id}_n$, hence $1-f_i=0$ for all $i\in[1,n]$, so (ii) is true.

Case $\kappa>0$. We have $b=(\lambda_\kappa,\varepsilon,\rho_\kappa)\cdot(\lambda_{\kappa-1},\varepsilon,\rho_{\kappa-1})\cdots(\lambda_1,\varepsilon,\rho_1)\in\mathrm{Ass}_0(n-\kappa,n)$.

By induction, (i) and (ii) are true for $b'=(\lambda_{\kappa-1},\varepsilon,\rho_{\kappa-1})\cdots(\lambda_1,\varepsilon,\rho_1)\in\mathrm{Ass}_0(n-\kappa+1,n)$. So denoting the fibre sizes of b' by $f'_i:=|b^{-1}(i)|$ for $i\in[1,n]$, then (i′) and (ii′) hold.

(i′) We have $\sum_{i\in[1,\lambda_s+s-1]}(1-f'_i)<s$ and $\sum_{i\in[1,\lambda_s+s]}(1-f'_i)=s$ for $s\in[1,\kappa-1]$.

(ii′) We have $1-f'_i=0$ for $i\in[\lambda_{\kappa-1}+\kappa,n]$.

Note that for $i\in[1,n]$ the fibre sizes of b satisfy

$$1-f_i=\begin{cases}1-f'_i & \text{if } i\in[1,n-\rho_\kappa-1]=[1,\lambda_\kappa+\kappa-1]\\ 1 & \text{if } i=n-\rho_\kappa=\lambda_\kappa+\kappa\\ 1-f'_i & \text{if } i\in[n-\rho_\kappa+1,n]=[\lambda_\kappa+\kappa+1,n].\end{cases}$$

Ad (ii). Let $i\in[\lambda_\kappa+\kappa+1,n]\subseteq[\lambda_{\kappa-1}+\kappa,n]$. Then $1-f_i=1-f'_i=0$ by (ii′).

Ad (i). Let $s\in[1,\kappa]$. We have to show that $\sum_{i\in[1,\lambda_s+s-1]}(1-f_i)\overset{!}{<}s$ and $\sum_{i\in[1,\lambda_s+s]}(1-f_i)\overset{!}{=}s$.

Case 1: $s\in[1,\kappa-1]$. Since $s<\kappa$ we have $\lambda_s\leqslant\lambda_\kappa$ and $\lambda_s+s<\lambda_\kappa+\kappa$. Hence

$$\sum_{i\in[1,\lambda_s+s-1]}(1-f_i)=\sum_{i\in[1,\lambda_s+s-1]}(1-f'_i)<s$$

by (i′) and

$$\sum_{i\in[1,\lambda_s+s]}(1-f_i) = \sum_{i\in[1,\lambda_s+s]}(1-f'_i) = s$$

also by (i′).

Case 2: $s=\kappa$. Note that $\sum_{i\in[1,n]} f_i = n-\kappa$. Now with (ii) we have

$$\begin{aligned}\sum_{i\in[1,\lambda_\kappa+\kappa]}(1-f_i) &= \Big(\sum_{i\in[1,n]}(1-f_i)\Big) - \Big(\sum_{i\in[\lambda_\kappa+\kappa+1,n]}(1-f_i)\Big)\\ &= (n-(n-\kappa))-0\\ &= \kappa\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in[1,\lambda_\kappa+\kappa-1]}(1-f_i) &= \Big(\sum_{i\in[1,\lambda_\kappa+\kappa]}(1-f_i)\Big) - (1-f_{\lambda_\kappa+\kappa})\\ &= \kappa-1,\end{aligned}$$

which shows (i).

This completes the proof of *Claim 2.3.*

Claim 2.4. Let

$$\begin{aligned}v &= q_0(l_1,\hat\mu,r_1)q_1\cdots q_{k-1}(l_k,\hat\mu,r_k)q_k(\lambda_\kappa,\hat\varepsilon,\rho_\kappa)p_{\kappa-1}\cdots p_2(\lambda_1,\hat\varepsilon,\rho_1)p_1 \in \mathrm{Words}(X)(m,n)\\ v' &= q'_0(l'_1,\hat\mu,r'_1)q'_1\cdots q'_{k'-1}(l'_{k'},\hat\mu,r'_{k'})q'_{k'}(\lambda'_{\kappa'},\hat\varepsilon,\rho'_{\kappa'})p'_{\kappa'-1}\cdots p'_2(\lambda'_1,\hat\varepsilon,\rho'_1)p'_1 \in \mathrm{Words}(X)(m',n')\end{aligned}$$

for $m,n,m',n'\in\mathbb{Z}_{\geqslant 0}$ be two words in standard form such that $\big[[v]\big]_Y\bar\phi_0 = \big[[v']\big]_Y\bar\phi_0$.

Then we have $v'=v$, that is, $m'=m$, $n'=n$, $k'=k$, $\kappa'=\kappa$, $l'_i=l_i$ for $i\in[1,k]$ and $\lambda'_j=\lambda_j$ for $j\in[1,\kappa]$.

Proof of Claim 2.4. First note that $\big[[v]\big]_Y\bar\phi_0 = \big[[v']\big]_Y\bar\phi_0$ implies $m'=m$ and $n'=n$.

Case 1: $m=0$. If $k>0$ then we have $q_0 = l_1+\hat\mu s+r_1 = l_1+2+r_1$, a contradiction, since $l_1,r_1\in\mathbb{Z}_{\geqslant 0}$. So $m=0$ implies $k=0$, so v is of the form $v=p_\kappa(\lambda_\kappa,\hat\varepsilon,\rho_\kappa)p_{\kappa-1}\cdots p_2(\lambda_1,\hat\varepsilon,\rho_1)p_1\in\mathrm{Words}(X)(0,n)$. Note that this implies $\kappa=n$. Hence, if $\kappa=n=0$, the only word in $\mathrm{Words}(X)(0,0)$ in standard form is $v=p_0=p_\kappa$.

So suppopse that $n>0$. But since $0=p_\kappa=\lambda_\kappa+\hat\varepsilon s+\rho_\kappa=\lambda_\kappa+\rho_\kappa$, we have $\lambda_\kappa=0$. Furthermore, since $\lambda_i\leqslant\lambda_{i+1}$ for $i\in[1,\kappa-1]$, we have $\lambda_{\kappa-j}\leqslant\lambda_k=0$ for $j\in[1,\kappa-1]$, so $\lambda_i=0$ for $i\in[1,\kappa]$.

So for $m=0$ and $n\in\mathbb{Z}_{\geqslant 0}$ the only word in $\mathrm{Words}(X)(0,n)$ in standard form is the word

$$v = 0(0,\hat\varepsilon,0)1(0,\hat\varepsilon,1)2\cdots(n-1)(0,\hat\varepsilon,n-1)n.$$

Hence in this case $v=v'$ is uniquely determined.

Case 2: $m\in\mathbb{Z}_{\geqslant 1}$. Then Claim 2.1. states that there is a unique way of writing $\big[[v]\big]_Y\bar\phi_0=ab$ and $\big[[v']\big]_Y\bar\phi_0=a'b'$ such that a and a' are surjective and b and b' are injective. In particular, we have

$$\begin{array}{rll} a &= (l_1,\mu,r_1)\cdots(l_k,\mu,r_k) & \in \mathrm{Ass}_0(m,m-k)\\ b &= (\lambda_\kappa,\varepsilon,\rho_\kappa)\cdots(\lambda_1,\varepsilon,\rho_1) & \in \mathrm{Ass}_0(n-\kappa,n)\\ a' &= (l'_1,\mu,r'_1)\cdots(l'_{k'},\mu,r'_{k'}) & \in \mathrm{Ass}_0(m,m-k)\\ b' &= (\lambda'_{\kappa'},\varepsilon,\rho'_{\kappa'})\cdots(\lambda'_1,\varepsilon,\rho'_1) & \in \mathrm{Ass}_0(n-\kappa,n).\end{array}$$

This implies that $a' = a$ and $b' = b$, hence $k' = k$ and $\kappa' = \kappa$.

Now using Claim 2.2. we get that $l'_i = l_i$ for $i \in [1,k]$ and with Claim 2.3. we get that $\lambda'_j = \lambda_j$ for $j \in [1,\kappa]$.

This completes the proof of *Claim 2.4*.

Step 3: Injectivity of $\bar{\phi}_0(m,n)$ for $m,n \in \mathbb{Z}_{\geqslant 0}$. Suppose given $m,n \in \mathbb{Z}_{\geqslant 0}$ and two words

$$\begin{aligned} w &= \tilde{q}_0(\tilde{l}_1, x_1, \tilde{r}_1)\tilde{q}_1 \cdots \tilde{q}_{u-1}(\tilde{l}_u, x_u, \tilde{r}_u)\tilde{q}_u \in \mathrm{Words}(X)(m,n) \\ w' &= \tilde{q}'_0(\tilde{l}'_1, x'_1, \tilde{r}'_1)\tilde{q}'_1 \cdots \tilde{q}'_{u'-1}(\tilde{l}'_{u'}, x'_{u'}, \tilde{r}'_{u'})\tilde{q}'_{u'} \in \mathrm{Words}(X)(m,n) \end{aligned}$$

such that $[[w]]_Y \bar{\phi}_0 = [[w']]_Y \bar{\phi}_0$.

By Claim 1.1. there exist words $z \in \mathrm{Words}(X)(m,n)$ and $z' \in \mathrm{Words}(X)(m',n')$ in standard form such that $[z] \equiv_Y [w]$ and $[z'] \equiv_Y [w']$. Hence we have

$$[[z]]_Y \bar{\phi}_0 = [[w]]_Y \bar{\phi}_0 = [[w']]_Y \bar{\phi}_0 = [[z']]_Y \bar{\phi}_0 .$$

By Claim 2.4 we have $[z] = [z']$. Hence $\bar{\phi}_0(m,n)$ is injective for $m,n \in \mathbb{Z}_{\geqslant 0}$.

This completes the proof of the Theorem. □

We can get a similar result for the linear preoperad Ass.

4.7 A presentation for Ass

Theorem 4.33. *We have the presentation of linear preoperads*

$$\mathrm{Ass} \cong {}_{\mathrm{lpo}}\langle \hat{\varepsilon}, \hat{\mu} \mid \big((\hat{\mu} \boxtimes \mathrm{id}_1)\cdot\hat{\mu} - (\mathrm{id}_1 \boxtimes \hat{\mu})\cdot\hat{\mu}\big), \big((\mathrm{id}_1 \boxtimes \hat{\varepsilon})\cdot\hat{\mu} - \mathrm{id}_1\big), \big((\hat{\varepsilon} \boxtimes \mathrm{id}_1)\cdot\hat{\mu} - \mathrm{id}_1\big) \rangle,$$

where we abbreviate $(\boxtimes) := (\boxtimes_{\mathrm{Free}})$, $(\cdot) := (\cdot_{\mathrm{Free}})$ *and* $\mathrm{id}_1 := \mathrm{id}_{\mathrm{Free},1}$.

More precisely, defining $D := \big\{\big((\hat{\mu}\boxtimes\mathrm{id}_1)\cdot\hat{\mu} - (\mathrm{id}_1 \boxtimes \hat{\mu})\cdot\hat{\mu}\big), \big((\mathrm{id}_1 \boxtimes \hat{\varepsilon})\cdot\hat{\mu} - \mathrm{id}_1\big), \big((\hat{\varepsilon}\boxtimes\mathrm{id}_1)\cdot\hat{\mu} - \mathrm{id}_1\big)\big\}$, $X := \{\hat{\varepsilon}, \hat{\mu}\}$ *and* $\mathcal{I} := {}_{\mathrm{ideal}}\langle D \rangle \subseteq \mathrm{Free}(X)$, *then we have the isomorphism of linear preoperads*

$$\bar{\phi} : {}_{\mathrm{lpo}}\langle X \mid D \rangle \longrightarrow \mathrm{Ass}$$

defined by $[\hat{\varepsilon}]_{\mathcal{I}}\bar{\phi} = \varepsilon$ *and* $[\hat{\mu}]_{\mathcal{I}}\bar{\phi} = \mu$.

Proof. By Theorem 4.32 we have the presentation $\mathrm{Ass}_0 \cong {}_{\mathrm{spo}}\langle X \mid Y \rangle$, where $X = \{\hat{\varepsilon}, \hat{\mu}\}$ and

$$Y = \big\{\big((\hat{\mu} \boxtimes \mathrm{id}_1)\cdot\hat{\mu}\,,\ (\mathrm{id}_1 \boxtimes \hat{\mu})\cdot\hat{\mu}\big), \big((\mathrm{id}_1 \boxtimes \hat{\varepsilon})\cdot\hat{\mu}\,,\ \mathrm{id}_1\big), \big((\hat{\varepsilon} \boxtimes \mathrm{id}_1)\cdot\hat{\mu}\,,\ \mathrm{id}_1\big)\big\}.$$

So by Lemma 4.27 we have the presentation $\mathrm{Ass} = R\,\mathrm{Ass}_0 \cong {}_{\mathrm{lpo}}\langle X \mid D_Y \rangle$, where for $m,n \in \mathbb{Z}_{\geqslant 0}$ we have $D_Y(m,n) = \{f - \tilde{f} : (f,\tilde{f}) \in Y(m,n)\}$. So in our case we have

$$D_Y = \big\{\big((\hat{\mu} \boxtimes \mathrm{id}_1)\cdot\hat{\mu} - (\mathrm{id}_1 \boxtimes \hat{\mu})\cdot\hat{\mu}\big), \big((\mathrm{id}_1 \boxtimes \hat{\varepsilon})\cdot\hat{\mu} - \mathrm{id}_1\big), \big((\hat{\varepsilon} \boxtimes \mathrm{id}_1)\cdot\hat{\mu} - \mathrm{id}_1\big)\big\} = D.$$

More precisely, by Lemma 4.27, we have the isomorphism of linear preoperads $\bar{\phi} : {}_{\mathrm{lpo}}\langle X \mid D \rangle \longrightarrow \mathrm{Ass}$ that satisfies $\beta_{\frac{\mathrm{Free}_0(X)}{(\equiv_Y)}} \chi \bar{\phi} = \bar{\phi}_0 \beta_{\mathrm{Ass}_0}$. So in particular, we have

$$\begin{aligned} [\hat{\varepsilon}]_{\mathcal{I}}\bar{\phi} &= [\hat{\varepsilon}]_Y \chi\bar{\phi} = [\hat{\varepsilon}]_Y \beta_{\frac{\mathrm{Free}_0(X)}{(\equiv_Y)}} \chi\bar{\phi} = [\hat{\varepsilon}]_Y \bar{\phi}_0 \beta_{\mathrm{Ass}_0} = \varepsilon\beta_{\mathrm{Ass}_0} = \varepsilon \\ [\hat{\mu}]_{\mathcal{I}}\bar{\phi} &= [\hat{\mu}]_Y \chi\bar{\phi} = [\hat{\mu}]_Y \beta_{\frac{\mathrm{Free}_0(X)}{(\equiv_Y)}} \chi\bar{\phi} = [\hat{\mu}]_Y \bar{\phi}_0 \beta_{\mathrm{Ass}_0} = \mu\beta_{\mathrm{Ass}_0} = \mu. \end{aligned}$$

□

5 Algebras over preoperads

5.1 General notion

Definition 5.1. Let $\mathcal{P}_0$ be a set-preoperad. A $\mathcal{P}_0$-algebra (X, ϱ_0) consists of a set X and a morphism of set-preoperads

$$\varrho_0 : \mathcal{P}_0 \longrightarrow \mathrm{End}_0(X).$$

Definition 5.2. Let $\mathcal{P}$ be a linear preoperad over R. A $\mathcal{P}$-algebra (V, ϱ) consists of an R-module V and a morphism of linear preoperads

$$\varrho : \mathcal{P} \longrightarrow \mathrm{End}(V).$$

The name "algebra" may be surprising here, but we will see that given an Ass-algebra (V, ϱ), then the R-module V in fact can be turned into an algebra in the usual sense. Algebras over the set-preoperad Ass_0 , however, will eventually yield monoids.

5.2 Associative monoids and Ass_0-algebras

Recall that the set-preoperad Ass_0 has $\mathrm{Ass}_0(m, n)$ consisting of monotone maps $[1, m] \longrightarrow [1, n]$ for $m, n \in \mathbb{Z}_{\geqslant 0}$. Recall the unique elements $\varepsilon \in \mathrm{Ass}_0(0, 1)$ and $\mu \in \mathrm{Ass}_0(2, 1)$; cf. Definition 4.28. Furthermore, recall that in Ass_0 we have $(\mu \boxtimes \mathrm{id}_1)\mu = (\mathrm{id}_1 \boxtimes \mu)\mu$ and $(\mathrm{id}_1 \boxtimes \varepsilon)\mu = (\varepsilon \boxtimes \mathrm{id}_1)\mu = \mathrm{id}_1$.

Proposition 5.3. *Let (X, ψ_0) be an Ass_0-algebra, that is, X is a set and $\psi_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{End}_0(X)$ is a morphism of set-preoperads.*

Define $\mu_X := \mu\psi_0 : X \times X \longrightarrow X$ and $\varepsilon_X := \varepsilon\psi_0 : X^{\times 0} = \{()\} \longrightarrow X$. Then $(X, \mu_X, \varepsilon_X)$ is an (associative) monoid.

Proof. First note that since in Ass_0 we have $(\mathrm{id}_1 \boxtimes \mu)\mu = (\mu \boxtimes \mathrm{id}_1)\mu$ and since ψ_0 is a morphism of set-preoperads, we have

$$\begin{aligned}
(\mathrm{id}_X \times \mu_X)\mu_X &= (\mathrm{id}_{\mathrm{End}_0,1} \boxtimes_{\mathrm{End}_0} \mu_X) \cdot_{\mathrm{End}_0} \mu_X \\
&= (\mathrm{id}_{\mathrm{Ass}_0,1}\, \psi_0 \boxtimes_{\mathrm{End}_0} \mu\psi_0) \cdot_{\mathrm{End}_0} \mu\psi_0 \\
&= ((\mathrm{id}_{\mathrm{Ass}_0,1} \boxtimes_{\mathrm{Ass}_0} \mu) \cdot_{\mathrm{Ass}_0} \mu)\psi_0 \\
&= ((\mu \boxtimes_{\mathrm{Ass}_0} \mathrm{id}_{\mathrm{Ass}_0,1}) \cdot_{\mathrm{Ass}_0} \mu)\psi_0 \\
&= (\mu\psi_0 \boxtimes_{\mathrm{End}_0} \mathrm{id}_{\mathrm{Ass}_0,1}\, \psi_0) \cdot_{\mathrm{End}_0} \mu\psi_0 \\
&= (\mu_X \boxtimes_{\mathrm{End}_0} \mathrm{id}_{\mathrm{End}_0,1}) \cdot_{\mathrm{End}_0} \mu_X \\
&= (\mu_X \times \mathrm{id}_X)\mu_X .
\end{aligned}$$

Furthermore, since in Ass_0 we have $(\mathrm{id}_1 \boxtimes \varepsilon)\mu = \mathrm{id}_1$ and since ψ_0 is a morphism of set-preoperads, we have

$$\begin{aligned}
(\mathrm{id}_X \times \varepsilon_X)\mu_X &= (\mathrm{id}_{\mathrm{End}_0,1} \boxtimes_{\mathrm{End}_0} \varepsilon_X) \cdot_{\mathrm{End}_0} \mu_X \\
&= (\mathrm{id}_{\mathrm{Ass}_0,1}\, \psi_0 \boxtimes_{\mathrm{End}_0} \varepsilon\psi_0) \cdot_{\mathrm{End}_0} \mu\psi_0 \\
&= ((\mathrm{id}_{\mathrm{Ass}_0,1} \boxtimes_{\mathrm{Ass}_0} \varepsilon) \cdot_{\mathrm{Ass}_0} \mu)\psi_0 \\
&= \mathrm{id}_{\mathrm{Ass}_0,1}\, \psi_0 \\
&= \mathrm{id}_{\mathrm{End}_0,1} \\
&= \mathrm{id}_X .
\end{aligned}$$

Finally, in the same way we see that $(\varepsilon \boxtimes \mathrm{id}_1)\mu = \mathrm{id}_1$ in Ass_0 implies $(\varepsilon_X \times \mathrm{id}_X)\mu_X = \mathrm{id}_X$.

This shows that $(X, \mu_X, \varepsilon_X)$ is an associative monoid. $\square$

Our next aim will be to show the converse, that is, every (associative) monoid can be turned into an Ass_0-algebra. That is, we will show that for each monoid $(X, \mu_X, \varepsilon_X)$ we can define a morphism of set-preoperads $\psi_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{End}_0(X)$ satisfying $\mu\psi_0 = \mu_X$ and $\varepsilon\psi_0 = \varepsilon_X$.

Recall that any element $f \in \mathrm{Ass}_0(m,n)$ is a monotone map $[1,m] \longrightarrow [1,n]$. That means that the fibres $f^{-1}(i) = \{j \in [1,m] : jf = i\}$ yield a partition of $[1,m]$ into n (possibly empty) subintervals.

For an interval $[a,b] \subseteq \mathbb{Z}$ and $s \in \mathbb{Z}_{\geqslant 0}$ define $[a,b]_{+s} := [a+s, b+s]$, the interval shifted to the right by s and $[a,b]_{-s} := [a-s, b-s]$, the interval shifted to the left by s. Since the fibres of a monotone map $f : [1,m] \longrightarrow [1,n]$ are intervals we can use that notation as follows. Given $[a,b] = f^{-1}(i)$ for some $i \in [1,n]$, then for $s,t \in \mathbb{Z}_{\geqslant 0}$ we can define $f^{-1}(i)_{-s} := [a,b]_{-s}$ and $f^{-1}(i)_{+t} := [a,b]_{+t}$.

Proposition 5.4. *Let $(X, \mu_X, \varepsilon_X)$ be an (associative) monoid. For $x, y \in X$ define $x \cdot y := (x,y)\mu_X$.*

Using the usual convention of dropping brackets where associativity is known, we may define

$$x_{[a,b]} := x_a \cdot x_{a+1} \cdots x_b =: \prod_{i \in [a,b]} x_i$$

for $m \in \mathbb{Z}_{\geqslant 0}$, $x_1, \ldots, x_m \in X$ and for any interval $[a,b] \subseteq [1,m]$. In particular, for $[a,b] = \varnothing$ we have $x_{[a,b]} = 1_X := (\,)\varepsilon_X \in X$.

Consider the biindexed map $\psi_0 = (\psi_0(m,n))_{m,n \geqslant 0} : \mathrm{Ass}_0 \longrightarrow \mathrm{End}_0(X)$ that maps $f \in \mathrm{Ass}_0(m,n)$ to $f\psi_0 \in \mathrm{End}_0(X)(m,n)$, defined by

$$\begin{aligned} f\psi_0 : \qquad X^{\times m} &\longrightarrow X^{\times n} \\ (x_1, \ldots, x_m) &\longmapsto (x_{f^{-1}(1)}, \ldots, x_{f^{-1}(n)}). \end{aligned}$$

Then ψ_0 is a morphism of set-preoperads satisfying $\mu\psi_0 = \mu_X$ and $\varepsilon\psi_0 = \varepsilon_X$.

So in particular, (X, ψ_0) is an Ass_0-algebra.

Proof. First note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have

$$(x_1, \ldots, x_m)(\mathrm{id}_{\mathrm{Ass}_0, m}\, \psi_0) = (x_1, \ldots, x_m)$$

for $x_1, \ldots, x_m \in x$, hence $\mathrm{id}_{\mathrm{Ass}_0, m}\, \psi_0 = \mathrm{id}_{\mathrm{End}_0, m}$.

From now on we will abbreviate $(\boxtimes) := (\boxtimes_{\mathrm{Ass}_0})$ and $(\cdot) := (\cdot_{\mathrm{Ass}_0})$.

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Ass}_0(m,n)$, $f' \in \mathrm{Ass}_0(m',n')$. Consider the fibres of $f \boxtimes f'$.

For $i \in [1,n]$ we have $(f \boxtimes f')^{-1}(i) = f^{-1}(i)$.

For $i \in [n+1, n+n']$ we have $(f \boxtimes f')^{-1}(i) = f'^{-1}(i-n)_{+m}$, since for $j \in [1, m+m']$ we have $j \in f'^{-1}(i-n)_{+m}$ if and only if $j - m \in f'^{-1}(i-n)$, i.e. $j(f \boxtimes f') = (j-m)f' + n = i - n + n = i$, i.e. $j \in (f \boxtimes f')^{-1}(i)$.

So for $x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+m'} \in X$ we have

$$\begin{aligned} (x_1, \ldots, x_{m+m'})\big((f \boxtimes f')\psi_0\big) &= x_{(f \boxtimes f')^{-1}(1)}, \ldots, x_{(f \boxtimes f')^{-1}(n+n')} \\ &= (x_{f^{-1}(1)}, \ldots, x_{f^{-1}(n)}, x_{f'^{-1}(1)_{+m}}, \ldots, x_{f'^{-1}(n')_{+m}}). \end{aligned}$$

On the other hand, defining $y_i := x_{n+i}$ for $i \in [1, m']$, we obtain

$$\begin{aligned} (x_1, \ldots, x_{m+m'})(f\psi_0 \boxtimes_{\mathrm{End}_0} f'\psi_0) &= (x_1, \ldots, x_m)f\psi_0 \times (x_{m+1}, \ldots, x_{m+m'})f'\psi_0 \\ &= (x_1, \ldots, x_m)f\psi_0 \times (y_1, \ldots, y_{m'})f'\psi_0 \\ &= (x_{f^{-1}(1)}, \ldots, x_{f^{-1}(n)}, y_{f'^{-1}(1)}, \ldots, y_{f'^{-1}(n')}) \\ &= (x_{f^{-1}(1)}, \ldots, x_{f^{-1}(n)}, x_{f'^{-1}(1)_{+m}}, \ldots, x_{f'^{-1}(n')_{+m}}). \end{aligned}$$

Hence we have $(f \boxtimes f')\psi_0 = f\psi_0 \boxtimes_{\mathrm{End}_0} f'\psi_0$.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Ass}_0(m,n)$, $g \in \mathrm{Ass}_0(n,k)$. In order to show that $(f \cdot g)\psi_0 = (f\psi_0) \cdot_{\mathrm{End}_0} (g\psi_0)$ we will have to take a look at $(fg)^{-1}(i)$ for $i \in [1,k]$.

Claim: We have $(fg)^{-1}(i) \overset{!}{=} \bigsqcup_{u \in g^{-1}(i)} f^{-1}(u)$, as a subset of $[1,m]$.

Proof of the Claim. First we remark that the right hand side is a disjoint union.

Suppose given $j \in [1,m]$. We have $j \in (fg)^{-1}(i)$ if and only if $(jf)g = i$, i.e. $jf \in g^{-1}(i)$, i.e. there exists $u \in g^{-1}(i)$ such that $j \in f^{-1}(u)$. This proves the *Claim.*

Now we can show that ψ_0 is compatible with composition and hence a morphism of set-preoperads.

For $x_1, \ldots, x_m \in X$ we have

$$\begin{aligned}(x_1,\ldots,x_m)(f\psi_0 \cdot_{\mathrm{End}_0} g\psi_0) &= \big((x_1,\ldots,x_m)f\psi_0\big)g\psi_0 \\ &= (x_{f^{-1}(1)},\ldots,x_{f^{-1}(n)})g\psi_0 .\end{aligned}$$

By writing $y_i := x_{f^{-1}(i)} \in X$ for $i \in [1,n]$, we obtain

$$\begin{aligned}(x_1,\ldots,x_m)(f\psi_0 \cdot_{\mathrm{End}_0} g\psi_0) &= (y_1,\ldots,y_n)g\psi_0 \\ &= (y_{g^{-1}(1)},\ldots,y_{g^{-1}(k)}) \\ &= \Bigg(\Big(\prod_{u_1 \in g^{-1}(1)} y_{u_1}\Big),\ldots,\Big(\prod_{u_k \in g^{-1}(k)} y_{u_k}\Big)\Bigg) \\ &= \Bigg(\Big(\prod_{u_1 \in g^{-1}(1)} x_{f^{-1}(u_1)}\Big),\ldots,\Big(\prod_{u_k \in g^{-1}(k)} x_{f^{-1}(u_k)}\Big)\Bigg) \\ &= \Bigg(x_{\big(\bigsqcup_{u_1 \in g^{-1}(1)} f^{-1}(u_1)\big)},\ldots,x_{\big(\bigsqcup_{u_k \in g^{-1}(k)} f^{-1}(u_k)\big)}\Bigg) \\ &= (x_{(fg)^{-1}(1)},\ldots,x_{(fg)^{-1}(k)}) \\ &= (x_1,\ldots,x_m)\big((fg)\psi_0\big).\end{aligned}$$

Hence we have $f\psi_0 \cdot_{\mathrm{End}_0} g\psi_0 = (fg)\psi_0$.

This shows that $\psi_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{End}_0(X)$ is a morphism of set-preoperads.

Hence (X, ψ_0) is an Ass_0-algebra.

Moreover, for $x_1, x_2 \in X$ we have $(x_1,x_2)(\mu\psi_0) = x_{\mu^{-1}(1)} = x_{[1,2]} = x_1 \cdot x_2 = (x_1,x_2)\mu_X$, so $\mu\psi_0 = \mu_X$. Finally, we have $(\,)(\varepsilon\psi_0) = x_{\varepsilon^{-1}(1)} = x_{[1,0]} = 1_X = (\,)\varepsilon_X$, hence $\varepsilon\psi_0 = \varepsilon_X$. □

Remark 5.5. Recall that

$$\mathrm{Ass}_0 \cong {}_{\mathrm{spo}}\big\langle \hat\varepsilon, \hat\mu \mid \big((\hat\mu \boxtimes \mathrm{id}_1)\cdot\hat\mu\,,\, (\mathrm{id}_1 \boxtimes \hat\mu)\cdot\hat\mu\big), \big((\mathrm{id}_1 \boxtimes \hat\varepsilon)\cdot\hat\mu\,,\, \mathrm{id}_1\big), \big((\hat\varepsilon \boxtimes \mathrm{id}_1)\cdot\hat\mu\,,\, \mathrm{id}_1\big)\big\rangle = \mathrm{Ass}_{0,\mathrm{P}},$$

where $\hat\mu s = 2$, $\hat\mu t = 1$, $\hat\varepsilon s = 0$ and $\hat\varepsilon t = 1$ and where we abbreviate $(\boxtimes) = (\boxtimes_{\mathrm{Free}_0})$, $(\cdot) = (\cdot_{\mathrm{Free}_0})$ and $\mathrm{id}_1 = \mathrm{id}_{\mathrm{Free}_0,1}$; cf. Theorem 4.32.

More precisely, if we write

$$Y = \{\big((\hat\mu \boxtimes \mathrm{id}_1)\hat\mu\,,\, (\mathrm{id}_1 \boxtimes \hat\mu)\hat\mu\big), \big((\mathrm{id}_1 \boxtimes \hat\varepsilon)\hat\mu\,,\, \mathrm{id}_1\big), \big((\hat\varepsilon \boxtimes \mathrm{id}_1)\hat\mu\,,\, \mathrm{id}_1\big)\} \subseteq \mathrm{Free}_0(\hat\varepsilon,\hat\mu) \times \mathrm{Free}_0(\hat\varepsilon,\hat\mu),$$

then, writing $[f]_Y$ for the congruence class of $f \in \mathrm{Free}_0(\hat\varepsilon,\hat\mu)(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$, we have the isomorphism of set-preoperads

$$\begin{aligned}\bar\phi_0 : \mathrm{Ass}_{0,\mathrm{P}} &\longrightarrow \mathrm{Ass}_0 \\ [\hat\varepsilon]_Y &\longmapsto \varepsilon \\ [\hat\mu]_Y &\longmapsto \mu .\end{aligned}$$

Let $\tilde{\psi}_0 : \mathrm{Ass}_{0,\mathrm{P}} \longrightarrow \mathrm{End}_0(X)$ be the morphism of set-preoperads with

$$[\hat{\varepsilon}]_Y \tilde{\psi}_0 := \varepsilon_X \in \mathrm{End}_0(X)(0,1)$$
$$[\hat{\mu}]_Y \tilde{\psi}_0 := \mu_X \in \mathrm{End}_0(X)(2,1).$$

Since $(\mu_X \times \mathrm{id}_X)\mu_X = (\mathrm{id}_X \times \mu_X)\mu_X$ and $(\mathrm{id}_X \times \varepsilon_X)\mu_X = \mathrm{id}_X = (\varepsilon_X \times \mathrm{id}_X)\mu_X$, this morphism of set-preoperads is well-defined; cf. Lemma 4.24.

Note that we have $[\hat{\varepsilon}]_Y \bar{\phi}_0 \psi_0 = \varepsilon\psi_0 = \varepsilon_X = [\hat{\varepsilon}]_Y \tilde{\psi}_0$ and $[\hat{\mu}]_Y \bar{\phi}_0 \psi_0 = \mu\psi_0 = \mu_X = [\hat{\mu}]_Y \tilde{\psi}_0$. So we have $\bar{\phi}_0 \psi_0 = \tilde{\psi}_0$.

So using the presentation of Ass_0 from Theorem 4.32, by defining $\psi_0 := \bar{\phi}_0^{-1} \tilde{\psi}_0$ we obtain the same morphism of set-preoperads $\psi_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{End}_0(X)$ as in Proposition 5.4, turning X into an Ass_0-algebra.

5.3 Associative algebras and Ass-algebras

Proposition 5.6. *Let (V, ψ) be an* Ass*-algebra, that is, V is an R-module and $\psi : \mathrm{Ass} \longrightarrow \mathrm{End}(V)$ is a morphism of linear preoperads over R.*

Define $\mu_V := \mu\psi : V \otimes V \longrightarrow V$ and $\varepsilon_V := \varepsilon\psi : R = V^{\otimes 0} \longrightarrow V$. Then $(V, \mu_V, \varepsilon_V)$ is an associative algebra.

Proof. As in the non-linear case, since in Ass we have $(\mathrm{id}_1 \boxtimes \mu)\mu = (\mu \boxtimes \mathrm{id}_1)\mu$ and since ψ is a morphism of linear operads, we have

$$(\mathrm{id}_V \otimes \mu_V)\mu_V = (\mu_V \otimes \mathrm{id}_V)\mu_V \,.$$

Furthermore, again with the same calculations as in the non-linear case, the fact that ψ is a morphism of linear preoperads and the equations $(\mathrm{id}_1 \boxtimes \varepsilon)\mu = \mathrm{id}_1 = (\varepsilon \boxtimes \mathrm{id}_1)\mu$ in Ass imply

$$(\mathrm{id}_V \otimes \varepsilon_V)\mu_V = \mathrm{id}_V = (\varepsilon_V \otimes \mathrm{id}_V)\mu_V \,.$$

This shows that $(V, \mu_V, \varepsilon_V)$ is an associative R-algebra. □

Our next aim will be to show that every associative algebra can be turned into an Ass-algebra. Again, we will construct the morphism $\mathrm{Ass} \longrightarrow \mathrm{End}(V)$ in two different ways. The first will use analogous calculations to the first way for Ass_0-algebras and give an explicit formula for the images of $f \in \mathrm{Ass}(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$. The second will use the presentation of Ass as seen in Theorem 4.33.

Proposition 5.7. *Let $(V, \mu_V, \varepsilon_V)$ be an associative R-algebra. For $v, w \in V$ define $v \cdot w := (v \otimes w)\mu_V$.*

Using the usual convention of dropping brackets where associativity is known, we may define

$$v_{[a,b]} := v_a \cdot v_{a+1} \cdots v_b =: \prod_{i \in [a,b]} v_i$$

for $m \in \mathbb{Z}_{\geqslant 0}$, $v_1, \ldots, v_m \in V$ and for any interval $[a,b] \subseteq [1,m]$. In particular, if $[a,b] = \varnothing$ we have $v_{[a,b]} = 1_V := 1_R \varepsilon_V \in V$.

Consider the biindexed map $\psi = (\psi(m,n))_{m,n \geqslant 0} : \mathrm{Ass} \longrightarrow \mathrm{End}(V)$ that maps $f \in \mathrm{Ass}_0(m,n)$ to $f\psi \in \mathrm{End}(V)(m,n)$, defined by

$$\begin{aligned} f\psi : \qquad V^{\otimes m} &\longrightarrow V^{\otimes n} \\ v_1 \otimes \ldots \otimes v_m &\longmapsto v_{f^{-1}(1)} \otimes \ldots \otimes v_{f^{-1}(n)}. \end{aligned}$$

Then ψ is a morphism of linear preoperads over R satisfying $\mu\psi = \mu_V$ and $\varepsilon\psi = \varepsilon_V$.

So in particular, (V, ψ) is an Ass*-algebra.*

Proof. First we will show that this map is well-defined. In order to do this, consider the corresponding map

$$\begin{aligned}(\widehat{f\psi})(m,n): \qquad V^{\times m} &\longrightarrow V^{\otimes n}\\ (v_1,\ldots,v_m) &\longmapsto v_{f^{-1}(1)}\otimes\ldots\otimes v_{f^{-1}(n)}\end{aligned}$$

for $m,n \in \mathbb{Z}_{\geqslant 0}$. We have to show that this is an R-multilinear map.

Let $i \in [1,m]$, $v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_m \in V$ and let $k \in \mathbb{Z}_{\geqslant 0}$ and $v_{i,j} \in V$ and $r_j \in R$ for $j \in [1,k]$. Let $v_i := \sum_{j\in[1,k]} r_j v_{i,j}$.

Let $l := if$. Then the factor v_i occurs only in $v_{f^{-1}(l)}$. Moreover, we know that $f^{-1}(l)$ is an interval. So we can write $f^{-1}(l) =: [a,b] \subseteq [1,m]$ for some $a \in [1,i]$ and $b \in [i,m]$. We have

$$\begin{aligned}v_{f^{-1}(l)} &= \prod_{j\in[a,b]} v_j\\ &= v_a \cdots v_i \cdots v_b\\ &= v_a \cdots v_{i-1}\cdot \Big(\sum_{j\in[1,k]} r_j v_{i,j}\Big)\cdot v_{i+1}\cdots v_b\\ &= \sum_{j\in[1,k]} r_j (v_a\cdots v_{i-1}\cdot v_{i,j}\cdot v_{i+1}\cdots v_b).\end{aligned}$$

So we have

$$\begin{aligned}&\big(v_1,\ldots,v_{i-1},\sum_{j\in[1,k]} r_j v_{i,j}, v_{i+1},\ldots,v_m\big)(\widehat{f\psi})\\ &= v_{f^{-1}(1)}\otimes\ldots\otimes v_{f^{-1}(l)}\otimes\ldots\otimes v_{f^{-1}(n)}\\ &= v_{f^{-1}(1)}\otimes\ldots\otimes v_{f^{-1}(l-1)}\otimes\Big(\sum_{j\in[1,k]} r_j(v_a\cdots v_{i-1}\cdot v_{i,j}\cdot v_{i+1}\cdots v_b)\Big)\otimes v_{f^{-1}(l+1)}\ldots\otimes v_{f^{-1}(n)}\\ &= \sum_{j\in[1,k]} r_j\big(v_{f^{-1}(1)}\otimes\ldots\otimes v_{f^{-1}(l-1)}\otimes(v_a\cdots v_{i-1}\cdot v_{i,j}\cdot v_{i+1}\cdots v_b)\otimes v_{f^{-1}(l+1)}\otimes\ldots\otimes v_{f^{-1}(n)}\big)\\ &= \sum_{j\in[1,k]} r_j\big((v_1,\ldots,v_{i-1},v_{i,j},v_{i+1},\ldots,v_m)(\widehat{f\psi})\big).\end{aligned}$$

This shows that ψ is well-defined.

Note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have

$$(v_1\otimes\ldots\otimes v_m)(\mathrm{id}_{\mathrm{Ass},m}\,\psi) = v_1\otimes\ldots\otimes v_m$$

for $v_1,\ldots,v_m \in V$, hence $\mathrm{id}_{\mathrm{Ass},m}\,\psi = \mathrm{id}_{\mathrm{End},m}$.

From now on we will abbreviate $(\boxtimes) := (\boxtimes_{\mathrm{Ass}})$ and $(\cdot) := (\cdot_{\mathrm{Ass}})$.

Now suppose given $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Ass}_0(m,n)$, $f' \in \mathrm{Ass}_0(m',n')$. As in the non-linear case, for $i \in [1,n+n']$ we have

$$(f\boxtimes f')^{-1}(i) = \begin{cases} f^{-1}(i) & \text{if } i\in[1,n]\\ f'^{-1}(i-n)_{+m} & \text{if } i \in [n+1,n+n'],\end{cases}$$

where $f'^{-1}(i-n)_{+m}$ is the interval $f'^{-1}(i-n)$ shifted to the right by m.

So for $v_1, \ldots, v_{m+m'} \in V$ we have

$$\begin{aligned}(v_1 \otimes \ldots \otimes v_{m+m'})\big((f \boxtimes f')\psi\big) &= v_{(f \boxtimes f')^{-1}(1)} \otimes \ldots \otimes v_{(f \boxtimes f')^{-1}(n+n')} \\ &= v_{f^{-1}(1)} \otimes \ldots \otimes v_{f^{-1}(n)} \otimes v_{f'^{-1}(1)+m} \otimes \ldots \otimes v_{f'^{-1}(n')+m} \,.\end{aligned}$$

On the other hand, defining $w_i := v_{m+i}$ for $i \in [1, m']$ we get

$$\begin{aligned}(v_1, \ldots, v_{m+m'})(f\psi \otimes f'\psi) &= \big((v_1 \otimes \ldots \otimes v_m) f\psi\big) \otimes \big((v_{m+1} \otimes \ldots \otimes v_{m+m'}) f'\psi\big) \\ &= \big((v_1 \otimes \ldots \otimes v_m) f\psi\big) \otimes \big((w_1 \otimes \ldots \otimes w_{m'}) f'\psi\big) \\ &= v_{f^{-1}(1)} \otimes \ldots \otimes v_{f^{-1}(n)} \otimes w_{f'^{-1}(1)} \otimes \ldots \otimes w_{f'^{-1}(n')} \\ &= v_{f^{-1}(1)} \otimes \ldots v_{f^{-1}(n)} \otimes v_{f'^{-1}(1)+m} \otimes \ldots \otimes v_{f'^{-1}(n')+m} \,.\end{aligned}$$

Hence we have $(f \boxtimes f')\psi = f\psi \otimes f'\psi$ for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Ass}_0(m, n)$, $f' \in \mathrm{Ass}_0(m', n')$.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Ass}_0(m, n)$, $g \in \mathrm{Ass}_0(n, k)$. Recall from the proof of Lemma 5.4 that $(fg)^{-1}(i) = \bigsqcup_{u \in g^{-1}(i)} f^{-1}(u)$ for $i \in [1, k]$.

Now for $v_1, \ldots, v_m \in V$ we have

$$\begin{aligned}(v_1 \otimes \ldots \otimes v_m)(f\psi \cdot_{\mathrm{End}} g\psi) &= \big((v_1 \otimes \ldots \otimes v_m) f\psi\big) g\psi \\ &= (v_{f^{-1}(1)} \otimes \ldots \otimes v_{f^{-1}(n)}) g\psi \,.\end{aligned}$$

By writing $w_i := v_{f^{-1}(i)} \in V$ for $i \in [1, n]$ we obtain

$$\begin{aligned}(v_1 \otimes \ldots \otimes v_m)(f\psi \cdot_{\mathrm{End}} g\psi) &= (w_1 \otimes \ldots \otimes w_n) g\psi \\ &= w_{g^{-1}(1)} \otimes \ldots \otimes w_{g^{-1}(k)} \\ &= \Big(\prod_{u_1 \in g^{-1}(1)} w_{u_1}\Big) \otimes \ldots \otimes \Big(\prod_{u_k \in g^{-1}(k)} w_{u_k}\Big) \\ &= \Big(\prod_{u_1 \in g^{-1}(1)} v_{f^{-1}(u_1)}\Big) \otimes \ldots \otimes \Big(\prod_{u_k \in g^{-1}(k)} v_{f^{-1}(u_k)}\Big) \\ &= v_{\left(\bigsqcup_{u_1 \in g^{-1}(1)} f^{-1}(u_1)\right)} \otimes \ldots \otimes v_{\left(\bigsqcup_{u_k \in g^{-1}(k)} f^{-1}(u_k)\right)} \\ &= v_{(fg)^{-1}(1)} \otimes \ldots \otimes v_{(fg)^{-1}(k)} \\ &= (v_1 \otimes \ldots \otimes v_m)\big((fg)\psi\big).\end{aligned}$$

Hence we have $f\psi \cdot_{\mathrm{End}} g\psi = (fg)\psi$ for $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Ass}_0(m, n)$, $g \in \mathrm{Ass}_0(n, p)$.

This shows that $\psi : \mathrm{Ass} \longrightarrow \mathrm{End}(V)$ is a morphism of linear preoperads.

Hence (V, ψ) is an Ass-algebra.

Moreover, fo $v_1, v_2 \in V$ we have $(v_1 \otimes v_2)\mu\psi = v_{\mu^{-1}(1)} = v_{[1,2]} = v_1 \cdot v_2 = (v_1 \otimes v_2)\mu_V$, so $\mu\psi = \mu_V$. Finally, we have $1_R\varepsilon\psi = v_{\varepsilon^{-1}(1)} = v_{[1,0]} = 1_V = 1_R\varepsilon_V$, hence $\varepsilon\psi = \varepsilon_V$. □

Remark 5.8. Recall that

$$\mathrm{Ass} \cong {}_{\mathrm{lpo}}\langle \hat\mu, \hat\varepsilon \mid \big((\hat\mu \boxtimes \mathrm{id}_1)\hat\mu - (\mathrm{id}_1 \boxtimes \hat\mu)\hat\mu\big), \big((\mathrm{id}_1 \boxtimes \hat\varepsilon)\hat\mu - \mathrm{id}_1\big), \big((\hat\varepsilon \boxtimes \mathrm{id}_1)\hat\mu - \mathrm{id}_1\big) \rangle = \mathrm{Ass}_{\mathrm{P}} \,,$$

where $\hat\mu s = 2$, $\hat\mu t = 1$, $\hat\varepsilon s = 0$ and $\hat\varepsilon t = 1$ and where we abbreviate $(\boxtimes) = (\boxtimes_{\mathrm{Free}})$, $(\cdot) = (\cdot_{\mathrm{Free}})$ and $\mathrm{id}_1 = \mathrm{id}_{\mathrm{Free},1}$; cf. Theorem 4.33.

More precisely, if we write

$$D = \{\big((\hat\mu \boxtimes \mathrm{id}_1)\hat\mu - (\mathrm{id}_1 \boxtimes \hat\mu)\hat\mu\big), \big((\mathrm{id}_1 \boxtimes \hat\varepsilon)\hat\mu - \mathrm{id}_1\big), \big((\hat\varepsilon \boxtimes \mathrm{id}_1)\hat\mu - \mathrm{id}_1\big)\} \subseteq \mathrm{Free}(\hat\varepsilon, \hat\mu) = R\,\mathrm{Free}_0(\hat\varepsilon, \hat\mu)$$

and $\mathcal{I} := {}_{\text{ideal}}\langle\, D\, \rangle$, then we have the isomorphism of linear preoperads

$$\begin{aligned} \bar{\phi} : \mathrm{Ass}_{\mathrm{P}} &\longrightarrow \mathrm{Ass} \\ [\hat{\varepsilon}]_{\mathcal{I}} &\longmapsto \varepsilon \\ [\hat{\mu}]_{\mathcal{I}} &\longmapsto \mu\,. \end{aligned}$$

Let $\tilde{\psi} : \mathrm{Ass}_{\mathrm{P}} \longrightarrow \mathrm{End}(V)$ be the morphism of set-preoperads with

$$\begin{aligned} [\hat{\varepsilon}]_{\mathcal{I}}\tilde{\psi} &:= \varepsilon_V \in \mathrm{End}(V)(0,1) \\ [\hat{\mu}]_{\mathcal{I}}\tilde{\psi} &:= \mu_V \in \mathrm{End}(V)(2,1). \end{aligned}$$

Since $(\mu_V \otimes \mathrm{id}_V)\mu_V = (\mathrm{id}_V \otimes \mu_V)\mu_V$ and $(\mathrm{id}_V \otimes \varepsilon_V)\mu_V = \mathrm{id}_V = (\varepsilon_V \otimes \mathrm{id}_V)\mu_V$, this morphism of linear preoperads is well-defined; cf. Lemma 4.26.

Note that we have $[\hat{\varepsilon}]_{\mathcal{I}}\bar{\phi}\psi = \varepsilon\psi = \varepsilon_V = [\hat{\varepsilon}]_{\mathcal{I}}\tilde{\psi}$ and $[\hat{\mu}]_{\mathcal{I}}\bar{\phi}\psi = \mu\psi = \mu_V = [\hat{\mu}]_{\mathcal{I}}\tilde{\psi}$. So we have $\bar{\phi}\psi = \tilde{\psi}$.

So using the presentation of Ass from Theorem 4.33, by defining $\psi := \bar{\phi}^{-1}\tilde{\psi}$ we obtain the same morphism of set-preoperads $\psi : \mathrm{Ass} \longrightarrow \mathrm{End}(X)$ as in Proposition 5.7, turning V into an Ass-algebra.

6 Operads and algebras over operads

In this chapter we will define set-operads and linear operads and consider some basic examples.

In order to so, we will need certain elements of the set-preoperad Map_0.

Definition 6.1. For $m, n \in \mathbb{Z}_{\geqslant 0}$ define $s_{m,n} \in \mathrm{Map}_0(m+n, m+n)$ by

$$is_{m,n} = \begin{cases} i+n & \text{if } i \in [1,m] \\ i-m & \text{if } i \in [m+1, m+n] \end{cases}$$

for $i \in [1, m+n]$. We get an element $s_{m,n}^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m+n, m+n)$.

Note that for $m, n \in \mathbb{Z}_{\geqslant 0}$ we have $s_{m,n} \cdot s_{n,m} = \mathrm{id}_{m+n} = s_{n,m} \cdot s_{m,n}$. So $s_{m,n}$ is a bijective map.

Definition 6.2. For $n, l \in \mathbb{Z}_{\geqslant 0}$ and $n \in [1, kl]$ we can uniquely write $n = l \cdot u + v$ with $v \in [1, l]$ and $u \in \mathbb{Z}$. By defining $\underline{n} := u$ and $\bar{n} := v$ we can also define a map

$$\begin{aligned} h_{k,l} : [1, k \cdot l] &\longrightarrow [1, l] \\ n &\longmapsto \bar{n}. \end{aligned}$$

So we get an element $h_{k,l} \in \mathrm{Map}_0(kl, l)$.

Note that we have $h_{1,l} = \mathrm{id}_{[1,l]} = \mathrm{id}_{\mathrm{Map}_0, l}$ for $l \in \mathbb{Z}_{\geqslant 0}$.

6.1 Set-operads

Definition 6.3. A *set-operad* $(\mathcal{P}_0, \mathfrak{p}_0)$ is given by a set-preoperad $(\mathcal{P}_0, \boxtimes, \cdot)$ and a morphism of set-preoperads $\mathfrak{p}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathcal{P}_0$ such that (so1) and (so2) hold.

(so1) We have $(s_{m,m'}^{\mathrm{op}} \mathfrak{p}_0) \cdot (f \boxtimes f') = (f' \boxtimes f) \cdot (s_{n,n'}^{\mathrm{op}} \mathfrak{p}_0) \in \mathcal{P}_0(m+m', n+n')$ for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$, $f \in \mathcal{P}_0(m,n)$, $f' \in \mathcal{P}_0(m', n')$.

(so2) We have $(h_{k,m}^{\mathrm{op}} \mathfrak{p}_0) \cdot f^{\boxtimes k} = f \cdot (h_{k,n}^{\mathrm{op}} \mathfrak{p}_0) \in \mathcal{P}_0(m, kn)$ for $k, m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m,n)$.

For brevity, we refer to the set-operad $(\mathcal{P}_0, \mathfrak{p}_0)$ simply by $\mathcal{P}_0$. We then denote by $\mathcal{P}_0^{\mathrm{pre}}$ the underlying set-preoperad of $\mathcal{P}_0$.

So we have $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ for the morphism of set-preoperads $\mathfrak{p}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathcal{P}_0$ belonging to $\mathcal{P}_0$.

Whenever necessary, the multiplication in $\mathcal{P}_0^{\mathrm{pre}}$ is written $(\boxtimes_{\mathcal{P}_0^{\mathrm{pre}}})$ or simply $(\boxtimes_{\mathcal{P}_0})$ and the composition is written $(\cdot_{\mathcal{P}_0^{\mathrm{pre}}})$ or simply $(\cdot_{\mathcal{P}_0})$. Moreover, we usually denote the identity elements of $\mathcal{P}_0^{\mathrm{pre}}$ by id_m or $\mathrm{id}_{\mathcal{P}_0, m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

Remark 6.4. *Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad. Then for $m \in \mathbb{Z}_{\geqslant 0}$ we have $|\mathcal{P}_0^{\mathrm{pre}}(m, 0)| = 1$.*

In other words, if we view $\mathcal{P}_0^{\mathrm{pre}}$ as a category, then 0 is a terminal element.

Proof. Let $m \in \mathbb{Z}_{\geqslant 0}$. First note that $h_{0,m}^{\mathrm{op}} \mathfrak{p}_0 \in \mathcal{P}_0(m, 0)$, so $|\mathcal{P}_0(m,0)| \geqslant 1$.

Suppose given $f \in \mathcal{P}_0(m, 0)$. We will show that $f = h_{0,m}^{\mathrm{op}} \mathfrak{p}_0$.

Note that $f^{\boxtimes 0} = \mathrm{id}_0 \in \mathcal{P}_0(0,0)$ and that $h_{0,0}^{\mathrm{op}} \mathfrak{p}_0 = \mathrm{id}_0 \in \mathcal{P}_0(0,0)$ since $h_{0,0} \in \mathrm{Map}_0(0,0)$ and $\mathrm{Map}_0(0,0) = \{\mathrm{id}_{\mathrm{Map}_0, 0}\}$. So by (so2) we have

$$(h_{0,m}^{\mathrm{op}} \mathfrak{p}_0) = (h_{0,m}^{\mathrm{op}} \mathfrak{p}_0) \cdot \mathrm{id}_0 = (h_{0,m}^{\mathrm{op}} \mathfrak{p}_0) \cdot f^{\boxtimes 0} \overset{\text{(so2)}}{=} f \cdot (h_{0,0}^{\mathrm{op}} \mathfrak{p}_0) = f \cdot \mathrm{id}_0 = f.$$

□

Example 6.5. Let X be a set. Recall the morphism of set-preoperads $\mathfrak{e}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{End}_0(X)$ that maps an element $a^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$ (given by a map $a \in \mathrm{Map}_0(n,m)$) to $a^{\mathrm{op}}\mathfrak{e}_0 \in \mathrm{End}_0(X)(m,n)$ defined by

$$(x_1,\ldots,x_m)(a^{\mathrm{op}}\mathfrak{e}_0) = (x_{1a},\ldots,x_{na})$$

for $x_1,\ldots,x_m \in X$; cf. Definition 2.61.

Then $\mathrm{END}_0(X) := (\mathrm{End}_0(X),\mathfrak{e}_0)$ is a set-operad.

Ad (so1). Note that for $m,m' \in \mathbb{Z}_{\geqslant 0}$ and $x_1,\ldots,x_{m+m'} \in X$ we have

$$(x_1,\ldots,x_{m+m'})\,(s^{\mathrm{op}}_{m,m'}\mathfrak{e}_0) = (x_{1s_{m,m'}},\ldots,x_{(m+m')s_{m,m'}}) = (x_{m'+1},\ldots,x_{m'+m},x_1,\ldots,x_{m'}).$$

Let $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}_0(X)(m,n)$, $f' \in \mathrm{End}_0(X)(m',n')$. Then for $x_1,\ldots,x_{m+m'} \in X$ we have

$$\begin{aligned}
(x_1,\ldots,x_{m+m'})\big((s^{\mathrm{op}}_{m,m'}\mathfrak{e}_0)\cdot(f\boxtimes f')\big) &= \big((x_1,\ldots,x_m,x_{m+1},\ldots,x_{m+m'})(s^{\mathrm{op}}_{m,m'}\mathfrak{e}_0)\big)(f\boxtimes f')\\
&= (x_{m'+1},\ldots,x_{m'+m},x_1,\ldots,x_{m'})(f\boxtimes f')\\
&= (x_{m'+1},\ldots,x_{m'+m})f \times (x_1,\ldots,x_{m'})f'\\
&= \big((x_1,\ldots,x_{m'})f' \times (x_{m'+1},\ldots,x_{m'+m})f\big)s^{\mathrm{op}}_{n,n'}\mathfrak{e}_0\\
&= \big((x_1,\ldots,x_{m'},x_{m'+1},\ldots,x_{m'+m})(f'\boxtimes f)\big)(s^{\mathrm{op}}_{n,n'}\mathfrak{e}_0)\\
&= (x_1,\ldots,x_{m'+m})\big((f'\boxtimes f)\cdot(s^{\mathrm{op}}_{n,n'}\mathfrak{e}_0)\big).
\end{aligned}$$

Hence we have $(s^{\mathrm{op}}_{m,m'}\mathfrak{e}_0)\cdot(f\boxtimes f') = (f'\boxtimes f)\cdot(s^{\mathrm{op}}_{n,n'}\mathfrak{e}_0)$.

Ad (so2). Let $m,n,k \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{End}_0(X)(m,n)$. Then for $x_1,\ldots,x_m \in X$ by defining $(y_1,\ldots,y_n) := (x_1,\ldots,x_m)f$ we have

$$\begin{aligned}
(x_1,\ldots,x_l)\big((h^{\mathrm{op}}_{k,m}\mathfrak{e}_0)\cdot f^{\boxtimes k}\big) &= \big((x_1,\ldots,x_l)(h^{\mathrm{op}}_{k,m}\mathfrak{e}_0)\big)f^{\boxtimes k}\\
&= (x_{1h_{k,m}},x_{2h_{k,m}},\ldots,x_{(km)h_{k,m}})f^{\boxtimes k}\\
&= (x_1,\ldots,x_m,x_1,\ldots,x_m,\ldots,x_1,\ldots,x_m)f^{\boxtimes k}\\
&= (x_1,\ldots,x_m)f \times (x_1,\ldots,x_m)f \times \ldots \times (x_1,\ldots,x_m)f\\
&= (y_1,\ldots,y_n) \times (y_1,\ldots,y_n) \times \ldots \times (y_1,\ldots,y_n)\\
&= (y_{1h_{k,n}},y_{2h_{k,n}},\ldots,y_{(kn)h_{k,n}})\\
&= (y_1,\ldots,y_n)(h^{\mathrm{op}}_{k,n}\mathfrak{e}_0)\\
&= \big((x_1,\ldots,x_m)f\big)(h^{\mathrm{op}}_{k,n}\mathfrak{e}_0)\\
&= (x_1,\ldots,x_m)\big(f\cdot(h^{\mathrm{op}}_{k,n}\mathfrak{e}_0)\big).
\end{aligned}$$

So we have $(h^{\mathrm{op}}_{k,m}\mathfrak{e}_0)\cdot f^{\boxtimes k} = f\cdot(h^{\mathrm{op}}_{k,n}\mathfrak{e}_0)$.

This shows that $\mathrm{END}_0(X) = (\mathrm{End}_0(X),\mathfrak{e}_0)$ is a set-operad. We have $\mathrm{END}_0(X)^{\mathrm{pre}} = \mathrm{End}_0(X)$.

Lemma 6.6. *Let $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Map}_0(m,n)$, $a' \in \mathrm{Map}_0(m',n')$. Then we have*

$$(a \boxtimes_{\mathrm{Map}_0} a') \cdot_{\mathrm{Map}_0} s_{m,m'} = s_{n,n'} \cdot_{\mathrm{Map}_0} (a' \boxtimes_{\mathrm{Map}_0} a).$$

Proof. Let $i \in [1,n+n']$. Then we have

$$\begin{aligned}
i((a\boxtimes_{\mathrm{Map}_0} a')\cdot_{\mathrm{Map}_0} s_{m,m'} &= \begin{cases}(ia)s_{m,m'} & \text{if } i\in[1,n]\\ ((i-n)a'+m)s_{m,m'} & \text{if } i\in[n+1,n+n']\end{cases}\\
&= \begin{cases}ia+m' & \text{if } i\in[1,n]\\ ((i-n)a'+m)-m & \text{if } i\in[n+1,n+n']\end{cases}\\
&= \begin{cases}ia+m' & \text{if } i\in[1,n]\\ (i-n)a' & \text{if } i\in[n+1,n+n']\end{cases}
\end{aligned}$$

and on the other hand

$$\begin{aligned}
i(s_{n,n'} \cdot_{\mathrm{Map}_0} (a' \boxtimes_{\mathrm{Map}_0} a)) &= \begin{cases} (i+n')(a' \boxtimes_{\mathrm{Map}_0} a) & \text{if } i \in [1,n] \\ (i-n)(a' \boxtimes_{\mathrm{Map}_0} a) & \text{if } i \in [n+1, n+n'] \end{cases} \\
&= \begin{cases} ((i+n')-n')a + m' & \text{if } i \in [1,n] \\ (i-n)a' & \text{if } i \in [n+1, n+n'] \end{cases} \\
&= \begin{cases} ia + m' & \text{if } i \in [1,n] \\ (i-n)a' & \text{if } i \in [n+1, n+n']. \end{cases}
\end{aligned}$$

So the maps are the same. □

Example 6.7. We have the set-operad $\mathrm{MAP}_0^{\mathrm{op}} := (\mathrm{Map}_0^{\mathrm{op}}, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}})$, where $\mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}$ is the identity morphism

$$\mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}} : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{Map}_0^{\mathrm{op}} .$$

For the proof denote by $(\boxtimes)$ and $(\cdot)$ multiplication and composition in Map_0 and by $(\boxtimes_{\mathrm{op}})$ and $(\cdot_{\mathrm{op}})$ multiplication and composition in $\mathrm{Map}_0^{\mathrm{op}}$.

Ad (so1). We have to show that for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $a^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$ and $a'^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m',n')$ we have

$$(s_{m,m'}^{\mathrm{op}} \, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}) \cdot_{\mathrm{op}} (a^{\mathrm{op}} \boxtimes_{\mathrm{op}} a'^{\mathrm{op}}) \stackrel{!}{=} (a'^{\mathrm{op}} \boxtimes_{\mathrm{op}} a^{\mathrm{op}}) \cdot_{\mathrm{op}} (s_{n,n'}^{\mathrm{op}} \, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}).$$

Using the definition of $(\cdot_{\mathrm{op}})$ and $(\boxtimes_{\mathrm{op}})$ and Lemma 6.6 above, we get

$$\begin{aligned}
(s_{m,m'}^{\mathrm{op}} \, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}) \cdot_{\mathrm{op}} (a^{\mathrm{op}} \boxtimes_{\mathrm{op}} a'^{\mathrm{op}}) &= s_{m,m'}^{\mathrm{op}} \cdot_{\mathrm{op}} (a \boxtimes a')^{\mathrm{op}} \\
&= ((a \boxtimes a') \cdot s_{m,m'})^{\mathrm{op}} \\
&= (s_{n,n'} \cdot (a' \boxtimes a))^{\mathrm{op}} \\
&= (a' \boxtimes a)^{\mathrm{op}} \cdot_{\mathrm{op}} s_{n,n'}^{\mathrm{op}} \\
&= (a'^{\mathrm{op}} \boxtimes_{\mathrm{op}} a^{\mathrm{op}}) \cdot_{\mathrm{op}} (s_{n,n'}^{\mathrm{op}} \, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}).
\end{aligned}$$

Ad (so2). Let $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$. We have to show that

$$(h_{k,m}^{\mathrm{op}} \, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}) \cdot_{\mathrm{op}} (f^{\mathrm{op}})^{\boxtimes k} \stackrel{!}{=} f^{\mathrm{op}} \cdot_{\mathrm{op}} (h_{k,n}^{\mathrm{op}} \, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}).$$

Since

$$\begin{aligned}
(h_{k,m}^{\mathrm{op}} \, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}) \cdot_{\mathrm{op}} (f^{\mathrm{op}})^{\boxtimes k} &= h_{k,m}^{\mathrm{op}} \cdot_{\mathrm{op}} (f^{\boxtimes k})^{\mathrm{op}} = (f^{\boxtimes k} \cdot h_{k,m})^{\mathrm{op}} \\
f^{\mathrm{op}} \cdot_{\mathrm{op}} (h_{k,n}^{\mathrm{op}} \, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}) &= (h_{k,n} \cdot f)^{\mathrm{op}},
\end{aligned}$$

it suffices to show that

$$f^{\boxtimes k} \cdot h_{k,m} \stackrel{!}{=} h_{k,n} \cdot f.$$

We will show this via induction on $k \geqslant 0$.

Let $k = 0$. Then $h_{0,m} \in \mathrm{Map}_0(0,m)$ and $h_{0,n} \in \mathrm{Map}_0(0,n)$ are the unique elements in $\mathrm{Map}_0(0,m)$ and $\mathrm{Map}_0(0,n)$. Furthermore, $f^{\boxtimes 0} = \mathrm{id}_{\mathrm{Map}_0, 0}$ is the unique map in $\mathrm{Map}_0(0,0)$. So the statement is true for $k = 0$ since on both sides of the equation we have a map from $[1,0]$ to $[1,m]$ and this is uniquely determined.

Now let $k > 0$ and assume that the statement is true for $k-1$. Note that for $r \in \mathbb{Z}_{\geqslant 0}$ and $j \in [1, kr]$ we have

$$\begin{aligned} jh_{k,r} &= \begin{cases} jh_{k-1,r} & \text{if } j \in [1,(k-1)r] \\ (j-(k-1)r)h_{1,r} & \text{if } j \in [(k-1)r+1, kr] \end{cases} \\ &= \begin{cases} jh_{k-1,r} & \text{if } j \in [1,(k-1)r] \\ j-(k-1)r & \text{if } j \in [(k-1)r+1, kr]; \end{cases} \end{aligned}$$

cf. Definition 6.2. Now let $i \in [1, kn]$. Then we have

$$\begin{aligned} i(f^{\boxtimes k} \cdot h_{k,m}) &= i((f^{\boxtimes(k-1)} \boxtimes f) \cdot h_{k,m} \\ &= \begin{cases} (if^{\boxtimes(k-1)})h_{k,m} & \text{if } i \in [1,(k-1)n] \\ \big((i-(k-1)n)f + (k-1)m\big)h_{k,m} & \text{if } i \in [(k-1)n+1, kn] \end{cases} \\ &= \begin{cases} (if^{\boxtimes(k-1)})h_{k-1,m} & \text{if } i \in [1,(k-1)n] \\ ((i-(k-1)n)f + (k-1)m) - (k-1)m & \text{if } i \in [(k-1)n+1, kn] \end{cases} \\ &= \begin{cases} i(f^{\boxtimes(k-1)} \cdot h_{k-1,m}) & \text{if } i \in [1,(k-1)n] \\ (i-(k-1)n)f & \text{if } i \in [(k-1)n+1, kn] \end{cases} \\ &\overset{\text{ind.}}{=} \begin{cases} i(h_{k-1,n} \cdot f) & \text{if } i \in [1,(k-1)n] \\ (i-(k-1)n)f & \text{if } i \in [(k-1)n+1, kn] \end{cases} \\ &= \begin{cases} (ih_{k-1,n})f & \text{if } i \in [1,(k-1)n] \\ (i-(k-1)n)f & \text{if } i \in [(k-1)n+1, kn] \end{cases} \\ &= (ih_{k,n})f \\ &= i(h_{k,n} \cdot f). \end{aligned}$$

This shows that $\mathrm{MAP}_0^{\mathrm{op}} = (\mathrm{Map}_0^{\mathrm{op}}, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}})$ is in fact a set-operad.

Now recall that for $n \in \mathbb{Z}_{\geqslant 0}$ and for a tuple $k = (k_i)_{i \in [1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$ there exists the bijective map $\varphi_k : \Big[1, \sum\limits_{i \in [1,n]} k_i\Big] \longrightarrow \bigsqcup\limits_{i \in [1,n]} [1, k_i]$ given by

$$\varphi_k : t \longmapsto \Big(t\chi_k \,,\, t - \sum_{s \in [1, t\chi_k - 1]} k_s\Big),$$

where

$$\begin{aligned} \chi_k : \Big[1, \sum_{i \in [1,n]} k_i\Big] &\longrightarrow [1,n] \\ t &\longmapsto \min\Big\{u \in [1,n] : \sum_{s \in [1,u]} k_s \geqslant t\Big\}. \end{aligned}$$

Its inverse map is

$$\begin{aligned} \varphi_k^{-1} : \bigsqcup_{i \in [1,n]} [1, k_i] &\longrightarrow \Big[1, \sum_{i \in [1,n]} k_i\Big] \\ (i, x) &\longmapsto \Big(\sum_{s \in [1, i-1]} k_s\Big) + x; \end{aligned}$$

cf. Definition 1.18.

Definition 6.8. Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and let $a \in \mathrm{Map}_0(m, n)$ be a map. Let $k = (k_i)_{i \in [1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$. We write $ka^* := (k_{ja})_{j \in [1,m]} \in (\mathbb{Z}_{\geqslant 0})^{\times m}$. Then define

$$\begin{aligned} \tilde{a}_{[k]} : \bigsqcup_{j \in [1,m]} [1, k_{ja}] &\longrightarrow \bigsqcup_{i \in [1,n]} [1, k_i] \\ (j, x) &\longmapsto (ja, x) \end{aligned}$$

and define the map $a_{[k]}$ by

$$a_{[k]} = \varphi_{ka^*} \cdot \tilde{a}_{[k]} \cdot \varphi_k^{-1}.$$

We have

$$\Big[1, \sum_{j \in [1,m]} k_{ja}\Big] \xrightarrow{\varphi_{ka^*}} \bigsqcup_{j \in [1,m]} [1, k_{ja}] \xrightarrow{\tilde{a}_{[k]}} \bigsqcup_{i \in [1,n]} [1, k_i] \xrightarrow{\varphi_k^{-1}} \Big[1, \sum_{i \in [1,n]} k_i\Big],$$

so $a_{[k]} \in \mathrm{Map}_0\big(\sum_{j \in [1,m]} k_{ja}, \sum_{i \in [1,n]} k_i\big)$.

Example 6.9.

(i) Let $m = n$ and let $a = \mathrm{id}_m$, the identity map. Then for $k = (k_i)_{i \in [1,m]} \in (\mathbb{Z}_{\geqslant 0})^{\times m}$ we have $k\,\mathrm{id}_m^* = (k_{j\,\mathrm{id}_m})_{j \in [1,m]} = (k_j)_{j \in [1,m]} = k$, hence $\varphi_{k\,\mathrm{id}_m^*} = \varphi_k$.

Moreover, $(\widetilde{\mathrm{id}}_m)_{[k]}$ is the identity map, hence $(\mathrm{id}_m)_{[k]} = \mathrm{id}_{\Sigma k}$, where $\Sigma k := \sum_{i \in [1,m]} k_i$.

(ii) Let $m = n = 2$ and let $a = (1, 2)$ be the transposition. Then for $k = (k_1, k_2) \in (\mathbb{Z}_{\geqslant 0})^{\times 2}$ we have $(1, 2)_{[k]} = s_{k_2,k_1}$ since for $i \in [1, k_2 + k_1]$ we have

$$\begin{aligned} i\big(\varphi_{(k_2,k_1)} \cdot \widetilde{(1,2)}_{[(k_1,k_2)]} \cdot \varphi^{-1}_{(k_1,k_2)}\big) &= \begin{cases} (1, i)\big(\widetilde{(1,2)}_{[(k_1,k_2)]} \cdot \varphi^{-1}_{(k_1,k_2)}\big) & \text{if } i \in [1, k_2] \\ (2, i - k_2)\big(\widetilde{(1,2)}_{[(k_1,k_2)]} \cdot \varphi^{-1}_{(k_1,k_2)}\big) & \text{if } i \in [k_2 + 1, k_2 + k_1] \end{cases} \\ &= \begin{cases} (2, i)\varphi^{-1}_{(k_1,k_2)} & \text{if } i \in [1, k_2] \\ (1, i - k_2)\varphi^{-1}_{(k_1,k_2)} & \text{if } i \in [k_2 + 1, k_2 + k_1] \end{cases} \\ &= \begin{cases} k_1 + i & \text{if } i \in [1, k_2] \\ i - k_2 & \text{if } i \in [k_2 + 1, k_2 + k_1] \end{cases} \\ &= i s_{k_2,k_1}. \end{aligned}$$

(iii) Let $m \in \mathbb{Z}_{\geqslant 0}$ and let $a = \mu_m \in \mathrm{Map}_0(m, 1)$; cf. Definition 4.29. Then for $k = (k_1) \in (\mathbb{Z}_{\geqslant 0})^{\times 1}$ we have $k\mu_m^* = (k_{j\mu_m})_{j \in [1,m]} = (k_1, \ldots, k_1) \in (\mathbb{Z}_{\geqslant 0})^{\times m}$. We can write $i = \underline{i} \cdot k_1 + \overline{i}$ for $i \in [1, m \cdot k_1]$ with $\overline{i} \in [1, k_1]$ and $\underline{i} \in [0, m - 1]$ in a unique way. Then we have $i\varphi_{k\mu_m^*} = (\underline{i} + 1, \overline{i})$ and thus

$$\begin{aligned} i(\varphi_{k\mu_m^*} \cdot (\widetilde{\mu_m})_{[k]} \cdot \varphi_k^{-1}) &= (\underline{i} + 1, \overline{i})((\widetilde{\mu_m})_{[k]} \cdot \varphi_k^{-1}) \\ &= (1, \overline{i})\varphi_k^{-1} \\ &= \overline{i} \\ &= i h_{m,k_1}. \end{aligned}$$

Hence we have $(\mu_m)_{[(k_1)]} = h_{m,k_1}$.

(iv) Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and let $a \in \mathrm{Map}_0(m, n)$. Let $k = (1, \ldots, 1) \in (\mathbb{Z}_{\geqslant 0})^{\times n}$. Then we have $ka^* = (1, \ldots, 1) = k$ and φ_k is the map

$$\begin{aligned} \varphi_k : [1, n] &\longrightarrow \bigsqcup_{i \in [1,n]} [1, 1] \\ i &\longmapsto (i, 1). \end{aligned}$$

So for $i \in [1, n]$ we have

$$ia_{[k]} = i(\varphi_k \cdot \tilde{a}_{[k]} \cdot \varphi_k^{-1}) = (i, 1)(\tilde{a}_{[k]} \cdot \varphi_k^{-1}) = (ia, 1)\varphi_k^{-1} = (ia - 1) \cdot 1 + a = ia.$$

Hence we have $a_{[k]} = a$.

Lemma 6.10. *Let $(\mathcal{P}_0, \boxtimes, \cdot)$ be a set-preoperad and let $\mathfrak{p}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathcal{P}_0$ be a morphism of set-preoperads.*

For $m, n \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Map}_0(m, n)$ consider the following assertion $\mathrm{B}_0(a)$.

$\mathrm{B}_0(a)$: *For $l_i, r_i \in \mathbb{Z}_{\geqslant 0}$ and $f_i \in \mathcal{P}_0(l_i, r_i)$ for $i \in [1, n]$ and for $l = (l_i)_{i \in [1,n]}$ and $r = (r_i)_{i \in [1,n]}$ we have*

$$\left(\boxtimes_{i \in [1,n]} f_i \right) \cdot \left(a_{[r]}^{\mathrm{op}} \mathfrak{p}_0 \right) = \left(a_{[l]}^{\mathrm{op}} \mathfrak{p}_0 \right) \cdot \left(\boxtimes_{j \in [1,m]} f_{ja} \right).$$

We have

(1) *The morphism of set-preoperads $\mathfrak{p}_0$ satisfies condition* (so1) *from Definition* 6.3 *if and only if* $\mathrm{B}_0(a)$ *holds for all $m, n \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \mathrm{Map}_0(m, n)$ such that a is bijective.*

(2) *The morphism of set-preoperads $\mathfrak{p}_0$ satisfies conditions* (so1) *and* (so2) *from Definition* 6.3 *if and only if* $\mathrm{B}_0(a)$ *holds for all $m, n \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \mathrm{Map}_0(m, n)$.*

So $(\mathcal{P}_0, \mathfrak{p}_0)$ is a set-operad if and only if $\mathrm{B}_0(a)$ *holds for all $m, n \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \mathrm{Map}_0(m, n)$.*

Proof. We will show this using a couple of steps. Note that to prove (1) we shall only need to apply $\mathfrak{p}_0$ to bijective maps, using the the Claims 1.1, 1.2, 2.1, 2.2, 3.1 and 3.2.

During this proof we denote by $(\boxtimes_{\mathcal{P}_0})$ and $(\cdot_{\mathcal{P}_0})$ the multiplication and composition in $\mathcal{P}_0$, by $(\boxtimes)$ and $(\cdot)$ the multiplication and composition in Map_0.

Our first aim will be to show that $\mathrm{B}_0(a')$ and $\mathrm{B}_0(a'')$ for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Map}_0(m, n)$, $a' \in \mathrm{Map}_0(m', n')$ imply $\mathrm{B}_0(a \boxtimes a')$.

Claim 1.1. Let $m', n', m'', n'' \in \mathbb{Z}_{\geqslant 0}$ and $a' \in \mathrm{Map}_0(m', n')$, $a'' \in \mathrm{Map}_0(m'', n'')$. Then for $k = (k_i)_{i \in [1, n'+n'']} \in (\mathbb{Z}_{\geqslant 0})^{\times (n+n')}$ we have

$$\left(a' \boxtimes a'' \right)_{[k]} = a'_{[k']} \boxtimes a''_{[k'']},$$

where $k' = (k_i)_{i \in [1,n']} \in (\mathbb{Z}_{\geqslant 0})^{\times n'}$ and $k'' = (k_{i+n'})_{i \in [1,n'']} \in (\mathbb{Z}_{\geqslant 0})^{\times n''}$.

Proof of Claim 1.1. We will use the following abbreviations.

$$\begin{aligned} \Sigma k' &:= \sum_{i \in [1,n']} k'_i = \sum_{i \in [1,n']} k_i & \Sigma k' a'^* &:= \sum_{j \in [1,m']} k'_{ja'} = \sum_{j \in [1,m']} k_{ja'} \\ \Sigma k'' &:= \sum_{i \in [1,n'']} k''_i = \sum_{i \in [1,n'']} k_{i+n'} & \Sigma k'' a''^* &:= \sum_{j \in [1,m'']} k''_{ja''} = \sum_{j \in [1,m'']} k_{ja''+m'} \\ \Sigma k &:= \sum_{i \in [1,n'+n'']} k_i = \Sigma k' + \Sigma k'' & \Sigma k''(a' \boxtimes a'')^* &:= \sum_{j \in [1,m'+m'']} k_{j(a' \boxtimes a'')} = \Sigma k' a'^* + \Sigma k'' a''^* \end{aligned}$$

First note that for $t \in [1, \Sigma k']$ we have

$$t\chi_k = \min\{u \in [1, n'+n''] : \sum_{s\in[1,u]} k_s \geqslant t\} = \min\{u \in [1, n'] : \sum_{s\in[1,u]} k_s \geqslant t\} = t\chi_{k'} \in [1, n']$$

and for $t \in [(\Sigma k')+1, \Sigma k]$ we have

$$\begin{aligned} t\chi_k &= \min\{u \in [1, n'+n''] : \sum_{s\in[1,u]} k_s \geqslant t\} \\ &= n' + \min\{u \in [1, n''] : (\Sigma k') + \Big(\sum_{s\in[1,u]} k_{s+n'}\Big) \geqslant t\} \\ &= n' + \min\{u \in [1, n''] : \sum_{s\in[1,u]} k_{s+n'} \geqslant (t - \Sigma k')\} \\ &= n' + (t - \Sigma k')\chi_{k''} \in [n'+1, n'+n'']. \end{aligned}$$

Now suppose given $t \in [1, \Sigma k(a' \boxtimes a'')^*]$.

Case $t \in [1, \Sigma k' a'^*]$. So $t\chi_{k(a'\boxtimes a'')^*} = t\chi_{k'a'^*} \in [1, m']$. Then we have

$$\begin{aligned} t\left(a' \boxtimes a''\right)_{[k]} &= t\big(\varphi_{k(a'\boxtimes a'')^*} \cdot (\widetilde{a' \boxtimes a''})_{[k]} \cdot \varphi_k^{-1}\big) \\ &= \Bigg(t\chi_{k(a'\boxtimes a'')^*}\,,\; t - \sum_{s\in[1, t\chi_{k(a'\boxtimes a'')^*}-1]} k_{s(a'\boxtimes a'')}\Bigg)\big((\widetilde{a' \boxtimes a''})_{[k]} \cdot \varphi_k^{-1}\big) \\ &= \Bigg(t\chi_{k'a'^*}\,,\; t - \sum_{s\in[1, t\chi_{k'a'^*}-1]} k_{s(a'\boxtimes a'')}\Bigg)\big((\widetilde{a' \boxtimes a''})_{[k]} \cdot \varphi_k^{-1}\big) \\ &= \Bigg((t\chi_{k'a'^*})(a' \boxtimes a'')\,,\; t - \sum_{s\in[1, t\chi_{k'a'^*}-1]} k_{sa'}\Bigg)\varphi_k^{-1} \\ &= \Bigg((t\chi_{k'a'^*})a'\,,\; t - \sum_{s\in[1, t\chi_{k'a'^*}-1]} k_{sa'}\Bigg)\varphi_k^{-1} \\ &= \Bigg(\sum_{i\in[1, (t\chi_{k'a'^*})a'-1]} k_i\Bigg) + t - \sum_{s\in[1, t\chi_{k'a'^*}-1]} k_{sa'} \end{aligned}$$

and

$$\begin{aligned} t\big(a'_{[k']} \boxtimes a''_{[k'']}\big) &= ta'_{[k']} \\ &= t\big(\varphi_{k'a'^*} \cdot \tilde{a}'_{[k']} \cdot \varphi_{k'}^{-1}\big) \\ &= \Bigg(t\chi_{k'a'^*}\,,\; t - \sum_{s\in[1, t\chi_{k'a'^*}-1]} k_{sa'}\Bigg)\big(\tilde{a}'_{[k']} \cdot \varphi_{k'}^{-1}\big) \\ &= \Bigg((t\chi_{k'a'^*})a'\,,\; t - \sum_{s\in[1, t\chi_{k'a'^*}-1]} k_{sa'}\Bigg)\varphi_{k'}^{-1} \\ &= \Bigg(\sum_{i\in[1, (t\chi_{k'a'^*})a'-1]} k_i\Bigg) + t - \sum_{s\in[1, t\chi_{k'a'^*}-1]} k_{sa'}\,. \end{aligned}$$

So we have

$$t\left(a'_{[k']} \boxtimes a''_{[k'']}\right) = t\left(a' \boxtimes a''\right)_{[k]}\,.$$

Case $t \in [\Sigma k' a'^* + 1, \Sigma k(a' \boxtimes a'')^*]$. Define $t' := t - \Sigma k' a'^*$. So from the above calculations we know $t\chi_{k(a'\boxtimes a'')^*} = m' + t'\chi_{k''a''^*} \in [m'+1, m'+m'']$. Then we have

$$\begin{aligned}
t\big(a' \boxtimes a''\big)_{[k]} &= t\big(\varphi_{k(a'\boxtimes a'')^*} \cdot (\widetilde{a' \boxtimes a''})_{[k]} \cdot \varphi_k^{-1}\big) \\
&= \Bigg(t\chi_{k(a'\boxtimes a'')^*}\,,\ t - \sum_{s \in [1, t\chi_{k(a'\boxtimes a'')^*} - 1]} k_{s(a'\boxtimes a'')}\Bigg)\big((\widetilde{a' \boxtimes a''})_{[k]} \cdot \varphi_k^{-1}\big) \\
&= \Bigg(t'\chi_{k''a''^*} + m'\,,\ t - \sum_{s \in [1, t'\chi_{k''a''^*} + m' - 1]} k_{s(a'\boxtimes a'')}\Bigg)\big((\widetilde{a' \boxtimes a''})_{[k]} \cdot \varphi_k^{-1}\big) \\
&= \Bigg(t'\chi_{k''a''^*} + m'\,,\ t - \sum_{s \in [1, m']} k_{s(a'\boxtimes a'')} - \sum_{s \in [m'+1, t'\chi_{k''a''^*} + m' - 1]} k_{s(a'\boxtimes a'')}\Bigg)\big((\widetilde{a' \boxtimes a''})_{[k]} \cdot \varphi_k^{-1}\big) \\
&= \Bigg((t'\chi_{k''a''^*} + m')(a' \boxtimes a'')\,,\ t - \Sigma k' a'^* - \sum_{s \in [m'+1, t'\chi_{k''a''^*} + m' - 1]} k_{(s-m')a''+n'}\Bigg)\varphi_k^{-1} \\
&= \Bigg((t'\chi_{k''a''^*})a'' + n'\,,\ t' - \sum_{s \in [1, t'\chi_{k''a''^*} - 1]} k_{sa''+n'}\Bigg)\varphi_k^{-1} \\
&= \Bigg(\sum_{i \in [1, (t'\chi_{k''a''^*})a'' + n' - 1]} k_i\Bigg) + t' - \sum_{s \in [1, t'\chi_{k''(a'')^*} - 1]} k_{sa''+n'}
\end{aligned}$$

and

$$\begin{aligned}
t\big(a'_{[k']} \boxtimes a''_{[k'']}\big) &= t' a''_{[k'']} + \Sigma k' \\
&= t'\big(\varphi_{k''a''^*} \cdot \tilde{a}''_{[k'']} \cdot \varphi_{k''}^{-1}\big) + \Sigma k' \\
&= \Bigg(t'\chi_{k''a''^*}\,,\ t' - \sum_{s \in [1, t'\chi_{k''a''^*} - 1]} k_{sa''+n'}\Bigg)\big(\tilde{a}''_{[k'']} \cdot \varphi_{k''}^{-1}\big) + \Sigma k' \\
&= \Bigg((t'\chi_{k''a''^*})a''\,,\ t' - \sum_{s \in [1, t'\chi_{k''a''^*} - 1]} k_{sa''+n'}\Bigg)\varphi_{k''}^{-1} + \Sigma k' \\
&= \Bigg(\sum_{i \in [1, (t'\chi_{k''a''^*})a'' - 1]} k_{n'+i}\Bigg) + t' - \Bigg(\sum_{s \in [1, t'\chi_{k''a''^*} - 1]} k_{sa''+n'}\Bigg) + \Sigma k' \\
&= \Bigg(\sum_{i \in [n'+1, (n' + t'\chi_{k''a''^*})a'' - 1]} k_i\Bigg) + t' - \Bigg(\sum_{s \in [1, t'\chi_{k''a''^*} - 1]} k_{sa''+n'}\Bigg) + \Sigma k' \\
&= \Bigg(\sum_{i \in [1, (t'\chi_{k''a''^*})a'' + n' - 1]} k_i\Bigg) + t' - \Bigg(\sum_{s \in [1, t'\chi_{k''(a'')^*} - 1]} k_{sa''+n'}\Bigg).
\end{aligned}$$

So we have

$$t(a'_{[k']} \boxtimes a''_{[k'']}) = t\left(a' \boxtimes a''\right)_{[k]}\,.$$

This proves *Claim* 1.1.

Claim 1.2. Let $m', n', m'', n'' \in \mathbb{Z}_{\geqslant 0}$ and $a' \in \mathrm{Map}_0(m', n')$, $a'' \in \mathrm{Map}_0(m'', n'')$. If $\mathrm{B}_0(a')$ and $\mathrm{B}_0(a'')$ are true then so is $\mathrm{B}_0(a' \boxtimes a'')$.

Proof of Claim 1.2. We have to show that for tuples $l = (l_i)_{i \in [1, n'+n'']} \in (\mathbb{Z}_{\geqslant 0})^{\times (n'+n'')}$, $r = (r_i)_{i \in [1, n'+n'']} \in (\mathbb{Z}_{\geqslant 0})^{\times (n'+n'')}$ and for $f_i \in \mathcal{P}_0(l_i, r_i)$ for $i \in [1, n'+n'']$ we have

$$\Bigg(\mathop{\boxtimes_{\mathcal{P}_0}}_{i \in [1, n'+n'']} f_i\Bigg) \cdot_{\mathcal{P}_0} \Big(\big((a' \boxtimes a'')_{[r]}\big)^{\mathrm{op}} \mathfrak{p}_0\Big) \overset{!}{=} \Big(\big((a' \boxtimes a'')_{[l]}\big)^{\mathrm{op}} \mathfrak{p}_0\Big) \cdot_{\mathcal{P}_0} \Bigg(\mathop{\boxtimes_{\mathcal{P}_0}}_{j \in [1, m'+m'']} k_{j(a'\boxtimes a'')}\Bigg).$$

Define $f'_i := f_i$, $l'_i := l_i$ and $r'_i := r_i$ for $i \in [1,n']$ and $f''_i := f_{i+n'}$, $l''_i := l_{i+n'}$ and $r''_i := r_{i+n'}$ for $i \in [1,n'']$. So we have $f'_i \in \mathcal{P}_0(l'_i, r'_i)$ for $i \in [1,n']$ and $f''_i \in \mathcal{P}_0(l''_i, r''_i)$ for $i \in [1,n'']$. So we have

$$
\begin{aligned}
&\Big(\underset{i\in[1,n'+n'']}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \cdot_{\mathcal{P}_0} \big(((a' \boxtimes a'')_{[r]})^{\mathrm{op}}\mathfrak{p}_0\big)\\
&\overset{\text{Cl. 1.1}}{=} \bigg(\Big(\underset{i\in[1,n']}{\boxtimes_{\mathcal{P}_0}} f'_i\Big) \boxtimes_{\mathcal{P}_0} \Big(\underset{i\in[1,n'']}{\boxtimes_{\mathcal{P}_0}} f''_i\Big)\bigg) \cdot_{\mathcal{P}_0} \big((a'_{[r']} \boxtimes a''_{[r'']})^{\mathrm{op}}\mathfrak{p}_0\big)\\
&= \bigg(\Big(\underset{i\in[1,n']}{\boxtimes_{\mathcal{P}_0}} f'_i\Big) \boxtimes_{\mathcal{P}_0} \Big(\underset{i\in[1,n'']}{\boxtimes_{\mathcal{P}_0}} f''_i\Big)\bigg) \cdot_{\mathcal{P}_0} \big((a'_{[r']})^{\mathrm{op}}\mathfrak{p}_0 \boxtimes_{\mathcal{P}_0} (a''_{[r'']})^{\mathrm{op}}\mathfrak{p}_0\big)\\
&\overset{\text{(mc2)}}{=} \bigg(\Big(\underset{i\in[1,n']}{\boxtimes_{\mathcal{P}_0}} f'_i\Big) \cdot_{\mathcal{P}_0} \big((a'_{[r']})^{\mathrm{op}}\mathfrak{p}_0\big)\bigg) \boxtimes_{\mathcal{P}_0} \bigg(\Big(\underset{i\in[1,n'']}{\boxtimes_{\mathcal{P}_0}} f''_i\Big) \cdot_{\mathcal{P}_0} \big((a''_{[r'']})^{\mathrm{op}}\mathfrak{p}_0\big)\bigg)\\
&\overset{\mathrm{B}_0(a'),\,\mathrm{B}_0(a'')}{=} \bigg(\big((a'_{[l']})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{j\in[1,m']}{\boxtimes_{\mathcal{P}_0}} f'_{ja'}\Big)\bigg) \boxtimes_{\mathcal{P}_0} \bigg(\big((a''_{[l'']})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{j\in[1,m'']}{\boxtimes_{\mathcal{P}_0}} f''_{ja''}\Big)\bigg)\\
&\overset{\text{(mc2)}}{=} \big((a'_{[l']})^{\mathrm{op}}\mathfrak{p}_0 \boxtimes_{\mathcal{P}_0} (a''_{[l'']})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \bigg(\Big(\underset{j\in[1,m']}{\boxtimes_{\mathcal{P}_0}} f'_{ja'}\Big) \boxtimes_{\mathcal{P}_0} \Big(\underset{j\in[1,m'']}{\boxtimes_{\mathcal{P}_0}} f''_{ja''}\Big)\bigg)\\
&= \big((a'_{[l']} \boxtimes a''_{[l'']})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \bigg(\Big(\underset{j\in[1,m']}{\boxtimes_{\mathcal{P}_0}} f_{ja'}\Big) \boxtimes_{\mathcal{P}_0} \Big(\underset{j\in[1,m'']}{\boxtimes_{\mathcal{P}_0}} f_{ja''+n'}\Big)\bigg)\\
&\overset{\text{Cl. 1.1}}{=} \big((a' \boxtimes a'')^{\mathrm{op}}_{[l]}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{j\in[1,m'+m'']}{\boxtimes_{\mathcal{P}_0}} f_{j(a'\boxtimes a'')}\Big).
\end{aligned}
$$

This proves *Claim* 1.2.

Our next step will be to show that given $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Map}_0(m,n)$, $b \in \mathrm{Map}_0(n,p)$ such that $\mathrm{B}_0(a)$ and $\mathrm{B}_0(b)$ hold, then $\mathrm{B}_0(a \cdot b)$ holds.

Claim 2.1. Let $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Map}_0(m,n)$, $b \in \mathrm{Map}_0(n,p)$. For $k = (k_i)_{i\in[1,p]}$ with $k_i \in \mathbb{Z}_{\geqslant 0}$ for $i \in [1,p]$ we have

$$a_{[kb^*]}b_{[k]} = (ab)_{[k]}\,.$$

Proof of Claim 2.1. First note that $(kb^*)a^* = (k_{uab})_{u\in[1,p]} = k(ab)^*$. So we have

$$
\begin{array}{ccccc}
\bigsqcup\limits_{u\in[1,m]} [1, k_{u(ab)}] & \xrightarrow{\tilde{a}_{[kb^*]}} & \bigsqcup\limits_{j\in[1,n]} [1, k_{ja}] & \xrightarrow{\tilde{b}_{[k]}} & \bigsqcup\limits_{i\in[1,p]} [1, k_i]\\
& & \xrightarrow[\widetilde{(ab)}_{[k]}]{\hspace{6em}} & &
\end{array}
$$

which is a commutative diagram since for $(u,x) \in \bigsqcup\limits_{u\in[1,m]} [1, k_{u(ab)}]$ we have

$$(u,x)(\tilde{a}_{[kb^*]}\tilde{b}_{[k]}) = (ua, x)\tilde{b}_{[k]} = (u(ab), x) = (u,x)\widetilde{(ab)}_{[k]}\,.$$

Hence we have $\tilde{a}_{[kb^*]}\tilde{b}_{[k]} = \widetilde{(ab)}_{[k]}$.

This implies that

$$
\begin{aligned}
a_{[kb^*]}b_{[k]} &= (\varphi_{(kb^*)a^*} \cdot \tilde{a}_{[k]} \cdot \varphi^{-1}_{kb^*}) \cdot (\varphi_{kb^*} \cdot \tilde{b}_{[k]} \cdot \varphi^{-1}_k)\\
&= \varphi_{k(ab)^*} \cdot (\widetilde{ab})_{[k]} \cdot \varphi^{-1}_k\\
&= (ab)_{[k]}\,.
\end{aligned}
$$

This proves *Claim* 2.1.

Claim 2.2. Let $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Map}_0(m, n)$, $b \in \mathrm{Map}_0(n, p)$. If $\mathrm{B}_0(a)$ and $\mathrm{B}_0(b)$ are true then so is $\mathrm{B}_0(a \cdot b)$.

Proof of Claim 2.2. We have to show that for $l = (l_i)_{i \in [1,p]} \in (\mathbb{Z}_{\geqslant 0})^{\times p}$ and $r = (r_i)_{i \in [1,p]} \in (\mathbb{Z}_{\geqslant 0})^{\times p}$ and for $f_i \in \mathcal{P}_0(l_i, r_i)$ for $i \in [1, p]$ we have

$$\Big(\underset{i\in[1,p]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \cdot_{\mathcal{P}_0} \big(((ab)_{[r]})^{\mathrm{op}}\mathfrak{p}_0\big) \overset{!}{=} \big(((ab)_{[l]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot \Big(\underset{u\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_{u(ab)}\Big).$$

Note that $f_{jb} \in \mathcal{P}_0(l_{jb}, r_{jb})$ for $j \in [1, n]$ and $\mathrm{B}_0(a)$ imply that

$$\Big(\underset{j\in[1,n]}{\boxtimes_{\mathcal{P}_0}} f_{jb}\Big) \cdot_{\mathcal{P}_0} \big((a_{[rb*]})^{\mathrm{op}}\mathfrak{p}_0\big) = \big((a_{[lb*]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{u\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_{u(ab)}\Big).$$

So we have

$$\begin{aligned}
\Big(\underset{i\in[1,p]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \cdot_{\mathcal{P}_0} \big(((ab)_{[r]})^{\mathrm{op}}\mathfrak{p}_0\big) &\overset{\text{Cl. 2.1}}{=} \Big(\underset{i\in[1,p]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \cdot_{\mathcal{P}_0} \big((a_{[rb*]} \cdot b_{[r]})^{\mathrm{op}}\mathfrak{p}_0\big) \\
&= \Big(\underset{i\in[1,p]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \cdot_{\mathcal{P}_0} \big((b_{[r]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \big((a_{[rb*]})^{\mathrm{op}}\mathfrak{p}_0\big) \\
&\overset{\mathrm{B}_0(b)}{=} \big((b_{[l]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{j\in[1,n]}{\boxtimes_{\mathcal{P}_0}} f_{jb}\Big) \cdot_{\mathcal{P}_0} \big((a_{[rb*]})^{\mathrm{op}}\mathfrak{p}_0\big) \\
&\overset{\mathrm{B}_0(a)}{=} \big((b_{[l]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \big((a_{[lb*]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{u\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_{u(ab)}\Big) \\
&= \big((a_{[lb*]} \cdot b_{[l]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{u\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_{u(ab)}\Big) \\
&\overset{\text{Cl. 2.1}}{=} \big(((ab)_{[l]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{u\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_{u(ab)}\Big).
\end{aligned}$$

This proves *Claim* 2.2.

Claim 3.1. The assertion $\mathrm{B}_0(\mathrm{id}_m)$ is true for $m \in \mathbb{Z}_{\geqslant 0}$.

Proof of Claim 3.1. From Example 6.9 (i) we know that for $k = (k_i)_{i \in [1,m]} \in (\mathbb{Z}_{\geqslant 0})^{\times m}$ we have $(\mathrm{id}_m)_{[k]} = \mathrm{id}_{\Sigma k}$, where we abbreviate $\Sigma k := \Big(\sum_{i \in [1,m]} k_i\Big)$.

So for $l = (l_i)_{i \in [1,m]} \in (\mathbb{Z}_{\geqslant 0})^{\times m}$ and $r = (r_i)_{i \in [1,m]} \in (\mathbb{Z}_{\geqslant 0})^{\times m}$ and for $f_i \in \mathcal{P}_0(l_i, r_i)$ for $i \in [1, m]$ we have

$$\begin{aligned}
\Big(\underset{i\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \cdot_{\mathcal{P}_0} \big((\mathrm{id}_{[r]})^{\mathrm{op}}\mathfrak{p}_0\big) &= \Big(\underset{i\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \cdot_{\mathcal{P}_0} \big((\mathrm{id}_{\Sigma r})^{\mathrm{op}}\mathfrak{p}_0\big) \\
&= \Big(\underset{i\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \cdot_{\mathcal{P}_0} \big(\mathrm{id}_{\mathcal{P}_0, \Sigma r}\big) \\
&\overset{\text{(c2)}}{=} \Big(\underset{i\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \\
&\overset{\text{(c2)}}{=} \big(\mathrm{id}_{\mathcal{P}_0, \Sigma l}\big) \cdot_{\mathcal{P}_0} \Big(\underset{i\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_i\Big) \\
&= \big((\mathrm{id}_{\Sigma l})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{i\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_{i\,\mathrm{id}_m}\Big) \\
&= \big((\mathrm{id}_{[l]})^{\mathrm{op}}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \Big(\underset{i\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_{i\,\mathrm{id}_m}\Big).
\end{aligned}$$

This shows that $\mathrm{B}_0(\mathrm{id}_m)$ is true for $m \in \mathbb{Z}_{\geqslant 0}$ and proves *Claim* 3.1.

Claim 3.2. Let $(1,2) \in \mathrm{Sym}_0(2,2)$ be the transposition. The assertion $\mathrm{B}_0((1,2))$ is true if and only of $\mathfrak{p}_0$ satisfies condition (so1).

Proof of Claim 3.2. From Example 6.9 (ii) we know that $(1,2)_{[(k_1,k_2)]} = s_{k_2,k_1}$ for $k_1, k_2 \in \mathbb{Z}_{\geqslant 0}$.

So $\mathrm{B}_0((1,2))$ holds if and only if for $l_1, l_2, r_1, r_2 \in \mathbb{Z}_{\geqslant 0}$ and $f_1 \in \mathcal{P}_0(l_1, r_1)$ and $f_2 \in \mathcal{P}_0(l_2, r_2)$ we have

$$\Big(\underset{i\in[1,2]}{\boxtimes_{\mathcal{P}_0}} f_i \Big) \cdot_{\mathcal{P}_0} \big(((1,2)_{[(r_1,r_2)]})^{\mathrm{op}} \mathfrak{p}_0 \big) = \big(((1,2)_{[(l_1,l_2)]})^{\mathrm{op}} \mathfrak{p}_0 \big) \cdot_{\mathcal{P}_0} \Big(\underset{j\in[1,2]}{\boxtimes_{\mathcal{P}_0}} f_{j(1,2)} \Big),$$

i.e. if and only if

$$(f_1 \boxtimes_{\mathcal{P}_0} f_2) \cdot_{\mathcal{P}_0} (s^{\mathrm{op}}_{r_2,r_1} \mathfrak{p}_0) = (s^{\mathrm{op}}_{l_2,l_1} \mathfrak{p}_0) \cdot_{\mathcal{P}_0} (f_2 \boxtimes_{\mathcal{P}_0} f_1)$$

for $l_1, l_2, r_1, r_2 \in \mathbb{Z}_{\geqslant 0}$ and $f_1 \in \mathcal{P}_0(l_1, r_1)$ and $f_2 \in \mathcal{P}_0(l_2, r_2)$, which is equivalent to condition (so1).

This proves *Claim* 3.2.

Claim 3.3. The assertion $\mathrm{B}_0(\mu_m)$ is true for all $m \in \mathbb{Z}_{\geqslant 0}$ if and only if $\mathfrak{p}_0$ satisfies condition (so2).

Proof of Claim 3.3. From Example 6.9 (iii) we know that $(\mu_m)_{[(k_1)]} = h_{m,k_1}$ for $k_1 \in \mathbb{Z}_{\geqslant 0}$.

So for $m \in \mathbb{Z}_{\geqslant 0}$ the assertion $\mathrm{B}_0(\mu_m)$ is true if and only if for $l_1, r_1 \in \mathbb{Z}_{\geqslant 0}$ and $f_1 \in \mathcal{P}_0(l_1, r_1)$ we have

$$\Big(\underset{i\in[1,1]}{\boxtimes_{\mathcal{P}_0}} f_i \Big) \cdot_{\mathcal{P}_0} \big(((\mu_m)_{[(r_1)]})^{\mathrm{op}} \mathfrak{p}_0 \big) = \big(((\mu_m)_{[(l_1)]})^{\mathrm{op}} \mathfrak{p}_0 \big) \cdot_{\mathcal{P}_0} \Big(\underset{j\in[1,m]}{\boxtimes_{\mathcal{P}_0}} f_{j\mu_m} \Big),$$

i.e. if and only if

$$f_1 \cdot_{\mathcal{P}_0} (h^{\mathrm{op}}_{m,r_1} \mathfrak{p}_0) = (h^{\mathrm{op}}_{m,l_1} \mathfrak{p}_0) \cdot_{\mathcal{P}_0} f_1^{\boxtimes_{\mathcal{P}_0} m}$$

for $l_1, r_1 \in \mathbb{Z}_{\geqslant 0}$ and $f_1 \in \mathcal{P}_0(l_1, r_1)$, which is equivalent to condition (so2).

This proves *Claim* 3.3.

Now we can show (1) and (2).

Ad (1). First note that if $\mathrm{B}_0(a)$ is true for all $m, n \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \mathrm{Map}_0(m,n)$ such that a is bijective, then in particular $\mathrm{B}_0((1,2))$ is true for the transposition $(1,2) \in \mathrm{Map}_0(2,2)$. So by Claim 3.2, the condition (so1) is satisfied.

Now suppose that $\mathfrak{p}_0$ satisfies condition (so1) from Definition 6.3. Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Map}_0(m,n)$ such that a is bijective. So $m = n$.

The map a is the composite of elementary transpositions, so there exist $s \in \mathbb{Z}_{\geqslant 0}$, $i_1, \ldots, i_s \in [1, m-1]$ such that

$$\begin{aligned} a &= (i_1, i_1 + 1) \cdot (i_2, i_2 + 1) \cdots (i_s, i_s + 1) \\ &= (\mathrm{id}_{i_1-1} \boxtimes (1,2) \boxtimes \mathrm{id}_{m-i_1-1}) \cdot (\mathrm{id}_{i_2-1} \boxtimes (1,2) \boxtimes \mathrm{id}_{m-i_2-1}) \cdots (\mathrm{id}_{i_s-1} \boxtimes (1,2) \boxtimes \mathrm{id}_{m-i_s-1}). \end{aligned}$$

By Claim 3.1 and Claim 3.2 we know that $\mathrm{B}_0(\mathrm{id}_j)$ is true for $j \in \mathbb{Z}_{\geqslant 0}$ and that $\mathrm{B}_0((1,2))$ is true. By Claim 1.2, the assertion $\mathrm{B}_0(\mathrm{id}_{i_j-1} \boxtimes (1,2) \boxtimes \mathrm{id}_{m-i_j-1})$ is true for $j \in [1,s]$. By Claim 2.2, the assertion $\mathrm{B}_0(a)$ is true. This shows (1).

Ad (2). First note that if $\mathrm{B}_0(a)$ is true for all $m, n \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \mathrm{Map}_0(m,n)$, then in particular $\mathrm{B}_0((1,2))$ and $\mathrm{B}_0(\mu_m)$ for $m \in \mathbb{Z}_{\geqslant 0}$ are true. By Claim 3.2, the condition (so1) is satisfied since $\mathrm{B}_0((1,2))$ is true and by Claim 3.3, the condition (so2) is satisfied since $\mathrm{B}_0(\mu_m)$ is true for $m \in \mathbb{Z}_{\geqslant 0}$. So in particular $(\mathcal{P}, \mathfrak{p}_0)$ is a set-operad; cf. Definition 6.3.

Now suppose that $\mathfrak{p}_0$ satisfies the conditions (so1) and (so2) from Definition 6.3.

Reordering $[1, m]$, we see that there exist a monotone map $a_{\text{mon}} \in \text{Map}_0(m, n)$ and a bijective map $a_{\text{bij}} \in \text{Map}_0(m, m)$ such that $a_{\text{bij}} \cdot a_{\text{mon}} = a$.

There exist $j_1, \ldots, j_n \in [0, m]$ such that

$$a_{\text{mon}} = \mu_{j_1} \boxtimes \ldots \boxtimes \mu_{j_n} \, ;$$

cf. Remark 4.30. By Claim 3.3, the assertion $\text{B}_0(\mu_{j_i})$ is true for $i \in [1, n]$. So by Claim 1.2, the assertion $\text{B}_0(a_{\text{mon}})$ is true.

By (1) we know that $\text{B}_0(a_{\text{bij}})$ is true. Hence by Claim 2.2, $\text{B}_0(a)$ is also true. □

Remark 6.11. It would also be sufficient to ask $\text{B}_0(\mu_0) = \text{B}_0(\varepsilon)$ and $\text{B}_0(\mu_2) = \text{B}_0(\mu)$ to be true instead of $\text{B}_0(\mu_m)$ for all $m \in \mathbb{Z}_{\geqslant 0}$ since for $m \geqslant 3$ every μ_m can be written as a composite and product of $\mu_2 = \mu$ and id_1. Then by Claim 1.2 and Claim 2.2 from the proof of Lemma 6.10, $\text{B}_0(\mu_m)$ is also true.

Lemma 6.12. *Let $\mathcal{P}_0$ be a set-preoperad and let $\mathfrak{p}_0 : \text{Sym}_0^{\text{op}} \longrightarrow \mathcal{P}_0$ be a morphism of set-preoperads. Consider the following condition* (lo_0).

(lo_0) *We have $(s_{m,m'}^{\text{op}} \mathfrak{p}_0) \cdot (f \boxtimes f') = (f' \boxtimes f) \cdot (s_{n,n'}^{\text{op}} \mathfrak{p}_0) \in \mathcal{P}_0(m + m', n + n')$ for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}_0(m, n)$, $f' \in \mathcal{P}_0(m', n')$.*

Furthermore, for $m \in \mathbb{Z}_{\geqslant 0}$ and $a \in \text{Sym}_0(m, m)$ consider the following assertion $\widetilde{\text{B}}(a)$.

$\widetilde{\text{B}}(a)$*: For $l_i, r_i \in \mathbb{Z}_{\geqslant 0}$ and $f_i \in \mathcal{P}_0(l_i, r_i)$ for $i \in [1, m]$ and $l := (l_i)_{i \in [1,m]}$, $r := (r_i)_{i \in [1,m]}$ we have*

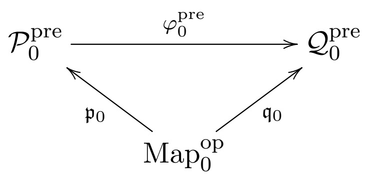
$$\left(\boxtimes_{i \in [1,m]} f_i \right) \cdot \left(a_{[r]}^{\text{op}} \mathfrak{p}_0 \right) = \left(a_{[l]}^{\text{op}} \mathfrak{p}_0 \right) \cdot \left(\boxtimes_{j \in [1,m]} f_{ja} \right) .$$

Then the morphism of set-preoperads $\mathfrak{p}_0$ satisfies (lo_0) *if and only if $\widetilde{\text{B}}(a)$ is true for all $m \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \text{Sym}_0(m, m)$.*

Proof. This holds since in the proof of Lemma 6.10 (1) the morphism of set-preoperads $\mathfrak{p}_0$ has only been applied to bijective maps. □

6.2 Morphisms and suboperads of set-operads

Definition 6.13. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\text{pre}}, \mathfrak{p}_0)$, $\mathcal{Q}_0 = (\mathcal{Q}_0^{\text{pre}}, \mathfrak{q}_0)$ be set-operads. A *morphism* $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ *of set-operads* is given by a morphism $\varphi_0^{\text{pre}} : \mathcal{P}_0^{\text{pre}} \longrightarrow \mathcal{Q}_0^{\text{pre}}$ of set-preoperads such that $\mathfrak{p}_0 \varphi_0^{\text{pre}} = \mathfrak{q}_0$.

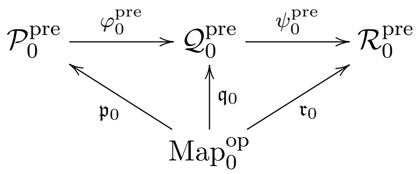


Whether we use φ_0 or φ_0^{pre} to denote it depends on whether we are in the context of set-operads or set-preoperads.

Note that the source of φ_0^{pre} is $\mathcal{P}_0^{\text{pre}}$, whereas the source of φ_0 is $\left(\text{Map}_0^{\text{op}} \xrightarrow{\mathfrak{p}_0} \mathcal{P}_0 \right)$. Similarly, the target of φ_0^{pre} is $\mathcal{Q}_0^{\text{pre}}$, whereas the target of φ_0 is $\left(\text{Map}_0^{\text{op}} \xrightarrow{\mathfrak{q}_0} \mathcal{Q}_0 \right)$. This prevents us from formally equating φ_0^{pre} and φ_0 .

Example 6.14. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad. The identity morphism $\mathrm{id}_{\mathcal{P}_0} : \mathcal{P}_0 \longrightarrow \mathcal{P}_0$ is given by $\mathrm{id}_{\mathcal{P}_0}^{\mathrm{pre}} = \mathrm{id}_{\mathcal{P}_0^{\mathrm{pre}}} : \mathcal{P}_0^{\mathrm{pre}} \longrightarrow \mathcal{P}_0^{\mathrm{pre}}$.

Definition 6.15. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$, $\mathcal{Q}_0 = (\mathcal{Q}_0^{\mathrm{pre}}, \mathfrak{q}_0)$ and $\mathcal{R}_0 = (\mathcal{R}_0^{\mathrm{pre}}, \mathfrak{r}_0)$ be set-operads. Let $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ and $\psi_0 : \mathcal{Q}_0 \longrightarrow \mathcal{R}_0$ be morphisms of set-operads, that is, we have the following commutative diagram.



The composition $\varphi_0\psi_0$ is given by the composition $\varphi_0^{\mathrm{pre}}\psi_0^{\mathrm{pre}}$ of morphisms of set-preoperads; cf. Definition 2.18 (2).

This defines a morphism $\varphi_0\psi_0$ of set-operads since $\mathfrak{p}_0(\varphi_0^{\mathrm{pre}}\psi_0^{\mathrm{pre}}) = (\mathfrak{p}_0\varphi_0^{\mathrm{pre}})\psi_0^{\mathrm{pre}} = \mathfrak{q}_0\psi_0^{\mathrm{pre}} = \mathfrak{r}_0$.

Definition 6.16. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad. A set-operad $\mathcal{Q}_0 = (\mathcal{Q}_0^{\mathrm{pre}}, \mathfrak{q}_0)$ is called a *set-suboperad* of $\mathcal{P}_0$ if $\mathcal{Q}_0^{\mathrm{pre}} \subseteq \mathcal{P}_0^{\mathrm{pre}}$ is a set-subpreoperad and if $\mathrm{Im}(\mathfrak{p}_0) \subseteq \mathcal{Q}_0^{\mathrm{pre}}$ and $\mathfrak{q}_0 = \mathfrak{p}_0\big|^{\mathcal{Q}_0^{\mathrm{pre}}}$.

Remark 6.17. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad. Suppose given a set-subpreoperad $\mathcal{T}_0 \subseteq \mathcal{P}_0^{\mathrm{pre}}$ such that $\mathrm{Im}(\mathfrak{p}_0) \subseteq \mathcal{T}_0$. Then $(\mathcal{T}_0, \mathfrak{p}_0\big|^{\mathcal{T}_0})$ is a set-suboperad of $\mathcal{P}_0$.

Definition 6.18. Let I be a set. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad. Let $\mathcal{Q}_{0,i} = (\mathcal{Q}_{0,i}^{\mathrm{pre}}, \mathfrak{p}_0\big|^{\mathcal{Q}_{0,i}}) \subseteq \mathcal{P}_0$ be a set-suboperad for $i \in I$. Then $\bigcap_{i\in I} \mathcal{Q}_{0,i} := \Big(\bigcap_{i\in I} \mathcal{Q}_{0,i}^{\mathrm{pre}}, \mathfrak{p}_0\big|^{\bigcap_{i\in I} \mathcal{Q}_{0,i}^{\mathrm{pre}}}\Big)$ is a set-suboperad of $\mathcal{P}_0$ since $\bigcap_{i\in I} \mathcal{Q}_{0,i}^{\mathrm{pre}} \subseteq \mathcal{P}_0^{\mathrm{pre}}$ is a set-subpreoperad by Lemma 2.31 and since $\mathrm{Im}(\mathfrak{p}_0) \subseteq \mathcal{Q}_{0,i}^{\mathrm{pre}}$ for $i \in I$ implies $\mathrm{Im}(\mathfrak{p}_0) \subseteq \Big(\bigcap_{i\in I} \mathcal{Q}_{0,i}\Big)^{\mathrm{pre}} = \bigcap_{i\in I} \mathcal{Q}_{0,i}^{\mathrm{pre}}$.

Definition 6.19. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad. Let $X \subseteq \mathcal{P}_0^{\mathrm{pre}}$ be a biindexed subset. We define the set-suboperad of $\mathcal{P}_0$ *generated by* X by

$$_{\mathrm{op}}\langle\, X \,\rangle := \bigcap \{\mathcal{Q}_0 : \mathcal{Q}_0 \text{ is a set-suboperad of } \mathcal{P}_0 \text{ with } X \subseteq \mathcal{Q}_0^{\mathrm{pre}}\}.$$

Lemma 6.20. *Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad and let $X \subseteq \mathcal{P}_0^{\mathrm{pre}}$ be a biindexed subset. Then we have*

$$_{\mathrm{preop}}\langle\, X \cup \mathrm{Im}(\mathfrak{p}_0) \,\rangle = \Big(_{\mathrm{op}}\langle\, X \,\rangle\Big)^{\mathrm{pre}}.$$

Equivalently, writing this as an equation of set-operads, we have

$$\Big(_{\mathrm{preop}}\langle\, X \cup \mathrm{Im}(\mathfrak{p}_0) \,\rangle\,,\ \mathfrak{p}_0\big|^{_{\mathrm{preop}}\langle\, X\cup \mathrm{Im}(\mathfrak{p}_0) \,\rangle}\Big) = {}_{\mathrm{op}}\langle\, X \,\rangle\,.$$

Proof. Note that for a set-subpreoperad $\mathcal{T}_0 \subseteq \mathcal{P}_0^{\mathrm{pre}}$ we have that $(\mathcal{T}_0, \mathfrak{p}_0\big|^{\mathcal{T}_0})$ is a set-suboperad of $\mathcal{P}_0$ if and only if $\mathrm{Im}(\mathfrak{p}_0) \subseteq \mathcal{T}_0$. So we have

$$\begin{aligned}
_{\mathrm{preop}}\langle\, X \cup \mathrm{Im}(\mathfrak{p}_0) \,\rangle &= \bigcap \{\mathcal{T}_0 : \mathcal{T}_0 \text{ is a set-subpreoperad of } \mathcal{P}_0^{\mathrm{pre}} \text{ with } X \cup \mathrm{Im}(\mathfrak{p}_0) \subseteq \mathcal{T}_0\} \\
&= \bigcap \{\mathcal{T}_0 : \mathcal{T}_0 \text{ is a set-subpreoperad of } \mathcal{P}_0^{\mathrm{pre}} \text{ with } X \subseteq \mathcal{T}_0 \text{ and } \mathrm{Im}(\mathfrak{p}_0) \subseteq \mathcal{T}_0\} \\
&= \bigcap \{\mathcal{Q}_0^{\mathrm{pre}} : (\mathcal{Q}_0^{\mathrm{pre}}, \mathfrak{p}_0\big|^{\mathcal{Q}_0^{\mathrm{pre}}}) \text{ is a set-suboperad of } \mathcal{P}_0 \text{ with } X \subseteq \mathcal{Q}_0^{\mathrm{pre}}\} \\
&= \Big(\bigcap \{\mathcal{Q}_0 : \mathcal{Q}_0 \text{ is a set-suboperad of } \mathcal{P}_0 \text{ with } X \subseteq \mathcal{Q}_0^{\mathrm{pre}}\}\Big)^{\mathrm{pre}} \\
&= \Big(_{\mathrm{op}}\langle\, X \,\rangle\Big)^{\mathrm{pre}}.
\end{aligned}$$

□

6.3 Algebras over set-operads

Definition 6.21. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\text{pre}}, \mathfrak{p}_0)$ be a set-operad. A $\mathcal{P}_0$-algebra (X, ϱ_0) is a set X together with a morphism $\varrho_0 : \mathcal{P}_0 \longrightarrow \text{END}_0(X)$ of set-operads.

$$\begin{array}{ccc} \mathcal{P}_0^{\text{pre}} & \xrightarrow{\varrho_0^{\text{pre}}} & \text{END}_0(X)^{\text{pre}} \\ {\scriptstyle \mathfrak{p}_0} \uparrow & \nearrow {\scriptstyle \mathfrak{e}_0} & \\ \text{Map}_0^{\text{op}} & & \end{array}$$

6.4 Linear operads

Definition 6.22. A *linear operad* $(\mathcal{P}, \mathfrak{p})$ *over* R is given by a linear preoperad $(\mathcal{P}_0, \boxtimes, \cdot)$ over R and a morphism of linear preoperads $\mathfrak{p} : \text{Sym}^{\text{op}} \longrightarrow \mathcal{P}$ such that (lo) holds.

(lo) We have $(s^{\text{op}}_{m,m'}\mathfrak{p}) \cdot (f \boxtimes f') = (f' \boxtimes f) \cdot (s^{\text{op}}_{n,n'}\mathfrak{p}) \in \mathcal{P}(m+m', n+n')$ for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathcal{P}(m,n)$, $f' \in \mathcal{P}(m', n')$.

For brevity, we refer to the linear operad $(\mathcal{P}, \mathfrak{p})$ simply by $\mathcal{P}$. We then denote by $\mathcal{P}^{\text{pre}}$ the underlying linear preoperad of $\mathcal{P}$.

So we have $\mathcal{P} = (\mathcal{P}^{\text{pre}}, \mathfrak{p})$ for the morphism of linear preoperads $\mathfrak{p} : \text{Sym}^{\text{op}} \longrightarrow \mathcal{P}$ belonging to $\mathcal{P}$.

Whenever necessary, the multiplication in $\mathcal{P}^{\text{pre}}$ is written $(\boxtimes_{\mathcal{P}^{\text{pre}}})$ or simply $(\boxtimes_{\mathcal{P}})$ and the composition is written $(\cdot_{\mathcal{P}^{\text{pre}}})$ or simply $(\cdot_{\mathcal{P}})$. Moreover, we usually denote the identity elements of $\mathcal{P}^{\text{pre}}$ by id_m or $\text{id}_{\mathcal{P},m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

Example 6.23. Let V be an R-module. Recall the morphism $\mathfrak{e} : \text{Sym}^{\text{op}} \longrightarrow \text{End}(V)$ of linear preoperads that maps an element $a^{\text{op}} \in \text{Sym}_0^{\text{op}}(m,m)$ to $a^{\text{op}}\mathfrak{e} \in \text{End}(V)(m,m)$ defined by

$$(v_1 \otimes \ldots \otimes v_m)(a^{\text{op}}\mathfrak{e}) = v_{1a} \otimes \ldots \otimes v_{ma}$$

for $v_1, \ldots, v_m \in V$; cf. Definition 2.64.

Then $\text{END}(V) := (\text{End}(V), \mathfrak{e})$ is a linear operad.

In order to show that this is true first note that for $m, m' \in \mathbb{Z}_{\geqslant 0}$ and $v_1, \ldots, v_{m+m'} \in V$ we have

$$(v_1 \otimes \ldots \otimes v_{m+m'})\,(s^{\text{op}}_{m,m'}\mathfrak{e}) = v_{1s_{m,m'}} \otimes \ldots \otimes v_{(m+m')s_{m,m'}} = v_{m'+1} \otimes \ldots \otimes v_{m'+m} \otimes v_1 \otimes \ldots \otimes v_{m'} \, .$$

Now let $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \text{End}(V)(m,n)$, $f' \in \text{End}_0(V)(m', n')$. Then for $v_1, \ldots, v_{m+m'} \in V$ we have

$$\begin{aligned}
(v_1 \otimes \ldots \otimes v_{m+m'}) \Big((s^{\text{op}}_{m,m'}\mathfrak{e}) \cdot \big(f \otimes f'\big) \Big) &= \Big((v_1 \otimes \ldots \otimes v_m \otimes v_{m+1} \otimes \ldots \otimes v_{m+m'})\,(s^{\text{op}}_{m,m'}\mathfrak{e}) \Big) \big(f \otimes f'\big) \\
&= (v_{m'+1} \otimes \ldots \otimes v_{m'+m} \otimes v_1 \otimes \ldots \otimes v_{m'}) \big(f \otimes f'\big) \\
&= (v_{m'+1} \otimes \ldots \otimes v_{m'+m})\, f \otimes (v_1 \otimes \ldots \otimes v_{m'})\, f' \\
&= \Big((v_1 \otimes \ldots \otimes v_{m'})\, f' \otimes (v_{m'+1} \otimes \ldots \otimes v_{m'+m})\, f \Big)\, s^{\text{op}}_{n,n'}\mathfrak{e} \\
&= \Big((v_1 \otimes \ldots \otimes v_{m'} \otimes v_{m'+1} \otimes \ldots \otimes v_{m'+m}) \big(f' \otimes f\big) \Big)\,(s^{\text{op}}_{n,n'}\mathfrak{e}) \\
&= (v_1 \otimes \ldots \otimes v_{m'+m}) \Big(\big(f' \otimes f\big) \cdot (s^{\text{op}}_{n,n'}\mathfrak{e}) \Big) \, .
\end{aligned}$$

Hence we have $(s^{\text{op}}_{m,m'}\mathfrak{e}) \cdot (f \otimes f') = (f' \otimes f) \cdot (s^{\text{op}}_{n,n'}\mathfrak{e})$.

This shows that $\text{END}(V) = (\text{End}(V), \mathfrak{e})$ is a linear operad. We have $\text{END}(V)^{\text{pre}} = \text{End}(V)$.

Example 6.24. We have the linear operad $\mathrm{SYM}^{\mathrm{op}} := (\mathrm{Sym}^{\mathrm{op}}, \mathrm{id}_{\mathrm{Sym}^{\mathrm{op}}})$ where

$$\mathrm{id}_{\mathrm{Sym}^{\mathrm{op}}} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{Sym}^{\mathrm{op}}$$

is the identity morphism.

For $m, n \in \mathbb{Z}_{\geqslant 0}$, a map $a \in \mathrm{Map}_0(m,n)$ and a tuple $k = (k_i)_{i\in[1,n]}$ with $k_i \in \mathbb{Z}_{\geqslant 0}$ for $i \in [1,n]$ recall the map

$$a_{[k]} : \Big[1, \sum_{j\in[1,m]} k_{ja}\Big] \longrightarrow \Big[1, \sum_{i\in[1,n]} k_i\Big];$$

cf. Definition 6.8.

Similar to the characterization of set-operads given in Lemma 6.10 we are now going to state an equivalent characterization of linear operads.

Lemma 6.25. *Let $(\mathcal{P}, \boxtimes, \cdot)$ be a linear preoperad over R and let $\mathfrak{p} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathcal{P}$ be a morphism of linear preoperads over R. For $m \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Sym}_0(m,m)$ consider the following assertion $\mathrm{B}(a)$.*

$\mathrm{B}(a)$*: For $l_i, r_i \in \mathbb{Z}_{\geqslant 0}$ and $f_i \in \mathcal{P}(l_i, r_i)$ for $i \in [1,m]$ and for $l = (l_i)_{i\in[1,m]}$ and $r = (r_i)_{i\in[1,m]}$ we have*

$$\Big(\boxtimes_{i\in[1,m]} f_i\Big) \cdot \big(a^{\mathrm{op}}_{[r]}\mathfrak{p}\big) = \big(a^{\mathrm{op}}_{[l]}\mathfrak{p}\big) \cdot \Big(\boxtimes_{j\in[1,m]} f_{ja}\Big).$$

Then $(\mathcal{P}, \mathfrak{p})$ is a linear operad if and only if $\mathrm{B}(a)$ holds for all for $m \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Sym}_0(m,m)$.

Proof. First recall that we can view $\mathcal{P}$ as a set-preoperad and $\mathfrak{p}|_{\mathrm{Sym}_0^{\mathrm{op}}} : \mathrm{Sym}_0^{\mathrm{op}} \longrightarrow \mathcal{P}$ as a morphism of set-preoperads; cf. Remarks 2.10 and 2.17. Since for $m \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)$ we have $f^{\mathrm{op}}\mathfrak{p} = f^{\mathrm{op}}\mathfrak{p}|_{\mathrm{Sym}_0^{\mathrm{op}}}$, the morphism of set-preoperads $\mathfrak{p}|_{\mathrm{Sym}_0^{\mathrm{op}}}$ satisfies the condition (lo_0) from Lemma 6.12 if and only if the morphism of linear preoperads $\mathfrak{p} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathcal{P}$ satisfies the condition (lo).

Hence by Lemma 6.12, the morphism of linear preoperads $\mathfrak{p} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathcal{P}$ satisfies the condition (lo) if and only if for all $m \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \mathrm{Sym}_0(m,m)$ and for $l_i, r_i \in \mathbb{Z}_{\geqslant 0}$ and $f_i \in \mathcal{P}(l_i, r_i)$ for $i \in [1,m]$ and for $l = (l_i)_{i\in[1,m]}$ and $r = (r_i)_{i\in[1,m]}$ we have

$$\Big(\boxtimes_{i\in[1,m]} f_i\Big) \cdot \big(a^{\mathrm{op}}_{[r]}\mathfrak{p}|_{\mathrm{Sym}_0^{\mathrm{op}}}\big) = \big(a^{\mathrm{op}}_{[l]}\mathfrak{p}|_{\mathrm{Sym}_0^{\mathrm{op}}}\big) \cdot \Big(\boxtimes_{j\in[1,m]} f_{ja}\Big),$$

i.e. if and only if we have

$$\Big(\boxtimes_{i\in[1,m]} f_i\Big) \cdot \big(a^{\mathrm{op}}_{[r]}\mathfrak{p}\big) = \big(a^{\mathrm{op}}_{[l]}\mathfrak{p}\big) \cdot \Big(\boxtimes_{j\in[1,m]} f_{ja}\Big)$$

for all $m \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \mathrm{Sym}_0(m,m)$ for $l_i, r_i \in \mathbb{Z}_{\geqslant 0}$ and $f_i \in \mathcal{P}(l_i, r_i)$ for $i \in [1,m]$ and for $l = (l_i)_{i\in[1,m]}$ and $r = (r_i)_{i\in[1,m]}$, i.e. if and only if the assertion $\mathrm{B}(a)$ is true for all $m \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \mathrm{Sym}_0(m,m)$. □

Remark 6.26. *Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad. Let $\mathcal{Q}_0 \subseteq \mathcal{P}_0^{\mathrm{pre}}$ be a set-subpreoperad such that $\mathrm{Im}(\mathfrak{p}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}) \subseteq \mathcal{Q}_0$, that is, for $m \in \mathbb{Z}_{\geqslant 0}$ we have*

$$\{a^{\mathrm{op}}\mathfrak{p}_0 : a^{\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)\} \subseteq \mathcal{Q}_0(m,m).$$

Then $\Big(R\mathcal{Q}_0, R\big(\mathfrak{p}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}^{\mathcal{Q}_0}\big)\Big)$ is a linear operad.

Proof. First note that $\mathrm{Im}(\mathfrak{p}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}) \subseteq \mathcal{Q}_0$ ensures that we can define the morphism of linear preoperads $R\big(\mathfrak{p}_0|^{\mathcal{Q}_0}_{\mathrm{Sym}_0^{\mathrm{op}}}\big) : \mathrm{Sym}^{\mathrm{op}} \longrightarrow R\mathcal{Q}_0$.

We have to show that $\Big(R\mathcal{Q}_0, R\big(\mathfrak{p}_0|^{\mathcal{Q}_0}_{\mathrm{Sym}_0^{\mathrm{op}}}\big)\Big)$ satisfies condition (lo).

Let $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $\xi := \sum_{f \in \mathcal{Q}_0(m,n)} r_f\, f \in R\mathcal{Q}_0(m,n)$, $\xi' := \sum_{f' \in \mathcal{Q}_0(m',n')} r'_{f'}\, f' \in R\mathcal{Q}_0(m',n')$. Since $\mathcal{P}_0$ is a set-operad and since $\mathcal{Q}_0 \subseteq \mathcal{P}_0^{\mathrm{pre}}$ is a set-subpreoperad, we have, using the definition of $R\big(\mathfrak{p}_0|^{\mathcal{Q}_0}_{\mathrm{Sym}_0^{\mathrm{op}}}\big)$, of $(\boxtimes_{R\mathcal{Q}_0})$ and $(\cdot_{R\mathcal{Q}_0})$,

$$\begin{aligned}
\Big(s^{\mathrm{op}}_{m,m'}\big(R\big(\mathfrak{p}_0|^{\mathcal{Q}_0}_{\mathrm{Sym}_0^{\mathrm{op}}}\big)\big)\Big) \cdot_{R\mathcal{Q}_0} \big(\xi \boxtimes_{R\mathcal{Q}_0} \xi'\big) &= \big(s^{\mathrm{op}}_{m,m'}\mathfrak{p}_0\big) \cdot_{R\mathcal{Q}_0} \Bigg(\sum_{\substack{f \in \mathcal{Q}_0(m,n)\\ f' \in \mathcal{Q}_0(m',n')}} r_f\, r'_{f'} \big(f \boxtimes_{\mathcal{Q}_0} f'\big)\Bigg)\\
&= \sum_{\substack{f \in \mathcal{Q}_0(m,n)\\ f' \in \mathcal{Q}_0(m',n')}} r_f\, r'_{f'} \Big(\big(s^{\mathrm{op}}_{m,m'}\mathfrak{p}_0\big) \cdot_{\mathcal{Q}_0} \big(f \boxtimes_{\mathcal{Q}_0} f'\big)\Big)\\
&= \sum_{\substack{f \in \mathcal{Q}_0(m,n)\\ f' \in \mathcal{Q}_0(m',n')}} r_f\, r'_{f'} \Big(\big(s^{\mathrm{op}}_{m,m'}\mathfrak{p}_0\big) \cdot_{\mathcal{P}_0} \big(f \boxtimes_{\mathcal{P}_0} f'\big)\Big)\\
&= \sum_{\substack{f \in \mathcal{Q}_0(m,n)\\ f' \in \mathcal{Q}_0(m',n')}} r_f\, r'_{f'} \Big(\big(f' \boxtimes_{\mathcal{P}_0} f\big) \cdot_{\mathcal{P}_0} \big(s^{\mathrm{op}}_{n,n'}\mathfrak{p}_0\big)\Big)\\
&= \sum_{\substack{f \in \mathcal{Q}_0(m,n)\\ f' \in \mathcal{Q}_0(m',n')}} r_f\, r'_{f'} \Big(\big(f' \boxtimes_{\mathcal{Q}_0} f\big) \cdot_{\mathcal{Q}_0} \big(s^{\mathrm{op}}_{n,n'}\mathfrak{p}_0\big)\Big)\\
&= \Bigg(\sum_{\substack{f \in \mathcal{Q}_0(m,n)\\ f' \in \mathcal{Q}_0(m',n')}} r_f\, r'_{f'} \big(f' \boxtimes_{\mathcal{Q}_0} f\big)\Bigg) \cdot_{R\mathcal{Q}_0} \big(s^{\mathrm{op}}_{n,n'}\mathfrak{p}_0\big)\\
&= \big(\xi' \boxtimes_{R\mathcal{Q}_0} \xi\big) \cdot_{R\mathcal{Q}_0} \Big(s^{\mathrm{op}}_{n,n'}\big(R\big(\mathfrak{p}_0|^{\mathcal{Q}_0}_{\mathrm{Sym}_0^{\mathrm{op}}}\big)\big)\Big).
\end{aligned}$$

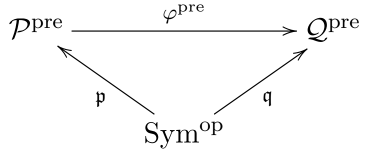
□

Remark 6.27. Let $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ be a set-operad. Then $\mathcal{P}_0^{\mathrm{pre}}$ is a set-subpreoperad of $\mathcal{P}_0^{\mathrm{pre}}$ with $\mathrm{Im}(\mathfrak{p}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}) \subseteq \mathcal{P}_0^{\mathrm{pre}}$. So by Remark 6.26 we have the linear operad

$$\Big(R\mathcal{P}_0^{\mathrm{pre}}, R\big(\mathfrak{p}_0|^{\mathcal{P}_0^{\mathrm{pre}}}_{\mathrm{Sym}_0^{\mathrm{op}}}\big)\Big) = \Big(R\mathcal{P}_0^{\mathrm{pre}}, R\big(\mathfrak{p}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}\big)\Big)\,.$$

6.5 Morphisms and suboperads of linear operads

Definition 6.28. Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$, $\mathcal{Q} = (\mathcal{Q}^{\mathrm{pre}}, \mathfrak{q})$ be linear operads. A *morphism* $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ *of linear operads* is given by a morphism $\varphi^{\mathrm{pre}} : \mathcal{P}^{\mathrm{pre}} \longrightarrow \mathcal{Q}^{\mathrm{pre}}$ of linear preoperads such that $\mathfrak{p}\varphi^{\mathrm{pre}} = \mathfrak{q}$.

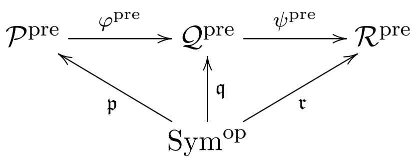


Whether we use φ or φ^{pre} to denote it depends on whether we are in the context of linear operads or linear preoperads.

Note that the source of φ^{pre} is $\mathcal{P}^{\mathrm{pre}}$, whereas the source of φ is $\big(\mathrm{Sym}^{\mathrm{op}} \xrightarrow{\mathfrak{p}} \mathcal{P}\big)$. Similarly, the target of φ^{pre} is $\mathcal{Q}^{\mathrm{pre}}$, whereas the target of φ is $\big(\mathrm{Sym}^{\mathrm{op}} \xrightarrow{\mathfrak{q}} \mathcal{Q}\big)$. This prevents us from formally equating φ^{pre} and φ .

Example 6.29. Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ be a linear operad. The identity morphism $\mathrm{id}_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P}$ is given by $\mathrm{id}_{\mathcal{P}}^{\mathrm{pre}} = \mathrm{id}_{\mathcal{P}^{\mathrm{pre}}} : \mathcal{P}^{\mathrm{pre}} \longrightarrow \mathcal{P}^{\mathrm{pre}}$.

Definition 6.30. Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$, $\mathcal{Q} = (\mathcal{Q}^{\mathrm{pre}}, \mathfrak{q})$ and $\mathcal{R} = (\mathcal{R}^{\mathrm{pre}}, \mathfrak{r})$ be linear operads. Let $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ and $\psi : \mathcal{Q} \longrightarrow \mathcal{R}$ be morphisms of linear operads, that is, we have the following commutative diagram.



The composition $\varphi\psi$ is given by the composition $\varphi^{\mathrm{pre}}\psi^{\mathrm{pre}}$ of morphisms of linear preoperads; cf. Definition 2.18 (2).

This defines a morphism $\varphi\psi$ of linear operads since $\mathfrak{p}(\varphi^{\mathrm{pre}}\psi^{\mathrm{pre}}) = (\mathfrak{p}\varphi^{\mathrm{pre}})\psi^{\mathrm{pre}} = \mathfrak{q}\psi^{\mathrm{pre}} = \mathfrak{r}$.

Definition 6.31. Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ be a linear operad. A linear operad $\mathcal{Q} = (\mathcal{Q}^{\mathrm{pre}}, \mathfrak{q})$ is called a *linear suboperad* of $\mathcal{P}$ if $\mathcal{Q}^{\mathrm{pre}} \subseteq \mathcal{P}^{\mathrm{pre}}$ is a linear subpreoperad and if $\mathrm{Im}(\mathfrak{p}) \subseteq \mathcal{Q}^{\mathrm{pre}}$ and $\mathfrak{q} = \mathfrak{p}\big|^{\mathcal{Q}^{\mathrm{pre}}}$.

Remark 6.32. Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ be a linear operad. Suppose given a linear subpreoperad $\mathcal{T} \subseteq \mathcal{P}^{\mathrm{pre}}$ such that $\mathrm{Im}(\mathfrak{p}) \subseteq \mathcal{T}$. Then $(\mathcal{T}, \mathfrak{p}\big|^{\mathcal{T}})$ is a linear suboperad of $\mathcal{P}$.

Definition 6.33. Let I be a set. Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ be a linear operad. Let $\mathcal{Q}_i = (\mathcal{Q}_i^{\mathrm{pre}}, \mathfrak{p}\big|^{\mathcal{Q}_i}) \subseteq \mathcal{P}$ be a linear suboperad for $i \in I$. Then $\bigcap_{i\in I} \mathcal{Q}_i := \Big(\bigcap_{i\in I} \mathcal{Q}_i^{\mathrm{pre}}, \mathfrak{p}\big|^{\bigcap_{i\in I} \mathcal{Q}_i^{\mathrm{pre}}}\Big)$ is a linear suboperad of $\mathcal{P}$.

Definition 6.34. Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ be a linear operad. Let $X \subseteq \mathcal{P}^{\mathrm{pre}}$ be a biindexed subset. We define the linear suboperad of $\mathcal{P}$ *generated by* X by

$$_{\mathrm{op}}\langle\, X \,\rangle := \bigcap \{\mathcal{Q} : \mathcal{Q} \text{ is a linear suboperad of } \mathcal{P} \text{ with } X \subseteq \mathcal{Q}^{\mathrm{pre}}\}\,.$$

Lemma 6.35. *Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ be a linear operad and let $X \subseteq \mathcal{P}^{\mathrm{pre}}$ be a biindexed subset. Then we have*

$$_{\mathrm{preop}}\langle\, X \cup \mathrm{Im}(\mathfrak{p}) \,\rangle = \Big(_{\mathrm{op}}\langle\, X \,\rangle\Big)^{\mathrm{pre}}\,.$$

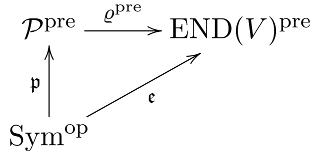
Equivalently, writing this as an equation of linear operads, we have

$$\Big(_{\mathrm{preop}}\langle\, X \cup \mathrm{Im}(\mathfrak{p}) \,\rangle\,,\ \mathfrak{p}\big|^{_{\mathrm{preop}}\langle\, X \cup \mathrm{Im}(\mathfrak{p}) \,\rangle}\Big) = {}_{\mathrm{op}}\langle\, X \,\rangle\,.$$

Proof. This can be proven in the same way as the analogous assertion for set-operads in Lemma 6.20. □

6.6 Algebras over linear operads

Definition 6.36. Let $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ be a linear operad. A $\mathcal{P}$-algebra (V, ϱ) is an R-module V together with a morphism $\varrho : \mathcal{P} \longrightarrow \mathrm{END}(V)$ of linear operads.



7 The set-operad ASS_0 and the linear operad ASS

7.1 Sorted pullbacks

In order to define the set-operad ASS_0 we first will have to define the sorted pullback of maps.

Recall that a map is called *isotone* if it is monotone and bijective.

Furthermore, note that given a commutative diagram

$$\begin{CD} [1,s] @>{\hat{a}}>> [1,l] \\ @V{\hat{g}}VV @VV{g}V \\ [1,k] @>>{a}> [1,n] \end{CD}$$

of maps, where $s, k, l, n \in \mathbb{Z}_{\geqslant 0}$, then for $i \in [1,l]$ we have that $((\hat{a}^{-1}(i))\hat{g} \subseteq a^{-1}(ig)$ since for $j \in \hat{a}^{-1}(i)$ we have $(j\hat{g})a = j(\hat{a}g) = (j\hat{a})g = ig$.

Definition 7.1. A commutative diagram

$$\begin{CD} [1,s] @>{\hat{a}}>> [1,l] \\ @V{\hat{g}}VV @VV{g}V \\ [1,k] @>>{a}> [1,n] \end{CD}$$

of maps, where $s, k, l, n \in \mathbb{Z}_{\geqslant 0}$, is called a *sorted pullback* if a and $\hat{a}$ are monotone and if $\hat{g}|^{a^{-1}(ig)}_{\hat{a}^{-1}(i)}$ is isotone for $i \in [1,l]$.

We will indicate this by writing

$$\begin{CD} [1,s] @>{\hat{a}}>> [1,l] \\ @V{\hat{g}}VV @VV{g}V \\ [1,k] @>>{a}> [1,n] \end{CD}$$

and we will also often say that the tuple $([1,s], \hat{g}, \hat{a})$ is a sorted pullback of a and g.

Lemma 7.2. *Let $k, l, n \in \mathbb{Z}_{\geqslant 0}$ and let $a : [1,k] \longrightarrow [1,n]$ be a monotone map and $g : [1,l] \longrightarrow [1,n]$ be a map. Consider the standard pullback $(S, \check{g}, \check{a})$ of a and g; cf. Lemma 1.29.*

$$\begin{CD} S @>{\check{a}}>> [1,l] \\ @V{\check{g}}VV @VV{g}V \\ [1,k] @>>{a}> [1,n] \end{CD}$$

So we have $S = \{(i,j) \in [1,k] \times [1,l] : ia = jg\}$ and

$$\begin{array}{rrcl} \check{g}: & S & \longrightarrow & [1,k] \\ & (i,j) & \longmapsto & i \end{array} \qquad \begin{array}{rrcl} \check{a}: & S & \longrightarrow & [1,l] \\ & (i,j) & \longmapsto & j. \end{array}$$

Then (1) and (2) hold.

(1) *There exists exactly one linear order on S such that $\check{a}$ is monotone and $\check{g}|^{a^{-1}(jg)}_{\check{a}^{-1}(j)}$ is monotone for $j \in [1,l]$. This order is the colexicographic order, i.e. for $(i,j), (i',j') \in S$ we have $(i,j) \leqslant (i',j')$ if and only if $j < j'$ or ($j = j'$ and $i \leqslant i'$).*

(2) *The map* $\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)}$ *is isotone for* $j \in [1, l]$ *with respect to the linear order from (1).*

Proof. *Ad* (1). *Existence.* We endow S with the colexicographic order.

Then $(i, j) \leqslant (i', j')$ implies $(i, j)\check{a} = j \leqslant j' = (i', j')\check{a}$, hence $\check{a}$ is monotone.

Now suppose given $j \in [1, l]$.

Then for $(i, j), (i', j) \in \check{a}^{-1}(j)$ with $(i, j) \leqslant (i', j)$ we have $i \leqslant i'$. Hence we have

$$(i, j)\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)} = (i, j)\check{g} = i \leqslant i' = (i', j)\check{g} = (i', j)\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)} .$$

So $\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)}$ is monotone.

Uniqueness. Now assume that $(\preceq)$ is a linear order on S such that $\check{a}$ is monotone and $\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)}$ is monotone for $j \in [1, l]$.

We have to show that for $(i, j), (i', j') \in S$ we have that $(i, j) \preceq (i', j')$ implies $(i, j) \leqslant (i', j)$. Then, since $(\leqslant)$ and $(\preceq)$ are both linear orders, they have to be the same.

So suppose given $(i, j), (i', j') \in S$ with $(i, j) \preceq (i', j')$. Then, since $\check{a}$ is monotone with respect to $(\preceq)$, we have $j = (i, j)\check{a} \leqslant (i', j')\check{a} = j'$.

Now if $j = j'$, then we have $(i, j), (i', j) \in \check{a}^{-1}(j)$. So since $j \in [1, l]$ and since $\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)}$ is monotone with respect to $(\preceq)$, we have $i = (i, j)\check{g} = (i, j)\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)} \leqslant (i', j)\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)} = (i', j)\check{g} = i'$.

So $(i, j) \preceq (i', j')$ implies $j \leqslant j'$, and if $j = j'$ then it implies $i \leqslant i'$, hence $(i, j) \leqslant (i', j')$.

Ad (2). Suppose given $j \in [1, l]$. By Lemma 1.32, the map $\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)}$ is bijective since $(S, \check{g}, \check{a})$ is a pullback of a and g. By (1), it is a monotone map. So $\check{g}|_{\check{a}^{-1}(j)}^{a^{-1}(jg)}$ is isotone. □

Remark 7.3. *A sorted pullback as defined in Definition* 7.1 *is in particular a pullback of sets; cf. Definition* 1.26.

Proof. Suppose given $k, l, n \in \mathbb{Z}_{\geqslant 0}$ and maps $a : [1, k] \longrightarrow [1, n]$ and $g : [1, l] \longrightarrow [1, n]$. Let $([1, s], \hat{g}, \hat{a})$ be a sorted pullback of a and g, cf. Definition 7.1. So in particular $\hat{g}|_{\hat{a}^{-1}(i)}^{a^{-1}(ig)}$ is bijective for $i \in [1, l]$. By Lemma 1.32, this implies that $([1, s], \hat{g}, \hat{a})$ is a pullback of a and g. □

Lemma 7.4. *Let* $k, l, n \in \mathbb{Z}_{\geqslant 0}$. *Let* $a : [1, k] \longrightarrow [1, n]$ *be a monotone map. Let* $g : [1, l] \longrightarrow [1, n]$ *be a map. There exists a uniquely determined* $s \in \mathbb{Z}_{\geqslant 0}$ *and uniquely determined maps* $\hat{a} : [1, s] \longrightarrow [1, l]$ *and* $\hat{g} : [1, s] \longrightarrow [1, k]$ *such that*

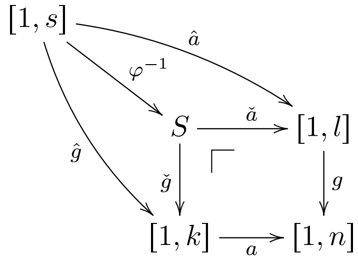
$$\begin{array}{ccc} [1, s] & \xrightarrow{\hat{a}} & [1, l] \\ {\scriptstyle \hat{g}}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ [1, k] & \xrightarrow[a]{} & [1, n] . \end{array}$$

Proof. *Existence.* Recall the standard pullback $(S, \check{g}, \check{a})$ of a and g; cf. Definition 1.29. In Lemma 7.2 we showed that $\check{a}$ is monotone and $\check{g}|_{a^{-\check{1}}(j)}^{a^{-1}(jg)}$ is isotone for $j \in [1, l]$ with respect to the colexicographic order $(\leqslant)$ on S. The map

$$\begin{aligned} \varphi : S &\longrightarrow [1, |S|] \\ x &\longmapsto |\{y \in S : y \leqslant x\}| \end{aligned}$$

is strictly monotone, since for $x_1, s_2 \in S$ with $x_1 < x_2$ we have $\{y \in S : y \leqslant x_1\} \subset \{y \in S : y \leqslant x_2\}$, hence $x_1\varphi < x_2\varphi$. Since $|S| = |[1, |S|]|$, the map φ is isotone. This implies that $\varphi^{-1} : [1, |S|] \longrightarrow S$ is also isotone.

Now define $s := |S|$ and $\hat{a} := \varphi^{-1}\check{a}$ and $\hat{g} := \varphi^{-1}\check{g}$.



Then we have $\hat{a}g = \varphi^{-1}\check{a}g = \varphi^{-1}\check{g}a = \hat{g}a$ and $\hat{a}$ is monotone as the composite of monotone maps.

Now suppose given $j \in [1, l]$. Suppose given $i, i' \in \hat{a}^{-1}(j)$ with $i < i'$. Then we have $\left(i\varphi^{-1}\right)\check{a} = i\hat{a} = j = i'\hat{a} = \left(i'\varphi^{-1}\right)\check{a}$, hence $i\varphi^{-1}, i'\varphi^{-1} \in \check{a}^{-1}(j)$. Moreover, since φ^{-1} is isotone, we have $i\varphi^{-1} < i'\varphi^{-1}$.

Now by construction we have

$$i\hat{g}\big|_{\hat{a}^{-1}(j)}^{a^{-1}(jg)} = i\hat{g} = (i\varphi^{-1})\check{g} = (i\varphi^{-1})\check{g}\big|_{\check{a}^{-1}(j)}^{a^{-1}(jg)} < (i'\varphi^{-1})\check{g}\big|_{\check{a}^{-1}(j)}^{a^{-1}(jg)} = (i'\varphi^{-1})\check{g} = i'\hat{g} = i'\hat{g}\big|_{\hat{a}^{-1}(j)}^{a^{-1}(jg)},$$

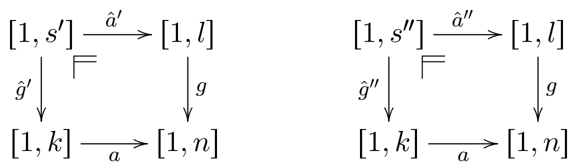
hence $\hat{g}\big|_{\hat{a}^{-1}(j)}^{a^{-1}(jg)}$ is strictly monotone.

So what remains to show is that $\hat{g}\big|_{\hat{a}^{-1}(j)}^{a^{-1}(jg)}$ is surjective.

Suppose given $i \in a^{-1}(jg)$. We have to show that there exists $m \in \hat{a}^{-1}(j)$ such that $m\hat{g}\big|_{\hat{a}^{-1}(j)}^{a^{-1}(jg)} = i$. But by Lemma 7.2 (2) there exists $x \in S$ with $x \in \check{a}^{-1}(j)$ such that $x\check{g}\big|_{\check{a}^{-1}(j)}^{a^{-1}(jg)} = x\check{g} = i$. By defining $m := x\varphi \in [1, s]$, we get $m\hat{a} = (x\varphi)\varphi^{-1}\check{a} = x\check{a} = j$, hence we have $m \in \hat{a}^{-1}(j)$ and $m\hat{g}\big|_{\hat{a}^{-1}(j)}^{a^{-1}(jg)} = m\hat{g} = (x\varphi)\varphi^{-1}\check{g} = x\check{g} = i$. This shows that $\hat{g}\big|_{\hat{a}^{-1}(j)}^{a^{-1}(jg)}$ is surjective.

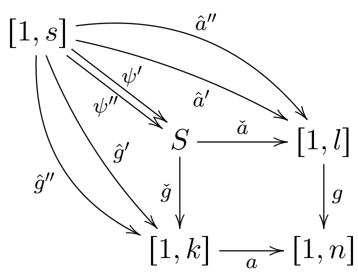
Hence $\hat{g}\big|_{\hat{a}^{-1}(j)}^{a^{-1}(jg)}$ is isotone for $j \in [1, l]$.

Uniqueness. Suppose given the following sorted pullbacks.



Then, since $(S, \check{g}, \check{a})$ as well as $([1, s'], \hat{g}', \hat{a}')$ and $([1, s''], \hat{g}'', \hat{a}'')$ are pullbacks of a and g, there exist uniquely determined bijective maps $\psi' : [1, s'] \longrightarrow S$ and $\psi'' : [1, s''] \longrightarrow S$ such that $\psi'\check{a} = \hat{a}'$, $\psi'\check{g} = \hat{g}'$, $\psi''\check{a} = \hat{a}''$ and $\psi''\check{g} = \hat{g}''$; cf. Remark 7.3 and Remark 1.27. In particular, $s' = s'' = s := |S|$.

So we have the following diagram.



Note that since for $x = (x_1, x_2) \in S$ we have $x\check{a} = x_2$ and $x\check{g} = x_1$, for $u \in [1,s]$ we have $u\psi' = (u\hat{g}', u\hat{a}')$ and $u\psi'' = (u\hat{g}'', u\hat{a}'')$.

Suppose given $u, v \in [1,s]$ with $u \leqslant v$. Since $\hat{a}'$ is monotone we have $u\hat{a}' \leqslant v\hat{a}'$.

Case 1. If $u\hat{a}' < v\hat{a}'$ then we have $u\psi' = (u\hat{g}', u\hat{a}') < (v\hat{g}', v\hat{a}') = v\psi'$.

Case 2. If $u\hat{a}' = v\hat{a}' =: j$ then we have $j \in [1,l]$ and $u, v \in \hat{a}'^{-1}(j)$ with $u \leqslant v$. Since $\hat{g}'|_{\hat{a}'^{-1}(j)}^{a^{-1}(jg)}$ is isotone, hence in particular monotone, we have $u\hat{g}' = u\hat{g}'|_{\hat{a}'^{-1}(j)}^{a^{-1}(jg)} \leqslant v\hat{g}'|_{\hat{a}'^{-1}(j)}^{a^{-1}(jg)} = v\hat{g}'$, hence $u\psi' = (u\hat{g}', u\hat{a}') \leqslant (v\hat{g}', v\hat{a}') = v\psi'$.

This shows that $\psi' : [1,s] \longrightarrow S$ is monotone. Since ψ' is bijective, it is an isotone map. In the same way we see that $\psi'' : [1,s] \longrightarrow S$ is isotone. So we have two isotone maps $\psi', \psi'' : [1,s] \longrightarrow S$, so they have to be the same.

Hence we have $\hat{a}'' = \psi''\check{a} = \psi'\check{a} = \hat{a}'$ and $\hat{g}'' = \psi''\check{g} = \psi'\check{g} = \hat{g}'$.

This shows that s, $\hat{a}$ and $\hat{g}$ are uniquely determined. □

Now we will need some properties for the sorted pullback.

Lemma 7.5. *Suppose given*

$$\begin{array}{ccccc} [1,t] & \xrightarrow{\hat{b}} & [1,s] & \xrightarrow{\hat{a}} & [1,l] \\ \downarrow{\scriptstyle \hat{\hat{g}}} & & \downarrow{\scriptstyle \hat{g}} & & \downarrow{\scriptstyle g} \\ [1,m] & \xrightarrow[b]{} & [1,k] & \xrightarrow[a]{} & [1,n]. \end{array}$$

Then the quadrangle $([1,t],[1,m],[1,l],[1,n])$ *is also a sorted pullback.*

Proof. We need to verify that $[1,t]$, $\hat{b}\hat{a}$ and $\hat{\hat{g}}$ satisfy the following conditions.

- We have $\hat{b}\hat{a}g = \hat{\hat{g}}ba$.
- The map $\hat{b}\hat{a}$ is monotone.
- For $j \in [1,l]$ the map $\hat{\hat{g}}|_{(\hat{b}\hat{a})^{-1}(j)}^{(ba)^{-1}(jg)}$ is isotone.

We have that $([1,s], \hat{a}, \hat{g})$ is a sorted pullback of a and g and that $([1,t], \hat{b}, \hat{\hat{g}})$ is a sorted pullback of b and $\hat{g}$. By Remark 7.3 this implies that $([1,s], \hat{a}, \hat{g})$ is a pullback of a and g and that $([1,t], \hat{b}, \hat{\hat{g}})$ is a pullback of b and $\hat{g}$.

By Lemma 1.37, $([1,t], \hat{b}\hat{a}, \hat{\hat{g}})$ is a pullback of ba and g.

So from Lemma 1.32, we know that $[1,t]$, $\hat{b}\hat{a}$ and $\hat{\hat{g}}$ satisfy the following conditions.

- We have $\hat{b}\hat{a}g = \hat{\hat{g}}ba$.
- For $j \in [1,l]$ the map $\hat{\hat{g}}|_{(\hat{b}\hat{a})^{-1}(j)}^{(ba)^{-1}(jg)}$ is bijective.

So it remains to show that $\hat{b}\hat{a}$ is monotone and that $\hat{\hat{g}}|_{(\hat{b}\hat{a})^{-1}(j)}^{(ba)^{-1}(jg)}$ is monotone for $j \in [1,l]$.

First note that $\hat{b}\hat{a}$ is monotone as the composite of monotone maps.

Now suppose given $j \in [1,l]$.

Suppose given $i, i' \in (\hat{b}\hat{a})^{-1}(j)$, that is, $i\hat{b}\hat{a} = i'\hat{b}\hat{a} = j$, and suppose $i \leqslant i'$. We have to show that

$$i\hat{\hat{g}}\big|^{(ba)^{-1}(jg)}_{(\hat{b}\hat{a})^{-1}(j)} = i\hat{\hat{g}} \overset{!}{\leqslant} i'\hat{\hat{g}} = i'\hat{\hat{g}}\big|^{(ba)^{-1}(jg)}_{(\hat{b}\hat{a})^{-1}(j)} .$$

Since $\hat{b}$ is monotone, we have two possible cases: either $i\hat{b} = i'\hat{b}$ or $i\hat{b} < i'\hat{b}$.

Case $i\hat{b} = i'\hat{b} =: r \in [1, s]$. Then we have $i, i' \in \hat{b}^{-1}(r)$. Since $([1, t], \hat{\hat{g}}, \hat{b})$ is a sorted pullback of b and $\hat{g}$, we know that $\hat{\hat{g}}\big|^{b^{-1}(r\hat{g})}_{\hat{b}^{-1}(r)}$ is isotone. This implies

$$i\hat{\hat{g}} = i\hat{\hat{g}}\big|^{b^{-1}(r\hat{g})}_{\hat{b}^{-1}(r)} \leqslant i'\hat{\hat{g}}\big|^{b^{-1}(r\hat{g})}_{\hat{b}^{-1}(r)} = i'\hat{\hat{g}}.$$

Case $i\hat{b} < i'\hat{b}$. Still we have $(i\hat{b})\hat{a} = (i'\hat{b})\hat{a} = j$, hence $i\hat{b}, i'\hat{b} \in \hat{a}^{-1}(j)$. Since $([1, s], \hat{g}, \hat{a})$ is a sorted pullback of a and g, we know that $\hat{g}\big|^{a^{-1}(jg)}_{\hat{a}^{-1}(j)}$ is isotone. This implies that

$$(i\hat{\hat{g}})b = (i\hat{b})\hat{g} = (i\hat{b})\hat{g}\big|^{a^{-1}(jg)}_{\hat{a}^{-1}(j)} < (i'\hat{b})\hat{g}\big|^{a^{-1}(jg)}_{\hat{a}^{-1}(j)} = (i'\hat{b})\hat{g} = (i'\hat{\hat{g}})b.$$

Now since b is monotone we have to have $i\hat{\hat{g}} < i'\hat{\hat{g}}$.

Hence $\hat{\hat{g}}\big|^{(ba)^{-1}(jg)}_{(\hat{b}\hat{a})^{-1}(j)}$ is monotone. □

Lemma 7.6. *Suppose given*

$$\begin{array}{ccc}
[1,t] & \xrightarrow{\hat{\hat{a}}} & [1,m] \\
{\scriptstyle \hat{h}}\downarrow & & \downarrow{\scriptstyle h} \\
[1,s] & \xrightarrow{\hat{a}} & [1,l] \\
{\scriptstyle \hat{g}}\downarrow & & \downarrow{\scriptstyle g} \\
[1,k] & \xrightarrow[a]{} & [1,n] .
\end{array}$$

Then the quadrangle $([1, t], [1, k], [1, m], [1, n])$ *is also a sorted pullback.*

Proof. We need to verify that $[1, t]$, $\hat{\hat{a}}$ and $\hat{h}\hat{g}$ satisfy the following conditions.

- We have $\hat{\hat{a}}hg = \hat{h}\hat{g}a$.
- $\hat{\hat{a}}$ is a monotone map.
- $(\hat{h}\hat{g})\big|^{a^{-1}(jhg)}_{\hat{\hat{a}}^{-1}(j)}$ is isotone for $j \in [1, m]$.

First note that we have $\hat{\hat{a}}hg = \hat{h}\hat{a}g = \hat{h}\hat{g}a$, since $([1, t], \hat{h}, \hat{\hat{a}})$ is a sorted pullback of $\hat{a}$ and h and since $([1, s], \hat{g}, \hat{a})$ is a sorted pullback of g and a.

Furthermore, $\hat{\hat{a}}$ is monotone since $([1, t], \hat{h}, \hat{\hat{a}})$ is a sorted pullback of $\hat{a}$ and h.

Finally, for $j \in [1, m]$ we have

$$(\hat{h}\hat{g})\big|^{a^{-1}(jhg)}_{\hat{\hat{a}}^{-1}(j)} = \left(\hat{h}\big|^{\hat{a}^{-1}(jh)}_{\hat{\hat{a}}^{-1}(j)}\right)\left(\hat{g}\big|^{a^{-1}((jh)g)}_{\hat{a}^{-1}(jh)}\right).$$

Since $([1, t], \hat{h}, \hat{\hat{a}})$ is a sorted pullback of $\hat{a}$ and h, we know that $\hat{h}\big|^{\hat{a}^{-1}(jh)}_{\hat{\hat{a}}^{-1}(j)}$ is isotone. Since $([1, s], \hat{g}, \hat{a})$ is a sorted pullback of g and a and since $jh \in [1, l]$, we know that $\hat{g}\big|^{a^{-1}((jh)g)}_{\hat{a}^{-1}(jh)}$ is isotone.

So $(\hat{h}\hat{g})\big|^{a^{-1}(jhg)}_{\hat{\hat{a}}^{-1}(j)}$ is isotone as the composite of isotone maps. □

Finally, we will see that multiplying sorted pullbacks via $(\boxtimes_{\mathrm{Map}_0})$ yields a sorted pullback.

Lemma 7.7. *Suppose given sorted pullbacks*

$$\begin{array}{ccc} [1,s] & \xrightarrow{\hat a} & [1,l] \\ {\scriptstyle \hat g}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ [1,k] & \xrightarrow[a]{} & [1,n] \end{array} \quad \text{and} \quad \begin{array}{ccc} [1,s'] & \xrightarrow{\hat a'} & [1,l'] \\ {\scriptstyle \hat g'}\downarrow & \ulcorner & \downarrow{\scriptstyle g'} \\ [1,k'] & \xrightarrow[a']{} & [1,n'] \end{array}.$$

Then we have the sorted pullback

$$\begin{array}{ccc} [1,s+s'] & \xrightarrow{\hat a \boxtimes \hat a'} & [1,l+l'] \\ {\scriptstyle \hat g \boxtimes \hat g'}\downarrow & \ulcorner & \downarrow{\scriptstyle g \boxtimes g'} \\ [1,k+k'] & \xrightarrow[a \boxtimes a']{} & [1,n+n'] \end{array},$$

where $(\boxtimes) := (\boxtimes_{\mathrm{Map}_0})$*; cf. Definition* 2.57.

So informally, stacking two sorted pullbacks yields a sorted pullback.

Proof. We have to verify that $[1, s+s']$, $\hat g \boxtimes \hat g'$ and $\hat a \boxtimes \hat a'$ satisfy the following conditions.

- We have $(\hat a \boxtimes \hat a')(g \boxtimes g') = (\hat g \boxtimes \hat g')(a \boxtimes a')$.
- The map $\hat a \boxtimes \hat a'$ is monotone.
- For $j \in [1, l+l']$ the map $(\hat g \boxtimes \hat g')|^{(a\boxtimes a')^{-1}(j(g\boxtimes g'))}_{(\hat a \boxtimes \hat a')^{-1}(j)}$ is isotone.

Since Map_0 is a set-preoperad; cf. Definition 2.57 and since $([1,s], \hat g, \hat a)$ is a sorted pullback of a and g and since $([1,s'], \hat g', \hat a')$ is a sorted pullback of a' and g', we have

$$\left(\hat a \boxtimes \hat a'\right) \cdot \left(g \boxtimes g'\right) \overset{\text{(mc2)}}{=} (\hat a \cdot g) \boxtimes \left(\hat a' \cdot g'\right) = (\hat g \cdot a) \boxtimes \left(\hat g' \cdot a'\right) \overset{\text{(mc2)}}{=} \left(\hat g \boxtimes \hat g'\right) \cdot \left(\hat a \boxtimes \hat a'\right).$$

Furthermore, since Ass_0 is a set-subpreoperad of Map_0; cf. Definition 2.58, the maps $a \boxtimes a'$ and $\hat a \boxtimes \hat a'$ are monotone as the product of monotone maps.

Suppose given $j \in [1, l+l']$. Consider $(\hat g \boxtimes \hat g')|^{(a\boxtimes a')^{-1}(j(g\boxtimes g'))}_{(\hat a \boxtimes \hat a')^{-1}(j)}$.

Strict monotonicity. Suppose given $i, i' \in (\hat a \boxtimes \hat a')^{-1}(j)$, that is, $i(\hat a \boxtimes \hat a') = j = i'(\hat a \boxtimes \hat a')$, and suppose $i < i'$. By the definition of $(\boxtimes)$, either $i, i' \in [1, s]$ or $i, i' \in [s+1, s+s']$; cf. Definition 2.57.

Case 1: $i, i' \in [1, s]$. Then $i\hat a = i(\hat a \boxtimes \hat a') = j = i'(\hat a \boxtimes \hat a') = i'\hat a$, so $i, i' \in \hat a^{-1}(j)$. Since $\hat g|^{a^{-1}(jg)}_{\hat a^{-1}(j)}$ is isotone, we have $i(\hat g \boxtimes \hat g') = i\hat g = i\hat g|^{a^{-1}(jg)}_{\hat a^{-1}(j)} < i'\hat g|^{a^{-1}(jg)}_{\hat a^{-1}(j)} = i'\hat g = i'(\hat g \boxtimes \hat g')$.

Case 2: $i, i' \in [s+1, s+s']$. Then $(i-s)\hat a' + l = i(\hat a \boxtimes \hat a') = j = i'(\hat a \boxtimes \hat a') = (i'-s)\hat a' + l$, so $i-s, i'-s \in (\hat a')^{-1}(j-l)$ and $j - l \in [1, l']$. Since $(\hat g')|^{(a')^{-1}((j-l)g')}_{(\hat a')^{-1}(j-l)}$ is isotone, we have

$$\begin{aligned} i(\hat g \boxtimes \hat g') &= (i-s)\hat g' + k \\ &= (i-s)(\hat g')|^{(a')^{-1}((j-l)g')}_{(\hat a')^{-1}(j-l)} + k \\ &< (i'-s)(\hat g')|^{(a')^{-1}((j-l)g')}_{(\hat a')^{-1}(j-l)} + k \\ &= (i'-s)\hat g' + k \\ &= i'(\hat g \boxtimes \hat g'). \end{aligned}$$

So $(\hat{g} \boxtimes \hat{g}')\big|^{(a \boxtimes a')^{-1}(j(g \boxtimes g'))}_{(\hat{a} \boxtimes \hat{a}')^{-1}(j)}$ is strictly monotone for $j \in [1, l + l']$.

Surjectivity. Suppose given $x \in (a \boxtimes a')^{-1}(j(g \boxtimes g'))$, that is, $x(a \boxtimes a') = j(g \boxtimes g')$. Again we only have the possibilities ($x \in [1,k]$ and $j \in [1,l]$) or ($x \in [k+1, k+k']$ and $j \in [l+1, l+l']$); cf. Definition 2.57.

Case 1: $x \in [1,k]$ and $j \in [1,l]$. Then we have $xa = x(a \boxtimes a') = j(g \boxtimes g') = jg \in [1,n]$, hence $x \in a^{-1}(jg)$. Since $\hat{g}\big|^{a^{-1}(jg)}_{\hat{a}^{-1}(j)}$ is isotone, there exists $y \in \hat{a}^{-1}(j) \subseteq [1,s]$ such that $y\hat{g} = y\hat{g}\big|^{a^{-1}(jg)}_{\hat{a}^{-1}(j)} = x$. Moreover, since $y \in [1,s]$ we have $y(\hat{a} \boxtimes \hat{a}') = y\hat{a} = j$, hence $y \in (\hat{a} \boxtimes \hat{a}')^{-1}(j)$. So we have $y(\hat{g} \boxtimes \hat{g}')\big|^{(a \boxtimes a')^{-1}(j(g \boxtimes g'))}_{(\hat{a} \boxtimes \hat{a}')^{-1}(j)} = y(\hat{g} \boxtimes \hat{g}') = y\hat{g} = x$.

Case 2: $x \in [k+1, k+k']$ and $j \in [l+1, l+l']$. Then we have $x - k \in (a')^{-1}((j-l)g')$ since $(x-k)a' + n = x(a \boxtimes a') = j(g \boxtimes g') = (j-l)g' + n$. Since $j - l \in [1, l']$, we know that $(\hat{g}')\big|^{(a')^{-1}((j-l)g')}_{(\hat{a}')^{-1}(j-l)}$ is isotone. So there exists $y \in (\hat{a}')^{-1}(j-l) \subseteq [1, s']$ such that $y\hat{g}' = y(\hat{g}')\big|^{(a')^{-1}((j-l)g')}_{(\hat{a}')^{-1}(j-l)} = x - k$. So $y\hat{a} = j - l$.

Now let $z := y + s \in [s+1, s+s']$. Then $z(\hat{a} \boxtimes \hat{a}') = (z-s)\hat{a}' + l = y\hat{a}' + l = (j-l) + l = j \in [l+1, l+l']$, hence $z \in (\hat{a} \boxtimes \hat{a}')^{-1}(j)$. So we have

$$z(\hat{g} \boxtimes \hat{g}')\big|^{(a \boxtimes a')^{-1}(j(g \boxtimes g'))}_{(\hat{a} \boxtimes \hat{a}')^{-1}(j)} = z(\hat{g} \boxtimes \hat{g}') = (z-s)\hat{g}' + k = y\hat{g}' + k = (x-k) + k = x.$$

This shows that $(\hat{g} \boxtimes \hat{g}')\big|^{(a \boxtimes a')^{-1}(j(g \boxtimes g'))}_{(\hat{a} \boxtimes \hat{a}')^{-1}(j)}$ is surjective.

Altogether, $(\hat{g} \boxtimes \hat{g}')\big|^{(a \boxtimes a')^{-1}(j(g \boxtimes g'))}_{(\hat{a} \boxtimes \hat{a}')^{-1}(j)}$ is isotone. □

We will now consider a special case of a sorted pullback.

Lemma 7.8. *Let $k, l, n \in \mathbb{Z}_{\geqslant 0}$. Let $g : [1,l] \longrightarrow [1,n]$ be a map and $a : [1,k] \longrightarrow [1,n]$ be a monotone map. We have the sorted pullbacks*

$$\text{(i)} \quad \begin{array}{ccc} [1,k] & \xrightarrow{a} & [1,n] \\ {\scriptstyle \mathrm{id}_k}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{id}_n} \\ [1,k] & \xrightarrow[a]{} & [1,n] \end{array} \qquad \text{and (ii)} \quad \begin{array}{ccc} [1,l] & \xrightarrow{\mathrm{id}_l} & [1,l] \\ {\scriptstyle g}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ [1,n] & \xrightarrow[\mathrm{id}_n]{} & [1,n] \end{array},$$

where we abbreviate $\mathrm{id}_k := \mathrm{id}_{\mathrm{Map}_0, k}$, $\mathrm{id}_n := \mathrm{id}_{\mathrm{Map}_0, n}$ and $\mathrm{id}_l := \mathrm{id}_{\mathrm{Map}_0, l}$.

Proof. Ad (i). The map a is monotone. The map $\mathrm{id}_k\big|^{a^{-1}(j\,\mathrm{id}_n)}_{a^{-1}(j)} = \mathrm{id}_{a^{-1}(j)}$ is isotone for $j \in [1,n]$.

Ad (ii). The map id_l is monotone. The map $g\big|^{\mathrm{id}_n^{-1}(ig)}_{\mathrm{id}_l^{-1}(i)} = g\big|^{\{ig\}}_{\{i\}}$ is isotone for $i \in [1,l]$. □

Lemma 7.9. *Let $k, n, t \in \mathbb{Z}_{\geqslant 0}$. Let $a : [1,k] \longrightarrow [1,n]$ be a monotone map and let $g : [1,t] \longrightarrow [1,n]$ be a map. We can write*

$$a = a_1 \boxtimes_{\mathrm{Ass}_0} \ldots \boxtimes_{\mathrm{Ass}_0} a_n =: \mathop{\boxtimes}_{i \in [1,n]} a_i ,$$

where we omit the index "Ass_0" and where $a_i := \mu_{l_i} \in \mathrm{Ass}_0(l_i, 1)$ with $l_i := |a^{-1}(i)| \in \mathbb{Z}_{\geqslant 0}$ is the unique monotone map $[1, l_i] \longrightarrow [1,1]$ for $i \in [1,n]$; cf. Definition 4.29 *and Remark* 4.30*. We have $\sum_{i \in [1,n]} l_i = k$. Define $l := (l_i)_{i \in [1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$.*

Then we have the following sorted pullback.

$$\begin{array}{ccc}
\Big[1, \sum\limits_{j\in[1,t]} l_{jg}\Big] & \xrightarrow{\ \boxtimes_{j\in[1,t]} a_{jg}\ } & [1,t] \\
{\scriptstyle g_{[l]}}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\
[1,k] & \xrightarrow[\ a\ ]{} & [1,n],
\end{array}$$

where $g_{[l]}$ *is defined as in Definition* 6.8.

Proof. We have to verify that $\Big[1, \sum\limits_{j\in[1,t]} l_{jg}\Big]$, $g_{[l]}$ and $\boxtimes_{j\in[1,t]} a_{jg}$ satisfy the following conditions (1), (2) and (3).

(1) We have $\Big(\boxtimes_{j\in[1,t]} a_{jg}\Big) g = g_{[l]}\, a$.

(2) The map $\boxtimes_{j\in[1,t]} a_{jg}$ is monotone.

(3) The map $g_{[l]}\big|^{a^{-1}(xg)}_{\left(\boxtimes_{j\in[1,t]} a_{jg}\right)^{-1}(x)}$ is isotone for $x \in [1,t]$.

During this proof we will write $(\boxtimes)$ and $(\cdot)$ for multiplication and composition in Map_0 and $(\boxtimes_{\mathrm{op}})$ and $(\cdot_{\mathrm{op}})$ for multiplication and composition in $\mathrm{Map}_0^{\mathrm{op}}$. Recall that Ass_0 is a set-subpreoperad of Map_0; cf. Definition 2.58.

Ad (1). Define $\tilde{r} = (\tilde{r}_i)_{i\in[1,n]} := (l_i)_{i\in[1,n]} = l \in (\mathbb{Z}_{\geqslant 0})^{\times n}$ and $\tilde{l} = (1,\ldots,1) \in (\mathbb{Z}_{\geqslant 0})^{\times n}$. So we have $\tilde{l}_i = 1$ and $a_i^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(\tilde{l}_i, \tilde{r}_i)$ for $i \in [1,n]$. Since $(\mathrm{Map}_0^{\mathrm{op}}, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}})$ is a set-operad, by Lemma 6.10 we have

$$\begin{aligned}
\big(g_{[l]} \cdot a\big)^{\mathrm{op}} &= \Big(g_{[l]} \cdot \Big(\boxtimes_{i\in[1,n]} a_i\Big)\Big)^{\mathrm{op}} \\
&= \Big(\boxtimes_{i\in[1,n]} a_i\Big)^{\mathrm{op}} \cdot_{\mathrm{op}} \big(g_{[l]}\big)^{\mathrm{op}} \\
&= \Big(\boxtimes_{\mathrm{op}\,i\in[1,n]} a_i^{\mathrm{op}}\Big) \cdot_{\mathrm{op}} \Big(\big(g_{[\tilde{r}]}\big)^{\mathrm{op}}\, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}\Big) \\
&\overset{\mathrm{B}_0(g)}{=} \Big(\big(g_{[\tilde{l}]}\big)^{\mathrm{op}}\, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}}\Big) \cdot_{\mathrm{op}} \Big(\boxtimes_{\mathrm{op}\,j\in[1,t]} a_{jg}^{\mathrm{op}}\Big) \\
&= \big(g_{[\tilde{l}]}\big)^{\mathrm{op}} \cdot_{\mathrm{op}} \Big(\boxtimes_{j\in[1,t]} a_{jg}\Big)^{\mathrm{op}} \\
&= \Big(\Big(\boxtimes_{j\in[1,t]} a_{jg}\Big) \cdot g_{[\tilde{l}]}\Big)^{\mathrm{op}} \\
&\overset{6.9\ \mathrm{(iv)}}{=} \Big(\Big(\boxtimes_{j\in[1,t]} a_{jg}\Big) \cdot g\Big)^{\mathrm{op}}.
\end{aligned}$$

Hence we have $g_{[l]} \cdot a = \Big(\boxtimes_{j\in[1,t]} a_{jg}\Big) \cdot g$, which shows (1).

Ad (2). The map $\boxtimes_{j\in[1,t]} a_{jg}$ is monotone since a_{jg} is a monotone map for $j \in [1,m]$.

Ad (3). Suppose given $x \in [1,t]$. Consider $g_{[l]}|^{a^{-1}(xg)}_{\left(\underset{j\in[1,t]}{\boxtimes} a_{jg}\right)^{-1}(x)}$.

First note that we can write

$$\begin{aligned}\Big(\underset{j\in[1,t]}{\boxtimes} a_{jg}\Big)^{-1}(x) &= \Big[\Big(\sum_{j\in[1,x-1]} l_{jg}\Big) + 1\,,\ \sum_{j\in[1,x]} l_{jg}\Big] \\ a^{-1}(xg) = \Big(\underset{i\in[1,n]}{\boxtimes} a_i\Big)^{-1}(xg) &= \Big[\Big(\sum_{i\in[1,xg-1]} l_i\Big) + 1\,,\ \sum_{i\in[1,xg]} l_i\Big].\end{aligned}$$

Since $|a^{-1}(xg)| = l_{xg} = \Big|\Big(\underset{j\in[1,t]}{\boxtimes} a_{jg}\Big)^{-1}(x)\Big|$, it suffices to show strict monotonicity.

Given $u \in \Big(\underset{j\in[1,t]}{\boxtimes} a_{jg}\Big)^{-1}(x)$, we can write $u = \Big(\sum_{j\in[1,x-1]} l_{jg}\Big) + \tilde{u}$ in a unique way, where $\tilde{u} \in [1, l_{xg}]$. Then we have $lg^* = (l_{jg})_{j\in[1,t]}$ and $u = (x,\tilde{u})\varphi_{lg^*}^{-1}$; cf. Definition 1.18.

This means that we have

$$\begin{aligned} ug_{[l]}|^{a^{-1}(xg)}_{\left(\underset{j\in[1,t]}{\boxtimes} a_{jg}\right)^{-1}(x)} &= ug_{[l]} \\ &= u(\varphi_{lg^*} \cdot \tilde{g}_{[l]} \cdot \varphi_r^{-1}) \\ &= ((x,\tilde{u})\varphi_{rg^*}^{-1})(\varphi_{lg^*} \cdot \tilde{g}_{[l]} \cdot \varphi_l^{-1}) \\ &= (x,\tilde{u})(\tilde{g}_{[l]} \cdot \varphi_l^{-1}) \\ &= (xg,\tilde{u})\varphi_l^{-1} \\ &= \Big(\sum_{i\in[1,xg-1]} l_i\Big) + \tilde{u}. \end{aligned}$$

Now suppose given $u, v \in \Big(\underset{j\in[1,t]}{\boxtimes} a_{jg}\Big)^{-1}(x)$ and suppose $u < v$.

Write $u = \Big(\sum_{j\in[1,x-1]} l_{jg}\Big) + \tilde{u}$ and $v = \Big(\sum_{j\in[1,x-1]} l_{jg}\Big) + \tilde{v}$ where $\tilde{u}, \tilde{v} \in [1, l_{xg}]$. Since $u < v$, we have $\tilde{u} < \tilde{v}$. So we have

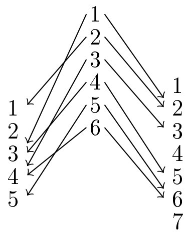
$$\begin{aligned} ug_{[l]}|^{a^{-1}(xg)}_{\left(\underset{j\in[1,t]}{\boxtimes} a_{jg}\right)^{-1}(x)} &= ug_{[l]} \\ &= \Big(\sum_{i\in[1,xg-1]} l_i\Big) + \tilde{u} \\ &< \Big(\sum_{i\in[1,xg-1]} l_i\Big) + \tilde{v} \\ &= vg_{[l]} \\ &= vg_{[l]}|^{a^{-1}(xg)}_{\left(\underset{j\in[1,t]}{\boxtimes} a_{jg}\right)^{-1}(x)}. \end{aligned}$$

This shows that $g_{[l]}|^{a^{-1}(xg)}_{\left(\underset{j\in[1,t]}{\boxtimes} a_{jg}\right)^{-1}(x)}$ is strictly monotone. □

7.2 The set-operad ASS_0

Definition 7.10. Define the set-preoperad $\mathrm{ASS}_0^{\mathrm{pre}}$ as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ the set $\mathrm{ASS}_0^{\mathrm{pre}}(m,n)$ consists of tuples (f,a) where $k \in \mathbb{Z}_{\geqslant 0}$, where $f : [1,k] \longrightarrow [1,m]$ is a map and where $a : [1,k] \longrightarrow [1,n]$ is a monotone map. We will also write $f \backslash a := (f,a) \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$.

Pictorially we have for example



$$\in \mathrm{ASS}_0^{\mathrm{pre}}(5,7).$$

Recall that $a \boxtimes_{\mathrm{Ass}_0} a' = a \boxtimes_{\mathrm{Map}_0} a'$ for $a \in \mathrm{Ass}_0(m,n)$, $a' \in \mathrm{Ass}_0(m',n')$ and $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$.

Define

$$\begin{aligned}(\boxtimes) := \big(\boxtimes_{\mathrm{ASS}_0^{\mathrm{pre}}}\big) : \mathrm{ASS}_0^{\mathrm{pre}}(m,n) \times \mathrm{ASS}_0^{\mathrm{pre}}(m',n') &\longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}(m+m',n+n') \\ \big(f \backslash a\,,\ f' \backslash a'\big) &\longmapsto \big(f \boxtimes_{\mathrm{Map}_0} f'\big) \setminus \big(a \boxtimes_{\mathrm{Ass}_0} a'\big) \\ &\qquad = \big(f \boxtimes_{\mathrm{Map}_0} f'\big) \setminus \big(a \boxtimes_{\mathrm{Map}_0} a'\big)\end{aligned}$$

for $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$.

The composition is defined using the sorted pullback: Given $m,n,p \in \mathbb{Z}_{\geqslant 0}$ and $f \backslash a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$, $g \backslash b \in \mathrm{ASS}_0^{\mathrm{pre}}(n,p)$ and $k,l \in \mathbb{Z}_{\geqslant 0}$ such that $f : [1,k] \longrightarrow [1,m]$, $a : [1,k] \longrightarrow [1,n]$ and such that $g : [1,l] \longrightarrow [1,n]$, $b : [1,l] \longrightarrow [1,p]$, then by Lemma 7.4 there exists a uniquely determined $s \in \mathbb{Z}_{\geqslant 0}$ and uniquely determined maps $\hat{a} : [1,s] \longrightarrow [1,l]$ and $\hat{g} : [1,s] \longrightarrow [1,k]$ such that

$$\begin{array}{ccc} [1,s] & \xrightarrow{\hat{a}} & [1,l] \\ {\scriptstyle \hat{g}}\downarrow & & \downarrow{\scriptstyle g} \\ [1,k] & \xrightarrow[a]{} & [1,n] \end{array}$$

is a sorted pullback.

So define

$$(f \backslash a) \cdot (g \backslash b) := (f \backslash a) \cdot_{\mathrm{ASS}_0^{\mathrm{pre}}} (g \backslash b) := \big(\hat{g} \cdot_{\mathrm{Map}_0} f\big) \setminus \big(\hat{a} \cdot_{\mathrm{Map}_0} b\big) = (\hat{g}f) \backslash (\hat{a}b),$$

where $\hat{a}b$ is a monotone map since both $\hat{a}$ and b are monotone. We have

$$\begin{array}{ccccccc} & & & [1,s] & & & \\ & & \swarrow^{\hat{g}} & & \searrow^{\hat{a}} & & \\ & [1,k] & & & & [1,l] & \\ \swarrow^{f} & & \searrow^{a} & & \swarrow^{g} & & \searrow^{b} \\ [1,m] & & & [1,n] & & & [1,p]. \end{array}$$

For $m \in \mathbb{Z}_{\geqslant 0}$ define $\mathrm{id}_m := \mathrm{id}_{\mathrm{ASS}_0^{\mathrm{pre}},m} := \mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Ass}_0,m}$.

Now we have to show that this actually defines a set-preoperad.

Ad (m1). Let $m,n,m',n',m'',n'' \in \mathbb{Z}_{\geqslant 0}$ and $f \backslash a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$, $f' \backslash a' \in \mathrm{ASS}_0^{\mathrm{pre}}(m',n')$ and $f'' \backslash a'' \in \mathrm{ASS}_0^{\mathrm{pre}}(m'',n'')$. Then we have

$$\begin{aligned}\big((f \backslash a) \boxtimes (f' \backslash a')\big) \boxtimes (f'' \backslash a'') &= \big((f \boxtimes_{\mathrm{Map}_0} f') \backslash (a \boxtimes_{\mathrm{Ass}_0} a')\big) \boxtimes (f'' \backslash a'') \\ &= \big((f \boxtimes_{\mathrm{Map}_0} f') \boxtimes_{\mathrm{Map}_0} f''\big) \backslash \big((a \boxtimes_{\mathrm{Ass}_0} a') \boxtimes_{\mathrm{Ass}_0} a''\big) \\ &= \big(f \boxtimes_{\mathrm{Map}_0} (f' \boxtimes_{\mathrm{Map}_0} f'')\big) \backslash \big(a \boxtimes_{\mathrm{Ass}_0} (a' \boxtimes_{\mathrm{Ass}_0} a'')\big) \\ &= (f \backslash a) \boxtimes \big((f' \boxtimes_{\mathrm{Map}_0} f'') \backslash (a' \boxtimes_{\mathrm{Ass}_0} a'')\big) \\ &= (f \backslash a) \boxtimes \big((f' \backslash a') \boxtimes (f'' \backslash a'')\big).\end{aligned}$$

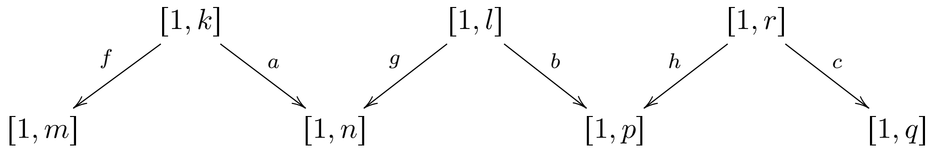
Hence the multiplication is associative.

Ad (m2). Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $f : [1,k] \longrightarrow [1,m]$, $a : [1,k] \longrightarrow [1,n]$. Then

$$\begin{aligned}(f \setminus a) \boxtimes \mathrm{id}_0 &= (f \setminus a) \boxtimes (\mathrm{id}_{\mathrm{Map}_0,0} \setminus \mathrm{id}_{\mathrm{Ass}_0,0}) \\ &= (f \boxtimes_{\mathrm{Map}_0} \mathrm{id}_{\mathrm{Map}_0,0}) \setminus (a \boxtimes_{\mathrm{Ass}_0} \mathrm{id}_{\mathrm{Ass}_0,0}) \\ &= f \setminus a \\ &= (\mathrm{id}_{\mathrm{Map}_0,0} \boxtimes_{\mathrm{Map}_0} f) \setminus (\mathrm{id}_{\mathrm{Ass}_0,0} \boxtimes_{\mathrm{Ass}_0} a) \\ &= (\mathrm{id}_{\mathrm{Map}_0,0} \setminus \mathrm{id}_{\mathrm{Ass}_0,0}) \boxtimes (f \setminus a) \\ &= \mathrm{id}_0 \boxtimes (f \setminus a).\end{aligned}$$

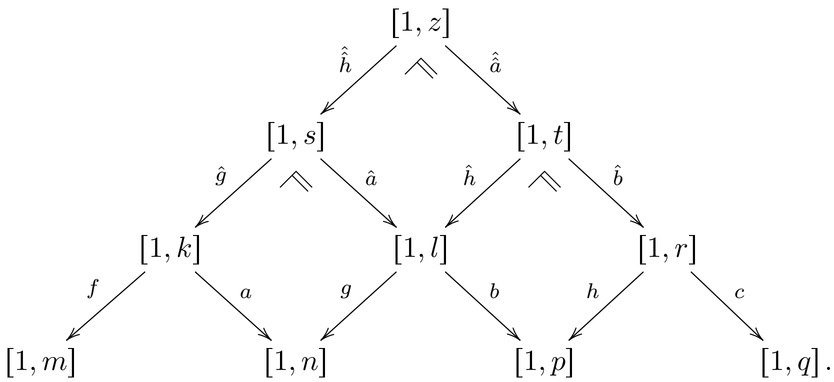
This shows that id_0 is neutral with respect to multiplication.

Ad (c1). Let $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$, $g \setminus b \in \mathrm{ASS}_0^{\mathrm{pre}}(n,p)$, $h \setminus c \in \mathrm{ASS}_0(p,q)^{\mathrm{pre}}$. That is, we have



for some $k, l, r \in \mathbb{Z}_{\geqslant 0}$.

Consider



Then by Lemma 7.6 the quadrangle $([1,z],[1,k],[1,t],[1,n])$ is a sorted pullback and by Lemma 7.5 the quadrangle $([1,z],[1,s],[1,r],[1,p])$ is a sorted pullback.

So we have

$$\begin{aligned}\big((f \setminus a) \cdot (g \setminus b)\big) \cdot (h \setminus c) &= (\hat{g}f \setminus \hat{a}b) \cdot (h \setminus c) \\ &= \big(\hat{\hat{h}}(\hat{g}f)\big) \setminus \big((\hat{\hat{a}}\hat{b})c\big) \\ &= \big((\hat{\hat{h}}\hat{g})f\big) \setminus \big(\hat{\hat{a}}(\hat{b}c)\big) \\ &= (f \setminus a) \cdot (\hat{h}g \setminus \hat{b}c) \\ &= (f \setminus a) \cdot \big((g \setminus b) \cdot (h \setminus c)\big).\end{aligned}$$

This shows that the composition is associative.

Ad (c2): Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $f : [1,k] \longrightarrow [1,m]$ and $a : [1,k] \longrightarrow [1,n]$. We have to show that

$$(f \setminus a) \cdot (\mathrm{id}_{\mathrm{Map},n} \setminus \mathrm{id}_{\mathrm{Map},n}) \overset{!}{=} f \setminus a \overset{!}{=} (\mathrm{id}_{\mathrm{Map},m} \setminus \mathrm{id}_{\mathrm{Map},m}) \cdot (f \setminus a)\,.$$

By Lemma 7.8 we have the following sorted pullbacks.

$$
\begin{array}{ccc}
[1,k] & \xrightarrow{\ a\ } & [1,n] \\
{\scriptstyle \mathrm{id}_{\mathrm{Map}_0,k}}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{Map}_0,n}} \\
[1,k] & \xrightarrow[\ a\ ]{} & [1,n]
\end{array}
\qquad\qquad
\begin{array}{ccc}
[1,k] & \xrightarrow{\mathrm{id}_{\mathrm{Map}_0,k}} & [1,k] \\
{\scriptstyle f}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\
[1,m] & \xrightarrow[\mathrm{id}_{\mathrm{Map}_0,m}]{} & [1,m]
\end{array}
$$

So we have

$$
\begin{array}{ccccccc}
 & & & [1,k] & & & \\
 & & {\scriptstyle \mathrm{id}_{\mathrm{Map}_0,k}}\swarrow & \wedge & \searrow{\scriptstyle a} & & \\
 & [1,k] & & & & [1,n] & \\
{\scriptstyle f}\swarrow & & \searrow{\scriptstyle a} & & {\scriptstyle \mathrm{id}_{\mathrm{Map}_0,n}}\swarrow & & \searrow{\scriptstyle \mathrm{id}_{\mathrm{Map}_0,n}} \\
[1,m] & & & [1,n] & & & [1,n]
\end{array}
$$

and

$$
\begin{array}{ccccccc}
 & & & [1,k] & & & \\
 & & {\scriptstyle f}\swarrow & \wedge & \searrow{\scriptstyle \mathrm{id}_{\mathrm{Map}_0,k}} & & \\
 & [1,m] & & & & [1,k] & \\
{\scriptstyle \mathrm{id}_{\mathrm{Map}_0,m}}\swarrow & & \searrow{\scriptstyle \mathrm{id}_{\mathrm{Map}_0,m}} & & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle a} \\
[1,m] & & & [1,m] & & & [1,n]\,.
\end{array}
$$

Hence we have

$$
\begin{aligned}
(f \setminus a) \cdot (\mathrm{id}_{\mathrm{Map}_0,n} \setminus \mathrm{id}_{\mathrm{Map}_0,n}) &= (\mathrm{id}_{\mathrm{Map}_0,k}\, f) \setminus (a\, \mathrm{id}_{\mathrm{Map}_0,n}) = f \setminus a \\
(\mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Map}_0,m}) \cdot (f \setminus a) &= (f\, \mathrm{id}_{\mathrm{Map}_0,m}) \setminus (\mathrm{id}_{\mathrm{Map}_0,k}\, a)) = f \setminus a.
\end{aligned}
$$

This shows (c2).

Ad (mc1). Let $m, n, p, m', n', p' \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$, $f' \setminus a' \in \mathrm{ASS}_0(m',n')^{\mathrm{pre}}$ and $g \setminus b \in \mathrm{ASS}_0^{\mathrm{pre}}(n,p)$, $g' \setminus b' \in \mathrm{ASS}_0^{\mathrm{pre}}(n',p')$. Let $k, l, k', l' \in \mathbb{Z}_{\geqslant 0}$ such that

$$
\begin{array}{ccccccc}
 & [1,k] & & & & [1,l] & \\
{\scriptstyle f}\swarrow & & \searrow{\scriptstyle a} & & {\scriptstyle g}\swarrow & & \searrow{\scriptstyle b} \\
[1,m] & & & [1,n] & & & [1,p]
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccc}
 & [1,k'] & & & & [1,l'] & \\
{\scriptstyle f'}\swarrow & & \searrow{\scriptstyle a'} & & {\scriptstyle g'}\swarrow & & \searrow{\scriptstyle b'} \\
[1,m'] & & & [1,n'] & & & [1,p'],
\end{array}
$$

We have to show that

$$
\big((f \setminus a) \boxtimes (f' \setminus a')\big) \cdot \big((g \setminus b) \boxtimes (g' \setminus b')\big) \overset{!}{=} \big((f \setminus a) \cdot (g \setminus b)\big) \boxtimes \big((f' \setminus a') \cdot (g' \setminus b')\big).
$$

From Lemma 7.7 we know that stacking sorted pullbacks yields a sorted pullback, that is, given

$$
\begin{array}{ccccccc}
 & & & [1,s] & & & \\
 & & {\scriptstyle \hat{g}}\swarrow & \wedge & \searrow{\scriptstyle \hat{a}} & & \\
 & [1,k] & & & & [1,l] & \\
{\scriptstyle f}\swarrow & & \searrow{\scriptstyle a} & & {\scriptstyle g}\swarrow & & \searrow{\scriptstyle b} \\
[1,m] & & & [1,n] & & & [1,p]
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccc}
 & & & [1,s'] & & & \\
 & & {\scriptstyle \hat{g}'}\swarrow & \wedge & \searrow{\scriptstyle \hat{a}'} & & \\
 & [1,k'] & & & & [1,l'] & \\
{\scriptstyle f'}\swarrow & & \searrow{\scriptstyle a'} & & {\scriptstyle g'}\swarrow & & \searrow{\scriptstyle b'} \\
[1,m'] & & & [1,n'] & & & [1,p'],
\end{array}
$$

we obtain

$$
\begin{array}{ccccc}
 & & [1, s+s'] & & \\
 & \swarrow^{\hat{g} \boxtimes_{\mathrm{Map}_0} \hat{g}'} & \Leftarrow & \searrow^{\hat{a} \boxtimes_{\mathrm{Ass}_0} \hat{a}'} & \\
 & [1, k+k'] & & [1, l+l'] & \\
\swarrow^{f \boxtimes_{\mathrm{Map}_0} f'} & & \searrow^{a \boxtimes_{\mathrm{Ass}_0} a'} \quad \swarrow^{g \boxtimes_{\mathrm{Map}_0} g'} & & \searrow^{b \boxtimes_{\mathrm{Ass}_0} b'} \\
[1, m+m'] & & [1, n+n'] & & [1, p+p'] \, .
\end{array}
$$

So we have

$$
\begin{aligned}
\big((f \setminus a) \boxtimes (f' \setminus a')\big) \cdot \big((g \setminus b) \boxtimes (g' \setminus b')\big) &= \big((f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Ass}_0} a')\big) \cdot \big((g \boxtimes_{\mathrm{Map}_0} g') \setminus (b \boxtimes_{\mathrm{Ass}_0} b')\big) \\
&= \big((\hat{g} \boxtimes_{\mathrm{Map}_0} \hat{g}')(f \boxtimes_{\mathrm{Map}_0} f')\big) \setminus \big((\hat{a} \boxtimes_{\mathrm{Ass}_0} \hat{a}')(b \boxtimes_{\mathrm{Ass}_0} b')\big) \\
&= \big((\hat{g} f) \boxtimes_{\mathrm{Map}_0} (\hat{g}' f')\big) \setminus \big((\hat{a} b) \boxtimes_{\mathrm{Ass}_0} (\hat{a}' b')\big) \\
&= \big((\hat{g} f) \setminus (\hat{a} b)\big) \boxtimes \big((\hat{g}' f') \setminus (\hat{a}' b')\big) \\
&= \big((f \setminus a) \cdot (g \setminus b)\big) \boxtimes \big((f' \setminus a') \cdot (g' \setminus b')\big).
\end{aligned}
$$

Ad (mc2). Let $m \in \mathbb{Z}_{\geqslant 0}$. By induction on $m \geqslant 0$ we see that

$$
\begin{aligned}
\mathrm{id}_m &= \mathrm{id}_{\mathrm{Map}_0, m} \setminus \mathrm{id}_{\mathrm{Ass}_0, m} \\
&= \big(\mathrm{id}_{\mathrm{Map}_0, 1}^{\boxtimes_{\mathrm{Map}_0} m}\big) \setminus \big(\mathrm{id}_{\mathrm{Ass}_0, 1}^{\boxtimes_{\mathrm{Ass}_0} m}\big) \\
&= \big(\mathrm{id}_{\mathrm{Map}_0, m-1} \boxtimes_{\mathrm{Map}_0} \mathrm{id}_{\mathrm{Map}_0, 1}\big) \setminus \big(\mathrm{id}_{\mathrm{Ass}_0, m-1} \boxtimes_{\mathrm{Ass}_0} \mathrm{id}_{\mathrm{Ass}_0, 1}\big) \\
&= \big(\mathrm{id}_{\mathrm{Map}_0, m-1} \setminus \mathrm{id}_{\mathrm{Ass}_0, m-1}\big) \boxtimes \big(\mathrm{id}_{\mathrm{Map}_0, 1} \setminus \mathrm{id}_{\mathrm{Ass}_0, 1}\big) \\
&\overset{\text{ind.}}{=} \mathrm{id}_1^{\boxtimes (m-1)} \boxtimes \mathrm{id}_1 \\
&= \mathrm{id}_1^{\boxtimes m} \, .
\end{aligned}
$$

This completes the proof that $\mathrm{ASS}_0^{\mathrm{pre}}$ is a set-preoperad.

Remark 7.11. Note that Lemma 7.8 implies that given $m, n, p, k, l \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Map}_0(k, m)$, $g \in \mathrm{Map}_0(l, k)$, $a \in \mathrm{Ass}_0(k, n)$, $b \in \mathrm{Ass}_0(l, p)$ and $c \in \mathrm{Ass}_0(n, p)$, then we have

$$
\begin{aligned}
(f \setminus \mathrm{id}_{\mathrm{Ass}_0, k}) \cdot (g \setminus b) &= (gf) \setminus (\mathrm{id}_{\mathrm{Ass}_0, l}\, b) = (gf) \setminus b \\
(f \setminus a) \cdot (\mathrm{id}_{\mathrm{Map}_0, l} \setminus c) &= (\mathrm{id}_{\mathrm{Map}_0, k}\, f) \setminus (ac) = f \setminus (ac).
\end{aligned}
$$

Definition 7.12. Define the biindexed map $\mathfrak{a}_0 = (\mathfrak{a}_0(m, n))_{m, n \geqslant 0} : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}$ as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ let

$$
\begin{aligned}
\mathfrak{a}_0(m, n) : \mathrm{Map}_0^{\mathrm{op}}(m, n) &\longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}(m, n) \\
f^{\mathrm{op}} &\longmapsto f \setminus \mathrm{id}_{\mathrm{Ass}_0, n} \, .
\end{aligned}
$$

Lemma 7.13. *The biindexed map* $\mathfrak{a}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}$*; cf. Definition* 7.12*, is a morphism of set-preoperads.*

Proof. First let $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and let $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m, n)$ and $f'^{\,\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m', n')$. We have

$$
\begin{aligned}
(f^{\mathrm{op}} \boxtimes_{\mathrm{Map}_0^{\mathrm{op}}} f'^{\,\mathrm{op}})\mathfrak{a}_0 &= (f \boxtimes_{\mathrm{Map}_0} f')^{\mathrm{op}} \mathfrak{a}_0 \\
&= (f \boxtimes_{\mathrm{Map}_0} f') \setminus \mathrm{id}_{\mathrm{Ass}_0, n+n'} \\
&= (f \boxtimes_{\mathrm{Map}_0} f') \setminus (\mathrm{id}_{\mathrm{Ass}_0, n} \boxtimes_{\mathrm{Ass}_0} \mathrm{id}_{\mathrm{Ass}_0, n'}) \\
&= (f \setminus \mathrm{id}_{\mathrm{Ass}_0, n}) \boxtimes (f' \setminus \mathrm{id}_{\mathrm{Ass}_0, n'}) \\
&= (f^{\mathrm{op}} \mathfrak{a}_0) \boxtimes (f'^{\,\mathrm{op}} \mathfrak{a}_0).
\end{aligned}
$$

For the second property let $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and let $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$ and $g \in \mathrm{Map}_0^{\mathrm{op}}(n,p)$. By Remark 7.11 we have

$$\begin{aligned}(f^{\mathrm{op}}\mathfrak{a}_0)\cdot(g^{\mathrm{op}}\mathfrak{a}_0) &= (f \setminus \mathrm{id}_{\mathrm{Ass}_0,n})\cdot(g \setminus \mathrm{id}_{\mathrm{Ass}_0,p}) \\ &= (gf)\setminus \mathrm{id}_{\mathrm{Ass}_0,p} \\ &= (g \cdot_{\mathrm{Map}_0} f)^{\mathrm{op}}\mathfrak{a}_0 \\ &= (f^{\mathrm{op}} \cdot_{\mathrm{Map}_0^{\mathrm{op}}} g^{\mathrm{op}})\mathfrak{a}_0\,.\end{aligned}$$

□

Lemma 7.14. *We have the set-operad* $\mathrm{ASS}_0 := (\mathrm{ASS}_0^{\mathrm{pre}}, \mathfrak{a}_0)$*; cf. Definitions* 7.10 *and* 7.12 *and Definition* 6.3*.*

Proof. Since $\mathfrak{a}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}$ is a morphism of set-preoperads we have to verify the conditions (so1) and (so2); cf. Definition 6.3.

Ad (so1). Let $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$, $f' \setminus a' \in \mathrm{ASS}_0^{\mathrm{pre}}(m',n')$. Let $k, k' \in \mathbb{Z}_{\geqslant 0}$ such that $f : [1,k] \longrightarrow [1,m]$, $a : [1,k] \longrightarrow [1,n]$, $f' : [1,k'] \longrightarrow [1,m']$ and $a' : [1,k'] \longrightarrow [1,n']$.

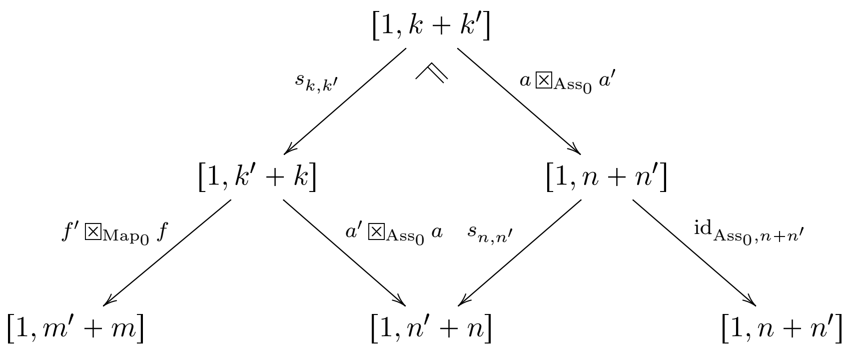
We have to show that

$$(s_{m,m'}^{\mathrm{op}}\mathfrak{a}_0)\cdot\big((f\setminus a)\boxtimes(f'\setminus a')\big) \overset{!}{=} \big((f'\setminus a')\boxtimes(f\setminus a)\big)\cdot(s_{n,n'}^{\mathrm{op}}\mathfrak{a}_0).$$

By Remark 7.11 we have

$$\begin{aligned}(s_{m,m'}^{\mathrm{op}}\mathfrak{a}_0)\cdot\big((f\setminus a)\boxtimes(f'\setminus a')\big) &= (s_{m,m'}\setminus \mathrm{id}_{\mathrm{Ass}_0,m+m'})\cdot\big((f\boxtimes_{\mathrm{Map}_0} f')\setminus(a\boxtimes_{\mathrm{Ass}_0} a')\big) \\ &= \big((f\boxtimes_{\mathrm{Map}_0} f')s_{m,m'}\big)\setminus(a\boxtimes_{\mathrm{Ass}_0} a').\end{aligned}$$

Claim. We have the following diagram.



Proof of the Claim. Since Ass_0 is a set-subpreoperad of Map_0 we have

$$(a\boxtimes_{\mathrm{Ass}_0} a')s_{n,n'} = s_{k,k'}(a'\boxtimes_{\mathrm{Ass}_0} a)$$

by Lemma 6.6. Moreover, $a\boxtimes_{\mathrm{Ass}_0} a'$ is monotone as a product of monotone maps. So it remains to show that $s_{k,k'}\big|_{(a\boxtimes_{\mathrm{Ass}_0} a')^{-1}(j)}^{(a'\boxtimes_{\mathrm{Ass}_0} a)^{-1}(js_{n,n'})}$ is isotone for $j \in [1, n+n']$.

So suppose given $j \in [1, n+n']$. Suppose given $i, i' \in (a\boxtimes_{\mathrm{Ass}} a')^{-1}(j)$, that is, we have $i(a\boxtimes_{\mathrm{Ass}} a') = j = i'(a\boxtimes_{\mathrm{Ass}} a')$, and suppose $i < i'$.

By the definition of $(\boxtimes_{\mathrm{Ass}_0})$ either $i, i' \in [1,k]$ or $i, i' \in [k+1, k+k']$; cf. Definitions 2.57 and 2.58.

If $i, i' \in [1,k]$, then we have

$$is_{k,k'}\big|_{(a\boxtimes_{\mathrm{Ass}_0} a')^{-1}(j)}^{(a'\boxtimes_{\mathrm{Ass}_0} a)^{-1}(js_{n,n'})} = is_{k,k'} = i + k' < i' + k' = i's_{k+k'} = i's_{k,k'}\big|_{(a\boxtimes_{\mathrm{Ass}_0} a')^{-1}(j)}^{(a'\boxtimes_{\mathrm{Ass}_0} a)^{-1}(js_{n,n'})}.$$

If $i, i' \in [k+1, k+k']$, then we have

$$is_{k,k'}\big|^{(a' \boxtimes_{\mathrm{Ass}_0} a)^{-1}(js_{n,n'})}_{(a \boxtimes_{\mathrm{Ass}_0} a')^{-1}(j)} = is_{k,k'} = i - k < i' - k = i's_{k,k'} = i's_{k,k'}\big|^{(a' \boxtimes_{\mathrm{Ass}_0} a)^{-1}(js_{n,n'})}_{(a \boxtimes_{\mathrm{Ass}_0} a')^{-1}(j)}.$$

This shows that $s_{k,k'}\big|^{(a' \boxtimes_{\mathrm{Ass}_0} a)^{-1}(js_{n,n'})}_{(a \boxtimes_{\mathrm{Ass}} a')^{-1}(j)}$ is strictly monotone.

Now let $i \in (a' \boxtimes_{\mathrm{Ass}} a)^{-1}(js_{n,n'})$, that is, $i(a' \boxtimes_{\mathrm{Ass}} a) = js_{n,n'}$. We have to show that there exists $x \in [1, k+k']$ with $x \in (a \boxtimes_{\mathrm{Ass}} a')^{-1}(j)$ such that $xs_{k,k'} = i$.

Define $x := is_{k',k}$. Then by Lemma 6.6 we have

$$x(a \boxtimes_{\mathrm{Ass}_0} a') = is_{k',k}(a \boxtimes_{\mathrm{Ass}_0} a') = i(a' \boxtimes_{\mathrm{Ass}_0} a)s_{n',n} = js_{n,n'}s_{n',n} = j\,\mathrm{id}_{\mathrm{Map}_0, n+n'} = j,$$

so $x \in (a \boxtimes_{\mathrm{Ass}_0} a')^{-1}(j)$.

Moreover, we have $xs_{k,k'}\big|^{(a' \boxtimes_{\mathrm{Ass}_0} a)^{-1}(js_{n,n'})}_{(a \boxtimes_{\mathrm{Ass}_0} a')^{-1}(j)} = xs_{k,k'} = i(s_{k',k}s_{k,k'}) = i\,\mathrm{id}_{\mathrm{Map}_0, k+k'} = i$.

So $s_{k,k'}\big|^{(a' \boxtimes_{\mathrm{Ass}_0} a)^{-1}(js_{n,n'})}_{(a \boxtimes_{\mathrm{Ass}_0} a')^{-1}(j)}$ is surjective.

This completes the proof of the *Claim.*

So we have

$$\begin{aligned} \big((f' \setminus a') \boxtimes (f \setminus a)\big) \cdot (s^{\mathrm{op}}_{n,n'}\mathfrak{a}_0) &= \big((f' \boxtimes_{\mathrm{Map}_0} f) \setminus (a' \boxtimes_{\mathrm{Ass}_0} a)\big) \cdot (s_{n,n'} \setminus \mathrm{id}_{\mathrm{Ass}_0, n+n'}) \\ &\overset{7.11}{=} \big(s_{k,k'}(f' \boxtimes_{\mathrm{Map}_0} f)\big) \setminus (a \boxtimes_{\mathrm{Ass}_0} a') \\ &\overset{6.6}{=} \big((f \boxtimes_{\mathrm{Map}_0} f')s_{m,m'}\big) \setminus (a \boxtimes_{\mathrm{Ass}_0} a') \\ &= (s^{\mathrm{op}}_{m,m'}\mathfrak{a}_0) \cdot \big((f \setminus a) \boxtimes (f' \setminus a')\big). \end{aligned}$$

Ad (so2). Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}^{\mathrm{pre}}_0(m,n)$. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $f : [1,k] \longrightarrow [1,m]$ and $a : [1,k] \longrightarrow [1,n]$. Let $l \in \mathbb{Z}_{\geqslant 0}$. We have to show that

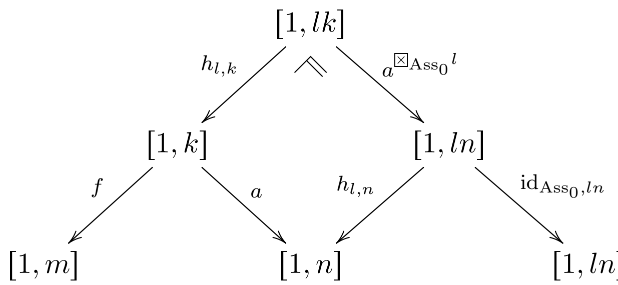
$$(*) \qquad (h^{\mathrm{op}}_{l,m}\mathfrak{a}_0) \cdot (f \setminus a)^{\boxtimes l} \overset{!}{=} (f \setminus a) \cdot (h^{\mathrm{op}}_{l,n}\mathfrak{a}_0).$$

By Remark 7.11 we know that

$$\begin{aligned} (h^{\mathrm{op}}_{l,m}\mathfrak{a}_0) \cdot (f \setminus a)^{\boxtimes l} &= (h_{l,m} \setminus \mathrm{id}_{\mathrm{Ass}_0, lm}) \cdot (f^{\boxtimes_{\mathrm{Map}_0} l} \setminus a^{\boxtimes_{\mathrm{Ass}_0} l}) \\ &= (f^{\boxtimes_{\mathrm{Map}_0} l} h_{l,m}) \setminus a^{\boxtimes_{\mathrm{Ass}_0} l}. \end{aligned}$$

Now consider the right hand side of $(*)$.

Claim. We have the following diagram.



Proof of the Claim. Since $\mathrm{Ass}_0 \subseteq \mathrm{Map}_0$ is a set-subpreoperad and since $(\mathrm{Map}^{\mathrm{op}}_0, \mathrm{id}_{\mathrm{Map}^{\mathrm{op}}_0})$ is a set-operad, we have $a^{\boxtimes_{\mathrm{Ass}_0} l} h_{l,n} = a^{\boxtimes_{\mathrm{Map}_0} l} h_{l,n} = h_{l,k} a$; cf. Lemma 6.7.

Moreover, $a^{\boxtimes_{\mathrm{Ass}_0} l}$ is a monotone map, since a is monotone.

So we have to show that $h_{l,k}|^{a^{-1}(jh_{l,n})}_{\left(a^{\boxtimes_{\mathrm{Ass}_0} l}\right)^{-1}(j)}$ is isotone for $j \in [1, ln]$.

So suppose given $j \in [1, ln]$.

Recall that we can write $j = \underline{j}n + \bar{j}$ in a unique way, where $\underline{j} \in [0, l-1]$ and $\bar{j} \in [1, n]$.

Strict monotonicity. Let $u, v \in (a^{\boxtimes_{\mathrm{Ass}_0} l})^{-1}(j)$ and suppose $u < v$. Write $u = \underline{u}k + \bar{u}$ and $v = \underline{v}k + \bar{v}$, $\underline{u}, \underline{v} \in [0, l-1]$ and $\bar{u}, \bar{v} \in [1, k]$. Now since $u, v \in (a^{\boxtimes_{\mathrm{Ass}_0} l})^{-1}(j)$ we have

$$\bar{u}a + \underline{u}n = ua^{\boxtimes_{\mathrm{Ass}_0} l} = j = va^{\boxtimes_{\mathrm{Ass}_0} l} = \bar{v}a + \underline{v}n$$

and since $\underline{j}, \bar{j}$ are uniquely determined, we have $\underline{u} = \underline{j} = \underline{v}$. Now since $u < v$ this implies $\bar{u} < \bar{v}$.

So we have

$$uh_{l,k}|^{a^{-1}(jh_{l,n})}_{\left(a^{\boxtimes_{\mathrm{Ass}_0} l}\right)^{-1}(j)} = (\underline{u}k + \bar{u})h_{l,k} = \bar{u} < \bar{v} = (\underline{v}k + \bar{v})h_{l,k} = vh_{l,k}|^{a^{-1}(jh_{l,n})}_{\left(a^{\boxtimes_{\mathrm{Ass}_0} l}\right)^{-1}(j)}.$$

This shows that $h_{l,k}|^{a^{-1}(jh_{l,n})}_{\left(a^{\boxtimes_{\mathrm{Ass}_0} l}\right)^{-1}(j)}$ is strictly monotone.

Surjectivity. Let $y \in a^{-1}(jh_{l,n}) = a^{-1}(\bar{j}) \subseteq [1, k]$.

We have to show that there exists $x \in \left(a^{\boxtimes_{\mathrm{Ass}_0} l}\right)^{-1}(j)$ such that $xh_{l,k}|^{a^{-1}(jh_{l,n})}_{\left(a^{\boxtimes_{\mathrm{Ass}_0} l}\right)^{-1}(j)} = xh_{l,k} = y$.

Let $x := y + \underline{j}k$. Then we have $xa^{\boxtimes_{\mathrm{Ass}_0} l} = ya + \underline{j}n = \bar{j} + \underline{j}n = j$ and $xh_{l,k}|^{a^{-1}(jh_{l,n})}_{\left(a^{\boxtimes_{\mathrm{Ass}_0} l}\right)^{-1}(j)} = xh_{l,k} = y$.

This shows that $h_{l,k}|^{a^{-1}(jh_{l,n})}_{\left(a^{\boxtimes_{\mathrm{Ass}_0} l}\right)^{-1}(j)}$ is surjective and completes the proof of the *Claim.*

We now have

$$(f \setminus a) \cdot (h^{\mathrm{op}}_{l,n}\mathfrak{a}_0) = (f \setminus a) \cdot (h_{l,n} \setminus \mathrm{id}_{\mathrm{Ass}_0, ln}) = (h_{l,k}f) \setminus a^{\boxtimes_{\mathrm{Ass}_0} l}.$$

Now since $(\mathrm{Map}_0^{\mathrm{op}}, \mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}})$ is a set-operad by Lemma 6.7, we have $h_{l,k}f = f^{\boxtimes_{\mathrm{Map}_0} l}h_{l,m}$. Hence we have

$$(f \setminus a) \cdot (h^{\mathrm{op}}_{l,n}\mathfrak{a}_0) = (h_{l,k}f) \setminus a^{\boxtimes_{\mathrm{Ass}_0} l} = (f^{\boxtimes_{\mathrm{Map}_0} l}h_{l,m}) \setminus a^{\boxtimes_{\mathrm{Ass}_0} l} = (h^{\mathrm{op}}_{l,m}\mathfrak{a}_0) \cdot (f \setminus a)^{\boxtimes l}.$$

□

Definition 7.15. We define the morphism $\alpha_0 = (\alpha_0(m,n))_{m,n \geqslant 0} : \mathrm{Ass}_0 \longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}$ of set-preoperads as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ we let

$$\begin{aligned} \alpha_0(m,n) : \mathrm{Ass}_0(m,n) &\longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}(m,n) \\ a &\longmapsto \mathrm{id}_{\mathrm{Map}_0, m} \setminus a. \end{aligned}$$

In order to show that this in fact is a morphism of set-preoperads, first note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have $\mathrm{id}_{\mathrm{Ass}_0, m}\, \alpha_0 = \mathrm{id}_{\mathrm{Map}_0, m} \setminus \mathrm{id}_{\mathrm{Ass}_0, m} = \mathrm{id}_{\mathrm{ASS}_0, m}$.

Moreover, for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Ass}_0(m,n)$ and $a' \in \mathrm{Ass}_0(m', n')$ we have

$$\begin{aligned} (a \boxtimes_{\mathrm{Ass}_0} a')\alpha_0 &= \mathrm{id}_{\mathrm{Map}_0, m+m'} \setminus (a \boxtimes_{\mathrm{Ass}_0} a') \\ &= (\mathrm{id}_{\mathrm{Map}_0, m} \boxtimes_{\mathrm{Map}_0} \mathrm{id}_{\mathrm{Map}_0, m'}) \setminus (a \boxtimes_{\mathrm{Ass}_0} a') \\ &= (\mathrm{id}_{\mathrm{Map}_0, m} \setminus a) \boxtimes (\mathrm{id}_{\mathrm{Map}_0, m'} \setminus a') \\ &= a\alpha_0 \boxtimes a'\alpha_0 . \end{aligned}$$

Furthermore, for $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Ass}_0(m,n)$ and $b \in \mathrm{Ass}_0(n,k)$ we have

$$(a \cdot_{\mathrm{Ass}_0} b)\alpha_0 = \mathrm{id}_{\mathrm{Map}_0, m} \setminus (a \cdot_{\mathrm{Ass}_0} b) \overset{7.11}{=} (\mathrm{id}_{\mathrm{Map}_0, m} \setminus a) \cdot (\mathrm{id}_{\mathrm{Map}_0, n} \setminus b) = a\alpha_0 \cdot b\alpha_0 .$$

7.3 Associative monoids and ASS_0-algebras

Proposition 7.16. *Let X be a set and let $\Psi_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{END}_0(X)$ be a morphism of set-operads. So X is an ASS_0-algebra.*

$$\begin{array}{ccc} \mathrm{ASS}_0^{\mathrm{pre}} & \xrightarrow{\Psi_0^{\mathrm{pre}}} & \mathrm{End}_0(X) \\ \uparrow{\scriptstyle \mathfrak{a}_0} & \nearrow{\scriptstyle \mathfrak{e}_0} & \\ \mathrm{Map}_0^{\mathrm{op}} & & \end{array}$$

Define $\mu_X := (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}}$ and $\varepsilon_X := (\mathrm{id}_0 \setminus \varepsilon)\Psi_0^{\mathrm{pre}}$, where we abbreviate $\mathrm{id}_m := \mathrm{id}_{\mathrm{Map}_0,m}$ for $m \in \mathbb{Z}_{\geqslant 0}$. Then $(X, \mu_X, \varepsilon_X)$ is an (associative) monoid.

Proof. We have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Ass}_0 & & \\ \downarrow{\scriptstyle \alpha_0} & \searrow{\scriptstyle \alpha_0\Psi_0^{\mathrm{pre}}} & \\ \mathrm{ASS}_0^{\mathrm{pre}} & \xrightarrow{\Psi_0^{\mathrm{pre}}} & \mathrm{End}_0(X) \\ \uparrow{\scriptstyle \mathfrak{a}_0} & \nearrow{\scriptstyle \mathfrak{e}_0} & \\ \mathrm{Map}_0^{\mathrm{op}} & & \end{array}$$

So $(X, \alpha_0\Psi_0^{\mathrm{pre}})$ is an Ass_0-algebra. Furthermore, we have $\mu_X = (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}} = \mu(\alpha_0\Psi_0^{\mathrm{pre}})$ and $\varepsilon_X = (\mathrm{id}_0 \setminus \varepsilon)\Psi_0^{\mathrm{pre}} = \varepsilon(\alpha_0\Psi_0^{\mathrm{pre}})$.

So according to Proposition 5.3, $(X, \mu_X, \varepsilon_X)$ is an (associative) monoid. □

> We aim to show the converse statement that every (associative) monoid can be turned into an ASS_0-algebra. Instead of showing this directly we will give a more general statement.

Lemma 7.17. *Let $\mathcal{T}_0 = (\mathcal{T}_0^{\mathrm{pre}}, \mathfrak{t}_0)$ be a set-operad. Let $\tau_0 : \mathrm{Ass}_0 \longrightarrow \mathcal{T}_0^{\mathrm{pre}}$ be a morphism of set-preoperads. Then there exists a uniquely determined morphism $\hat{\tau}_0^{\mathrm{pre}} : \mathrm{ASS}_0^{\mathrm{pre}} \longrightarrow \mathcal{T}_0^{\mathrm{pre}}$ of set-preoperads such that the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Ass}_0 & & \\ \downarrow{\scriptstyle \alpha_0} & \searrow{\scriptstyle \tau_0} & \\ \mathrm{ASS}_0^{\mathrm{pre}} & \xrightarrow{\exists!\hat{\tau}_0^{\mathrm{pre}}} & \mathcal{T}_0\,; \\ \uparrow{\scriptstyle \mathfrak{a}_0} & \nearrow{\scriptstyle \mathfrak{t}_0} & \\ \mathrm{Map}_0^{\mathrm{op}} & & \end{array}$$

cf. Definition 7.15.

In particular, the commutativity of the lower triangle means that $\hat{\tau}_0^{\mathrm{pre}}$ defines a morphism $\hat{\tau}_0$ of set-operads.

We have $(f \setminus a)\hat{\tau}_0^{\mathrm{pre}} = (f^{\mathrm{op}}\mathfrak{t}_0) \cdot_{\mathcal{T}_0} (a\tau_0)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m, n)$.

Proof. Uniqueness. First assume that $\tilde{\tau}_0^{\mathrm{pre}} : \mathrm{ASS}_0^{\mathrm{pre}} \longrightarrow \mathcal{T}_0^{\mathrm{pre}}$ is a morphism of set-preoperads such that $\alpha_0\tilde{\tau}_0^{\mathrm{pre}} = \tau_0$ and $\mathfrak{a}_0\tilde{\tau}_0^{\mathrm{pre}} = \mathfrak{t}_0$. Then for $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m, n)$, where $m, n \in \mathbb{Z}_{\geqslant 0}$ and where $f : [1, k] \longrightarrow [1, m]$ and $a : [1, k] \longrightarrow [1, n]$ for some $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$f \setminus a = (f \setminus \mathrm{id}_{\mathrm{Ass}_0,k}) \cdot (\mathrm{id}_{\mathrm{Map}_0,k} \setminus a);$$

cf. Remark 7.11. So we have

$$(f \setminus a)\tilde{\tau}_0^{\text{pre}} = \big((f \setminus \text{id}_{\text{Ass}_0,k})\tilde{\tau}_0^{\text{pre}}\big) \cdot_{\mathcal{T}_0} \big((\text{id}_{\text{Map}_0,k} \setminus a)\tilde{\tau}_0^{\text{pre}}\big) = \big(f^{\text{op}}(\mathfrak{a}_0\tilde{\tau}_0^{\text{pre}})\big) \cdot_{\mathcal{T}_0} \big(a(\alpha_0\tilde{\tau}_0)\big) = (f^{\text{op}}\mathfrak{t}_0) \cdot_{\mathcal{T}_0} (a\tau_0),$$

hence such a morphism $\tilde{\tau}_0^{\text{pre}}$ is uniquely determined by $\mathfrak{t}_0$ and τ_0.

Existence. We have to show that

$$\begin{aligned} \hat{\tau}_0^{\text{pre}} : \text{ASS}_0^{\text{pre}} &\longrightarrow \mathcal{T}_0^{\text{pre}} \\ (f \setminus a) &\longmapsto (f^{\text{op}}\mathfrak{t}_0) \cdot_{\mathcal{T}_0} (a\tau_0) \end{aligned}$$

is in fact a morphism of set-preoperads satisfying $\alpha_0\hat{\tau}_0^{\text{pre}} = \tau_0$ and $\mathfrak{a}_0\hat{\tau}_0^{\text{pre}} = \mathfrak{t}_0$.

First note that given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \text{ASS}_0^{\text{pre}}(m, n)$, say $f : [1, k] \longrightarrow [1, m]$ and $a : [1, k] \longrightarrow [1, n]$, where $k \in \mathbb{Z}_{\geqslant 0}$, then we have $f^{\text{op}}\mathfrak{t}_0 \in \mathcal{T}_0^{\text{pre}}(m, k)$ and $a\alpha_0 \in \mathcal{T}_0^{\text{pre}}(k, n)$, so in fact $(f^{\text{op}}\mathfrak{t}_0) \cdot_{\mathcal{T}_0} (a\alpha_0) \in \mathcal{T}_0^{\text{pre}}(m, n)$.

Furthermore, note that given $m \in \mathbb{Z}_{\geqslant 0}$ then we have

$$\text{id}_m\, \hat{\tau}_0^{\text{pre}} = \big(\text{id}_{\text{Map}_0,m} \setminus \text{id}_{\text{Ass}_0,m}\big)\hat{\tau}_0^{\text{pre}} = \big(\text{id}_{\text{Map}_0,m}^{\text{op}}\, \mathfrak{t}_0\big) \cdot_{\mathcal{T}_0} \big(\text{id}_{\text{Ass}_0,m}\, \tau_0\big) = \text{id}_{\mathcal{T}_0,m} \cdot_{\mathcal{T}_0} \text{id}_{\mathcal{T}_0,m} = \text{id}_{\mathcal{T}_0,m}\,.$$

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \text{ASS}_0^{\text{pre}}(m, n)$, $f' \setminus a' \in \text{ASS}_0^{\text{pre}}(m', n')$. Let $k, k' \in \mathbb{Z}_{\geqslant 0}$ be such that $f : [1, k] \longrightarrow [1, m]$, $a : [1, k] \longrightarrow [1, n]$, $f' : [1, k'] \longrightarrow [1, m']$ and $a : [1, k'] \longrightarrow [1, n']$.

We have

$$\begin{aligned} \big((f \setminus a) \boxtimes (f' \setminus a')\big)\hat{\tau}_0^{\text{pre}} &= \big((f \boxtimes_{\text{Map}_0} f') \setminus (a \boxtimes_{\text{Ass}_0} a')\big)\hat{\tau}_0^{\text{pre}} \\ &= \big((f \boxtimes_{\text{Map}_0} f')^{\text{op}}\mathfrak{t}_0\big) \cdot_{\mathcal{T}_0} \big((a \boxtimes_{\text{Ass}_0} a')\tau_0\big) \\ &= \big((f^{\text{op}}\mathfrak{t}_0) \boxtimes_{\mathcal{T}_0} (f'^{\text{op}}\mathfrak{t}_0)\big) \cdot_{\mathcal{T}_0} \big((a\tau_0) \boxtimes_{\mathcal{T}_0} (a'\tau_0)\big) \\ &= \big((f^{\text{op}}\mathfrak{t}_0) \cdot_{\mathcal{T}_0} (a\tau_0)\big) \boxtimes_{\mathcal{T}_0} \big((f'^{\text{op}}\mathfrak{t}_0) \cdot_{\mathcal{T}_0} (a'\tau_0)\big) \\ &= \big((f \setminus a)\hat{\tau}_0^{\text{pre}}\big) \boxtimes_{\mathcal{T}_0} \big((f' \setminus a')\hat{\tau}_0^{\text{pre}}\big). \end{aligned}$$

Now suppose given $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \text{ASS}_0^{\text{pre}}(m, n)$, $g \setminus b \in \text{ASS}_0^{\text{pre}}(n, p)$. Let $k, t \in \mathbb{Z}_{\geqslant 0}$ be such that $f : [1, k] \longrightarrow [1, m]$, $a : [1, k] \longrightarrow [1, n]$, $g : [1, t] \longrightarrow [1, n]$ and $b : [1, t] \longrightarrow [1, p]$.

Since a is monotone we can write

$$a = a_1 \boxtimes_{\text{Ass}_0} \ldots \boxtimes_{\text{Ass}_0} a_n = \mathop{\boxtimes}_{i \in [1,n]} a_i\,,$$

where we omit the index "Ass_0" and where $a_i = \mu_{l_i} \in \text{Ass}_0(l_i, 1)$ with $l_i = |a^{-1}(i)| \in [0, k]$ for $i \in [1, n]$ and $\sum_{i \in [1,n]} l_i = k$; cf. Remark 4.30. Write $l := (l_i)_{i \in [1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$ and $r = (1, \ldots, 1) \in (\mathbb{Z}_{\geqslant 0})^{\times n}$.

By Lemma 7.9 we have the following sorted pullback.

$$\begin{array}{ccc} \Big[1, \sum_{j \in [1,t]} l_{jg}\Big] & \xrightarrow{\ \mathop{\boxtimes}_{j \in [1,t]} a_{jg}\ } & [1, t] \\ \Big\downarrow{\scriptstyle g_{[l]}} & \ulcorner & \Big\downarrow{\scriptstyle g} \\ [1, k] & \xrightarrow[\ a\ ]{} & [1, n] \end{array}$$

So by the definition of composition in ASS_0 we have

$$(f \setminus a) \cdot (g \setminus b) = \big(g_{[l]}f\big) \setminus \Big(\Big(\mathop{\boxtimes}_{j \in [1,t]} a_{jg}\Big)b\Big).$$

So we have

$$
\begin{aligned}
\big((f\setminus a)\cdot(g\setminus b)\big)\hat{\tau}_0^{\text{pre}} &= \Big((g_{[l]}f)\setminus\Big(\Big(\boxtimes_{j\in[1,t]} a_{jg}\Big)b\Big)\Big)\hat{\tau}_0^{\text{pre}} \\
&= \big((g_{[l]}f)^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\Big(\Big(\boxtimes_{j\in[1,t]} a_{jg}\Big)b\Big)\tau_0 \\
&= \big(f^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\big((g_{[l]})^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\Big(\Big(\boxtimes_{j\in[1,t]} a_{jg}\Big)\tau_0\Big)\cdot_{\mathcal{T}_0}\big(b\tau_0\big) \\
&= \big(f^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\big((g_{[l]})^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\Big(\boxtimes_{j\in[1,t]}(a_{jg}\tau_0)\Big)\cdot_{\mathcal{T}_0}\big(b\tau_0\big) \\
&\overset{6.10}{=} \big(f^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\Big(\boxtimes_{i\in[1,n]}(a_i\tau_0)\Big)\cdot_{\mathcal{T}_0}\big((g_{[r]})^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\big(b\tau_0\big) \\
&\overset{6.9\,\text{(iv)}}{=} \big(f^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\Big(\Big(\boxtimes_{i\in[1,n]} a_i\Big)\tau_0\Big)\cdot_{\mathcal{T}_0}\big(g^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\big(b\tau_0\big) \\
&= \big(f^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\big(a\tau_0\big)\cdot_{\mathcal{T}_0}\big(g^{\text{op}}\mathfrak{t}_0\big)\cdot_{\mathcal{T}_0}\big(b\tau_0\big) \\
&= (f\setminus a)\hat{\tau}_0^{\text{pre}}\cdot_{\mathcal{T}_0}(g\setminus b)\hat{\tau}_0^{\text{pre}}\,.
\end{aligned}
$$

We have now shown that $\hat{\tau}_0^{\text{pre}}$ is in fact a morphism of set-preoperads.

Since by construction $f^{\text{op}}\mathfrak{a}_0\hat{\tau}_0^{\text{pre}} = (f\setminus \text{id}_{\text{Ass}_0,n})\hat{\tau}_0^{\text{pre}} = (f^{\text{op}}\mathfrak{t}_0)\cdot_{\mathcal{T}_0}\text{id}_{\mathcal{T}_0,n} = f^{\text{op}}\mathfrak{t}_0$ for $m,n\in\mathbb{Z}_{\geqslant 0}$ and $f^{\text{op}}\in\text{Map}_0^{\text{op}}(m,n)$, we have $\mathfrak{a}_0\hat{\tau}_0^{\text{pre}} = \mathfrak{t}_0$. So $\hat{\tau}_0 : \text{ASS}_0 \longrightarrow \mathcal{T}_0$ is a morphism of set-operads.

Moreover, we have $a\alpha_0\hat{\tau}_0^{\text{pre}} = (\text{id}_{\text{Map}_0,m}\setminus a)\hat{\tau}_0^{\text{pre}} = \text{id}_{\mathcal{T}_0,m}\cdot_{\mathcal{T}_0}(a\tau_0) = a\tau_0$ for $m,n\in\mathbb{Z}_{\geqslant 0}$ and $a\in\text{Ass}_0(m,n)$. Hence $\alpha\hat{\tau}_0^{\text{pre}} = \tau_0$. □

We can now use this to show that an (associative) monoid X can be turned into an ASS_0-algebra.

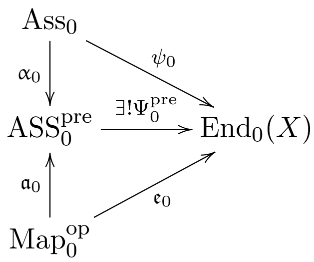
Proposition 7.18. *Let (X,μ_X,ε_X) be a monoid.*

Then there exists a morphism of set-operads $\Psi_0 : \text{ASS}_0 \longrightarrow \text{END}_0(X)$ such that $\mu_X = (\text{id}_2\setminus\mu)\Psi_0^{\text{pre}}$ and $\varepsilon_X = (\text{id}_0\setminus\varepsilon)\Psi_0^{\text{pre}}$, where again we abbreviate $\text{id}_m := \text{id}_{\text{Map}_0,m}$ for $m\in\mathbb{Z}_{\geqslant 0}$.

In particular, (X,Ψ_0) is an ASS_0-algebra.

Proof. By Proposition 5.4, we can turn X into an Ass_0-algebra using the morphism of set-preoperads $\psi_0 : \text{Ass}_0 \longrightarrow \text{End}_0(X)$ that satisfies $\mu_X = \mu\psi_0$ and $\varepsilon_X = \varepsilon\psi_0$.

Then by Lemma 7.17, there exists a uniquely determined morphism $\Psi_0 : \text{ASS}_0 \longrightarrow \text{END}_0(X)$ of set-operads such that the following diagram commutes.



The morphism Ψ_0 satisfies $(f\setminus a)\Psi_0^{\text{pre}} = (f^{\text{op}}\mathfrak{e}_0)\cdot(a\psi_0)$ for $m,n\in\mathbb{Z}_{\geqslant 0}$ and $f\setminus a\in\text{ASS}_0^{\text{pre}}(m,n)$.

So (X,Ψ_0) is an ASS_0-algebra.

We have $(\text{id}_2\setminus\mu)\Psi_0^{\text{pre}} = \mu\alpha_0\Psi_0^{\text{pre}} = \mu\psi_0 = \mu_X$ and $(\text{id}_0\setminus\varepsilon)\Psi_0^{\text{pre}} = \varepsilon\alpha_0\Psi_0^{\text{pre}} = \varepsilon\psi_0 = \varepsilon_X$. □

7.4 The linear operad ASS

Definition 7.19. Recall the set-preoperad $\mathrm{ASS}_0^{\mathrm{pre}}$; cf. Definition 7.10. Define the set-subpreoperad $\mathrm{ASS}_0^{\mathrm{pre,bij}} \subseteq \mathrm{ASS}_0^{\mathrm{pre}}$ as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ let

$$\mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n) = \big\{ f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n) : [1,m] \xrightarrow{f} [1,m] \text{ is a bijective map} \big\}.$$

We have to show that $\mathrm{ASS}_0^{\mathrm{pre,bij}}$ is closed under multiplication and composition of $\mathrm{ASS}_0^{\mathrm{pre}}$ and that $\mathrm{id}_{\mathrm{ASS}_0^{\mathrm{pre}},m} \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,m)$ for $m \in \mathbb{Z}_{\geqslant 0}$; cf. Lemma 2.27.

First note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have $\mathrm{id}_{\mathrm{ASS}_0^{\mathrm{pre}},m} = \mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Ass}_0,m} \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,m)$, since $\mathrm{id}_{\mathrm{Map}_0,m}$ is a bijective map.

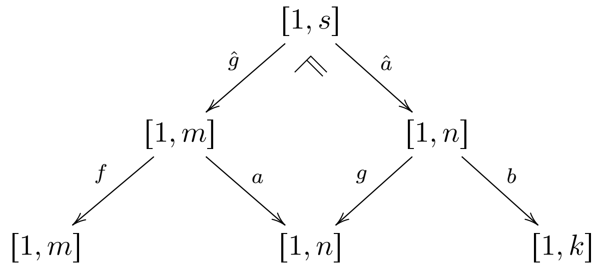
Now since $\mathrm{Sym}_0 \subseteq \mathrm{Map}_0$ is a set-subpreoperad, we know that $f \boxtimes_{\mathrm{Map}_0} f' \in \mathrm{Sym}_0(m+m', m+m')$ for $m, m' \in \mathbb{Z}_{\geqslant 0}$ and $f \in \mathrm{Sym}_0(m,m)$, $f' \in \mathrm{Sym}_0(m',m')$; cf. Lemma 2.27.

So for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)$, $f' \setminus a' \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m',n')$ we have

$$(f \setminus a) \boxtimes_{\mathrm{ASS}_0} (f' \setminus a') = (f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Ass}_0} a') \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m+m', n+n'),$$

since $f \boxtimes_{\mathrm{Map}_0} f'$ is a bijective map.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)$, $g \setminus b \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(n,k)$. Consider the following diagram.



So $([1,s], \hat{g}, \hat{a})$ is a sorted pullback of a and g. By Lemma 7.3, we know that $([1,s], \hat{g}, \hat{a})$ is in particular a pullback of a and g. Since g is bijective, by Lemma 1.35 (iii), the map $\hat{g}$ is also bijective. So we have

$$(f \setminus a) \cdot_{\mathrm{ASS}_0} (g \setminus b) = (\hat{g} f) \setminus (\hat{a} b) \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,k),$$

since $\hat{g} f$ is bijective as the composite of bijective maps.

This completes the proof that $\mathrm{ASS}_0^{\mathrm{pre,bij}}$ is a set-subpreoperad of $\mathrm{ASS}_0^{\mathrm{pre}}$.

Definition 7.20. We define the linear operad $\mathrm{ASS} := (\mathrm{ASS}^{\mathrm{pre}}, \mathfrak{a})$ as follows.

- Let $\mathrm{ASS}^{\mathrm{pre}} := R\,\mathrm{ASS}_0^{\mathrm{pre,bij}}$.
- Let $\mathfrak{a} := R\Big(\mathfrak{a}_0\big|_{\mathrm{Sym}_0^{\mathrm{op}}}^{\mathrm{ASS}_0^{\mathrm{pre,bij}}}\Big) : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{ASS}^{\mathrm{pre}}$.

Note that since we have $\mathrm{Im}(\mathfrak{a}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}) \subseteq \mathrm{ASS}_0^{\mathrm{pre,bij}}$, by Remark 6.26 this is in fact a linear operad. Recall that this definition means the following.

- We have $\mathrm{ASS}^{\mathrm{pre}}(m,n) := R\,\mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.
- We have $\mathrm{id}_{\mathrm{ASS},m} = \mathrm{id}_{\mathrm{ASS}_0,m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

- Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$.
  Then for $\sum\limits_{\xi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)} r_\xi \, \xi \in \mathrm{ASS}^{\mathrm{pre}}(m,n)$ and $\sum\limits_{\xi' \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m',n')} r'_{\xi'} \, \xi' \in \mathrm{ASS}^{\mathrm{pre}}(m',n')$ we have

$$\Bigg(\sum_{\xi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)} r_\xi \, \xi \Bigg) \boxtimes_{\mathrm{ASS}} \Bigg(\sum_{\xi' \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m',n')} r'_{\xi'} \, \xi' \Bigg) = \sum_{\substack{\xi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n) \\ \xi' \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m',n')}} r_\xi \, r'_{\xi'} \big(\xi \boxtimes_{\mathrm{ASS}_0} \xi' \big).$$

- Suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$.
  Then for $\sum\limits_{\xi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)} r_\xi \, \xi \in \mathrm{ASS}^{\mathrm{pre}}(m,n)$ and $\sum\limits_{\chi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(n,k)} s_{\chi'} \, \chi \in \mathrm{ASS}^{\mathrm{pre}}(n,k)$ we have

$$\Bigg(\sum_{\xi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)} r_\xi \, \xi \Bigg) \cdot_{\mathrm{ASS}} \Bigg(\sum_{\chi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(n,k)} s_\chi \, \chi \Bigg) = \sum_{\substack{\xi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n) \\ \chi \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(n,k)}} r_\xi \, s_\chi \big(\xi \cdot_{\mathrm{ASS}_0} \chi \big).$$

- For $m \in \mathbb{Z}_{\geqslant 0}$ and $\sum\limits_{f \in \mathrm{Sym}_0(m,m)} r_f \, f^{\mathrm{op}} \in \mathrm{Sym}^{\mathrm{op}}(m,m)$ we have

$$\Bigg(\sum_{f \in \mathrm{Sym}_0(m,m)} r_f \, f^{\mathrm{op}} \Bigg) \mathfrak{a} = \sum_{f \in \mathrm{Sym}_0(m,m)} r_f \, (f^{\mathrm{op}} \mathfrak{a}_0) = \sum_{f \in \mathrm{Sym}_0(m,m)} r_f \, (f \setminus \mathrm{id}_{\mathrm{Ass}_0, m}).$$

Definition 7.21. Recall the morphism of set-preoperads $\alpha_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}$; cf. Definition 7.15. For $m, n \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Ass}_0(m,n)$ we have $a\alpha_0 = \mathrm{id}_{\mathrm{Map}_0, m} \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)$, hence the restriction $\alpha_0\big|^{\mathrm{ASS}_0^{\mathrm{pre,bij}}} : \mathrm{Ass}_0 \longrightarrow \mathrm{ASS}_0^{\mathrm{pre,bij}}$ is well-defined, so it is a morphism of set-preoperads.

So we can define the morphism of linear preoperads $\alpha := R\Big(\alpha_0\big|^{\mathrm{ASS}_0^{\mathrm{pre,bij}}}\Big) : \mathrm{Ass} \longrightarrow \mathrm{ASS}^{\mathrm{pre}}$.

7.5 Associative algebras and ASS-algebras

Proposition 7.22. *Let V be an R-module and let $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ be a morphism of linear operads.*

$$\begin{array}{ccc} \mathrm{ASS}^{\mathrm{pre}} & \xrightarrow{\ \Psi^{\mathrm{pre}}\ } & \mathrm{End}(V) \\ & {\nwarrow^{\mathfrak{a}}} \quad {\nearrow_{\mathfrak{e}}} & \\ & \mathrm{Sym}^{\mathrm{op}} & \end{array}$$

That is, (V, Ψ) is an ASS-algebra.

Define $\mu_V := (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} \in \mathrm{End}(V)(2,1)$ and $\varepsilon_V := (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}} \in \mathrm{End}(V)(0,1)$, where we abbreviate $\mathrm{id}_m := \mathrm{id}_{\mathrm{Map}_0, m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

Then $(V, \mu_V, \varepsilon_V)$ is an associative R-algebra.

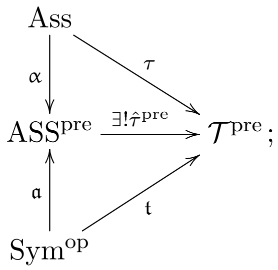
Proof. We have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Ass} & & \\ \downarrow{\scriptstyle \alpha} & \searrow^{\alpha\Psi^{\mathrm{pre}}} & \\ \mathrm{ASS}^{\mathrm{pre}} & \xrightarrow{\ \Psi^{\mathrm{pre}}\ } & \mathrm{End}(V) \\ \uparrow{\scriptstyle \mathfrak{a}} & \nearrow_{\mathfrak{e}} & \\ \mathrm{Sym}^{\mathrm{op}} & & \end{array}$$

So $\alpha\Psi^{\mathrm{pre}} : \mathrm{Ass} \longrightarrow \mathrm{END}(V)^{\mathrm{pre}} = \mathrm{End}(V)$ is a morphism of linear preoperads satisfying $\mu(\alpha\Psi^{\mathrm{pre}}) = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} = \mu_V$ and $\varepsilon(\alpha\Psi^{\mathrm{pre}}) = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}} = \varepsilon_V$.

So $(V, \alpha\Psi^{\mathrm{pre}})$ is an Ass-algebra. By Proposition 5.6 we know that $(V, \mu_V, \varepsilon_V)$ is an associative R-algebra. □

Lemma 7.23. *Let $\mathcal{T} = (\mathcal{T}, \mathfrak{t})$ be a linear operad. Let $\tau : \mathrm{Ass} \longrightarrow \mathcal{T}^{\mathrm{pre}}$ be a morphism of linear preoperads. Then there exists a uniquely determined morphism $\hat{\tau}^{\mathrm{pre}} : \mathrm{ASS}^{\mathrm{pre}} \longrightarrow \mathcal{T}^{\mathrm{pre}}$ of linear preoperads such that the following diagram commutes.*



cf. Definition 7.21.

In particular, the commutativity of the lower triangle means that $\hat{\tau}^{\mathrm{pre}}$ defines a morphism $\hat{\tau}$ of linear operads.

We have $(f \setminus a)\hat{\tau}^{\mathrm{pre}} = (f^{\mathrm{op}}\mathfrak{t}) \cdot_{\mathcal{T}} (a\tau)$ for $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m, n)$ and $m, n \in \mathbb{Z}_{\geqslant 0}$.

Proof. Uniqueness. Let $\tilde{\tau}^{\mathrm{pre}} : \mathrm{ASS}^{\mathrm{pre}} \longrightarrow \mathcal{T}^{\mathrm{pre}}$ be a morphism of linear preoperads such that $\alpha\tilde{\tau}^{\mathrm{pre}} = \tau$ and $\mathfrak{a}\tilde{\tau}^{\mathrm{pre}} = \mathfrak{t}$. Then for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m, n)$, where $f : [1, m] \longrightarrow [1, m]$ and $a : [1, m] \longrightarrow [1, n]$, we have

$$(f \setminus a)\tilde{\tau}^{\mathrm{pre}} = \big((f \setminus \mathrm{id}_m) \cdot (\mathrm{id}_m \setminus a)\big)\tilde{\tau}^{\mathrm{pre}} = \big((f^{\mathrm{op}}\mathfrak{a}) \cdot (a\alpha)\big)\tilde{\tau}^{\mathrm{pre}} = (f^{\mathrm{op}}\mathfrak{a}\tilde{\tau}^{\mathrm{pre}}) \cdot_{\mathcal{T}} (a\alpha\tilde{\tau}^{\mathrm{pre}}) = (f^{\mathrm{op}}\mathfrak{t}) \cdot_{\mathcal{T}} (a\tau).$$

So such a morphism of linear preoperads $\tilde{\tau}^{\mathrm{pre}}$ is uniquely determined by $\mathfrak{t}$ and τ.

Existence. To define $\hat{\tau}^{\mathrm{pre}} : \mathrm{ASS}^{\mathrm{pre}} \longrightarrow \mathcal{T}^{\mathrm{pre}}$ it suffices to define its restriction to $\mathrm{ASS}_0^{\mathrm{pre,bij}}$ as a morphism of set-preoperads; cf. Remark 2.23. So let

$$(f \setminus a)\hat{\tau}^{\mathrm{pre}} := (f^{\mathrm{op}}\mathfrak{t}) \cdot_{\mathcal{T}} (a\tau)$$

for $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m, n)$ and $m, n \in \mathbb{Z}_{\geqslant 0}$.

First note that for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m, n)$ we have $f^{\mathrm{op}}\mathfrak{t} \in \mathcal{T}(m, m)$ and $a\tau \in \mathcal{T}(m, n)$, so $(f^{\mathrm{op}}\mathfrak{t}) \cdot_{\mathcal{T}} (a\tau) \in \mathcal{T}(m, n)$.

Furthermore, note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have

$$\mathrm{id}_m \hat{\tau}^{\mathrm{pre}} = \big(\mathrm{id}_{\mathrm{Map}_0, m} \setminus \mathrm{id}_{\mathrm{Ass}_0, m}\big)\hat{\tau}^{\mathrm{pre}} = \big(\mathrm{id}^{\mathrm{op}}_{\mathrm{Map}_0, m}\,\mathfrak{t}\big) \cdot_{\mathcal{T}} \big(\mathrm{id}_{\mathrm{Ass}_0, m}\,\tau\big) = \mathrm{id}_{\mathcal{T}, m} \cdot_{\mathcal{T}} \mathrm{id}_{\mathcal{T}, m} = \mathrm{id}_{\mathcal{T}, m} \ .$$

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m, n)$, $f' \setminus a' \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m', n')$. We have

$$\begin{aligned}
\big((f \setminus a) \boxtimes (f' \setminus a')\big)\hat{\tau}^{\mathrm{pre}} &= \big((f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Ass}_0} a')\big)\hat{\tau}^{\mathrm{pre}} \\
&= \big((f \boxtimes_{\mathrm{Sym}_0} f') \setminus (a \boxtimes_{\mathrm{Ass}_0} a')\big)\hat{\tau}^{\mathrm{pre}} \\
&= \big((f \boxtimes_{\mathrm{Sym}_0} f')^{\mathrm{op}}\mathfrak{t}\big) \cdot_{\mathcal{T}} \big((a \boxtimes_{\mathrm{Ass}_0} a')\tau\big) \\
&= \big((f^{\mathrm{op}} \boxtimes_{\mathrm{Sym}_0^{\mathrm{op}}} f'^{\,\mathrm{op}})\mathfrak{t}\big) \cdot_{\mathcal{T}} \big((a \boxtimes_{\mathrm{Ass}_0} a')\tau\big) \\
&= \big((f^{\mathrm{op}}\mathfrak{t}) \boxtimes_{\mathcal{T}} (f'^{\,\mathrm{op}}\mathfrak{t})\big) \cdot_{\mathcal{T}} \big((a\tau) \boxtimes_{\mathcal{T}} (a'\tau)\big) \\
&= \big((f^{\mathrm{op}}\mathfrak{t}) \cdot_{\mathcal{T}} (a\tau)\big) \boxtimes_{\mathcal{T}} \big((f'^{\,\mathrm{op}}\mathfrak{t}) \cdot_{\mathcal{T}} (a'\tau)\big) \\
&= (f \setminus a)\hat{\tau}^{\mathrm{pre}} \boxtimes_{\mathcal{T}} (f' \setminus a')\hat{\tau}^{\mathrm{pre}}.
\end{aligned}$$

Now suppose given $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m, n)$, $g \setminus b \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(n, p)$. So we have $f : [1, m] \longrightarrow [1, m]$, $a : [1, m] \longrightarrow [1, n]$, $g : [1, n] \longrightarrow [1, n]$ and $b : [1, n] \longrightarrow [1, p]$.

Since a is monotone we can write

$$a = a_1 \boxtimes_{\mathrm{Ass}_0} \ldots \boxtimes_{\mathrm{Ass}_0} a_n = \underset{i \in [1,n]}{\boxtimes} a_i \,,$$

where we omit the index "Ass_0" and where $a_i = \mu_{l_i} \in \mathrm{Ass}_0(l_i, 1)$ where $l_i \in [0, m]$ for $i \in [1, n]$ and $\sum\limits_{i \in [1,n]} l_i = m$; cf. Remark 4.30. Write $l = (l_i)_{i \in [1,n]} \in (\mathbb{Z}_{\geqslant 0})^{\times n}$ and $r = (1, \ldots, 1) \in (\mathbb{Z}_{\geqslant 0})^{\times n}$.

By Lemma 7.9 we have the following sorted pullback.

$$\begin{array}{ccc}
\Big[1, \sum\limits_{j \in [1,n]} l_{jg}\Big] & \xrightarrow{\;\underset{j \in [1,n]}{\boxtimes} a_{jg}\;} & [1, n] \\
{\scriptstyle g_{[l]}} \Big\downarrow & & \Big\downarrow {\scriptstyle g} \\
[1, m] & \xrightarrow[\;a\;]{} & [1, n]
\end{array}$$

So by the definition of composition in $\mathrm{ASS}_0^{\mathrm{pre,bij}}$ we have

$$(f \setminus a) \cdot (g \setminus b) = \big(g_{[l]} f\big) \setminus \Big(\Big(\underset{j \in [1,n]}{\boxtimes} a_{jg}\Big) b\Big);$$

cf. Definition 7.10. So we have

$$\begin{array}{rcl}
\big((f \setminus a) \cdot (g \setminus b)\big)\hat{\tau}^{\mathrm{pre}} & = & \Big(\big(g_{[l]} f\big) \setminus \Big(\Big(\underset{j \in [1,n]}{\boxtimes} a_{jg}\Big) b\Big)\Big)\hat{\tau}^{\mathrm{pre}} \\
& = & \big(g_{[l]} f\big)^{\mathrm{op}} \mathfrak{t} \cdot_{\mathcal{T}} \Big(\Big(\underset{j \in [1,n]}{\boxtimes} a_{jg}\Big) b\Big)\tau \\
& = & \big(f^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \big(\big(g_{[l]}\big)^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \Big(\Big(\underset{j \in [1,n]}{\boxtimes} a_{jg}\Big)\tau\Big) \cdot_{\mathcal{T}} \big(b\tau\big) \\
& = & \big(f^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \big(\big(g_{[l]}\big)^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \Big(\underset{j \in [1,n]}{\boxtimes} (a_{jg}\tau)\Big) \cdot_{\mathcal{T}} \big(b\tau\big) \\
& \overset{6.25}{=} & \big(f^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \Big(\underset{i \in [1,n]}{\boxtimes} (a_i\tau)\Big) \cdot_{\mathcal{T}} \big(\big(g_{[r]}\big)^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \big(b\tau\big) \\
& \overset{6.9\,\mathrm{(iv)}}{=} & \big(f^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \Big(\Big(\underset{i \in [1,n]}{\boxtimes} a_i\Big)\tau\Big) \cdot_{\mathcal{T}} \big(g^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \big(b\tau\big) \\
& = & \big(f^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \big(a\tau\big) \cdot_{\mathcal{T}} \big(g^{\mathrm{op}} \mathfrak{t}\big) \cdot_{\mathcal{T}} \big(b\tau\big) \\
& = & (f \setminus a)\hat{\tau}^{\mathrm{pre}} \cdot_{\mathcal{T}} (g \setminus b)\hat{\tau}^{\mathrm{pre}} \,.
\end{array}$$

This shows that $\hat{\tau}^{\mathrm{pre}}$ is in fact a morphism of linear preoperads.

Since by construction $f^{\mathrm{op}} \mathfrak{a} \hat{\tau}^{\mathrm{pre}} = (f \setminus \mathrm{id}_{\mathrm{Ass}_0, m})\hat{\tau}^{\mathrm{pre}} = (f^{\mathrm{op}} \mathfrak{t}) \cdot_{\mathcal{T}} \mathrm{id}_{\mathcal{T}, m} = f^{\mathrm{op}} \mathfrak{t}$ for $m \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m, m)$, we have $\mathfrak{a} \hat{\tau}^{\mathrm{pre}} = \mathfrak{t}$. So $\hat{\tau} : \mathrm{ASS} \longrightarrow \mathcal{T}$ is a morphism of linear operads.

Moreover, note that we have $a \alpha \hat{\tau}^{\mathrm{pre}} = (\mathrm{id}_{\mathrm{Map}_0, m} \setminus a)\hat{\tau}^{\mathrm{pre}} = \mathrm{id}_{\mathcal{T}, m} \cdot_{\mathcal{T}} (a\tau) = a\tau$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $a \in \mathrm{Ass}_0(m, n)$. Hence $\alpha \hat{\tau}^{\mathrm{pre}} = \tau$. □

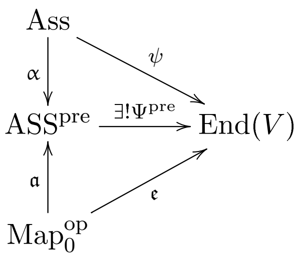
Proposition 7.24. *Let $(V, \mu_V, \varepsilon_V)$ be an associative R-algebra.*

Then there exists a morphism of linear operads $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ such that $\mu_V = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}}$ and $\varepsilon_V = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}}$.

So (V, Ψ) is an ASS*-algebra.*

Proof. By Proposition 5.7 there exists the morphism of linear preoperads $\psi : \mathrm{Ass} \longrightarrow \mathrm{End}(V)$ that satisfies $\mu_V = \mu\psi$ and $\varepsilon_V = \varepsilon\psi$. So (V, ψ) is an Ass-algebra.

Then by Lemma 7.23, there exists a uniquely determined morphism $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ of linear operads such that the following diagram commutes.



The morphism Ψ satisfies $(f \setminus a)\Psi^{\mathrm{pre}} = (f^{\mathrm{op}}\mathfrak{e}) \cdot (a\psi)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m, n)$.

So (V, Ψ) is an ASS-algebra. Furthermore, we have $(\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} = \mu\alpha\Psi^{\mathrm{pre}} = \mu\psi = \mu_V$ and $(\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}} = \varepsilon\alpha\Psi^{\mathrm{pre}} = \varepsilon\psi = \varepsilon_V$. □

8 The linear operad BIALG

Our aim in this chapter will be to define a linear operad BIALG with the property that BIALG-algebras are R-bialgebras.

Definition 8.1. Recall the set-operad $\mathrm{ASS}_0 = (\mathrm{ASS}_0^{\mathrm{pre}}, \mathfrak{a}_0)$; cf. Definitions 7.10 and 7.12. We define the linear operad $\mathrm{BIALG} := (\mathrm{BIALG}^{\mathrm{pre}}, \mathfrak{b})$ as follows. We let $\mathrm{BIALG}^{\mathrm{pre}} := R\,\mathrm{ASS}_0^{\mathrm{pre}}$ and $\mathfrak{b} := R\big(\mathfrak{a}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}\big) : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{BIALG}^{\mathrm{pre}}$. This is a linear operad by Remark 6.27.

Recall that this definition means that we have $\mathrm{BIALG}(m,n) = R\,\mathrm{ASS}_0(m,n)$ for $m,n \in \mathbb{Z}_{\geqslant 0}$ and that for $m \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)$ we have $f^{\mathrm{op}}\mathfrak{b} = f^{\mathrm{op}}\mathfrak{a}_0 = f \setminus \mathrm{id}_{\mathrm{Ass}_0,m}$. Furthermore, recall that this means that multiplication and composition in BIALG work as follows.

- For $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $\sum\limits_{\xi \in \mathrm{ASS}_0(m,n)} r_\xi\, \xi \in \mathrm{BIALG}(m,n)$, $\sum\limits_{\xi' \in \mathrm{ASS}_0(m',n')} r'_{\xi'}\, \xi' \in \mathrm{BIALG}(m',n')$ we have
$$\Big(\sum_{\xi \in \mathrm{ASS}_0(m,n)} r_\xi\, \xi\Big) \boxtimes_{\mathrm{BIALG}} \Big(\sum_{\xi' \in \mathrm{ASS}_0(m',n')} r'_{\xi'}\, \xi'\Big) = \sum_{\substack{\xi \in \mathrm{ASS}_0(m,n)\\ \xi' \in \mathrm{ASS}_0(m',n')}} r_\xi\, r'_{\xi'}\, \big(\xi \boxtimes_{\mathrm{ASS}_0} \xi'\big).$$

- For $m,n,p \in \mathbb{Z}_{\geqslant 0}$ and $\sum\limits_{\xi \in \mathrm{ASS}_0(m,n)} r_\xi\, \xi \in \mathrm{BIALG}(m,n)$, $\sum\limits_{\chi \in \mathrm{ASS}_0(n,p)} s_\chi\, \chi \in \mathrm{BIALG}(n,p)$ we have
$$\Big(\sum_{\xi \in \mathrm{ASS}_0(m,n)} r_\xi\, \xi\Big) \cdot_{\mathrm{BIALG}} \Big(\sum_{\chi \in \mathrm{ASS}_0(n,p)} s_\chi\, \chi\Big) = \sum_{\substack{\xi \in \mathrm{ASS}_0(m,n)\\ \chi \in \mathrm{ASS}_0(n,p)}} r_\xi\, s_\chi\, \big(\xi \cdot_{\mathrm{ASS}_0} \chi\big).$$

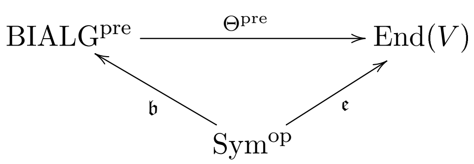
Remark 8.2. Recall the linear operad $\mathrm{ASS} = (\mathrm{ASS}^{\mathrm{pre}}, \mathfrak{a})$; cf. Definition 7.20. Note that $\mathrm{ASS} \subseteq \mathrm{BIALG}$ is a linear suboperad with $\mathfrak{a} = R\Big(\mathfrak{a}_0\big|_{\mathrm{Sym}_0^{\mathrm{op}}}^{\mathrm{ASS}_0^{\mathrm{pre,bij}}}\Big) = \mathfrak{b}\big|^{\mathrm{ASS}^{\mathrm{pre}}}$.

Both ASS and BIALG arise from ASS_0. The linear preoperad $\mathrm{BIALG}^{\mathrm{pre}}$ consists of formal linear combinations of elements from $\mathrm{ASS}_0^{\mathrm{pre}}$, whereas in $\mathrm{ASS}^{\mathrm{pre}}$ we only allow formal linear combinations of fractions from $\mathrm{ASS}_0^{\mathrm{pre}}$ with bijective denominators.

Proposition 8.3. *Let V be an R-module. Suppose given a morphism of linear operads*

$$\Theta : \mathrm{BIALG} \longrightarrow \mathrm{END}(V),$$

that is, (V, Θ) is a BIALG*-algebra.*



Define

$$\begin{aligned}
\mu_V &:= (\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}} \in \mathrm{End}(V)(2,1)\\
\varepsilon_V &:= (\mathrm{id}_0 \setminus \varepsilon)\Theta^{\mathrm{pre}} \in \mathrm{End}(V)(0,1)\\
\Delta_V &:= (\mu \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} \in \mathrm{End}(V)(1,2)\\
\eta_V &:= (\varepsilon \setminus \mathrm{id}_0)\Theta^{\mathrm{pre}} \in \mathrm{End}(V)(1,0),
\end{aligned}$$

where we abbreviate $\mathrm{id}_m := \mathrm{id}_{\mathrm{Map}_0,m} = \mathrm{id}_{\mathrm{Ass}_0,m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

Then V is an R bialgebra with multiplication μ_V, unit ε_V, comultiplication Δ_V and counit η_V; cf. **[4,** Definition 4.1.3**]**.

Proof. During this proof we will denote by $(\boxtimes)$ and $(\cdot)$ the multiplication and composition in BIALG and in ASS_0.

Recall that we have $(\mu \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} \mu = (\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} \mu) \cdot_{\mathrm{Map}_0} \mu$, $(\varepsilon \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} \mu = \mathrm{id}_1$ and $(\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} \varepsilon) \cdot_{\mathrm{Map}_0} \mu = \mathrm{id}_1$. Furthermore, recall the morphism of set-preoperads $\alpha_0 : \mathrm{Ass}_0 \longrightarrow \mathrm{ASS}_0^{\mathrm{pre}}$ that maps $a \in \mathrm{Ass}_0(m,n)$ to $\mathrm{id}_m \setminus a \in \mathrm{ASS}_0(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$; cf. Definition 7.15.

So since Θ is a morphism of linear operads and since α_0 is a morphism of set-preoprads, we have

$$
\begin{aligned}
(\mu_V \otimes \mathrm{id}_V)\mu_V &= \big((\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}} \otimes (\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}}\big) \cdot_{\mathrm{END}} (\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}} \\
&= \big(((\mathrm{id}_2 \setminus \mu) \boxtimes (\mathrm{id}_1 \setminus \mathrm{id}_1)) \cdot (\mathrm{id}_2 \setminus \mu)\big)\Theta^{\mathrm{pre}} \\
&= \big((\mu\alpha_0 \boxtimes_{\mathrm{ASS}_0} \mathrm{id}_1\, \alpha_0) \cdot \mu\alpha_0\big)\Theta^{\mathrm{pre}} \\
&= \big(((\mu \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} \mu)\alpha_0\big)\Theta^{\mathrm{pre}} \\
&= \big(((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} \mu) \cdot_{\mathrm{Map}_0} \mu)\alpha_0\big)\Theta^{\mathrm{pre}} \\
&= \big((\mathrm{id}_1\, \alpha_0 \boxtimes \mu\alpha_0) \cdot \mu\alpha_0\big)\Theta^{\mathrm{pre}} \\
&= \big(((\mathrm{id}_1 \setminus \mathrm{id}_1) \boxtimes (\mathrm{id}_2 \setminus \mu)) \cdot (\mathrm{id}_2 \setminus \mu)\big)\Theta^{\mathrm{pre}} \\
&= \big((\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}} \otimes (\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}}\big) \cdot_{\mathrm{END}} (\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}} \\
&= (\mathrm{id}_V \otimes \mu_V)\mu_V \\
(\varepsilon_V \otimes \mathrm{id}_V)\mu_V &= \big((\mathrm{id}_0 \setminus \varepsilon)\Theta^{\mathrm{pre}} \otimes (\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}}\big) \cdot_{\mathrm{END}} (\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}} \\
&= \big((\varepsilon\alpha_0 \boxtimes \mathrm{id}_1\, \alpha_0) \cdot \mu\alpha_0\big)\Theta^{\mathrm{pre}} \\
&= \big(((\varepsilon \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} \mu)\alpha_0\big)\Theta^{\mathrm{pre}} \\
&= (\mathrm{id}_1\, \alpha_0)\Theta^{\mathrm{pre}} \\
&= (\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}} \\
&= \mathrm{id}_V \\
(\mathrm{id}_V \otimes \varepsilon_V)\mu_V &= \big((\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}} \otimes (\mathrm{id}_0 \setminus \varepsilon)\Theta^{\mathrm{pre}}\big) \cdot_{\mathrm{END}} (\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}} \\
&= \big((\mathrm{id}_1\, \alpha_0 \boxtimes \varepsilon\alpha_0) \cdot \mu\alpha_0\big)\Theta^{\mathrm{pre}} \\
&= \big(((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} \varepsilon) \cdot_{\mathrm{Map}_0} \mu)\alpha_0\big)\Theta^{\mathrm{pre}} \\
&= (\mathrm{id}_1\, \alpha_0)\Theta^{\mathrm{pre}} \\
&= (\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}} \\
&= \mathrm{id}_V\,.
\end{aligned}
$$

This shows that $(V, \mu_V, \varepsilon_V)$ is an associative R-algebra.

Moreover, since Θ is a morphism of linear operads and since $\mathfrak{a}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{ASS}_0$ is a morphism of set-preoperads, we have

$$
\begin{aligned}
\Delta_V(\mathrm{id}_V \otimes \Delta_V) &= (\mu \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} \cdot_{\mathrm{END}} \big((\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}} \otimes (\mu \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}}\big) \\
&= \big(\mu^{\mathrm{op}}\mathfrak{a}_0 \cdot (\mathrm{id}_1^{\mathrm{op}}\, \mathfrak{a}_0 \boxtimes \mu^{\mathrm{op}}\mathfrak{a}_0)\big)\Theta^{\mathrm{pre}} \\
&= \big((\mu^{\mathrm{op}} \cdot_{\mathrm{Map}^{\mathrm{op}}} (\mathrm{id}_1^{\mathrm{op}} \boxtimes_{\mathrm{Map}^{\mathrm{op}}} \mu^{\mathrm{op}}))\mathfrak{a}_0\big)\Theta^{\mathrm{pre}} \\
&= \big(((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} \mu) \cdot_{\mathrm{Map}_0} \mu)^{\mathrm{op}}\mathfrak{a}_0\big)\Theta^{\mathrm{pre}} \\
&= \big(((\mu \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} \mu)^{\mathrm{op}}\mathfrak{a}_0\big)\Theta^{\mathrm{pre}} \\
&= \big((\mu^{\mathrm{op}} \cdot_{\mathrm{Map}_0^{\mathrm{op}}} (\mu^{\mathrm{op}} \boxtimes_{\mathrm{Map}_0^{\mathrm{op}}} \mathrm{id}_1^{\mathrm{op}}))\mathfrak{a}_0\big)\Theta^{\mathrm{pre}} \\
&= \big(\mu^{\mathrm{op}}\mathfrak{a}_0 \cdot (\mu^{\mathrm{op}}\mathfrak{a}_0 \boxtimes \mathrm{id}_1^{\mathrm{op}}\, \mathfrak{a}_0)\big)\Theta^{\mathrm{pre}} \\
&= (\mu \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} \cdot_{\mathrm{END}} \big((\mu \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} \otimes (\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}}\big) \\
&= \Delta_V(\Delta_V \otimes \mathrm{id}_V)
\end{aligned}
$$

and

$$\begin{aligned}
\Delta_V(\eta_V \otimes \mathrm{id}_V) &= (\mu \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} \cdot_{\mathrm{END}} \big((\varepsilon \setminus \mathrm{id}_0)\Theta^{\mathrm{pre}} \otimes (\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}}\big)\\
&= \big(\mu^{\mathrm{op}}\mathfrak{a}_0 \cdot (\varepsilon^{\mathrm{op}}\mathfrak{a}_0 \boxtimes \mathrm{id}_1^{\mathrm{op}}\,\mathfrak{a}_0)\big)\Theta^{\mathrm{pre}}\\
&= \big((\mu^{\mathrm{op}} \cdot_{\mathrm{Map}_0^{\mathrm{op}}} (\varepsilon^{\mathrm{op}} \boxtimes_{\mathrm{Map}_0^{\mathrm{op}}} \mathrm{id}_1^{\mathrm{op}}))\mathfrak{a}_0\big)\Theta^{\mathrm{pre}}\\
&= \big(((\varepsilon \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} \mu)^{\mathrm{op}}\mathfrak{a}_0\big)\Theta^{\mathrm{pre}}\\
&= (\mathrm{id}_1^{\mathrm{op}}\,\mathfrak{a}_0)\Theta^{\mathrm{pre}}\\
&= (\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}}\\
&= \mathrm{id}_V\\
\Delta_V(\mathrm{id}_V \otimes \eta_V) &= (\mu \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} \cdot_{\mathrm{END}} \big((\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}} \otimes (\varepsilon \setminus \mathrm{id}_0)\Theta^{\mathrm{pre}}\big)\\
&= \big(\mu^{\mathrm{op}}\mathfrak{a}_0 \cdot (\mathrm{id}_1^{\mathrm{op}}\,\mathfrak{a}_0 \boxtimes \varepsilon^{\mathrm{op}}\mathfrak{a}_0)\big)\Theta^{\mathrm{pre}}\\
&= \big((\mu^{\mathrm{op}} \cdot_{\mathrm{Map}_0^{\mathrm{op}}} (\mathrm{id}_1^{\mathrm{op}} \boxtimes_{\mathrm{Map}_0^{\mathrm{op}}} \varepsilon^{\mathrm{op}}))\mathfrak{a}_0\big)\Theta^{\mathrm{pre}}\\
&= \big(((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} \varepsilon) \cdot_{\mathrm{Map}_0} \mu)^{\mathrm{op}}\mathfrak{a}_0\big)\Theta^{\mathrm{pre}}\\
&= (\mathrm{id}_1^{\mathrm{op}}\,\mathfrak{a}_0)\Theta^{\mathrm{pre}}\\
&= (\mathrm{id}_1 \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}}\\
&= \mathrm{id}_V\,.
\end{aligned}$$

This shows that (V, Δ_V, η_V) is a coassociative R-coalgebra.

In order to complete the proof that $(V, \mu_V, \varepsilon_V, \Delta_V, \eta_V)$ is a bialgebra, we have to show that the following compatibility conditions (i) – (iv) are satisfied.

(i) We have
$$\mu_V \cdot_{\mathrm{END}} \Delta_V = (\Delta_V \otimes \Delta_V) \cdot_{\mathrm{END}} (\mathrm{id}_V \otimes \tau_V \otimes \mathrm{id}_V) \cdot_{\mathrm{END}} (\mu_V \otimes \mu_V),$$
where $\tau_V \in \mathrm{END}(V)(2,2)$ is the linear map defined by
$$\begin{aligned}\tau_V :\quad V^{\otimes 2} &\longrightarrow V^{\otimes 2}\\ v \otimes w &\longmapsto w \otimes v\end{aligned}$$
for $v, w \in V$.

(ii) We have $\mu_V \cdot_{\mathrm{END}} \eta_V = \eta_V \otimes \eta_V$.

(iii) We have $\varepsilon_V \cdot_{\mathrm{END}} \Delta_V = \varepsilon_V \otimes \varepsilon_V$.

(iv) We have $\varepsilon_V \cdot_{\mathrm{END}} \eta_V = \mathrm{id}_R$.

Ad (i). Consider the transposition $(1,2) \in \mathrm{Sym}_0(2,2)$. Since Θ is a morphism of linear operads, we have
$$((1,2) \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} = (1,2)^{\mathrm{op}}\mathfrak{b}\Theta^{\mathrm{pre}} = (1,2)^{\mathrm{op}}\mathfrak{e}.$$
Moreover, we know that $(1,2)^{\mathrm{op}}\mathfrak{e} \in \mathrm{END}(V)(2,2)$ is the map
$$\begin{aligned}(1,2)^{\mathrm{op}}\mathfrak{e} :\quad V^{\otimes 2} &\longrightarrow V^{\otimes 2}\\ v \otimes w &\longmapsto w \otimes v;\end{aligned}$$
cf. Example 2.66. So we have
$$((1,2) \setminus \mathrm{id}_1)\Theta^{\mathrm{pre}} = (1,2)^{\mathrm{op}}\mathfrak{e} = \tau_V\,.$$

We have to show that
$$\mu_V \cdot_{\mathrm{END}} \Delta_V \overset{!}{=} (\Delta_V \otimes \Delta_V) \cdot_{\mathrm{END}} (\mathrm{id}_V \otimes \tau_V \otimes \mathrm{id}_V) \cdot_{\mathrm{END}} (\mu_V \otimes \mu_V).$$

On the one hand we have

$$\begin{aligned}\mu_V \cdot_{\mathrm{END}} \Delta_V &= (\mathrm{id}_2 \setminus \mu)\Theta^{\mathrm{pre}} \cdot_{\mathrm{END}} (\mu \setminus \mathrm{id}_2)\Theta^{\mathrm{pre}} \\ &= ((\mathrm{id}_2 \setminus \mu) \cdot (\mu \setminus \mathrm{id}_2))\Theta.\end{aligned}$$

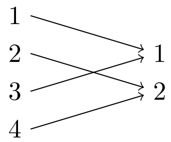
On the other hand we have

$$\begin{aligned}&(\Delta_V \otimes \Delta_V) \cdot_{\mathrm{END}} (\mathrm{id}_V \otimes \tau_V \otimes \mathrm{id}_V) \cdot_{\mathrm{END}} (\mu_V \otimes \mu_V) \\ &\quad = (((\mu \setminus \mathrm{id}_2) \boxtimes (\mu \setminus \mathrm{id}_2)) \cdot ((\mathrm{id}_1 \setminus \mathrm{id}_1) \boxtimes ((1,2) \setminus \mathrm{id}_2) \boxtimes (\mathrm{id}_1 \setminus \mathrm{id}_1)) \cdot ((\mathrm{id}_2 \setminus \mu) \boxtimes (\mathrm{id}_2 \setminus \mu)))\Theta^{\mathrm{pre}}.\end{aligned}$$

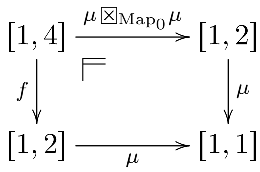
So since Θ^{pre} is a morphism of linear preoperads it suffices to show that

$$(\mathrm{id}_2 \setminus \mu) \cdot (\mu \setminus \mathrm{id}_2) \overset{!}{=} ((\mu \setminus \mathrm{id}_2) \boxtimes (\mu \setminus \mathrm{id}_2)) \cdot ((\mathrm{id}_1 \setminus \mathrm{id}_1) \boxtimes ((1,2) \setminus \mathrm{id}_2) \boxtimes (\mathrm{id}_1 \setminus \mathrm{id}_1)) \cdot ((\mathrm{id}_2 \setminus \mu) \boxtimes (\mathrm{id}_2 \setminus \mu)).$$

Let $f \in \mathrm{Map}_0(4,2)$ be the map defined by $1f = 1$, $2f = 2$, $3f = 1$, $4f = 2$.



Claim. We have the following sorted pullback.



Proof of the Claim. First note that we have $f \cdot_{\mathrm{Map}_0} \mu, (\mu \boxtimes_{\mathrm{Map}_0} \mu) \cdot_{\mathrm{Map}_0} \mu \in \mathrm{Map}_0(4,1) = \{\mu_4\}$; cf. Definition 4.29. So the maps have to be the same and the diagram commutes. Furthermore, $\mu \boxtimes_{\mathrm{Map}_0} \mu$ is monotone as the product of monotone maps.

Finally, $f\big|^{\mu^{-1}(1\mu)}_{(\mu \boxtimes_{\mathrm{Map}_0} \mu)^{-1}(1)} = f\big|^{[1,2]}_{[1,2]}$ and $f\big|^{\mu^{-1}(2\mu)}_{(\mu \boxtimes_{\mathrm{Map}_0} \mu)^{-1}(2)} = f\big|^{[1,2]}_{[3,4]}$ are isotone.

This proves the *Claim.*

So we have

$$(\mathrm{id}_2 \setminus \mu) \cdot (\mu \setminus \mathrm{id}_2) \overset{7.10}{=} (f \, \mathrm{id}_2) \setminus \big((\mu \boxtimes_{\mathrm{Map}_0} \mu) \, \mathrm{id}_2\big) = f \setminus \big(\mu \boxtimes_{\mathrm{Map}_0} \mu\big).$$

On the other hand, by Remark 7.11 we have

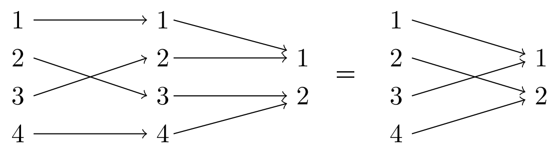
$$\begin{aligned}&\big((\mu \setminus \mathrm{id}_2) \boxtimes (\mu \setminus \mathrm{id}_2)\big) \cdot \big((\mathrm{id}_1 \setminus \mathrm{id}_1) \boxtimes ((1,2) \setminus \mathrm{id}_2) \boxtimes (\mathrm{id}_1 \setminus \mathrm{id}_1)\big) \cdot \big((\mathrm{id}_2 \setminus \mu) \boxtimes (\mathrm{id}_2 \setminus \mu)\big) \\ &\qquad = \big((\mu \boxtimes_{\mathrm{Map}_0} \mu) \setminus \mathrm{id}_4\big) \cdot \big((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} (1,2) \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \setminus \mathrm{id}_4\big) \cdot \big(\mathrm{id}_4 \setminus (\mu \boxtimes_{\mathrm{Map}_0} \mu)\big) \\ &\qquad = \big(((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} (1,2) \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} (\mu \boxtimes_{\mathrm{Map}_0} \mu)) \setminus \mathrm{id}_4\big) \cdot \big(\mathrm{id}_4 \setminus (\mu \boxtimes_{\mathrm{Map}_0} \mu)\big) \\ &\qquad = \big((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} (1,2) \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} (\mu \boxtimes_{\mathrm{Map}_0} \mu)\big) \setminus (\mu \boxtimes_{\mathrm{Map}_0} \mu).\end{aligned}$$

So it suffices to show that

$$(\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} (1,2) \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} (\mu \boxtimes_{\mathrm{Map}_0} \mu) \overset{!}{=} f.$$

We have

$$\begin{aligned}1\big((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} (1,2) \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} (\mu \boxtimes_{\mathrm{Map}_0} \mu)\big) &= 1(\mu \boxtimes_{\mathrm{Map}_0} \mu) = 1\mu = 1 = 1f \\ 2\big((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} (1,2) \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} (\mu \boxtimes_{\mathrm{Map}_0} \mu)\big) &= 3(\mu \boxtimes_{\mathrm{Map}_0} \mu) = 1\mu + 1 = 2 = 2f \\ 3\big((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} (1,2) \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} (\mu \boxtimes_{\mathrm{Map}_0} \mu)\big) &= 2(\mu \boxtimes_{\mathrm{Map}_0} \mu) = 2\mu = 1 = 3f \\ 4\big((\mathrm{id}_1 \boxtimes_{\mathrm{Map}_0} (1,2) \boxtimes_{\mathrm{Map}_0} \mathrm{id}_1) \cdot_{\mathrm{Map}_0} (\mu \boxtimes_{\mathrm{Map}_0} \mu)\big) &= 4(\mu \boxtimes_{\mathrm{Map}_0} \mu) = 2\mu + 1 = 2 = 4f.\end{aligned}$$



This shows that compatibility condition (i) is satisfied.

Ad (ii). We have to show that

$$\big((\mathrm{id}_2 \setminus \mu) \cdot (\varepsilon \setminus \mathrm{id}_0)\big)\Theta^{\mathrm{pre}} = \mu_V \cdot_{\mathrm{END}} \eta_V \overset{!}{=} \eta_V \otimes \eta_V = \big((\varepsilon \setminus \mathrm{id}_0) \boxtimes (\varepsilon \setminus \mathrm{id}_0)\big)\Theta^{\mathrm{pre}}.$$

So it suffices to show that

$$(\mathrm{id}_2 \setminus \mu) \cdot (\varepsilon \setminus \mathrm{id}_0) \overset{!}{=} (\varepsilon \setminus \mathrm{id}_0) \boxtimes (\varepsilon \setminus \mathrm{id}_0).$$

Claim. We have the following sorted pullback.

$$\begin{array}{ccc} [1,0] & \xrightarrow{\mathrm{id}_0} & [1,0] \\ {\scriptstyle \varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon}\downarrow & \ulcorner & \downarrow{\scriptstyle \varepsilon} \\ [1,2] & \xrightarrow[\mu]{} & [1,1] \end{array}$$

Proof of the Claim. First note that $(\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon) \cdot_{\mathrm{Map}_0} \mu$, $\mathrm{id}_0 \cdot_{\mathrm{Map}_0} \varepsilon \in \mathrm{Map}_0(0,1) = \{\varepsilon\}$, so the diagram commutes. Furthermore, the map id_0 is monotone.

Finally, $(\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon)\big|^{\mu^{-1}(i\varepsilon)}_{\mathrm{id}_0^{-1}(i)}$ is isotone for $i \in [1,0] = \varnothing$.

This completes the proof of the *Claim.*

So we have

$$(\mathrm{id}_2 \setminus \mu) \cdot (\varepsilon \setminus \mathrm{id}_0) \overset{7.10}{=} \big((\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon) \cdot_{\mathrm{Map}_0} \mathrm{id}_2\big) \setminus (\mathrm{id}_0 \cdot_{\mathrm{Map}_0} \mathrm{id}_0) = (\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon) \setminus \mathrm{id}_0 = (\varepsilon \setminus \mathrm{id}_0) \boxtimes (\varepsilon \setminus \mathrm{id}_0).$$

This shows that compatibility condition (ii) is satisfied.

Ad (iii). We have to show that

$$\big((\mathrm{id}_0 \setminus \varepsilon) \cdot (\mu \setminus \mathrm{id}_2)\big)\Theta^{\mathrm{pre}} = \varepsilon_V \cdot_{\mathrm{END}} \Delta_V \overset{!}{=} \varepsilon_V \otimes \varepsilon_V = \big((\mathrm{id}_0 \setminus \varepsilon) \boxtimes (\mathrm{id}_0 \setminus \varepsilon)\big)\Theta^{\mathrm{pre}}.$$

So it suffices to show that

$$(\mathrm{id}_0 \setminus \varepsilon) \cdot (\mu \setminus \mathrm{id}_2) \overset{!}{=} (\mathrm{id}_0 \setminus \varepsilon) \boxtimes (\mathrm{id}_0 \setminus \varepsilon).$$

Claim. We have the following sorted pullback.

$$\begin{array}{ccc} [1,0] & \xrightarrow{\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon} & [1,2] \\ {\scriptstyle \mathrm{id}_0}\downarrow & \ulcorner & \downarrow{\scriptstyle \mu} \\ [1,0] & \xrightarrow[\varepsilon]{} & [1,1] \end{array}$$

Proof of the Claim. Again $\mathrm{id}_0 \cdot_{\mathrm{Map}_0} \varepsilon$, $(\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon) \cdot_{\mathrm{Map}_0} \mu \in \mathrm{Map}_0(0,1) = \{\varepsilon\}$, so the diagram commutes. Moreover, $\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon$ is monotone as the product of monotone maps.

Finally, the map $\mathrm{id}_0\big|^{\varepsilon^{-1}(i)}_{(\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon)^{-1}(i)}$ is isotone for $i \in [1,2]$, since $(\varepsilon \boxtimes_{\mathrm{Map}_0} \varepsilon)^{-1}(i) = [1,0]$ and $\varepsilon^{-1}(i\mu) = \varepsilon^{-1}(1) = [1,0]$ for $i \in [1,2]$.

This completes the proof of the *Claim.*

So we have

$$(\mathrm{id}_0 \backslash \varepsilon)\cdot(\mu \backslash \mathrm{id}_2) \overset{7.10}{=} (\mathrm{id}_0 \cdot_{\mathrm{Map}_0} \mathrm{id}_0)\backslash\big((\varepsilon\boxtimes_{\mathrm{Map}_0}\varepsilon)\cdot_{\mathrm{Map}_0}\mathrm{id}_2\big) = \mathrm{id}_0\backslash(\varepsilon\boxtimes_{\mathrm{Map}_0}\varepsilon) = (\mathrm{id}_0\backslash\varepsilon)\boxtimes(\mathrm{id}_0\backslash\varepsilon).$$

This shows that compatibility condition (iii) is satisfied.

Ad (iv). We have to show that

$$\big((\mathrm{id}_0 \backslash \varepsilon)\cdot(\varepsilon \backslash \mathrm{id}_0)\big)\Theta^{\mathrm{pre}} = \varepsilon_V \cdot_{\mathrm{END}} \eta_V \overset{!}{=} \mathrm{id}_R = (\mathrm{id}_0 \backslash \mathrm{id}_0)\Theta^{\mathrm{pre}}.$$

So it suffices to show that

$$(\mathrm{id}_0 \backslash \varepsilon)\cdot(\varepsilon \backslash \mathrm{id}_0) \overset{!}{=} \mathrm{id}_0 \backslash \mathrm{id}_0\ .$$

Claim. We have the following sorted pullback.

$$\begin{array}{ccc} [1,0] & \xrightarrow{\mathrm{id}_0} & [1,0] \\ {\scriptstyle \mathrm{id}_0}\downarrow & \ulcorner & \downarrow{\scriptstyle \varepsilon} \\ [1,0] & \xrightarrow[\varepsilon]{} & [1,1] \end{array}$$

Proof of the Claim. The diagram is commutative. Furthermore, the map id_0 is monotone. Finally, $\mathrm{id}_0\big|_{\mathrm{id}_0^{-1}(i)}^{\varepsilon^{-1}(i\varepsilon)}$ is isotone for $i \in [1,0] = \varnothing$.

This completes the proof of the *Claim.*

So we have

$$(\mathrm{id}_0 \backslash \varepsilon)\cdot(\varepsilon \backslash \mathrm{id}_0) \overset{7.10}{=} (\mathrm{id}_0 \cdot_{\mathrm{Map}_0} \mathrm{id}_0)\backslash(\mathrm{id}_0 \cdot_{\mathrm{Map}_0} \mathrm{id}_0) = \mathrm{id}_0 \backslash \mathrm{id}_0\ .$$

This shows that compatibility condition (iv) is satisfied

This completes the proof that $(V, \mu_V, \varepsilon_V, \Delta_V, \eta_V)$ is a bialgebra. □

Question 8.4. *Is a bialgebra a* BIALG*-algebra?*

That is, given a bialgebra $(V, \mu_V, \varepsilon_V, \Delta_V, \eta_V)$, we ask if it is possible to define a morphism of linear operads $\Theta : \mathrm{BIALG} \longrightarrow \mathrm{END}(V)$ such that $\mu_V = (\mathrm{id}_2 \backslash \mu)\Theta^{\mathrm{pre}}$, $\varepsilon_V = (\mathrm{id}_0 \backslash \varepsilon)\Theta^{\mathrm{pre}}$ as well as $\Delta_V = (\mu \backslash \mathrm{id}_2)\Theta^{\mathrm{pre}}$ and $\eta_V = (\varepsilon \backslash \mathrm{id}_0)\Theta^{\mathrm{pre}}$.

9 The set-operad COM_0 and the linear operad COM

Our aim in this chapter will be to define a linear operad COM with the property that for an R-module V, giving a morphism of linear operads $\mathrm{COM} \longrightarrow \mathrm{END}(V)$ is equivalent to giving the structure of a commutative R-algebra on V.

9.1 Construction of the set-operad COM_0

Recall the disjoint union of sets and maps; cf. Definitions 1.13 and 1.14. Suppose given sets X, X'. Then

$$X \sqcup X' := \{(1,x) : x \in X\} \cup \{(2,x') : x' \in X'\}.$$

Suppose given sets X, Y, X', Y' and maps $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$. Then

$$\begin{aligned} f \sqcup f' : X \sqcup X' &\longrightarrow Y \sqcup Y' \\ (i,z) &\longmapsto \begin{cases} (i, zf) & \text{if } i = 1 \\ (i, zf') & \text{if } i = 2. \end{cases} \end{aligned}$$

Definition 9.1. Define

$$\mathcal{E}_0 := \{[1,m] : m \in \mathbb{Z}_{\geqslant 0}\}$$

and for $k \geqslant 0$ recursively define

$$\begin{aligned} \mathcal{E}'_{k+1} &:= \{X \sqcup X', X \times X' : X, X' \in \mathcal{E}_k\} \cup \mathcal{E}_k \\ \mathcal{E}_{k+1} &:= \{Y : \text{there exists } X \in \mathcal{E}'_{k+1} \text{ such that } Y \subseteq X\}. \end{aligned}$$

Note that we have $\mathcal{E}_k \subseteq \mathcal{E}_{k+1}$ for $k \in \mathbb{Z}_{\geqslant 0}$. Finally, define

$$\mathcal{E} := \bigcup_{k \in \mathbb{Z}_{\geqslant 0}} \mathcal{E}_k .$$

Remark 9.2. The set $\mathcal{E}$ has the following properties.

(1) Given $X, X' \in \mathcal{E}$, then, since $\mathcal{E}_i \subseteq \mathcal{E}_{i+1}$ for $i \in \mathbb{Z}_{\geqslant 0}$, there exists $k \in \mathbb{Z}_{\geqslant 0}$ such that $X, X' \in \mathcal{E}_k$. Then we have $X \sqcup X', X \times X' \in \mathcal{E}'_{k+1} \subseteq \mathcal{E}_{k+1}$. Hence $X \sqcup X', X \times X' \in \mathcal{E}$.

(2) Given $X \in \mathcal{E}$ and $Y \subseteq X$. Then there exists $k \in \mathbb{Z}_{\geqslant 0}$ with $X \in \mathcal{E}_k \subseteq \mathcal{E}'_{k+1}$. So we have $Y \in \mathcal{E}_{k+1}$, hence $Y \in \mathcal{E}$.

(3) Given $X, Y, Z \in \mathcal{E}$ and given maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, then recall the standard pullback $(S, \check{g}, \check{f})$ of f and g, where $S = \{(x,y) \in X \times Y : xf = yg\}$; cf. Lemma 1.29. By (1) we have $X \times Y \in \mathcal{E}$ and since $S \subseteq X \times Y$, we have $S \in \mathcal{E}$ by (2).

(4) All elements of $\mathcal{E}$ are finite sets.

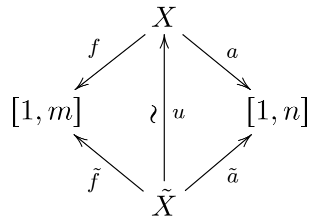
For $n \in \mathbb{Z}_{\geqslant 0}$ and a tuple $k = (k_i)_{i \in [1,n]}$ with $k_i \in \mathbb{Z}_{\geqslant 0}$ for $i \in [1,n]$ recall the bijective map $\varphi_k : \left[1, \sum_{i \in [1,n]} k_i\right] \longrightarrow \bigsqcup_{i \in [1,n]} [1, k_i]$; cf. Definition 1.18.

Definition 9.3. We define the set-preoperad $\mathrm{COM}_0^{\mathrm{pre}}$ as follows.

First consider the set

$$C(m,n) := \left\{(f,a) : \text{there exists } X \in \mathcal{E} \text{ such that } [1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]\right\}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$. Now define an equivalence relation $(\sim)$ on the biindexed set $C = (C(m,n))_{m,n \geqslant 0}$ as follows. Let $X, \tilde{X} \in \mathcal{E}$, $m, n \in \mathbb{Z}_{\geqslant 0}$, $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,m] \xleftarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{a}} [1,n]$. Then $(f,a) \sim (\tilde{f}, \tilde{a})$ if and only if there exists a bijective map $u : \tilde{X} \longrightarrow X$ such that $uf = \tilde{f}$ and $ua = \tilde{a}$.



We denote the equivalence class of $(f,a) \in C$ with respect to $(\sim)$ by $f \setminus a$ and define

$$\mathrm{COM}_0^{\mathrm{pre}}(m,n) := \frac{C(m,n)}{(\sim)} = \left\{ f \setminus a : \exists X \in \mathcal{E} \text{ such that } [1,m] \xleftarrow{f} X \xrightarrow{a} [1,n] \right\}$$

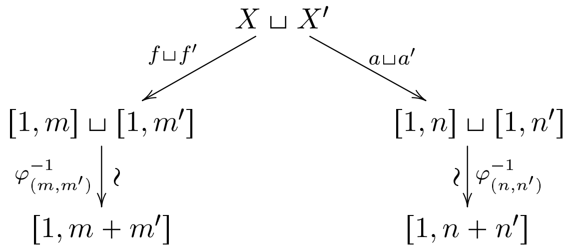
for $m, n \in \mathbb{Z}_{\geqslant 0}$.

The multiplication in $\mathrm{COM}_0^{\mathrm{pre}}$ is given by

$$\begin{aligned} (\boxtimes) := (\boxtimes_{\mathrm{COM}_0}) : \mathrm{COM}_0^{\mathrm{pre}}(m,n) \times \mathrm{COM}_0^{\mathrm{pre}}(m',n') &\longrightarrow \mathrm{COM}_0^{\mathrm{pre}}(m+m', n+n') \\ (f \setminus a \,,\, f' \setminus a') &\longmapsto ((f \sqcup f')\varphi^{-1}_{(m,m')}) \setminus ((a \sqcup a')\varphi^{-1}_{(n,n')}) \\ &=: (f \setminus a) \boxtimes_{\mathrm{COM}_0} (f' \setminus a') \end{aligned}$$

for $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$.

Multiplication can be illustrated as follows.

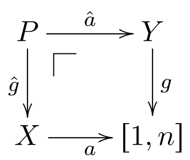


Composition is defined using the pullback $(*)$ below, letting

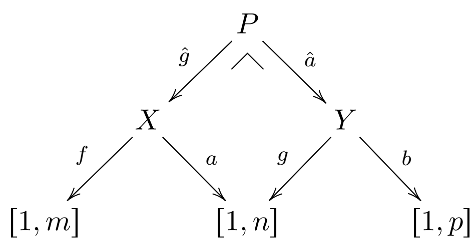
$$\begin{aligned} (\cdot) := (\cdot_{\mathrm{COM}_0}) : \mathrm{COM}_0^{\mathrm{pre}}(m,n) \times \mathrm{COM}_0^{\mathrm{pre}}(n,p) &\longrightarrow \mathrm{COM}_0^{\mathrm{pre}}(m,p) \\ (f \setminus a \,,\, g \setminus b) &\longmapsto (\hat{g} f \setminus \hat{a} b) =: (f \setminus a) \cdot_{\mathrm{COM}_0} (g \setminus b) \end{aligned}$$

for $m, n, p \in \mathbb{Z}_{\geqslant 0}$, where $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,n] \xleftarrow{g} X \xrightarrow{b} [1,p]$ and where

$(*)$



is a pullback, arbitrarily chosen. Composition can be illustrated as follows.



The identity elements are $\mathrm{id}_m := \mathrm{id}_{\mathrm{COM}_0^{\mathrm{pre}}, m} := \mathrm{id}_{\mathrm{Map}_0, m} \setminus \mathrm{id}_{\mathrm{Map}_0, m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

We now have to verify that this defines a set-preoperad.

First we will show that the multiplication map is well-defined.

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $X, \tilde{X}, X', \tilde{X}' \in \mathcal{E}$. Suppose given $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,m] \xleftarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{a}} [1,n]$ such that $f \setminus a = \tilde{f} \setminus \tilde{a}$ and suppose given $[1,m'] \xleftarrow{f'} X' \xrightarrow{a'} [1,n']$ and $[1,m'] \xleftarrow{\tilde{f}'} \tilde{X}' \xrightarrow{\tilde{a}'} [1,n']$ such that $f' \setminus a' = \tilde{f}' \setminus \tilde{a}'$. This means that there exist bijective maps $u : \tilde{X} :\longrightarrow X$ and $u' : \tilde{X}' \longrightarrow X'$ such that $uf = \tilde{f}$, $ua = \tilde{a}$, $u'f' = \tilde{f}'$ and $u'a' = \tilde{a}'$.

Since $u \sqcup u' : \tilde{X} \sqcup \tilde{X}' \longrightarrow X \sqcup X'$ is a bijective map we have

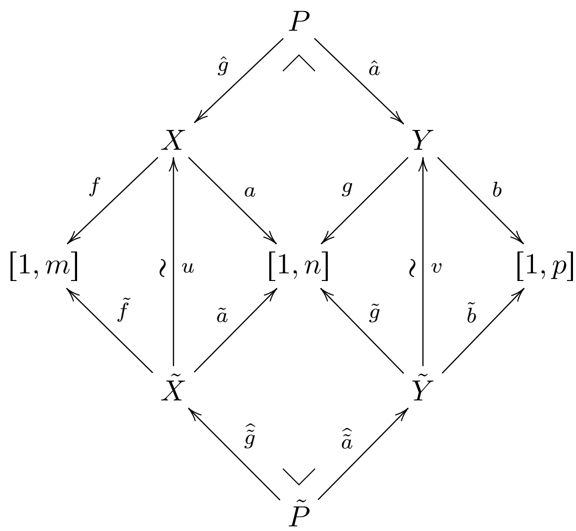
$$\begin{aligned}
\big((\tilde{f} \sqcup \tilde{f}')\varphi^{-1}_{(m,m')}\big) \setminus \big((\tilde{a} \sqcup \tilde{a}')\varphi^{-1}_{(n,n')}\big) &= \big((uf \sqcup u'f')\varphi^{-1}_{(m,m')}\big) \setminus \big((ua \sqcup u'a')\varphi^{-1}_{(n,n')}\big) \\
&\overset{1.24(\mathrm{i})}{=} \big((u \sqcup u')(f \sqcup f')\varphi^{-1}_{(m,m')}\big) \setminus \big((u \sqcup u')(a \sqcup a')\varphi^{-1}_{(n,n')}\big) \\
&= \big((f \sqcup f')\varphi^{-1}_{(m,m')}\big) \setminus \big((a \sqcup a')\varphi^{-1}_{(n,n')}\big).
\end{aligned}$$

Hence the multiplication map is well-defined.

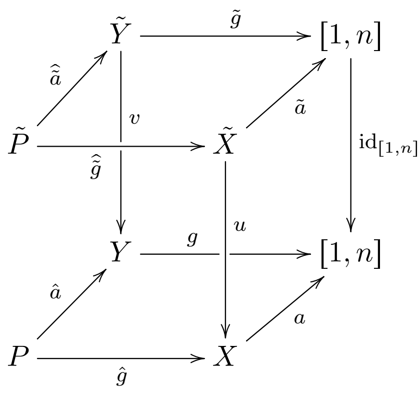
Now we will show that the composition map is well-defined.

Let $m, n, p \in \mathbb{Z}_{\geqslant 0}$, $X, \tilde{X}, Y, \tilde{Y} \in \mathcal{E}$ and let $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,m] \xleftarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{a}} [1,n]$ such that $f \setminus a = \tilde{f} \setminus \tilde{a}$ and $[1,n] \xleftarrow{g} Y \xrightarrow{b} [1,p]$ and $[1,n] \xleftarrow{\tilde{g}} \tilde{Y} \xrightarrow{\tilde{b}} [1,p]$ such that $g \setminus b = \tilde{g} \setminus \tilde{b}$. This means that there exist bijective maps $u : \tilde{X} \longrightarrow X$ and $v : \tilde{Y} \longrightarrow Y$ such that $uf = \tilde{f}$, $ua = \tilde{a}$, $vg = \tilde{g}$ and $vb = \tilde{b}$.

Consider the following commutative diagram.



Including the identity map $\mathrm{id}_{[1,n]} = \mathrm{id}_{\mathrm{Map}_0, n}$, this translates to the following commutative diagram.



By applying Lemma 1.28 (2) to $(u, v, \mathrm{id}_{[1,n]})$, we know that there exists a unique bijective map $w : \tilde{P} \longrightarrow P$ such that $w\hat{g} = \hat{\tilde{g}}u$ and $w\hat{a} = \hat{\tilde{a}}v$. Hence we have

$$(\hat{\tilde{g}}\tilde{f}) \setminus (\hat{\tilde{a}}\tilde{b}) = (\hat{\tilde{g}}uf) \setminus (\hat{\tilde{a}}vb) = (w\hat{g}f) \setminus (w\hat{a}b) = (\hat{g}f) \setminus (\hat{a}b).$$

This shows that the composition map is well-defined.

Hence the claimed image of $(f \setminus a, g \setminus b)$ under the composition map is independent from the choice of representative of their equivalence classes and, letting $u = \mathrm{id}_X$ and $v = \mathrm{id}_Y$, also from the choice of a pullback in $(*)$.

Ad (m1). Suppose given $m, n, m', n', m'', n'' \in \mathbb{Z}_{\geqslant 0}$, $X, X', X'' \in \mathcal{E}$ and $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$, $[1,m'] \xleftarrow{f'} X' \xrightarrow{a'} [1,n']$ and $[1,m''] \xleftarrow{f''} X'' \xrightarrow{a''} [1,n'']$.

We have to show that

$$((f \setminus a) \boxtimes (f' \setminus a')) \boxtimes (f'' \setminus a'') \overset{!}{=} (f \setminus a) \boxtimes ((f' \setminus a') \boxtimes (f'' \setminus a'')).$$

We have

$$\begin{aligned}
&\big((f \setminus a) \boxtimes (f' \setminus a')\big) \boxtimes (f'' \setminus a'') \\
&\quad = \big(((f \sqcup f')\varphi^{-1}_{(m,m')}) \setminus (a \sqcup a')\varphi^{-1}_{(n,n')})\big) \boxtimes (f'' \setminus a'') \\
&\quad = \big((((f \sqcup f')\varphi^{-1}_{(m,m')}) \sqcup f'')\varphi^{-1}_{(m+m',m'')}\big) \setminus \big((((a \sqcup a')\varphi^{-1}_{(n,n')}) \sqcup a'')\varphi^{-1}_{(n+n',n'')}\big) \\
&\quad = \big(((f \sqcup f') \sqcup f'')(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')}\big) \setminus \big(((a \sqcup a') \sqcup a'')(\varphi^{-1}_{(n,n')} \sqcup \mathrm{id}_{[1,n'']})\varphi^{-1}_{(n+n',n'')}\big).
\end{aligned}$$

In the same way we get

$$\begin{aligned}
&(f \setminus a) \boxtimes \big((f' \setminus a') \boxtimes (f'' \setminus a'')\big) \\
&\quad = \big((f \sqcup (f' \sqcup f''))(\mathrm{id}_{[1,m]} \sqcup \varphi^{-1}_{(m',m'')})\varphi^{-1}_{(m,m'+m'')}\big) \setminus \big((a \sqcup (a' \sqcup a''))(\mathrm{id}_{[1,n]} \sqcup \varphi^{-1}_{(n',n'')})\varphi^{-1}_{(n,n'+n'')}\big).
\end{aligned}$$

So by the definition of the equivalence relation defining $\mathrm{COM}_0^{\mathrm{pre}}$ it suffices to show that there exists a bijective map $\tilde{\gamma} : X \sqcup (X' \sqcup X'') \longrightarrow (X \sqcup X') \sqcup X''$ such that

$$\begin{aligned}
\tilde{\gamma}((f \sqcup f') \sqcup f'')(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} &\overset{!}{=} (f \sqcup (f' \sqcup f''))(\mathrm{id}_{[1,m]} \sqcup \varphi^{-1}_{(m',m'')})\varphi^{-1}_{(m,m'+m'')} \\
\tilde{\gamma}((a \sqcup a') \sqcup a'')(\varphi^{-1}_{(n,n')} \sqcup \mathrm{id}_{[1,n'']})\varphi^{-1}_{(n+n',n'')} &\overset{!}{=} (a \sqcup (a' \sqcup a''))(\mathrm{id}_{[1,n]} \sqcup \varphi^{-1}_{(n',n'')})\varphi^{-1}_{(n,n'+n'')} .
\end{aligned}$$

Consider the following diagram; cf. Definition 1.16.

$$\begin{array}{ccccc}
X \sqcup X' \sqcup X'' & \xrightarrow{f \sqcup f' \sqcup f''} & [1,m] \sqcup [1,m'] \sqcup [1,m''] & \xrightarrow{\varphi^{-1}_{(m,m',m'')}} & [1, m+m', m''] \\
\downarrow{\scriptstyle \gamma_{(X,X'),X''}} & & \downarrow{\scriptstyle \gamma_{([1,m],[1,m']),[1,m'']}} & & \uparrow{\scriptstyle \varphi^{-1}_{(m+m'),m''}} \\
(X \sqcup X') \sqcup X'' & \xrightarrow[(f \sqcup f') \sqcup f'']{} & ([1,m] \sqcup [1,m']) \sqcup [1,m''] & \xrightarrow[\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']}]{} & [1, m+m'] \sqcup [1,m'']
\end{array}$$

By Lemma 1.17 the first quadrangle in the diagram commutes and by Lemma 1.20 the second quadrangle commutes. This shows that

$$\gamma_{(X,X'),X''}((f \sqcup f') \sqcup f'')(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_m)\varphi^{-1}_{(m+m',m'')} = (f \sqcup f' \sqcup f'')\varphi^{-1}_{(m,m',m'')} .$$

By replacing (f, f', f'') by (a, a', a'') we see that

$$\gamma_{(X,X'),X''}((a \sqcup a') \sqcup a'')(\varphi^{-1}_{(n,n')} \sqcup \mathrm{id}_n)\varphi^{-1}_{(n+n',n'')} = (a \sqcup a' \sqcup a'')\varphi^{-1}_{(n,n',n'')} .$$

Furthermore, consider the following diagram.

$$\begin{array}{ccccc}
X \sqcup X' \sqcup X'' & \xrightarrow{f \sqcup f' \sqcup f''} & [1,m] \sqcup [1,m'] \sqcup [1,m''] & \xrightarrow{\varphi^{-1}_{(m,m',m'')}} & [1,m+m',m''] \\
\Big\downarrow{\scriptstyle \gamma_{X,(X',X'')}} & & \Big\downarrow{\scriptstyle \gamma_{[1,m],([1,m'],[1,m''])}} & & \Big\uparrow{\scriptstyle \varphi^{-1}_{m,(m'+m'')}} \\
X \sqcup (X' \sqcup X'') & \xrightarrow[f \sqcup (f' \sqcup f'')]{} & [1,m] \sqcup ([1,m'] \sqcup [1,m'']) & \xrightarrow[\mathrm{id}_{[1,m]} \sqcup \varphi^{-1}_{(m',m'')}]{} & [1,m] \sqcup [1,m'+m'']
\end{array}$$

By Lemma 1.17 the first quadrangle in the diagram commutes and by Lemma 1.20 the second quadrangle commutes. This shows that

$$\gamma_{X,(X',X'')}(f \sqcup (f' \sqcup f''))(\mathrm{id}_m \sqcup \varphi^{-1}_{(m',m'')})\varphi^{-1}_{(m,m'+m'')} = (f \sqcup f' \sqcup f'')\varphi^{-1}_{(m,m',m'')}$$

and by replacing (f, f', f'') by (a, a', a'') we see that

$$\gamma_{X,(X',X'')}(a \sqcup (a' \sqcup a''))(\mathrm{id}_n \sqcup \varphi^{-1}_{(n',n'')})\varphi^{-1}_{(n,n'+n'')} = (a \sqcup a' \sqcup a'')\varphi^{-1}_{(n,n',n'')} .$$

So define $\tilde{\gamma} := \gamma^{-1}_{X,(X',X'')}\gamma_{(X,X'),X''} : X \sqcup (X' \sqcup X'') \longrightarrow (X \sqcup X') \sqcup X''$. This is a bijective map. We have

$$\begin{aligned}
&\tilde{\gamma}((f \sqcup f') \sqcup f'')(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} \\
&\qquad = \gamma^{-1}_{X,(X',X'')}\gamma_{(X,X'),X''}((f \sqcup f') \sqcup f'')(\varphi^{-1}_{(m,m')} \sqcup \mathrm{id}_{[1,m'']})\varphi^{-1}_{(m+m',m'')} \\
&\qquad = \gamma^{-1}_{X,(X',X'')}(f \sqcup f' \sqcup f'')\varphi^{-1}_{(m,m',m'')} \\
&\qquad = (f \sqcup (f' \sqcup f''))(\mathrm{id}_{[1,m]} \sqcup \varphi^{-1}_{(m',m'')})\varphi^{-1}_{(m,m'+m'')} \\
&\tilde{\gamma}((a \sqcup a') \sqcup a'')(\varphi^{-1}_{(n,n')} \sqcup \mathrm{id}_{[1,n'']})\varphi^{-1}_{(n+n',n'')} \\
&\qquad = \gamma^{-1}_{X,(X',X'')}\gamma_{(X,X'),X''}((a \sqcup a') \sqcup a'')(\varphi^{-1}_{(n,n')} \sqcup \mathrm{id}_{[1,n'']})\varphi^{-1}_{(n+n',n'')} \\
&\qquad = \gamma^{-1}_{X,(X',X'')}(a \sqcup a' \sqcup a'')\varphi^{-1}_{(n,n',n'')} \\
&\qquad = (a \sqcup (a' \sqcup a''))(\mathrm{id}_{[1,n]} \sqcup \varphi^{-1}_{(n',n'')})\varphi^{-1}_{(n,n'+n'')} .
\end{aligned}$$

This completes the proof that the multiplication ($\boxtimes$) is associative.

Ad (m2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$. We have to show that

$$(f \setminus a) \boxtimes (\mathrm{id}_{\mathrm{Map}_0,0} \setminus \mathrm{id}_{\mathrm{Map}_0,0}) \overset{!}{=} f \overset{!}{=} (\mathrm{id}_{\mathrm{Map}_0,0} \setminus \mathrm{id}_{\mathrm{Map}_0,0}) \boxtimes (f \setminus a).$$

We have $(f \setminus a) \boxtimes (\mathrm{id}_{\mathrm{Map}_0,0} \setminus \mathrm{id}_{\mathrm{Map}_0,0}) = ((f \sqcup \mathrm{id}_{\mathrm{Map}_0,0})\varphi^{-1}_{(m,0)}) \setminus ((a \sqcup \mathrm{id}_{\mathrm{Map}_0,0})\varphi^{-1}_{(n,0)})$.

Recall the bijective map $u_Z : Z \sqcup [1,0] \longrightarrow Z$, $(1,z) \mapsto z$ for a set Z; cf. Lemma 1.24. Furthermore, recall that in the case of intervals we have

$$u_{[1,m]} = \varphi^{-1}_{(m,0)}$$

for $m \in \mathbb{Z}_{\geqslant 0}$.

Hence we have

$$\begin{array}{rcl}
(f \setminus a) \boxtimes (\mathrm{id}_{\mathrm{Map}_0,0} \setminus \mathrm{id}_{\mathrm{Map}_0,0}) & = & ((f \sqcup \mathrm{id}_{\mathrm{Map}_0,0})\varphi^{-1}_{(m,0)}) \setminus ((a \sqcup \mathrm{id}_{\mathrm{Map}_0,0})\varphi^{-1}_{(n,0)}) \\
& = & ((f \sqcup \mathrm{id}_{\mathrm{Map}_0,0})u_{[1,m]}) \setminus ((a \sqcup \mathrm{id}_{\mathrm{Map}_0,0})u_{[1,n]}) \\
& \overset{1.24(\mathrm{ii})}{=} & (u_X f) \setminus (u_X a) \\
& = & f \setminus a.
\end{array}$$

On the other hand recall the map $v_Z : [1,0] \sqcup Z \longrightarrow Z$, $(2,z) \mapsto z$ for a set Z; cf. Lemma 1.24 and that in the case of intervals we have

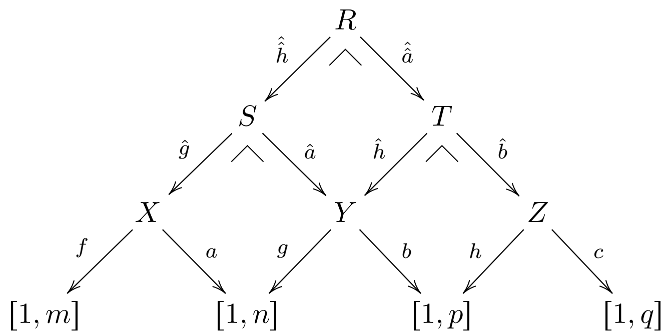
$$v_{[1,m]} = \varphi^{-1}_{(0,m)}$$

for $m \in \mathbb{Z}_{\geqslant 0}$.

Thus we have

$$\begin{aligned}(\mathrm{id}_{\mathrm{Map}_0,0} \setminus \mathrm{id}_{\mathrm{Map}_0,0}) \boxtimes (f \setminus a) &= ((\mathrm{id}_{\mathrm{Map}_0,0} \sqcup f)\varphi^{-1}_{(0,m)}) \setminus ((\mathrm{id}_{\mathrm{Map}_0,0} \sqcup a)\varphi^{-1}_{(0,n)}) \\ &= ((\mathrm{id}_{\mathrm{Map}_0,0} \sqcup f)v_{[1,m]}) \setminus ((\mathrm{id}_{\mathrm{Map}_0,0} \sqcup a)v_{[1,n]}) \\ &\overset{1.24(\mathrm{iii})}{=} (v_X f) \setminus (v_X a) \\ &= f \setminus a.\end{aligned}$$

Ad (c1). Suppose given $m, n, p, q \in \mathbb{Z}_{\geqslant 0}$ and $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$, $[1,n] \xleftarrow{g} Y \xrightarrow{b} [1,p]$ and $[1,p] \xleftarrow{h} Z \xrightarrow{c} [1,q]$. Consider the following diagram.

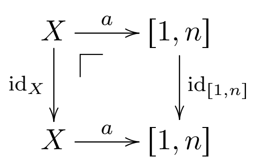


So $(S, \hat{g}, \hat{a})$ is a pullback of a and g, $(T, \hat{h}, \hat{b})$ is a pullback of b and h and $(R, \hat{\hat{h}}, \hat{\hat{a}})$ is a pullback of $\hat{a}$ and $\hat{h}$. Then by Lemma 1.37, $(R, \hat{\hat{h}}\hat{g}, \hat{\hat{a}})$ is a pullback of a and $\hat{h}g$ and $(R, \hat{\hat{h}}, \hat{\hat{a}}\hat{b})$ is a pullback of $\hat{a}b$ and h. So by the definition of composition in $\mathrm{COM}_0^{\mathrm{pre}}$ we have

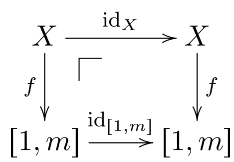
$$\begin{aligned}\big((f \setminus a) \cdot (g \setminus b)\big) \cdot (h \setminus c) &= (\hat{g}f \setminus \hat{a}b) \cdot (h \setminus c) \\ &= (\hat{\hat{h}}\hat{g}f) \setminus (\hat{\hat{a}}\hat{b}c) \\ &= (f \setminus a) \cdot (\hat{h}g \setminus \hat{b}c) \\ &= (f \setminus a) \cdot \big((g \setminus b) \cdot (h \setminus c)\big).\end{aligned}$$

This shows that the composition $(\cdot)$ is associative.

Ad (c2). Suppose given $m, n \in \mathbb{Z}_{\geqslant 0}$ and $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$. By Corollary 1.34 we know that



and

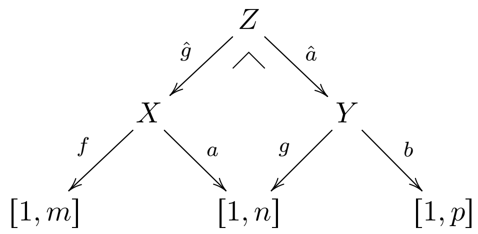


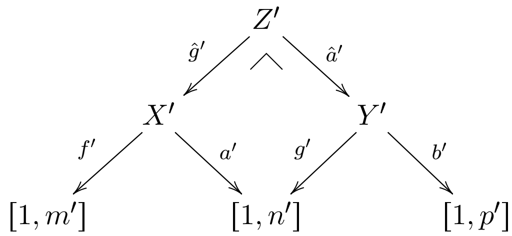
are pullbacks. Hence we have

$$\begin{aligned}(f \setminus a) \cdot (\mathrm{id}_{\mathrm{Map}_0,n} \setminus \mathrm{id}_{\mathrm{Map}_0,n}) &= (\mathrm{id}_X\, f) \setminus (a\, \mathrm{id}_{\mathrm{Map}_0,n}) = (f \setminus a) \\ (\mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Map}_0,m}) \cdot (f \setminus a) &= (f\, \mathrm{id}_{\mathrm{Map}_0,m}) \setminus (\mathrm{id}_X\, a) = (f \setminus a).\end{aligned}$$

Ad (mc1). Suppose given $m, n, p, m', n', p' \in \mathbb{Z}_{\geqslant 0}$ and $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$, $[1,n] \xleftarrow{g} Y \xrightarrow{b} [1,p]$, $[1,m'] \xleftarrow{f'} X' \xrightarrow{a'} [1,n']$ and $[1,n'] \xleftarrow{g'} Y' \xrightarrow{b'} [1,p']$.

Consider the following diagrams.

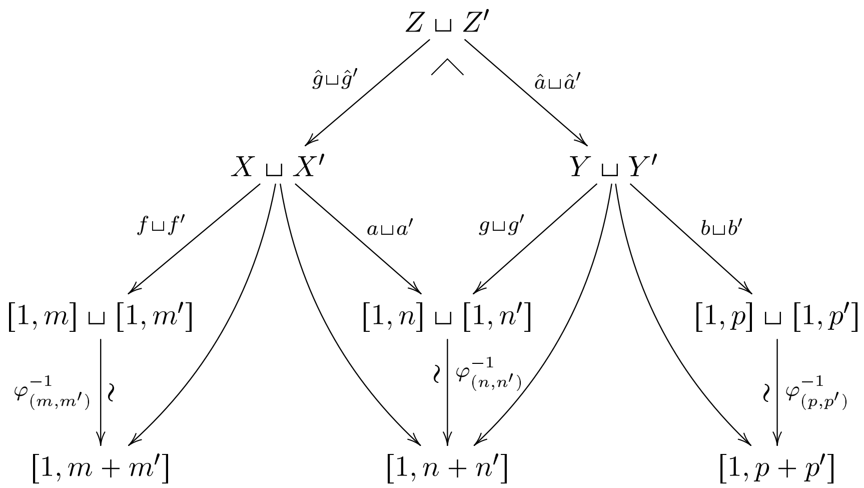




So $(Z, \hat{g}, \hat{a})$ is a pullback of a and g and $(Z', \hat{g}', \hat{a}')$ is a pullback of a' and g' and we have

$$(f \setminus a) \cdot (g \setminus b) = \hat{g}f \setminus \hat{a}b$$
$$(f' \setminus a') \cdot (g' \setminus b') = \hat{g}'f' \setminus \hat{a}'b'.$$

By Lemma 1.38 we know that $(Z \sqcup Z', \hat{g} \sqcup \hat{g}', \hat{a} \sqcup \hat{a}')$ is a pullback of $a \sqcup a'$ and $g \sqcup g'$. By Lemma 1.36, $(Z \sqcup Z', \hat{g} \sqcup \hat{g}', \hat{a} \sqcup \hat{a}')$ is also a pullback of $(a \sqcup a')\varphi^{-1}_{(n,n')}$ and $(g \sqcup g')\varphi^{-1}_{(n,n')}$ since $\varphi^{-1}_{(n,n')}$ is injective. So we have the following diagram.



So since by Lemma 1.24 (i) we have $(\hat{g} \sqcup \hat{g}')(f \sqcup f') = \hat{g}f \sqcup \hat{g}'f'$ and $(\hat{a} \sqcup \hat{a}')(b \sqcup b') = \hat{a}b \sqcup \hat{a}'b'$, we have

$$\begin{aligned}
&\big((f \setminus a) \boxtimes (f' \setminus a')\big) \cdot \big((g \setminus b) \boxtimes (g' \setminus b')\big) \\
&\qquad = \Big(\big((f \sqcup f')\varphi^{-1}_{(m,m')}\big) \setminus \big((a \sqcup a')\varphi^{-1}_{(n,n')}\big)\Big) \cdot \Big(\big((g \sqcup g')\varphi^{-1}_{(n,n')}\big) \setminus \big((b \sqcup b')\varphi^{-1}_{(p,p')}\big)\Big) \\
&\qquad = \Big((\hat{g} \sqcup \hat{g}')(f \sqcup f')\varphi^{-1}_{(m,m')}\Big) \setminus \Big((\hat{a} \sqcup \hat{a}')(b \sqcup b')\varphi^{-1}_{(p,p')}\Big) \\
&\qquad = \Big((\hat{g}f \sqcup \hat{g}'f')\varphi^{-1}_{(m,m')}\Big) \setminus \Big((\hat{a}b \sqcup \hat{a}'b')\varphi^{-1}_{(p,p')}\Big) \\
&\qquad = (\hat{g}f \setminus \hat{a}b) \boxtimes (\hat{g}'f' \setminus \hat{a}'b') \\
&\qquad = \big((f \setminus a) \cdot (g \setminus b)\big) \boxtimes \big((f' \setminus a') \cdot (g' \setminus b')\big).
\end{aligned}$$

Ad (mc2). Suppose given $m \in \mathbb{Z}_{\geqslant 0}$. We have to show that

$$(\mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Map}_0,m}) \overset{!}{=} (\mathrm{id}_{\mathrm{Map}_0,1} \setminus \mathrm{id}_{\mathrm{Map}_0,1})^{\boxtimes m}.$$

We prove this via induction on $m \geqslant 0$. In the case $m = 0$ this is the definition. So let $m \geqslant 1$ and suppose that we already know $(\mathrm{id}_{\mathrm{Map}_0,m-1} \setminus \mathrm{id}_{\mathrm{Map}_0,m-1}) = (\mathrm{id}_{\mathrm{Map}_0,1} \setminus \mathrm{id}_{\mathrm{Map}_0,1})^{\boxtimes(m-1)}$. Then we have

$$\begin{aligned}(\mathrm{id}_{\mathrm{Map}_0,1} \setminus \mathrm{id}_{\mathrm{Map}_0,1})^{\boxtimes m} &= (\mathrm{id}_{\mathrm{Map}_0,1} \setminus \mathrm{id}_{\mathrm{Map}_0,1})^{\boxtimes (m-1)} \boxtimes (\mathrm{id}_{\mathrm{Map}_0,1} \setminus \mathrm{id}_{\mathrm{Map}_0,1}) \\ &\overset{\text{ind.}}{=} (\mathrm{id}_{\mathrm{Map}_0,m-1} \setminus \mathrm{id}_{\mathrm{Map}_0,m-1}) \boxtimes (\mathrm{id}_{\mathrm{Map}_0,1} \setminus \mathrm{id}_{\mathrm{Map}_0,1}) \\ &= \big((\mathrm{id}_{\mathrm{Map}_0,m-1} \sqcup \mathrm{id}_{\mathrm{Map}_0,1})\varphi^{-1}_{(m-1,1)}\big) \setminus \big((\mathrm{id}_{\mathrm{Map}_0,m-1} \sqcup \mathrm{id}_{\mathrm{Map}_0,1})\varphi^{-1}_{(m-1,1)}\big) \\ &= \big((\mathrm{id}_{[1,m-1]\sqcup[1,1]})\varphi^{-1}_{(m-1,1)}\big) \setminus \big((\mathrm{id}_{[1,m-1]\sqcup[1,1]})\varphi^{-1}_{(m-1,1)}\big),\end{aligned}$$

where $\mathrm{id}_{[1,m-1]\sqcup[1,1]}$ is the identity map on the disjoint union $[1, m-1] \sqcup [1, 1]$.

The map $(\mathrm{id}_{[1,m-1]\sqcup[1,1]})\varphi^{-1}_{(m-1,1)} : [1, m-1] \sqcup [1, 1] \longrightarrow [1, m]$ is bijective. So we have

$$\begin{aligned}\mathrm{id}_m &= \mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Map}_0,m} \\ &= \big(((\mathrm{id}_{[1,m-1]\sqcup[1,1]})\varphi^{-1}_{(m-1,1)})\, \mathrm{id}_{\mathrm{Map}_0,m}\big) \setminus \big(((\mathrm{id}_{[1,m-1]\sqcup[1,1]})\varphi^{-1}_{(m-1,1)}))\, \mathrm{id}_{\mathrm{Map}_0,m}\big) \\ &= \big((\mathrm{id}_{[1,m-1]\sqcup[1,1]})\varphi^{-1}_{(m-1,1)}\big) \setminus \big((\mathrm{id}_{[1,m-1]\sqcup[1,1]})\varphi^{-1}_{(m-1,1)}\big) \\ &= (\mathrm{id}_{\mathrm{Map}_0,1} \setminus \mathrm{id}_{\mathrm{Map}_0,1})^{\boxtimes m}.\end{aligned}$$

This proves (mc2) and completes the proof that $\mathrm{COM}_0^{\mathrm{pre}}$ is a set-preoperad.

Remark 9.4. We sometimes refer to $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m, n)$ as a *fraction*. Then *expanding* by a bijective map $u : \tilde{X} \longrightarrow X$ for some $\tilde{X} \in \mathcal{E}$ yields the same fraction $uf \setminus ua = f \setminus a$.

Remark 9.5.

(i) *Let $X \in \mathcal{E}$ and let $m, k, n \in \mathbb{Z}_{\geqslant 0}$. Let $[1, m] \xleftarrow{f} [1, k] \xrightarrow{\mathrm{id}_k} [1, k]$ and $[1, k] \xleftarrow{g} X \xrightarrow{b} [1, n]$, where we abbreviate $\mathrm{id}_k := \mathrm{id}_{\mathrm{Map}_0,k}$. Then we have*

$$(f \setminus \mathrm{id}_k) \cdot (g \setminus b) = (gf) \setminus b.$$

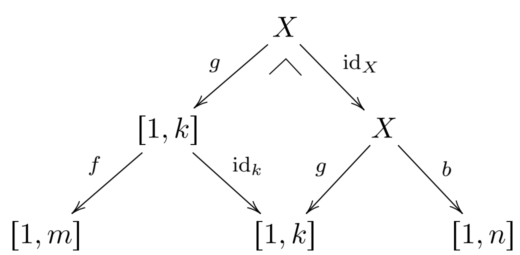
(ii) *Let $X \in \mathcal{E}$ and let $m, n, p \in \mathbb{Z}_{\geqslant 0}$. Let $[1, m] \xleftarrow{f} X \xrightarrow{a} [1, n]$ and $[1, n] \xleftarrow{\mathrm{id}_n} [1, n] \xrightarrow{b} [1, p]$, where we abbreviate $\mathrm{id}_n := \mathrm{id}_{\mathrm{Map}_0,n}$. Then we have*

$$(f \setminus a) \cdot (\mathrm{id}_n \setminus b) = f \setminus (ab).$$

(iii) *Let $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and let $f \in \mathrm{Map}_0(m, n)$, $f' \in \mathrm{Map}_0(m', n')$. Then we have*

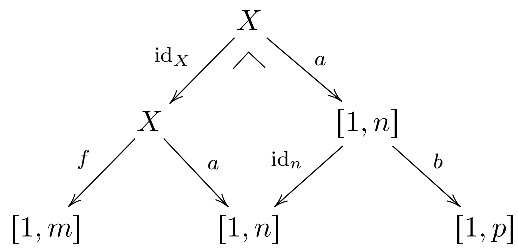
$$f \boxtimes_{\mathrm{Map}_0} f' = \varphi_{(m,m')}(f \sqcup f')\varphi^{-1}_{(n,n')}.$$

Proof. *Ad* (i). By Corollary 1.34 we have the following diagram.



So we have $(f \setminus \mathrm{id}_k) \cdot (g \setminus b) = (gf) \setminus (\mathrm{id}_X\, b) = (gf) \setminus b$.

Ad (ii). By Corollary 1.34 we have the following diagram.



So we have $(f \setminus a) \cdot (\mathrm{id}_X \setminus b) = (\mathrm{id}_n\, f) \setminus (ab) = f \setminus (ab)$.

Ad (iii). For $i \in [1, m + m']$ we have

$$i(f \boxtimes_{\mathrm{Map}_0} f') = \begin{cases} if & \text{if } i \in [1, m] \\ (i-m)f' + n & \text{if } i \in [m+1, m+m']; \end{cases}$$

cf. Definition 2.57. On the other hand we have

$$\begin{aligned} i(\varphi_{(m,m')}(f \sqcup f')\varphi^{-1}_{(n,n')}) &= \begin{cases} (1,i)(f \sqcup f')\varphi^{-1}_{(n,n')} & \text{if } i \in [1, m] \\ (2, i-m)(f \sqcup f')\varphi^{-1}_{(n,n')} & \text{if } i \in [m+1, m+m'] \end{cases} \\ &= \begin{cases} (1, if)\varphi^{-1}_{(n,n')} & \text{if } i \in [1, m] \\ (2, (i-m)f')\varphi^{-1}_{(n,n')} & \text{if } i \in [m+1, m+m'] \end{cases} \\ &= \begin{cases} if & \text{if } i \in [1, m] \\ (i-m)f' + n & \text{if } i \in [m+1, m+m'] \end{cases} \end{aligned}$$

for $i \in [1, m + m']$. So $(f \boxtimes_{\mathrm{Map}_0} f') = \varphi_{(m,m')}(f \sqcup f')\varphi^{-1}_{(n,n')}$. □

Definition 9.6. We define the biindexed map $\mathfrak{c}_0 = (\mathfrak{c}_0(m,n))_{m,n \geqslant 0} : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{COM}_0^{\mathrm{pre}}$ by

$$\begin{aligned} \mathfrak{c}_0(m,n) : \mathrm{Map}_0^{\mathrm{op}}(m,n) &\longrightarrow \mathrm{COM}_0^{\mathrm{pre}}(m,n) \\ f^{\mathrm{op}} &\longmapsto f \setminus \mathrm{id}_{\mathrm{Map}_0, n} \end{aligned}$$

for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Lemma 9.7. *The biindexed map* $\mathfrak{c}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{COM}_0^{\mathrm{pre}}$ *is a morphism of set-preoperads.*

Proof. First let $m \in \mathbb{Z}_{\geqslant 0}$. We have $\mathrm{id}_{\mathrm{Map}_0^{\mathrm{op}}, m} = (\mathrm{id}_{\mathrm{Map}_0, m})^{\mathrm{op}}$ and hence

$$(\mathrm{id}_{\mathrm{Map}_0, m})^{\mathrm{op}} \mathfrak{c}_0 = \mathrm{id}_{\mathrm{Map}_0, m} \setminus \mathrm{id}_{\mathrm{Map}_0, m} = \mathrm{id}_m \,.$$

Now suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$ and $f'^{\,\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m',n')$, that is, $f : [1,n] \longrightarrow [1,m]$ and $f' : [1,n'] \longrightarrow [1,m']$ are maps. We have

$$\begin{aligned} f^{\mathrm{op}} \boxtimes_{\mathrm{Map}_0^{\mathrm{op}}} f'^{\,\mathrm{op}} &\overset{2.13}{=} (f \boxtimes_{\mathrm{Map}_0} f')^{\mathrm{op}} \\ &\overset{9.5\,(\mathrm{iii})}{=} (\varphi_{(n,n')} \cdot_{\mathrm{Map}_0} (f \sqcup f') \cdot_{\mathrm{Map}_0} \varphi^{-1}_{(m,m')})^{\mathrm{op}} \\ &= (\varphi_{(n,n')}(f \sqcup f')\varphi^{-1}_{(m,m')})^{\mathrm{op}}. \end{aligned}$$

We also know that

$$\begin{aligned} \mathrm{id}_{\mathrm{Map}_0, n+n'} &= \mathrm{id}_{\mathrm{Map}_0, n} \boxtimes_{\mathrm{Map}_0} \mathrm{id}_{\mathrm{Map}_0, n'} \\ &= \varphi_{(n,n')}(\mathrm{id}_{\mathrm{Map}_0, n} \sqcup \mathrm{id}_{\mathrm{Map}_0, n'})\varphi^{-1}_{(n,n')}. \end{aligned}$$

So since $\varphi_{(n,n')}$ is a bijective map we have

$$\begin{aligned}
(f^{\mathrm{op}} \boxtimes_{\mathrm{Map}_0^{\mathrm{op}}} f'^{\,\mathrm{op}})\mathfrak{c}_0 &= (\varphi_{(n,n')}(f \sqcup f')\varphi^{-1}_{(m,m')}) \setminus (\varphi_{(n,n')}(\mathrm{id}_{\mathrm{Map}_0,n} \sqcup \mathrm{id}_{\mathrm{Map}_0,n'})\varphi^{-1}_{(n,n')}) \\
&= ((f \sqcup f')\varphi^{-1}_{(m,m')}) \setminus ((\mathrm{id}_{\mathrm{Map}_0,n} \sqcup \mathrm{id}_{\mathrm{Map}_0,n'})\varphi^{-1}_{(n,n')}) \\
&= (f \setminus \mathrm{id}_{\mathrm{Map}_0,n}) \boxtimes (f' \setminus \mathrm{id}_{\mathrm{Map}_0,n'}) \\
&= f^{\mathrm{op}}\mathfrak{c}_0 \boxtimes f'^{\,\mathrm{op}}\mathfrak{c}_0 \,.
\end{aligned}$$

Finally, suppose given $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$ and $g^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(n,p)$, that is, $f : [1,n] \longrightarrow [1,m]$ and $g : [1,p] \longrightarrow [1,n]$ are maps. By Remark 9.5 (i) we have

$$\begin{aligned}
f^{\mathrm{op}}\mathfrak{c}_0 \cdot g^{\mathrm{op}}\mathfrak{c}_0 &= (f \setminus \mathrm{id}_{\mathrm{Map}_0,n}) \cdot (g \setminus \mathrm{id}_{\mathrm{Map}_0,p}) \\
&= (gf) \setminus \mathrm{id}_{\mathrm{Map}_0,p} \\
&= (gf)^{\mathrm{op}}\mathfrak{c}_0 \\
&= (f^{\mathrm{op}} \cdot_{\mathrm{Map}_0^{\mathrm{op}}} g^{\mathrm{op}})\mathfrak{c}_0 \,.
\end{aligned}$$

□

Lemma 9.8. *We have the set-operad* $\mathrm{COM}_0 := (\mathrm{COM}_0^{\mathrm{pre}}, \mathfrak{c}_0)$.

Proof. Ad (so1). Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$. Suppose given $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$ and $f' \setminus a' \in \mathrm{COM}_0^{\mathrm{pre}}(m',n')$, where $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,m'] \xleftarrow{f'} X' \xrightarrow{a'} [1,n']$ and $X, X' \in \mathcal{E}$. We have to show that

$$(*) \qquad (s^{\mathrm{op}}_{m,m'}\mathfrak{c}_0) \cdot \big((f \setminus a) \boxtimes (f' \setminus a')\big) \overset{!}{=} \big((f' \setminus a') \boxtimes (f \setminus a)\big) \cdot (s^{\mathrm{op}}_{n,n'}\mathfrak{c}_0).$$

By Remark 9.5 (i) we have

$$\begin{aligned}
(s^{\mathrm{op}}_{m,m'}\mathfrak{c}_0) \cdot \big((f \setminus a) \boxtimes (f' \setminus a')\big) &= (s_{m,m'} \setminus \mathrm{id}_{\mathrm{Map}_0,m+m'}) \cdot \big(((f \sqcup f')\varphi^{-1}_{(m,m')}) \setminus ((a \sqcup a')\varphi^{-1}_{(n,n')})\big) \\
&= \big((f \sqcup f')\varphi^{-1}_{(m,m')}s_{m,m'}\big) \setminus \big((a \sqcup a')\varphi^{-1}_{(n,n')}\big).
\end{aligned}$$

In order to calculate the right hand side of $(*)$, define the map

$$\begin{aligned}
\sigma_{Y,Y'} : Y \sqcup Y' &\longrightarrow Y' \sqcup Y \\
(1,y) &\longmapsto (2,y) \\
(2,y') &\longmapsto (1,y')
\end{aligned}$$

for sets Y, Y'.

Claim 1. Suppose given $k, k' \in \mathbb{Z}_{\geqslant 0}$, $Y, Y' \in \mathcal{E}$ and maps $g : Y \longrightarrow [1,k]$ and $g' : Y' \longrightarrow [1,k']$. We have the following pullback.

$$\begin{array}{ccc}
Y \sqcup Y' & \xrightarrow{(g \sqcup g')\varphi^{-1}_{(k,k')}} & [1, k+k'] \\
{\scriptstyle \sigma_{Y,Y'}}\downarrow & \ulcorner & \downarrow{\scriptstyle s_{k,k'}} \\
Y' \sqcup Y & \xrightarrow[(g' \sqcup g)\varphi^{-1}_{(k',k)}]{} & [1, k'+k]
\end{array}$$

Proof of Claim 1. First note that the diagram is commutative, since for $y \in Y$ we have

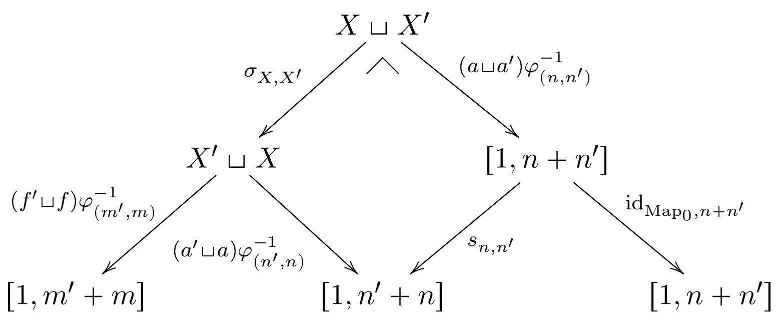
$$\begin{aligned}
(1,y)\sigma_{Y,Y'}(g' \sqcup g)\varphi^{-1}_{(k',k)} &= (2,y)(g' \sqcup g)\varphi^{-1}_{(k',k)} \\
&= (2,yg)\varphi^{-1}_{(k',k)} \\
&= k' + yg \\
(1,y)(g \sqcup g')\varphi^{-1}_{(k,k')}s_{k,k'} &= (1,yg)\varphi^{-1}_{(k,k')}s_{k,k'} \\
&= (yg)s_{k,k'} \\
&= k' + yg
\end{aligned}$$

and for $y' \in Y'$ we have

$$\begin{aligned}
(2,y')\sigma_{Y,Y'}(g' \sqcup g)\varphi^{-1}_{(k',k)} &= (1,y')(g' \sqcup g)\varphi^{-1}_{(k',k)} \\
&= (1,y'g')\varphi^{-1}_{(k',k)} \\
&= y'g' \\
(2,y')(g \sqcup g')\varphi^{-1}_{(k,k')}s_{k,k'} &= (2,y'g')\varphi^{-1}_{(k,k')}s_{k,k'} \\
&= (k + y'g')s_{k,k'} \\
&= (k + y'g') - k \\
&= y'g'.
\end{aligned}$$

Now $\sigma_{Y,Y'}$ and $s_{k,k'}$ are bijective maps. So by Lemma 1.33 the commutative diagram is a pullback. This completes the proof of *Claim* 1.

Applying Claim 1 to $a : X \longrightarrow [1,n]$ and $a' : X' \longrightarrow [1,n']$ yields the folllowing diagram.



So we have

$$\begin{aligned}
\big((f' \setminus a') \boxtimes (f \setminus a)\big) \cdot (s^{\mathrm{op}}_{n,n'}\mathfrak{c}_0) \quad &= \quad \big(((f' \sqcup f)\varphi^{-1}_{(m',m)}) \setminus ((a' \sqcup a)\varphi^{-1}_{(n',n)})\big) \cdot (s_{n,n'} \setminus \mathrm{id}_{\mathrm{Map}_0,n+n'}) \\
&= \quad \big(\sigma_{X,X'}(f' \sqcup f)\varphi^{-1}_{(m',m)}\big) \setminus \big((a \sqcup a')\varphi^{-1}_{(n,n')}\big) \\
&\overset{\text{Claim 1}}{\underset{\text{for } f,\, f'}{=}} \quad \big((f \sqcup f')\varphi^{-1}_{(m,m')}s_{m,m'}\big) \setminus \big((a \sqcup a')\varphi^{-1}_{(n,n')}\big) \\
&= \quad (s^{\mathrm{op}}_{m,m'}\mathfrak{c}_0) \cdot \big((f \setminus a) \boxtimes (f' \setminus a')\big).
\end{aligned}$$

This shows (so1).

Ad (so2). Suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{COM}_0(m,n)$, where $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $X \in \mathcal{E}$. We have to show that

$$(**) \qquad (h^{\mathrm{op}}_{k,m}\mathfrak{c}_0) \cdot (f \setminus a)^{\boxtimes k} \overset{!}{=} (f \setminus a) \cdot (h^{\mathrm{op}}_{k,n}\mathfrak{c}_0).$$

Define $s_i := m$ for $i \in [1,k]$ and $t_i := n$ for $i \in [1,k]$. Then let $s := (s_i)_{i \in [1,k]} \in (\mathbb{Z}_{\geqslant 0})^{\times k}$, $t := (t_i)_{i \in [1,k]} \in (\mathbb{Z}_{\geqslant 0})^{\times k}$.

First we will show that $(f \setminus a)^{\boxtimes k} \overset{!}{=} (f^{\sqcup k}\varphi_s^{-1}) \setminus (a^{\sqcup k}\varphi_t^{-1})$.

We will show this via induction on $k \geqslant 0$. If $k = 0$ then we have $s = t = () \in (\mathbb{Z}_{\geqslant 0})^{\times 0}$. So we have $(f \setminus a)^{\boxtimes 0} = \mathrm{id}_0$ and $f^{\sqcup 0}\varphi_{()}^{-1} = \mathrm{id}_{\mathrm{Map}_0,0}$ and $a^{\sqcup 0}\varphi_{()}^{-1} = \mathrm{id}_{\mathrm{Map}_0,0}$, hence

$$(f \setminus a)^{\boxtimes 0} = \mathrm{id}_0 = \mathrm{id}_{\mathrm{Map}_0,0} \setminus \mathrm{id}_{\mathrm{Map}_0,0} = (f^{\sqcup 0}\varphi_{()}^{-1}) \setminus (a^{\sqcup 0}\varphi_{()}^{-1}).$$

Assume now that $k \geqslant 1$ and that we already know that $(f \setminus a)^{\boxtimes (k-1)} = (f^{\sqcup (k-1)}\varphi_{\hat{s}}^{-1}) \setminus (a^{\sqcup (k-1)}\varphi_{\hat{t}}^{-1})$ for $\hat{s} = (s_i)_{i \in [1,k-1]} \in (\mathbb{Z}_{\geqslant 0})^{\times (k-1)}$ and $\hat{t} = (t_i)_{i \in [1,k-1]} \in (\mathbb{Z}_{\geqslant 0})^{\times (k-1)}$.

Recall the bijective map $\gamma_{k,X} : X^{\sqcup k} \longrightarrow X^{\sqcup (k-1)} \sqcup X$; cf. Definition 1.21. We have

$$\begin{array}{rl}
(f \setminus a)^{\boxtimes k} = & (f \setminus a)^{\boxtimes (k-1)} \boxtimes (f \setminus a) \\
= & \big((f^{\sqcup (k-1)}\varphi_{\hat{s}}^{-1}) \setminus (a^{\sqcup (k-1)}\varphi_{\hat{t}}^{-1})\big) \boxtimes (f \setminus a) \\
= & \big(((f^{\sqcup (k-1)}\varphi_{\hat{s}}^{-1}) \sqcup f)\varphi_{((k-1)m,m)}^{-1}\big) \setminus \big(((a^{\sqcup (k-1)}\varphi_{\hat{t}}^{-1}) \sqcup a)\varphi_{((k-1)n,n)}^{-1}\big) \\
= & \big(\gamma_{k,X}((f^{\sqcup (k-1)}\varphi_{\hat{s}}^{-1}) \sqcup f)\varphi_{((k-1)m,m)}^{-1}\big) \setminus \big(\gamma_{k,X}((a^{\sqcup (k-1)}\varphi_{\hat{t}}^{-1}) \sqcup a)\varphi_{((k-1)n,n)}^{-1}\big) \\
\overset{1.24\text{(i)}}{=} & \big(\gamma_{k,X}(f^{\sqcup (k-1)} \sqcup f)(\varphi_{\hat{s}}^{-1} \sqcup \mathrm{id}_m)\varphi_{((k-1)m,m)}^{-1}\big) \setminus \big(\gamma_{k,X}(a^{\sqcup (k-1)} \sqcup a)(\varphi_{\hat{t}}^{-1} \sqcup \mathrm{id}_n)\varphi_{((k-1)n,n)}^{-1}\big) \\
\overset{1.22}{=} & \big((f^{\sqcup k}\gamma_{k,[1,m]})(\varphi_{\hat{s}}^{-1} \sqcup \mathrm{id}_m)\varphi_{((k-1)m,m)}^{-1})\big) \setminus \big((a^{\sqcup k}\gamma_{k,[1,n]})(\varphi_{\hat{t}}^{-1} \sqcup \mathrm{id}_n)\varphi_{((k-1)n,n)}^{-1})\big) \\
\overset{1.23}{=} & (f^{\sqcup k}\varphi_s^{-1}) \setminus (a^{\sqcup k}\varphi_t^{-1}).
\end{array}$$

Now by Remark 9.5 (i) we have

$$\begin{aligned}
(h_{k,m}^{\mathrm{op}}\mathfrak{c}_0) \cdot (f \setminus a)^{\boxtimes k} &= (h_{k,m} \setminus \mathrm{id}_{\mathrm{Map}_0,km}) \cdot (f^{\sqcup k}\varphi_s^{-1}) \setminus (a^{\sqcup k}\varphi_t^{-1}) \\
&= (f^{\sqcup k}\varphi_s^{-1}h_{k,m}) \setminus (a^{\sqcup k}\varphi_t^{-1}).
\end{aligned}$$

In order to calculate the right hand side of $(**)$ we define the map

$$\begin{aligned}
\eta_{l,Y} : Y^{\sqcup l} &\longrightarrow Y \\
(j,y) &\longmapsto y
\end{aligned}$$

for a set Y and $l \in \mathbb{Z}_{\geqslant 0}$.

Claim 2. Suppose given $p, l \in \mathbb{Z}_{\geqslant 0}$, $Y \in \mathcal{E}$ and a map $g : Y \longrightarrow [1,p]$. Let $r_i := p$ for $i \in [1,l]$ and $r := (r_i)_{i \in [1,l]} \in (\mathbb{Z}_{\geqslant 0})^{\times l}$. We have the following pullback.

$$\begin{array}{ccc}
Y^{\sqcup l} & \xrightarrow{\ g^{\sqcup l}\varphi_r^{-1}\ } & [1,lp] \\
{\scriptstyle \eta_{l,Y}}\downarrow & \ulcorner & \downarrow{\scriptstyle h_{l,p}} \\
Y & \xrightarrow[\ g\ ]{} & [1,p]
\end{array}$$

Proof of Claim 2. Note that for $(i,j) \in [1,p]^{\sqcup l}$ we have

$$(i,j)\varphi_r^{-1}h_{l,p} = ((i-1)p + j)h_{l,p} = j = (i,j)\eta_{l,[1,p]},$$

hence we have the following commutative diagram.

$$\begin{array}{ccc}
[1,p]^{\sqcup l} & \xrightarrow{\ \varphi_r^{-1}\ } & [1,lp] \\
{\scriptstyle \eta_{l,[1,p]}}\downarrow & & \downarrow{\scriptstyle h_{l,p}} \\
[1,p] & \xrightarrow[\ \mathrm{id}_{[1,p]}\ ]{} & [1,p]
\end{array}$$

Since φ_r^{-1} and $\mathrm{id}_{[1,p]}$ are bijective, this is a pullback by Lemma 1.33.

So by Lemma 1.37 it suffices to show that we have the following pullback.

$$\begin{array}{ccc} Y^{\sqcup l} & \xrightarrow{g^{\sqcup l}} & [1,p]^{\sqcup l} \\ {\scriptstyle \eta_{l,Y}}\downarrow & \ulcorner & \downarrow{\scriptstyle \eta_{l,[1,p]}} \\ Y & \xrightarrow[g]{} & [1,p] \end{array}$$

First note that for $(i,y) \in Y^{\sqcup l}$ we have

$$(i,y)(g^{\sqcup l}\eta_{l,[1,p]}) = (i,yg)\eta_{l,[1,p]} = yg = (i,y)\eta_{l,Y}g,$$

hence the diagram commutes.

Now suppose given $(i,j) \in [1,p]^{\sqcup l}$. We have to show that $\eta_{l,Y}\big|_{(g^{\sqcup l})^{-1}(i,j)}^{g^{-1}((i,j)\eta_{l,[1,p]})}$ is bijective.

Note that $(g^{\sqcup l})^{-1}(i,j) = \{(i,y) : y \in g^{-1}(j)\}$ and $g^{-1}((i,j)\eta_{l,[1,p]}) = g^{-1}(j)$.

Injectivity. Suppose given $(i,y), (i,y') \in (g^{\sqcup l})^{-1}(i,j)$ such that

$$(i,y)\eta_{l,Y}\big|_{(g^{\sqcup l})^{-1}(i,j)}^{g^{-1}((i,j)\eta_{l,[1,p]})} = (i,y')\eta_{l,Y}\big|_{(g^{\sqcup l})^{-1}(i,j)}^{g^{-1}((i,j)\eta_{l,[1,p]})}.$$

Then we have

$$y = (i,y)\eta_{l,Y} = (i,y)\eta_{l,Y}\big|_{(g^{\sqcup l})^{-1}(i,j)}^{g^{-1}((i,j)\eta_{l,[1,p]})} = (i,y')\eta_{l,Y}\big|_{(g^{\sqcup l})^{-1}(i,j)}^{g^{-1}((i,j)\eta_{l,[1,p]})} = (i,y')\eta_{l,Y} = y'.$$

This shows that $\eta_{l,Y}\big|_{(g^{\sqcup l})^{-1}(i,j)}^{g^{-1}((i,j)\eta_{l,[1,p]})}$ is injective.

Surjectivity. Suppose given $y \in g^{-1}((i,j)\eta_{l,[1,p]}) = g^{-1}(j)$. We have $(i,y)g^{\sqcup l} = (i,yg) = (i,j)$, so $(i,y) \in (g^{\sqcup l})^{-1}(i,j)$ and

$$(i,y)\eta_{l,Y}\big|_{(g^{\sqcup l})^{-1}(i,j)}^{g^{-1}((i,j)\eta_{l,[1,p]})} = (i,y)\eta_{l,Y} = y.$$

This shows that $\eta_{l,Y}\big|_{(g^{\sqcup l})^{-1}(i,j)}^{g^{-1}((i,j)\eta_{l,[1,p]})}$ is surjective.

This completes the proof of *Claim* 2.

Recall that we have defined $s = (s_i)_{i\in[1,k]}, t = (t_i)_{i\in[1,k]} \in (\mathbb{Z}_{\geqslant 0})^{\times k}$ with $s_i = m$ and $t_i = n$ for $i \in [1,k]$.

Applying Claim 2 to the map $a : X \longrightarrow [1,n]$ and $k \in \mathbb{Z}_{\geqslant 0}$ we get the following diagram

$$\begin{array}{ccccccc} & & & X^{\sqcup k} & & & \\ & & \swarrow^{\eta_{k,X}} & \wedge & \searrow^{a^{\sqcup k}\varphi_t^{-1}} & & \\ & X & & & & [1,kn] & \\ \swarrow^{f} & & \searrow^{a} & & \swarrow^{h_{k,n}} & & \searrow^{\mathrm{id}_{\mathrm{Map}_0,kn}} \\ [1,m] & & & [1,n] & & & [1,kn] \end{array}$$

So we have

$$\begin{array}{rcl} (f \setminus a)\cdot(h_{k,n}^{\mathrm{op}}\mathfrak{c}_0) & = & (f\setminus a)\cdot(h_{k,n}\setminus \mathrm{id}_{\mathrm{Map}_0,kn}) \\ & = & (\eta_{k,X}f)\setminus(a^{\sqcup k}\varphi_t^{-1}) \\ & \overset{\text{Claim 2}}{\underset{\text{for } f,\, k}{=}} & (f^{\sqcup k}\varphi_s^{-1}h_{k,m})\setminus(a^{\sqcup k}\varphi_t^{-1}). \end{array}$$

So the left hand side and the right hand side of $(**)$ coincide.

This shows (so2) and completes the proof that $\mathrm{COM}_0 = (\mathrm{COM}_0^{\mathrm{pre}}, \mathfrak{c}_0)$ is a set-operad. $\square$

Remark 9.9. *Let $m,n,k,m',n',k' \in \mathbb{Z}_{\geqslant 0}$ and let $f \in \mathrm{Map}_0(k,m)$, $a \in \mathrm{Map}_0(k,n)$, $f' \in \mathrm{Map}_0(k',m')$ and $a' \in \mathrm{Map}_0(k',n')$. So we have $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$ and $f' \setminus a' \in \mathrm{COM}_0^{\mathrm{pre}}(m',n')$.*

Then we have

$$(f \setminus a) \boxtimes (f' \setminus a') = (f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Map}_0} a').$$

Proof. We have

$$\begin{aligned}(f \setminus a) \boxtimes (f' \setminus a') &= \big((f \sqcup f')\varphi^{-1}_{(m,m')}\big) \setminus \big((a \sqcup a')\varphi^{-1}_{(n,n')}\big) \\ &= (\varphi_{(k,k')}(f \sqcup f')\varphi^{-1}_{(m,m')}) \setminus (\varphi_{(k,k')}(a \sqcup a')\varphi^{-1}_{(n,n')}) \\ &\overset{9.5\,\text{(iii)}}{=} (f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Map}_0} a').\end{aligned}$$

□

Lemma 9.10. *Define the biindexed map $\kappa_0^{\mathrm{pre}} = (\kappa_0^{\mathrm{pre}}(m,n))_{m,n \geqslant 0} : \mathrm{ASS}_0^{\mathrm{pre}} \longrightarrow \mathrm{COM}_0^{\mathrm{pre}}$ as follows. For $m,n \in \mathbb{Z}_{\geqslant 0}$ let*

$$\begin{aligned}\kappa_0^{\mathrm{pre}}(m,n) : \mathrm{ASS}_0^{\mathrm{pre}}(m,n) &\longrightarrow \mathrm{COM}_0^{\mathrm{pre}}(m,n) \\ f \setminus a &\longmapsto f \setminus a.\end{aligned}$$

Then $\kappa_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{COM}_0$ is a morphism of set-operads.

Proof. First note that for $m \in \mathbb{Z}_{\geqslant 0}$ we have

$$\mathrm{id}_{\mathrm{ASS}_0,m}\, \kappa_0^{\mathrm{pre}} = (\mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Ass}_0,m})\kappa_0^{\mathrm{pre}} = \mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Ass}_0,m} = \mathrm{id}_{\mathrm{Map}_0,m} \setminus \mathrm{id}_{\mathrm{Map}_0,m} = \mathrm{id}_{\mathrm{COM}_0,m}\,.$$

Suppose given $m,n,m',n' \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$, $f' \setminus a' \in \mathrm{ASS}_0^{\mathrm{pre}}(m',n')$. Let $k,k' \in \mathbb{Z}_{\geqslant 0}$ such that

$$[1,m] \xleftarrow{\;f\;} [1,k] \xrightarrow{\;a\;} [1,n] \quad\text{and}\quad [1,m'] \xleftarrow{\;f'\;} [1,k'] \xrightarrow{\;a'\;} [1,n'].$$

Then we have

$$\begin{aligned}\big((f \setminus a) \boxtimes_{\mathrm{ASS}_0} (f' \setminus a')\big)\kappa_0^{\mathrm{pre}} &= \big((f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Ass}_0} a')\big)\kappa_0^{\mathrm{pre}} \\ &= \big((f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Map}_0} a')\big)\kappa_0^{\mathrm{pre}} \\ &= (f \boxtimes_{\mathrm{Map}_0} f') \setminus (a \boxtimes_{\mathrm{Map}_0} a') \\ &\overset{9.9}{=} (f \setminus a) \boxtimes_{\mathrm{COM}_0} (f' \setminus a').\end{aligned}$$

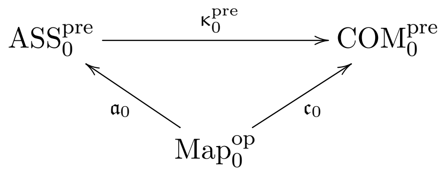
Now suppose given $m,n,p \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$ and $g \setminus b \in \mathrm{ASS}_0^{\mathrm{pre}}(n,p)$. Let $k,l \in \mathbb{Z}_{\geqslant 0}$ such that

$$[1,m] \xleftarrow{\;f\;} [1,k] \xrightarrow{\;a\;} [1,n] \quad\text{and}\quad [1,n] \xleftarrow{\;g\;} [1,l] \xrightarrow{\;b\;} [1,p].$$

Let $([1,s], \hat{g}, \hat{a})$ be the sorted pullback of a and g; cf. Definition 7.1. By Remark 7.3 this is a pullback of a and g. So we have

$$\begin{aligned}\big((f \setminus a) \cdot_{\mathrm{ASS}_0} (g \setminus b)\big)\kappa_0^{\mathrm{pre}} &= (\hat{g}f \setminus \hat{a}b)\kappa_0^{\mathrm{pre}} \\ &= \hat{g}f \setminus \hat{a}b \\ &= (f \setminus a) \cdot_{\mathrm{COM}_0} (g \setminus b).\end{aligned}$$

This shows that κ_0^{pre} is a morphism of set-preoperads. It remains to show that we have the following commutative diagram.



But for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m, n)$ we have

$$f^{\mathrm{op}}\mathfrak{a}_0\kappa_0^{\mathrm{pre}} = (f \setminus \mathrm{id}_{\mathrm{Ass}_0,n})\kappa_0^{\mathrm{pre}} = f \setminus \mathrm{id}_{\mathrm{Ass}_0,n} = f \setminus \mathrm{id}_{\mathrm{Map}_0,n} = f^{\mathrm{op}}\mathfrak{c}_0.$$

Hence $\mathfrak{a}_0\kappa_0^{\mathrm{pre}} = \mathfrak{c}_0$ and the diagram commutes. □

9.2 Commutative monoids and COM_0-algebras

> We will now see that given a COM_0-algebra (S, Φ_0), then S can be turned into a commutative monoid and that, given a commutative monoid $(S, \mu_S, \varepsilon_S)$, then there exists a morphism of set-operads $\Phi_0 : \mathrm{COM}_0 \longrightarrow \mathrm{END}_0(S)$ with $\mu_S = (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}}$ and $\varepsilon_S = (\mathrm{id}_0 \setminus \varepsilon)\Phi_0^{\mathrm{pre}}$, yielding a COM_0-algebra (S, Φ_0).

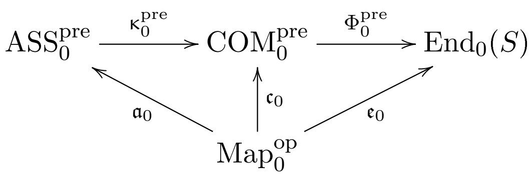
During §9.2 we will denote by id_m for $m \in \mathbb{Z}_{\geqslant 0}$ the identity elements in Map_0 and Ass_0 and by $(\boxtimes)$ and $(\cdot)$ the multiplication and composition in Map_0 and Ass_0 .

Proposition 9.11. *Let S be a set and let $\Phi_0 : \mathrm{COM}_0 \longrightarrow \mathrm{END}_0(S)$ be a morphism of set-operads, that is, (S, Φ_0) is a COM_0-algebra.*

Define $\mu_S := (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}} : S^{\times 2} \longrightarrow S^{\times 1} = S$ and $\varepsilon_S := (\mathrm{id}_0 \setminus \varepsilon)\Phi_0^{\mathrm{pre}} : \{()\} = S^{\times 0} \longrightarrow S$.

Then $(S, \mu_S, \varepsilon_S)$ is a commutative monoid.

Proof. We have the following commutative diagram of set-preoperads.



So $\kappa_0\Phi_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{END}_0(S)$ is a morphism of set-operads. This means that $(S, \kappa_0\Phi_0)$ is an ASS_0-algebra.

By Proposition 7.16, the set S is an associative monoid when equipped with multiplication $(\mathrm{id}_2 \setminus \mu)(\kappa_0^{\mathrm{pre}}\Phi_0^{\mathrm{pre}}) = (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}} = \mu_S$ and unit $(\mathrm{id}_0 \setminus \varepsilon)(\kappa_0^{\mathrm{pre}}\Phi_0^{\mathrm{pre}}) = (\mathrm{id}_0 \setminus \varepsilon)\Phi_0^{\mathrm{pre}} = \varepsilon_S$.

Note that $((1, 2) \setminus \mathrm{id}_2)\Phi_0^{\mathrm{pre}} = ((1, 2)^{\mathrm{op}}\mathfrak{c}_0)\Phi_0^{\mathrm{pre}} = (1, 2)^{\mathrm{op}}\mathfrak{e}_0 \in \mathrm{End}_0(S)(2, 2)$ is the map

$$\begin{aligned} \tau_S : S^{\times 2} &\longrightarrow S^{\times 2} \\ (s, t) &\longmapsto (t, s); \end{aligned}$$

cf. Example 2.63. In COM_0 , expansion by the transposition $(1, 2) : [1, 2] \longrightarrow [1, 2]$ yields

$$(1, 2) \setminus \mu = ((1, 2)(1, 2)) \setminus ((1, 2)\mu) = \mathrm{id}_2 \setminus \mu;$$

cf. Definition 9.3 and Remark 9.4.

So by the definition of composition in COM_0 we have

$$\begin{aligned}\tau_S\mu_S &= ((1,2)\setminus \mathrm{id}_2)\Phi_0^{\mathrm{pre}} \cdot_{\mathrm{End}_0} (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}} \\ &= \big((1,2)\setminus \mathrm{id}_2) \cdot_{\mathrm{COM}_0} (\mathrm{id}_2 \setminus \mu)\big)\Phi_0^{\mathrm{pre}} \\ &= \big((\mathrm{id}_2(1,2))\setminus (\mathrm{id}_2\,\mu)\big)\Phi_0^{\mathrm{pre}} \\ &= ((1,2)\setminus \mu)\Phi_0^{\mathrm{pre}} \\ &= (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}} \\ &= \mu_S\,.\end{aligned}$$

So $(S, \mu_S, \varepsilon_S)$ is a commutative monoid. □

Lemma 9.12. *Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and let $a : [1,m] \longrightarrow [1,n]$ and $t : [1,m] \longrightarrow [1,m]$ be maps such that t is bijective and both a and ta are monotone.*

Then we have $ta = a$.

Proof. Assume that there exists $i \in [1,m]$ such that $ita \neq ia$.

Case 1: $ita < ia$. Since ta is monotone we have $j(ta) \leqslant i(ta) < ia$ for $j \in [1,i]$. Since t is bijective, the restricted map

$$\begin{aligned}[1,i] &\longrightarrow \{x \in [1,m] : xa < ia\} \\ j &\longmapsto jt\end{aligned}$$

is injective. Since a is monotone, we have $\{x \in [1,m] : xa < ia\} \subseteq [1, i-1]$. So we have an injective map $[1,i] \longrightarrow \{x \in [1,m] : xa < ia\} \subseteq [1,i-1]$, a *contradiction.*

Case 2: $ita > ia$. Since ta is monotone we have $j(ta) \geqslant i(ta) > ia$ for $j \in [i,m]$. Since t is bijective, the restricted map

$$\begin{aligned}[i,m] &\longrightarrow \{x \in [1,m] : xa > ia\} \\ j &\longmapsto jt\end{aligned}$$

is injective. Since a is monotone, we have $\{x \in [1,m] : xa > ia\} \subseteq [i+1, m]$. So we have an injective map $[1,i] \longrightarrow \{x \in [1,m] : xa > ia\} \subseteq [i+1,m]$, a *contradiction.* □

Lemma 9.13. *Let $\mathcal{A}_0 \subseteq \mathrm{ASS}_0^{\mathrm{pre}}$ be a set-subpreoperad such that $\mathrm{ASS}_0^{\mathrm{pre,bij}} \subseteq \mathcal{A}_0$. Let $\mathcal{T}_0$ be a set-preoperad and let $\tau_0 : \mathcal{A}_0 \longrightarrow \mathcal{T}_0$ be a morphism of set-preoperads such that* $(\mathrm{id}_2 \setminus \mu)\tau_0 = ((1,2)\setminus \mu)\tau_0$.

Then we have $(s \setminus \mu_l)\tau_0 = (\mathrm{id}_l \setminus \mu_l)\tau_0$ for $l \in \mathbb{Z}_{\geqslant 0}$ and $s \in \mathrm{Sym}_0(l,l)$.

Proof. We prove this via induction on $l \geqslant 0$.

If $l = 0$, then we have $\mathrm{Sym}_0(0,0) = \{\mathrm{id}_0\}$, so there is nothing to show.

Now let $l \geqslant 1$ and suppose that the statement is true for $r \in [1, l-1]$. Let $s : [1,l] \longrightarrow [1,l]$ be a bijective map. Let $i \in [1, l+1]$ be maximal with the property that $js = j$ for $j \in [1, i-1]$.

Case 1: $i = l+1$. Then $s = \mathrm{id}_l$ and there is nothing to show.

Case 2: $i \in [2,l]$. Note that by the choice of i and since s is injective, we know that $is > i$. Since $js = j$ for $j \in [1, i-1]$ we can write $s = \mathrm{id}_{i-1} \boxtimes \tilde{s}$ for a bijective map $\tilde{s} : [1, l-i+1] \longrightarrow [1, l-i+1]$. So by induction we have $(\tilde{s} \setminus \mu_{l-i+1})\tau_0 = (\mathrm{id}_{l-i+1} \setminus \mu_{l-i+1})\tau_0$.

So we have

$$
\begin{aligned}
(s \setminus \mu_l)\tau_0 &= \big((\mathrm{id}_{i-1} \boxtimes \tilde{s}) \setminus \mu_l\big)\tau_0 \\
&= \big((\mathrm{id}_{i-1} \boxtimes \tilde{s}) \setminus ((\mathrm{id}_{i-1} \boxtimes \mu_{l-i+1})\mu_i)\big)\tau_0 \\
&\overset{7.11}{=} \big(((\mathrm{id}_{i-1} \boxtimes \tilde{s}) \setminus (\mathrm{id}_{i-1} \boxtimes \mu_{l-i+1})) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_i \setminus \mu_i)\big)\tau_0 \\
&= \big(((\mathrm{id}_{i-1} \setminus \mathrm{id}_{i-1}) \boxtimes_{\mathrm{ASS}_0} (\tilde{s} \setminus \mu_{l-i+1})) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_i \setminus \mu_i)\big)\tau_0 \\
&= \big((\mathrm{id}_{i-1} \setminus \mathrm{id}_{i-1})\tau_0 \boxtimes_{\mathcal{T}_0} (\tilde{s} \setminus \mu_{l-i+1})\tau_0\big) \cdot_{\mathcal{T}_0} (\mathrm{id}_i \setminus \mu_i)\,\tau_0 \\
&\overset{\mathrm{ind.}}{=} \big((\mathrm{id}_{i-1} \setminus \mathrm{id}_{i-1})\tau_0 \boxtimes_{\mathcal{T}_0} (\mathrm{id}_{l-i+1} \setminus \mu_{l-i+1})\tau_0\big) \cdot_{\mathcal{T}_0} (\mathrm{id}_i \setminus \mu_i)\,\tau_0 \\
&= \big(((\mathrm{id}_{i-1} \setminus \mathrm{id}_{i-1}) \boxtimes_{\mathrm{ASS}_0} (\mathrm{id}_{l-i+1} \setminus \mu_{l-i+1})) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_i \setminus \mu_i)\big)\tau_0 \\
&= \big(((\mathrm{id}_{i-1} \boxtimes \mathrm{id}_{l-i+1}) \setminus (\mathrm{id}_{i-1} \boxtimes \mu_{l-i+1})) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_i \setminus \mu_i)\big)\tau_0 \\
&\overset{7.11}{=} \big(\mathrm{id}_l \setminus ((\mathrm{id}_{i-1} \boxtimes \mu_{l-i+1})\mu_i)\big)\tau_0 \\
&= (\mathrm{id}_l \setminus \mu_l)\,\tau_0 \,.
\end{aligned}
$$

Case 3: $i = 1$. Let $j := 1s^{-1} > 1$. Consider the set of inversions of s.

$$\mathrm{inv}(s) := \{(u, v) \in [1, l] \times [1, l] : \; u < v \text{ and } us > vs\}$$

We proceed via induction on $|\,\mathrm{inv}(s)|$.

For $u \in [1, j-1]$ we have $(u, j) \in \mathrm{inv}(s)$, since $u < j$ and $us > 1 = js$.

Define the bijective map $t := \mathrm{id}_{j-2} \boxtimes (1,2) \boxtimes \mathrm{id}_{l-j} = (j-1, j) \in \mathrm{Sym}_0(l, l)$. So $t^2 = \mathrm{id}_l$. We have

$$
\begin{aligned}
\big(t \setminus (\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j})\big)\tau_0 &= \big((\mathrm{id}_{j-2} \boxtimes (1,2) \boxtimes \mathrm{id}_{l-j}) \setminus (\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j})\big)\tau_0 \\
&= \big((\mathrm{id}_{j-2} \setminus \mathrm{id}_{j-2}) \boxtimes ((1,2) \setminus \mu) \boxtimes_{\mathrm{ASS}_0} (\mathrm{id}_{l-j} \setminus \mathrm{id}_{l-j})\big)\tau_0 \\
&= (\mathrm{id}_{j-2} \setminus \mathrm{id}_{j-2})\,\tau_0 \boxtimes_{\mathcal{T}_0} ((1,2) \setminus \mu)\,\tau_0 \boxtimes_{\mathcal{T}_0} (\mathrm{id}_{l-j} \setminus \mathrm{id}_{l-j})\,\tau_0 \\
&= (\mathrm{id}_{j-2} \setminus \mathrm{id}_{j-2})\,\tau_0 \boxtimes_{\mathcal{T}_0} (\mathrm{id}_2 \setminus \mu)\,\tau_0 \boxtimes_{\mathcal{T}_0} (\mathrm{id}_{l-j} \setminus \mathrm{id}_{l-j})\,\tau_0 \\
&= \big((\mathrm{id}_{j-2} \setminus \mathrm{id}_{j-2}) \boxtimes_{\mathrm{ASS}_0} (\mathrm{id}_2 \setminus \mu) \boxtimes_{\mathrm{ASS}_0} (\mathrm{id}_{l-j} \setminus \mathrm{id}_{l-j})\big)\tau_0 \\
&= \big((\mathrm{id}_{j-2} \boxtimes \mathrm{id}_2 \boxtimes \mathrm{id}_{l-j}) \setminus (\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j})\big)\tau_0 \\
&= \big(\mathrm{id}_l \setminus (\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j})\big)\tau_0 \,.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(t \setminus \mu_l)\tau_0 &= \big(t \setminus ((\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j})\mu_{l-1})\big)\tau_0 \\
&\overset{7.11}{=} \big((t \setminus (\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j})) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_{l-1} \setminus \mu_{l-1})\big)\tau_0 \\
&= (t \setminus (\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j}))\,\tau_0 \cdot_{\mathcal{T}_0} (\mathrm{id}_{l-1} \setminus \mu_{l-1})\,\tau_0 \\
&= (\mathrm{id}_l \setminus (\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j}))\,\tau_0 \cdot_{\mathcal{T}_0} (\mathrm{id}_{l-1} \setminus \mu_{l-1})\,\tau_0 \\
&= \big((\mathrm{id}_l \setminus (\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j})) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_{l-1} \setminus \mu_{l-1})\big)\tau_0 \\
&\overset{7.11}{=} \big(\mathrm{id}_l \setminus ((\mathrm{id}_{j-2} \boxtimes \mu \boxtimes \mathrm{id}_{l-j})\mu_{l-1})\big)\tau_0 \\
&= (\mathrm{id}_l \setminus \mu_l)\,\tau_0 \,.
\end{aligned}
$$

Note that we may alternatively apply *Case* 2 only if $j \geqslant 3$.

Note that $ts : [1, l] \longrightarrow [1, l]$ is a bijective map and that we have $\mathrm{inv}(ts) = \mathrm{inv}(s) \backslash \{(j-1, j)\}$, hence $|\,\mathrm{inv}(ts)| < |\,\mathrm{inv}(s)|$.

If ts falls under *Case* 1 or *Case* 2, we have already shown that $(ts \setminus \mu_l)\tau_0 = (\mathrm{id}_l \setminus \mu_l)\tau_0$.

If ts falls under *Case* 3, then we have $(ts \setminus \mu_l)\tau_0 = (\mathrm{id}_l \setminus \mu_l)\tau_0$ by induction on the number of inversions.

So we have

$$\begin{aligned}
(s \setminus \mu_l)\,\tau_0 &= \left(t^2 s \setminus \mu_l\right)\tau_0 \\
&\overset{7.11}{=} \left((ts \setminus \mathrm{id}_l) \cdot_{\mathrm{ASS}_0} (t \setminus \mu_l)\right)\tau_0 \\
&= (ts \setminus \mathrm{id}_l)\,\tau_0 \cdot_{\mathcal{T}_0} (t \setminus \mu_l)\,\tau_0 \\
&= (ts \setminus \mathrm{id}_l)\,\tau_0 \cdot_{\mathcal{T}_0} (\mathrm{id}_l \setminus \mu_l)\,\tau_0 \\
&= \left((ts \setminus \mathrm{id}_l) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_l \setminus \mu_l)\right)\tau_0 \\
&\overset{7.11}{=} (ts \setminus \mu_l)\,\tau_0 \\
&= (\mathrm{id}_l \setminus \mu_l)\,\tau_0\,.
\end{aligned}$$

□

Lemma 9.14. *Let $(\mathcal{T}_0, \mathrm{t}_0)$ be a set-operad and $\tau_0 : \mathrm{ASS}_0 \longrightarrow \mathcal{T}_0$ be a morphism of set-operads satisfying $(\mathrm{id}_2 \setminus \mu)\tau_0^{\mathrm{pre}} = ((1,2) \setminus \mu)\tau_0^{\mathrm{pre}}$. Then there exists a unique morphism of set-operads $\bar{\tau}_0$ such that $\kappa_0 \bar{\tau}_0 = \tau_0$.*

$$\begin{array}{ccc}
\mathrm{ASS}_0 & \xrightarrow{\tau_0} & \mathcal{T}_0 \\
{\scriptstyle \kappa_0}\big\downarrow & \nearrow{\scriptstyle \bar{\tau}_0} & \\
\mathrm{COM}_0 & &
\end{array}$$

Proof. First note that for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$, where $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ for some finite set $X \in \mathcal{E}$, there exists a bijective map $s : [1,|X|] \longrightarrow X$ such that sa is a monotone map.

Uniqueness. Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$, where $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ for some finite set $X \in \mathcal{E}$. Let $s : [1,|X|] \longrightarrow X$ be a bijective map such that sa is monotone. Then we have $sf \setminus sa \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$. Given a morphism $\tilde{\tau}_0 : \mathrm{COM}_0 \longrightarrow \mathcal{T}_0$ of set-operads satisfying $\kappa_0 \tilde{\tau}_0 = \tau_0$, then we have

$$(sf \setminus sa)\tau_0^{\mathrm{pre}} = (sf \setminus sa)\kappa_0^{\mathrm{pre}} \tilde{\tau}_0^{\mathrm{pre}} = (sf \setminus sa)\tilde{\tau}_0^{\mathrm{pre}} = (f \setminus a)\tilde{\tau}_0^{\mathrm{pre}},$$

so such a morphism of set-operads $\tilde{\tau}_0$ is uniquely determined by the requirement $\kappa_0 \tilde{\tau}_0 = \tau_0$.

Existence. Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and let $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$, where $X \in \mathcal{E}$ is a finite set, $f : X \longrightarrow [1,m]$ and $a : X \longrightarrow [1,n]$. Let $s : [1,|X|] \longrightarrow X$ be a bijective map such that sa is a monotone map. Define $\bar{\tau}_0$ by

$$(f \setminus a)\bar{\tau}_0^{\mathrm{pre}} := (sf \setminus sa)\tau_0^{\mathrm{pre}}.$$

First we have to show that this is well-defined.

Let $\tilde{s} : [1,|X|] \longrightarrow X$ also be a map such that $\tilde{s}a$ is monotone. Then define $t := \tilde{s}s^{-1}$. So $t : [1,|X|] \longrightarrow [1,|X|]$ is a bijective map.

Now define $f' := sf : [1,|X|] \longrightarrow [1,m]$ and $a' := sa : [1,|X|] \longrightarrow [1,n]$.

Then we have $\tilde{s}f = tsf = tf'$ and $\tilde{s}a = tsa = ta'$. So we have a bijective map $t : [1,|X|] \longrightarrow [1,|X|]$ such that both $a' = sa$ and $ta' = \tilde{s}a$ are monotone. So by Lemma 9.12 we know that

$$\tilde{s}a = ta' = a' = sa.$$

We have to show that

$$(f' \setminus a')\tau_0^{\mathrm{pre}} = (sf \setminus sa)\tau_0^{\mathrm{pre}} \overset{!}{=} (\tilde{s}f \setminus \tilde{s}a)\tau_0^{\mathrm{pre}} = (tf' \setminus a')\tau_0^{\mathrm{pre}}\,.$$

Since $ta' = a'$, we know that $(a'^{-1}(j))t = a'^{-1}(j)$ for $j \in [1,n]$. Therefore, if we write

$$a' = \mu_{l_1} \boxtimes_{\mathrm{Ass}_0} \ldots \boxtimes_{\mathrm{Ass}_0} \mu_{l_n}\,,$$

where $l_j := |a'^{-1}(j)| \geqslant 0$ and μ_{l_j} is the unique monotone map $\mu_{l_j} : [1, l_j] \longrightarrow [1,1]$ for $j \in [1,n]$; cf. Remark 4.30, then we can write

$$t = t_1 \boxtimes_{\mathrm{Map}_0} \dots \boxtimes_{\mathrm{Map}_0} t_n \,,$$

where $t_j : [1, l_j] \longrightarrow [1, l_j]$ is a bijective map for $j \in [1,n]$. Moreover, we have $\sum_{j\in[1,n]} l_j = |X|$.

Since $\tau_0^{\mathrm{pre}} : \mathrm{ASS}_0^{\mathrm{pre}} \longrightarrow \mathcal{T}_0^{\mathrm{pre}}$ is a morphism of set-preoperads and $\mathrm{ASS}_0^{\mathrm{pre,bij}} \subseteq \mathrm{ASS}_0^{\mathrm{pre}}$ is a set-subpreoperad, we can apply Lemma 9.13. So for $j \in [1,n]$ we have $(t_j \backslash \mu_{l_j})\tau_0^{\mathrm{pre}} = (\mathrm{id}_{l_j} \backslash \mu_{l_j})\tau_0^{\mathrm{pre}}$.

So we have

$$\begin{aligned}
(tf' \backslash a')\tau_0^{\mathrm{pre}} &\overset{7.11}{=} \big((f' \backslash \mathrm{id}_{|X|}) \cdot_{\mathrm{ASS}_0} (t \backslash a')\big)\tau_0^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau_0^{\mathrm{pre}} \cdot_{\mathcal{T}_0} (t \backslash a')\tau_0^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau_0^{\mathrm{pre}} \cdot_{\mathcal{T}_0} \big((t_1 \boxtimes \dots \boxtimes t_n) \backslash (\mu_{l_1} \boxtimes \dots \boxtimes \mu_{l_n})\big)\tau_0^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau_0^{\mathrm{pre}} \cdot_{\mathcal{T}_0} \big((t_1 \backslash \mu_{l_1}) \boxtimes_{\mathrm{ASS}_0} \dots \boxtimes_{\mathrm{ASS}_0} (t_n \backslash \mu_{l_n})\big)\tau_0^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau_0^{\mathrm{pre}} \cdot_{\mathcal{T}_0} \big((t_1 \backslash \mu_{l_1})\tau_0^{\mathrm{pre}} \boxtimes_{\mathcal{T}_0} \dots \boxtimes_{\mathcal{T}_0} (t_n \backslash \mu_{l_n})\tau_0^{\mathrm{pre}}\big) \\
&\overset{9.13}{=} (f' \backslash \mathrm{id}_{|X|})\tau_0^{\mathrm{pre}} \cdot_{\mathcal{T}_0} \big((\mathrm{id}_{l_1} \backslash \mu_{l_1})\tau_0^{\mathrm{pre}} \boxtimes_{\mathcal{T}_0} \dots \boxtimes_{\mathcal{T}_0} (\mathrm{id}_{l_n} \backslash \mu_{l_n})\tau_0^{\mathrm{pre}}\big) \\
&= (f' \backslash \mathrm{id}_{|X|})\tau_0^{\mathrm{pre}} \cdot_{\mathcal{T}_0} \big((\mathrm{id}_{l_1} \backslash \mu_{l_1}) \boxtimes_{\mathrm{ASS}_0} \dots \boxtimes_{\mathrm{ASS}_0} (\mathrm{id}_{l_n} \backslash \mu_{l_n})\big)\tau_0^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau_0^{\mathrm{pre}} \cdot_{\mathcal{T}_0} \big((\mathrm{id}_{l_1} \boxtimes \dots \boxtimes \mathrm{id}_{l_n}) \backslash (\mu_{l_1} \boxtimes \dots \boxtimes \mu_{l_n})\big)\tau_0^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau_0^{\mathrm{pre}} \cdot_{\mathcal{T}_0} (\mathrm{id}_{|X|} \backslash a')\tau_0^{\mathrm{pre}} \\
&= \big((f' \backslash \mathrm{id}_{|X|}) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_{|X|} \backslash a')\big)\tau_0^{\mathrm{pre}} \\
&\overset{7.11}{=} (f' \backslash a')\tau_0^{\mathrm{pre}} \,.
\end{aligned}$$

We have now shown that given maps $f : X \longrightarrow [1,m]$, $a : X \longrightarrow [1,n]$ and maps $s : [1,|X|] \longrightarrow X$ and $\tilde{s} : [1,|X|] \longrightarrow X$ such that both sa and $\tilde{s}a$ are monotone we have

$$(sf \backslash sa)\tau_0^{\mathrm{pre}} = (\tilde{s}f \backslash \tilde{s}a)\tau_0^{\mathrm{pre}} \,,$$

hence the image of $f \backslash a$ does not depend on the choice of s.

Suppose given $f \backslash a, \tilde{f} \backslash \tilde{a} \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$, where $X, \tilde{X} \in \mathcal{E}$ and where $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,m] \xleftarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{a}} [1,n]$ such that $f \backslash a = \tilde{f} \backslash \tilde{a}$. That is, there exists a bijective map $u : \tilde{X} \longrightarrow X$ such that $uf = \tilde{f}$ and $ua = \tilde{a}$. Let $s : [1,|X|] \longrightarrow X$ be a bijective map such that sa is monotone and let $\tilde{s} : [1,|X|] \longrightarrow \tilde{X}$ be a bijective map such that $\tilde{s}\tilde{a}$ is monotone. But then both $(\tilde{s}u)a = \tilde{s}\tilde{a}$ and sa are monotone, so we have

$$(sf \backslash sa)\tau_0^{\mathrm{pre}} = ((\tilde{s}u)f \backslash (\tilde{s}u)a)\tau_0^{\mathrm{pre}} = (\tilde{s}\tilde{f} \backslash \tilde{s}\tilde{a})\tau_0^{\mathrm{pre}} \,.$$

Hence the image of $f \backslash a$ does not depend on the choice of f, a and s.

Now we have to show that $\bar{\tau}_0$ is in fact a morphism of set-operads. For $\xi = f \backslash a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$ there exists $\hat{\xi} \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$ such that $\hat{\xi}\kappa_0^{\mathrm{pre}} = \xi$, since we may take $\hat{\xi} = sf \backslash sa$ for some bijective map s such that sa is monotone.

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $\xi \in \mathrm{COM}_0^{\mathrm{pre}}(m,n)$, $\xi' \in \mathrm{COM}_0^{\mathrm{pre}}(m',n')$. Let $\hat{\xi} \in \mathrm{ASS}_0^{\mathrm{pre}}(m,n)$

such that $\hat{\xi}\kappa_0^{\mathrm{pre}} = \xi$ and let $\hat{\xi}' \in \mathrm{ASS}_0^{\mathrm{pre}}(m', n')$ such that $\hat{\xi}'\kappa_0^{\mathrm{pre}} = \hat{\xi}'$. Hence we have

$$\begin{aligned}
(\xi\bar{\tau}_0^{\mathrm{pre}}) \boxtimes_{\mathcal{T}_0} (\xi'\bar{\tau}_0^{\mathrm{pre}}) &= (\hat{\xi}\kappa_0^{\mathrm{pre}}\bar{\tau}_0^{\mathrm{pre}}) \boxtimes_{\mathcal{T}_0} (\hat{\xi}'\kappa_0^{\mathrm{pre}}\bar{\tau}_0^{\mathrm{pre}}) \\
&= (\hat{\xi}\tau_0^{\mathrm{pre}}) \boxtimes_{\mathcal{T}_0} (\hat{\xi}'\tau_0^{\mathrm{pre}}) \\
&= (\hat{\xi} \boxtimes_{\mathrm{ASS}_0} \hat{\xi}')\tau_0^{\mathrm{pre}} \\
&= (\hat{\xi} \boxtimes_{\mathrm{ASS}_0} \hat{\xi}')(\kappa_0^{\mathrm{pre}}\bar{\tau}_0^{\mathrm{pre}}) \\
&= (\hat{\xi}\kappa_0^{\mathrm{pre}} \boxtimes_{\mathrm{COM}_0} \hat{\xi}'\kappa_0^{\mathrm{pre}})\bar{\tau}_0^{\mathrm{pre}} \\
&= (\xi \boxtimes_{\mathrm{COM}_0} \xi')\bar{\tau}_0^{\mathrm{pre}}\,.
\end{aligned}$$

Now suppose given $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $\xi \in \mathrm{COM}_0(m, n)$, $\eta \in \mathrm{COM}_0(n, p)$. Let $\hat{\xi} \in \mathrm{ASS}_0(m, n)$ such that $\hat{\xi}\kappa_0^{\mathrm{pre}} = \xi$ and let $\hat{\eta} \in \mathrm{ASS}_0(n, p)$ such that $\hat{\eta}\kappa_0^{\mathrm{pre}} = \eta$. We have

$$\begin{aligned}
(\xi\bar{\tau}_0) \cdot_{\mathcal{T}_0} (\eta\bar{\tau}_0) &= (\hat{\xi}\kappa_0\bar{\tau}_0) \cdot_{\mathcal{T}_0} (\hat{\eta}\kappa_0\bar{\tau}_0) \\
&= (\hat{\xi}\tau_0) \cdot_{\mathcal{T}_0} (\hat{\eta}\tau_0) \\
&= (\hat{\xi} \cdot_{\mathrm{ASS}_0} \hat{\eta})\tau_0 \\
&= (\hat{\xi} \cdot_{\mathrm{ASS}_0} \hat{\eta})(\kappa_0\bar{\tau}_0) \\
&= (\hat{\xi}\kappa_0 \cdot_{\mathrm{COM}_0} \hat{\eta}\kappa_0)\bar{\tau}_0 \\
&= (\xi \cdot_{\mathrm{COM}_0} \eta)\bar{\tau}_0\,.
\end{aligned}$$

This shows that $\bar{\tau}_0^{\mathrm{pre}}$ is a morphism of set-preoperads. Furthermore, since κ_0 and τ_0 are morphisms of set-operads, we have $\mathfrak{a}_0\kappa_0^{\mathrm{pre}} = \mathfrak{c}_0$ and $\mathfrak{a}_0\tau_0^{\mathrm{pre}} = \mathfrak{t}_0$. So we have

$$\mathfrak{c}_0\bar{\tau}_0^{\mathrm{pre}} = \mathfrak{a}_0\kappa_0^{\mathrm{pre}}\bar{\tau}_0^{\mathrm{pre}} = \mathfrak{a}_0\tau_0^{\mathrm{pre}} = \mathfrak{t}_0\,.$$

Hence $\bar{\tau}_0 : \mathrm{COM}_0 \longrightarrow \mathcal{T}_0$ is a morphism of set-operads. □

We can now use this to show that a commutative monoid can be turned into a COM_0-algebra.

Proposition 9.15. *Let $(S, \mu_S, \varepsilon_S)$ be a commutative (and associative) monoid. Then there exists a morphism $\Phi_0 : \mathrm{COM}_0 \longrightarrow \mathrm{END}_0(S)$ of set-operads such that (S, Φ_0) is a COM_0-algebra with $\mu_S = (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}}$ and $\varepsilon_S = (\mathrm{id}_0 \setminus \varepsilon)\Phi_0^{\mathrm{pre}}$.*

Proof. Since S is in particular an associative monoid, by Proposition 7.18 we get a morphism $\Psi_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{END}_0(S)$ of set-operads such that (S, Ψ_0) is an ASS_0-algebra with $\mu_S = (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}}$ and $\varepsilon_S = (\mathrm{id}_0 \setminus \varepsilon)\Psi_0^{\mathrm{pre}}$. We already know that $((1,2) \setminus \mathrm{id}_2)\Psi_0^{\mathrm{pre}} = (1,2)^{\mathrm{op}}\mathfrak{e}_0$ is the map

$$\begin{aligned}
\tau_S : S^{\times 2} &\longrightarrow S^{\times 2} \\
(s, t) &\longmapsto (t, s);
\end{aligned}$$

cf. Example 2.63. Since $(S, \mu_S, \varepsilon_S)$ is a commutative monoid, we have $\tau_S\mu_S = \mu_S$, hence

$$\begin{aligned}
((1,2) \setminus \mu)\Psi_0^{\mathrm{pre}} &= \big(((1,2) \setminus \mathrm{id}_2) \cdot_{\mathrm{COM}_0} (\mathrm{id}_2 \setminus \mu)\big)\Psi_0^{\mathrm{pre}} \\
&= ((1,2) \setminus \mathrm{id}_2)\Psi_0^{\mathrm{pre}} \cdot_{\mathrm{End}_0} (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}} \\
&= \tau_S\mu_S \\
&= \mu_S \\
&= (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}}.
\end{aligned}$$

Hence by Lemma 9.14 there exists a uniquely determined morphism of set-operads

$$\Phi_0 := \bar{\Psi}_0 : \mathrm{COM}_0 \longrightarrow \mathrm{END}_0(S)$$

such that $\kappa_0\Phi_0 = \Psi_0$.

Hence (S, Φ_0) is a COM_0-algebra with $\mu_S = (\mathrm{id}_2 \setminus \mu)\Psi_0^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\kappa_0^{\mathrm{pre}}\Phi_0^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\Phi_0^{\mathrm{pre}}$ and $\varepsilon_S = (\mathrm{id}_0 \setminus \varepsilon)\Psi_0^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\kappa_0^{\mathrm{pre}}\Phi_0^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\Phi_0^{\mathrm{pre}}$. □

9.3 The linear operad COM

We aim to define a linear operad COM such that COM-algebras are commutative R-algebras. This is developed in parallel to §9.1 and §9.2 on COM_0 and COM_0-algebras. Some repetitions occur, which seemed hard to avoid.

Definition 9.16. We define the set-preoperad $\mathrm{COM}_0^{\mathrm{pre,bij}} \subseteq \mathrm{COM}_0^{\mathrm{pre}}$ as follows. For $m, n \in \mathbb{Z}_{\geqslant 0}$ let

$$\mathrm{COM}_0^{\mathrm{pre,bij}}(m,n) := \{f \setminus a \in \mathrm{COM}_0^{\mathrm{pre}}(m,n) : f \text{ is a bijective map}\}\,.$$

We have to show that $\mathrm{COM}_0^{\mathrm{pre,bij}} \subseteq \mathrm{COM}_0^{\mathrm{pre}}$ is a set-subpreoperad.

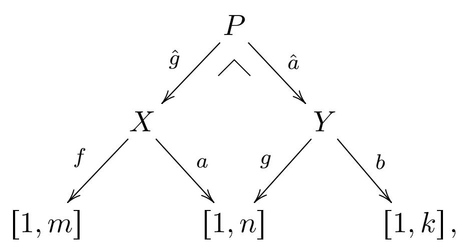
First note that $\mathrm{COM}_0^{\mathrm{pre,bij}}(m,n) \subseteq \mathrm{COM}_0^{\mathrm{pre}}(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)$, $f' \setminus a' \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m',n')$, where $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,m'] \xleftarrow{f'} X' \xrightarrow{a'} [1,n']$ and where $X, X' \in \mathcal{E}$ are finite sets and f, f' are bijective maps.

Then we have $(f \setminus a) \boxtimes (f' \setminus a') = ((f \sqcup f')\varphi^{-1}_{(m,m')}) \setminus ((a \sqcup a')\varphi^{-1}_{(n,n')} \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m+m', n+n')$, since $(f \sqcup f')\varphi^{-1}_{(m,m')}$ is bijective as the disjoint union and composite of bijective maps.

Now suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)$, $g \setminus b \in \mathrm{COM}_0^{\mathrm{pre,bij}}(n,k)$, where $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,n] \xleftarrow{g} Y \xrightarrow{b} [1,k]$ and where $X, Y \in \mathcal{E}$ are finite sets and f, g are bijective maps.

Let $(P, \hat{g}, \hat{a})$ be a pullback of a and g. Since g is bijective, so is $\hat{g}$ by Lemma 1.35. So we have



hence $(f \setminus a) \cdot (g \setminus b) = \hat{g}f \setminus \hat{a}b \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,k)$, since $\hat{g}f$ is bijective as the composite of bijective maps.

Definition 9.17. We define the linear operad $\mathrm{COM} = (\mathrm{COM}^{\mathrm{pre}}, \mathfrak{c})$ as follows.

- Let $\mathrm{COM}^{\mathrm{pre}} := R\,\mathrm{COM}_0^{\mathrm{pre,bij}}$.
- Let $\mathfrak{c} := R\Big(\mathfrak{c}_0\big|_{\mathrm{Sym}_0^{\mathrm{op}}}^{\mathrm{COM}_0^{\mathrm{pre,bij}}}\Big) : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{COM}^{\mathrm{pre}}$.

Note that since we have $\mathrm{Im}(\mathfrak{c}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}) \subseteq \mathrm{COM}_0^{\mathrm{pre,bij}}$, by Remark 6.26 this is in fact a linear operad. Recall that this definition means the following.

- We have $\mathrm{COM}^{\mathrm{pre}}(m,n) = R\,\mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$.
- We have $\mathrm{id}_{\mathrm{COM},m} = \mathrm{id}_{\mathrm{COM}_0,m}$ for $m \in \mathbb{Z}_{\geqslant 0}$.

- Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$.

  Then for $\sum\limits_{\xi \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)} r_\xi\, \xi \in \mathrm{COM}(m,n)$ and $\sum\limits_{\xi' \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m',n')} r'_{\xi'}\, \xi' \in \mathrm{COM}(m',n')$ we have

$$\Bigg(\sum_{\xi \in \mathrm{COM}_0^{\mathrm{bij}}(m,n)} r_\xi\, \xi \Bigg) \boxtimes_{\mathrm{COM}} \Bigg(\sum_{\xi' \in \mathrm{COM}_0^{\mathrm{bij}}(m',n')} r'_{\xi'}\, \xi' \Bigg) = \sum_{\substack{\xi \in \mathrm{COM}_0^{\mathrm{bij}}(m,n) \\ \xi' \in \mathrm{COM}_0^{\mathrm{bij}}(m',n')}} r_\xi\, r'_{\xi'} \big(\xi \boxtimes_{\mathrm{COM}_0} \xi'\big).$$

- Suppose given $m, n, k \in \mathbb{Z}_{\geqslant 0}$.

  Then for $\sum\limits_{\xi \in \mathrm{COM}_0^{\mathrm{bij}}(m,n)} r_\xi\, \xi \in \mathrm{COM}(m,n)$ and $\sum\limits_{\eta \in \mathrm{COM}_0^{\mathrm{bij}}(n,k)} s_\eta\, \eta \in \mathrm{COM}(n,k)$ we have

$$\Bigg(\sum_{\xi \in \mathrm{COM}_0^{\mathrm{bij}}(m,n)} r_\xi\, \xi \Bigg) \cdot_{\mathrm{COM}} \Bigg(\sum_{\eta \in \mathrm{COM}_0^{\mathrm{bij}}(n,k)} s_\eta\, \eta \Bigg) = \sum_{\substack{\xi \in \mathrm{COM}_0^{\mathrm{bij}}(m,n) \\ \eta \in \mathrm{COM}_0^{\mathrm{bij}}(n,k)}} r_\xi\, s_\eta \big(\xi \cdot_{\mathrm{COM}_0} \eta\big).$$

- For $m \in \mathbb{Z}_{\geqslant 0}$ and $\sum\limits_{f \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)} r_f\, f^{\mathrm{op}} \in \mathrm{Sym}^{\mathrm{op}}(m,m)$ we have

$$\Bigg(\sum_{f \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)} r_f\, f^{\mathrm{op}} \Bigg) \mathfrak{c} = \sum_{f \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)} r_f\, (f^{\mathrm{op}} \mathfrak{c}_0) = \sum_{f \in \mathrm{Sym}_0^{\mathrm{op}}(m,m)} r_f\, (f \setminus \mathrm{id}_{\mathrm{Map}_0, m}).$$

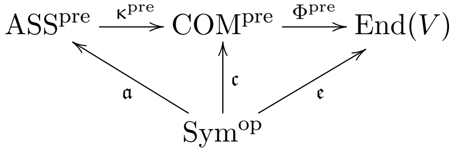
Definition 9.18. Recall the morphism of set-operads $\kappa_0 : \mathrm{ASS}_0 \longrightarrow \mathrm{COM}_0$; cf. Lemma 9.10. Since $(f \setminus a)\kappa_0^{\mathrm{pre}} \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and $f \setminus a \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)$ we may define the morphism of linear operads $\kappa : \mathrm{ASS} \longrightarrow \mathrm{COM}$ by

$$\kappa^{\mathrm{pre}} := R\Big(\kappa_0^{\mathrm{pre}} \big|_{\mathrm{ASS}_0^{\mathrm{pre,bij}}}^{\mathrm{COM}_0^{\mathrm{pre,bij}}} \Big).$$

9.4 Commutative algebras and COM-algebras

Proposition 9.19. *Let V be an R-module and let $\Phi : \mathrm{COM} \longrightarrow \mathrm{END}(V)$ be a morphism of linear operads, that is, (V, Φ) is a COM-algebra. Define $\mu_V := (\mathrm{id}_2 \setminus \mu)\Phi^{\mathrm{pre}} \in \mathrm{End}(V)(2,1)$ and $\varepsilon_V := (\mathrm{id}_0 \setminus \varepsilon)\Phi^{\mathrm{pre}} \in \mathrm{END}(V)(0,1)$. Then $(V, \mu_V, \varepsilon_V)$ is a commutative R-algebra.*

Proof. We have the following commutative diagram of linear preoperads.



So $\kappa\Phi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ is a morphism of linear operads. Hence $(V, \kappa\Phi)$ is an ASS-algebra.

By Proposition 7.22, the R-module V is an associative algebra when equipped with multiplication $(\mathrm{id}_2 \setminus \mu)(\kappa^{\mathrm{pre}}\Phi^{\mathrm{pre}}) = (\mathrm{id}_2 \setminus \mu)\Phi^{\mathrm{pre}} = \mu_V$ and unit $(\mathrm{id}_0 \setminus \varepsilon)(\kappa^{\mathrm{pre}}\Phi^{\mathrm{pre}}) = (\mathrm{id}_0 \setminus \varepsilon)\Phi^{\mathrm{pre}} = \varepsilon_V$.

Note that $((1,2) \setminus \mathrm{id}_2)\Phi^{\mathrm{pre}} = ((1,2)^{\mathrm{op}}\mathfrak{c})\Phi^{\mathrm{pre}} = (1,2)^{\mathrm{op}}\mathfrak{e} \in \mathrm{End}(V)(2,2)$ is the map defined by

$$\begin{aligned} \tau_V : V^{\otimes 2} &\longrightarrow V^{\otimes 2} \\ v \otimes w &\longmapsto w \otimes v \end{aligned}$$

for $v, w \in V$; cf. Example 2.66. In COM, expansion by the transposition $(1,2) : [1,2] \longrightarrow [1,2]$ yields

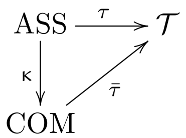
$$(1,2) \setminus \mu = ((1,2)(1,2)) \setminus ((1,2)\mu) = \mathrm{id}_2 \setminus \mu;$$

cf. Definition 9.3 and Remark 9.4. So by the definition of composition in COM we have

$$\begin{aligned} \tau_V \mu_V &= ((1,2) \setminus \mathrm{id}_2)\Phi^{\mathrm{pre}} \cdot_{\mathrm{End}_0} (\mathrm{id}_2 \setminus \mu)\Phi^{\mathrm{pre}} \\ &= \big(((1,2) \setminus \mathrm{id}_2) \cdot_{\mathrm{COM}} (\mathrm{id}_2 \setminus \mu)\big)\Phi^{\mathrm{pre}} \\ &= ((1,2) \setminus \mu)\Phi^{\mathrm{pre}} \\ &= (\mathrm{id}_2 \setminus \mu)\Phi^{\mathrm{pre}} \\ &= \mu_V \,. \end{aligned}$$

Hence $(V, \mu_V, \varepsilon_V)$ is a commutative algebra. □

Lemma 9.20. *Let $(\mathcal{T}, \mathfrak{t})$ be a linear operad and let $\tau : \mathrm{ASS} \longrightarrow \mathcal{T}$ be a morphism of linear operads satisfying $(\mathrm{id}_2 \setminus \mu)\tau^{\mathrm{pre}} = ((1,2) \setminus \mu)\tau^{\mathrm{pre}}$. Then there exists a unique morphism of linear operads $\bar{\tau} : \mathrm{COM} \longrightarrow \mathcal{T}$ such that $\kappa\bar{\tau} = \tau$.*



Proof. For $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre,bij}}$, where $f : X \longrightarrow [1,m]$, $a : X \longrightarrow [1,n]$ for some set X and $m, n \in \mathbb{Z}_{\geqslant 0}$ there exists a bijective map $s : [1, |X|] \longrightarrow X$ such that sa is a monotone map.

Uniqueness. Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and X be a set, $f : X \longrightarrow [1,m]$, $a : X \longrightarrow [1,n]$ such that $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)$ and let $s : [1,|X|] \longrightarrow X$ be a bijective map such that $sa = a$. Then we have $sf \setminus sa \in \mathrm{ASS}_0^{\mathrm{pre,bij}}(m,n)$. Given a morphism $\tilde{\tau} : \mathrm{COM} \longrightarrow \mathcal{T}$ of linear operads satisfying $\kappa\tilde{\tau} = \tau$, then we have

$$(sf \setminus sa)\tau^{\mathrm{pre}} = (sf \setminus sa)\kappa^{\mathrm{pre}}\tilde{\tau}^{\mathrm{pre}} = (sf \setminus sa)\tilde{\tau}^{\mathrm{pre}} = (f \setminus a)\tilde{\tau}^{\mathrm{pre}},$$

so such a morphism $\tilde{\tau}$ is uniquely determined by the requirement $\kappa\tilde{\tau} = \tau$.

Existence. Let $f \setminus a \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)$, let $m, n \in \mathbb{Z}_{\geqslant 0}$ and X be a set, $f : X \longrightarrow [1,m]$ and $a : X \longrightarrow [1,n]$ and let $s : [1,|X|] \longrightarrow X$ be a bijective map such that sa is a monotone map. Define $\bar{\tau}$ by

$$(f \setminus a)\bar{\tau}^{\mathrm{pre}} := (sf \setminus sa)\tau^{\mathrm{pre}}$$

and the usual linear extension on $\mathrm{COM}^{\mathrm{pre}} = R\,\mathrm{COM}_0^{\mathrm{pre,bij}}$. First we have to show that this is well-defined.

Let $\tilde{s} : [1,|X|] \longrightarrow X$ also be a map such that $\tilde{s}a$ is monotone. Then define $t := \tilde{s}s^{-1}$. So $t : [1,|X|] \longrightarrow [1,|X|]$ is a bijective map. Now define $f' := sf$ and $a' := sa$.

Then we have $\tilde{s}f = tsf = tf'$ and $\tilde{s}a = tsa = ta'$. So we have a bijective map $t : [1,|X|] \longrightarrow [1,|X|]$ such that both $a' = sa$ and $ta' = \tilde{s}a$ are monotone. So by Lemma 9.12 we know that

$$\tilde{s}a = ta' = a' = sa.$$

We have to show that

$$(f' \setminus a')\tau^{\mathrm{pre}} = (sf \setminus sa)\tau^{\mathrm{pre}} \overset{!}{=} (\tilde{s}f \setminus \tilde{s}a)\tau^{\mathrm{pre}} = (tf' \setminus a')\tau^{\mathrm{pre}}.$$

Since $ta' = a'$, we know that $(a'^{-1}(j))t = a'^{-1}(j)$ for $j \in [1,n]$. Therefore, if we write

$$a' = \mu_{l_1} \boxtimes_{\mathrm{Ass}_0} \ldots \boxtimes_{\mathrm{Ass}_0} \mu_{l_n}\,,$$

where $l_j = |a'^{-1}(j)| \geqslant 0$ and μ_{l_j} is the unique monotone map $[1, l_j] \longrightarrow [1,1]$ for $j \in [1,n]$; cf. Remark 4.30, then we can write

$$t = t_1 \boxtimes_{\mathrm{Map}_0} \dots \boxtimes_{\mathrm{Map}_0} t_n \,,$$

where $t_j : [1, l_j] \longrightarrow [1, l_j]$ is a bijective map for $j \in [1,n]$. Moreover, we have $\sum\limits_{i \in [1,n]} l_j = |X|$.

Note that we can view $\mathrm{ASS}^{\mathrm{pre}}$ and $\mathcal{T}^{\mathrm{pre}}$ as set-preoperads and $\tau^{\mathrm{pre}} : \mathrm{ASS}^{\mathrm{pre}} \longrightarrow \mathcal{T}^{\mathrm{pre}}$ as a morphism of set-preoperads; cf. Remark 2.10. Since $\mathrm{ASS}_0^{\mathrm{pre,bij}} \subseteq R\,\mathrm{ASS}_0^{\mathrm{pre,bij}}$, we can apply Lemma 9.13.

So for $j \in [1,n]$ we have $(t_j \backslash \mu_{l_j})\tau^{\mathrm{pre}} = (\mathrm{id}_{l_j} \backslash \mu_{l_j})\tau^{\mathrm{pre}}$.

So we have

$$\begin{aligned}
(tf' \backslash a')\tau^{\mathrm{pre}} &\overset{7.11}{=} \big((a' \backslash \mathrm{id}_{|X|}) \cdot_{\mathrm{ASS}} (t \backslash a')\big)\tau^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau^{\mathrm{pre}} \cdot_{\mathcal{T}} (t \backslash a')\tau^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau^{\mathrm{pre}} \cdot_{\mathcal{T}} \big((t_1 \boxtimes \dots \boxtimes t_n) \backslash (\mu_{l_1} \boxtimes \dots \boxtimes \mu_{l_n})\big)\tau^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau^{\mathrm{pre}} \cdot_{\mathcal{T}} \big((t_1 \backslash \mu_{i_1}) \boxtimes_{\mathrm{ASS}} \dots \boxtimes_{\mathrm{ASS}} (t_n \backslash \mu_{i_n})\big)\tau^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau^{\mathrm{pre}} \cdot_{\mathcal{T}} \big((t_1 \backslash \mu_{l_1})\tau^{\mathrm{pre}} \boxtimes_{\mathcal{T}} \dots \boxtimes_{\mathcal{T}} (t_n \backslash \mu_{l_n})\tau^{\mathrm{pre}}\big) \\
&\overset{9.13}{=} (f' \backslash \mathrm{id}_{|X|})\tau^{\mathrm{pre}} \cdot_{\mathcal{T}} \big((\mathrm{id}_{l_1} \backslash \mu_{l_1})\tau^{\mathrm{pre}} \boxtimes_{\mathcal{T}} \dots \boxtimes_{\mathcal{T}} (\mathrm{id}_{l_n} \backslash \mu_{l_n})\tau^{\mathrm{pre}}\big) \\
&= (f' \backslash \mathrm{id}_{|X|})\tau^{\mathrm{pre}} \cdot_{\mathcal{T}} \big((\mathrm{id}_{l_1} \backslash \mu_{l_1}) \boxtimes_{\mathrm{ASS}} \dots \boxtimes_{\mathrm{ASS}} (\mathrm{id}_{l_n} \backslash \mu_{l_n})\big)\tau^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau^{\mathrm{pre}} \cdot_{\mathcal{T}} \big((\mathrm{id}_{l_1} \boxtimes \dots \boxtimes \mathrm{id}_{l_n}) \backslash (\mu_{l_1} \boxtimes \dots \boxtimes \mu_{l_n})\big)\tau^{\mathrm{pre}} \\
&= (f' \backslash \mathrm{id}_{|X|})\tau^{\mathrm{pre}} \cdot_{\mathcal{T}} (\mathrm{id}_{|X|} \backslash a')\tau^{\mathrm{pre}} \\
&= \big((f' \backslash \mathrm{id}_{|X|}) \cdot_{\mathrm{ASS}} (\mathrm{id}_{|X|} \backslash a')\big)\tau^{\mathrm{pre}} \\
&\overset{7.11}{=} (f' \backslash a')\tau^{\mathrm{pre}} \,.
\end{aligned}$$

We have now shown that given maps $f : X \longrightarrow [1,m]$, $a : X \longrightarrow [1,n]$ and maps $s : [1, |X|] \longrightarrow X$ and $\tilde{s} : [1, |X|] \longrightarrow X$ such that both sa and $\tilde{s}a$ are monotone we have

$$(sf \backslash sa)\tau^{\mathrm{pre}} = (\tilde{s}f \backslash \tilde{s}a)\tau^{\mathrm{pre}} \,,$$

hence the image of $f \backslash a$ does not depend on the choice of s.

Let $f \backslash a, \tilde{f} \backslash \tilde{a} \in \mathrm{COM}_0^{\mathrm{pre,bij}}(m,n)$, $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ and $[1,m] \xleftarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{a}} [1,n]$ such that $f \backslash a = \tilde{f} \backslash \tilde{a}$, that is, there exists a bijective map $u : \tilde{X} \longrightarrow X$ such that $uf = \tilde{f}$ and $ua = \tilde{a}$. Let $s : [1, |X|] \longrightarrow X$ be a bijective map such that sa is monotone and let $\tilde{s} : [1, |X|] \longrightarrow \tilde{X}$ be a bijective map such that $\tilde{s}\tilde{a}$ is monotone. But then both $\tilde{s}\tilde{a} = (\tilde{s}u)a$ and sa are monotone maps from $[1, |X|]$ to $[1,n]$, so we have

$$(sf \backslash sa)\tau^{\mathrm{pre}} = ((\tilde{s}u)f \backslash (\tilde{s}u)a)\tau^{\mathrm{pre}} = (\tilde{s}\tilde{f} \backslash \tilde{s}\tilde{a})\tau^{\mathrm{pre}}.$$

Hence $\bar{\tau}^{\mathrm{pre}}$ is well-defined.

Now we have to show that $\bar{\tau}^{\mathrm{pre}}$ in fact defines a morphism of linear operads $\bar{\tau} : \mathrm{COM} \longrightarrow \mathcal{T}$. Note that given $\xi \in \mathrm{COM}^{\mathrm{pre}}(m,n)$, then there exists $\hat{\xi} \in \mathrm{ASS}^{\mathrm{pre}}(m,n)$ such that $\hat{\xi}\kappa^{\mathrm{pre}} = \xi$.

Suppose given $m, n, m', n' \in \mathbb{Z}_{\geqslant 0}$ and $\xi \in \mathrm{COM}^{\mathrm{pre}}(m,n)$, $\xi' \in \mathrm{COM}^{\mathrm{pre}}(m',n')$. Let $\hat{\xi} \in \mathrm{ASS}^{\mathrm{pre}}(m,n)$

such that $\hat{\xi}\kappa = \xi$ and let $\hat{\xi}' \in \mathrm{ASS}^{\mathrm{pre}}(m', n')$ such that $\hat{\xi}'\kappa = \hat{\xi}'$. Hence we have

$$\begin{aligned}(\xi\bar{\tau}^{\mathrm{pre}}) \boxtimes_{\mathcal{T}} (\xi'\bar{\tau}^{\mathrm{pre}}) &= (\hat{\xi}\kappa^{\mathrm{pre}}\bar{\tau}^{\mathrm{pre}}) \boxtimes_{\mathcal{T}} (\hat{\xi}'\kappa^{\mathrm{pre}}\bar{\tau}^{\mathrm{pre}}) \\ &= (\hat{\xi}\tau^{\mathrm{pre}}) \boxtimes_{\mathcal{T}} (\hat{\xi}'\tau^{\mathrm{pre}}) \\ &= (\hat{\xi} \boxtimes_{\mathrm{ASS}} \hat{\xi}')\tau^{\mathrm{pre}} \\ &= (\hat{\xi} \boxtimes_{\mathrm{ASS}} \hat{\xi}')(\kappa^{\mathrm{pre}}\bar{\tau}^{\mathrm{pre}}) \\ &= (\hat{\xi}\kappa^{\mathrm{pre}} \boxtimes_{\mathrm{COM}} \hat{\xi}'\kappa^{\mathrm{pre}})\bar{\tau}^{\mathrm{pre}} \\ &= (\xi \boxtimes_{\mathrm{COM}} \xi')\bar{\tau}^{\mathrm{pre}} .\end{aligned}$$

Now suppose given $m, n, p \in \mathbb{Z}_{\geqslant 0}$ and $\xi \in \mathrm{COM}^{\mathrm{pre}}(m, n)$, $\eta \in \mathrm{COM}^{\mathrm{pre}}(n, p)$. Let $\hat{\xi} \in \mathrm{ASS}^{\mathrm{pre}}(m, n)$ such that $\hat{\xi}\kappa^{\mathrm{pre}} = \xi$ and let $\hat{\eta} \in \mathrm{ASS}^{\mathrm{pre}}(n, p)$ such that $\hat{\eta}\kappa^{\mathrm{pre}} = \eta$. We have

$$\begin{aligned}(\xi\bar{\tau}^{\mathrm{pre}}) \cdot_{\mathcal{T}} (\eta\bar{\tau}^{\mathrm{pre}}) &= (\hat{\xi}\kappa^{\mathrm{pre}}\bar{\tau}^{\mathrm{pre}}) \cdot_{\mathcal{T}} (\hat{\eta}\kappa^{\mathrm{pre}}\bar{\tau}^{\mathrm{pre}}) \\ &= (\hat{\xi}\tau^{\mathrm{pre}}) \cdot_{\mathcal{T}} (\hat{\eta}\tau^{\mathrm{pre}}) \\ &= (\hat{\xi} \cdot_{\mathrm{ASS}} \hat{\eta}^{\mathrm{pre}})\tau^{\mathrm{pre}} \\ &= (\hat{\xi} \cdot_{\mathrm{ASS}} \hat{\eta})(\kappa^{\mathrm{pre}}\bar{\tau}^{\mathrm{pre}}) \\ &= (\hat{\xi}\kappa^{\mathrm{pre}} \cdot_{\mathrm{COM}} \hat{\eta}\kappa^{\mathrm{pre}})\bar{\tau}^{\mathrm{pre}} \\ &= (\xi \cdot_{\mathrm{COM}} \eta)\bar{\tau}^{\mathrm{pre}} .\end{aligned}$$

This shows that $\bar{\tau}^{\mathrm{pre}}$ is a morphism of linear preoperads. Furthermore, since κ and τ are morphisms of linear operads, we have $\mathfrak{a}\kappa^{\mathrm{pre}} = \mathfrak{c}$ and $\mathfrak{a}\tau^{\mathrm{pre}} = \mathfrak{t}$. So we have

$$\mathfrak{c}\bar{\tau}^{\mathrm{pre}} = \mathfrak{a}\kappa^{\mathrm{pre}}\bar{\tau}^{\mathrm{pre}} = \mathfrak{a}\tau^{\mathrm{pre}} = \mathfrak{t}.$$

Hence $\bar{\tau} : \mathrm{COM} \longrightarrow \mathcal{T}$ is a morphism of linear operads. □

We can now use this to show that a commutative R-algebra can be turned into a COM-algebra.

Proposition 9.21. *Let $(V, \mu_V, \varepsilon_V)$ be a commutative (and associative) R-algebra. Then there exists a morphism of linear operads $\Phi : \mathrm{COM} \longrightarrow \mathrm{END}(V)$ such that (V, Φ) is a* COM*-algebra with $(\mathrm{id}_2 \setminus \mu)\Phi^{\mathrm{pre}} = \mu_V$ and $(\mathrm{id}_0 \setminus \varepsilon)\Phi^{\mathrm{pre}} = \varepsilon_V$.*

Proof. Since V in particular is an associative R-algebra, by Proposition 7.24 we get a morphism of linear operads $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ such that (V, Ψ) is an ASS-algebra with $\mu_V = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}}$ and $\varepsilon_V = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}}$.

Note that $((1, 2) \setminus \mathrm{id}_2)\Psi^{\mathrm{pre}} = (1, 2)^{\mathrm{op}}\mathfrak{c}$ is the map defined by

$$\begin{aligned}\tau_V : \quad V^{\otimes 2} &\longrightarrow V^{\otimes 2} \\ v \otimes w &\longmapsto w \otimes v\end{aligned}$$

for $v, w \in V$; cf. Example 2.66. Since $(V, \mu_V, \varepsilon_V)$ is a commutative R-algebra, we have $\tau_V \mu_V = \mu_V$, hence

$$\begin{aligned}((1, 2) \setminus \mu)\Psi^{\mathrm{pre}} &= \big(((1, 2) \setminus \mu) \cdot_{\mathrm{ASS}} (\mathrm{id}_2 \setminus \mu)\big)\Psi^{\mathrm{pre}} \\ &= ((1, 2) \setminus \mathrm{id}_2)\Psi^{\mathrm{pre}} \cdot_{\mathrm{End}_0} (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} \\ &= \tau_V \mu_V \\ &= \mu_V \\ &= (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}}.\end{aligned}$$

Hence by Lemma 9.20 there exists a uniquely determined morphism of linear operads

$$\Phi := \bar{\Psi} : \mathrm{COM} \longrightarrow \mathrm{END}(V)$$

such that $\kappa\bar{\Psi} = \Psi$.

Hence (V, Φ) is a COM-algebra with $(\mathrm{id}_2 \setminus \mu)\Phi^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\kappa^{\mathrm{pre}}\Phi^{\mathrm{pre}} = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} = \mu_V$ and $(\mathrm{id}_0 \setminus \varepsilon)\Phi^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\kappa^{\mathrm{pre}}\Phi^{\mathrm{pre}} = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}} = \varepsilon_V$. □

10 The linear operad LIE

Definition 10.1. Recall the linear operad ASS $= (\mathrm{ASS}^{\mathrm{pre}}, \mathfrak{a})$ over R; cf. Definition 7.20. Let

$$\lambda := \mathrm{id}_2 \setminus \mu - (1,2) \setminus \mu \in \mathrm{ASS}^{\mathrm{pre}}(2,2),$$

where $(1,2) \in \mathrm{Sym}_0(2,2)$ is the transposition, μ is the unique element in $\mathrm{Ass}_0(2,1)$ and where we abbreviate $\mathrm{id}_2 := \mathrm{id}_{\mathrm{Map}_0, 2}$. We define the linear suboperad LIE of ASS as follows; cf. Definition 6.34 and Lemma 6.35.

$$\mathrm{LIE} := {}_{\mathrm{op}}\langle\, \lambda \,\rangle .$$

Recall that LIE being a linear suboperad means that $\mathrm{Im}(\mathfrak{a}) \subseteq \mathrm{LIE}^{\mathrm{pre}}$ and that LIE $= (\mathrm{LIE}^{\mathrm{pre}}, \mathfrak{l})$ is a linear operad, where $\mathfrak{l} = \mathfrak{a}|^{\mathrm{LIE}^{\mathrm{pre}}} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathrm{LIE}^{\mathrm{pre}}$; cf. Definition 6.31.

Proposition 10.2. *Suppose $2 \in \mathcal{U}(R)$, so 2 is invertible in R. Let V be an R-module and let $\Lambda : \mathrm{LIE} \longrightarrow \mathrm{END}(V)$ be a morphism of linear operads. So (V, Λ) is a* LIE-*algebra.*

$$\begin{array}{ccc} \mathrm{LIE}^{\mathrm{pre}} & \xrightarrow{\quad \Lambda^{\mathrm{pre}} \quad} & \mathrm{End}(V) \\ & {}_{\mathfrak{l}}\nwarrow \qquad \nearrow_{\mathfrak{e}} & \\ & \mathrm{Sym}^{\mathrm{op}} & \end{array}$$

Define $\lambda_V := \lambda\Lambda^{\mathrm{pre}} = \big((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu)\big)\Lambda^{\mathrm{pre}} \in \mathrm{End}(V)(2,1)$ *and define the bilinear map*

$$\begin{aligned} [-,=] : V \times V &\longrightarrow V \\ (v, w) &\longmapsto [v, w] := (v \otimes w)\lambda_V . \end{aligned}$$

Then $(V, [-,=])$ is a Lie algebra over R; cf. **[2,** Definition 1.1**]**.

Proof. During this proof we denote by $(\boxtimes)$ and $(\cdot)$ the multiplication and composition in Map_0. We have to show the following.

(1) For $u, v, w \in V$ we have $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] \overset{!}{=} 0$.

(2) For $v \in V$ we have $[v,v] \overset{!}{=} 0$.

Ad (1). Suppose given $u, v, w \in V$. We have

$$[u,[v,w]] = (u \otimes [v,w])\lambda_V = (u \otimes (v \otimes w)\lambda_V)\lambda_V = (u \otimes v \otimes w)((\mathrm{id}_V \otimes \lambda_V)\lambda_V),$$

hence

$$\begin{aligned} [v,[w,u]] &= (v \otimes w \otimes u)((\mathrm{id}_V \otimes \lambda_V)\lambda_V) = (u \otimes v \otimes w)((1,2,3)^{\mathrm{op}}\mathfrak{e})((\mathrm{id}_V \otimes \lambda_V)\lambda_V) \\ [w,[u,v]] &= (w \otimes u \otimes v)((\mathrm{id}_V \otimes \lambda_V)\lambda_V) = (u \otimes v \otimes w)((1,3,2)^{\mathrm{op}}\mathfrak{e})((\mathrm{id}_V \otimes \lambda_V)\lambda_V); \end{aligned}$$

cf. Definition 2.64. So we have to show that

$$(\mathrm{id}_{V^{\otimes 3}} + (1,2,3)^{\mathrm{op}}\mathfrak{e} + (1,3,2)^{\mathrm{op}}\mathfrak{e})((\mathrm{id}_V \otimes \lambda_V)\lambda_V) \overset{!}{=} 0_{\mathrm{End}(V)} .$$

We have

$$\begin{aligned} \mathrm{id}_{V^{\otimes 3}} + (1,2,3)^{\mathrm{op}}\mathfrak{e} + (1,3,2)^{\mathrm{op}}\mathfrak{e} &= \mathrm{id}_3^{\mathrm{op}}\, \mathfrak{l}\Lambda^{\mathrm{pre}} + (1,2,3)^{\mathrm{op}}\mathfrak{l}\Lambda^{\mathrm{pre}} + (1,3,2)^{\mathrm{op}}\mathfrak{l}\Lambda^{\mathrm{pre}} \\ &= (\mathrm{id}_3 \setminus \mathrm{id}_3)\Lambda^{\mathrm{pre}} + ((1,2,3) \setminus \mathrm{id}_3)\Lambda^{\mathrm{pre}} + ((1,3,2) \setminus \mathrm{id}_3)\Lambda^{\mathrm{pre}} \end{aligned}$$

and

$$\begin{aligned}((\mathrm{id}_V \otimes \lambda_V)\lambda_V) &= (\mathrm{id}_{\mathrm{LIE},1}\, \Lambda^{\mathrm{pre}} \otimes \lambda\Lambda^{\mathrm{pre}})(\lambda\Lambda^{\mathrm{pre}}) \\ &= ((\mathrm{id}_{\mathrm{LIE},1} \boxtimes_{\mathrm{LIE}} \lambda) \cdot_{\mathrm{LIE}} \lambda)\Lambda^{\mathrm{pre}}.\end{aligned}$$

So since Λ is a morphism of linear operads, it suffices to show that

$$((\mathrm{id}_3 \backslash \mathrm{id}_3) + ((1,2,3) \backslash \mathrm{id}_3) + ((1,3,2) \backslash \mathrm{id}_3)) \cdot_{\mathrm{LIE}} (\mathrm{id}_{\mathrm{LIE},1} \boxtimes_{\mathrm{LIE}} \lambda) \cdot_{\mathrm{LIE}} \lambda \overset{!}{=} 0_{\mathrm{LIE}}\,.$$

Using the definition of multiplication and composition in ASS_0 and ASS; cf. Definitions 7.10 and 7.20, we obtain

$$\begin{aligned}
&(\,\mathrm{id}_{\mathrm{LIE},1} \boxtimes_{\mathrm{LIE}} \lambda) \cdot_{\mathrm{LIE}} \lambda \\
&\quad = \big((\mathrm{id}_1 \backslash \mathrm{id}_1) \boxtimes_{\mathrm{LIE}} ((\mathrm{id}_2 \backslash \mu) - ((1,2) \backslash \mu))\big) \cdot_{\mathrm{LIE}} \big((\mathrm{id}_2 \backslash \mu) - ((1,2) \backslash \mu)\big) \\
&\quad = \big((\mathrm{id}_1 \backslash \mathrm{id}_1) \boxtimes_{\mathrm{ASS}} ((\mathrm{id}_2 \backslash \mu) - ((1,2) \backslash \mu))\big) \cdot_{\mathrm{ASS}} \big((\mathrm{id}_2 \backslash \mu) - ((1,2) \backslash \mu)\big) \\
&\quad = \Big(\big((\mathrm{id}_1 \backslash \mathrm{id}_1) \boxtimes_{\mathrm{ASS}_0} (\mathrm{id}_2 \backslash \mu)\big) - \big((\mathrm{id}_1 \backslash \mathrm{id}_1) \boxtimes_{\mathrm{ASS}_0} ((1,2) \backslash \mu)\big)\Big) \cdot_{\mathrm{ASS}} \big((\mathrm{id}_2 \backslash \mu) - ((1,2) \backslash \mu)\big) \\
&\quad = \big((\mathrm{id}_3 \backslash (\mathrm{id}_1 \boxtimes \mu)) - ((\mathrm{id}_1 \boxtimes (1,2)) \backslash (\mathrm{id}_1 \boxtimes \mu))\big) \cdot_{\mathrm{ASS}} \big((\mathrm{id}_2 \backslash \mu) - ((1,2) \backslash \mu)\big) \\
&\quad = \big((\mathrm{id}_3 \backslash (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_2 \backslash \mu)\big) - \big((\mathrm{id}_3 \backslash (\mathrm{id}_1 \boxtimes \mu) \cdot_{\mathrm{ASS}_0} ((1,2) \backslash \mu)\big) \\
&\qquad\quad - \big(((\mathrm{id}_1 \boxtimes (1,2)) \backslash (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_2 \backslash \mu)\big) + \big(((\mathrm{id}_1 \boxtimes (1,2)) \backslash (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} ((1,2) \backslash \mu)\big) \\
&\quad = \big((\mathrm{id}_3 \backslash (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_2 \backslash \mu)\big) - \big((\mathrm{id}_3 \backslash (\mathrm{id}_1 \boxtimes \mu) \cdot_{\mathrm{ASS}_0} ((1,2) \backslash \mu)\big) \\
&\qquad\quad - \big((2,3) \backslash (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_2 \backslash \mu)\big) + \big((2,3) \backslash (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} ((1,2) \backslash \mu)\big),
\end{aligned}$$

where $(2,3) = \mathrm{id}_1 \boxtimes (1,2) \in \mathrm{Sym}_0(3,3)$.

By Remark 7.11 we know that

$$\begin{aligned}(\mathrm{id}_3 \backslash (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_2 \backslash \mu) &= \mathrm{id}_3 \backslash ((\mathrm{id}_1 \boxtimes \mu)\mu) = \mathrm{id}_3 \backslash \mu_3 \\ ((2,3) \backslash (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_2 \backslash \mu) &= (2,3) \backslash ((\mathrm{id}_1 \boxtimes \mu)\mu) = (2,3) \backslash \mu_3\,,\end{aligned}$$

where $\mu_3 = (\mathrm{id}_1 \boxtimes \mu) \cdot \mu = (\mu \boxtimes \mathrm{id}_1) \cdot \mu$ is the unique element in $\mathrm{Ass}_0(3,1)$.

Claim. We have the following sorted pullback.

$$\begin{array}{ccc} [1,3] & \xrightarrow{\mu \boxtimes \mathrm{id}_1} & [1,2] \\ {\scriptstyle (1,2,3)}\downarrow & \ulcorner & \downarrow{\scriptstyle (1,2)} \\ [1,3] & \xrightarrow[\mathrm{id}_1 \boxtimes \mu]{} & [1,2] \end{array}$$

Proof of the Claim. Note that

$$\begin{aligned}1(\mu \boxtimes \mathrm{id}_1)(1,2) &= 2 = 1(1,2,3)(\mathrm{id}_1 \boxtimes \mu) \\ 2(\mu \boxtimes \mathrm{id}_1)(1,2) &= 2 = 2(1,2,3)(\mathrm{id}_1 \boxtimes \mu) \\ 3(\mu \boxtimes \mathrm{id}_1)(1,2) &= 1 = 3(1,2,3)(\mathrm{id}_1 \boxtimes \mu).\end{aligned}$$

So the diagram is commutative. Moreover, $\mu \boxtimes \mathrm{id}_1$ is monotone. Finally, the restricted maps

$$\begin{aligned}(1,2,3)\Big|^{(\mathrm{id}_1 \boxtimes \mu)^{-1}(1(1,2))}_{(\mu \boxtimes \mathrm{id}_1)^{-1}(1)} &= (1,2,3)\Big|^{(\mathrm{id}_1 \boxtimes \mu)^{-1}(2)}_{(\mu \boxtimes \mathrm{id}_1)^{-1}(1)} = (1,2,3)\Big|^{[2,3]}_{[1,2]} \\ (1,2,3)\Big|^{(\mathrm{id}_1 \boxtimes \mu)^{-1}(2(1,2))}_{(\mu \boxtimes \mathrm{id}_1)^{-1}(2)} &= (1,2,3)\Big|^{(\mathrm{id}_1 \boxtimes \mu)^{-1}(1)}_{(\mu \boxtimes \mathrm{id}_1)^{-1}(2)} = (1,2,3)\Big|^{[1,1]}_{[3,3]}\end{aligned}$$

are isotone.

This proves the *Claim.*

So by the definition of composition in ASS_0 in Definition 7.10, we have

$$(\mathrm{id}_3 \setminus (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} ((1,2) \setminus \mu) = ((1,2,3)\, \mathrm{id}_3) \setminus ((\mu \boxtimes \mathrm{id}_1)\mu) = (1,2,3) \setminus \mu_3$$
$$((2,3) \setminus (\mathrm{id}_1 \boxtimes \mu)) \cdot_{\mathrm{ASS}_0} ((1,2) \setminus \mu) = ((1,2,3)(2,3)) \setminus ((\mu \boxtimes \mathrm{id}_1)\mu) = (1,3) \setminus \mu_3\,.$$

So we have

$$(\mathrm{id}_{\mathrm{LIE},1} \boxtimes_{\mathrm{LIE}} \lambda) \cdot_{\mathrm{LIE}} \lambda = (\mathrm{id}_3 \setminus \mu_3) - ((1,2,3) \setminus \mu_3) - ((2,3) \setminus \mu_3) + ((1,3) \setminus \mu_3).$$

Altogether, by the defintion of $(\cdot_{\mathrm{ASS}})$ and $(\cdot_{\mathrm{ASS}_0})$ and by Remark 7.11, we have

$$\begin{aligned}
&\big((\mathrm{id}_3 \setminus \mathrm{id}_3) + ((1,2,3) \setminus \mathrm{id}_3) + ((1,3,2) \setminus \mathrm{id}_3)\big) \cdot_{\mathrm{LIE}} (\mathrm{id}_{\mathrm{LIE},1} \boxtimes_{\mathrm{LIE}} \lambda) \cdot_{\mathrm{LIE}} \lambda \\
&\quad = \big((\mathrm{id}_3 \setminus \mathrm{id}_3) + ((1,2,3) \setminus \mathrm{id}_3) + ((1,3,2) \setminus \mathrm{id}_3)\big) \\
&\qquad\qquad \cdot_{\mathrm{ASS}} \big((\mathrm{id}_3 \setminus \mu_3) - ((1,2,3) \setminus \mu_3) - ((2,3) \setminus \mu_3) + ((1,3) \setminus \mu_3)\big) \\
&\quad = (\mathrm{id}_3 \setminus \mu_3) - ((1,2,3) \setminus \mu_3) - ((2,3) \setminus \mu_3) + ((1,3) \setminus \mu_3) \\
&\qquad\qquad + ((1,2,3) \setminus \mu_3) - (((1,2,3)(1,2,3)) \setminus \mu_3) - (((2,3)(1,2,3)) \setminus \mu_3) + (((1,3)(1,2,3)) \setminus \mu_3) \\
&\qquad\qquad + ((1,3,2) \setminus \mu_3) - (((1,2,3)(1,3,2)) \setminus \mu_3) - (((2,3)(1,3,2)) \setminus \mu_3) + (((1,3)(1,3,2)) \setminus \mu_3) \\
&\quad = (\mathrm{id}_3 \setminus \mu_3) - ((1,2,3) \setminus \mu_3) - ((2,3) \setminus \mu_3) + ((1,3) \setminus \mu_3) \\
&\qquad\qquad + ((1,2,3) \setminus \mu_3) - ((1,3,2) \setminus \mu_3) - ((1,2) \setminus \mu_3) + ((2,3) \setminus \mu_3) \\
&\qquad\qquad + ((1,3,2) \setminus \mu_3) - (\mathrm{id}_3 \setminus \mu_3) - ((1,3) \setminus \mu_3) + ((1,2) \setminus \mu_3) \\
&\quad = 0_{\mathrm{ASS}} \\
&\quad = 0_{\mathrm{LIE}}\,.
\end{aligned}$$

This completes the proof of (1).

Ad (2). Suppose given $v \in V$. Note that

$$v \otimes v = (v \otimes v)((1,2)^{\mathrm{op}}\mathfrak{e}) = (v \otimes v)((1,2)^{\mathrm{op}}\mathfrak{l}\Lambda^{\mathrm{pre}}) = (v \otimes v)\big((1,2) \setminus \mathrm{id}_2\big)\Lambda^{\mathrm{pre}}.$$

So we have

$$\begin{aligned}
[v,v] &= (v \otimes v)\lambda_V \\
&= (v \otimes v)\big((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu)\big)\Lambda^{\mathrm{pre}} \\
&= (v \otimes v)\big(((1,2) \setminus \mathrm{id}_2\big)\Lambda^{\mathrm{pre}} \cdot_{\mathrm{END}} \big((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu)\big)\Lambda^{\mathrm{pre}} \\
&= (v \otimes v)\big(((1,2) \setminus \mathrm{id}_2\big) \cdot_{\mathrm{LIE}} \big((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu)\big)\big)\Lambda^{\mathrm{pre}} \\
&= (v \otimes v)\big(((1,2) \setminus \mathrm{id}_2\big) \cdot_{\mathrm{ASS}} \big((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu)\big)\big)\Lambda^{\mathrm{pre}} \\
&= (v \otimes v)\big(\big(((1,2) \setminus \mathrm{id}_2) \cdot_{\mathrm{ASS}_0} (\mathrm{id}_2 \setminus \mu)\big) - \big(((1,2) \setminus \mathrm{id}_2) \cdot_{\mathrm{ASS}_0} ((1,2) \setminus \mu)\big)\big)\Lambda^{\mathrm{pre}} \\
&\overset{7.11}{=} (v \otimes v)\big(((1,2) \setminus \mu) - (((1,2)(1,2)) \setminus \mu)\big)\Lambda^{\mathrm{pre}} \\
&= (v \otimes v)\big(((1,2) \setminus \mu) - (\mathrm{id}_2 \setminus \mu)\big)\Lambda^{\mathrm{pre}} \\
&= -(v \otimes v)\big((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu)\big)\Lambda^{\mathrm{pre}} \\
&= -(v \otimes v)\lambda_V \\
&= -[v,v]\,.
\end{aligned}$$

So $2[v,v] = 0$ and since $2 \in \mathcal{U}(R)$ we have $[v,v] = 0$.

This completes the proof of (2). □

Remark 10.3. Note that we may not drop the condition $2 \in \mathcal{U}(R)$. If $2 \notin \mathcal{U}(R)$, the last step in the proof of Proposition 10.2 may fail. For instance, suppose $R = \mathbb{F}_2$, the finite field with two elements, and suppose $V = \mathbb{F}_2 \oplus \mathbb{F}_2 = {}_{\mathbb{F}_2}\langle\, x \,\rangle \oplus {}_{\mathbb{F}_2}\langle\, y \,\rangle$. Then we can define the $\mathbb{F}_2$-bilinear map $[-,=] : V \otimes V \longrightarrow V$ by

$$\begin{aligned} [x,x] &:= y \\ [x,y] := [y,x] &:= 0 \\ [y,y] &:= 0. \end{aligned}$$

So $[V, {}_{\mathbb{F}_2}\langle\, y \,\rangle] = 0$ and $[V,V] \subseteq {}_{\mathbb{F}_2}\langle\, y \,\rangle$.

So for $u,v,w \in V$ we have $[u,[v,w]] = 0$, since $[v,w] \in {}_{\mathbb{F}_2}\langle\, y \,\rangle$. So in particular

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

for $u,v,w \in V$.

Moreover, for $v,w \in V$ we have $[v,w] = [w,v]$, hence $[v,w] + [w,v] = 0$.

So $V = \mathbb{F}_2 \oplus \mathbb{F}_2$ satisfies condition (1) for a Lie algebra and satisfies $[v,w] + [w,v] = 0$ for $v,w \in V$, but does not satisfy condition (2) for a Lie algebra since $[x,x] = y \neq 0$.

In particular, we can not conclude from $2[v,v] = 0$ that $[v,v] = 0$ for $v \in V$, as was needed in the proof of Proposition 10.2.

Question 10.4. *Is a Lie algebra a* LIE*-algebra?*

That is, we ask if, given a Lie algebra $(V, [-,=])$*, there exists a morphism* $\Lambda : \mathrm{LIE} \longrightarrow \mathrm{END}(V)$ *such that* $(v \otimes w)\big(\lambda\Lambda^{\mathrm{pre}}\big) = [v,w]$ *for* $v,w \in V$.

Remark 10.5. Suppose given a morphism of linear operads $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$, i.e. suppose given an ASS-algebra (V, Ψ). By Lemma 7.22, $(V, \mu_V, \varepsilon_V)$ is an associative R-algebra with multiplication $\mu_V = (\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}}$ and unit $\varepsilon_V = (\mathrm{id}_0 \setminus \varepsilon)\Psi^{\mathrm{pre}}$. Moreover, we already know that for $v,w \in V$ we have

$$(v \otimes w)\big(((1,2) \setminus \mu)\Psi^{\mathrm{pre}}\big) = (w \otimes v)\mu_V\,;$$

cf. the proof of Proposition 9.21. The restriction $\Psi\big|_{\mathrm{LIE}} : \mathrm{LIE} \longrightarrow \mathrm{END}(V)$ is a morphism of linear operads, since $\mathrm{LIE} \subseteq \mathrm{ASS}$ is a linear suboperad. Hence $(V, \Psi\big|_{\mathrm{LIE}})$ is a LIE-algebra.

We define the bilinear map $[-,=] : V^{\otimes 2} \longrightarrow V$ as in Proposition 10.2, i.e. for $v,w \in V$ we let

$$[v,w] := (v \otimes w)(\lambda\Psi^{\mathrm{pre}}\big|_{\mathrm{LIE}}) = (v \otimes w)\big(\big((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu)\big)\Psi^{\mathrm{pre}}\big|_{\mathrm{LIE}}\big)\,.$$

For $v,w \in V$ we have

$$\begin{aligned} [v,w] &= (v \otimes w)(\lambda\Psi^{\mathrm{pre}}\big|_{\mathrm{LIE}}) \\ &= (v \otimes w)(\lambda\Psi^{\mathrm{pre}}) \\ &= (v \otimes w)\big(\big((\mathrm{id}_2 \setminus \mu) - ((1,2) \setminus \mu)\big)\Psi^{\mathrm{pre}}\big) \\ &= (v \otimes w)\big((\mathrm{id}_2 \setminus \mu)\Psi^{\mathrm{pre}} - ((1,2) \setminus \mu)\Psi^{\mathrm{pre}}\big) \\ &= (v \otimes w)\mu_V - (v \otimes w)\big(((1,2) \setminus \mu)\Psi^{\mathrm{pre}}\big) \\ &= (v \otimes w)\mu_V - (w \otimes v)\mu_V\,. \end{aligned}$$

So $(V, [-,=])$ is the commutator Lie algebra of the associative algebra $(V, \mu_V, \varepsilon_V)$.

Note that given a LIE-algebra (V, Λ), that is, given a morphism $\Lambda : \mathrm{LIE} \longrightarrow \mathrm{END}(V)$ of linear operads, we do not know whether there exists a morphism $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ of linear operads with $\Lambda = \Psi\big|_{\mathrm{LIE}}$.

References

[1] M. Aguiar and M. Livernet. The associative operad and the weak order on the symmetric groups. *Journal of Homotopy and Related Structures*, 2(1):57–84, 2007.

[2] N. Bourbaki. *Éléments de mathématique: Chapitre 1. Groupes et algèbres de Lie.* Springer, 2007.

[3] F. Chapoton. Introduction to operads. *Preprint*, 2002.

[4] S. Dăscălescu, C. Năstăsescu, and Ş. Raianu. *Hopf algebras: An introduction.* CRC Press, 2000.

[5] B. Fresse. *Modules over Operads and Functors.* Springer Lecture Notes 1967, 2009.

[6] B. Fresse. *Homotopy of Operads and Grothendieck-Teichmüller Groups*, 2016. Research Monograph, `http://math.univ-lille1.fr/~fresse/OperadHomotopyBook`.

[7] F. W. Lawvere. Functorial semantics of algebraic theories. *Proceedings of the National Academy of Sciences*, 50(5):869–872, 1963.

[8] S. Mac Lane. Natural associativity and commutativity. *Rice University Studies*, 49(4), 1963.

[9] S. Mac Lane. Categorical algebra. *Bulletin AMS*, 71(1):40–106, 1965.

[10] S. Mac Lane. *Categories for the Working Mathematician.* Springer Graduate Texts in Mathematics 5, 1998.

[11] M. Markl. Operads and PROPs. *Handbook of algebra*, 5:87–140, 2008.

[12] M. Markl, S. Shnider, and J. D. Stasheff. *Operads in Algebra, Topology and Physics.* Number 96. AMS, 2007.

[13] J. P. May. *The Geometry of Iterated Loop Spaces.* Springer, 1972.

[14] M. Ritter. *On universal properties of preadditive and additive categories.* Bachelor's thesis, University of Stuttgart, Germany, 2016.

[15] J. E. Vatne. Introduction to operads. *Lecture notes, University of Bergen*, 2004.

Zusammenfassung

In der vorliegenden Arbeit werden Operaden vorgestellt und einige Beispiele behandelt. Der verwendete Operadenbegriff weicht dabei vom Begriff einer Operade in der Literatur ab. Klassische Operaden, wie zum Beispiel von May (cf. **[13,** Definition 1.1**]**) definiert, tauchen in dieser Arbeit in ähnlicher Form unter dem Begriff „absolute Operaden" auf; cf. §3. Gleichzeitig sind die in §6 definierten *linearen Operaden* eng verwandt mit den von S. Mac Lane 1963 eingeführten PROPs (Abkürzung für "product and permutation category"; cf. **[8,** §6**]**), was auch der Grund dafür ist, dass der Titel dieser Arbeit "Operads in the sense of Mac Lane" lautet.

Präoperaden

Eine erste Version der Operaden sind die Präoperaden. Eine Präoperade in Mengen ist im Wesentlichen eine strikte monoidale Kategorie mit $\mathbb{Z}_{\geqslant 0}$ als Menge der Objekte. Genauer gesagt ist eine Präoperade in Mengen $(\mathcal{P}_0, \boxtimes, \cdot)$ gegeben durch eine biindizierte Menge $\mathcal{P}_0 = (\mathcal{P}_0(m,n))_{m,n\geqslant 0}$, mit einer assoziativen Multiplikation, gegeben durch Multiplikationsabbildungen

$$(\boxtimes) : \mathcal{P}_0(m,n) \times \mathcal{P}(m',n') \longrightarrow \mathcal{P}_0(m+m', n+n'),$$

und einer assoziativen Komposition, gegeben durch Kompositionsabbildungen

$$(\cdot) : \mathcal{P}_0(m,n) \times \mathcal{P}(n,k) \longrightarrow \mathcal{P}_0(m,k),$$

sowie Identitäten $\mathrm{id}_m \in \mathcal{P}_0(m,m)$ für alle m, wobei noch bestimmte Kompatibilitätsbedingungen erfüllt sind; cf. Definition 2.6.

Eine lineare Präoperade $(\mathcal{P}, \boxtimes, \cdot)$ (über R) ist fast genauso definiert, mit dem Unterschied, dass $\mathcal{P}(m,n)$ stets ein R-Modul ist und dass die Multiplikations- und Kompositionsabbildungen

$$(\boxtimes) : \mathcal{P}(m,n) \otimes \mathcal{P}(m',n') \longrightarrow \mathcal{P}(m+m', n+n')$$
$$(\cdot) : \mathcal{P}(m,n) \otimes \mathcal{P}(n,k) \longrightarrow \mathcal{P}(m,k)$$

stets R-linear sein müssen.

Es können dann grundlegende algebraische Strukturen in den Präoperaden definiert und untersucht werden. Zum Beispiel ist ein Morphismus von Präoperaden eine biindizierte Abbildung (im R-linearen Fall von R-linearen Abbildungen), die mit der Struktur der Operade verträglich ist, also mit Multiplikation und Komposition vertauscht und Identitäten auf Identitäten abbildet. Des Weiteren werden Teilpräoperaden und Faktorpräoperaden definiert.

Elementare Beispiele sind die Präoperade Map_0 in Mengen, die aus Abbildungen $f : [1,m] \longrightarrow [1,n]$ zwischen endlichen ganzzahligen Intervallen besteht, sowie ihre Teilpräoperaden Ass_0, bestehend aus monotonen Abbildungen, und Sym_0, bestehend aus bijektiven Abbildungen. Durch das Bilden freier R-Moduln und durch lineares Fortsetzen der Multiplikations- und Kompositionsabbildungen ergeben sich außerdem die lineare Präoperade Map und ihre linearen Teilpräoperaden Ass und Sym.

Für eine Menge X kann zudem die Präoperade $\mathrm{End}_0(X)$ definiert werden, die alle Abbildungen $f : X^{\times m} \longrightarrow X^{\times n}$ enthält, ausgestattet mit dem kartesischen Produkt von Abbildungen als Multiplikation und der üblichen Komposition von Abbildungen als Komposition.

In gleicher Weise kann für einen R-Modul V eine lineare Präoperade $\mathrm{End}(V)$ definiert werden, die R-lineare Abbildungen $f : V^{\otimes m} \longrightarrow V^{\otimes n}$ enthält, ausgestattet mit dem Tensorprodukt von Abbildungen als Multiplikation und der gewöhnlichen Komposition von Abbildungen als Komposition.

Durch Einschränkung auf $\mathcal{P}(m,1)$ für $m \in \mathbb{Z}_{\geqslant 0}$ geht aus einer linearen Präoperade eine nicht-symmetrische Operade im klassischen Sinne (zum Beispiel bei May, cf. [13]) hervor, bei uns „absolute Operade“ genannt.

In §4 wird für eine biindizierte Menge $X = (X(m,n))_{m,n\geqslant 0}$ eine freie Präoperade $\mathrm{Free}_0(X)$ definiert, bestehend aus Äquivalenzklassen von Wörtern, deren Buchstaben Elemente aus X sind, die künstlich mit Identitäten multipliziert werden. Des Weiteren definieren wir Präsentationen von Präoperaden und finden eine Präsentation für die Präoperade Ass_0 und für die lineare Präoperade Ass.

Theorem (cf. Theorem 4.32, Theorem 4.33)**.** Es ist

$$\begin{aligned}\mathrm{Ass}_0 &\xleftarrow{\sim} {}_{\mathrm{spo}}\langle \hat{\varepsilon}, \hat{\mu} \mid \big((\hat{\mu}\boxtimes \mathrm{id}_1)\hat{\mu}\,,\,(\mathrm{id}_1\boxtimes\hat{\mu})\hat{\mu}\big), \big((\mathrm{id}_1\boxtimes\hat{\varepsilon})\hat{\mu}\,,\,\mathrm{id}_1\big), \big((\hat{\varepsilon}\boxtimes\mathrm{id}_1)\hat{\mu}\,,\,\mathrm{id}_1\big)\rangle\\ \mathrm{Ass} &\xleftarrow{\sim} {}_{\mathrm{lpo}}\langle \hat{\varepsilon}, \hat{\mu} \mid \big((\hat{\mu}\boxtimes \mathrm{id}_1)\hat{\mu} - (\mathrm{id}_1\boxtimes\hat{\mu})\hat{\mu}\big), \big((\mathrm{id}_1\boxtimes\hat{\varepsilon})\hat{\mu} - \mathrm{id}_1\big), \big((\hat{\varepsilon}\boxtimes\mathrm{id}_1)\hat{\mu} - \mathrm{id}_1\big)\rangle,\end{aligned}$$

wobei $\hat{\mu}$ auf μ, das eindeutig bestimmte Element von $\mathrm{Ass}_0(2,1)$, und $\hat{\varepsilon}$ auf ε, das eindeutig bestimmte Element von $\mathrm{Ass}_0(0,1)$, abgebildet wird.

Für eine Präoperade $\mathcal{P}_0$ in Mengen ist eine *$\mathcal{P}_0$-Algebra* gegeben durch ein Tupel (X,ϱ_0), wobei X eine Menge und $\varrho_0 : \mathcal{P}_0 \longrightarrow \mathrm{End}_0(X)$ ein Morphismus von Präoperaden in Mengen ist. Für eine lineare Präoperade $\mathcal{P}$ ist eine *$\mathcal{P}$-Algebra* gegeben durch ein Tupel (V,ϱ), wobei V ein R-Modul und $\varrho : \mathcal{P} \longrightarrow \mathrm{End}(V)$ ein Morphismus von linearen Präoperaden ist.

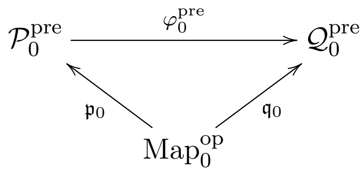
Wir zeigen dann, dass Ass_0-Algebren zu (assoziativen) Monoiden korrespondieren. Das heißt einerseits kann bei gegebener Ass_0-Algebra (X,ψ_0) eine Multiplikationsabbildung $\mu_X : X \times X \longrightarrow X$ und eine Einsabbildung $\varepsilon_X : \{()\} = X^{\times 0} \longrightarrow X$ so definiert werden, dass (X,μ_X,ε_X) ein Monoid ist, andererseits kann zu einem Monoid (X,μ_X,ε_X) ein Morphismus von Präoperaden angegeben werden, der diese Konstruktion umkehrt.

In ähnlicher Weise zeigen wir, dass Ass-Algebren zu assoziativen R-Algebren korrespondieren.

Da sich dieses Resultat nicht in naheliegender Weise auf kommutative Monoide ausdehnen lässt, werden Operaden eingeführt.

Operaden

Eine Operade $\mathcal{P}_0 = (\mathcal{P}_0^{\mathrm{pre}}, \mathfrak{p}_0)$ in Mengen besteht aus einer Präoperade in Mengen $\mathcal{P}_0^{\mathrm{pre}}$ und einem Morphismus von Präoperaden $\mathfrak{p}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathcal{P}_0^{\mathrm{pre}}$ so, dass gewisse Kompatibilitätsbedingungen mit den Bildern ausgewählter Elemente von $\mathrm{Map}_0^{\mathrm{op}}$ erfüllt sind; cf. Definition 6.3. Ein Morphismus von Operaden in Mengen $\varphi_0 : \mathcal{P}_0 \longrightarrow \mathcal{Q}_0$ ist dann gegeben durch einen Morphismus $\varphi_0^{\mathrm{pre}} : \mathcal{Q}_0^{\mathrm{pre}} \longrightarrow \mathcal{Q}_0^{\mathrm{pre}}$ von Präoperaden, der verträglich ist mit den zu $\mathcal{P}_0$ und $\mathcal{Q}_0$ gehörenden Strukturmorphismen.



So kann zum Beispiel für eine Menge X die Präoperade $\mathrm{End}_0(X)$ zu einer Operade in Mengen $\mathrm{END}_0(X) = (\mathrm{End}_0(X), \mathfrak{e}_0)$ gemacht werden.

Für eine Oprade $\mathcal{P}_0$ in Mengen ist eine *$\mathcal{P}_0$-Algebra* dann gegeben durch ein Tupel (X,ϱ_0), wobei X eine Menge und $\varrho_0 : \mathcal{P}_0 \longrightarrow \mathrm{END}_0(X)$ ein Morphismus von Operaden in Mengen ist.

Eine lineare Operade $\mathcal{P} = (\mathcal{P}^{\mathrm{pre}}, \mathfrak{p})$ besteht aus einer linearen Präoperade $\mathcal{P}^{\mathrm{pre}}$ und einem Morphismus von Präoperaden $\mathfrak{p} : \mathrm{Sym}^{\mathrm{op}} \longrightarrow \mathcal{P}^{\mathrm{pre}}$ so, dass Multiplikation und Komposition in $\mathcal{P}^{\mathrm{pre}}$ in geeigneter Weise verträglich sind mit den Bildern vorgegebener bijektiver Abbildungen unter $\mathfrak{p}$; cf.

Definition 6.22. Ein Morphismus von linearen Operaden $\varphi : \mathcal{P} \longrightarrow \mathcal{Q}$ ist dann gegeben durch einen Morphismus $\varphi^{\text{pre}} : \mathcal{P}^{\text{pre}} \longrightarrow \mathcal{Q}^{\text{pre}}$ von linearen Präoperaden, der verträglich ist mit den zu $\mathcal{P}$ und $\mathcal{Q}$ gehörenden Strukturmorphismen.

Wir können so, ähnlich wie im Mengen-Fall, für einen R-Modul V aus der linearen Präoperade $\operatorname{End}(V)$ eine lineare Operade $\operatorname{END}(V) = (\operatorname{End}(V), \mathfrak{e})$ definieren. Für eine lineare Operade $\mathcal{P}$ ist dann eine *$\mathcal{P}$-Algebra* gegeben durch ein Tupel (V, ϱ), wobei V ein R-Modul und $\varrho : \mathcal{P} \longrightarrow \operatorname{END}(V)$ ein Morphismus von linearen Operaden ist.

In den Kapiteln 7 – 10 werden einige Beispiele behandelt.

Die Operade ASS_0 in Mengen besteht aus Tupeln der Form $f \backslash a$, meist als Bruch bezeichnet, mit einem Element a aus Ass_0 im Zähler und einem Element f aus Map_0 im Nenner. Die Strukturabbildung $\mathfrak{a}_0 : \ \mathrm{Map}_0^{\mathrm{op}} \ \longrightarrow \ \mathrm{ASS}_0^{\mathrm{pre}}$ bildet $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$ auf den Bruch $f \backslash \mathrm{id}_{\mathrm{Ass}_0,n}$ ab.

Theorem (cf. Proposition 7.16 und Proposition 7.18)**.** ASS_0-Algebren korrespondieren zu assoziativen Monoiden.

Die Operade COM_0 in Mengen besteht, ähnlich wie ASS_0 , aus Brüchen von Abbildungen. Allerdings ist ein Bruch $f \backslash a$ in COM_0 eine Äquivalenzklasse eines Tupels (f, a), wobei $[1,m] \xleftarrow{f} X \xrightarrow{a} [1,n]$ für eine endliche Menge X. Brüche können hier mit bijektiven Abbildungen erweitert werden. Die Strukturabbildung $\mathfrak{c}_0 : \mathrm{Map}_0^{\mathrm{op}} \longrightarrow \mathrm{COM}_0^{\mathrm{pre}}$ bildet ein Element $f^{\mathrm{op}} \in \mathrm{Map}_0^{\mathrm{op}}(m,n)$ auf den Bruch $f \backslash \mathrm{id}_{\mathrm{Map}_0,n}$ ab.

Theorem (cf. Proposition 9.11 und Proposition 9.15)**.** COM_0-Algebren korrespondieren zu kommutativen Monoiden.

Um diese Operaden in Mengen nun zu linearen Operaden zu erweitern, schränken wir diese auf Brüche mit bijektiven Nennern ein. Das liefert die Teilpräoperaden in Mengen $\mathrm{ASS}_0^{\mathrm{pre,bij}} \subseteq \mathrm{ASS}_0^{\mathrm{pre}}$ und $\mathrm{COM}_0^{\mathrm{pre,bij}} \subseteq \mathrm{COM}_0^{\mathrm{pre}}$. Durch lineare Fortsetzung erhalten wir die linearen Operaden

$$\mathrm{ASS} = (\mathrm{ASS}^{\mathrm{pre}}, \mathfrak{a}) := \big(R\,\mathrm{ASS}_0^{\mathrm{pre,bij}}, R\big(\mathfrak{a}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}^{\mathrm{ASS}_0^{\mathrm{pre,bij}}}\big)\big)$$
$$\mathrm{COM} = (\mathrm{COM}^{\mathrm{pre}}, \mathfrak{c}) := \big(R\,\mathrm{COM}_0^{\mathrm{pre,bij}}, R\big(\mathfrak{c}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}^{\mathrm{COM}_0^{\mathrm{pre,bij}}}\big)\big)$$

Theorem (cf. Proposition 7.22 und Proposition 7.24)**.** ASS-Algebren korrespondieren zu assoziativen Algebren.

Theorem (cf. Proposition 9.19 und Proposition 9.21)**.** COM-Algebren korrespondieren zu kommutativen Algebren.

Schränken wir $\mathrm{ASS}_0^{\mathrm{pre}}$ nicht auf bijektive Nenner ein, so erhalten wir die lineare Operade

$$\mathrm{BIALG} = (\mathrm{BIALG}^{\mathrm{pre}}, \mathfrak{b}) := \big(R\,\mathrm{ASS}_0^{\mathrm{pre}}, R\big(\mathfrak{a}_0|_{\mathrm{Sym}_0^{\mathrm{op}}}\big)\big).$$

Es ist $\mathrm{ASS} \subseteq \mathrm{BIALG}$ eine lineare Teiloperade.

Proposition (cf. Proposition 8.3)**.** Sei (V, Θ) eine BIALG-Algebra. Dann ist $(V, \mu_V, \varepsilon_V, \Delta_V, \eta_V)$ eine Bialgebra mit

$$\begin{array}{rcl}
\text{Multiplikation} & \mu_V & := (\mathrm{id}_2 \backslash \mu)\Theta^{\mathrm{pre}} \in \operatorname{End}(V)(2,1) \\
\text{Eins} & \varepsilon_V & := (\mathrm{id}_0 \backslash \varepsilon)\Theta^{\mathrm{pre}} \in \operatorname{End}(V)(0,1) \\
\text{Komultiplikation} & \Delta_V & := (\mu \backslash \mathrm{id}_2)\Theta^{\mathrm{pre}} \in \operatorname{End}(V)(1,2) \\
\text{Koeins} & \eta_V & := (\varepsilon \backslash \mathrm{id}_0)\Theta^{\mathrm{pre}} \in \operatorname{End}(V)(1,0).
\end{array}$$

Die Tatsache, dass $\mathrm{ASS} \subseteq \mathrm{BIALG}$ eine Teiloperade ist, ist hier in der Tatsache wiederzufinden, dass eine Bialgebra insbesondere eine assoziative Algebra ist.

Das letzte Beispiel ist die lineare Operade LIE. Diese ist die von dem Element

$$\lambda := \mathrm{id}_2 \setminus \mu - (1,2) \setminus \mu \in \mathrm{ASS}^{\mathrm{pre}}(2,1)$$

erzeugte Teiloperade von ASS, wobei $(1,2) \in \mathrm{Sym}_0(2,2)$ die Transposition ist.

Proposition (cf. Proposition 10.2)**.** Sei nun $2 \in \mathcal{U}(R)$ vorausgesetzt. Sei (V, Λ) eine LIE-Algebra. Dann ist $(V, [-,=])$ eine Liealgebra mit Lieklammer definiert durch

$$[v, w] := (v \otimes w)\big(\lambda \Lambda^{\mathrm{pre}}\big)$$

für $v, w \in V$.

Jede assoziative Algebra $(V, \mu_V, \varepsilon_V)$ kann zu einer ASS-Algebra (V, Ψ) gemacht werden. Da nach Definition LIE $\subseteq$ ASS eine lineare Teiloperade ist, erhalten wir durch Einschränkung eine LIE-Algebra $(V, \Psi\big|^{\mathrm{LIE}^{\mathrm{pre}}})$. Die daraus entstehende Liealgebra $(V, [-,=])$ ist genau die Kommutator-Liealgebra zur assoziativen Algebra $(V, \mu_V, \varepsilon_V)$.

Allerdings muss ein Morphismus von linearen Operaden $\Lambda : \mathrm{LIE} \longrightarrow \mathrm{END}(V)$ nicht unbedingt Einschränkung eines Morphismus $\Psi : \mathrm{ASS} \longrightarrow \mathrm{END}(V)$ linearer Operaden sein. Dies entspricht der bekannten Tatsache, dass nicht jede Lielagebra die Kommutator-Liealgebra einer assoziativen Algebra ist.

Offen bleibt die Frage, ob die Konstruktionen für BIALG und LIE auch umgekehrt werden können, das heißt, ob zu einer gegebenen Bialgebra $(V, \mu_V, \varepsilon_V, \Delta_V, \eta_V)$ ein zugehöriger Morphismus linearer Operaden $\Theta : \mathrm{BIALG} \longrightarrow \mathrm{END}(V)$ existiert und ob zu einer gegebenen Liealgebra $(V, [-,=])$ stets ein zugehöriger Morphismus $\Lambda : \mathrm{LIE} \longrightarrow \mathrm{END}(V)$ linearer Operaden existiert.

Ebenso unbeantwortet bleibt die Frage nach einer freien Operade FREE(X) und folglich nach Präsentationen für die behandelten Operaden, was im Falle von BIALG und LIE auch bei der Konstruktion obiger zugehöriger Morphismen helfen würde.

Erklärung

Ich versichere, dass ich die vorliegende Masterarbeit selbstständig und lediglich unter Benutzung der angegebenen Quellen verfasst habe. Alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen wurden als solche gekennzeichnet. Ich erkläre weiterhin, dass diese Arbeit weder vollständig noch in wesentlichen Teilen im Rahmen eines anderen Prüfungsverfahrens eingereicht wurde. Das elektronische Exemplar stimmt mit dieser Arbeit überein.

Ort, Datum	Unterschrift