Adjoint functors between crossed squares and $\lfloor 2, 0 \rfloor$-simplicial groups

Master Thesis

$$\begin{tikzcd}
\lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp} \arrow[rrr, bend left, "\mathrm{Sq}"] \arrow[rrr, phantom, "\perp"] & & & \mathcal{CrSq} \arrow[dl, "\mathrm{Tr}", "\sim"'] \\
& 2\text{-}\mathcal{CrMod} \arrow[ul, "\mathrm{Rec}", "\sim"'] & \mathcal{CrSq} \arrow[l, bend left=10, "\mathrm{To}"] &
\end{tikzcd}$$

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Contents

Abstract

We consider the following diagram.

$$\begin{tikzcd}
\lfloor 2,0\rfloor\text{-}\mathit{SimpGrp} \arrow[rrr, bend left, "\mathrm{Sq}"] & & & \mathit{CrSq} \arrow[dl, "\mathrm{Tr}", "\sim"'] \\
& 2\text{-}\mathit{CrMod} \arrow[ul, "\mathrm{Rec}", "\sim"'] & \mathit{CrSq} \arrow[l, "\mathrm{To}"] &
\end{tikzcd}$$

Conduché has defined the category $\lfloor 2,0\rfloor$-*SimpGrp* of $\lfloor 2,0\rfloor$-simplicial groups, and a truncation functor $\mathrm{Trunc}_{\lfloor 2,0\rfloor} : \mathit{SimpGrp} \to \lfloor 2,0\rfloor\text{-}\mathit{SimpGrp}$.

The category *CrSq* of crossed squares has been introduced by Loday. It carries the transposition functor Tr.

The functor $\mathrm{Sq} : \lfloor 2,0\rfloor\text{-}\mathit{SimpGrp} \to \mathit{CrSq}$ has been constructed by Porter.

The total 2-crossed module functor $\mathrm{To} : \mathit{CrSq} \to 2\text{-}\mathit{CrMod}$ has been constructed by Conduché, as well as the reconstruction equivalence $\mathrm{Rec} : 2\text{-}\mathit{CrMod} \to \lfloor 2,0\rfloor\text{-}\mathit{SimpGrp}$.

We show that the functor Sq is left-adjoint to the composite $\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}$.

0 Introduction

0.1 Simplicial groups

The simplex category $\triangle$ is the category of linearly ordered finite sets of the form $\{0, \ldots, n\}$, where $n \in \mathbb{Z}_{\geqslant 0}$, and monotone maps between them.

A *simplicial group* is a functor from $\triangle^{\mathrm{op}}$ to the category of groups.

This means, a simplicial group G is a diagram of groups of the form

$$\begin{tikzcd}
\ldots\ G_3 \arrow[r, shift left=18, "\mathrm{d}_3^{G,3}"] \arrow[r, leftarrow, shift left=12, "\mathrm{s}_2^{G,2}"] \arrow[r, shift left=6, "\mathrm{d}_2^{G,3}"] \arrow[r, leftarrow, "\mathrm{s}_1^{G,2}"] \arrow[r, shift right=6, "\mathrm{d}_1^{G,3}"] \arrow[r, leftarrow, shift right=12, "\mathrm{s}_0^{G,2}"] \arrow[r, shift right=18, "\mathrm{d}_0^{G,3}"] & G_2 \arrow[r, shift left=12, "\mathrm{d}_2^{G,2}"] \arrow[r, leftarrow, shift left=6, "\mathrm{s}_1^{G,1}"] \arrow[r, "\mathrm{d}_1^{G,2}"] \arrow[r, leftarrow, shift right=6, "\mathrm{s}_0^{G,1}"] \arrow[r, shift right=12, "\mathrm{d}_0^{G,2}"] & G_1 \arrow[r, shift left=6, "\mathrm{d}_1^{G,1}"] \arrow[r, leftarrow, "\mathrm{s}_0^{G,0}"] \arrow[r, shift right=6, "\mathrm{d}_0^{G,1}"] & G_0\ ,
\end{tikzcd}$$

where the group morphisms $\mathrm{d}_i^{G,n} : G_n \to G_{n-1}$ and $\mathrm{s}_j^{G,n} : G_n \to G_{n+1}$ satisfy the simplicial relations.

The *category of simplicial groups* is denoted by $\mathcal{S}imp\mathcal{G}rp$.

Connected topological spaces can be modelled by simplicial groups as follows.

We have an equivalence between the homotopy category of pointed connected topological spaces and the homotopy category of reduced simplicial sets; cf. [**4,** Th. 3.2, Th. 3.4, see also Th. 3.4(ii)].

Moreover, we have an equivalence between the homotopy category of reduced simplicial sets and the homotopy category of simplicial groups; cf. [**5,** V, Cor. 6.4].

Altogether, simplicial groups model pointed connected topological spaces up to homotopy equivalence.

0.2 $\lfloor 2, 0\rfloor$-simplicial groups

A $\lfloor 2, 0\rfloor$-*simplicial group* G consists of groups G_2, G_1, G_0 and face and degeneracy morphisms as follows

$$\begin{tikzcd}
G_2 \arrow[r, shift left=12, "\mathrm{d}_2^{G,2}"] \arrow[r, <-, shift left=6, "\mathrm{s}_1^{G,1}"] \arrow[r, "\mathrm{d}_1^{G,2}"] \arrow[r, <-, shift right=6, "\mathrm{s}_0^{G,1}"] \arrow[r, shift right=12, "\mathrm{d}_0^{G,2}"] & G_1 \arrow[r, shift left=6, "\mathrm{d}_1^{G,1}"] \arrow[r, <-, "\mathrm{s}_0^{G,0}"] \arrow[r, shift right=6, "\mathrm{d}_0^{G,1}"] & G_0 \ ,
\end{tikzcd}$$

satisfying the simplicial relations and the Conduché condition

$$\begin{array}{rcl} [G_{2;0}, G_{2;1,2}] & = & 1 \\ [G_{2;1}, G_{2;0,2}] & = & 1 \\ [G_{2;2}, G_{2;0,1}] & = & 1 \end{array}$$

in which we write

$$\begin{array}{rcl} G_{2;i} & := & \ker(G_2 \xrightarrow{\mathrm{d}_i^{G,2}} G_1) \trianglelefteq G_2 \ , \\ G_{2;j,k} & := & \ker(G_2 \xrightarrow{\mathrm{d}_j^{G,2}} G_1) \cap \ker(G_2 \xrightarrow{\mathrm{d}_k^{G,2}} G_1) \trianglelefteq G_2 \end{array}$$

for i, j, $k \in \{0, 1, 2\}$, where $j < k$.

The *category of* $\lfloor 2, 0\rfloor$-*simplicial groups* is denoted by $\lfloor 2, 0\rfloor$-$\mathit{SimpGrp}$.

There is a *truncation functor*

$$\mathit{SimpGrp} \xrightarrow{\mathrm{Trunc}_{\lfloor 2,0\rfloor}} \lfloor 2, 0\rfloor\text{-}\mathit{SimpGrp}$$

respecting homotopy groups; cf. [**1,** Def. 52]. So if one wants to study the $\lfloor 2, 0\rfloor$-part of a simplicial group G, one can study $G\,\mathrm{Trunc}_{\lfloor 2,0\rfloor}$.

0.3 Crossed Squares

A *crossed square* is a commutative quadrangle of groups

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & M' \arrow[d] \\
M \arrow[r] & P \ ,
\end{tikzcd}$$

where in addition M and M' act on L, where P acts on M, on M' and on L, together with the Loday bracket

$$\begin{array}{rcl} M \times M' & \xrightarrow{\chi} & L \\ (m, m') & \longmapsto & \lceil m, m' \rceil , \end{array}$$

satisfying a list of properties; cf. Definition 15.

This notion is due to Loday [**7,** Def. 5.1].

The *category of crossed squares* is denoted by CrSq.

In a sense, crossed squares may be thought of as crossed modules of crossed modules.

0.4 A functor Sq from $\lfloor 2,0\rfloor$-simplicial groups to crossed squares

Following PORTER [**8,** Prop. 7, Proof of Lem. A], we define the crossed square

$$G\,\mathrm{Sq} := \left(\begin{tikzcd} G_{2;1,2} \arrow[r, "\lambda^{0,1}_{G\,\mathrm{Sq}}"] \arrow[d, "\lambda^{1,0}_{G\,\mathrm{Sq}}"'] & G_{2;2}/G_{2;0,2} \arrow[d, "\mu^{G\,\mathrm{Sq}}_{0,1}"] \\ G_{2;1}/G_{2;0,1} \arrow[r, "\mu^{G\,\mathrm{Sq}}_{1,0}"'] & G_2/G_{2;0} \end{tikzcd}\right),$$

in which all maps are identical on representatives, in which all actions are induced by conjugation and for which $\chi^{G\,\mathrm{Sq}}$ is given by the commutator of the representatives.

This yields the functor

$$\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp \xrightarrow{\mathrm{Sq}} \mathcal{C}r\mathcal{S}q\ .$$

It has been investigated in [**1,** §5.2.3].

The functor Sq is not dense and therefore not an equivalence; cf. [**1,** Rem. 83].

We have the group isomorphisms

$$\begin{array}{rcl} \psi_{1,1} = \mathrm{id}_{G_{2;1,2}} & : & G_{2;1,2} \xrightarrow{\sim} G_{2;1,2} : g_2 \longmapsto g_2 \\ \psi_{1,0} & : & G_{2;1}/G_{2;0,1} \xrightarrow{\sim} G_{1;0} : g_2 G_{2;0,1} \longmapsto g_2\,\mathrm{d}_0 \\ \psi_{0,1} & : & G_{2;2}/G_{2;0,2} \xrightarrow{\sim} G_{1;1} : g_2 G_{2;0,2} \longmapsto g_2\,\mathrm{d}_0 \\ \psi_{0,0} & : & G_2/G_{2;0} \xrightarrow{\sim} G_1 : g_2 G_{2;0} \longmapsto g_2\,\mathrm{d}_0\ ; \end{array}$$

cf. Remark 84.

By transport of structure, we build a crossed square $G\check{\mathrm{Sq}}$ fitting into the isomorphism of crossed squares; cf. Definition 85.

$$\begin{tikzcd} G\,\mathrm{Sq} : \arrow[dd, "G\psi"'] & & G_{2;1,2} \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\psi_{1,1}\,\wr"' near start] & & G_{2;2}/G_{2;0,2} \arrow[dl, "\mu'"] \arrow[dd, "\wr\,\psi_{0,1}"] \\ & G_{2;1}/G_{2;0,1} \arrow[rr, "\mu" near start] \arrow[dd, "\psi_{1,0}\,\wr"'] & & G_2/G_{2;0} \arrow[dd, "\wr\,\psi_{0,0}" near start] & \\ G\check{\mathrm{Sq}} : & & G_{2;1,2} \arrow[rr, "\tilde{\lambda}'" near start] \arrow[dl, "\tilde{\lambda}"'] & & G_{1;1} \arrow[dl, "\tilde{\mu}'"] \\ & G_{1;0} \arrow[rr, "\tilde{\mu}"] & & G_1 & \end{tikzcd}$$

We turn $\check{\mathrm{Sq}}$ into a functor $\check{\mathrm{Sq}} : \lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp \to \mathcal{C}r\mathcal{S}q$ such that we obtain the isotransformation $\psi = (G\psi)_{G\in\mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp)} : \mathrm{Sq} \xrightarrow{\sim} \check{\mathrm{Sq}}$.

0.5 A transposition functor Tr from crossed squares to crossed squares

The symmetry with respect to the diagonal in the definition of a crossed square can be expressed by constructing a transposition functor as follows.

Suppose given a crossed square

$$\begin{tikzcd}
 & L \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & M' \arrow[d, "\mu'"] \\
C: & M \arrow[r, "\mu"'] & P\, .
\end{tikzcd}$$

We define the map

$$\begin{array}{ccl} M' \times M & \xrightarrow{\chi^{\mathrm{tr}}} & L \\ (m', m) & \longmapsto & (m', m)\chi^{\mathrm{tr}} := \lceil m', m \rceil^{\mathrm{tr}} := \lceil m, m' \rceil^{-}. \end{array}$$

Using χ^{tr} and interchanging the other data, we obtain the transposed crossed square

$$\begin{tikzcd}
 & L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
C^{\mathrm{tr}}: & M' \arrow[r, "\mu'"'] & P\, .
\end{tikzcd}$$

This yields the *transposition functor*

$$\mathcal{CrSq} \xrightarrow{\mathrm{Tr}} \mathcal{CrSq}\, .$$

0.6 A total 2-crossed module functor To from crossed squares to 2-crossed modules

Following Conduché [2, Déf. 2.2], a 2-*crossed module* is a diagram of groups of the form

$$N_2 \xrightarrow{\partial_2} N_1 \xrightarrow{\partial_1} N_0\, ,$$

where furthermore N_0 acts on N_1 via β_1 and on N_2 via β_2, together with the Conduché bracket

$$\begin{array}{ccl} N_1 \times N_1 & \xrightarrow{\zeta} & N_2 \\ (n_1, n_1') & \longmapsto & \lfloor n_1, n_1' \rfloor, \end{array}$$

satisfying certain properties; cf. Definition 36.

The *category of* 2-*crossed modules* is denoted by $2\text{-}\mathcal{CrMod}$.

Suppose given a crossed square

$$\begin{tikzcd}
 & L \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & M' \arrow[d, "\mu'"] \\
C: & M \arrow[r, "\mu"'] & P\, .
\end{tikzcd}$$

We form the semidirect product $M \ltimes_\alpha M'$, in which

$$(m, m') \cdot (\tilde{m}, \tilde{m}') = (m \cdot \tilde{m}, m'^{\tilde{m}\mu} \cdot \tilde{m}'),$$

for (m, m'), $(\tilde{m}, \tilde{m}') \in M \ltimes_\alpha M'$.

Following Conduché [**3,** Cor. 3.5], refining a construction of Loday [**7,** Def. and Lem. 3.1, Def. 5.1], we obtain the total 2-crossed module of C

$$C\,\mathrm{To} := \left(L \xrightarrow{\partial_2} M \ltimes_\alpha M' \xrightarrow{\partial_1} P \right),$$

in which the Conduché bracket of $C\,\mathrm{To}$ is defined via the Loday bracket of C as

$$\lfloor (m, m'), (\tilde{m}, \tilde{m}') \rfloor := \lceil m^{\tilde{m}}, \tilde{m}' \rceil,$$

where (m, m'), $(\tilde{m}, \tilde{m}') \in M \ltimes_\alpha M'$, and where the action of P on $M \ltimes_\alpha M'$ and on L is taken from C.

This yields the *total* 2-*crossed module functor*

$$\mathit{CrSq} \xrightarrow{\mathrm{To}} 2\text{-}\mathit{CrMod}.$$

The functor To is not full and therefore not an equivalence; cf. Remark 98.

0.7 A reconstruction functor Rec from 2-crossed modules to $\lfloor 2, 0 \rfloor$-simplicial groups, inverse to the functor $\hat{\mathrm{N}}$

Suppose given a 2-crossed module N.

We form the semidirect product $(N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2)$ with the help of the group morphisms

$$\begin{array}{rcl} N_1 & \xrightarrow{\varepsilon_2} & \mathrm{Aut}(N_2) \\ n_1 & \longmapsto & (n_2 \mapsto n_2^{n_1} := n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor) \end{array}$$

and

$$\begin{array}{rcl} N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\varepsilon_{1,2}} & \mathrm{Aut}(N_1 \ltimes_{\varepsilon_2} N_2) \\ (n_0, n_1) & \longmapsto & ((\check{n}_1, n_2) \mapsto (\check{n}_1, n_2)^{(n_0, n_1)} := ((\check{n}_1^{n_0})^{n_1}, \lfloor \check{n}_1^{n_0}, n_1 \rfloor \cdot n_2^{n_0 \cdot n_1 \partial_1})). \end{array}$$

We then obtain the $\lfloor 2, 0 \rfloor$-simplicial group

$$\begin{tikzcd}[column sep=huge] N\,\mathrm{Rec}: & (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) \arrow[r, shift left=8, "\mathrm{d}_2^{N\,\mathrm{Rec},2}"] \arrow[r, <-, shift left=4, "\mathrm{s}_1^{N\,\mathrm{Rec},1}"] \arrow[r, "\mathrm{d}_1^{N\,\mathrm{Rec},2}"] \arrow[r, <-, shift right=4, "\mathrm{s}_0^{N\,\mathrm{Rec},1}"] \arrow[r, shift right=8, "\mathrm{d}_0^{N\,\mathrm{Rec},2}"] & N_0 \ltimes_{\beta_1} N_1 \arrow[r, shift left=4, "\mathrm{d}_1^{N\,\mathrm{Rec},1}"] \arrow[r, <-, "\mathrm{s}_0^{N\,\mathrm{Rec},0}"] \arrow[r, shift right=4, "\mathrm{d}_0^{N\,\mathrm{Rec},1}"] & N_0 . \end{tikzcd}$$

This yields the *reconstruction functor*

$$2\text{-}\mathit{CrMod} \xrightarrow{\mathrm{Rec}} \lfloor 2, 0 \rfloor\text{-}\mathit{SimpGrp}\,.$$

Suppose given a $\lfloor 2,0\rfloor$-simplicial group G.

Let

$$\begin{array}{lcllcl} G\mathrm{N}_0 & := & G_0 & & & \\ G\mathrm{N}_1 & := & G_{1;1} & = & \ker(\mathrm{d}_1^{G,1}) & \trianglelefteq G_1 \\ G\mathrm{N}_2 & := & G_{2;1,2} & = & \ker(\mathrm{d}_1^{G,2}) \cap \ker(\mathrm{d}_2^{G,2}) & \trianglelefteq G_2 \,. \end{array}$$

Then we obtain the 2-crossed module

$$G\hat{\mathrm{N}} := \left(G\mathrm{N}_2 \xrightarrow{\partial_2} G\mathrm{N}_1 \xrightarrow{\partial_1} G\mathrm{N}_0 \right),$$

in which the "hat" on "$\hat{\mathrm{N}}$" is meant to indicate the extra data attached to the diagram needed to yield a 2-crossed module; cf. Lemma 51.

This yields the functor

$$\lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp} \xrightarrow{\hat{\mathrm{N}}} 2\text{-}\mathcal{CrMod}\,.$$

The functors Rec and $\hat{\mathrm{N}}$ are mutually inverse equivalences; cf. [**2,** Th. 2.6] and Proposition 78.

0.8 The adjoint functors $\mathrm{Sq} \dashv (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})$

We construct the isotransformation

$$\vartheta : \hat{\mathrm{N}} \blacktriangle \mathrm{Rec} \xrightarrow{\sim} \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp}};$$

cf. Lemma 76 and Lemma 77.

We construct the transformation

$$\nu : \hat{\mathrm{N}} \to \mathrm{Sq} \blacktriangle \mathrm{Tr} \blacktriangle \mathrm{To}\ ;$$

cf. Lemma 100 and Lemma 102.

We let

$$\varepsilon := \vartheta^{-} \blacktriangle \nu\, \mathrm{Rec} : \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp}} \to \mathrm{Sq} \blacktriangle (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})\ ;$$

cf. Lemma 104.

Given the crossed square

$$\begin{tikzcd} C : & L \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & M' \arrow[d, "\mu'"] \\ & M \arrow[r, "\mu"'] & P\ , \end{tikzcd}$$

we have the $\lfloor 2,0\rfloor$-simplicial group

$$\mathrm{G}^C := C\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec}$$

consisting of the groups

$$\begin{array}{lcl} \mathrm{G}_0^C & := & P \\ \mathrm{G}_1^C & := & P \ltimes_{\beta_1} (M' \ltimes_\alpha M) \\ \mathrm{G}_2^C & := & (P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L)\ ; \end{array}$$

cf. Remark 105.

We have the group morphisms

$$\begin{array}{rcl} \mathrm{G}^{C}_{2;1,2} & \xrightarrow{C\xi_{1,1}} & L \\ ((1,(1,1)),((1,1),l)) & \longmapsto & l \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^{C}_{1;0} & \xrightarrow{C\xi_{1,0}} & M \\ (m^{-}\mu \cdot m'^{-}\mu', (m',m)) & \longmapsto & m^{-} \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^{C}_{1;1} & \xrightarrow{C\xi_{0,1}} & M' \\ (1,(m',m)) & \longmapsto & m' \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^{C}_{1} & \xrightarrow{C\xi_{0,0}} & P \\ (p,(m',m)) & \longmapsto & p \cdot m'\mu' \ ; \end{array}$$

cf. Lemma 108.

This yields the transformation

$$\xi = (C\xi)_{C\in\mathrm{Ob}(\mathit{CrSq})} : (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \check{\mathrm{Sq}} \to \mathrm{Id}_{\mathit{CrSq}} \ ;$$

cf. Lemma 110.

We let

$$\eta := (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})\psi \blacktriangle \xi : (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \mathrm{Sq} \to \mathrm{Id}_{\mathit{CrSq}} \ ;$$

cf. Lemma 111.

Altogether, we have the following functors

$$\begin{tikzcd} \lfloor 2,0 \rfloor\text{-}\mathit{SimpGrp} \arrow[r, bend left, "\mathrm{Sq}"] & \mathit{CrSq} \arrow[l, bend left, "\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}"] \end{tikzcd} .$$

Proving commutativity of the adjunction triangles, we obtain that

$$(\mathrm{Sq},\ \mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec},\ \varepsilon,\ \eta)$$

is an adjunction. In particular, we have

$$\mathrm{Sq} \dashv \mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec} \, .$$

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$$\begin{tikzcd}
\underbrace{\lfloor 2,0\rfloor\text{-}\mathit{SimpGrp}}_{\S\S 4.1,\,4.2} \arrow[rrrr, bend left, "\overbrace{\mathrm{Sq}}^{\S 5.3}"] \arrow[dr, bend left, "\overbrace{\hat{\mathrm{N}}}^{\S 4.5.3}", "\sim"'] & & \underbrace{\perp}_{\S 7.3} & & \underbrace{\mathit{CrSq}}_{\S\S 3.1,\,3.2} \arrow[dl, bend left, "\underbrace{\mathrm{Tr}}_{\S 3.3}", "\sim"'] \\
& \underbrace{2\text{-}\mathit{CrMod}}_{\S\S 4.3,\,4.4} \arrow[ul, bend left, "\underbrace{\mathrm{Rec}}_{\S 4.6.3}"] & & \underbrace{\mathit{CrSq}}_{\S\S 3.1,\,3.2} \arrow[ll, bend left, "\underbrace{\mathrm{To}}_{\S 6.3}"] &
\end{tikzcd}$$

1 Conventions

Suppose given a group G.

- We use $(\,\blacktriangle\,)$ to denote composition in a category.

  We compose on the right, i.e. the composite of $X \xrightarrow{u} Y \xrightarrow{v} Z$ is written $X \xrightarrow{u \blacktriangle v} Z$.

  Here, $u \blacktriangle v$ reads "u comp v".
- Categories are supposed to be small with respect to a suitable universe.
- For $n \in \mathbb{Z}_{\geqslant 0}$, we write $[0, n] = \{a \in \mathbb{Z} : 0 \leqslant a \leqslant n\}$.
- For $n \in \mathbb{Z}_{\geqslant 0}$, we write $\lfloor n, 0 \rfloor := \{a \in \mathbb{Z} : 0 \leqslant a \leqslant n\}$, ordered decreasingly.
- We write $g^{-} := g^{-1}$ for the inverse element of $g \in G$.
- Suppose given $g,\, h \in G$.

  We write $h^g := g^{-} \cdot h \cdot g$ and $[g, h] := g^{-} \cdot h^{-} \cdot g \cdot h$.
- Suppose given a group H acting on G.

  Suppose given a group K acting on H.

  Suppose given $g \in G$, $h \in H$ and $k \in K$.

  We write $g^{h^k} := g^{(h^k)}$.
- Suppose given subgroups $M,\, N \leqslant G$.

  We write $[M, N] := \langle [m,\, n] : m \in M,\, n \in N \rangle \trianglelefteqslant G$.

  Note that if $M,\, N \trianglelefteqslant G$, then $[M,\, N] \trianglelefteqslant G$.
- Given a group isomorphism $\varphi : G \xrightarrow{\sim} H$, we sometimes write

$$
\begin{array}{rcl}
\varphi : G & \longrightarrow & H \\
g & \longmapsto & g\varphi \\
h\varphi^{-} & \longleftarrow\!\shortmid & h.
\end{array}
$$

2 Preliminaries

2.1 A remark on functors

Remark 1 Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$.

Suppose given a map $\check{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.

Suppose given an isomorphism $Xa : XF \xrightarrow{\sim} X\check{F}$ for $X \in \mathrm{Ob}(\mathcal{C})$.

For $X \xrightarrow{u} X'$ in $\mathcal{C}$ we define $u\check{F} := (Xa)^- \blacktriangle uF \blacktriangle X'a$.

Then $\check{F} : \mathcal{C} \to \mathcal{D}$ is a functor and $a = (Xa)_{X \in \mathrm{Ob}(\mathcal{C})}$ is an isotransformation from F to $\check{F}$.

Proof. We have

$$\begin{aligned} \mathrm{id}_X \check{F} &= (Xa)^- \blacktriangle \mathrm{id}_X F \blacktriangle Xa \\ &= (Xa)^- \blacktriangle Xa \\ &= \mathrm{id}_{X\check{F}} . \end{aligned}$$

Suppose given morphisms $X \xrightarrow{u} X' \xrightarrow{v} X''$ in $\mathcal{C}$.

Then we have

$$\begin{aligned} (u\check{F}) \blacktriangle (v\check{F}) &= ((Xa)^- \blacktriangle uF \blacktriangle X'a) \blacktriangle ((X'a)^- \blacktriangle vF \blacktriangle X''a) \\ &= (Xa)^- \blacktriangle uF \blacktriangle X'a \blacktriangle (X'a)^- \blacktriangle vF \blacktriangle X''a \\ &= (Xa)^- \blacktriangle uF \blacktriangle vF \blacktriangle X''a \\ &= (Xa)^- \blacktriangle (uF \blacktriangle vF) \blacktriangle X''a \\ &= (Xa)^- \blacktriangle (u \blacktriangle v)F \blacktriangle X''a \\ &= (u \blacktriangle v)\check{F}. \end{aligned}$$

So $\check{F} : \mathcal{C} \to \mathcal{D}$ is a functor.

By construction, a is an isotransformation from F to $\check{F}$:

$$\begin{tikzcd}
XF \arrow[r, "Xa", "\sim"'] \arrow[d, "uF"'] \arrow[dr, phantom, "\circlearrowleft"] & X\check{F} \arrow[d, "u\check{F} = (Xa)^- \blacktriangle uF \blacktriangle X'a"] \\
X'F \arrow[r, "X'a", "\sim"'] & X'\check{F}
\end{tikzcd}$$

□

2.2 Remarks on transformations

Remark 2 Suppose given functors $F : \mathcal{C} \to \mathcal{D}$, $\tilde{F} : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$.

Suppose given a transformation $a : F \to \tilde{F}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "\tilde{F}"'{name=L}] & \mathcal{D} \arrow[r, "G"] & \mathcal{E}
\arrow[from=U, to=L, "a"]
\end{tikzcd}$$

Then we have that

$$aG : F \blacktriangle G \to \tilde{F} \blacktriangle G,$$

defined by $X(aG) := (Xa)G$ for $X \in \mathrm{Ob}(\mathcal{C})$, is also a transformation.

We often write $XaG := X(aG) = (Xa)G$.

Proof. We have that $X(F \blacktriangle G) \xrightarrow{X(aG)} X(\tilde{F} \blacktriangle G)$ for $X \in \mathrm{Ob}(\mathcal{C})$.

Given a morphism $u : X \to X'$ in $\mathcal{C}$, we get

$$\begin{aligned}
(u(F \blacktriangle G)) \blacktriangle (X'(aG)) &= ((uF)G) \blacktriangle ((X'a)G) \\
&= ((uF) \blacktriangle (X'a))G \\
&\overset{a \text{ trans-}}{\underset{\text{formation}}{=}} ((Xa) \blacktriangle (u\tilde{F}))G \\
&= ((Xa)G) \blacktriangle ((u\tilde{F})G) \\
&= (X(aG)) \blacktriangle (u(\tilde{F} \blacktriangle G)).
\end{aligned}$$

So we have the following commutative quadrangle.

$$\begin{tikzcd}
X(F \blacktriangle G) \arrow[r, "X(aG)"] \arrow[d, "u(F \blacktriangle G)"'] \arrow[dr, phantom, "\circlearrowleft"] & X(\tilde{F} \blacktriangle G) \arrow[d, "u(\tilde{F} \blacktriangle G)"] \\
X'(F \blacktriangle G) \arrow[r, "X'(aG)"'] & X'(\tilde{F} \blacktriangle G)
\end{tikzcd}$$

□

Remark 3 Suppose given functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ and $\tilde{G} : \mathcal{D} \to \mathcal{E}$.

Suppose given a transformation $b : G \to \tilde{G}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "\tilde{G}"'] \arrow[r, phantom, "\downarrow b"] & \mathcal{E}
\end{tikzcd}$$

Then we have that

$$Fb : F \blacktriangle G \to F \blacktriangle \tilde{G},$$

defined by $X(Fb) := (XF)b$ for $X \in \mathrm{Ob}(\mathcal{C})$, is also a transformation.

We often write $XFb := X(Fb) = (XF)b$.

Proof. We have that $X(F \blacktriangle G) \xrightarrow{X(Fb)} X(F \blacktriangle \tilde{G})$ for $X \in \mathrm{Ob}(\mathcal{C})$.

Given a morphism $u : X \to X'$ in $\mathcal{C}$, we get

$$\begin{aligned}
(u(F \blacktriangle G)) \blacktriangle (X'(Fb)) &= ((uF)G) \blacktriangle ((X'F)b) \\
&\overset{b \text{ trans-}}{\underset{\text{formation}}{=}} ((XF)b) \blacktriangle ((uF)\tilde{G}) \\
&= (X(Fb)) \blacktriangle (u(F \blacktriangle \tilde{G})).
\end{aligned}$$

So we have the following commutative quadrangle.

$$\begin{tikzcd}
X(F \blacktriangle G) \arrow[r, "X(Fb)"] \arrow[d, "u(F \blacktriangle G)"'] \arrow[dr, phantom, "\circlearrowleft"] & X(F \blacktriangle \tilde{G}) \arrow[d, "u(F \blacktriangle \tilde{G})"] \\
X'(F \blacktriangle G) \arrow[r, "X'(Fb)"'] & X'(F \blacktriangle \tilde{G})
\end{tikzcd}$$

□

2.3 Adjoint functors

Definition 4 Suppose given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

Suppose given transformations $\varepsilon : \mathrm{Id}_{\mathcal{C}} \to F \blacktriangle G$ and $\eta : G \blacktriangle F \to \mathrm{Id}_{\mathcal{D}}$ such that the following *adjunction triangles* are commutative.

$$\begin{tikzcd}
F \arrow[r, "\varepsilon F"] \arrow[dr, "\mathrm{id}_F"'] & F \blacktriangle G \blacktriangle F \arrow[d, "F\eta"] \\
& F
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "G\varepsilon"] \arrow[dr, "\mathrm{id}_G"'] & G \blacktriangle F \blacktriangle G \arrow[d, "\eta G"] \\
& G
\end{tikzcd}$$

Then

$$(F,\, G,\, \varepsilon,\, \eta)$$

is an *adjunction.*

We call F a *left adjoint* to G and G a *right adjoint* to F.

We call ε the *unit* of the adjunction.

We call η the *counit* of the adjunction.

Notation. We often write $F \dashv G$ to express that there exists an adjunction $(F,\, G,\, \varepsilon,\, \eta)$.

Remark 5 Suppose given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

Suppose given transformations $\varepsilon : \mathrm{Id}_{\mathcal{C}} \to F \blacktriangle G$ and $\eta : G \blacktriangle F \to \mathrm{Id}_{\mathcal{D}}$ such that $(F,\, G,\, \varepsilon,\, \eta)$ is an adjunction.

Then we have the bijection

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{D}}(XF, Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(X, YG) \\
u & \longmapsto & (X\varepsilon) \blacktriangle (uG)
\end{array}$$

with inverse

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{C}}(X, YG) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}}(XF, Y) \\
v & \longmapsto & (vF) \blacktriangle (Y\eta)
\end{array}$$

for $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$.

2.4 Actions of groups

Remark 6 Suppose given groups G and H.

Suppose given a map

$$\begin{array}{rcl}
H \times G & \longrightarrow & H \\
(h, g) & \longmapsto & h^g.
\end{array}$$

Suppose that for g, $\tilde{g} \in G$ and h, $\tilde{h} \in H$ the following properties hold.

(1) $h^1 = h$

(2) $h^{g \cdot \tilde{g}} = (h^g)^{\tilde{g}}$

(3) $(h \cdot \tilde{h})^g = h^g \cdot \tilde{h}^g$

Then we have the group morphism

$$\begin{array}{rcl} G & \longrightarrow & \mathrm{Aut}(H) \\ g & \longmapsto & (g\gamma : h \mapsto h^g). \end{array}$$

Proof. We have that

$$\begin{array}{rcl} g\gamma : H & \longrightarrow & H \\ h & \longmapsto & h^g \end{array}$$

is a group morphism for $g \in G$, due to (3).

Furthermore, for $g \in G$ and $h \in H$, we have

$$\begin{array}{rcl} (h)(g\gamma \blacktriangle g^{-}\gamma) & = & (h^g)^{g^-} \\ & \overset{(2)}{=} & h^{g \cdot g^-} \\ & = & h^1 \\ & \overset{(1)}{=} & h \end{array}$$

and

$$\begin{array}{rcl} (h)(g^{-}\gamma \blacktriangle g\gamma) & = & (h^{g^-})^g \\ & \overset{(2)}{=} & h^{g^- \cdot g} \\ & = & h^1 \\ & \overset{(1)}{=} & h. \end{array}$$

So we get

$$g\gamma \blacktriangle g^{-}\gamma = \mathrm{id}$$

and

$$g^{-}\gamma \blacktriangle g\gamma = \mathrm{id}\,.$$

Therefore, $g\gamma \in \mathrm{Aut}(H)$ for $g \in G$, having $(g\gamma)^- = g^-\gamma$.

Moreover, for g, $\tilde{g} \in G$ and $h \in H$, we have

$$\begin{array}{rcl} h((g \cdot \tilde{g})\gamma) & = & h^{g \cdot \tilde{g}} \\ & \overset{(2)}{=} & (h^g)^{\tilde{g}} \\ & = & (h(g\gamma))\tilde{g}\gamma \\ & = & h(g\gamma \blacktriangle \tilde{g}\gamma). \end{array}$$

So we get

$$(g \cdot \tilde{g})\gamma = g\gamma \blacktriangle \tilde{g}\gamma.$$

Altogether, we get that

$$\begin{array}{rcl} G & \longrightarrow & \mathrm{Aut}(H) \\ g & \longmapsto & g\gamma \end{array}$$

is a group morphism. ☐

Remark 7 Suppose given groups G and H.

Suppose given a group morphism

$$\begin{array}{rcl} \gamma : G & \longrightarrow & \mathrm{Aut}(H) \\ g & \longmapsto & (g\gamma : h \mapsto h^g). \end{array}$$

We form the semidirect product $G \ltimes_\gamma H = \{(g,h) : g \in G,\ h \in H\}$, carrying the multiplication

$$(g,h) \cdot (g',h') \;=\; (g \cdot g', h^{g'} \cdot h'),$$

for (g,h), $(g',h') \in G \ltimes_\gamma H$.

Then $G \ltimes_\gamma H$ is a group.

Furthermore, the following maps are group morphisms.

$$\begin{array}{rcl} G \ltimes_\gamma H & \longrightarrow & G \\ (g,h) & \longmapsto & g \end{array}$$

$$\begin{array}{rcl} G & \longrightarrow & G \ltimes_\gamma H \\ g & \longmapsto & (g,1) \end{array}$$

$$\begin{array}{rcl} H & \longrightarrow & G \ltimes_\gamma H \\ h & \longmapsto & (1,h) \end{array}$$

Remark 8 Suppose given a semidirect product $G \ltimes_\gamma H$.

(1) We have

$$(g,h)^- = (g^-, (h^-)^{g^-})$$

for $(g,h) \in G \ltimes_\gamma H$.

(2) We have

$$(g,h)^{(g',h')} = (g^{g'}, (h'^-)^{g^{g'}} \cdot h^{g'} \cdot h')$$

for (g,h), $(g',h') \in G \ltimes_\gamma H$.

In particular,

$$(1,h)^{(g',1)} = (1, h^{g'})$$

for $(1,h)$, $(g',1) \in G \ltimes_\gamma H$.

(3) We have

$$[(g,h),(1,h')] = (1, ((h'^-)^g)^h \cdot h')$$

for (g,h), $(1,h') \in G \ltimes_\gamma H$.

Proof.

Ad (1). We have

$$\begin{array}{rcl} (g,h) \cdot (g^-, (h^-)^{g^-}) & = & (g \cdot g^-, h^{g^-} \cdot (h^-)^{g^-}) \\ & = & (1, (h \cdot h^-)^{g^-}) \\ & = & (1,1). \end{array}$$

Ad (2). We have

$$\begin{array}{rcl} (g,h)^{(g',h')} & = & (g',h')^- \cdot (g,h) \cdot (g',h') \\ & = & (g'^-, (h'^-)^{g'^-}) \cdot (g,h) \cdot (g',h') \\ & = & (g'^- \cdot g, (h'^-)^{g'^- \cdot g} \cdot h) \cdot (g',h') \\ & = & (g'^- \cdot g \cdot g', ((h'^-)^{g'^- \cdot g} \cdot h)^{g'} \cdot h') \\ & = & (g'^- \cdot g \cdot g', (h'^-)^{g'^- \cdot g \cdot g'} \cdot h^{g'} \cdot h') \\ & = & (g^{g'}, (h'^-)^{g^{g'}} \cdot h^{g'} \cdot h'). \end{array}$$

Ad (3). We have

$$\begin{array}{rcl}
[(g,h),(1,h')] & = & (g,h)^- \cdot (1,h')^- \cdot (g,h) \cdot (1,h') \\
& = & (g^-, (h^-)^{g^-}) \cdot (1, h'^-) \cdot (g,h) \cdot (1,h') \\
& = & (g^-, (h^-)^{g^-} \cdot h'^-) \cdot (g, h \cdot h') \\
& = & (g^- \cdot g, ((h^-)^{g^-} \cdot h'^-)^g \cdot h \cdot h') \\
& = & (1, (h^-)^{g^- \cdot g} \cdot (h'^-)^g \cdot h \cdot h') \\
& = & (1, h^- \cdot (h'^-)^g \cdot h \cdot h') \\
& = & (1, ((h'^-)^g)^h \cdot h').
\end{array}$$

□

Remark 9 Suppose given a group isomorphism

$$\mathfrak{q} : G \xrightarrow{\sim} H.$$

Then we have the group isomorphism

$$\begin{array}{rcl}
\hat{\mathfrak{q}} : \mathrm{Aut}(G) & \xrightarrow{\sim} & \mathrm{Aut}(H) \\
\alpha & \overset{\hat{\mathfrak{q}}}{\longmapsto} & \mathfrak{q}^- \blacktriangle \alpha \blacktriangle \mathfrak{q} \\
\mathfrak{q} \blacktriangle \beta \blacktriangle \mathfrak{q}^- & \overset{\hat{\mathfrak{q}}^-}{\longleftarrow\!\shortmid} & \beta.
\end{array}$$

Cf. also [**1,** Rem. 18].

2.5 Crossed modules

2.5.1 The notion of a crossed module

We recall e.g. from [**1,** Def. 23] the notion of a crossed module.

Definition 10 Suppose given groups M and B.

Suppose given a group morphism

$$\begin{array}{rcl}
\gamma : B & \longrightarrow & \mathrm{Aut}(M) \\
b & \longmapsto & b\gamma.
\end{array}$$

We write

$$\begin{array}{rcl}
b\gamma : M & \longrightarrow & M \\
m & \longmapsto & m(b\gamma) =: m^b.
\end{array}$$

We have $(m \cdot \tilde{m})^b = m^b \cdot \tilde{m}^b$ and $m^{b \cdot \tilde{b}} = (m^b)^{\tilde{b}}$ for m, $\tilde{m} \in M$, b, $\tilde{b} \in B$.

Suppose given a group morphism $f : M \to B$.

Suppose the following properties (CM 1, 2) to hold.

(CM 1) $(m^b)f = (mf)^b$ for $m \in M$ and $b \in B$.

(CM 2) $m^n = m^{nf}$ for m, $n \in M$ (Peiffer identity).

Then the quadruple

$$(M,\, B,\, \gamma,\, f)$$

is called a *crossed module*.

Notation. We sometimes refer to $(M,\, B,\, \gamma,\, f)$ just by f. E.g. we may say that (CM 2) holds for f.

2.5.2 The category of crossed modules

We recall e.g. from [**1,** Def. 26] the notion of a morphism of crossed modules.

Definition 11 Suppose given crossed modules $(M,\, B,\, \gamma,\, f)$ and $(M',\, B',\, \gamma',\, f')$.

Suppose given group morphisms $\mu : M \to M'$ and $\beta : B \to B'$ such that $f \blacktriangle \beta = \mu \blacktriangle f'$ and such that $(m^b)\mu = (m\mu)^{b\beta}$ for $m \in M$ and $b \in B$.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\beta"] \\
M' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Then we call

$$(\mu,\, \beta) : (M,\, B,\, \gamma,\, f) \to (M',\, B',\, \gamma',\, f')$$

a *morphism of crossed modules.*

Remark 12 Suppose given morphisms of crossed modules

$$(M,\, B,\, \gamma,\, f) \xrightarrow{(\mu,\beta)} (M',\, B',\, \gamma',\, f') \xrightarrow{(\mu',\beta')} (M'',\, B'',\, \gamma'',\, f'').$$

Then the composite

$$(\mu, \beta) \blacktriangle (\mu', \beta') := (\mu \blacktriangle \mu',\, \beta \blacktriangle \beta')$$

is a morphism of crossed modules from $(M,\, B,\, \gamma,\, f)$ to $(M'',\, B'',\, \gamma'',\, f'')$.

Cf. [**1,** Rem. 27].

Remark 13 Suppose given a crossed module $(M,\, B,\, \gamma,\, f)$.

Then its identity, given by

$$\mathrm{id}_{(M,\, B,\, \gamma,\, f)} := (\mathrm{id}_M, \mathrm{id}_B) : (M,\, B,\, \gamma,\, f) \to (M,\, B,\, \gamma,\, f)$$

is a morphism of crossed modules.

Cf. [**1,** Rem. 28].

Definition 14 We have the *category of crossed modules*, written $\mathscr{CrMod}$.

It has crossed modules as objects and crossed modules morphisms as morphisms; cf. Definition 10 and Definition 11.

The composite of morphisms is described in Remark 12.

The identity on an object is described in Remark 13.

Cf. [**1,** Rem. 29].

3 Crossed Squares

3.1 The notion of a crossed square

We shall recall the notion of a crossed square; cf. e.g. from [**1,** Def. 57]. Originally, the notion of a crossed square is due to Guin-Waléry, Loday [**6**] and Loday [**7,** Def. 5.1].

Definition 15 Suppose given groups L, M, M', P and group morphisms

$$\begin{tikzcd}
L \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & M' \arrow[d, "\mu'"] \\
M \arrow[r, "\mu"'] & P \, .
\end{tikzcd}$$

Suppose given group morphisms

$$\begin{array}{lcllcl}
M & \xrightarrow{\gamma_{M,L}} & \mathrm{Aut}(L) & : \; m & \longmapsto & (l \mapsto (l)(m\gamma_{M,L}) =: l^m) \\
M' & \xrightarrow{\gamma_{M',L}} & \mathrm{Aut}(L) & : \; m' & \longmapsto & (l \mapsto (l)(m'\gamma_{M',L}) =: l^{m'}) \\
P & \xrightarrow{\gamma_{P,L}} & \mathrm{Aut}(L) & : \; p & \longmapsto & (l \mapsto (l)(p\gamma_{P,L}) =: l^p) \\
P & \xrightarrow{\gamma_{P,M}} & \mathrm{Aut}(M) & : \; p & \longmapsto & (m \mapsto (m)(p\gamma_{P,M}) =: m^p) \\
P & \xrightarrow{\gamma_{P,M'}} & \mathrm{Aut}(M') & : \; p & \longmapsto & (m' \mapsto (m')(p\gamma_{P,M'}) =: m'^p)
\end{array}$$

and a map

$$\begin{array}{rcl}
M \times M' & \xrightarrow{\chi} & L \\
(m, m') & \longmapsto & (m, m')\chi =: \lceil m, m' \rceil ,
\end{array}$$

also called *Loday bracket.*

Suppose that the following properties (CS $1, 2, 3, 4$) hold.

(CS 1) We have $\lambda \blacktriangle \mu = \lambda' \blacktriangle \mu'$.

We write $\kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : L \to P$.

(CS 2) We have the following crossed modules; cf. Definition 10.

(1) $(L,\, M,\, \gamma_{M,L},\, \lambda)$

(2) $(L,\, M',\, \gamma_{M',L},\, \lambda')$

(3) $(L,\, P,\, \gamma_{P,L},\, \kappa)$

(4) $(M,\, P,\, \gamma_{P,M},\, \mu)$

(5) $(M',\, P,\, \gamma_{P,M'},\, \mu')$

(CS 3) We have the following morphisms of crossed modules; cf. Definition 11.

(1) $(\mathrm{id}_L,\, \mu) : (L,\, M,\, \gamma_{M,L},\, \lambda) \to (L,\, P,\, \gamma_{P,L},\, \kappa)$

(2) $(\mathrm{id}_L,\, \mu') : (L,\, M',\, \gamma_{M',L},\, \lambda') \to (L,\, P,\, \gamma_{P,L},\, \kappa)$

(3) $(\lambda,\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M,\, P,\, \gamma_{P,M},\, \mu)$

(4) $(\lambda',\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M',\, P,\, \gamma_{P,M'},\, \mu')$

(CS 4)

(1) $m \cdot \lceil m, m' \rceil \lambda = m^{m'\mu'}$ for $m \in M$ and $m' \in M'$

(2) $m'^{m\mu} \cdot \lceil m, m' \rceil \lambda' = m'$ for $m \in M$ and $m' \in M'$

(3) $l \cdot \lceil l\lambda, m' \rceil = l^{m'}$ for $l \in L$ and $m' \in M'$

(4) $l^m \cdot \lceil m, l\lambda' \rceil = l$ for $m \in M$ and $l \in L$

(5) $\lceil m \cdot m^*,\, m' \rceil = \lceil m, m' \rceil^{m^*} \cdot \lceil m^*, m' \rceil$ for $m,\, m^* \in M$ and $m' \in M'$

(6) $\lceil m,\, m' \cdot m^{*\prime} \rceil = \lceil m, m^{*\prime} \rceil \cdot \lceil m, m' \rceil^{m^{*\prime}}$ for $m \in M$ and $m',\, m^{*\prime} \in M'$

(7) $\lceil m, m' \rceil^p = \lceil m^p, m'^p \rceil$ for $m \in M$, $m' \in M'$ and $p \in P$

Then

$$C := (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

is called a *crossed square.*

Notation. We write

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
C_{1,1} & := & L & \gamma^{C}_{1,0} & := & \gamma_{M,L} & \lambda^{1,0}_{C} & := & \lambda \\
C_{1,0} & := & M & \gamma^{C}_{0,1} & := & \gamma_{M',L} & \lambda^{0,1}_{C} & := & \lambda' \\
C_{0,1} & := & M' & \gamma^{1,1}_{C} & := & \gamma_{P,L} & \mu^{C}_{1,0} & := & \mu \\
C_{0,0} & := & P & \gamma^{1,0}_{C} & := & \gamma_{P,M} & \mu^{C}_{0,1} & := & \mu' \\
 & & & \gamma^{0,1}_{C} & := & \gamma_{P,M'} & \chi_{C} & := & \chi \\
 & & & & & & \kappa_{C} & := & \kappa.
\end{array}$$

Notation. We often write just $C = (L,\, M,\, M',\, P)$ to denote this crossed square.

This definition of a crossed square is equivalent to the one in [**1,** Def. 57]; cf. Remark 17 below.

Remark 16 Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

Suppose given $l \in L$, $m \in M$, $m' \in M'$ and $p \in P$.

Then we have

$$l^m \;=\; l^{m\mu}$$

and

$$l^{m'} \;=\; l^{m'\mu'}.$$

Furthermore, we have

$$(l^p)\lambda \;=\; (l\lambda)^p$$

and

$$(l^p)\lambda' \;=\; (l\lambda')^p.$$

Proof. We have the following commutative diagram of groups and group morphisms.

$$\begin{tikzcd}
L \arrow[r, "\mathrm{id}_L"] \arrow[d, "\mathrm{id}_L"'] & L \arrow[r, "\lambda'"] \arrow[d, "\kappa"] & M' \arrow[d, "\mu'"] \\
L \arrow[r, "\kappa"'] \arrow[d, "\lambda"'] & P \arrow[r, "\mathrm{id}_P"'] \arrow[d, "\mathrm{id}_P"] & P \\
M \arrow[r, "\mu"'] & P &
\end{tikzcd}$$

Due to (CS 3.1) and (CS 3.2) we have the following morphism of crossed modules.

$$\begin{aligned}
(\mathrm{id}_L,\, \mu) &: (L,\, M,\, \gamma_{M,L},\, \lambda) \to (L,\, P,\, \gamma_{P,L},\, \kappa) \\
(\mathrm{id}_L,\, \mu') &: (L,\, M',\, \gamma_{M',L},\, \lambda') \to (L,\, P,\, \gamma_{P,L},\, \kappa)
\end{aligned}$$

So we have

$$(l^m)\,\mathrm{id}_L = (l\,\mathrm{id}_L)^{m\mu}$$

and

$$(l^{m'})\,\mathrm{id}_L = (l\,\mathrm{id}_L)^{m'\mu'}.$$

Altogether, we obtain

$$l^m = l^{m\mu}$$

and

$$l^{m'} = l^{m'\mu'}.$$

Due to (CS 3.3) and (CS 3.4) we have the following morphism of crossed modules.

$$\begin{aligned}
(\lambda,\, \mathrm{id}_P) &: (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M,\, P,\, \gamma_{P,M},\, \mu) \\
(\lambda',\, \mathrm{id}_P) &: (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M',\, P,\, \gamma_{P,M'},\, \mu')
\end{aligned}$$

So we have

$$\begin{aligned}
(l^p)\lambda &= (l\lambda)^{p\,\mathrm{id}_P} \\
&= (l\lambda)^p
\end{aligned}$$

and

$$\begin{aligned}
(l^p)\lambda' &= (l\lambda')^{p\,\mathrm{id}_P} \\
&= (l\lambda')^p.
\end{aligned}$$

□

Remark 17 Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

The property (CS 4.8) from [**1,** Def. 57] can be derived from (CS 2.1, 2.4, 3.1, 3.2, 4.1). That is, given $l \in L$, $m \in M$ and $m' \in M'$, we have

$$((l^{m'})^m)^{\lceil m,m' \rceil} = (l^m)^{m'}.$$

Proof. We obtain

$$\begin{aligned}
((l^{m'})^{m})^{\lceil m,m'\rceil} &\overset{\text{(CM 2)}}{\underset{\text{for } \lambda}{=}} ((l^{m'})^{m})^{\lceil m,m'\rceil\lambda} \\
&\overset{\text{(CS 4.1)}}{=} ((l^{m'})^{m})^{m^{-}\cdot\, m^{m'\mu'}} \\
&= ((l^{m'})^{m\cdot\, m^{-}})^{m^{m'\mu'}} \\
&= (l^{m'})^{m^{m'\mu'}} \\
&\overset{\text{Rem. 16}}{=} (l^{m'\mu'})^{(m^{m'\mu'})\mu} \\
&\overset{\text{(CM 1)}}{\underset{\text{for } \mu}{=}} (l^{m'\mu'})^{(m\mu)^{m'\mu'}} \\
&= (l^{m'\mu'})^{(m'\mu')^{-}\cdot\, m\mu\,\cdot\, m'\mu'} \\
&= l^{m'\mu'\cdot\,(m'\mu')^{-}\cdot\, m\mu\,\cdot\, m'\mu'} \\
&= (l^{m\mu})^{m'\mu'} \\
&\overset{\text{Rem. 16}}{=} (l^{m})^{m'}.
\end{aligned}$$

□

Remark 18 Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

Then we have

$$\lceil m, 1\rceil = 1$$

and

$$\lceil 1, m'\rceil = 1$$

for $m \in M$ and $m' \in M'$.

Proof. We have

$$\begin{aligned}
\lceil m, 1\rceil &= \lceil m, 1\cdot 1\rceil \\
&\overset{\text{(CS 4.6)}}{=} \lceil m, 1\rceil \cdot \lceil m, 1\rceil.
\end{aligned}$$

So we have

$$\lceil m, 1\rceil = 1.$$

Moreover, we have

$$\begin{aligned}
\lceil 1, m'\rceil &= \lceil 1\cdot 1, m'\rceil \\
&\overset{\text{(CS 4.5)}}{=} \lceil 1, m'\rceil \cdot \lceil 1, m'\rceil.
\end{aligned}$$

Therefore, we have

$$\lceil 1, m'\rceil = 1.$$

□

3.2 The category of crossed squares

We shall recall the notion of a morphism of crossed squares; cf. e.g. from [**1,** Def. 59].

Definition 19 Suppose given crossed squares

$$(L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

and

$$(\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}}\,,\, \gamma_{\tilde{M}',\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{M}}\,,\, \gamma_{\tilde{P},\tilde{M}'}\,,\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi}).$$

We write $\kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : L \to P$ and $\tilde{\kappa} := \tilde{\lambda} \blacktriangle \tilde{\mu} = \tilde{\lambda}' \blacktriangle \tilde{\mu}' : \tilde{L} \to \tilde{P}$.

Suppose given group morphisms $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{m}'$, $\mathfrak{p}$ fitting into the following diagram.

$$\begin{tikzcd}
 & L \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\mathfrak{l}"' near end] & & M' \arrow[dl, "\mu'"] \arrow[dd, "\mathfrak{m}'"] \\
M \arrow[rr, "\mu" near end] \arrow[dd, "\mathfrak{m}"'] & & P \arrow[dd, "\mathfrak{p}" near start] & \\
 & \tilde{L} \arrow[rr, "\tilde{\lambda}'" near start] \arrow[dl, "\tilde{\lambda}"'] & & \tilde{M}' \arrow[dl, "\tilde{\mu}'"] \\
\tilde{M} \arrow[rr, "\tilde{\mu}"] & & \tilde{P} &
\end{tikzcd}$$

Suppose that the following properties (CSM 1, 2) hold.

(CSM 1) We have the following morphisms of crossed modules; cf. Definition 11.

 (1) $(\mathfrak{l},\, \mathfrak{m}) : (L,\, M,\, \gamma_{L,M},\, \lambda) \to (\tilde{L},\, \tilde{M},\, \gamma_{\tilde{L},\tilde{M}},\, \tilde{\lambda})$

 (2) $(\mathfrak{l},\, \mathfrak{m}') : (L,\, M',\, \gamma_{L,M'},\, \lambda') \to (\tilde{L},\, \tilde{M}',\, \gamma_{\tilde{L},\tilde{M}'},\, \tilde{\lambda}')$

 (3) $(\mathfrak{m},\, \mathfrak{p}) : (M,\, P,\, \gamma_{P,M},\, \mu) \to (\tilde{M},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}},\, \tilde{\mu})$

 (4) $(\mathfrak{m}',\, \mathfrak{p}) : (M',\, P,\, \gamma_{P,M'},\, \mu') \to (\tilde{M}',\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\mu}')$

 (5) $(\mathfrak{l},\, \mathfrak{p}) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (\tilde{L},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\kappa})$

(CSM 2) We have

$$\lceil m\mathfrak{m},\, m'\mathfrak{m}' \rceil = \lceil m, m' \rceil \mathfrak{l}$$

for $m \in M$ and $m' \in M'$.

Then we call

$$\mathfrak{c} := (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) : (L,\, M,\, M',\, P) \to (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P})$$

a *morphism of crossed squares.*

We also write this morphism of crossed squares $(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p})$ as follows.

$$\begin{tikzcd}
 & L \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\mathfrak{l}"' near end] & & M' \arrow[dl, "\mu'"] \arrow[dd, "\mathfrak{m}'"] \\
M \arrow[rr, "\mu" near end] \arrow[dd, "\mathfrak{m}"'] & & P \arrow[dd, "\mathfrak{p}" near start] & \\
 & \tilde{L} \arrow[rr, "\tilde{\lambda}'" near start] \arrow[dl, "\tilde{\lambda}"'] & & \tilde{M}' \arrow[dl, "\tilde{\mu}'"] \\
\tilde{M} \arrow[rr, "\tilde{\mu}"] & & \tilde{P} &
\end{tikzcd}$$

Notation. We write

$$\begin{array}{rcl}
\mathfrak{c}_{1,1} & := & \mathfrak{l} \\
\mathfrak{c}_{1,0} & := & \mathfrak{m} \\
\mathfrak{c}_{0,1} & := & \mathfrak{m}' \\
\mathfrak{c}_{0,0} & := & \mathfrak{p}.
\end{array}$$

So $\mathfrak{c} = (\mathfrak{c}_{1,1},\, \mathfrak{c}_{1,0},\, \mathfrak{c}_{0,1},\, \mathfrak{c}_{0,0}) = (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p})$.

Remark 20 Suppose given morphisms of crossed squares

$$(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) : (L,\, M,\, M',\, P) \to (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P})$$

and

$$(\tilde{\mathfrak{l}},\, \tilde{\mathfrak{m}},\, \tilde{\mathfrak{m}}',\, \tilde{\mathfrak{p}}) : (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P}) \to (\tilde{\tilde{L}},\, \tilde{\tilde{M}},\, \tilde{\tilde{M}}',\, \tilde{\tilde{P}}).$$

Then the composite

$$(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) \blacktriangle (\tilde{\mathfrak{l}},\, \tilde{\mathfrak{m}},\, \tilde{\mathfrak{m}}',\, \tilde{\mathfrak{p}}) := (\mathfrak{l} \blacktriangle \tilde{\mathfrak{l}},\, \mathfrak{m} \blacktriangle \tilde{\mathfrak{m}},\, \mathfrak{m}' \blacktriangle \tilde{\mathfrak{m}}',\, \mathfrak{p} \blacktriangle \tilde{\mathfrak{p}}) : (L,\, M,\, M',\, P) \to (\tilde{\tilde{L}},\, \tilde{\tilde{M}},\, \tilde{\tilde{M}}',\, \tilde{\tilde{P}})$$

is also a morphism of crossed squares.

Cf. [**1,** Rem. 60].

Remark 21 Suppose given a crossed square $(L,\, M,\, M',\, P)$.

Then its identity, given by

$$\mathrm{id}_{(L,M,M',P)} := (\mathrm{id}_L,\, \mathrm{id}_M,\, \mathrm{id}_{M'},\, \mathrm{id}_P) : (L,\, M,\, M',\, P) \to (L,\, M,\, M',\, P),$$

is a morphism of crossed squares.

Cf. [**1,** Rem. 61].

Definition 22 We have the *category of crossed squares*, written $\mathcal{C}r\mathcal{S}q$.

It has crossed squares as objects and morphisms of crossed squares as morphisms; cf. Definition 15 and Definition 19.

Composition of morphisms is described in Remark 20.

The identity on an object is described in Remark 21.

Cf. [**1,** Def. 62].

Example 23 Suppose given a group P and normal subgroups $M,\, M' \trianglelefteq P$.

Let $L \trianglelefteq P$ with $[M, M'] \leqslant L \leqslant M \cap M'$.

For instance, we may take $L := M \cap M'$ or $L := [M, M']$.

The inclusion morphisms yield the following commutative diagram.

$$\begin{tikzcd}
L \arrow[r, hook, "\lambda'"] \arrow[d, hook, "\lambda"'] \arrow[dr, phantom, "\circlearrowleft"] & M' \arrow[d, hook, "\mu'"] \\
M \arrow[r, hook, "\mu"'] & P \,.
\end{tikzcd}$$

Conjugation in P gives

$$\begin{array}{lclcccl}
M & \xrightarrow{\gamma_{M,L}} & \mathrm{Aut}(L) & : & m & \longmapsto & (l \mapsto l^m) \\
M' & \xrightarrow{\gamma_{M',L}} & \mathrm{Aut}(L) & : & m' & \longmapsto & (l \mapsto l^{m'}) \\
P & \xrightarrow{\gamma_{P,L}} & \mathrm{Aut}(L) & : & p & \longmapsto & (l \mapsto l^p) \\
P & \xrightarrow{\gamma_{P,M}} & \mathrm{Aut}(M) & : & p & \longmapsto & (m \mapsto m^p) \\
P & \xrightarrow{\gamma_{P,M'}} & \mathrm{Aut}(M') & : & p & \longmapsto & (m' \mapsto m'^p).
\end{array}$$

The commutator bracket in P gives

$$\begin{array}{rcl}
M \times M' & \xrightarrow{\chi} & L \\
(m, m') & \longmapsto & \left(\begin{array}{rcl} \lceil m, m' \rceil & := & [m, m'] \\ & = & m^- \cdot m'^- \cdot m \cdot m' \\ & = & m^- \cdot m^{m'} \\ & = & (m'^m)^- \cdot m' \end{array} \right).
\end{array}$$

Then

$$(L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

is a crossed square.

Cf. [**1,** Ex. 64]. For another example, see [**1,** Ex. 65].

3.3 The transposition functor Tr

Remark 24 Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

$$C: \quad \begin{tikzcd} L \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & M' \arrow[d, "\mu'"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd}$$

We define the map

$$\begin{array}{rcl} M' \times M & \xrightarrow{\chi^{\mathrm{tr}}} & L \\ (m', m) & \longmapsto & (m', m)\chi^{\mathrm{tr}} := \lceil m', m \rceil^{\mathrm{tr}} := \lceil m, m' \rceil^{-}. \end{array}$$

Then

$$\begin{array}{rcl} C^{\mathrm{tr}} & = & (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)^{\mathrm{tr}} \\ & := & (L,\, M',\, M,\, P,\, \gamma_{M',L}\,,\, \gamma_{M,L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M'}\,,\, \gamma_{P,M}\,,\, \lambda',\, \lambda,\, \mu',\, \mu,\, \chi^{\mathrm{tr}}) \end{array}$$

is also a crossed square.

$$C^{\mathrm{tr}}: \quad \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ M' \arrow[r, "\mu'"'] & P \end{tikzcd}$$

Proof.

Ad (CS 1). By (CS 1) for C, we have $\lambda' \blacktriangle \mu' = \lambda \blacktriangle \mu$.

We write $\kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : L \to P$.

Ad (CS 2.1). By (CS 2.2) for C, we have that $(L,\, M',\, \gamma_{M',L},\, \lambda')$ is a crossed module.

Ad (CS 2.2). By (CS 2.1) for C, we have that $(L,\, M,\, \gamma_{M,L},\, \lambda)$ is a crossed module.

Ad (CS 2.3). By (CS 2.3) for C, we have that $(L,\, P,\, \gamma_{P,L},\, \kappa)$ is a crossed module.

Ad (CS 2.4). By (CS 2.5) for C, we have that $(M',\, P,\, \gamma_{P,M'},\, \mu')$ is a crossed module.

Ad (CS 2.5). By (CS 2.4) for C, we have that $(M,\, P,\, \gamma_{P,M},\, \mu)$ is a crossed module.

Ad (CS 3.1). By (CS 3.2) for C, we have the morphism

$$(\mathrm{id}_L,\, \mu') : (L,\, M',\, \gamma_{M',L},\, \lambda') \to (L,\, P,\, \gamma_{P,L},\, \kappa)$$

of crossed modules.

Ad (CS 3.2). By (CS 3.1) for C, we have the morphism

$$(\mathrm{id}_L,\, \mu) : (L,\, M,\, \gamma_{M,L},\, \lambda) \to (L,\, P,\, \gamma_{P,L},\, \kappa)$$

of crossed modules.

Ad (CS 3.3). By (CS 3.4) for C, we have the morphism

$$(\lambda',\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M',\, P,\, \gamma_{P,M'},\, \mu')$$

of crossed modules.

Ad (CS 3.4). By (CS 3.3) for C, we have the morphism

$$(\lambda,\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M,\, P,\, \gamma_{P,M},\, \mu)$$

of crossed modules.

Ad (CS 4.1). Suppose given $m' \in M'$ and $m \in M$.

Then we have

$$\begin{array}{rcl} m' \cdot \lceil m', m \rceil^{\mathrm{tr}} \lambda' & = & m' \cdot \lceil m, m' \rceil^{-} \lambda' \\ & \underset{\text{for } C}{\overset{\text{(CS 4.2)}}{=}} & m' \cdot m'^{-} \cdot m'^{m\mu} \\ & = & m'^{m\mu}. \end{array}$$

Ad (CS 4.2). Suppose given $m' \in M'$ and $m \in M$.

Then we have

$$\begin{array}{rcl} m^{m'\mu'} \cdot \lceil m', m \rceil^{\mathrm{tr}} \lambda & = & m^{m'\mu'} \cdot \lceil m, m' \rceil^{-} \lambda \\ & \underset{\text{for } C}{\overset{\text{(CS 4.1)}}{=}} & m^{m'\mu'} \cdot (m^{m'\mu'})^{-} \cdot m \\ & = & m. \end{array}$$

Ad (CS 4.3). Suppose given $l \in L$ and $m \in M$.

Then we have

$$\begin{array}{rcl} l \cdot \lceil l\lambda', m \rceil^{\mathrm{tr}} & = & l \cdot \lceil m, l\lambda' \rceil^{-} \\ & \underset{\text{for } C}{\overset{\text{(CS 4.4)}}{=}} & l \cdot l^{-} \cdot l^{m} \\ & = & l^{m}. \end{array}$$

Ad (CS 4.4). Suppose given $l \in L$ and $m' \in M'$.

Then we have

$$\begin{array}{rcl} l^{m'} \cdot \lceil m', l\lambda \rceil^{\mathrm{tr}} & = & l^{m'} \cdot \lceil l\lambda, m' \rceil^{-} \\ & \underset{\text{for } C}{\overset{\text{(CS 4.3)}}{=}} & l^{m'} \cdot (l^{m'})^{-} \cdot l \\ & = & l. \end{array}$$

Ad (CS 4.5). Suppose given m', $m^{*\prime} \in M'$ and $m \in M$.

Then we have

$$\begin{array}{rcl} \lceil m' \cdot m^{*\prime}, m \rceil^{\mathrm{tr}} & = & \lceil m, m' \cdot m^{*\prime} \rceil^{-} \\ & \underset{\text{for } C}{\overset{\text{(CS 4.6)}}{=}} & (\lceil m, m' \rceil^{m^{*\prime}})^{-} \cdot \lceil m, m^{*\prime} \rceil^{-} \\ & = & (\lceil m, m' \rceil^{-})^{m^{*\prime}} \cdot \lceil m, m^{*\prime} \rceil^{-} \\ & = & (\lceil m', m \rceil^{\mathrm{tr}})^{m^{*\prime}} \cdot \lceil m^{*\prime}, m \rceil^{\mathrm{tr}}. \end{array}$$

Ad (CS 4.6). Suppose given $m' \in M'$ and m, $m^* \in M$.

Then we have

$$\begin{array}{rcl} \lceil m', m \cdot m^* \rceil^{\mathrm{tr}} & = & \lceil m \cdot m^*, m' \rceil^{-} \\ & \underset{\text{for } C}{\overset{\text{(CS 4.5)}}{=}} & \lceil m^*, m' \rceil^{-} \cdot (\lceil m, m' \rceil^{m^*})^{-} \\ & = & \lceil m^*, m' \rceil^{-} \cdot (\lceil m, m' \rceil^{-})^{m^*} \\ & = & \lceil m', m^* \rceil^{\mathrm{tr}} \cdot (\lceil m', m \rceil^{\mathrm{tr}})^{m^*}. \end{array}$$

Ad (CS 4.7). Suppose given $m' \in M'$, $m \in M$ and $p \in P$.

Then we have

$$\begin{aligned} \lceil m'^p, m^p \rceil^{\mathrm{tr}} &= \lceil m^p, m'^p \rceil^- \\ &\underset{\text{for } C}{\overset{\text{(CS 4.7)}}{=}} (\lceil m, m' \rceil^p)^- \\ &= (\lceil m, m' \rceil^-)^p \\ &= (\lceil m', m \rceil^{\mathrm{tr}})^p. \end{aligned}$$

□

Remark 25 Suppose given crossed squares

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

and

$$\tilde{C} = (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}}\,,\, \gamma_{\tilde{M}',\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{M}}\,,\, \gamma_{\tilde{P},\tilde{M}'}\,,\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi}).$$

We write $\kappa := \lambda' \blacktriangle \mu' = \lambda \blacktriangle \mu : L \to P$ and $\tilde{\kappa} := \tilde{\lambda}' \blacktriangle \tilde{\mu}' = \tilde{\lambda} \blacktriangle \tilde{\mu} : \tilde{L} \to \tilde{P}$.

Suppose given a morphism of crossed squares $\mathfrak{c} = (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) : C \to \tilde{C}$.

$$\begin{tikzcd}
& L \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\mathfrak{l}"] & & M' \arrow[dl, "\mu'"] \arrow[dd, "\mathfrak{m}'"] \\
M \arrow[rr, "\mu"] \arrow[dd, "\mathfrak{m}"'] & & P \arrow[dd, "\mathfrak{p}"] & \\
& \tilde{L} \arrow[rr, "\tilde{\lambda}'"] \arrow[dl, "\tilde{\lambda}"'] & & \tilde{M}' \arrow[dl, "\tilde{\mu}'"] \\
\tilde{M} \arrow[rr, "\tilde{\mu}"] & & \tilde{P} &
\end{tikzcd}$$

Then

$$\mathfrak{c}^{\mathrm{tr}} = (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p})^{\mathrm{tr}} := (\mathfrak{l},\, \mathfrak{m}',\, \mathfrak{m},\, \mathfrak{p}) : C^{\mathrm{tr}} \to \tilde{C}^{\mathrm{tr}}$$

is a morphism of crossed squares.

$$\begin{tikzcd}
& L \arrow[rr, "\lambda"] \arrow[dl, "\lambda'"'] \arrow[dd, "\mathfrak{l}"] & & M \arrow[dl, "\mu"] \arrow[dd, "\mathfrak{m}"] \\
M' \arrow[rr, "\mu'"] \arrow[dd, "\mathfrak{m}'"'] & & P \arrow[dd, "\mathfrak{p}"] & \\
& \tilde{L} \arrow[rr, "\tilde{\lambda}"] \arrow[dl, "\tilde{\lambda}'"'] & & \tilde{M} \arrow[dl, "\tilde{\mu}"] \\
\tilde{M}' \arrow[rr, "\tilde{\mu}'"] & & \tilde{P} &
\end{tikzcd}$$

Proof.

Ad (CSM 1.1). By (CSM 1.2) for $\mathfrak{c}$, we have the morphism of crossed modules

$$(\mathfrak{l},\, \mathfrak{m}') : (L,\, M',\, \gamma_{L,M'},\, \lambda') \to (\tilde{L},\, \tilde{M}',\, \gamma_{\tilde{L},\tilde{M}'},\, \tilde{\lambda}').$$

Ad (CSM 1.2). By (CSM 1.1) for $\mathfrak{c}$, we have the morphism of crossed modules

$$(\mathfrak{l},\, \mathfrak{m}) : (L,\, M,\, \gamma_{L,M},\, \lambda) \to (\tilde{L},\, \tilde{M},\, \gamma_{\tilde{L},\tilde{M}},\, \tilde{\lambda}).$$

Ad (CSM 1.3). By (CSM 1.4) for $\mathfrak{c}$, we have the morphism of crossed modules

$$(\mathfrak{m}', \mathfrak{p}) : (M', P, \gamma_{P,M'}, \mu') \to (\tilde{M}', \tilde{P}, \gamma_{\tilde{P},\tilde{M}'}, \tilde{\mu}').$$

Ad (CSM 1.4). By (CSM 1.3) for $\mathfrak{c}$, we have the morphism of crossed modules

$$(\mathfrak{m}, \mathfrak{p}) : (M, P, \gamma_{P,M}, \mu) \to (\tilde{M}, \tilde{P}, \gamma_{\tilde{P},\tilde{M}}, \tilde{\mu}).$$

Ad (CSM 1.5). By (CSM 1.5) for $\mathfrak{c}$, we have the morphism of crossed modules

$$(\mathfrak{l}, \mathfrak{p}) : (L, P, \gamma_{P,L}, \kappa) \to (\tilde{L}, \tilde{P}, \gamma_{\tilde{P},\tilde{L}}, \tilde{\kappa}).$$

Ad (CSM 2). Suppose given $m' \in M'$ and $m \in M$.

Then we have

$$\begin{array}{rcl} \lceil m'\mathfrak{m}', m\mathfrak{m} \rceil^{\mathrm{tr}} & = & \lceil m\mathfrak{m}, m'\mathfrak{m}' \rceil^{-} \\ & \overset{\text{(CSM 2)}}{\underset{\text{for } \mathfrak{c}}{=}} & (\lceil m, m' \rceil \mathfrak{l})^{-} \\ & = & (\lceil m, m' \rceil^{-})\mathfrak{l} \\ & = & \lceil m', m \rceil^{\mathrm{tr}} \mathfrak{l}. \end{array}$$

□

Remark 26

(1) Suppose given a crossed square C.

Then $(\mathrm{id}_C)^{\mathrm{tr}} = \mathrm{id}_{C^{\mathrm{tr}}}$.

(2) Suppose given morphisms of crossed squares $C \xrightarrow{\mathfrak{c}} \tilde{C} \xrightarrow{\tilde{\mathfrak{c}}} \tilde{\tilde{C}}$.

Then $(\mathfrak{c} \blacktriangle \tilde{\mathfrak{c}})^{\mathrm{tr}} = \mathfrak{c}^{\mathrm{tr}} \blacktriangle \tilde{\mathfrak{c}}^{\mathrm{tr}}$.

Proof.

Ad (1). With $C =: (L, M, M', P)$, we have

$$\begin{array}{rcl} (\mathrm{id}_C)^{\mathrm{tr}} & = & (\mathrm{id}_L, \mathrm{id}_M, \mathrm{id}_{M'}, \mathrm{id}_P)^{\mathrm{tr}} \\ & = & (\mathrm{id}_L, \mathrm{id}_{M'}, \mathrm{id}_M, \mathrm{id}_P) \\ & = & \mathrm{id}_{C^{\mathrm{tr}}}. \end{array}$$

Cf. Remark 21.

Ad (2). With $\mathfrak{c} =: (\mathfrak{l}, \mathfrak{m}, \mathfrak{m}', \mathfrak{p})$ and $\tilde{\mathfrak{c}} =: (\tilde{\mathfrak{l}}, \tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}', \tilde{\mathfrak{p}})$, we have

$$\begin{array}{rcl} (\mathfrak{c} \blacktriangle \tilde{\mathfrak{c}})^{\mathrm{tr}} & = & (\mathfrak{l} \blacktriangle \tilde{\mathfrak{l}}, \mathfrak{m} \blacktriangle \tilde{\mathfrak{m}}, \mathfrak{m}' \blacktriangle \tilde{\mathfrak{m}}', \mathfrak{p} \blacktriangle \tilde{\mathfrak{p}})^{\mathrm{tr}} \\ & = & (\mathfrak{l} \blacktriangle \tilde{\mathfrak{l}}, \mathfrak{m}' \blacktriangle \tilde{\mathfrak{m}}', \mathfrak{m} \blacktriangle \tilde{\mathfrak{m}}, \mathfrak{p} \blacktriangle \tilde{\mathfrak{p}}) \\ & = & \mathfrak{c}^{\mathrm{tr}} \blacktriangle \tilde{\mathfrak{c}}^{\mathrm{tr}}. \end{array}$$

Cf. Remark 20. □

Definition 27 We shall define the following functor, called *transposition*.

$$\begin{array}{ccc} \mathit{CrSq} & \xrightarrow{\mathrm{Tr}} & \mathit{CrSq} \\ \left(\begin{array}{c} C \\ \downarrow{\scriptstyle \mathfrak{c}} \\ \tilde{C} \end{array}\right) & \longmapsto & \left(\begin{array}{c} C^{\mathrm{tr}} \\ \downarrow{\scriptstyle \mathfrak{c}^{\mathrm{tr}}} \\ \tilde{C}^{\mathrm{tr}} \end{array}\right) \end{array}$$

Cf. Remarks 24 and 25.

By Remark 26, we have that $\mathrm{Tr} : \mathit{CrSq} \to \mathit{CrSq}$ is a functor.

So $C\,\mathrm{Tr} = C^{\mathrm{tr}}$ for $C \in \mathrm{Ob}(\mathit{CrSq})$ and $\mathfrak{c}\,\mathrm{Tr} = \mathfrak{c}^{\mathrm{tr}}$ for $\mathfrak{c} \in \mathrm{Mor}(\mathit{CrSq})$.

Remark 28 We have $\mathrm{Tr} \blacktriangle \mathrm{Tr} = \mathrm{id}_{\mathcal{CrSq}}$.

Proof. Suppose given $m \in M$ and $m' \in M'$. Then we have

$$\begin{aligned} (m, m')(\chi^{\mathrm{tr}})^{\mathrm{tr}} &= ((m', m)\chi^{\mathrm{tr}})^{-} \\ &= ((m, m')\chi)^{-})^{-} \\ &= (m, m')\chi. \end{aligned}$$

So it holds that $(\chi^{\mathrm{tr}})^{\mathrm{tr}} = \chi$.

Suppose given $C = (L,\, M,\, M',\, P) \in \mathrm{Ob}(\mathcal{CrSq})$.

Then we have

$$\begin{aligned} C(\mathrm{Tr} \blacktriangle \mathrm{Tr}) &= (C^{\mathrm{tr}})^{\mathrm{tr}} \\ &= ((L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)^{\mathrm{tr}})^{\mathrm{tr}} \\ &= (L,\, M',\, M,\, P,\, \gamma_{M',L},\, \gamma_{M,L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M'}\,,\, \gamma_{P,M}\,,\, \lambda',\, \lambda,\, \mu',\, \mu,\, \chi^{\mathrm{tr}})^{\mathrm{tr}} \\ &= (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi) \\ &= C. \end{aligned}$$

Suppose given $\mathfrak{c} = (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) \in \mathrm{Mor}(\mathcal{CrSq})$.

Then we have

$$\begin{aligned} \mathfrak{c}(\mathrm{Tr} \blacktriangle \mathrm{Tr}) &= (\mathfrak{c}^{\mathrm{tr}})^{\mathrm{tr}} \\ &= ((\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p})^{\mathrm{tr}})^{\mathrm{tr}} \\ &= (\mathfrak{l},\, \mathfrak{m}',\, \mathfrak{m},\, \mathfrak{p})^{\mathrm{tr}} \\ &= (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) \\ &= \mathfrak{c}. \end{aligned}$$

This shows $\mathrm{Tr} \blacktriangle \mathrm{Tr} = \mathrm{id}_{\mathcal{CrSq}}$. □

3.4 A construction of an isomorphism of crossed squares

We shall recall how to transport structure from a crossed square via four group isomorphisms; cf. [**1,** Rem. 67].

Remark 29 Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

Suppose given group isomorphisms $L \xrightarrow[\sim]{\mathfrak{l}} \tilde{L}$, $M \xrightarrow[\sim]{\mathfrak{m}} \tilde{M}$, $M' \xrightarrow[\sim]{\mathfrak{m}'} \tilde{M}'$ and $P \xrightarrow[\sim]{\mathfrak{p}} \tilde{P}$.

Let $\tilde{\lambda} := \mathfrak{l}^{-} \blacktriangle \lambda \blacktriangle \mathfrak{m} : \tilde{L} \to \tilde{M}$.

Let $\tilde{\lambda}' := \mathfrak{l}^{-} \blacktriangle \lambda' \blacktriangle \mathfrak{m}' : \tilde{L} \to \tilde{M}'$.

Let $\tilde{\mu} := \mathfrak{m}^{-} \blacktriangle \mu \blacktriangle \mathfrak{p} : \tilde{M} \to \tilde{P}$.

Let $\tilde{\mu}' := \mathfrak{m}'^{-} \blacktriangle \mu' \blacktriangle \mathfrak{p} : \tilde{M}' \to \tilde{P}$.

$$\begin{tikzcd}
 & L \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\mathfrak{l}"', "\sim"] & & M' \arrow[dl, "\mu'"] \arrow[dd, "\mathfrak{m}'", "\sim"'] \\
M \arrow[rr, "\mu"] \arrow[dd, "\mathfrak{m}"', "\sim"] & & P \arrow[dd, "\mathfrak{p}", "\sim"'] & \\
 & \tilde{L} \arrow[rr, "\tilde{\lambda}'"] \arrow[dl, "\tilde{\lambda}"] & & \tilde{M}' \arrow[dl, "\tilde{\mu}'"] \\
\tilde{M} \arrow[rr, "\tilde{\mu}"] & & \tilde{P} &
\end{tikzcd}$$

Recall from Remark 9 that a group isomorphism $\mathfrak{q} : G \xrightarrow{\sim} H$ yields the group isomorphism

$$\begin{array}{rcl} \hat{\mathfrak{q}} : \mathrm{Aut}(G) & \xrightarrow{\sim} & \mathrm{Aut}(H) \\ \alpha & \longmapsto & \mathfrak{q}^{-} \blacktriangle \alpha \blacktriangle \mathfrak{q}. \end{array}$$

Let $\gamma_{\tilde{M},\tilde{L}} := \mathfrak{m}^{-} \blacktriangle \gamma_{M,L} \blacktriangle \hat{\mathfrak{l}} : \tilde{M} \to \mathrm{Aut}(\tilde{L})$.

Let $\gamma_{\tilde{M}',\tilde{L}} := \mathfrak{m}'^{-} \blacktriangle \gamma_{M',L} \blacktriangle \hat{\mathfrak{l}} : \tilde{M}' \to \mathrm{Aut}(\tilde{L})$.

Let $\gamma_{\tilde{P},\tilde{L}} := \mathfrak{p}^{-} \blacktriangle \gamma_{P,L} \blacktriangle \hat{\mathfrak{l}} : \tilde{P} \to \mathrm{Aut}(\tilde{L})$.

Let $\gamma_{\tilde{P},\tilde{M}} := \mathfrak{p}^{-} \blacktriangle \gamma_{P,M} \blacktriangle \hat{\mathfrak{m}} : \tilde{P} \to \mathrm{Aut}(\tilde{M})$.

Let $\gamma_{\tilde{P},\tilde{M}'} := \mathfrak{p}^{-} \blacktriangle \gamma_{P,M'} \blacktriangle \hat{\mathfrak{m}}' : \tilde{P} \to \mathrm{Aut}(\tilde{M}')$.

Let $\tilde{\chi} := (\mathfrak{m}^{-} \times \mathfrak{m}'^{-}) \blacktriangle \chi \blacktriangle \mathfrak{l} : \tilde{M} \times \tilde{M}' \to \tilde{L}$.

(1) Then

$$\tilde{C} := (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}}\,,\, \gamma_{\tilde{M}',\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{M}}\,,\, \gamma_{\tilde{P},\tilde{M}'}\,,\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi}) tilde \chi)$$

is a crossed square.

(2) Moreover, $(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p})$ is an isomorphism of crossed squares from C to $\tilde{C}$.

Cf. [**1,** Rem. 67].

4 $\lfloor 2,0 \rfloor$-simplicial groups and 2-crossed modules

4.1 The notion of a $\lfloor 2,0 \rfloor$-simplicial group

We recall e.g. from [**1,** Def. 39] the notion of a $\lfloor 2,0 \rfloor$-simplicial group.

Definition 30 Suppose given groups G_0, G_1 and G_2.

Furthermore, suppose given the following group morphisms.

$$\begin{array}{rcl} \mathrm{d}_0^{G,2}, \mathrm{d}_1^{G,2}, \mathrm{d}_2^{G,2} : G_2 & \to & G_1 \\ \mathrm{s}_0^{G,1}, \mathrm{s}_1^{G,1} : G_1 & \to & G_2 \\ \mathrm{d}_0^{G,1}, \mathrm{d}_1^{G,1} : G_1 & \to & G_0 \\ \mathrm{s}_0^{G,0} : G_0 & \to & G_1 \end{array}$$

We display these data as follows.

$$\begin{tikzcd}
G_2 \arrow[r, shift left=8, "\mathrm{d}_2^{G,2}"] \arrow[r, shift right=8, "\mathrm{d}_0^{G,2}"'] \arrow[r, "\mathrm{d}_1^{G,2}"] & G_1 \arrow[l, shift left=4, "\mathrm{s}_0^{G,1}"] \arrow[l, shift right=4, "\mathrm{s}_1^{G,1}"'] \arrow[r, shift left=4, "\mathrm{d}_1^{G,1}"] \arrow[r, shift right=4, "\mathrm{d}_0^{G,1}"'] & G_0 \arrow[l, "\mathrm{s}_0^{G,0}"']
\end{tikzcd}$$

Suppose that the conditions $(1, 2)$ hold.

(1) We have

$$\begin{array}{rcl}
\mathrm{s}_0^{G,0} \blacktriangle \mathrm{d}_0^{G,1} & = & \mathrm{id}_{G_0} \\
\mathrm{s}_0^{G,0} \blacktriangle \mathrm{d}_1^{G,1} & = & \mathrm{id}_{G_0} \\
\mathrm{s}_0^{G,1} \blacktriangle \mathrm{d}_0^{G,2} & = & \mathrm{id}_{G_1} \\
\mathrm{s}_0^{G,1} \blacktriangle \mathrm{d}_1^{G,2} & = & \mathrm{id}_{G_1} \\
\mathrm{s}_1^{G,1} \blacktriangle \mathrm{d}_1^{G,2} & = & \mathrm{id}_{G_1} \\
\mathrm{s}_1^{G,1} \blacktriangle \mathrm{d}_2^{G,2} & = & \mathrm{id}_{G_1} \\
\mathrm{s}_0^{G,1} \blacktriangle \mathrm{d}_2^{G,2} & = & \mathrm{d}_1^{G,1} \blacktriangle \mathrm{s}_0^{G,0} \\
\mathrm{s}_1^{G,1} \blacktriangle \mathrm{d}_0^{G,2} & = & \mathrm{d}_0^{G,1} \blacktriangle \mathrm{s}_0^{G,0} \\
\mathrm{d}_1^{G,2} \blacktriangle \mathrm{d}_0^{G,1} & = & \mathrm{d}_0^{G,2} \blacktriangle \mathrm{d}_0^{G,1} \\
\mathrm{d}_2^{G,2} \blacktriangle \mathrm{d}_0^{G,1} & = & \mathrm{d}_0^{G,2} \blacktriangle \mathrm{d}_1^{G,1} \\
\mathrm{d}_2^{G,2} \blacktriangle \mathrm{d}_1^{G,1} & = & \mathrm{d}_1^{G,2} \blacktriangle \mathrm{d}_1^{G,1} \\
\mathrm{s}_0^{G,0} \blacktriangle \mathrm{s}_0^{G,1} & = & \mathrm{s}_0^{G,0} \blacktriangle \mathrm{s}_1^{G,1} .
\end{array}$$

(2) We have

$$\begin{aligned}
[\ker \mathrm{d}_0^{G,2},\, \ker \mathrm{d}_1^{G,2} \cap \ker \mathrm{d}_2^{G,2}] &= 1\\
[\ker \mathrm{d}_1^{G,2},\, \ker \mathrm{d}_0^{G,2} \cap \ker \mathrm{d}_2^{G,2}] &= 1\\
[\ker \mathrm{d}_2^{G,2},\, \ker \mathrm{d}_0^{G,2} \cap \ker \mathrm{d}_1^{G,2}] &= 1.
\end{aligned}$$

Then we call

$$G := (G_2,\, G_1,\, G_0,\, \mathrm{d}_0^{G,2},\, \mathrm{d}_1^{G,2},\, \mathrm{d}_2^{G,2},\, \mathrm{s}_0^{G,1},\, \mathrm{s}_1^{G,1},\, \mathrm{d}_0^{G,1},\, \mathrm{d}_1^{G,1},\, \mathrm{s}_0^{G,0})$$

a $\lfloor 2, 0 \rfloor$-*simplicial group*.

Often, we write the *face morphisms* $\mathrm{d}_i := \mathrm{d}_i^{G,n}$ for $n \in [1, 2]$ and $i \in [0, n]$ and the *degeneracy morphisms* $\mathrm{s}_i := \mathrm{s}_i^{G,n}$ for $n \in [0, 1]$ and $i \in [0,\, n]$.

Furthermore, we sometimes write just $G = (G_2,\, G_1,\, G_0)$ to denote this $\lfloor 2, 0 \rfloor$-simplicial group.

Condition (2) is also called the *Conduché condition*.

4.2 The category of $\lfloor 2, 0 \rfloor$-simplicial groups

We recall e.g. from **[1,** Def. 40] the notion of a morphism of $\lfloor 2, 0 \rfloor$-simplicial groups.

Definition 31 Suppose given $\lfloor 2, 0 \rfloor$-simplicial groups G and H and a tuple of group morphism $\varphi = (\varphi_n)_{n \in \lfloor 2, 0 \rfloor}$.

Then φ is called a *morphism of* $\lfloor 2, 0 \rfloor$-*simplicial groups* if

$$\begin{tikzcd}
G_n \arrow[r, "\varphi_n"] \arrow[d, "\mathrm{d}_i^{G,n}"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n \arrow[d, "\mathrm{d}_i^{H,n}"] \\
G_{n-1} \arrow[r, "\varphi_{n-1}"'] & H_{n-1}
\end{tikzcd}$$

for $n \in [1, 2]$, $i \in [0, n]$ and

$$\begin{tikzcd}
G_n \arrow[r, "\varphi_n"] \arrow[d, "\mathrm{s}_j^{G,n}"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n \arrow[d, "\mathrm{s}_j^{H,n}"] \\
G_{n+1} \arrow[r, "\varphi_{n+1}"'] & H_{n+1}
\end{tikzcd}$$

for $n \in [0, 1]$, $j \in [0, n]$.

Remark 32 Suppose given morphisms of $\lfloor 2, 0 \rfloor$-simplicial groups $\varphi = (\varphi_n)_{n \in \lfloor 2, 0 \rfloor} : G \to H$ and $\varphi' = (\varphi'_n)_{n \in \lfloor 2, 0 \rfloor} : H \to K$.

Then the composite

$$\varphi \blacktriangle \varphi' := (\varphi_n \blacktriangle \varphi'_n)_{n \in \lfloor 2, 0 \rfloor} : G \to K$$

is also a morphism of $\lfloor 2, 0 \rfloor$-simplicial groups.

Cf. **[1,** Rem. 42].

Remark 33 Suppose given a $\lfloor 2, 0 \rfloor$-simplicial group G.

Then its identity, given by

$$\mathrm{id}_G := (\mathrm{id}_{G_2},\, \mathrm{id}_{G_1},\, \mathrm{id}_{G_0}) : G \to G,$$

is a morphism of $\lfloor 2, 0 \rfloor$-simplicial groups.

Cf. **[1,** Rem. 43].

Definition 34 We have the *category of* $\lfloor 2,0 \rfloor$-*simplicial groups*, written $\lfloor 2,0 \rfloor$-$\mathcal{S}imp\mathcal{G}rp$.

It has $\lfloor 2,0 \rfloor$-simplicial groups as objects and $\lfloor 2,0 \rfloor$-simplicial group morphisms as morphisms; cf. Definition 30 and Definition 31.

The composite of morphisms is described in Remark 32.

The identity on an object is described in Remark 33.

Cf. [**1,** Def. 44].

Remark 35 There is a truncation functor from the category of simplicial groups to $\lfloor 2,0 \rfloor$-$\mathcal{S}imp\mathcal{G}rp$.

Cf. e.g. [**1,** Def. 52].

4.3 The notion of a 2-crossed module

The notion of a 2-crossed module is due to CONDUCHÉ [**2,** Déf. 2.2].

Definition 36 Suppose given groups and group morphisms

$$N_2 \xrightarrow{\partial_2} N_1 \xrightarrow{\partial_1} N_0 \ .$$

Suppose given group morphisms

$$\begin{array}{rcl} N_0 & \xrightarrow{\beta_1} & \mathrm{Aut}(N_1) \\ n_0 & \longmapsto & (n_1 \mapsto (n_1)(n_0\beta_1) =: n_1^{n_0}) \end{array}$$

and

$$\begin{array}{rcl} N_0 & \xrightarrow{\beta_2} & \mathrm{Aut}(N_2) \\ n_0 & \longmapsto & (n_2 \mapsto (n_2)(n_0\beta_2) =: n_2^{n_0}). \end{array}$$

Suppose given a map

$$\begin{array}{rcl} N_1 \times N_1 & \xrightarrow{\zeta} & N_2 \\ (n_1, n_1') & \longmapsto & (n_1, n_1')\zeta =: \lfloor n_1, n_1' \rfloor, \end{array}$$

also called *Conduché bracket*.

Suppose that the following properties (2CM 1–9) hold.

(2CM)

(1) $\lfloor n_2'\partial_2, n_2\partial_2 \rfloor = [n_2, n_2']$ for n_2, $n_2' \in N_2$

(2) $\lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor n_2\partial_2, n_1 \rfloor = n_2^- \cdot n_2^{n_1\partial_1}$ for $n_1 \in N_1$ and $n_2 \in N_2$

(3) $\lfloor n_1, n_1' \rfloor \partial_2 = (n_1^-)^{n_1'} \cdot n_1^{n_1'\partial_1}$ for n_1, $n_1' \in N_1$

(4) $\lfloor n_1, n_1' \rfloor^{n_0} = \lfloor n_1^{n_0}, {n_1'}^{n_0} \rfloor$ for $n_0 \in N_0$ and n_1, $n_1' \in N_1$

(5) $n_2\partial_2\partial_1 = 1$ for $n_2 \in N_2$

(6) $(n_1^{n_0})\partial_1 = (n_1\partial_1)^{n_0}$ for $n_0 \in N_0$ and $n_1 \in N_1$

(7) $(n_2^{n_0})\partial_2 = (n_2\partial_2)^{n_0}$ for $n_0 \in N_0$ and $n_2 \in N_2$

(8) $\lfloor n_1, n_1' \cdot n_1'' \rfloor = \lfloor n_1^{n_1'}, n_1'' \rfloor \cdot \lfloor n_1, n_1' \rfloor^{n_1''\partial_1}$ for n_1, n_1', $n_1'' \in N_1$

(9) $\lfloor n_1 \cdot n_1', n_1'' \rfloor = \lfloor n_1, n_1'' \rfloor \cdot \lfloor {n_1'}^{n_1''}, \lfloor n_1, n_1'' \rfloor \partial_2 \rfloor \cdot \lfloor n_1', n_1'' \rfloor$ for n_1, n_1', $n_1'' \in N_1$

Then

$$N := (N_2,\, N_1,\, N_0,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta)$$

is called a 2-*crossed module*.

Notation. We often write just $N = (N_2 \xrightarrow{\partial_2} N_1 \xrightarrow{\partial_1} N_0)$ or $N = (N_2,\, N_1,\, N_0)$ to denote this 2-crossed module. The group morphisms ∂_2 and ∂_1 are called the *differentials* of N.

Remark 37 Suppose given a 2-crossed module $(N_2,\, N_1,\, N_0)$.

Then we have

$$\lfloor n_1, 1 \rfloor = 1$$

and

$$\lfloor 1, n_1 \rfloor = 1$$

for $n_1 \in N_1$.

Proof. We have

$$\begin{array}{rcl} \lfloor n_1, 1 \rfloor & = & \lfloor n_1, 1 \cdot 1 \rfloor \\ & \overset{(2\text{CM } 8)}{=} & \lfloor n_1^1, 1 \rfloor \cdot \lfloor n_1, 1 \rfloor^{1\partial_1} \\ & = & \lfloor n_1, 1 \rfloor \cdot \lfloor n_1, 1 \rfloor. \end{array}$$

So we have

$$\lfloor n_1, 1 \rfloor \;=\; 1.$$

Moreover, we have

$$\begin{array}{rcl} \lfloor 1, n_1 \rfloor & = & \lfloor 1 \cdot 1, n_1 \rfloor \\ & \overset{(2\text{CM } 9)}{=} & \lfloor 1, n_1 \rfloor \cdot \lfloor 1^{n_1}, \lfloor 1, n_1 \rfloor \partial_2 \rfloor \cdot \lfloor 1, n_1 \rfloor \\ & = & \lfloor 1, n_1 \rfloor \cdot \lfloor 1, \lfloor 1, n_1 \rfloor \partial_2 \rfloor \cdot \lfloor 1, n_1 \rfloor \\ & \overset{(2\text{CM } 3)}{=} & \lfloor 1, n_1 \rfloor \cdot \lfloor 1, (1^-)^{n_1} \cdot 1^{n_1 \partial_1} \rfloor \cdot \lfloor 1, n_1 \rfloor \\ & = & \lfloor 1, n_1 \rfloor \cdot \lfloor 1, 1 \rfloor \cdot \lfloor 1, n_1 \rfloor \\ & \overset{\text{see above}}{=} & \lfloor 1, n_1 \rfloor \cdot 1 \cdot \lfloor 1, n_1 \rfloor \\ & = & \lfloor 1, n_1 \rfloor \cdot \lfloor 1, n_1 \rfloor. \end{array}$$

Therefore, we have

$$\lfloor 1, n_1 \rfloor = 1.$$

□

Example 38 Suppose given $m \geqslant 1$. Let $N_2 := \langle d : d^m = 1 \rangle$. So $N_2 = \langle d \rangle \simeq \mathrm{C}_m$.

Suppose given a finite group N_0.

Suppose given $c \in \mathrm{Z}(N_0)$. Let $N_1 := \langle c \rangle \trianglelefteqslant N_0$. Write $k := |\langle c \rangle|$. So $\langle c \rangle \simeq \mathrm{C}_k$.

We consider the group morphisms

$$\begin{array}{lclcl} N_2 & \xrightarrow{\partial_2 := !} & N_1 & : & n_2 \longmapsto 1 \\ N_1 & \xrightarrow{\partial_1} & N_0 & : & n_1 \longmapsto n_1 \\ N_0 & \xrightarrow{\beta_1} & \mathrm{Aut}(N_1) & : & n_0 \longmapsto (n_1 \mapsto n_1^{n_0} := n_1) \\ N_0 & \xrightarrow{\beta_2} & \mathrm{Aut}(N_2) & : & n_0 \longmapsto (n_2 \mapsto n_2^{n_0} := n_2). \end{array}$$

Suppose given $z \in \mathbb{Z}$ with $k \cdot z$ divisible by m.

We consider the map

$$\begin{array}{rcl} N_1 \times N_1 := \langle c \rangle \times \langle c \rangle & \xrightarrow{\zeta} & N_2 := \langle d \rangle \\ (c^i, c^j) & \longmapsto & \lfloor c^i, c^j \rfloor := d^{i \cdot j \cdot z}. \end{array}$$

Let $i' \equiv_k i$ and $j' \equiv_k j$.

So we have $c^i = c^{i'}$ and $c^j = c^{j'}$.

Furthermore, we get, for some x, y, $a \in \mathbb{Z}$,

$$\begin{aligned} d^{i \cdot j \cdot z} &= d^{(k \cdot x + i') \cdot \, (k \cdot y + j') \cdot z} \\ &= d^{i' \cdot j' \cdot z + k \cdot z \cdot (i' \cdot y + j' \cdot x + k \cdot x \cdot y)} \\ &= d^{i' \cdot j' \cdot z + m \cdot a \cdot (i' \cdot y + j' \cdot x + k \cdot x \cdot y)} \\ &= d^{i' \cdot j' \cdot z}. \end{aligned}$$

So the map ζ is well-defined.

Then

$$(N_2,\, N_1,\, N_0,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta)$$

is a 2-crossed module.

$$\begin{tikzcd} N_2 \arrow[r, "\partial_2 = !"] & N_1 \arrow[r, hook, "\partial_1"] & N_0 \end{tikzcd}$$

Proof. Suppose given n_2, $n_2' \in N_2$, n_1, n_1', $n_1'' \in N_1$ and $n_0 \in N_0$.

Ad (2CM 1). We have

$$\begin{aligned} \lfloor n_2' \partial_2, n_2 \partial_2 \rfloor &= \lfloor 1, 1 \rfloor \\ &= 1 \\ &= [n_2, n_2']. \end{aligned}$$

Ad (2CM 2). We have

$$\begin{aligned} \lfloor n_1, n_2 \partial_2 \rfloor \cdot \lfloor n_2 \partial_2, n_1 \rfloor &= \lfloor n_1, 1 \rfloor \cdot \lfloor 1, n_1 \rfloor \\ &= 1 \\ &= n_2^- \cdot n_2 \\ &= n_2^- \cdot n_2^{n_1 \partial_1}. \end{aligned}$$

Ad (2CM 3). We have

$$\begin{aligned} \lfloor n_1, n_1' \rfloor \partial_2 &= 1 \\ &= n_1^- \cdot n_1 \\ &= (n_1^-)^{n_1'} \cdot n_1 \\ &= (n_1^-)^{n_1'} \cdot n_1^{n_1' \partial_1}. \end{aligned}$$

Ad (2CM 4). We have

$$\begin{aligned} \lfloor n_1, n_1' \rfloor^{n_0} &= \lfloor n_1, n_1' \rfloor \\ &= \lfloor n_1^{n_0}, n_1'^{n_0} \rfloor. \end{aligned}$$

Ad (2CM 5). We have

$$\begin{aligned} n_2 \partial_2 \partial_1 &= 1 \partial_1 \\ &= 1. \end{aligned}$$

Ad (2CM 6). We have

$$\begin{aligned} (n_1^{n_0}) \partial_1 &= n_1^{n_0} \\ &= n_1 \\ &= n_1 \partial_1 \\ &= (n_1 \partial_1)^{n_0}. \end{aligned}$$

Ad (2CM 7). We have

$$\begin{aligned} (n_2)^{n_0} \partial_2 &= 1 \\ &= n_2 \partial_2 \\ &= (n_2 \partial_2)^{n_0}. \end{aligned}$$

Ad (2CM 8). We want to show that

$$\begin{array}{rcl} \lfloor n_1, n_1' \cdot n_1'' \rfloor & \overset{!}{=} & \lfloor n_1^{n_1'}, n_1'' \rfloor \cdot \lfloor n_1, n_1' \rfloor^{n_1''\partial_1} \\ & = & \lfloor n_1^{n_1'}, n_1'' \rfloor \cdot \lfloor n_1, n_1' \rfloor \\ & = & \lfloor n_1, n_1'' \rfloor \cdot \lfloor n_1, n_1' \rfloor. \end{array}$$

We write $c^i = n_1$, $c^j = n_1'$ and $c^l = n_1''$, for suitably chosen i, j, $l \in \mathbb{Z}$.

So we have to show that

$$\lfloor c^i, c^j \cdot c^l \rfloor \overset{!}{=} \lfloor c^i, c^l \rfloor \cdot \lfloor c^i \cdot c^j \rfloor.$$

We get

$$\begin{array}{rcl} \lfloor c^i, c^j \cdot c^l \rfloor & = & \lfloor c^i, c^{j+l} \rfloor \\ & = & d^{i\cdot(j+l)\cdot z} \\ & = & d^{i\cdot j\cdot z} \cdot d^{i\cdot l\cdot z} \\ & = & d^{i\cdot l\cdot z} \cdot d^{i\cdot j\cdot z} \\ & = & \lfloor c^i, c^l \rfloor \cdot \lfloor c^i \cdot c^j \rfloor. \end{array}$$

Ad (2CM 9). We want to show that

$$\begin{array}{rcl} \lfloor n_1 \cdot n_1', n_1'' \rfloor & \overset{!}{=} & \lfloor n_1, n_1'' \rfloor \cdot \lfloor {n_1'}^{n_1''}, \lfloor n_1, n_1'' \rfloor \partial_2 \rfloor \cdot \lfloor n_1', n_1'' \rfloor \\ & = & \lfloor n_1, n_1'' \rfloor \cdot \lfloor {n_1'}^{n_1''}, 1 \rfloor \cdot \lfloor n_1', n_1'' \rfloor \\ & = & \lfloor n_1, n_1'' \rfloor \cdot \lfloor n_1', n_1'' \rfloor. \end{array}$$

We write $c^i = n_1$, $c^j = n_1'$ and $c^l = n_1''$, for suitably chosen i, j, $l \in \mathbb{Z}$.

So we have to show that

$$\lfloor c^i \cdot c^j, c^l \rfloor \overset{!}{=} \lfloor c^i, c^l \rfloor \cdot \lfloor c^j, c^l \rfloor.$$

We get

$$\begin{array}{rcl} \lfloor c^i \cdot c^j, c^l \rfloor & = & \lfloor c^{i+j}, c^l \rfloor \\ & = & d^{(i+j)\cdot l\cdot z} \\ & = & d^{i\cdot l\cdot z} \cdot d^{j\cdot l\cdot z} \\ & = & \lfloor c^i, c^l \rfloor \cdot \lfloor c^j, c^l \rfloor. \end{array}$$

□

For another example, see Example 97 below in which we apply the functor To to the crossed square of Example 23.

4.4 The category of 2-crossed modules

Definition 39 Suppose given 2-crossed modules

$$(N_2,\ N_1,\ N_0,\ \partial_2,\ \partial_1,\ \beta_1,\ \beta_2,\ \zeta)$$

and

$$(\tilde{N}_2,\ \tilde{N}_1,\ \tilde{N}_0,\ \tilde{\partial}_2,\ \tilde{\partial}_1,\ \tilde{\beta}_1,\ \tilde{\beta}_2,\ \tilde{\zeta}).$$

Suppose given group morphisms ν_2, ν_1, ν_0 fitting into the following commutative diagram.

$$\begin{tikzcd} N_2 \arrow[r, "\partial_2"] \arrow[d, "\nu_2"'] & N_1 \arrow[r, "\partial_1"] \arrow[d, "\nu_1"'] & N_0 \arrow[d, "\nu_0"] \\ \tilde{N}_2 \arrow[r, "\tilde{\partial}_2"'] & \tilde{N}_1 \arrow[r, "\tilde{\partial}_1"'] & \tilde{N}_0 \end{tikzcd}$$

Suppose that the following properties (2CMM 1, 2) hold.

(2CMM 1)

(1) We have
$$(n_1^{n_0})\nu_1 = (n_1\nu_1)^{(n_0\nu_0)}$$
for $n_0 \in N_0$ and $n_1 \in N_1$.

(2) We have
$$(n_2^{n_0})\nu_2 = (n_2\nu_2)^{(n_0\nu_0)}$$
for $n_0 \in N_0$ and $n_2 \in N_2$.

(2CMM 2) We have
$$\lfloor n_1, n_1'\rfloor\nu_2 = \lfloor n_1\nu_1, n_1'\nu_1\rfloor$$
for n_1, $n_1' \in N_1$.

Then we call
$$\nu := (\nu_2,\, \nu_1,\, \nu_0) : (N_2,\, N_1,\, N_0) \to (\tilde{N}_2,\, \tilde{N}_1,\, \tilde{N}_0)$$
a morphism of 2-*crossed modules.*

We also write this morphism of 2-crossed modules $(\nu_2,\, \nu_1,\, \nu_0)$ as follows.

$$\begin{tikzcd}
N_2 \arrow[r, "\partial_2"] \arrow[d, "\nu_2"'] & N_1 \arrow[r, "\partial_1"] \arrow[d, "\nu_1"'] & N_0 \arrow[d, "\nu_0"] \\
\tilde{N}_2 \arrow[r, "\tilde{\partial}_2"'] & \tilde{N}_1 \arrow[r, "\tilde{\partial}_1"'] & \tilde{N}_0
\end{tikzcd}$$

Remark 40 Suppose given morphisms of 2-crossed modules
$$(\nu_2,\, \nu_1,\, \nu_0) : (N_2,\, N_1,\, N_0) \to (\tilde{N}_2,\, \tilde{N}_1,\, \tilde{N}_0)$$
and
$$(\tilde{\nu}_2,\, \tilde{\nu}_1,\, \tilde{\nu}_0) : (\tilde{N}_2,\, \tilde{N}_1,\, \tilde{N}_0) \to (\tilde{\tilde{N}}_2,\, \tilde{\tilde{N}}_1,\, \tilde{\tilde{N}}_0)$$
Then the composite
$$(\nu_2,\, \nu_1,\, \nu_0) \blacktriangle (\tilde{\nu}_2,\, \tilde{\nu}_1,\, \tilde{\nu}_0) := (\nu_2 \blacktriangle \tilde{\nu}_2,\, \nu_1 \blacktriangle \tilde{\nu}_1,\, \nu_0 \blacktriangle \tilde{\nu}_0) : (N_2,\, N_1,\, N_0) \to (\tilde{\tilde{N}}_2,\, \tilde{\tilde{N}}_1,\, \tilde{\tilde{N}}_0)$$
is also a morphism of 2-crossed modules.

Proof. First, we have the following commutative diagram.

$$\begin{tikzcd}
N_2 \arrow[r, "\partial_2"] \arrow[d, "\nu_2"'] & N_1 \arrow[r, "\partial_1"] \arrow[d, "\nu_1"'] & N_0 \arrow[d, "\nu_0"] \\
\tilde{N}_2 \arrow[r, "\tilde{\partial}_2"] \arrow[d, "\tilde{\nu}_2"'] & \tilde{N}_1 \arrow[r, "\tilde{\partial}_1"] \arrow[d, "\tilde{\nu}_1"'] & \tilde{N}_0 \arrow[d, "\tilde{\nu}_0"'] \\
\tilde{\tilde{N}}_2 \arrow[r, "\tilde{\tilde{\partial}}_2"'] & \tilde{\tilde{N}}_1 \arrow[r, "\tilde{\tilde{\partial}}_1"'] & \tilde{\tilde{N}}_0
\end{tikzcd}$$

Second, we have to show (2CMM $1,2$).

Ad (2CMM 1.1). Suppose given $n_0 \in N_0$ and $n_1 \in N_1$.

Then we have

$$\begin{aligned} (n_1^{n_0})(\nu_1 \blacktriangle \tilde{\nu}_1) &= (n_1^{n_0}\nu_1)\tilde{\nu}_1 \\ &= ((n_1\nu_1)^{(n_0\nu_0)})\tilde{\nu}_1 \\ &= ((n_1\nu_1)\tilde{\nu}_1)^{((n_0\nu_0)\tilde{\nu}_0)} \\ &= (n_1(\nu_1 \blacktriangle \tilde{\nu}_1))^{(n_0(\nu_0 \blacktriangle \tilde{\nu}_0))}. \end{aligned}$$

Ad (2CMM 1.2). Suppose given $n_0 \in N_0$ and $n_2 \in N_2$.

Then we have

$$\begin{aligned} (n_2^{n_0})(\nu_2 \blacktriangle \tilde{\nu}_2) &= (n_2^{n_0}\nu_2)\tilde{\nu}_2 \\ &= ((n_2\nu_2)^{(n_0\nu_0)})\tilde{\nu}_2 \\ &= ((n_2\nu_2)\tilde{\nu}_2)^{((n_0\nu_0)\tilde{\nu}_0)} \\ &= (n_2(\nu_2 \blacktriangle \tilde{\nu}_2))^{(n_0(\nu_0 \blacktriangle \tilde{\nu}_0))}. \end{aligned}$$

Ad (2CMM 2). Suppose given $n_1,\, n_1' \in N_1$.

Then we have

$$\begin{aligned} \lfloor n_1, n_1' \rfloor(\nu_2 \blacktriangle \tilde{\nu}_2) &= (\lfloor n_1, n_1' \rfloor \nu_2)\tilde{\nu}_2 \\ &= \lfloor n_1\nu_1, n_1'\nu_1 \rfloor \tilde{\nu}_2 \\ &= \lfloor (n_1\nu_1)\tilde{\nu}_1, (n_1'\nu_1)\tilde{\nu}_1 \rfloor \\ &= \lfloor n_1(\nu_1 \blacktriangle \tilde{\nu}_1), n_1'(\nu_1 \blacktriangle \tilde{\nu}_1) \rfloor. \end{aligned}$$

□

Remark 41 Suppose given a 2-crossed module $(N_2,\, N_1,\, N_0)$.

Then its identity, given by

$$\mathrm{id}_{(N_2,N_1,N_0)} := (\mathrm{id}_{N_2},\, \mathrm{id}_{N_1},\, \mathrm{id}_{N_0}) : (N_2,\, N_1,\, N_0) \to (N_2,\, N_1,\, N_0),$$

is a morphism of 2-crossed modules; cf. Definition 39.

Definition 42 We have the *category of 2-crossed modules*, written $2\text{-}\mathcal{C}r\mathcal{M}od$.

It has 2-crossed modules as objects and morphisms of 2-crossed modules as morphisms; cf. Definition 36 and Definition 39.

Composition of morphisms is described in Remark 40.

The identity on an object is described in Remark 41.

Remark 43 Suppose given a morphism of 2-crossed modules

$$(\nu_2,\, \nu_1,\, \nu_0) : (N_2,\, N_1,\, N_0) \to (\tilde{N}_2,\, \tilde{N}_1,\, \tilde{N}_0),$$

such that ν_2, ν_1 and ν_0 are bijective.

Then

$$(\nu_2^-,\, \nu_1^-,\, \nu_0^-) : (\tilde{N}_2,\, \tilde{N}_1,\, \tilde{N}_0) \to (N_2,\, N_1,\, N_0).$$

is also a morphism of 2-crossed modules.

Proof. We show that the tuple of group morphisms $(\nu_2^-,\, \nu_1^-,\, \nu_0^-)$ is a morphism of 2-crossed modules.

The required quadrangles commute; cf. Definition 39.

Ad (2CMM 1.1). Suppose given $\tilde{n}_0 \in \tilde{N}_0$ and $\tilde{n}_1 \in \tilde{N}_1$.

Then we have

$$\begin{array}{rcl} (\tilde{n}_1^{\tilde{n}_0})\nu_1^- & = & ((\tilde{n}_1\nu_1^-\nu_1)^{\tilde{n}_0\nu_0^-\nu_0})\nu_1^- \\ & \underset{\text{for } \nu_1}{\overset{\text{(2CMM 1.1)}}{=}} & ((\tilde{n}_1\nu_1^-)^{\tilde{n}_0\nu_0^-})\nu_1\nu_1^- \\ & = & (\tilde{n}_1\nu_1^-)^{\tilde{n}_0\nu_0^-} . \end{array}$$

Ad (2CMM 1.2). Suppose given $\tilde{n}_0 \in \tilde{N}_0$ and $\tilde{n}_2 \in \tilde{N}_2$.

Then we have

$$\begin{array}{rcl} (\tilde{n}_2^{\tilde{n}_0})\nu_2^- & = & ((\tilde{n}_2\nu_2^-\nu_2)^{\tilde{n}_0\nu_0^-\nu_0})\nu_2^- \\ & \underset{\text{for } \nu_2}{\overset{\text{(2CMM 1.2)}}{=}} & ((\tilde{n}_2\nu_2^-)^{\tilde{n}_0\nu_0^-})\nu_2\nu_2^- \\ & = & (\tilde{n}_2\nu_2^-)^{\tilde{n}_0\nu_0^-} . \end{array}$$

Ad (2CMM 2). Suppose given $\tilde{n}_1$, $\tilde{n}'_1 \in \tilde{N}_1$.

Then we have

$$\begin{array}{rcl} \lfloor \tilde{n}_1, \tilde{n}'_1 \rfloor \nu_2^- & = & \lfloor \tilde{n}_1\nu_1^-\nu_1, \tilde{n}'_1\nu_1^-\nu_1 \rfloor \nu_2^- \\ & \underset{\text{for } \nu_2}{\overset{\text{(2CMM 2)}}{=}} & \lfloor \tilde{n}_1\nu_1^-, \tilde{n}'_1\nu_1^- \rfloor \nu_2\nu_2^- \\ & = & \lfloor \tilde{n}_1\nu_1^-, \tilde{n}'_1\nu_1^- \rfloor . \end{array}$$

□

4.5 From $\lfloor 2,0 \rfloor$-simplicial groups to 2-crossed modules

4.5.1 The construction for the objects

Suppose given a $\lfloor 2,0 \rfloor$-simplicial group G; cf. Definition 30.

Definition 44 We write

(1) $G_{1;X} := \bigcap\limits_{i \in X} \ker \mathrm{d}_i^{G,1} \trianglelefteqslant G_1$

(2) $G_{2;Y} := \bigcap\limits_{i \in Y} \ker \mathrm{d}_i^{G,2} \trianglelefteqslant G_2$

for $X \subseteq \{0,\, 1\}$ and $Y \subseteq \{0,\, 1,\, 2\}$.

We usually omit the set-braces of X and Y when used as an index.

For example we write $G_{1;1} = G_{1;\{1\}}$, $G_{2;1,2} = G_{2;\{1,2\}}$, etc.

Note that for $X' \subseteq X \subseteq \{0,\, 1\}$ and $Y' \subseteq Y \subseteq \{0,\, 1,\, 2\}$, we have $G_{1,X} \trianglelefteqslant G_{1,X'}$ and $G_{2,Y} \trianglelefteqslant G_{2,Y'}$.

Cf. [**1,** Def. 53].

Definition 45 Let

$$\begin{array}{lclcl} G\mathrm{N}_0 & := & G_0 & & \\ G\mathrm{N}_1 & := & G_{1;1} & = & \ker(\mathrm{d}_1^{G,1}) \trianglelefteqslant G_1 \\ G\mathrm{N}_2 & := & G_{2;1,2} & = & \ker(\mathrm{d}_1^{G,2}) \cap \ker(\mathrm{d}_2^{G,2}) \trianglelefteqslant G_2 . \end{array}$$

Note that for $g \in G\mathrm{N}_2$ we have

$$\begin{array}{rcl} g\,\mathrm{d}_0\,\mathrm{d}_1 & = & g\,\mathrm{d}_2\,\mathrm{d}_0 \\ & = & 1 \cdot \mathrm{d}_0 \\ & = & 1. \end{array}$$

So $g\,\mathrm{d}_0 \in G\mathrm{N}_1$.

Let

$$\begin{aligned} \partial_2 &:= \mathrm{d}_0|_{G\mathrm{N}_2}^{G\mathrm{N}_1} \\ \partial_1 &:= \mathrm{d}_0|_{G\mathrm{N}_1}^{G\mathrm{N}_0} = \mathrm{d}_0|_{G\mathrm{N}_1}. \end{aligned}$$

Then we have

$$G\mathrm{N}_2 \xrightarrow{\partial_2} G\mathrm{N}_1 \xrightarrow{\partial_1} G\mathrm{N}_0 .$$

Cf. also [**1,** Def. 45].

Lemma 46 We have the following group morphism.

$$\begin{array}{rcl} G\mathrm{N}_0 & \xrightarrow{\beta_1} & \mathrm{Aut}(G\mathrm{N}_1) \\ n_0 & \longmapsto & (n_1 \mapsto n_1^{n_0} := n_1^{n_0\,\mathrm{s}_0}) \end{array}$$

Since $G\mathrm{N}_1 \trianglelefteq G_1$, the element $n_1^{n_0\,\mathrm{s}_0}$ is contained to $G\mathrm{N}_1$.

Proof. We use Remark 6.

Suppose given n_0, $\tilde{n}_0 \in G\mathrm{N}_0$ and n_1, $\tilde{n}_1 \in G\mathrm{N}_1$.

Then we have

$$n_1^1 = n_1.$$

Moreover, we obtain

$$\begin{aligned} n_1^{n_0 \cdot \tilde{n}_0} &= n_1^{(n_0 \cdot \tilde{n}_0)\,\mathrm{s}_0} \\ &= n_1^{n_0\,\mathrm{s}_0 \cdot \tilde{n}_0\,\mathrm{s}_0} \\ &= (n_1^{n_0\,\mathrm{s}_0})^{\tilde{n}_0\,\mathrm{s}_0} \\ &= (n_1^{n_0\,\mathrm{s}_0})^{\tilde{n}_0} \\ &= (n_1^{n_0})^{\tilde{n}_0}. \end{aligned}$$

Finally, we calculate

$$\begin{aligned} (n_1 \cdot \tilde{n}_1)^{n_0} &= (n_1 \cdot \tilde{n}_1)^{n_0\,\mathrm{s}_0} \\ &= n_1^{n_0\,\mathrm{s}_0} \cdot \tilde{n}_1^{n_0\,\mathrm{s}_0} \\ &= n_1^{n_0} \cdot \tilde{n}_1^{n_0}. \end{aligned}$$

□

Lemma 47 We have the following group morphism.

$$\begin{array}{rcl} G\mathrm{N}_0 & \xrightarrow{\beta_2} & \mathrm{Aut}(G\mathrm{N}_2) \\ n_0 & \longmapsto & (n_0 \mapsto n_2^{n_0} := n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0}) \end{array}$$

Since $G\mathrm{N}_2 \trianglelefteq G_2$, the element $n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0}$ is contained to $G\mathrm{N}_2$.

Proof. We use Remark 6.

Suppose given n_0, $\tilde{n}_0 \in G\mathrm{N}_0$ and n_2, $\tilde{n}_2 \in G\mathrm{N}_2$.

Then we have

$$n_2^1 = n_2.$$

Moreover, we obtain

$$\begin{aligned} n_2^{n_0 \cdot \tilde{n}_0} &= n_2^{(n_0 \cdot \tilde{n}_0)\,\mathrm{s}_0\,\mathrm{s}_0} \\ &= n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot \tilde{n}_0\,\mathrm{s}_0\,\mathrm{s}_0} \\ &= (n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0})^{\tilde{n}_0\,\mathrm{s}_0\,\mathrm{s}_0} \\ &= (n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0})^{\tilde{n}_0} \\ &= (n_2^{n_0})^{\tilde{n}_0}. \end{aligned}$$

Finally, we calculate

$$\begin{aligned} (n_2 \cdot \tilde{n}_2)^{n_0} &= (n_2 \cdot \tilde{n}_2)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0} \\ &= n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0} \cdot \tilde{n}_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0} \\ &= n_2^{n_0} \cdot \tilde{n}_2^{n_0}. \end{aligned}$$

□

Remark 48 Suppose given $n_1,\, \tilde{n}_1 \in G\mathrm{N}_1$.

Then $(n_1^- \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_1} \in G\mathrm{N}_2$.

Proof. We have

$$\begin{aligned}
((n_1^- \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_1})\,\mathrm{d}_1 &= (n_1^- \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_0}\,\mathrm{d}_1 \cdot (n_1 \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_1}\,\mathrm{d}_1 \\
&= (n_1^- \,\mathrm{s}_0\,\mathrm{d}_1)^{\tilde{n}_1 \mathrm{s}_0 \mathrm{d}_1} \cdot (n_1 \,\mathrm{s}_0\,\mathrm{d}_1)^{\tilde{n}_1 \mathrm{s}_1 \mathrm{d}_1} \\
&= (n_1^-)^{\tilde{n}_1} \cdot (n_1)^{\tilde{n}_1} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
((n_1^- \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_1})\,\mathrm{d}_2 &= (n_1^- \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_0}\,\mathrm{d}_2 \cdot (n_1 \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_1}\,\mathrm{d}_2 \\
&= (n_1^- \,\mathrm{s}_0\,\mathrm{d}_2)^{\tilde{n}_1 \mathrm{s}_0 \mathrm{d}_2} \cdot (n_1 \,\mathrm{s}_0\,\mathrm{d}_2)^{\tilde{n}_1 \mathrm{s}_1 \mathrm{d}_2} \\
&= (n_1^- \,\mathrm{d}_1\,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{d}_1 \mathrm{s}_0} \cdot (n_1 \,\mathrm{d}_1\,\mathrm{s}_0)^{\tilde{n}_1} \\
&= 1.
\end{aligned}$$

□

Lemma 49 Recall the Conduché condition of a $\lfloor 2,0\rfloor$-simplicial group G from Definition 30:

$$\begin{aligned}
[\ker \mathrm{d}_0^{G,2},\, \ker \mathrm{d}_1^{G,2} \cap \ker \mathrm{d}_2^{G,2}] &= 1 \\
[\ker \mathrm{d}_1^{G,2},\, \ker \mathrm{d}_0^{G,2} \cap \ker \mathrm{d}_2^{G,2}] &= 1 \\
[\ker \mathrm{d}_2^{G,2},\, \ker \mathrm{d}_0^{G,2} \cap \ker \mathrm{d}_1^{G,2}] &= 1\,.
\end{aligned}$$

Using the Conduché condition, we show that the following assertions (CC 1–4) hold.

(CC)

(1) $n_2^{n_1 \mathrm{d}_0} = n_2^{n_1 \mathrm{s}_1}$ for $n_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$

(2) $\tilde{n}_2^{n_2 \mathrm{d}_0 \mathrm{s}_0} = \tilde{n}_2^{n_2}$ for $n_2,\, \tilde{n}_2 \in G\mathrm{N}_2$

(3) $(n_2^- \,\mathrm{d}_0\,\mathrm{s}_0)^{n_1 \mathrm{s}_0} \cdot (n_2 \,\mathrm{d}_0\,\mathrm{s}_0)^{n_1 \mathrm{s}_1} = (n_2^-)^{n_1 \mathrm{s}_0} \cdot n_2^{n_1 \mathrm{s}_1}$ for $n_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$

(4) $(n_1 \,\mathrm{s}_0)^{n_2 \mathrm{d}_0 \mathrm{s}_1} = (n_1 \,\mathrm{s}_0)^{n_2^- \cdot\, n_2 \mathrm{d}_0 \mathrm{s}_0}$ for $n_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$

Proof.

Ad (CC 1). Suppose given $n_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$.

Note that

$$n_2 \in G\mathrm{N}_2 = \ker \mathrm{d}_1^{G,2} \cap \ker \mathrm{d}_2^{G,2}\,.$$

Moreover,

$$n_1 \,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1 \in \ker \mathrm{d}_0^{G,2},$$

since

$$\begin{aligned}
(n_1 \,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)\,\mathrm{d}_0 &= n_1 \,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0\,\mathrm{d}_0 \cdot n_1^- \,\mathrm{s}_1\,\mathrm{d}_0 \\
&= n_1 \,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_1^- \,\mathrm{d}_0\,\mathrm{s}_0 \\
&= 1.
\end{aligned}$$

So we have $[n_1 \,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1,\, n_2] = 1$.

Therefore

$$\begin{aligned}
& n_2^{n_1 \mathrm{d}_0 \mathrm{s}_0 \mathrm{s}_0 \cdot n_1^- \mathrm{s}_1} &= \;& n_2 \\
\Leftrightarrow\quad & n_2^{n_1 \mathrm{d}_0 \mathrm{s}_0 \mathrm{s}_0} &= \;& n_2^{n_1 \mathrm{s}_1} \\
\Leftrightarrow\quad & n_2^{n_1 \mathrm{d}_0} &= \;& n_2^{n_1 \mathrm{s}_1}.
\end{aligned}$$

Ad (CC 2). Suppose given $n_2,\, \tilde{n}_2 \in G\mathrm{N}_2$.

Note that

$$\tilde{n}_2 \in G\mathrm{N}_2 = \ker \mathrm{d}_1^{G,2} \cap \ker \mathrm{d}_2^{G,2}\,.$$

Moreover,

$$n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^- \in \ker \operatorname{d}_0^{G,2},$$

since

$$\begin{array}{rcl} (n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-) \operatorname{d}_0 & = & n_2 \operatorname{d}_0 \operatorname{s}_0 \operatorname{d}_0 \cdot n_2^- \operatorname{d}_0 \\ & = & n_2 \operatorname{d}_0 \cdot n_2^- \operatorname{d}_0 \\ & = & 1. \end{array}$$

So we have $[n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-, \, \tilde{n}_2] = 1$.

Therefore

$$\begin{array}{rrcl} & \tilde{n}_2^{\, n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-} & = & \tilde{n}_2 \\ \Leftrightarrow & \tilde{n}_2^{\, n_2 \operatorname{d}_0 \operatorname{s}_0} & = & \tilde{n}_2^{\, n_2}. \end{array}$$

Ad (CC 3). Suppose given $n_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$.

We have

$$n_1 \operatorname{s}_1 \cdot n_1^- \operatorname{s}_0 \in \ker \operatorname{d}_1^{G,2}.$$

since

$$\begin{array}{rcl} (n_1 \operatorname{s}_1 \cdot n_1^- \operatorname{s}_0) \operatorname{d}_1 & = & n_1 \operatorname{s}_1 \operatorname{d}_1 \cdot n_1^- \operatorname{s}_0 \operatorname{d}_1 \\ & = & n_1 \cdot n_1^- \\ & = & 1. \end{array}$$

Moreover,

$$n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^- \in \ker \operatorname{d}_0^{G,2} \cap \ker \operatorname{d}_2^{G,2},$$

since

$$\begin{array}{rcl} (n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-) \operatorname{d}_0 & = & n_2 \operatorname{d}_0 \operatorname{s}_0 \operatorname{d}_0 \cdot n_2^- \operatorname{d}_0 \\ & = & n_2 \operatorname{d}_0 \cdot n_2^- \operatorname{d}_0 \\ & = & 1 \end{array}$$

and

$$\begin{array}{rcl} (n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-) \operatorname{d}_2 & = & n_2 \operatorname{d}_0 \operatorname{s}_0 \operatorname{d}_2 \cdot n_2^- \operatorname{d}_2 \\ & = & n_2 \operatorname{d}_0 \operatorname{s}_0 \operatorname{d}_2 \\ & = & n_2 \operatorname{d}_0 \operatorname{d}_1 \operatorname{s}_0 \\ & = & n_2 \operatorname{d}_2 \operatorname{d}_0 \operatorname{s}_0 \\ & = & 1. \end{array}$$

So we have $[n_1 \operatorname{s}_1 \cdot n_1^- \operatorname{s}_0, \; n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-] = 1$.

Therefore

$$\begin{array}{rrcl} & (n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-)^{n_1 \operatorname{s}_1 \cdot n_1^- \operatorname{s}_0} & = & n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^- \\ \Leftrightarrow & (n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-)^{n_1 \operatorname{s}_1} & = & (n_2 \operatorname{d}_0 \operatorname{s}_0 \cdot n_2^-)^{n_1 \operatorname{s}_0} \\ \Leftrightarrow & (n_2 \operatorname{d}_0 \operatorname{s}_0)^{n_1 \operatorname{s}_1} \cdot (n_2^-)^{n_1 \operatorname{s}_1} & = & (n_2 \operatorname{d}_0 \operatorname{s}_0)^{n_1 \operatorname{s}_0} \cdot (n_2^-)^{n_1 \operatorname{s}_0} \\ \Leftrightarrow & (n_2^- \operatorname{d}_0 \operatorname{s}_0)^{n_1 \operatorname{s}_0} \cdot (n_2 \operatorname{d}_0 \operatorname{s}_0)^{n_1 \operatorname{s}_1} & = & (n_2^-)^{n_1 \operatorname{s}_0} \cdot n_2^{n_1 \operatorname{s}_1}. \end{array}$$

Ad (CC 4). Suppose given $n_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$.

We have

$$n_1 \operatorname{s}_0 \in \ker \operatorname{d}_2^{G,2}.$$

since

$$\begin{array}{rcl} (n_1 \operatorname{s}_0) \operatorname{d}_2 & = & n_1 \operatorname{d}_1 \operatorname{s}_0 \\ & = & 1. \end{array}$$

Moreover,

$$n_2\,\mathrm{d}_0\,\mathrm{s}_1 \cdot n_2^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_2 \in \ker \mathrm{d}_0^{G,2} \cap \ker \mathrm{d}_1^{G,2},$$

since

$$\begin{aligned}(n_2\,\mathrm{d}_0\,\mathrm{s}_1 \cdot n_2^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_2)\,\mathrm{d}_0 &= n_2\,\mathrm{d}_0\,\mathrm{s}_1\,\mathrm{d}_0 \cdot n_2^-\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{d}_0 \cdot n_2\,\mathrm{d}_0\\ &= n_2\,\mathrm{d}_0\,\mathrm{s}_1\,\mathrm{d}_0 \cdot n_2^-\,\mathrm{d}_0 \cdot n_2\,\mathrm{d}_0\\ &= n_2\,\mathrm{d}_0\,\mathrm{s}_1\,\mathrm{d}_0\\ &= n_2\,\mathrm{d}_0\,\mathrm{d}_0\,\mathrm{s}_0\\ &= n_2\,\mathrm{d}_1\,\mathrm{d}_0\,\mathrm{s}_0\\ &= 1\end{aligned}$$

and

$$\begin{aligned}(n_2\,\mathrm{d}_0\,\mathrm{s}_1 \cdot n_2^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_2)\,\mathrm{d}_1 &= n_2\,\mathrm{d}_0\,\mathrm{s}_1\,\mathrm{d}_1 \cdot n_2^-\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{d}_1 \cdot n_2\,\mathrm{d}_1\\ &= n_2\,\mathrm{d}_0\,\mathrm{s}_1\,\mathrm{d}_1 \cdot n_2^-\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{d}_1\\ &= n_2\,\mathrm{d}_0 \cdot n_2^-\,\mathrm{d}_0\\ &= 1.\end{aligned}$$

So we have $[n_1\,\mathrm{s}_0,\, n_2\,\mathrm{d}_0\,\mathrm{s}_1 \cdot n_2^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_2] = 1$.

Therefore

$$\begin{aligned} & (n_1\,\mathrm{s}_0)^{n_2\,\mathrm{d}_0\,\mathrm{s}_1 \cdot n_2^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_2} &= n_1\,\mathrm{s}_0\\ \Leftrightarrow\quad & (n_1\,\mathrm{s}_0)^{n_2\,\mathrm{d}_0\,\mathrm{s}_1} &= (n_1\,\mathrm{s}_0)^{n_2^- \cdot n_2\,\mathrm{d}_0\,\mathrm{s}_0}.\end{aligned}$$

□

Example 50 With the help of Lemma 49 we get that

$$\begin{aligned} n_2^{(\tilde{n}_2\,\mathrm{d}_0)\,\mathrm{s}_1} &\overset{(\mathrm{CC}\ 1)}{=} n_2^{(\tilde{n}_2\,\mathrm{d}_0)\,\mathrm{d}_0}\\ &= n_2^{\tilde{n}_2\,\mathrm{d}_1\,\mathrm{d}_0}\\ &= n_2\end{aligned}$$

for n_2, $\tilde{n}_2 \in G\mathrm{N}_2$.

Lemma 51 Consider the groups $G\mathrm{N}_2$, $G\mathrm{N}_1$ and $G\mathrm{N}_0$.

With the help of Definition 45, Lemma 46 and Lemma 47 we have the group morphisms

$$\begin{array}{lllclcl} G\mathrm{N}_2 & \xrightarrow{\partial_2} & G\mathrm{N}_1 & : & n_2 & \longmapsto & n_2\partial_2 = n_2\,\mathrm{d}_0\\ G\mathrm{N}_1 & \xrightarrow{\partial_1} & G\mathrm{N}_0 & : & n_1 & \longmapsto & n_1\partial_1 = n_1\,\mathrm{d}_0\\ G\mathrm{N}_0 & \xrightarrow{\beta_1} & \mathrm{Aut}(G\mathrm{N}_1) & : & n_1 & \longmapsto & (n_1 \mapsto n_1^{n_0} := n_1^{n_0\,\mathrm{s}_0})\\ G\mathrm{N}_0 & \xrightarrow{\beta_2} & \mathrm{Aut}(G\mathrm{N}_2) & : & n_2 & \longmapsto & (n_0 \mapsto n_2^{n_0} := n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0})\ . \end{array}$$

With the help of Remark 48 let

$$\begin{array}{rcl} G\mathrm{N}_1 \times G\mathrm{N}_1 & \xrightarrow{\zeta} & G\mathrm{N}_2\\ (n_1, \tilde{n}_1) & \longmapsto & \lfloor n_1, \tilde{n}_1 \rfloor := (n_1^-\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_0} \cdot (n_1\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_1}. \end{array}$$

Then

$$G\,\hat{\mathrm{N}} := (G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta)$$

is a 2-crossed module.

So $G\,\hat{\mathrm{N}}_i = G\mathrm{N}_i$ for $i \in [0,2]$.

The “hat” on “$\hat{\mathrm{N}}$” is meant to indicate the extra data attached to the diagram

$$G\mathrm{N}_2 \xrightarrow{\partial_2} G\mathrm{N}_1 \xrightarrow{\partial_1} G\mathrm{N}_0$$

needed to yield a 2-crossed module, viz. β_1, β_2, ζ.

Proof. Note that

$$\lfloor n_2 \,\mathrm{d}_0, n_1 \rfloor = (n_2^-)^{n_1 \mathrm{s}_0} \cdot n_2^{n_1 \mathrm{s}_1}$$

for $n_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$; cf. Lemma 49, (CC 3).

We verify the conditions (2CM 1–9) of Definition 36.

Ad (2CM 1)*.* Suppose given n_2, $\tilde{n}_2 \in G\mathrm{N}_2$.

Then we have

$$\begin{aligned}
\lfloor \tilde{n}_2 \partial_2, n_2 \partial_2 \rfloor &= \lfloor \tilde{n}_2 \,\mathrm{d}_0, n_2 \,\mathrm{d}_0 \rfloor \\
&\overset{\text{(CC 3)}}{=} (\tilde{n}_2^-)^{n_2 \mathrm{d}_0 \mathrm{s}_0} \cdot \tilde{n}_2^{n_2 \mathrm{d}_0 \mathrm{s}_1} \\
&\overset{\text{(CC 2)}}{=} (\tilde{n}_2^-)^{n_2} \cdot \tilde{n}_2^{n_2 \mathrm{d}_0 \mathrm{s}_1} \\
&\overset{\text{(CC 1)}}{=} (\tilde{n}_2^-)^{n_2} \cdot \tilde{n}_2^{n_2 \mathrm{d}_0 \mathrm{d}_0} \\
&= (\tilde{n}_2^-)^{n_2} \cdot \tilde{n}_2 \\
&= n_2^- \cdot \tilde{n}_2^- \cdot n_2 \cdot \tilde{n}_2 \\
&= [n_2, \tilde{n}_2].
\end{aligned}$$

Ad (2CM 2)*.* Suppose given $n_2 \in G\mathrm{N}_2$ and $n_1 \in G\mathrm{N}_1$.

Then we have $n_1^- \,\mathrm{s}_0 \cdot (n_1 \,\mathrm{s}_0)^{n_2^-} \in G\mathrm{N}_2$, since

$$\begin{aligned}
(n_1^- \,\mathrm{s}_0 \cdot (n_1 \,\mathrm{s}_0)^{n_2^-}) \,\mathrm{d}_1 &= n_1^- \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot (n_1 \,\mathrm{s}_0)^{n_2^-} \,\mathrm{d}_1 \\
&= n_1^- \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot (n_1 \,\mathrm{s}_0 \,\mathrm{d}_1)^{n_2^- \mathrm{d}_1} \\
&= n_1^- \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot n_1 \,\mathrm{s}_0 \,\mathrm{d}_1 \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
(n_1^- \,\mathrm{s}_0 \cdot (n_1 \,\mathrm{s}_0)^{n_2^-}) \,\mathrm{d}_2 &= n_1^- \,\mathrm{s}_0 \,\mathrm{d}_2 \cdot (n_1 \,\mathrm{s}_0)^{n_2^-} \,\mathrm{d}_2 \\
&= n_1^- \,\mathrm{s}_0 \,\mathrm{d}_2 \cdot (n_1 \,\mathrm{s}_0 \,\mathrm{d}_2)^{n_2^- \mathrm{d}_2} \\
&= n_1^- \,\mathrm{s}_0 \,\mathrm{d}_2 \cdot n_1 \,\mathrm{s}_0 \,\mathrm{d}_2 \\
&= 1.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\lfloor n_1, n_2 \partial_2 \rfloor \cdot \lfloor n_2 \partial_2, n_1 \rfloor &= \lfloor n_1, n_2 \,\mathrm{d}_0 \rfloor \cdot \lfloor n_2 \,\mathrm{d}_0, n_1 \rfloor \\
&= (n_1^- \,\mathrm{s}_0)^{n_2 \mathrm{d}_0 \mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_0)^{n_2 \mathrm{d}_0 \mathrm{s}_1} \cdot \lfloor n_2 \,\mathrm{d}_0, n_1 \rfloor \\
&\overset{\text{(CC 3)}}{=} (n_1^- \,\mathrm{s}_0)^{n_2 \mathrm{d}_0 \mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_0)^{n_2 \mathrm{d}_0 \mathrm{s}_1} \cdot (n_2^-)^{n_1 \mathrm{s}_0} \cdot n_2^{n_1 \mathrm{s}_1} \\
&\overset{\text{(CC 1)}}{=} (n_1^- \,\mathrm{s}_0)^{n_2 \mathrm{d}_0 \mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_0)^{n_2 \mathrm{d}_0 \mathrm{s}_1} \cdot (n_2^-)^{n_1 \mathrm{s}_0} \cdot n_2^{n_1 \mathrm{d}_0} \\
&\overset{\text{(CC 4)}}{=} (n_1^- \,\mathrm{s}_0)^{n_2 \mathrm{d}_0 \mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_0)^{n_2^- \cdot\, n_2 \mathrm{d}_0 \mathrm{s}_0} \cdot (n_2^-)^{n_1 \mathrm{s}_0} \cdot n_2^{n_1 \mathrm{d}_0} \\
&= (n_1^- \,\mathrm{s}_0 \cdot (n_1 \,\mathrm{s}_0)^{n_2^-})^{n_2 \mathrm{d}_0 \mathrm{s}_0} \cdot (n_2^-)^{n_1 \mathrm{s}_0} \cdot n_2^{n_1 \mathrm{d}_0} \\
&\overset{\text{(CC 2)}}{=} (n_1^- \,\mathrm{s}_0 \cdot (n_1 \,\mathrm{s}_0)^{n_2^-})^{n_2} \cdot (n_2^-)^{n_1 \mathrm{s}_0} \cdot n_2^{n_1 \mathrm{d}_0} \\
&= (n_1^- \,\mathrm{s}_0)^{n_2} \cdot n_1 \,\mathrm{s}_0 \cdot (n_2^-)^{n_1 \mathrm{s}_0} \cdot n_2^{n_1 \mathrm{d}_0} \\
&= n_2^- \cdot n_1^- \,\mathrm{s}_0 \cdot n_2 \cdot n_1 \,\mathrm{s}_0 \cdot (n_1 \,\mathrm{s}_0)^- \cdot n_2^- \cdot n_1 \,\mathrm{s}_0 \cdot n_2^{n_1 \mathrm{d}_0} \\
&= n_2^- \cdot n_2^{n_1 \mathrm{d}_0} \\
&= n_2^- \cdot n_2^{n_1 \partial_1}.
\end{aligned}$$

Ad (2CM 3)*.* Suppose given n_1, $\tilde{n}_1 \in G\mathrm{N}_1$.

Then we have

$$\begin{aligned}
\lfloor n_1, \tilde{n}_1 \rfloor \partial_2 &= ((n_1^- s_0)^{\tilde{n}_1 \mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_1}) \,\mathrm{d}_0 \\
&= (n_1^- s_0)^{\tilde{n}_1 \mathrm{s}_0} \,\mathrm{d}_0 \cdot (n_1 \,\mathrm{s}_0)^{\tilde{n}_1 \mathrm{s}_1} \,\mathrm{d}_0 \\
&= (n_1^- s_0 \,\mathrm{d}_0)^{\tilde{n}_1 \mathrm{s}_0 \mathrm{d}_0} \cdot (n_1 \,\mathrm{s}_0 \,\mathrm{d}_0)^{\tilde{n}_1 \mathrm{s}_1 \mathrm{d}_0} \\
&= (n_1^-)^{\tilde{n}_1} \cdot n_1^{\tilde{n}_1 \mathrm{d}_0 \mathrm{s}_0} \\
&= (n_1^-)^{\tilde{n}_1} \cdot n_1^{\tilde{n}_1 \mathrm{d}_0} \\
&= (n_1^-)^{\tilde{n}_1} \cdot n_1^{\tilde{n}_1 \partial_1}.
\end{aligned}$$

Ad (2CM 4). Suppose given n_1, $\tilde{n}_1 \in G\mathrm{N}_1$ and $n_0 \in G\mathrm{N}_0$.

Then we have

$$\begin{array}{rcl}
\lfloor n_1, \tilde{n}_1 \rfloor^{n_0} & = & \lfloor n_1, \tilde{n}_1 \rfloor^{n_0\,\mathrm{s}_0\,\mathrm{s}_0} \\
& = & ((n_1^- s_0)^{\tilde{n}_1\,\mathrm{s}_0} \cdot (n_1\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_1})^{n_0\,\mathrm{s}_0\,\mathrm{s}_0} \\
& = & (n_1^- s_0)^{\tilde{n}_1\,\mathrm{s}_0 \cdot n_0\,\mathrm{s}_0\,\mathrm{s}_0} \cdot (n_1\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_1 \cdot n_0\,\mathrm{s}_0\,\mathrm{s}_0} \\
& = & (n_1^- s_0)^{\tilde{n}_1\,\mathrm{s}_0 \cdot n_0\,\mathrm{s}_0\,\mathrm{s}_0} \cdot (n_1\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_1 \cdot n_0\,\mathrm{s}_0\,\mathrm{s}_1} \\
& = & (n_1^- s_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot (n_0\,\mathrm{s}_0\,\mathrm{s}_0)^- \cdot \tilde{n}_1\,\mathrm{s}_0 \cdot n_0\,\mathrm{s}_0\,\mathrm{s}_0} \cdot (n_1\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot (n_0\,\mathrm{s}_0\,\mathrm{s}_1)^- \cdot \tilde{n}_1\,\mathrm{s}_1 \cdot n_0\,\mathrm{s}_0\,\mathrm{s}_1} \\
& = & (n_1^- s_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot (\tilde{n}_1\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0}} \cdot (n_1\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot (\tilde{n}_1\,\mathrm{s}_1)^{n_0\,\mathrm{s}_0\,\mathrm{s}_1}} \\
& = & (n_1^- s_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot \tilde{n}_1^{n_0\,\mathrm{s}_0}\,\mathrm{s}_0} \cdot (n_1\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot \tilde{n}_1^{n_0\,\mathrm{s}_0}\,\mathrm{s}_1} \\
& = & (n_1^- s_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot \tilde{n}_1^{n_0\,\mathrm{s}_0}\,\mathrm{s}_0} \cdot (n_1\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot \tilde{n}_1^{n_0\,\mathrm{s}_0}\,\mathrm{s}_1} \\
& = & ((n_1^{n_0\,\mathrm{s}_0})^-\,\mathrm{s}_0)^{(\tilde{n}_1^{n_0\,\mathrm{s}_0})\,\mathrm{s}_0} \cdot ((n_1^{n_0\,\mathrm{s}_0})\,\mathrm{s}_0)^{(\tilde{n}_1^{n_0\,\mathrm{s}_0})\,\mathrm{s}_1} \\
& = & ((n_1^{n_0})^-\,\mathrm{s}_0)^{(\tilde{n}_1^{n_0})\,\mathrm{s}_0} \cdot ((n_1^{n_0})\,\mathrm{s}_0)^{(\tilde{n}_1^{n_0})\,\mathrm{s}_1} \\
& = & \lfloor n_1^{n_0}, \tilde{n}_1^{n_0} \rfloor .
\end{array}$$

Ad (2CM 5). Suppose given $n_2 \in G\mathrm{N}_2$.

Then we have

$$\begin{array}{rcl}
n_2\partial_2\partial_1 & = & n_2\,\mathrm{d}_0\,\mathrm{d}_0 \\
& = & n_2\,\mathrm{d}_1\,\mathrm{d}_0 \\
& = & 1.
\end{array}$$

Ad (2CM 6). Suppose given $n_1 \in G\mathrm{N}_1$ and $n_0 \in G\mathrm{N}_0$.

Then we have

$$\begin{array}{rcl}
(n_1^{n_0})\partial_1 & = & (n_1^{n_0\,\mathrm{s}_0})\,\mathrm{d}_0 \\
& = & (n_1\,\mathrm{d}_0)^{n_0\,\mathrm{s}_0\,\mathrm{d}_0} \\
& = & (n_1\,\mathrm{d}_0)^{n_0} \\
& = & (n_1\partial_1)^{n_0}.
\end{array}$$

Ad (2CM 7). Suppose given $n_2 \in G\mathrm{N}_2$ and $n_0 \in G\mathrm{N}_0$.

Then we have

$$\begin{array}{rcl}
(n_2^{n_0})\partial_2 & = & (n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0})\,\mathrm{d}_0 \\
& = & (n_2\,\mathrm{d}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0\,\mathrm{d}_0} \\
& = & (n_2\,\mathrm{d}_0)^{n_0\,\mathrm{s}_0} \\
& = & (n_2\,\mathrm{d}_0)^{n_0} \\
& = & (n_2\partial_2)^{n_0}.
\end{array}$$

Ad (2CM 8). Suppose given n_1, $\tilde{n}_1$, $\tilde{\tilde{n}}_1 \in G\mathrm{N}_1$.

We have to show that

$$\lfloor n_1, \tilde{n}_1 \cdot \tilde{\tilde{n}}_1 \rfloor \overset{!}{=} \lfloor n_1^{\tilde{n}_1}, \tilde{\tilde{n}}_1 \rfloor \cdot \lfloor n_1, \tilde{n}_1 \rfloor^{\tilde{\tilde{n}}_1\partial_1}.$$

In order to calculate, we write $a := n_1$, $b := \tilde{n}_1$ and $c := \tilde{\tilde{n}}_1$.

Then a, b, $c \in G\mathrm{N}_1$.

We have to show that

$$\lfloor a, b \cdot c \rfloor \overset{!}{=} \lfloor a^b, c \rfloor \cdot \lfloor a, b \rfloor^{c\partial_1}.$$

We obtain

$$\begin{aligned}
\lfloor a, b\cdot c\rfloor &= (a^-\,\mathrm{s}_0)^{(b\cdot c)\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{(b\cdot c)\,\mathrm{s}_1}\\
&= (a^-\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{b\,\mathrm{s}_1\cdot c\,\mathrm{s}_1}\\
&= (a^-\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_1}\cdot(a^-\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_1}\cdot(a\,\mathrm{s}_0)^{b\,\mathrm{s}_1\cdot c\,\mathrm{s}_1}\\
&= (a^-\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_1}\cdot((a^-\,\mathrm{s}_0)^{b\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{b\,\mathrm{s}_1})^{c\,\mathrm{s}_1}\\
&= (a^-\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_1}\cdot\lfloor a,b\rfloor^{c\,\mathrm{s}_1}\\
&\overset{\text{(CC 1)}}{=} (a^-\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{b\,\mathrm{s}_0\cdot c\,\mathrm{s}_1}\cdot\lfloor a,b\rfloor^{c\,\mathrm{d}_0}\\
&= ((a^b)^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot(a^b\,\mathrm{s}_0)^{c\,\mathrm{s}_1}\cdot\lfloor a,b\rfloor^{c\,\mathrm{d}_0}\\
&= \lfloor a^b, c\rfloor\cdot\lfloor a,b\rfloor^{c\,\mathrm{d}_0}\\
&= \lfloor a^b, c\rfloor\cdot\lfloor a,b\rfloor^{c\partial_1}.
\end{aligned}$$

Ad (2CM 9)*.* Suppose given $n_1,\ \tilde{n}_1,\ \tilde{\tilde{n}}_1\in G\mathrm{N}_1$.

We have to show that

$$\lfloor n_1\cdot\tilde{n}_1,\tilde{\tilde{n}}_1\rfloor \overset{!}{=} \lfloor n_1,\tilde{\tilde{n}}_1\rfloor\cdot\lfloor \tilde{n}_1^{\tilde{\tilde{n}}_1},\lfloor n_1,\tilde{\tilde{n}}_1\rfloor\partial_2\rfloor\cdot\lfloor\tilde{n}_1,\tilde{\tilde{n}}_1\rfloor.$$

In order to calculate, we write $a := n_1$, $b := \tilde{n}_1$ and $c := \tilde{\tilde{n}}_1$.

Then $a,\ b,\ c\in G\mathrm{N}_1$.

We have to show that

$$\lfloor a\cdot b, c\rfloor \overset{!}{=} \lfloor a,c\rfloor\cdot\lfloor b^c,\lfloor a,c\rfloor\partial_2\rfloor\cdot\lfloor b,c\rfloor.$$

We obtain

$$\begin{aligned}
&\lfloor a\cdot b, c\rfloor\\
=\ & ((a\cdot b)^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot((a\cdot b)\,\mathrm{s}_0)^{c\,\mathrm{s}_1}\\
=\ & (b^-\,\mathrm{s}_0\cdot a^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0\cdot b\,\mathrm{s}_0)^{c\,\mathrm{s}_1}\\
=\ & (b^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot(a^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{c\,\mathrm{s}_1}\cdot(b\,\mathrm{s}_0)^{c\,\mathrm{s}_1}\\
=\ & (b^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot(a^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot(a\,\mathrm{s}_0)^{c\,\mathrm{s}_1}\cdot(b\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot(b^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot(b\,\mathrm{s}_0)^{c\,\mathrm{s}_1}\\
=\ & (b^-\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot\lfloor a,c\rfloor\cdot(b\,\mathrm{s}_0)^{c\,\mathrm{s}_0}\cdot\lfloor b,c\rfloor\\
=\ & \lfloor a,c\rfloor^{(b\,\mathrm{s}_0)^{c\,\mathrm{s}_0}}\cdot\lfloor b,c\rfloor\\
=\ & \lfloor a,c\rfloor^{(b^c)\,\mathrm{s}_0}\cdot\lfloor b,c\rfloor\\
\overset{\text{(CC 3)}}{=}\ & \lfloor a,c\rfloor^{(b^c)\,\mathrm{s}_1}\cdot\lfloor\lfloor a,c\rfloor\,\mathrm{d}_0, b^c\rfloor^-\cdot\lfloor b,c\rfloor\\
=\ & \lfloor a,c\rfloor^{(b^c)\,\mathrm{s}_1}\cdot\lfloor\lfloor a,c\rfloor\partial_2, b^c\rfloor^-\cdot\lfloor b,c\rfloor\\
\overset{\text{(2CM 2)}}{=}\ & \lfloor a,c\rfloor^{(b^c)\,\mathrm{s}_1}\cdot(\lfloor a,c\rfloor^-)^{(b^c)\partial_1}\cdot\lfloor a,c\rfloor\cdot\lfloor b^c,\lfloor a,c\rfloor\partial_2\rfloor\cdot\lfloor b,c\rfloor\\
=\ & \lfloor a,c\rfloor^{(b^c)\,\mathrm{s}_1}\cdot(\lfloor a,c\rfloor^-)^{(b^c)\,\mathrm{d}_0}\cdot\lfloor a,c\rfloor\cdot\lfloor b^c,\lfloor a,c\rfloor\partial_2\rfloor\cdot\lfloor b,c\rfloor\\
\overset{\text{(CC 1)}}{=}\ & \lfloor a,c\rfloor^{(b^c)\,\mathrm{s}_1}\cdot(\lfloor a,c\rfloor^-)^{(b^c)\,\mathrm{s}_1}\cdot\lfloor a,c\rfloor\cdot\lfloor b^c,\lfloor a,c\rfloor\partial_2\rfloor\cdot\lfloor b,c\rfloor\\
=\ & \lfloor a,c\rfloor\cdot\lfloor b^c,\lfloor a,c\rfloor\partial_2\rfloor\cdot\lfloor b,c\rfloor.
\end{aligned}$$

□

4.5.2 The construction for the morphisms

Suppose given a morphism $\varphi : G\to H$ of $\lfloor 2,0\rfloor$-simplicial groups; cf. Definition 31.

Remark 52

(0) We have the group morphism

$$\begin{aligned}
\varphi\mathrm{N}_0 := \varphi_0 : G\mathrm{N}_0 &\to H\mathrm{N}_0\\
n_0 &\mapsto n_0\varphi_0.
\end{aligned}$$

Recall that $G\mathrm{N}_0 = G_0$ and $H\mathrm{N}_0 = H_0$.

(1.0) We have the group morphism

$$\begin{array}{rcl} \varphi_{1;0} := \varphi_1|_{G_{1;0}}^{H_{1;0}} : G_{1;0} & \to & H_{1;0} \\ n_1 & \mapsto & n_1\varphi_1 . \end{array}$$

(1.1) We have the group morphism

$$\begin{array}{rcl} \varphi_{1;1} := \varphi\mathrm{N}_1 := \varphi_1|_{G\mathrm{N}_1}^{H\mathrm{N}_1} : G\mathrm{N}_1 & \to & H\mathrm{N}_1 \\ n_1 & \mapsto & n_1\varphi_1 . \end{array}$$

(2) We have the group morphism.

$$\begin{array}{rcl} \varphi_{2;1,2} := \varphi\mathrm{N}_2 := \varphi_2|_{G\mathrm{N}_2}^{H\mathrm{N}_2} : G\mathrm{N}_2 & \to & H\mathrm{N}_2 \\ n_2 & \mapsto & n_2\varphi_2 . \end{array}$$

Proof.

Ad (1.0). Suppose given $x \in G_{1;0}$. This means $x\,\mathrm{d}_0 = 1$.

Therefore we have

$$\begin{array}{rcl} x\varphi_1\,\mathrm{d}_0 & = & x\,\mathrm{d}_0\,\varphi_0 \\ & = & 1. \end{array}$$

So we have $x\varphi_1 \in H_{1;0}$.

Altogether,

$$(G_{1;0})\varphi_1 \subseteq H_{1;0}\,.$$

Ad (1.1). Suppose given $x \in G\mathrm{N}_1$. This means $x\,\mathrm{d}_1 = 1$.

Therefore we have

$$\begin{array}{rcl} x\varphi_1\,\mathrm{d}_1 & = & x\,\mathrm{d}_1\,\varphi_0 \\ & = & 1. \end{array}$$

So we have $x\varphi_1 \in H\mathrm{N}_1$.

Altogether,

$$(G\mathrm{N}_1)\varphi_1 \subseteq H\mathrm{N}_1.$$

Ad (2). Suppose given $x \in G\mathrm{N}_2$. This means $x\,\mathrm{d}_1 = 1$ and $x\,\mathrm{d}_2 = 1$.

Therefore we have

$$\begin{array}{rcl} x\varphi_2\,\mathrm{d}_1 & = & x\,\mathrm{d}_1\,\varphi_1 \\ & = & 1 \end{array}$$

and

$$\begin{array}{rcl} x\varphi_2\,\mathrm{d}_2 & = & x\,\mathrm{d}_2\,\varphi_1 \\ & = & 1. \end{array}$$

So we have $x\varphi_2 \in H\mathrm{N}_2$.

Altogether,

$$(G\mathrm{N}_2)\varphi_2 \subseteq H\mathrm{N}_2.$$

Cf. also [**1,** Rem. 71]. □

Lemma 53 Recall that we have the 2-crossed modules

$$G\,\hat{\mathrm{N}} =: (G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta)$$

and

$$H\,\hat{\mathrm{N}} =: (H\mathrm{N}_2,\, H\mathrm{N}_1,\, H\mathrm{N}_0,\, \partial'_2,\, \partial'_1,\, \beta'_1,\, \beta'_2,\, \zeta').$$

Cf. Lemma 51.

Recall the group morphisms $\varphi_{2;1,2}$, $\varphi_{1;1}$, φ_0; cf. Definition 31 and Remark 52.

Then

$$\varphi\,\hat{\mathrm{N}} := (\varphi_{2;1,2},\, \varphi_{1;1},\, \varphi_0) = (\varphi\mathrm{N}_2,\, \varphi\mathrm{N}_1,\, \varphi\mathrm{N}_0) : (G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0) \to (H\mathrm{N}_2,\, H\mathrm{N}_1,\, H\mathrm{N}_0)$$

is a morphism of 2-crossed modules; cf. Definition 39.

$$\begin{tikzcd}
G\mathrm{N}_2 \arrow[r, "\partial_2"] \arrow[d, "\varphi_{2;1,2}"'] & G\mathrm{N}_1 \arrow[r, "\partial_1"] \arrow[d, "\varphi_{1;1}"] & G\mathrm{N}_0 \arrow[d, "\varphi_0"] \\
H\mathrm{N}_2 \arrow[r, "\partial'_2"] & H\mathrm{N}_1 \arrow[r, "\partial'_1"] & H\mathrm{N}_0
\end{tikzcd}$$

Proof. First, we have to show that the diagram is commutative.

Suppose given $n_2 \in G\mathrm{N}_2$ and $n_1 \in G\mathrm{N}_1$.

Then we have

$$\begin{array}{rcl}
n_2(\partial_2 \blacktriangle \varphi_{1;1}) & = & (n_2\partial_2)\varphi_{1;1} \\
& = & n_2\, \mathrm{d}_0\, \varphi_1 \\
& = & n_2\varphi_2\, \mathrm{d}_0 \\
& = & (n_2\varphi_{2;1,2})\partial'_2 \\
& = & n_2(\varphi_{2;1,2} \blacktriangle \partial'_2)
\end{array}$$

and

$$\begin{array}{rcl}
n_1(\partial_1 \blacktriangle \varphi_0) & = & (n_1\partial_1)\varphi_0 \\
& = & (n_1\, \mathrm{d}_0)\varphi_0 \\
& = & n_1\varphi_1\, \mathrm{d}_0 \\
& = & (n_1\varphi_{1;1})\partial'_1 \\
& = & n_1(\varphi_{1;1} \blacktriangle \partial'_1).
\end{array}$$

So we have

$$\partial_2 \blacktriangle \varphi_{1;1} = \varphi_{2;1,2} \blacktriangle \partial'_2$$

and

$$\partial_1 \blacktriangle \varphi_0 = \varphi_{1;1} \blacktriangle \partial'_1.$$

Second, we have to show (2CMM $1,2$).

Ad (2CMM 1.1). Suppose given $n_1 \in G\mathrm{N}_1$ and $n_0 \in G\mathrm{N}_0$.

Then we have

$$\begin{array}{rcl}
(n_1^{n_0})\varphi_{1;1} & = & (n_1^{n_0\, \mathrm{s}_0})\varphi_{1;1} \\
& = & n_1^{n_0\, \mathrm{s}_0}\varphi_1 \\
& = & (n_1\varphi_1)^{n_0\, \mathrm{s}_0\, \varphi_1} \\
& = & (n_1\varphi_1)^{n_0\varphi_0\, \mathrm{s}_0} \\
& = & (n_1\varphi_1)^{n_0\varphi_0} \\
& = & (n_1\varphi_{1;1})^{n_0\varphi_0}.
\end{array}$$

Ad (2CMM 1.2). Suppose given $n_2 \in G\mathrm{N}_2$ and $n_0 \in G\mathrm{N}_0$.

Then we have

$$\begin{array}{rcl} (n_2^{n_0})\varphi_{2;1,2} & = & (n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0})\varphi_{2;1,2} \\ & = & n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0}\varphi_2 \\ & = & (n_2\varphi_2)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0\,\varphi_2} \\ & = & (n_2\varphi_2)^{n_0\,\mathrm{s}_0\,\varphi_1\,\mathrm{s}_0} \\ & = & (n_2\varphi_2)^{n_0\varphi_0\,\mathrm{s}_0\,\mathrm{s}_0} \\ & = & (n_2\varphi_2)^{n_0\varphi_0} \\ & = & (n_2\varphi_{2;1,2})^{n_0\varphi_0}. \end{array}$$

Ad (2CMM 2). Suppose given n_1, $\tilde{n}_1 \in G\mathrm{N}_1$.

Then we have

$$\begin{array}{rcl} \lfloor n_1, \tilde{n}_1\rfloor\varphi_{2;1,2} & = & ((n_1^-\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_0}\cdot(n_1\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_1})\varphi_{2;1,2} \\ & = & ((n_1^-\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_0}\cdot(n_1\,\mathrm{s}_0)^{\tilde{n}_1\,\mathrm{s}_1})\varphi_2 \\ & = & (n_1^-\,\mathrm{s}_0\,\varphi_2)^{\tilde{n}_1\,\mathrm{s}_0\,\varphi_2}\cdot(n_1\,\mathrm{s}_0\,\varphi_2)^{\tilde{n}_1\,\mathrm{s}_1\,\varphi_2} \\ & = & (n_1^-\varphi_1\,\mathrm{s}_0)^{\tilde{n}_1\varphi_1\,\mathrm{s}_0}\cdot(n_1\varphi_1\,\mathrm{s}_0)^{\tilde{n}_1\varphi_1\,\mathrm{s}_1} \\ & = & ((n_1\varphi_{1;1})^-\,\mathrm{s}_0)^{\tilde{n}_1\varphi_{1;1}\,\mathrm{s}_0}\cdot(n_1\varphi_{1;1}\,\mathrm{s}_0)^{\tilde{n}_1\varphi_{1;1}\,\mathrm{s}_1} \\ & = & \lfloor n_1\varphi_{1;1}, \tilde{n}_1\varphi_{1;1}\rfloor. \end{array}$$

☐

4.5.3 The functor $\hat{\mathrm{N}}$

Definition 54 We shall define the following functor.

$$\begin{tikzcd} \hat{\mathrm{N}} : \lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp} \arrow[r] & 2\text{-}\mathcal{CrMod} \\ \left(\begin{array}{c} G \\ \downarrow{\scriptstyle\varphi} \\ H \end{array}\right) \arrow[r, mapsto] & \left(\begin{array}{c} G\,\hat{\mathrm{N}} \\ \downarrow{\scriptstyle\varphi\,\hat{\mathrm{N}}} \\ H\,\hat{\mathrm{N}} \end{array}\right) \end{tikzcd}$$

(1) Suppose given a $\lfloor 2,0\rfloor$-simplicial group G.

The 2-crossed module $G\,\hat{\mathrm{N}}$ has been defined in Lemma 51.

For short:

$$G\,\hat{\mathrm{N}} = \left(\begin{tikzcd} G\mathrm{N}_2 \arrow[r, "\partial_2"] & G\mathrm{N}_1 \arrow[r, "\partial_1"] & G\mathrm{N}_0 \end{tikzcd}\right)$$

(2) Suppose given a morphism of $\lfloor 2,0\rfloor$-simplicial groups $G \xrightarrow{\varphi} H$.

The following diagram morphism is a morphism of 2-crossed modules, by Lemma 53.

$$\left(\begin{array}{c} G \\ {\scriptstyle\varphi}\downarrow \\ H \end{array}\right)\hat{\mathrm{N}} := \left(\begin{tikzcd} G\mathrm{N}_2 \arrow[r, "\partial_2"] \arrow[d, "\varphi_{2;1,2}"] & G\mathrm{N}_1 \arrow[r, "\partial_1"] \arrow[d, "\varphi_{1;1}"] & G\mathrm{N}_0 \arrow[d, "\varphi_0"] \\ H\mathrm{N}_2 \arrow[r, "\partial_2'"] & H\mathrm{N}_1 \arrow[r, "\partial_1'"] & H\mathrm{N}_0 \end{tikzcd}\right)$$

In particular,

$$\begin{array}{rclcl} n_2(\varphi\,\hat{\mathrm{N}})_2 & = & n_2(\varphi\mathrm{N}_2) & = & n_2\varphi_2 \\ n_1(\varphi\,\hat{\mathrm{N}})_1 & = & n_1(\varphi\mathrm{N}_1) & = & n_1\varphi_1 \\ n_0(\varphi\,\hat{\mathrm{N}})_0 & = & n_0(\varphi\mathrm{N}_0) & = & n_0\varphi_0 \end{array}$$

for $n_2 \in N_2$, $n_1 \in N_1$ and $n_0 \in N_0$.

(3) Suppose given morphisms of $\lfloor 2,0\rfloor$-simplicial groups $G \xrightarrow{\varphi} H \xrightarrow{\varphi'} K$.

Then we have

$$\begin{array}{ll} \text{(a)} & (\mathrm{id}_G)\,\hat{\mathrm{N}} = \mathrm{id}_{(G\,\hat{\mathrm{N}})} \\ \text{(b)} & (\varphi \blacktriangle \varphi')\,\hat{\mathrm{N}} = \varphi\,\hat{\mathrm{N}} \blacktriangle \varphi'\,\hat{\mathrm{N}}\,. \end{array}$$

In particular, $\hat{\mathrm{N}} : \lfloor 2,0\rfloor\text{-}\mathscr{SimpGrp} \to 2\text{-}\mathscr{CrMod}$ is a functor.

Proof.

Ad (3.a). Write $G =: (G_2,\, G_1,\, G_0)$.

We have $\mathrm{id}_G = (\mathrm{id}_{G_2},\, \mathrm{id}_{G_1},\, \mathrm{id}_{G_0})$; cf. Remark 33.

We have $(\mathrm{id}_G)\,\hat{\mathrm{N}} = (\mathrm{id}_{G_2}|_{G_{2;1,2}}^{G_{2;1,2}},\, \mathrm{id}_{G_1}|_{G_{1;1}}^{G_{1;1}},\, \mathrm{id}_{G_0}) = (\mathrm{id}_{G_{2;1,2}},\, \mathrm{id}_{G_{1;1}},\, \mathrm{id}_{G_0})$; cf. Lemma 53 and Remark 52.

On the other hand, we have $G\,\hat{\mathrm{N}} = (G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0)$; cf. Lemma 51.

So we have $\mathrm{id}_{G\,\hat{\mathrm{N}}} = (\mathrm{id}_{G\mathrm{N}_2},\, \mathrm{id}_{G\mathrm{N}_1},\, \mathrm{id}_{G\mathrm{N}_0})$; cf. Remark 41.

But we already know that $G\mathrm{N}_2 = G_{2;1,2}$, $G\mathrm{N}_1 = G_{1;1}$ and $G\mathrm{N}_0 = G_0$; cf. Definition 44 and Definition 45.

So $(\mathrm{id}_G)\,\hat{\mathrm{N}} = \mathrm{id}_{G\,\hat{\mathrm{N}}}$.

Ad (3.b). Write $G =: (G_2,\, G_1,\, G_0)$, $H =: (H_2,\, H_1,\, H_0)$, $K =: (K_2,\, K_1,\, K_0)$, $\varphi =: (\varphi_2,\, \varphi_1,\, \varphi_0)$ and $\varphi' =: (\varphi'_2,\, \varphi'_1,\, \varphi'_0)$.

So

$$(G \xrightarrow{\varphi} H \xrightarrow{\varphi'} K) = ((G_2,\, G_1,\, G_0) \xrightarrow{(\varphi_2,\, \varphi_1,\, \varphi_0)} (H_2,\, H_1,\, H_0) \xrightarrow{(\varphi'_2,\, \varphi'_1,\, \varphi'_0)} (K_2,\, K_1,\, K_0)).$$

We have

$$\begin{array}{cl} & (G\,\hat{\mathrm{N}} \xrightarrow{\varphi\,\hat{\mathrm{N}}} H\,\hat{\mathrm{N}} \xrightarrow{\varphi'\,\hat{\mathrm{N}}} K\,\hat{\mathrm{N}}) \\ = & ((G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0) \xrightarrow{(\varphi_{2;1,2},\, \varphi_{1;1},\, \varphi_0)} (H\mathrm{N}_2,\, H\mathrm{N}_1,\, H\mathrm{N}_0) \xrightarrow{(\varphi'_{2;1,2},\, \varphi'_{1;1},\, \varphi'_0)} (K\mathrm{N}_2,\, K\mathrm{N}_1,\, K\mathrm{N}_0)); \end{array}$$

cf. Lemma 51 and Lemma 53.

So we have

$$(G\,\hat{\mathrm{N}} \xrightarrow{\varphi\,\hat{\mathrm{N}} \blacktriangle \varphi'\,\hat{\mathrm{N}}} K\,\hat{\mathrm{N}}) = ((G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0) \xrightarrow{(\varphi_{2;1,2} \blacktriangle \varphi'_{2;1,2},\, \varphi_{1;1} \blacktriangle \varphi'_{1;1},\, \varphi_0 \blacktriangle \varphi'_0)} (K\mathrm{N}_2,\, K\mathrm{N}_1,\, K\mathrm{N}_0));$$

cf. Remark 40.

On the other hand, we have

$$(G \xrightarrow{\varphi \blacktriangle \varphi'} K) = ((G_2,\, G_1,\, G_0) \xrightarrow{(\varphi_2 \blacktriangle \varphi'_2,\, \varphi_1 \blacktriangle \varphi'_1,\, \varphi_0 \blacktriangle \varphi'_0)} (K_2,\, K_1,\, K_0));$$

cf. Remark 32.

Hence

$$\begin{array}{rl} & (G\,\hat{\mathrm{N}} \xrightarrow{(\varphi \blacktriangle \varphi')\,\hat{\mathrm{N}}} K\,\hat{\mathrm{N}}) \\ = & ((G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0) \xrightarrow{((\varphi \blacktriangle \varphi')_{2;1,2},\, (\varphi \blacktriangle \varphi')_{1;1},\, (\varphi \blacktriangle \varphi')_0)} (K\mathrm{N}_2,\, K\mathrm{N}_1,\, K\mathrm{N}_0)) \\ = & ((G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0) \xrightarrow{((\varphi_2 \blacktriangle \varphi_2')|_{G_{2;1,2}}^{K_{2;1,2}},\, (\varphi_1 \blacktriangle \varphi_1')|_{G_{1;1}}^{K_{1;1}},\, \varphi_0 \blacktriangle \varphi_0')} (K\mathrm{N}_2,\, K\mathrm{N}_1,\, K\mathrm{N}_0)) \\ = & ((G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0) \xrightarrow{(\varphi_2|_{G_{2;1,2}}^{H_{2;1,2}} \blacktriangle \varphi_2'|_{H_{2;1,2}}^{K_{2;1,2}},\, \varphi_1|_{G_{1;1}}^{H_{1;1}} \blacktriangle \varphi_1'|_{H_{1;1}}^{K_{1;1}},\, \varphi_0 \blacktriangle \varphi_0')} (K\mathrm{N}_2,\, K\mathrm{N}_1,\, K\mathrm{N}_0)) \\ = & ((G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0) \xrightarrow{(\varphi_{2;1,2} \blacktriangle \varphi'_{2;1,2},\, \varphi_{1;1} \blacktriangle \varphi'_{1;1},\, \varphi_0 \blacktriangle \varphi_0')} (K\mathrm{N}_2,\, K\mathrm{N}_1,\, K\mathrm{N}_0)); \end{array}$$

cf. Lemma 53. □

4.6 From 2-crossed modules to $\lfloor 2,0 \rfloor$-simplicial groups

4.6.1 The construction for the objects

Suppose given a 2-crossed module

$$N = (N_2,\, N_1,\, N_0,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta);$$

cf. Definition 36.

Lemma 55 We have the following group morphism.

$$\begin{array}{rcl} N_1 & \xrightarrow{\varepsilon_2} & \mathrm{Aut}(N_2) \\ n_1 & \longmapsto & (n_2 \mapsto n_2^{n_1} := n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor) \end{array}$$

Proof. We use Remark 6.

Suppose given $n_1,\, n_1' \in N_1$ and $n_2,\, n_2' \in N_2$.

Then we have

$$\begin{array}{rcl} n_2^1 & = & n_2 \cdot \lfloor 1, n_2\partial_2 \rfloor \\ & \overset{\text{Rem. 37}}{=} & n_2. \end{array}$$

Moreover, we obtain

$$\begin{array}{rcl} (n_2^{n_1})^{n_1'} & = & (n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor)^{n_1'} \\ & = & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor n_1', (n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor)\partial_2 \rfloor \\ & = & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor n_1', n_2\partial_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \partial_2 \rfloor \\ & \overset{\text{(2CM 3)}}{=} & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor n_1', n_2\partial_2 \cdot (n_1^-)^{n_2\partial_2} \cdot n_1^{n_2\partial_2\partial_1} \rfloor \\ & \overset{\text{(2CM 5)}}{=} & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor n_1', n_2\partial_2 \cdot (n_1^-)^{n_2\partial_2} \cdot n_1 \rfloor \\ & \overset{\text{(2CM 8)}}{=} & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor {n_1'}^{n_2\partial_2}, (n_1^-)^{n_2\partial_2} \cdot n_1 \rfloor \cdot \lfloor n_1', n_2\partial_2 \rfloor^{((n_1^-)^{n_2\partial_2} \cdot n_1)\partial_1} \\ & = & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor {n_1'}^{n_2\partial_2}, (n_1^-)^{n_2\partial_2} \cdot n_1 \rfloor \cdot \lfloor n_1', n_2\partial_2 \rfloor^{(n_1^-\partial_1)^{n_2\partial_2\partial_1} \cdot n_1\partial_1} \\ & \overset{\text{(2CM 5)}}{=} & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor {n_1'}^{n_2\partial_2}, (n_1^-)^{n_2\partial_2} \cdot n_1^{n_2\partial_2\partial_1} \rfloor \cdot \lfloor n_1', n_2\partial_2 \rfloor^{n_1^-\partial_1 \cdot n_1\partial_1} \\ & = & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor {n_1'}^{n_2\partial_2}, (n_1^-)^{n_2\partial_2} \cdot n_1^{n_2\partial_2\partial_1} \rfloor \cdot \lfloor n_1', n_2\partial_2 \rfloor \\ & \overset{\text{(2CM 3)}}{=} & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot \lfloor {n_1'}^{n_2\partial_2}, \lfloor n_1, n_2\partial_2 \rfloor \partial_2 \rfloor \cdot \lfloor n_1', n_2\partial_2 \rfloor \\ & \overset{\text{(2CM 9)}}{=} & n_2 \cdot \lfloor n_1 \cdot n_1', n_2\partial_2 \rfloor \\ & = & n_2^{n_1 \cdot n_1'}. \end{array}$$

Finally, we calculate

$$\begin{array}{rl}
 & (n_2 \cdot n_2')^{n_1} \\
= & n_2 \cdot n_2' \cdot \lfloor n_1, (n_2 \cdot n_2')\partial_2 \rfloor \\
= & n_2 \cdot n_2' \cdot \lfloor n_1, n_2\partial_2 \cdot n_2'\partial_2 \rfloor \\
\overset{\text{(2CM 8)}}{=} & n_2 \cdot n_2' \cdot \lfloor n_1^{n_2\partial_2}, n_2'\partial_2 \rfloor \cdot \lfloor n_1, n_2\partial_2 \rfloor^{n_2'\partial_2\partial_1} \\
= & n_2 \cdot n_2' \cdot \lfloor n_1^{n_2\partial_2\partial_1} \cdot (n_1^-)^{n_2\partial_2\partial_1} \cdot n_1^{n_2\partial_2}, n_2'\partial_2 \rfloor \cdot \lfloor n_1, n_2\partial_2 \rfloor^{n_2'\partial_2\partial_1} \\
\overset{\text{(2CM 3)}}{=} & n_2 \cdot n_2' \cdot \lfloor n_1^{n_2\partial_2\partial_1} \cdot (\lfloor n_1, n_2\partial_2 \rfloor \partial_2)^-, n_2'\partial_2 \rfloor \cdot \lfloor n_1, n_2\partial_2 \rfloor^{n_2'\partial_2\partial_1} \\
\overset{\text{(2CM 5)}}{=} & n_2 \cdot n_2' \cdot \lfloor n_1 \cdot (\lfloor n_1, n_2\partial_2 \rfloor \partial_2)^-, n_2'\partial_2 \rfloor \cdot \lfloor n_1, n_2\partial_2 \rfloor \\
\overset{\text{(2CM 9)}}{=} & n_2 \cdot n_2' \cdot \lfloor n_1, n_2'\partial_2 \rfloor \cdot \lfloor ((\lfloor n_1, n_2\partial_2 \rfloor \partial_2)^-)^{n_2'\partial_2}, \lfloor n_1, n_2'\partial_2 \rfloor \partial_2 \rfloor \cdot \lfloor (\lfloor n_1, n_2\partial_2 \rfloor \partial_2)^-, n_2'\partial_2 \rfloor \cdot \lfloor n_1, n_2\partial_2 \rfloor \\
= & n_2 \cdot n_2' \cdot \lfloor n_1, n_2'\partial_2 \rfloor \cdot \lfloor ((\lfloor n_1, n_2\partial_2 \rfloor)^-)^{n_2'}\partial_2, \lfloor n_1, n_2'\partial_2 \rfloor \partial_2 \rfloor \cdot \lfloor \lfloor n_1, n_2\partial_2 \rfloor^- \partial_2, n_2'\partial_2 \rfloor \cdot \lfloor n_1, n_2\partial_2 \rfloor \\
\overset{\text{(2CM 1)}}{=} & n_2 \cdot n_2' \cdot \lfloor n_1, n_2'\partial_2 \rfloor \cdot [\lfloor n_1, n_2'\partial_2 \rfloor, ((\lfloor n_1, n_2\partial_2 \rfloor)^-)^{n_2'}] \cdot [n_2', \lfloor n_1, n_2\partial_2 \rfloor^-] \cdot \lfloor n_1, n_2\partial_2 \rfloor \\
= & n_2 \cdot n_2' \cdot \lfloor n_1, n_2'\partial_2 \rfloor \cdot \lfloor n_1, n_2'\partial_2 \rfloor^- \cdot \lfloor n_1, n_2\partial_2 \rfloor^{n_2'} \cdot \lfloor n_1, n_2'\partial_2 \rfloor \cdot ((\lfloor n_1, n_2\partial_2 \rfloor)^-)^{n_2'} \cdot n_2'^- \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot n_2' \\
 & \cdot \lfloor n_1, n_2\partial_2 \rfloor^- \cdot \lfloor n_1, n_2\partial_2 \rfloor \\
= & n_2 \cdot n_2' \cdot \lfloor n_1, n_2\partial_2 \rfloor^{n_2'} \cdot \lfloor n_1, n_2'\partial_2 \rfloor \cdot ((\lfloor n_1, n_2\partial_2 \rfloor)^-)^{n_2'} \cdot \lfloor n_1, n_2\partial_2 \rfloor^{n_2'} \\
= & n_2 \cdot n_2' \cdot \lfloor n_1, n_2\partial_2 \rfloor^{n_2'} \cdot \lfloor n_1, n_2'\partial_2 \rfloor \\
= & n_2 \cdot n_2' \cdot n_2'^- \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot n_2' \cdot \lfloor n_1, n_2'\partial_2 \rfloor \\
= & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \cdot n_2' \cdot \lfloor n_1, n_2'\partial_2 \rfloor \\
= & n_2^{n_1} \cdot n_2'^{n_1}.
\end{array}$$

□

Remark 56 We have

$$n_2^{n_2'\partial_2} = n_2^{n_2'}$$

for n_2, $n_2' \in N_2$.

Proof. We obtain

$$\begin{array}{rcl}
n_2^{n_2'\partial_2} & \overset{\text{Lem. 55}}{=} & n_2 \cdot \lfloor n_2'\partial_2, n_2\partial_2 \rfloor \\
 & \overset{\text{(2CM 1)}}{=} & n_2 \cdot [n_2, n_2'] \\
 & = & n_2 \cdot n_2^- \cdot n_2'^- \cdot n_2 \cdot n_2' \\
 & = & n_2^{n_2'}.
\end{array}$$

□

Remark 57 We have

$$n_2^{n_1}\partial_2 = (n_2\partial_2)^{n_1}$$

for $n_1 \in N_1$ and $n_2 \in N_2$.

Proof. We obtain

$$\begin{array}{rcl}
n_2^{n_1}\partial_2 & \overset{\text{Lem. 55}}{=} & (n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor)\partial_2 \\
 & = & n_2\partial_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \partial_2 \\
 & \overset{\text{(2CM 3)}}{=} & n_2\partial_2 \cdot (n_1^-)^{n_2\partial_2} \cdot n_1^{n_2\partial_2\partial_1} \\
 & \overset{\text{(2CM 5)}}{=} & n_2\partial_2 \cdot (n_1^-)^{n_2\partial_2} \cdot n_1 \\
 & = & n_2\partial_2 \cdot (n_2\partial_2)^- \cdot n_1^- \cdot n_2\partial_2 \cdot n_1 \\
 & = & n_1^- \cdot n_2\partial_2 \cdot n_1 \\
 & = & (n_2\partial_2)^{n_1}.
\end{array}$$

□

Remark 58 We have

$$\begin{aligned} n_2^{n_1} &= \lfloor n_1, n_2^- \partial_2 \rfloor^- \cdot n_2 \\ &= \lfloor n_2^- \partial_2, n_1 \rfloor \cdot n_2^{n_1 \partial_1} \end{aligned}$$

for $n_1 \in N_1$ and $n_2 \in N_2$.

Proof. We obtain

$$\begin{aligned} n_2^{n_1} &\overset{\text{Lem. 55}}{=} ((n_2^-)^{n_1})^- \\ &\overset{\text{Lem. 55}}{=} (n_2^- \cdot \lfloor n_1, n_2^- \partial_2 \rfloor)^- \\ &= \lfloor n_1, n_2^- \partial_2 \rfloor^- \cdot n_2 \\ &= \lfloor n_1, n_2^- \partial_2 \rfloor^- \cdot n_2 \cdot (n_2^-)^{n_1 \partial_1} \cdot n_2^{n_1 \partial_1} \\ &\overset{\text{(2CM 2)}}{=} \lfloor n_1, n_2^- \partial_2 \rfloor^- \cdot \lfloor n_1, n_2^- \partial_2 \rfloor \cdot \lfloor n_2^- \partial_2, n_1 \rfloor \cdot n_2^{n_1 \partial_1} \\ &= \lfloor n_2^- \partial_2, n_1 \rfloor \cdot n_2^{n_1 \partial_1}. \end{aligned}$$

□

Lemma 59 We have the following group morphism.

$$\begin{array}{rcl} N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\varepsilon_{1,2}} & \operatorname{Aut}(N_1 \ltimes_{\varepsilon_2} N_2) \\ (n_0, n_1) & \longmapsto & ((\check{n}_1, n_2) \mapsto (\check{n}_1, n_2)^{(n_0, n_1)} := ((\check{n}_1^{n_0})^{n_1}, \lfloor \check{n}_1^{n_0}, n_1 \rfloor \cdot n_2^{n_0 \cdot n_1 \partial_1})) \end{array}$$

Proof. We use Remark 6.

Suppose given n_0, $n_0' \in N_0$, $\check{n}_1$, n_1, $n_1' \in N_1$ and n_2, $n_2' \in N_2$.

In order to facilitate calculations, we write $a := n_0$, $b := n_0'$, $c := \check{n}_1$, $d := n_1$, $e := n_1'$, $f := n_2$ and $g := n_2'$.

Then a, $b \in N_0$, c, d, $e \in N_1$ and f, $g \in N_2$.

We have

$$\begin{aligned} (n_1, n_2)^1 &= (n_1, n_2)^{(1,1)} \\ &= ((n_1^1)^1, \lfloor n_1^1, 1 \rfloor \cdot n_2^{1 \cdot 1 \partial_1}) \\ &\overset{\text{Rem. 37}}{=} (n_1, n_2). \end{aligned}$$

We have to show that

$$((\check{n}_1, n_2)^{(n_0, n_1)})^{(n_0', n_1')} \overset{!}{=} (\check{n}_1, n_2)^{(n_0, n_1) \cdot (n_0', n_1')}.$$

So we have to show that

$$((c, f)^{(a,d)})^{(b,e)} \overset{!}{=} (c, f)^{(a,d) \cdot (b,e)}.$$

We obtain

$$\begin{aligned} ((c, f)^{(a,d)})^{(b,e)} &= ((c^a)^d, \lfloor c^a, d \rfloor \cdot f^{a \cdot d\partial_1})^{(b,e)} \\ &= ((((c^a)^d)^b)^e, \lfloor ((c^a)^d)^b, e \rfloor \cdot (\lfloor c^a, d \rfloor \cdot f^{a \cdot d\partial_1})^{b \cdot e\partial_1}) \\ &= ((((c^a)^d)^b)^e, \lfloor ((c^a)^d)^b, e \rfloor \cdot \lfloor c^a, d \rfloor^{b \cdot e\partial_1} \cdot f^{a \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &= (((d^- \cdot c^a \cdot d)^b)^e, \lfloor (d^- \cdot c^a \cdot d)^b, e \rfloor \cdot \lfloor c^a, d \rfloor^{b \cdot e\partial_1} \cdot f^{a \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &= (e^- \cdot (d^- \cdot c^a \cdot d)^b \cdot e, \lfloor (d^- \cdot c^a \cdot d)^b, e \rfloor \cdot \lfloor c^a, d \rfloor^{b \cdot e\partial_1} \cdot f^{a \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &= (e^- \cdot (d^b)^- \cdot c^{a \cdot b} \cdot d^b \cdot e, \lfloor (d^b)^- \cdot c^{a \cdot b} \cdot d^b, e \rfloor \cdot \lfloor c^a, d \rfloor^{b \cdot e\partial_1} \cdot f^{a \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &= ((c^{a \cdot b})^{d^b \cdot e}, \lfloor (c^{a \cdot b})^{d^b}, e \rfloor \cdot \lfloor c^a, d \rfloor^{b \cdot e\partial_1} \cdot f^{a \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &= ((c^{a \cdot b})^{d^b \cdot e}, \lfloor (c^{a \cdot b})^{d^b}, e \rfloor \cdot (\lfloor c^a, d \rfloor^b)^{e\partial_1} \cdot f^{a \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &\overset{\text{(2CM 4)}}{=} ((c^{a \cdot b})^{d^b \cdot e}, \lfloor (c^{a \cdot b})^{d^b}, e \rfloor \cdot \lfloor c^{a \cdot b}, d^b \rfloor^{e\partial_1} \cdot f^{a \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &\overset{\text{(2CM 8)}}{=} ((c^{a \cdot b})^{d^b \cdot e}, \lfloor c^{a \cdot b}, d^b \cdot e \rfloor \cdot f^{a \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &= ((c^{a \cdot b})^{d^b \cdot e}, \lfloor c^{a \cdot b}, d^b \cdot e \rfloor \cdot f^{a \cdot b \cdot b^- \cdot d\partial_1 \cdot b \cdot e\partial_1}) \\ &= ((c^{a \cdot b})^{d^b \cdot e}, \lfloor c^{a \cdot b}, d^b \cdot e \rfloor \cdot f^{a \cdot b \cdot (d\partial_1)^b \cdot e\partial_1}) \\ &\overset{\text{(2CM 6)}}{=} ((c^{a \cdot b})^{d^b \cdot e}, \lfloor c^{a \cdot b}, d^b \cdot e \rfloor \cdot f^{a \cdot b \cdot d^b\partial_1 \cdot e\partial_1}) \end{aligned}$$

$$\begin{aligned}
&= ((c^{a\cdot b})^{d^b\cdot e}, \lfloor c^{a\cdot b}, d^b\cdot e\rfloor \cdot f^{a\cdot b\cdot (d^b\cdot e)\partial_1})\\
&= (c,f)^{(a\cdot b, d^b\cdot e)}\\
&= (c,f)^{(a,d)\cdot(b,e)}.
\end{aligned}$$

Moreover, we have to show that

$$((\check{n}_1, n_2)\cdot(n_1', n_2'))^{(n_0,n_1)} \overset{!}{=} (\check{n}_1, n_2)^{(n_0,n_1)} \cdot (n_1', n_2')^{(n_0,n_1)}.$$

So we have to show that

$$((c,f)\cdot(e,g))^{(a,d)} \overset{!}{=} (c,f)^{(a,d)}\cdot(e,g)^{(a,d)}.$$

We obtain

$$\begin{aligned}
& ((c,f)\cdot(e,g))^{(a,d)}\\
= \;& (c\cdot e, f^e\cdot g)^{(a,d)}\\
= \;& (((c\cdot e)^a)^d, \lfloor (c\cdot e)^a, d\rfloor \cdot (f^e\cdot g)^{a\cdot d\partial_1})\\
\overset{\text{Lem. 55}}{=} \;& (((c\cdot e)^a)^d, \lfloor (c\cdot e)^a, d\rfloor \cdot (f\cdot \lfloor e, f\partial_2\rfloor \cdot g)^{a\cdot d\partial_1})\\
= \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a\cdot e^a, d\rfloor \cdot f^{a\cdot d\partial_1}\cdot \lfloor e, f\partial_2\rfloor^{a\cdot d\partial_1}\cdot g^{a\cdot d\partial_1})\\
\overset{\text{(2CM 9)}}{=} \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot f^{a\cdot d\partial_1}\cdot\lfloor e, f\partial_2\rfloor^{a\cdot d\partial_1}\cdot g^{a\cdot d\partial_1})\\
\overset{\text{(2CM 4)}}{=} \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot f^{a\cdot d\partial_1}\cdot\lfloor e^{a\cdot d\partial_1}, (f\partial_2)^{a\cdot d\partial_1}\rfloor\cdot g^{a\cdot d\partial_1})\\
\overset{\text{(2CM 7)}}{=} \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot f^{a\cdot d\partial_1}\cdot\lfloor e^{a\cdot d\partial_1}, f^{a\cdot d\partial_1}\partial_2\rfloor\cdot g^{a\cdot d\partial_1})\\
\overset{\text{Lem. 55}}{=} \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot (f^{a\cdot d\partial_1})^{e^{a\cdot d\partial_1}}\cdot g^{a\cdot d\partial_1})\\
= \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot (f^{a\cdot d\partial_1})^{(e^a)^d\cdot((e^a)^-)^d\cdot(e^a)^{d\partial_1}}\cdot g^{a\cdot d\partial_1})\\
\overset{\text{(2CM 3)}}{=} \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot (f^{a\cdot d\partial_1})^{(e^a)^d\cdot\lfloor e^a,d\rfloor\partial_2}\cdot g^{a\cdot d\partial_1})\\
= \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot ((f^{a\cdot d\partial_1})^{(e^a)^d})^{\lfloor e^a,d\rfloor\partial_2}\cdot g^{a\cdot d\partial_1})\\
\overset{\text{Rem. 56}}{=} \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot ((f^{a\cdot d\partial_1})^{(e^a)^d})^{\lfloor e^a,d\rfloor}\cdot g^{a\cdot d\partial_1})\\
= \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot\lfloor e^a, d\rfloor\cdot\lfloor e^a, d\rfloor^-\cdot (f^{a\cdot d\partial_1})^{(e^a)^d}\cdot\lfloor e^a, d\rfloor\cdot g^{a\cdot d\partial_1})\\
= \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor\cdot\lfloor (e^a)^d, \lfloor c^a,d\rfloor\partial_2\rfloor\cdot (f^{a\cdot d\partial_1})^{(e^a)^d}\cdot\lfloor e^a, d\rfloor\cdot g^{a\cdot d\partial_1})\\
\overset{\text{Lem. 55}}{=} \;& ((c^a)^d\cdot(e^a)^d, \lfloor c^a, d\rfloor^{(e^a)^d}\cdot (f^{a\cdot d\partial_1})^{(e^a)^d}\cdot\lfloor e^a, d\rfloor\cdot g^{a\cdot d\partial_1})\\
= \;& ((c^a)^d\cdot(e^a)^d, (\lfloor c^a, d\rfloor\cdot f^{a\cdot d\partial_1})^{(e^a)^d}\cdot\lfloor e^a, d\rfloor\cdot g^{a\cdot d\partial_1})\\
= \;& ((c^a)^d, \lfloor c^a, d\rfloor\cdot f^{a\cdot d\partial_1})\cdot((e^a)^d, \lfloor e^a, d\rfloor\cdot g^{a\cdot d\partial_1})\\
= \;& (c,f)^{(a,d)}\cdot(e,g)^{(a,d)}.
\end{aligned}$$

□

Remark 60 We have the following group morphism.

$$\begin{array}{ccc}
N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\mathrm{d}_0^{N\,\mathrm{Rec},1}} & N_0\\
(n_0, n_1) & \longmapsto & n_0\cdot n_1\partial_1
\end{array}$$

Proof. Suppose given n_0, $n_0' \in N_0$ and n_1, $n_1' \in N_1$.

Then we have

$$\begin{aligned}
(n_0,n_1)\,\mathrm{d}_0^{N\,\mathrm{Rec},1}\cdot(n_0',n_1')\,\mathrm{d}_0^{N\,\mathrm{Rec},1} &= n_0\cdot n_1\partial_1\cdot n_0'\cdot n_1'\partial_1\\
&= n_0\cdot n_0'\cdot n_0'^{-}\cdot n_1\partial_1\cdot n_0'\cdot n_1'\partial_1\\
&= n_0\cdot n_0'\cdot (n_1\partial_1)^{n_0'}\cdot n_1'\partial_1\\
&\overset{\text{(2CM 6)}}{=} n_0\cdot n_0'\cdot n_1^{n_0'}\partial_1\cdot n_1'\partial_1\\
&= n_0\cdot n_0'\cdot (n_1^{n_0'}\cdot n_1')\partial_1\\
&= (n_0\cdot n_0', n_1^{n_0'}\cdot n_1')\,\mathrm{d}_0^{N\,\mathrm{Rec},1}\\
&= ((n_0,n_1)\cdot(n_0',n_1'))\,\mathrm{d}_0^{N\,\mathrm{Rec},1}.
\end{aligned}$$

□

Remark 61 We have the following group morphism.

$$\begin{array}{rcl} N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\mathrm{s}_0^{N\,\mathrm{Rec},1}} & (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) \\ (n_0, n_1) & \longmapsto & ((n_0, 1), (n_1, 1)) \end{array}$$

Proof. Suppose given n_0, $n_0' \in N_0$ and n_1, $n_1' \in N_1$.

Then we have

$$\begin{array}{rcl} (n_0, n_1)\,\mathrm{s}_0^{N\,\mathrm{Rec},1} \cdot (n_0', n_1')\,\mathrm{s}_0^{N\,\mathrm{Rec},1} & = & ((n_0, 1), (n_1, 1)) \cdot ((n_0', 1), (n_1', 1)) \\ & = & ((n_0, 1) \cdot (n_0', 1), (n_1, 1)^{(n_0', 1)} \cdot (n_1', 1)) \\ & \overset{\text{Lem. 59}}{=} & ((n_0, 1) \cdot (n_0', 1), (n_1^{n_0'}, \lfloor n_1^{n_0'}, 1\rfloor) \cdot (n_1', 1)) \\ & \overset{\text{Rem. 37}}{=} & ((n_0, 1) \cdot (n_0', 1), (n_1^{n_0'}, 1) \cdot (n_1', 1)) \\ & = & ((n_0 \cdot n_0', 1), (n_1^{n_0'} \cdot n_1', 1)) \\ & = & (n_0 \cdot n_0', n_1^{n_0'} \cdot n_1')\,\mathrm{s}_0^{N\,\mathrm{Rec},1} \\ & = & ((n_0, n_1) \cdot (n_0', n_1'))\,\mathrm{s}_0^{N\,\mathrm{Rec},1}\,. \end{array}$$

□

Remark 62 We have the following group morphism.

$$\begin{array}{rcl} (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) & \xrightarrow{\mathrm{d}_0^{N\,\mathrm{Rec},2}} & N_0 \ltimes_{\beta_1} N_1 \\ ((n_0, n_1), (\check{n}_1, n_2)) & \longmapsto & (n_0 \cdot n_1\partial_1, \check{n}_1 \cdot n_2\partial_2) \end{array}$$

Proof. Suppose given n_0, $n_0' \in N_0$, n_1, n_1', $\check{n}_1$, $\check{n}_1' \in N_1$ and n_2, $n_2' \in N_2$.

We have to show that

$$((n_0, n_1), (\check{n}_1, n_2))\,\mathrm{d}_0^{N\,\mathrm{Rec},2} \cdot ((n_0', n_1'), (\check{n}_1', n_2'))\,\mathrm{d}_0^{N\,\mathrm{Rec},2} \overset{!}{=} (((n_0, n_1), (\check{n}_1, n_2)) \cdot ((n_0', n_1'), (\check{n}_1', n_2')))\,\mathrm{d}_0^{N\,\mathrm{Rec},2}\,.$$

In order to calculate, we write $a := n_0$, $b := n_0'$, $c := n_1$, $d := n_1'$, $e := \check{n}_1$, $f := \check{n}_1'$, $g := n_2$ and $h := n_2'$.

Then a, $b \in N_0$, c, d, e, $f \in N_1$ and g, $h \in N_2$.

We have to show that

$$((a, c), (e, g))\,\mathrm{d}_0^{N\,\mathrm{Rec},2} \cdot ((b, d), (f, h))\,\mathrm{d}_0^{N\,\mathrm{Rec},2} \overset{!}{=} (((a, c), (e, g)) \cdot ((b, d), (f, h)))\,\mathrm{d}_0^{N\,\mathrm{Rec},2}\,.$$

We obtain

$$\begin{array}{cl} & ((a, c), (e, g))\,\mathrm{d}_0^{N\,\mathrm{Rec},2} \cdot ((b, d), (f, h))\,\mathrm{d}_0^{N\,\mathrm{Rec},2} \\ = & (a \cdot c\partial_1, e \cdot g\partial_2) \cdot (b \cdot d\partial_1, f \cdot h\partial_2) \\ = & (a \cdot c\partial_1 \cdot b \cdot d\partial_1, (e \cdot g\partial_2)^{b \cdot d\partial_1} \cdot f \cdot h\partial_2) \\ = & (a \cdot c\partial_1 \cdot b \cdot d\partial_1, e^{b \cdot d\partial_1} \cdot (g\partial_2)^{b \cdot d\partial_1} \cdot f \cdot h\partial_2) \\ \overset{\text{(2CM 7)}}{=} & (a \cdot c\partial_1 \cdot b \cdot d\partial_1, e^{b \cdot d\partial_1} \cdot g^{b \cdot d\partial_1}\partial_2 \cdot f \cdot h\partial_2) \\ = & (a \cdot b \cdot b^- \cdot c\partial_1 \cdot b \cdot d\partial_1, (e^b)^d \cdot ((e^b)^-)^d \cdot e^{b \cdot d\partial_1} \cdot g^{b \cdot d\partial_1}\partial_2 \cdot f \cdot h\partial_2) \\ = & (a \cdot b \cdot (c\partial_1)^b \cdot d\partial_1, (e^b)^d \cdot ((e^b)^-)^d \cdot e^{b \cdot d\partial_1} \cdot g^{b \cdot d\partial_1}\partial_2 \cdot f \cdot h\partial_2) \\ \overset{\text{(2CM 6)}}{=} & (a \cdot b \cdot c^b\partial_1 \cdot d\partial_1, (e^b)^d \cdot ((e^b)^-)^d \cdot (e^b)^{d\partial_1} \cdot g^{b \cdot d\partial_1}\partial_2 \cdot f \cdot h\partial_2) \\ \overset{\text{(2CM 3)}}{=} & (a \cdot b \cdot c^b\partial_1 \cdot d\partial_1, (e^b)^d \cdot \lfloor e^b, d\rfloor\partial_2 \cdot g^{b \cdot d\partial_1}\partial_2 \cdot f \cdot h\partial_2) \\ = & (a \cdot b \cdot c^b\partial_1 \cdot d\partial_1, (e^b)^d \cdot f \cdot \lfloor e^b, d\rfloor\partial_2 \cdot g^{b \cdot d\partial_1}\partial_2 \cdot (g^{b \cdot d\partial_1}\partial_2)^- \cdot (\lfloor e^b, d\rfloor\partial_2)^- \cdot f^- \\ & \cdot \lfloor e^b, d\rfloor\partial_2 \cdot g^{b \cdot d\partial_1}\partial_2 \cdot f \cdot h\partial_2) \\ = & (a \cdot b \cdot c^b\partial_1 \cdot d\partial_1, (e^b)^d \cdot f \cdot \lfloor e^b, d\rfloor\partial_2 \cdot g^{b \cdot d\partial_1}\partial_2 \cdot (f^-)^{\lfloor e^b, d\rfloor\partial_2 \cdot g^{b \cdot d\partial_1}\partial_2} \cdot f \cdot h\partial_2) \end{array}$$

$$\begin{aligned}
&= && (a\cdot b\cdot (c^b\cdot d)\partial_1, (e^b)^d\cdot f\cdot \lfloor e^b, d\rfloor\partial_2\cdot g^{b\,\cdot\, d\partial_1}\partial_2\cdot (f^-)^{(\lfloor e^b,d\rfloor\cdot\, g^{b\,\cdot\, d\partial_1})\partial_2}\cdot f\cdot h\partial_2)\\
&\overset{\text{(2CM 5)}}{=} && (a\cdot b\cdot (c^b\cdot d)\partial_1, (e^b)^d\cdot f\cdot \lfloor e^b, d\rfloor\partial_2\cdot g^{b\,\cdot\, d\partial_1}\partial_2\cdot (f^-)^{(\lfloor e^b,d\rfloor\cdot\, g^{b\,\cdot\, d\partial_1})\partial_2}\cdot f^{(\lfloor e^b,d\rfloor\cdot\, g^{b\,\cdot\, d\partial_1})\partial_2\partial_1}\cdot h\partial_2)\\
&\overset{\text{(2CM 3)}}{=} && (a\cdot b\cdot (c^b\cdot d)\partial_1, (e^b)^d\cdot f\cdot \lfloor e^b, d\rfloor\partial_2\cdot g^{b\,\cdot\, d\partial_1}\partial_2\cdot \lfloor f, (\lfloor e^b, d\rfloor\cdot g^{b\,\cdot\, d\partial_1})\partial_2\rfloor\partial_2\cdot h\partial_2)\\
&= && (a\cdot b\cdot (c^b\cdot d)\partial_1, (e^b)^d\cdot f\cdot (\lfloor e^b, d\rfloor\cdot g^{b\,\cdot\, d\partial_1}\cdot \lfloor f, (\lfloor e^b, d\rfloor\cdot g^{b\,\cdot\, d\partial_1})\partial_2\rfloor)\partial_2\cdot h\partial_2)\\
&\overset{\text{Lem. 55}}{=} && (a\cdot b\cdot (c^b\cdot d)\partial_1, (e^b)^d\cdot f\cdot (\lfloor e^b, d\rfloor\cdot g^{b\,\cdot\, d\partial_1})^f\partial_2\cdot h\partial_2)\\
&= && (a\cdot b\cdot (c^b\cdot d)\partial_1, (e^b)^d\cdot f\cdot ((\lfloor e^b, d\rfloor\cdot g^{b\,\cdot\, d\partial_1})^f\cdot h)\partial_2)\\
&= && ((a\cdot b, c^b\cdot d), ((e^b)^d\cdot f, (\lfloor e^b, d\rfloor\cdot g^{b\,\cdot\, d\partial_1})^f\cdot h)\,\mathrm{d}_0^{N\,\mathrm{Rec},2}\\
&= && ((a,c)\cdot (b,d), ((e^b)^d, \lfloor e^b, d\rfloor\cdot g^{b\,\cdot\, d\partial_1})\cdot (f,h))\,\mathrm{d}_0^{N\,\mathrm{Rec},2}\\
&\overset{\text{Lem. 59}}{=} && ((a,c)\cdot (b,d), (e,g)^{(b,d)}\cdot (f,h))\,\mathrm{d}_0^{N\,\mathrm{Rec},2}\\
&= && (((a,c),(e,g))\cdot ((b,d),(f,h)))\,\mathrm{d}_0^{N\,\mathrm{Rec},2}\,.
\end{aligned}$$

□

Remark 63 We have the following group morphism.

$$\begin{array}{rcl}
(N_0 \ltimes_{\beta_1} N_1)\ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) & \xrightarrow{\mathrm{d}_1^{N\,\mathrm{Rec},2}} & N_0 \ltimes_{\beta_1} N_1\\
((n_0,n_1),(\check n_1,n_2)) & \longmapsto & (n_0, n_1\cdot \check n_1)
\end{array}$$

Proof. Suppose given n_0, $n_0' \in N_0$, n_1, n_1', $\check n_1$, $\check n_1' \in N_1$ and n_2, $n_2' \in N_2$.

We have to show that

$$((n_0,n_1),(\check n_1,n_2))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\cdot ((n_0',n_1'),(\check n_1',n_2'))\,\mathrm{d}_1^{N\,\mathrm{Rec},2} \overset{!}{=} (((n_0,n_1),(\check n_1,n_2))\cdot ((n_0',n_1'),(\check n_1',n_2')))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\,.$$

In order to calculate, we write $a := n_0$, $b := n_0'$, $c := n_1$, $d := n_1'$, $e := \check n_1$, $f := \check n_1'$, $g := n_2$ and $h := n_2'$.

Then a, $b \in N_0$, c, d, e, $f \in N_1$ and g, $h \in N_2$.

We have to show that

$$((a,c),(e,g))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\cdot ((b,d),(f,h))\,\mathrm{d}_1^{N\,\mathrm{Rec},2} \overset{!}{=} (((a,c),(e,g))\cdot ((b,d),(f,h)))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\,.$$

We obtain

$$\begin{aligned}
& && ((a,c),(e,g))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\cdot ((b,d),(f,h))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\\
&= && (a, c\cdot e)\cdot (b, d\cdot f)\\
&= && (a\cdot b, (c\cdot e)^b\cdot d\cdot f)\\
&= && (a\cdot b, c^b\cdot e^b\cdot d\cdot f)\\
&= && (a\cdot b, c^b\cdot d\cdot d^-\cdot e^b\cdot d\cdot f)\\
&= && (a\cdot b, c^b\cdot d\cdot (e^b)^d\cdot f)\\
&= && ((a\cdot b, c^b\cdot d), ((e^b)^d\cdot f, (\lfloor e^b, d\rfloor\cdot g^{b\cdot d\partial_1})^f\cdot h)\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\\
&= && ((a,c)\cdot (b,d), ((e^b)^d, \lfloor e^b, d\rfloor\cdot g^{b\cdot d\partial_1})\cdot (f,h))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\\
&\overset{\text{Lem. 59}}{=} && ((a,c)\cdot (b,d), (e,g)^{(b,d)}\cdot (f,h))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\\
&= && (((a,c),(e,g))\cdot ((b,d),(f,h)))\,\mathrm{d}_1^{N\,\mathrm{Rec},2}\,.
\end{aligned}$$

□

Lemma 64 Consider the groups $(N_0 \ltimes_{\beta_1} N_1)\ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2)$, $N_0 \ltimes_{\beta_1} N_1$ and N_0; cf. Lemma 55, Lemma 59 and Definition 36.

With the help of Remark 7 we have the group morphism

$$N_0 \xrightarrow{\mathrm{s}_0^{N\,\mathrm{Rec},0}} N_0 \ltimes_{\varepsilon_2} N_1 \;:\; n_0 \longmapsto (n_0, 1).$$

With the help of Remark 60 and Remark 7 we have the group morphisms

$$\begin{array}{lclclcl} N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\mathrm{d}_0^{N\,\mathrm{Rec},1}} & N_0 & : & (n_0, n_1) & \longmapsto & n_0 \cdot n_1 \partial_1 \\ N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\mathrm{d}_1^{N\,\mathrm{Rec},1}} & N_0 & : & (n_0, n_1) & \longmapsto & n_0 . \end{array}$$

With the help of Remark 61 and Remark 7 we have the group morphisms

$$\begin{array}{lclclcl} N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\mathrm{s}_0^{N\,\mathrm{Rec},1}} & (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) & : & (n_0, n_1) & \longmapsto & ((n_0, 1), (n_1, 1)) \\ N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\mathrm{s}_1^{N\,\mathrm{Rec},1}} & (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) & : & (n_0, n_1) & \longmapsto & ((n_0, n_1), (1, 1)). \end{array}$$

With the help of Remark 62, Remark 63 and Remark 7 we have the group morphisms

$$\begin{array}{lclclcl} (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) & \xrightarrow{\mathrm{d}_0^{N\,\mathrm{Rec},2}} & N_0 \ltimes_{\beta_1} N_1 & : & ((n_0, n_1), (\check{n}_1, n_2)) & \longmapsto & (n_0 \cdot n_1 \partial_1, \check{n}_1 \cdot n_2 \partial_2) \\ (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) & \xrightarrow{\mathrm{d}_1^{N\,\mathrm{Rec},2}} & N_0 \ltimes_{\beta_1} N_1 & : & ((n_0, n_1), (\check{n}_1, n_2)) & \longmapsto & (n_0, n_1 \cdot \check{n}_1) \\ (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) & \xrightarrow{\mathrm{d}_2^{N\,\mathrm{Rec},2}} & N_0 \ltimes_{\beta_1} N_1 & : & ((n_0, n_1), (\check{n}_1, n_2)) & \longmapsto & (n_0, n_1). \end{array}$$

Then

$$\begin{array}{rcl} N\,\mathrm{Rec} & := & ((N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2),\, N_0 \ltimes_{\beta_1} N_1,\, N_0,\, \mathrm{d}_0^{N\,\mathrm{Rec},2},\, \mathrm{d}_1^{N\,\mathrm{Rec},2},\, \mathrm{d}_2^{N\,\mathrm{Rec},2},\, \mathrm{s}_0^{N\,\mathrm{Rec},1},\, \mathrm{s}_1^{N\,\mathrm{Rec},1}, \\ & & \mathrm{d}_0^{N\,\mathrm{Rec},1},\, \mathrm{d}_1^{N\,\mathrm{Rec},1},\, \mathrm{s}_0^{N\,\mathrm{Rec},0}) \end{array}$$

is a $\lfloor 2,0 \rfloor$-simplicial group; cf. Definition 30.

$$\begin{tikzcd}[column sep=huge]
(N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2)
\arrow[r, shift left=12, "\mathrm{d}_2^{N\,\mathrm{Rec},2}"]
\arrow[r, "\mathrm{d}_1^{N\,\mathrm{Rec},2}"]
\arrow[r, shift right=12, "\mathrm{d}_0^{N\,\mathrm{Rec},2}"]
& N_0 \ltimes_{\beta_1} N_1
\arrow[l, shift right=6, "\mathrm{s}_1^{N\,\mathrm{Rec},1}"']
\arrow[l, shift left=6, "\mathrm{s}_0^{N\,\mathrm{Rec},1}"]
\arrow[r, shift left=6, "\mathrm{d}_1^{N\,\mathrm{Rec},1}"]
\arrow[r, shift right=6, "\mathrm{d}_0^{N\,\mathrm{Rec},1}"']
& N_0
\arrow[l, "\mathrm{s}_0^{N\,\mathrm{Rec},0}"']
\end{tikzcd}$$

So $N\,\mathrm{Rec}_2 = (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2)$, $N\,\mathrm{Rec}_1 = N_0 \ltimes_{\beta_1} N_1$ and $N\,\mathrm{Rec}_0 = N_0$.

Proof. We verify the conditions $(1, 2)$ of Definition 30.

In order to calculate, we abbreviate $\mathrm{d}_i := \mathrm{d}_i^{N\,\mathrm{Rec},k}$ and $\mathrm{s}_i := \mathrm{s}_i^{N\,\mathrm{Rec},k}$, for all occurring i, k.

Ad (1). Suppose given $n_0 \in N_0$, n_1, $\check{n}_1 \in N_1$ and $n_2 \in N_2$.

We calculate as follows.

$$\begin{array}{rcl} n_0(\mathrm{s}_0 \blacktriangle \mathrm{d}_0) & = & (n_0\, \mathrm{s}_0)\, \mathrm{d}_0 \\ & = & (n_0, 1)\, \mathrm{d}_0 \\ & = & n_0 \end{array}$$

$$\begin{array}{rcl} n_0(\mathrm{s}_0 \blacktriangle \mathrm{d}_1) & = & (n_0\, \mathrm{s}_0)\, \mathrm{d}_1 \\ & = & (n_0, 1)\, \mathrm{d}_1 \\ & = & n_0 \end{array}$$

$$\begin{array}{rcl} (n_0, n_1)(\mathrm{s}_0 \blacktriangle \mathrm{d}_0) & = & ((n_0, n_1)\, \mathrm{s}_0)\, \mathrm{d}_0 \\ & = & ((n_0, 1), (n_1, 1))\, \mathrm{d}_0 \\ & = & (n_0, n_1) \end{array}$$

$$\begin{aligned}
(n_0,n_1)(\mathrm{s}_0 \blacktriangle \mathrm{d}_1) &= ((n_0,n_1)\,\mathrm{s}_0)\,\mathrm{d}_1\\
&= ((n_0,1),(n_1,1))\,\mathrm{d}_1\\
&= (n_0,n_1)
\end{aligned}$$

$$\begin{aligned}
(n_0,n_1)(\mathrm{s}_1 \blacktriangle \mathrm{d}_1) &= ((n_0,n_1)\,\mathrm{s}_1)\,\mathrm{d}_1\\
&= ((n_0,n_1),(1,1))\,\mathrm{d}_1\\
&= (n_0,n_1)
\end{aligned}$$

$$\begin{aligned}
(n_0,n_1)(\mathrm{s}_1 \blacktriangle \mathrm{d}_2) &= ((n_0,n_1)\,\mathrm{s}_1)\,\mathrm{d}_2\\
&= ((n_0,n_1),(1,1))\,\mathrm{d}_2\\
&= (n_0,n_1)
\end{aligned}$$

$$\begin{aligned}
(n_0,n_1)(\mathrm{s}_0 \blacktriangle \mathrm{d}_2) &= ((n_0,n_1)\,\mathrm{s}_0)\,\mathrm{d}_2\\
&= ((n_0,1),(n_1,1))\,\mathrm{d}_2\\
&= (n_0,1)\\
&= n_0\,\mathrm{s}_0\\
&= ((n_0,n_1)\,\mathrm{d}_1)\,\mathrm{s}_0\\
&= (n_0,n_1)(\mathrm{d}_1 \blacktriangle \mathrm{s}_0)
\end{aligned}$$

$$\begin{aligned}
(n_0,n_1)(\mathrm{s}_1 \blacktriangle \mathrm{d}_0) &= ((n_0,n_1)\,\mathrm{s}_1)\,\mathrm{d}_0\\
&= ((n_0,n_1),(1,1))\,\mathrm{d}_0\\
&= (n_0 \cdot n_1\partial_1, 1)\\
&= (n_0 \cdot n_1\partial_1)\,\mathrm{s}_0\\
&= ((n_0,n_1)\,\mathrm{d}_0)\,\mathrm{s}_0\\
&= (n_0,n_1)(\mathrm{d}_0 \blacktriangle \mathrm{s}_0)
\end{aligned}$$

$$\begin{aligned}
((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_1 \blacktriangle \mathrm{d}_0) &= (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_1)\,\mathrm{d}_0\\
&= (n_0, n_1 \cdot \check{n}_1)\,\mathrm{d}_0\\
&= n_0 \cdot (n_1 \cdot \check{n}_1)\partial_1\\
&\overset{(2\text{CM } 5)}{=} n_0 \cdot (n_1 \cdot \check{n}_1)\partial_1 \cdot n_2\partial_2\partial_1\\
&= n_0 \cdot n_1\partial_1 \cdot (\check{n}_1 \cdot n_2\partial_2)\partial_1\\
&= (n_0 \cdot n_1\partial_1, \check{n}_1 \cdot n_2\partial_2)\,\mathrm{d}_0\\
&= (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_0)\,\mathrm{d}_0\\
&= ((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_0 \blacktriangle \mathrm{d}_0)
\end{aligned}$$

$$\begin{aligned}
((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_2 \blacktriangle \mathrm{d}_0) &= (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_2)\,\mathrm{d}_0\\
&= (n_0,n_1)\,\mathrm{d}_0\\
&= n_0 \cdot n_1\partial_1\\
&= (n_0 \cdot n_1\partial_1, \check{n}_1 \cdot n_2\partial_2)\,\mathrm{d}_1\\
&= (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_0)\,\mathrm{d}_1\\
&= ((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_0 \blacktriangle \mathrm{d}_1)
\end{aligned}$$

$$\begin{aligned}
((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_2 \blacktriangle \mathrm{d}_1) &= (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_2)\,\mathrm{d}_1\\
&= (n_0,n_1)\,\mathrm{d}_1\\
&= n_0\\
&= (n_0, n_1 \cdot \check{n}_1)\,\mathrm{d}_1\\
&= (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_1)\,\mathrm{d}_1\\
&= ((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_1 \blacktriangle \mathrm{d}_1)
\end{aligned}$$

$$\begin{aligned}
n_0(\mathrm{s}_0 \blacktriangle \mathrm{s}_0) &= (n_0\,\mathrm{s}_0)\,\mathrm{s}_0\\
&= (n_0,1)\,\mathrm{s}_0\\
&= ((n_0,1),(1,1))\\
&= (n_0,1)\,\mathrm{s}_1\\
&= (n_0\,\mathrm{s}_0)\,\mathrm{s}_1\\
&= n_0(\mathrm{s}_0 \blacktriangle \mathrm{s}_1)
\end{aligned}$$

Altogether, we have

$$\begin{aligned}
\mathrm{s}_0^{N\,\mathrm{Rec},0} \blacktriangle \mathrm{d}_0^{N\,\mathrm{Rec},1} &= \mathrm{id}_{N\,\mathrm{Rec}_0} \\
\mathrm{s}_0^{N\,\mathrm{Rec},0} \blacktriangle \mathrm{d}_1^{N\,\mathrm{Rec},1} &= \mathrm{id}_{N\,\mathrm{Rec}_0} \\
\mathrm{s}_0^{N\,\mathrm{Rec},1} \blacktriangle \mathrm{d}_0^{N\,\mathrm{Rec},2} &= \mathrm{id}_{N\,\mathrm{Rec}_1} \\
\mathrm{s}_0^{N\,\mathrm{Rec},1} \blacktriangle \mathrm{d}_1^{N\,\mathrm{Rec},2} &= \mathrm{id}_{N\,\mathrm{Rec}_1} \\
\mathrm{s}_1^{N\,\mathrm{Rec},1} \blacktriangle \mathrm{d}_1^{N\,\mathrm{Rec},2} &= \mathrm{id}_{N\,\mathrm{Rec}_1} \\
\mathrm{s}_1^{N\,\mathrm{Rec},1} \blacktriangle \mathrm{d}_2^{N\,\mathrm{Rec},2} &= \mathrm{id}_{N\,\mathrm{Rec}_1} \\
\mathrm{s}_0^{N\,\mathrm{Rec},1} \blacktriangle \mathrm{d}_2^{N\,\mathrm{Rec},2} &= \mathrm{d}_1^{N\,\mathrm{Rec},1} \blacktriangle \mathrm{s}_0^{N\,\mathrm{Rec},0} \\
\mathrm{s}_1^{N\,\mathrm{Rec},1} \blacktriangle \mathrm{d}_0^{N\,\mathrm{Rec},2} &= \mathrm{d}_0^{N\,\mathrm{Rec},1} \blacktriangle \mathrm{s}_0^{N\,\mathrm{Rec},0} \\
\mathrm{d}_1^{N\,\mathrm{Rec},2} \blacktriangle \mathrm{d}_0^{N\,\mathrm{Rec},1} &= \mathrm{d}_0^{N\,\mathrm{Rec},2} \blacktriangle \mathrm{d}_0^{N\,\mathrm{Rec},1} \\
\mathrm{d}_2^{N\,\mathrm{Rec},2} \blacktriangle \mathrm{d}_0^{N\,\mathrm{Rec},1} &= \mathrm{d}_0^{N\,\mathrm{Rec},2} \blacktriangle \mathrm{d}_1^{N\,\mathrm{Rec},1} \\
\mathrm{d}_2^{N\,\mathrm{Rec},2} \blacktriangle \mathrm{d}_1^{N\,\mathrm{Rec},1} &= \mathrm{d}_1^{N\,\mathrm{Rec},2} \blacktriangle \mathrm{d}_1^{N\,\mathrm{Rec},1} \\
\mathrm{s}_0^{N\,\mathrm{Rec},0} \blacktriangle \mathrm{s}_0^{N\,\mathrm{Rec},1} &= \mathrm{s}_0^{N\,\mathrm{Rec},0} \blacktriangle \mathrm{s}_1^{N\,\mathrm{Rec},1} .
\end{aligned}$$

Ad (2). We have

$$\begin{aligned}
\ker \mathrm{d}_0^{N\,\mathrm{Rec},2} &= \{((n_1^- \partial_1, n_1), (n_2^- \partial_2, n_2)) : n_1 \in N_1, n_2 \in N_2\} \\
\ker \mathrm{d}_1^{N\,\mathrm{Rec},2} &= \{((1, n_1), (n_1^-, n_2)) : n_1 \in N_1, n_2 \in N_2\} \\
\ker \mathrm{d}_2^{N\,\mathrm{Rec},2} &= \{((1, 1), (n_1, n_2)) : n_1 \in N_1, n_2 \in N_2\} \\
\ker \mathrm{d}_1^{N\,\mathrm{Rec},2} \cap \ker \mathrm{d}_2^{N\,\mathrm{Rec},2} &= \{((1, 1), (1, n_2)) : n_2 \in N_2\} \\
\ker \mathrm{d}_0^{N\,\mathrm{Rec},2} \cap \ker \mathrm{d}_2^{N\,\mathrm{Rec},2} &= \{((1, 1), (n_2^- \partial_2, n_2)) : n_2 \in N_2\} \\
\ker \mathrm{d}_0^{N\,\mathrm{Rec},2} \cap \ker \mathrm{d}_1^{N\,\mathrm{Rec},2} &\overset{(2\mathrm{CM}\ 5)}{=} \{((1, n_2 \partial_2), (n_2^- \partial_2, n_2)) : n_2 \in N_2\}.
\end{aligned}$$

We want to show that

$$[\ker \mathrm{d}_0^{N\,\mathrm{Rec},2},\ \ker \mathrm{d}_1^{N\,\mathrm{Rec},2} \cap \ker \mathrm{d}_2^{N\,\mathrm{Rec},2}] \overset{!}{=} 1.$$

Suppose given $n_1 \in N_1$ and n_2, $n_2' \in N_2$.

We have to show that

$$((n_1^- \partial_1, n_1), (n_2^- \partial_2, n_2)) \cdot ((1, 1), (1, n_2')) \overset{!}{=} ((1, 1), (1, n_2')) \cdot ((n_1^- \partial_1, n_1), (n_2^- \partial_2, n_2)).$$

In order to calculate, we write $a = n_1$, $b = n_2$ and $c = n_2'$.

Then $a \in N_1$ and b, $c \in N_2$.

So we have to show that

$$((a^- \partial_1, a), (b^- \partial_2, b)) \cdot ((1, 1), (1, c)) \overset{!}{=} ((1, 1), (1, c)) \cdot ((a^- \partial_1, a), (b^- \partial_2, b)).$$

We obtain

$$\begin{aligned}
((a^- \partial_1, a), (b^- \partial_2, b)) \cdot ((1, 1), (1, c)) &= ((a^- \partial_1, a), ((b^- \partial_2, b) \cdot (1, c))) \\
&= ((a^- \partial_1, a), (b^- \partial_2, b \cdot c)) \\
&= ((a^- \partial_1, a), (b^- \partial_2, b \cdot c \cdot b^- \cdot b)) \\
&= ((a^- \partial_1, a), (b^- \partial_2, c^{b^-} \cdot b)) \\
&\overset{\text{Rem. 56}}{=} ((a^- \partial_1, a), (b^- \partial_2, c^{b^- \partial_2} \cdot b)) \\
&= ((a^- \partial_1, a), (1, c) \cdot (b^- \partial_2, b)) \\
&\overset{\text{Rem. 37}}{=} ((a^- \partial_1, a), (1, \lfloor 1, a \rfloor \cdot c) \cdot (b^- \partial_2, b)) \\
&= ((a^- \partial_1, a), (1, \lfloor 1, a \rfloor \cdot c^{a^- \partial_1 \cdot a \partial_1}) \cdot (b^- \partial_2, b)) \\
&\overset{\text{Lem. 59}}{=} ((a^- \partial_1, a), (1, c)^{(a^- \partial_1, a)} \cdot (b^- \partial_2, b)) \\
&= ((1, 1), (1, c)) \cdot ((a^- \partial_1, a), (b^- \partial_2, b)).
\end{aligned}$$

We want to show that

$$[\ker \mathrm{d}_1^{N\,\mathrm{Rec},2},\, \ker \mathrm{d}_0^{N\,\mathrm{Rec},2} \cap \ker \mathrm{d}_2^{N\,\mathrm{Rec},2}] \overset{!}{=} 1.$$

Suppose given $n_1 \in N_1$ and n_2, $n_2' \in N_2$.

We have to show that

$$((1,n_1),(n_1^-,n_2)) \cdot ((1,1),(n_2'^-\partial_2, n_2')) \overset{!}{=} ((1,1),(n_2'^-\partial_2, n_2')) \cdot ((1,n_1),(n_1^-,n_2)).$$

In order to calculate, we write $a = n_1$, $b = n_2$ and $c = n_2'$.

Then $a \in N_1$ and b, $c \in N_2$.

So we have to show that

$$((1,a),(a^-,b)) \cdot ((1,1),(c^-\partial_2,c)) \overset{!}{=} ((1,1),(c^-\partial_2,c)) \cdot ((1,a),(a^-,b)).$$

We obtain

$$\begin{array}{cl}
 & ((1,a),(a^-,b)) \cdot ((1,1),(c^-\partial_2,c)) \\
= & ((1,a),(a^-,b) \cdot (c^-\partial_2,c)) \\
= & ((1,a),(a^- \cdot c^-\partial_2, b^{c^-\partial_2} \cdot c)) \\
\overset{\text{Rem. 56}}{=} & ((1,a),(a^- \cdot c^-\partial_2, b^{c^-} \cdot c)) \\
= & ((1,a),(a^- \cdot c^-\partial_2, c \cdot b \cdot c^- \cdot c)) \\
= & ((1,a),(a^- \cdot c^-\partial_2, c \cdot b)) \\
= & ((1,a),(a^- \cdot c^-\partial_2, (c^a)^{a^-} \cdot b)) \\
\overset{\text{Lem. 55}}{=} & ((1,a),(a^- \cdot c^-\partial_2, c^a \cdot \lfloor a^-, c^a\partial_2\rfloor \cdot b)) \\
\overset{\text{Rem. 58}}{=} & ((1,a),(a^- \cdot c^-\partial_2, \lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1} \cdot \lfloor a^-, c^a\partial_2\rfloor \cdot b)) \\
\overset{\text{Rem. 57}}{=} & ((1,a),(a^- \cdot c^-\partial_2, \lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1} \cdot \lfloor a^-, (c\partial_2)^a\rfloor \cdot b)) \\
= & ((1,a),(a^- \cdot c^-\partial_2 \cdot a \cdot a^-, \lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1} \cdot \lfloor a^-, (c\partial_2)^a \cdot (c^-\partial_2)^{a\partial_1} \cdot (c\partial_2)^{a\partial_1}\rfloor \cdot b)) \\
= & ((1,a),((c^-\partial_2)^a \cdot a^-, \lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1} \cdot \lfloor a^-, (c\partial_2)^a \cdot (c^-\partial_2)^{a\partial_1} \cdot (c\partial_2)^{a\partial_1}\rfloor \cdot b)) \\
\overset{\text{(2CM 7)}}{=} & ((1,a),((c^-\partial_2)^a \cdot a^-, \lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1} \cdot \lfloor a^-, (c\partial_2)^a \cdot (c^-\partial_2)^{a\partial_1} \cdot c^{a\partial_1}\partial_2\rfloor \cdot b))
\end{array}$$

$$\begin{array}{cl}
\overset{\text{(2CM 3)}}{=} & ((1,a),((c^-\partial_2)^a \cdot a^-, \lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1} \cdot \lfloor a^-, \lfloor c^-\partial_2, a\rfloor\partial_2 \cdot c^{a\partial_1}\partial_2\rfloor \cdot b)) \\
= & ((1,a),((c^-\partial_2)^a \cdot a^-, \lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1} \cdot \lfloor a^-, (\lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1})\partial_2\rfloor \cdot b)) \\
\overset{\text{Lem. 55}}{=} & ((1,a),((c^-\partial_2)^a \cdot a^-, (\lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1})^{a^-} \cdot b)) \\
= & ((1,a),((c^-\partial_2)^a, \lfloor c^-\partial_2, a\rfloor \cdot c^{a\partial_1}) \cdot (a^-,b)) \\
\overset{\text{Lem. 59}}{=} & ((1,a),(c^-\partial_2,c)^{(1,a)} \cdot (a^-,b)) \\
= & ((1,1),(c^-\partial_2,c)) \cdot ((1,a),(a^-,b)).
\end{array}$$

Finally, we want to show that

$$[\ker \mathrm{d}_2^{N\,\mathrm{Rec},2},\, \ker \mathrm{d}_0^{N\,\mathrm{Rec},2} \cap \ker \mathrm{d}_1^{N\,\mathrm{Rec},2}] \overset{!}{=} 1.$$

Suppose given $n_1 \in N_1$ and n_2, $n_2' \in N_2$.

We have to show that

$$((1,1),(n_1,n_2)) \cdot ((1,n_2'\partial_2),(n_2'^-\partial_2, n_2')) \overset{!}{=} ((1,n_2'\partial_2),(n_2'^-\partial_2, n_2')) \cdot ((1,1),(n_1,n_2)).$$

In order to calculate, we write $a = n_1$, $b = n_2$ and $c = n_2'$.

Then $a \in N_1$ and b, $c \in N_2$.

So we have to show that

$$((1,1),(a,b)) \cdot ((1,c\partial_2),(c^-\partial_2,c)) \overset{!}{=} ((1,c\partial_2),(c^-\partial_2,c)) \cdot ((1,1),(a,b)).$$

We obtain

$$\begin{array}{rcl}
((1,1),(a,b)) \cdot ((1,c\partial_2),(c^-\partial_2,c)) & = & ((1,c\partial_2),(a,b)^{(1,c\partial_2)} \cdot (c^-\partial_2,c)) \\
& \overset{\text{Lem. 59}}{=} & ((1,c\partial_2),(a^{c\partial_2},\lfloor a,c\partial_2 \rfloor \cdot b^{\,c\partial_2\partial_1}) \cdot (c^-\partial_2,c)) \\
& \overset{\text{(2CM 5)}}{=} & ((1,c\partial_2),(a^{c\partial_2},\lfloor a,c\partial_2 \rfloor \cdot b) \cdot (c^-\partial_2,c)) \\
& = & ((1,c\partial_2),(a^{c\partial_2} \cdot c^-\partial_2,(\lfloor a,c\partial_2 \rfloor \cdot b)^{\,c^-\partial_2} \cdot c)) \\
& = & ((1,c\partial_2),(c^-\partial_2 \cdot a \cdot c\partial_2 \cdot c^-\partial_2,(\lfloor a,c\partial_2 \rfloor \cdot b)^{c^-\partial_2} \cdot c)) \\
& = & ((1,c\partial_2),(c^-\partial_2 \cdot a,(\lfloor a,c\partial_2 \rfloor \cdot b)^{c^-\partial_2} \cdot c)) \\
& \overset{\text{Rem. 56}}{=} & ((1,c\partial_2),(c^-\partial_2 \cdot a,(\lfloor a,c\partial_2 \rfloor \cdot b)^{c^-} \cdot c)) \\
& = & ((1,c\partial_2),(c^-\partial_2 \cdot a,c \cdot \lfloor a,c\partial_2 \rfloor \cdot b \cdot c^- \cdot c)) \\
& = & ((1,c\partial_2),(c^-\partial_2 \cdot a,c \cdot \lfloor a,c\partial_2 \rfloor \cdot b)) \\
& \overset{\text{Lem. 55}}{=} & ((1,c\partial_2),(c^-\partial_2 \cdot a,c^{\,a} \cdot b)) \\
& = & ((1,c\partial_2),(c^-\partial_2,c) \cdot (a,b)) \\
& = & ((1,c\partial_2),(c^-\partial_2,c)) \cdot ((1,1),(a,b)).
\end{array}$$

□

4.6.2 The construction for the morphisms

Suppose given 2-crossed modules

$$N = (N_2,\, N_1,\, N_0,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta)$$

and

$$\tilde{N} = (\tilde{N}_2,\, \tilde{N}_1,\, \tilde{N}_0,\, \tilde{\partial}_2,\, \tilde{\partial}_1,\, \tilde{\beta}_1,\, \tilde{\beta}_2,\, \tilde{\zeta});$$

cf. Definition 36.

Recall that we have the group morphisms

$$\begin{array}{rcl}
N_1 & \xrightarrow{\varepsilon_2} & \mathrm{Aut}(N_2) \\
n_1 & \longmapsto & (n_2 \mapsto n_2^{n_1} := n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor)
\end{array}$$

and

$$\begin{array}{rcl}
\tilde{N}_1 & \xrightarrow{\tilde{\varepsilon}_2} & \mathrm{Aut}(\tilde{N}_2) \\
\tilde{n}_1 & \longmapsto & (\tilde{n}_2 \mapsto \tilde{n}_2^{\tilde{n}_1} := \tilde{n}_2 \cdot \lfloor \tilde{n}_1, \tilde{n}_2\tilde{\partial}_2 \rfloor);
\end{array}$$

cf. Lemma 55.

Recall that we have the group morphisms

$$\begin{array}{rcl}
N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\varepsilon_{1,2}} & \mathrm{Aut}(N_1 \ltimes_{\varepsilon_2} N_2) \\
(n_0,n_1) & \longmapsto & ((\check{n}_1,n_2) \mapsto (\check{n}_1,n_2)^{(n_0,n_1)} := ((\check{n}_1^{n_0})^{n_1}, \lfloor \check{n}_1^{n_0}, n_1 \rfloor \cdot n_2^{n_0 \cdot n_1\partial_1}))
\end{array}$$

and

$$\begin{array}{rcl}
\tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1 & \xrightarrow{\tilde{\varepsilon}_{1,2}} & \mathrm{Aut}(\tilde{N}_1 \ltimes_{\tilde{\varepsilon}_2} \tilde{N}_2) \\
(\tilde{n}_0,\tilde{n}_1) & \longmapsto & ((\tilde{\check{n}}_1,\tilde{n}_2) \mapsto (\tilde{\check{n}}_1,\tilde{n}_2)^{(\tilde{n}_0,\tilde{n}_1)} := ((\tilde{\check{n}}_1^{\tilde{n}_0})^{\tilde{n}_1}, \lfloor \tilde{\check{n}}_1^{\tilde{n}_0}, \tilde{n}_1 \rfloor \cdot \tilde{n}_2^{\tilde{n}_0 \cdot \tilde{n}_1\tilde{\partial}_1}));
\end{array}$$

cf. Lemma 59.

Suppose given a morphism

$$\nu = (\nu_2,\, \nu_1,\, \nu_0) : (N_2,\, N_1,\, N_0) \to (\tilde{N}_2,\, \tilde{N}_1,\, \tilde{N}_0)$$

of 2-crossed modules; cf. Definition 39.

Remark 65 We have the following group morphism.

$$\begin{array}{ccc} N_0 & \xrightarrow{\nu\,\mathrm{Rec}_0} & \tilde{N}_0 \\ n_0 & \longmapsto & n_0\nu_0 \end{array}$$

So $\nu\,\mathrm{Rec}_0 := \nu_0$. Cf. Definition 39.

Lemma 66 We have the following group morphism.

$$\begin{array}{ccc} N_0 \ltimes_{\beta_1} N_1 & \xrightarrow{\nu\,\mathrm{Rec}_1} & \tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1 \\ (n_0, n_1) & \longmapsto & (n_0\nu_0, n_1\nu_1) \end{array}$$

Proof. Suppose given n_0, $n_0' \in N_0$ and n_1, $n_1' \in N_1$.

We have

$$\begin{aligned} ((n_0, n_1)\cdot(n_0', n_1'))(\nu\,\mathrm{Rec}_1) &= (n_0\cdot n_0', n_1^{n_0'}\cdot n_1')(\nu\,\mathrm{Rec}_1) \\ &= ((n_0\cdot n_0')\nu_0, (n_1^{n_0'}\cdot n_1')\nu_1) \\ &= (n_0\nu_0\cdot n_0'\nu_0, (n_1^{n_0'})\nu_1\cdot n_1'\nu_1) \\ &\overset{\text{(2CMM 1.1)}}{=} (n_0\nu_0\cdot n_0'\nu_0, (n_1\nu_1)^{n_0'\nu_0}\cdot n_1'\nu_1) \\ &= (n_0\nu_0, n_1\nu_1)\cdot(n_0'\nu_0\cdot n_1'\nu_1) \\ &= (n_0, n_1)(\nu\,\mathrm{Rec}_1)\cdot\ (n_0', n_1')(\nu\,\mathrm{Rec}_1). \end{aligned}$$

□

Remark 67 We have

$$(n_2^{n_1})\nu_2 = (n_2\nu_2)^{n_1\nu_1}$$

for $n_1 \in N_1$ and $n_2 \in N_2$.

Proof. We obtain

$$\begin{aligned} (n_2^{n_1})\nu_2 &\overset{\text{Lem. 55}}{=} (n_2\cdot\lfloor n_1, n_2\partial_2\rfloor)\nu_2 \\ &= n_2\nu_2\cdot\lfloor n_1, n_2\partial_2\rfloor\nu_2 \\ &\overset{\text{(2CMM 2)}}{=} n_2\nu_2\cdot\lfloor n_1\nu_1, n_2\partial_2\nu_1\rfloor \\ &\overset{\text{Def. 39}}{=} n_2\nu_2\cdot\lfloor n_1\nu_1, n_2\nu_2\tilde{\partial}_2\rfloor \\ &\overset{\text{Lem. 55}}{=} (n_2\nu_2)^{n_1\nu_1}. \end{aligned}$$

□

Lemma 68 We have the following group morphism.

$$\begin{array}{ccc} N_1 \ltimes_{\varepsilon_2} N_2 & \xrightarrow{\nu\,\mathrm{Rec}_{1,2}} & \tilde{N}_1 \ltimes_{\tilde{\varepsilon}_2} \tilde{N}_2 \\ (n_1, n_2) & \longmapsto & (n_1\nu_1, n_2\nu_2) \end{array}$$

Proof. Suppose given n_1, $n_1' \in N_1$ and n_2, $n_2' \in N_2$.

We have

$$\begin{aligned} ((n_1, n_2)\cdot(n_1', n_2'))(\nu\,\mathrm{Rec}_{1,2}) &= (n_1\cdot n_1', n_2^{n_1'}\cdot n_2')(\nu\,\mathrm{Rec}_{1,2}) \\ &= ((n_1\cdot n_1')\nu_1, (n_2^{n_1'}\cdot n_2')\nu_2) \\ &= (n_1\nu_1\cdot n_1'\nu_1, (n_2^{n_1'})\nu_2\cdot n_2'\nu_2) \\ &\overset{\text{Rem. 67}}{=} (n_1\nu_1\cdot n_1'\nu_1, (n_2\nu_2)^{n_1'\nu_1}\cdot n_2'\nu_2) \\ &= (n_1\nu_1, n_2\nu_2)\cdot(n_1'\nu_1, n_2'\nu_2) \\ &= (n_1, n_2)(\nu\,\mathrm{Rec}_{1,2})\cdot\ (n_1', n_2')(\nu\,\mathrm{Rec}_{1,2}). \end{aligned}$$

□

Lemma 69 We have

$$((\check{n}_1, n_2)^{(n_0,n_1)})(\nu\,\mathrm{Rec}_{1,2}) = ((\check{n}_1, n_2)(\nu\,\mathrm{Rec}_{1,2}))^{(n_0,n_1)(\nu\,\mathrm{Rec}_1)}$$

for $n_0 \in N_0$, n_1, $\check{n}_1 \in N_1$ and $n_2 \in N_2$.

Proof. We obtain

$$\begin{array}{lcl}
((\check{n}_1, n_2)^{(n_0,n_1)})(\nu\,\mathrm{Rec}_{1,2}) & \overset{\text{Lem. 59}}{=} & ((\check{n}_1^{n_0})^{n_1}, \lfloor \check{n}_1^{n_0}, n_1\rfloor \cdot n_2^{n_0\cdot\, n_1\partial_1})(\nu\,\mathrm{Rec}_{1,2}) \\
& = & ((\check{n}_1^{n_0})^{n_1}\nu_1, (\lfloor \check{n}_1^{n_0}, n_1\rfloor \cdot n_2^{n_0\cdot\, n_1\partial_1})\nu_2) \\
& = & ((\check{n}_1^{n_0})^{n_1}\nu_1, \lfloor \check{n}_1^{n_0}, n_1\rfloor\nu_2 \cdot (n_2^{n_0\cdot\, n_1\partial_1})\nu_2) \\
& = & (((\check{n}_1^{n_0})\nu_1)^{n_1\nu_1}, \lfloor \check{n}_1^{n_0}, n_1\rfloor\nu_2 \cdot (n_2^{n_0\cdot\, n_1\partial_1})\nu_2) \\
& \overset{\text{(2CMM 2)}}{=} & (((\check{n}_1^{n_0})\nu_1)^{n_1\nu_1}, \lfloor (\check{n}_1^{n_0})\nu_1, n_1\nu_1\rfloor \cdot (n_2^{n_0\cdot\, n_1\partial_1})\nu_2) \\
& \overset{\text{(2CMM 1.1)}}{=} & (((\check{n}_1\nu_1)^{n_0\nu_0})^{n_1\nu_1}, \lfloor (\check{n}_1\nu_1)^{n_0\nu_0}, n_1\nu_1\rfloor \cdot (n_2^{n_0\cdot\, n_1\partial_1})\nu_2) \\
& \overset{\text{(2CMM 1.2)}}{=} & (((\check{n}_1\nu_1)^{n_0\nu_0})^{n_1\nu_1}, \lfloor (\check{n}_1\nu_1)^{n_0\nu_0}, n_1\nu_1\rfloor \cdot (n_2\nu_2)^{(n_0\cdot\, n_1\partial_1)\nu_0}) \\
& = & (((\check{n}_1\nu_1)^{n_0\nu_0})^{n_1\nu_1}, \lfloor (\check{n}_1\nu_1)^{n_0\nu_0}, n_1\nu_1\rfloor \cdot (n_2\nu_2)^{n_0\nu_0\cdot\, n_1\partial_1\nu_0}) \\
& \overset{\text{Def. 39}}{=} & (((\check{n}_1\nu_1)^{n_0\nu_0})^{n_1\nu_1}, \lfloor (\check{n}_1\nu_1)^{n_0\nu_0}, n_1\nu_1\rfloor \cdot (n_2\nu_2)^{n_0\nu_0\cdot\, n_1\nu_1\tilde{\partial}_1}) \\
& \overset{\text{Lem. 59}}{=} & (\check{n}_1\nu_1, n_2\nu_2)^{(n_0\nu_0, n_1\nu_1)} \\
& = & ((\check{n}_1, n_2)(\nu\,\mathrm{Rec}_{1,2}))^{(n_0,n_1)\nu\,\mathrm{Rec}_1}.
\end{array}$$

□

Lemma 70 We have the following group morphism.

$$\begin{array}{rcl}
(N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) & \xrightarrow{\nu\,\mathrm{Rec}_2} & (\tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1) \ltimes_{\tilde{\varepsilon}_{1,2}} (\tilde{N}_1 \ltimes_{\tilde{\varepsilon}_2} \tilde{N}_2) \\
((n_0,n_1),(\check{n}_1,n_2)) & \longmapsto & ((n_0\nu_0, n_1\nu_1),(\check{n}_1\nu_1, n_2\nu_2)) = ((n_0,n_1)(\nu\,\mathrm{Rec}_1), (\check{n}_1,n_2)(\nu\,\mathrm{Rec}_{1,2}))
\end{array}$$

Proof. Suppose given n_0, $n_0' \in N_0$, n_1, $\check{n}_1$, n_1', $\check{n}_1' \in N_1$ and n_2, $n_2' \in N_2$.

We have

$$\begin{array}{cl}
& (((n_0,n_1),(\check{n}_1,n_2)) \cdot ((n_0',n_1'),(\check{n}_1',n_2')))(\nu\,\mathrm{Rec}_2) \\
= & ((n_0,n_1)\cdot(n_0',n_1'), (\check{n}_1,n_2)^{(n_0',n_1')} \cdot (\check{n}_1',n_2'))(\nu\,\mathrm{Rec}_2) \\
= & (((n_0,n_1)\cdot(n_0',n_1'))(\nu\,\mathrm{Rec}_1), ((\check{n}_1,n_2)^{(n_0',n_1')} \cdot (\check{n}_1',n_2'))(\nu\,\mathrm{Rec}_{1,2})) \\
\overset{\substack{\text{Lem. 66}}}{\underset{\text{Lem. 68}}{=}} & ((n_0,n_1)(\nu\,\mathrm{Rec}_1) \cdot (n_0',n_1')(\nu\,\mathrm{Rec}_1), ((\check{n}_1,n_2)^{(n_0',n_1')})(\nu\,\mathrm{Rec}_{1,2}) \cdot (\check{n}_1',n_2')(\nu\,\mathrm{Rec}_{1,2})) \\
\overset{\text{Lem. 69}}{=} & ((n_0,n_1)(\nu\,\mathrm{Rec}_1) \cdot (n_0',n_1')(\nu\,\mathrm{Rec}_1), ((\check{n}_1,n_2)(\nu\,\mathrm{Rec}_{1,2}))^{(n_0',n_1')(\nu\,\mathrm{Rec}_1)} \cdot (\check{n}_1',n_2')(\nu\,\mathrm{Rec}_{1,2})) \\
= & ((n_0,n_1)(\nu\,\mathrm{Rec}_1), (\check{n}_1,n_2)(\nu\,\mathrm{Rec}_{1,2})) \cdot ((n_0',n_1')(\nu\,\mathrm{Rec}_1), (\check{n}_1',n_2')(\nu\,\mathrm{Rec}_{1,2})) \\
= & ((n_0,n_1),(\check{n}_1,n_2))(\nu\,\mathrm{Rec}_2) \cdot ((n_0',n_1'),(\check{n}_1',n_2'))(\nu\,\mathrm{Rec}_2).
\end{array}$$

□

Lemma 71 Recall that we have the $\lfloor 2,0\rfloor$-simplicial groups

$$\begin{array}{rcl}
N\,\mathrm{Rec} & := & ((N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2),\ N_0 \ltimes_{\beta_1} N_1,\ N_0,\ \mathrm{d}_0^{N\,\mathrm{Rec},2},\ \mathrm{d}_1^{N\,\mathrm{Rec},2},\ \mathrm{d}_2^{N\,\mathrm{Rec},2},\ \mathrm{s}_0^{N\,\mathrm{Rec},1}, \\
& & \mathrm{s}_1^{N\,\mathrm{Rec},1},\ \mathrm{d}_0^{N\,\mathrm{Rec},1},\ \mathrm{d}_1^{N\,\mathrm{Rec},1},\ \mathrm{s}_0^{N\,\mathrm{Rec},0})
\end{array}$$

and

$$\begin{array}{rcl}
\tilde{N}\,\mathrm{Rec} & := & ((\tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1) \ltimes_{\tilde{\varepsilon}_{1,2}} (\tilde{N}_1 \ltimes_{\tilde{\varepsilon}_2} \tilde{N}_2),\ \tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1,\ \tilde{N}_0,\ \mathrm{d}_0^{\tilde{N}\,\mathrm{Rec},2},\ \mathrm{d}_1^{\tilde{N}\,\mathrm{Rec},2},\ \mathrm{d}_2^{\tilde{N}\,\mathrm{Rec},2},\ \mathrm{s}_0^{\tilde{N}\,\mathrm{Rec},1}, \\
& & \mathrm{s}_1^{\tilde{N}\,\mathrm{Rec},1},\ \mathrm{d}_0^{\tilde{N}\,\mathrm{Rec},1},\ \mathrm{d}_1^{\tilde{N}\,\mathrm{Rec},1},\ \mathrm{s}_0^{\tilde{N}\,\mathrm{Rec},0}).
\end{array}$$

Cf. Lemma 64.

Recall the group morphisms $\nu\,\mathrm{Rec}_2$, $\nu\,\mathrm{Rec}_1$, $\nu\,\mathrm{Rec}_0$; cf. Lemma 70, Lemma 66 and Remark 65. Then

$$\nu\,\mathrm{Rec} := (\nu\,\mathrm{Rec}_2,\, \nu\,\mathrm{Rec}_1,\, \nu\,\mathrm{Rec}_0) : (N\,\mathrm{Rec}_2,\, N\,\mathrm{Rec}_1,\, N\,\mathrm{Rec}_0) \to (\tilde{N}\,\mathrm{Rec}_2,\, \tilde{N}\,\mathrm{Rec}_1,\, \tilde{N}\,\mathrm{Rec}_0)$$

is a morphism of $\lfloor 2, 0 \rfloor$-simplicial groups; cf. Definition 31.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) \arrow[r, shift left=8, "\mathrm{d}_2^{N\,\mathrm{Rec},2}"] \arrow[r, leftarrow, shift left=4, "\mathrm{s}_1^{N\,\mathrm{Rec},1}"] \arrow[r, "\mathrm{d}_1^{N\,\mathrm{Rec},2}"] \arrow[r, leftarrow, shift right=4, "\mathrm{s}_0^{N\,\mathrm{Rec},1}"] \arrow[r, shift right=8, "\mathrm{d}_0^{N\,\mathrm{Rec},2}"] \arrow[d, "\nu\,\mathrm{Rec}_2"] & N_0 \ltimes_{\beta_1} N_1 \arrow[r, shift left=4, "\mathrm{d}_1^{N\,\mathrm{Rec},1}"] \arrow[r, leftarrow, "\mathrm{s}_0^{N\,\mathrm{Rec},0}"] \arrow[r, shift right=4, "\mathrm{d}_0^{N\,\mathrm{Rec},1}"] \arrow[d, "\nu\,\mathrm{Rec}_1"] & N_0 \arrow[d, "\nu\,\mathrm{Rec}_0"] \\
(\tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1) \ltimes_{\tilde{\varepsilon}_{1,2}} (\tilde{N}_1 \ltimes_{\tilde{\varepsilon}_2} \tilde{N}_2) \arrow[r, shift left=8, "\mathrm{d}_2^{\tilde{N}\,\mathrm{Rec},2}"] \arrow[r, leftarrow, shift left=4, "\mathrm{s}_1^{\tilde{N}\,\mathrm{Rec},1}"] \arrow[r, "\mathrm{d}_1^{\tilde{N}\,\mathrm{Rec},2}"] \arrow[r, leftarrow, shift right=4, "\mathrm{s}_0^{\tilde{N}\,\mathrm{Rec},1}"] \arrow[r, shift right=8, "\mathrm{d}_0^{\tilde{N}\,\mathrm{Rec},2}"] & \tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1 \arrow[r, shift left=4, "\mathrm{d}_1^{\tilde{N}\,\mathrm{Rec},1}"] \arrow[r, leftarrow, "\mathrm{s}_0^{\tilde{N}\,\mathrm{Rec},0}"] \arrow[r, shift right=4, "\mathrm{d}_0^{\tilde{N}\,\mathrm{Rec},1}"] & \tilde{N}_0
\end{tikzcd}$$

Proof. Suppose given $n_0 \in N_0$, n_1, $\check{n}_1 \in N_1$ and $n_2 \in N_2$.

We calculate as follows.

$$\begin{aligned}
n_0(\nu\,\mathrm{Rec}_0 \blacktriangle \mathrm{s}_0) &= (n_0(\nu\,\mathrm{Rec}_0))\,\mathrm{s}_0 \\
&\overset{\text{Lem. 64}}{=} (n_0(\nu\,\mathrm{Rec}_0), 1) \\
&\overset{\text{Rem. 65}}{=} (n_0\nu_0, 1) \\
&\overset{\text{Lem. 66}}{=} (n_0, 1)(\nu\,\mathrm{Rec}_1) \\
&\overset{\text{Lem. 64}}{=} (n_0\,\mathrm{s}_0)(\nu\,\mathrm{Rec}_1) \\
&= n_0(\mathrm{s}_0 \blacktriangle \nu\,\mathrm{Rec}_1)
\end{aligned}$$

$$\begin{aligned}
(n_0, n_1)(\nu\,\mathrm{Rec}_1 \blacktriangle \mathrm{d}_0) &= ((n_0, n_1)(\nu\,\mathrm{Rec}_1))\,\mathrm{d}_0 \\
&\overset{\text{Lem. 66}}{=} (n_0\nu_0, n_1\nu_1)\,\mathrm{d}_0 \\
&\overset{\text{Lem. 64}}{=} n_0\nu_0 \cdot n_1\nu_1\tilde{\partial}_1 \\
&\overset{\text{Def. 39}}{=} n_0\nu_0 \cdot n_1\partial_1\nu_0 \\
&= (n_0 \cdot n_1\partial_1)\nu_0 \\
&\overset{\text{Rem. 65}}{=} (n_0 \cdot n_1\partial_1)(\nu\,\mathrm{Rec}_0) \\
&\overset{\text{Lem. 64}}{=} ((n_0, n_1)\,\mathrm{d}_0)(\nu\,\mathrm{Rec}_0) \\
&= (n_0, n_1)(\mathrm{d}_0 \blacktriangle \nu\,\mathrm{Rec}_0)
\end{aligned}$$

$$\begin{aligned}
(n_0,n_1)(\nu\operatorname{Rec}_1 \blacktriangle \mathrm{d}_1) &= ((n_0,n_1)(\nu\operatorname{Rec}_1))\,\mathrm{d}_1 \\
&\overset{\text{Lem. 66}}{=} (n_0\nu_0, n_1\nu_1)\,\mathrm{d}_1 \\
&\overset{\text{Lem. 64}}{=} n_0\nu_0 \\
&\overset{\text{Rem. 65}}{=} n_0(\nu\operatorname{Rec}_0) \\
&\overset{\text{Lem. 64}}{=} ((n_0,n_1)\,\mathrm{d}_1)(\nu\operatorname{Rec}_0) \\
&= (n_0,n_1)(\mathrm{d}_1 \blacktriangle \nu\operatorname{Rec}_0)
\end{aligned}$$

$$\begin{aligned}
(n_0,n_1)(\nu\operatorname{Rec}_1 \blacktriangle \mathrm{s}_0) &= ((n_0,n_1)(\nu\operatorname{Rec}_1))\,\mathrm{s}_0 \\
&\overset{\text{Lem. 66}}{=} (n_0\nu_0, n_1\nu_1)\,\mathrm{s}_0 \\
&\overset{\text{Lem. 64}}{=} ((n_0\nu_0,1),(n_1\nu_1,1)) \\
&\overset{\text{Lem. 70}}{=} ((n_0,1),(n_1,1))(\nu\operatorname{Rec}_2) \\
&\overset{\text{Lem. 64}}{=} ((n_0,n_1)\,\mathrm{s}_0)(\nu\operatorname{Rec}_2) \\
&= (n_0,n_1)(\mathrm{s}_0 \blacktriangle \nu\operatorname{Rec}_2)
\end{aligned}$$

$$\begin{aligned}
(n_0,n_1)(\nu\operatorname{Rec}_1 \blacktriangle \mathrm{s}_1) &= ((n_0,n_1)(\nu\operatorname{Rec}_1))\,\mathrm{s}_1 \\
&\overset{\text{Lem. 66}}{=} (n_0\nu_0, n_1\nu_1)\,\mathrm{s}_1 \\
&\overset{\text{Lem. 64}}{=} ((n_0\nu_0,n_1\nu_1),(1,1)) \\
&\overset{\text{Lem. 70}}{=} ((n_0,n_1),(1,1))(\nu\operatorname{Rec}_2) \\
&\overset{\text{Lem. 64}}{=} ((n_0,n_1)\,\mathrm{s}_1)(\nu\operatorname{Rec}_2) \\
&= (n_0,n_1)(\mathrm{s}_1 \blacktriangle \nu\operatorname{Rec}_2)
\end{aligned}$$

$$\begin{aligned}
((n_0,n_1),(\check{n}_1,n_2))(\nu\operatorname{Rec}_2 \blacktriangle \mathrm{d}_0) &= (((n_0,n_1),(\check{n}_1,n_2))(\nu\operatorname{Rec}_2))\,\mathrm{d}_0 \\
&\overset{\text{Lem. 70}}{=} ((n_0\nu_0,n_1\nu_1),(\check{n}_1\nu_1,n_2\nu_2))\,\mathrm{d}_0 \\
&\overset{\text{Lem. 64}}{=} (n_0\nu_0\cdot n_1\nu_1\tilde{\partial}_1, \check{n}_1\nu_1\cdot n_2\nu_2\tilde{\partial}_2) \\
&\overset{\text{Def. 39}}{=} (n_0\nu_0\cdot n_1\partial_1\nu_0, \check{n}_1\nu_1\cdot n_2\partial_2\nu_1) \\
&= ((n_0\cdot n_1\partial_1)\nu_0, (\check{n}_1\cdot n_2\partial_2)\nu_1) \\
&\overset{\text{Lem. 66}}{=} (n_0\cdot n_1\partial_1, \check{n}_1\cdot n_2\partial_2)(\nu\operatorname{Rec}_1) \\
&\overset{\text{Lem. 64}}{=} (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_0)(\nu\operatorname{Rec}_1) \\
&= ((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_0 \blacktriangle \nu\operatorname{Rec}_1)
\end{aligned}$$

$$\begin{aligned}
((n_0,n_1),(\check{n}_1,n_2))(\nu\operatorname{Rec}_2 \blacktriangle \mathrm{d}_1) &= (((n_0,n_1),(\check{n}_1,n_2))(\nu\operatorname{Rec}_2))\,\mathrm{d}_1 \\
&\overset{\text{Lem. 70}}{=} ((n_0\nu_0,n_1\nu_1),(\check{n}_1\nu_1,n_2\nu_2))\,\mathrm{d}_1 \\
&\overset{\text{Lem. 64}}{=} (n_0\nu_0, n_1\nu_1\cdot\check{n}_1\nu_1) \\
&= (n_0\nu_0, (n_1\cdot\check{n}_1)\nu_1) \\
&\overset{\text{Lem. 66}}{=} (n_0, n_1\cdot\check{n}_1)(\nu\operatorname{Rec}_1) \\
&\overset{\text{Lem. 64}}{=} (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_1)(\nu\operatorname{Rec}_1) \\
&= ((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_1 \blacktriangle \nu\operatorname{Rec}_1)
\end{aligned}$$

$$\begin{aligned}
((n_0,n_1),(\check{n}_1,n_2))(\nu\operatorname{Rec}_2 \blacktriangle \mathrm{d}_2) &= (((n_0,n_1),(\check{n}_1,n_2))(\nu\operatorname{Rec}_2))\,\mathrm{d}_2 \\
&\overset{\text{Lem. 70}}{=} ((n_0\nu_0,n_1\nu_1),(\check{n}_1\nu_1,n_2\nu_2))\,\mathrm{d}_2 \\
&\overset{\text{Lem. 64}}{=} (n_0\nu_0, n_1\nu_1) \\
&\overset{\text{Lem. 66}}{=} (n_0,n_1)(\nu\operatorname{Rec}_1) \\
&\overset{\text{Lem. 64}}{=} (((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_2)(\nu\operatorname{Rec}_1) \\
&= ((n_0,n_1),(\check{n}_1,n_2))(\mathrm{d}_2 \blacktriangle \nu\operatorname{Rec}_1)
\end{aligned}$$

So we have

$$\begin{array}{rcl}
\nu\,\mathrm{Rec}_0 \blacktriangle \mathrm{s}_0^{\tilde N\,\mathrm{Rec},0} & = & \mathrm{s}_0^{N\,\mathrm{Rec},0} \blacktriangle \nu\,\mathrm{Rec}_1 \\
\nu\,\mathrm{Rec}_1 \blacktriangle \mathrm{d}_0^{\tilde N\,\mathrm{Rec},1} & = & \mathrm{d}_0^{N\,\mathrm{Rec},1} \blacktriangle \nu\,\mathrm{Rec}_0 \\
\nu\,\mathrm{Rec}_1 \blacktriangle \mathrm{d}_1^{\tilde N\,\mathrm{Rec},1} & = & \mathrm{d}_1^{N\,\mathrm{Rec},1} \blacktriangle \nu\,\mathrm{Rec}_0 \\
\nu\,\mathrm{Rec}_1 \blacktriangle \mathrm{s}_0^{\tilde N\,\mathrm{Rec},1} & = & \mathrm{s}_0^{N\,\mathrm{Rec},1} \blacktriangle \nu\,\mathrm{Rec}_2 \\
\nu\,\mathrm{Rec}_1 \blacktriangle \mathrm{s}_1^{\tilde N\,\mathrm{Rec},1} & = & \mathrm{s}_1^{N\,\mathrm{Rec},1} \blacktriangle \nu\,\mathrm{Rec}_2 \\
\nu\,\mathrm{Rec}_2 \blacktriangle \mathrm{d}_0^{\tilde N\,\mathrm{Rec},2} & = & \mathrm{d}_0^{N\,\mathrm{Rec},2} \blacktriangle \nu\,\mathrm{Rec}_1 \\
\nu\,\mathrm{Rec}_2 \blacktriangle \mathrm{d}_1^{\tilde N\,\mathrm{Rec},2} & = & \mathrm{d}_1^{N\,\mathrm{Rec},2} \blacktriangle \nu\,\mathrm{Rec}_1 \\
\nu\,\mathrm{Rec}_2 \blacktriangle \mathrm{d}_2^{\tilde N\,\mathrm{Rec},2} & = & \mathrm{d}_2^{N\,\mathrm{Rec},2} \blacktriangle \nu\,\mathrm{Rec}_1\,.
\end{array}$$

Altogether, we have

$$\begin{tikzcd}
N\,\mathrm{Rec}_n \arrow[r, "\nu\,\mathrm{Rec}_n"] \arrow[d, "\mathrm{d}_i^{N\,\mathrm{Rec},n}"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde N\,\mathrm{Rec}_n \arrow[d, "\mathrm{d}_i^{\tilde N\,\mathrm{Rec},n}"] \\
N\,\mathrm{Rec}_{n-1} \arrow[r, "\nu\,\mathrm{Rec}_{n-1}"'] & \tilde N\,\mathrm{Rec}_{n-1}
\end{tikzcd}$$

for $n \in [1,2]$, $i \in [0,n]$ and

$$\begin{tikzcd}
N\,\mathrm{Rec}_n \arrow[r, "\nu\,\mathrm{Rec}_n"] \arrow[d, "\mathrm{s}_j^{N\,\mathrm{Rec},n}"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde N\,\mathrm{Rec}_n \arrow[d, "\mathrm{s}_j^{\tilde N\,\mathrm{Rec},n}"] \\
N\,\mathrm{Rec}_{n+1} \arrow[r, "\nu\,\mathrm{Rec}_{n+1}"'] & \tilde N\,\mathrm{Rec}_{n+1}
\end{tikzcd}$$

for $n \in [0,1]$, $j \in [0,n]$.

Cf. Definition 31. □

4.6.3 The functor Rec

Definition 72 We shall define the following functor.

$$\begin{array}{rcl}
\mathrm{Rec} : 2\text{-}\mathcal{CrMod} & \longrightarrow & \lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp} \\
\left(\begin{array}{c} N \\ \big\downarrow \nu \\ \tilde N \end{array}\right) & \longmapsto & \left(\begin{array}{c} N\,\mathrm{Rec} \\ \big\downarrow \nu\,\mathrm{Rec} \\ \tilde N\,\mathrm{Rec} \end{array}\right)
\end{array}$$

(1) Suppose given a 2-crossed module N.

The $\lfloor 2,0\rfloor$-simplicial group $N\,\mathrm{Rec}$ has been defined as follows in Lemma 64.

$$\begin{array}{rcl}
N\,\mathrm{Rec} & = & ((N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2),\ N_0 \ltimes_{\beta_1} N_1,\ N_0,\ \mathrm{d}_0^{N\,\mathrm{Rec},2},\ \mathrm{d}_1^{N\,\mathrm{Rec},2},\ \mathrm{d}_2^{N\,\mathrm{Rec},2},\ \mathrm{s}_0^{N\,\mathrm{Rec},1}, \\
& & \mathrm{s}_1^{N\,\mathrm{Rec},1},\ \mathrm{d}_0^{N\,\mathrm{Rec},1},\ \mathrm{d}_1^{N\,\mathrm{Rec},1},\ \mathrm{s}_0^{N\,\mathrm{Rec},0})
\end{array}$$

$$\begin{tikzcd}[column sep=huge]
(N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) \arrow[r, shift left=8, "\mathrm{d}_2^{N\,\mathrm{Rec},2}"] \arrow[r, "\mathrm{d}_1^{N\,\mathrm{Rec},2}"] \arrow[r, shift right=8, "\mathrm{d}_0^{N\,\mathrm{Rec},2}"'] & N_0 \ltimes_{\beta_1} N_1 \arrow[l, shift right=4, "\mathrm{s}_1^{N\,\mathrm{Rec},1}"'] \arrow[l, shift left=4, "\mathrm{s}_0^{N\,\mathrm{Rec},1}"] \arrow[r, shift left=4, "\mathrm{d}_1^{N\,\mathrm{Rec},1}"] \arrow[r, shift right=4, "\mathrm{d}_0^{N\,\mathrm{Rec},1}"'] & N_0 \arrow[l, "\mathrm{s}_0^{N\,\mathrm{Rec},0}"']
\end{tikzcd}$$

(2) Suppose given a morphism of 2-crossed modules $N \xrightarrow{\nu} \tilde{N}$.

We have the morphism

$$\nu\,\mathrm{Rec} = (\nu\,\mathrm{Rec}_2,\ \nu\,\mathrm{Rec}_1,\ \nu\,\mathrm{Rec}_0) : N\,\mathrm{Rec} \to \tilde{N}\,\mathrm{Rec}$$

of $\lfloor 2,0\rfloor$-simplicial groups; cf Lemma 71.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2) \arrow[r, shift left=8, "\mathrm{d}_2^{N\,\mathrm{Rec},2}"] \arrow[r, "\mathrm{d}_1^{N\,\mathrm{Rec},2}"] \arrow[r, shift right=8, "\mathrm{d}_0^{N\,\mathrm{Rec},2}"'] \arrow[d, "\nu\,\mathrm{Rec}_2"] & N_0 \ltimes_{\beta_1} N_1 \arrow[l, shift right=4, "\mathrm{s}_1^{N\,\mathrm{Rec},1}"'] \arrow[l, shift left=4, "\mathrm{s}_0^{N\,\mathrm{Rec},1}"] \arrow[r, shift left=4, "\mathrm{d}_1^{N\,\mathrm{Rec},1}"] \arrow[r, shift right=4, "\mathrm{d}_0^{N\,\mathrm{Rec},1}"'] \arrow[d, "\nu\,\mathrm{Rec}_1"] & N_0 \arrow[l, "\mathrm{s}_0^{N\,\mathrm{Rec},0}"'] \arrow[d, "\nu\,\mathrm{Rec}_0"] \\
(\tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1) \ltimes_{\tilde{\varepsilon}_{1,2}} (\tilde{N}_1 \ltimes_{\tilde{\varepsilon}_2} \tilde{N}_2) \arrow[r, shift left=8, "\mathrm{d}_2^{\tilde{N}\,\mathrm{Rec},2}"] \arrow[r, "\mathrm{d}_1^{\tilde{N}\,\mathrm{Rec},2}"] \arrow[r, shift right=8, "\mathrm{d}_0^{\tilde{N}\,\mathrm{Rec},2}"'] & \tilde{N}_0 \ltimes_{\tilde{\beta}_1} \tilde{N}_1 \arrow[l, shift right=4, "\mathrm{s}_1^{\tilde{N}\,\mathrm{Rec},1}"'] \arrow[l, shift left=4, "\mathrm{s}_0^{\tilde{N}\,\mathrm{Rec},1}"] \arrow[r, shift left=4, "\mathrm{d}_1^{\tilde{N}\,\mathrm{Rec},1}"] \arrow[r, shift right=4, "\mathrm{d}_0^{\tilde{N}\,\mathrm{Rec},1}"'] & \tilde{N}_0 \arrow[l, "\mathrm{s}_0^{\tilde{N}\,\mathrm{Rec},0}"']
\end{tikzcd}$$

(3) Suppose given morphisms of 2-crossed modules $N \xrightarrow{\nu} \tilde{N} \xrightarrow{\tilde{\nu}} \tilde{\tilde{N}}$.

Then we have

(a) $(\mathrm{id}_N)\,\mathrm{Rec} = \mathrm{id}_{(N\,\mathrm{Rec})}$

(b) $(\nu \blacktriangle \tilde{\nu})\,\mathrm{Rec} = \nu\,\mathrm{Rec} \blacktriangle \tilde{\nu}\,\mathrm{Rec}\,.$

In particular, $\mathrm{Rec} : 2\text{-}\mathit{CrMod} \to \lfloor 2,0\rfloor\text{-}\mathit{SimpGrp}$ is a functor, called *reconstruction functor.*

Proof.

Ad (3.a). Write $N =: (N_2,\ N_1,\ N_0)$.

We have $\mathrm{id}_N = (\mathrm{id}_{N_2},\ \mathrm{id}_{N_1},\ \mathrm{id}_{N_0})$; cf. Remark 41.

We have $(\mathrm{id}_N)\,\mathrm{Rec} = (\mathrm{id}_{N\,\mathrm{Rec}_2},\ \mathrm{id}_{N\,\mathrm{Rec}_1},\ \mathrm{id}_{N\,\mathrm{Rec}_0})$; cf. Remark 65, Lemma 66, Lemma 68, Lemma 70 and Lemma 71.

We have $\mathrm{id}_{N\,\mathrm{Rec}} = (\mathrm{id}_{N\,\mathrm{Rec}_2},\ \mathrm{id}_{N\,\mathrm{Rec}_1},\ \mathrm{id}_{N\,\mathrm{Rec}_0})$; cf. Remark 33.

So $(\mathrm{id}_N)\,\mathrm{Rec} = \mathrm{id}_{N\,\mathrm{Rec}}\,.$

Ad (3.b). Write $\nu =: (\nu_2,\, \nu_1,\, \nu_0)$ and $\tilde{\nu} =: (\tilde{\nu}_2,\, \tilde{\nu}_1,\, \tilde{\nu}_0)$.

We have

$$(\nu \operatorname{Rec} \blacktriangle \tilde{\nu} \operatorname{Rec})_k = \nu \operatorname{Rec}_k \blacktriangle \tilde{\nu} \operatorname{Rec}_k$$

for $k \in [0, 2]$; cf. Remark 32.

So we have to show that

$$((\nu \blacktriangle \tilde{\nu}) \operatorname{Rec})_k \overset{!}{=} \nu \operatorname{Rec}_k \blacktriangle \tilde{\nu} \operatorname{Rec}_k$$

for $k \in [0, 2]$.

Note that $\nu \blacktriangle \tilde{\nu} = (\nu_2 \blacktriangle \tilde{\nu}_2, \nu_1 \blacktriangle \tilde{\nu}_1, \nu_0 \blacktriangle \tilde{\nu}_0)$; cf. Remark 40.

Suppose given $n_0 \in N_0$, n_1, $\check{n}_1 \in N_1$ and $n_2 \in N_2$.

We have

$$\begin{aligned}
n_0((\nu \blacktriangle \tilde{\nu}) \operatorname{Rec}_0) &\overset{\text{Rem. 65}}{=} n_0(\nu_0 \blacktriangle \tilde{\nu}_0) \\
&= n_0 \nu_0 \tilde{\nu}_0 \\
&\overset{\text{Rem. 65}}{=} n_0 \nu_0 (\tilde{\nu} \operatorname{Rec}_0) \\
&\overset{\text{Rem. 65}}{=} n_0 (\nu \operatorname{Rec}_0)(\tilde{\nu} \operatorname{Rec}_0) \\
&= n_0 (\nu \operatorname{Rec}_0 \blacktriangle \tilde{\nu} \operatorname{Rec}_0).
\end{aligned}$$

We have

$$\begin{aligned}
(n_0, n_1)((\nu \blacktriangle \tilde{\nu}) \operatorname{Rec}_1) &\overset{\text{Lem. 66}}{=} (n_0(\nu_0 \blacktriangle \tilde{\nu}_0), n_1(\nu_1 \blacktriangle \tilde{\nu}_1)) \\
&= (n_0 \nu_0 \tilde{\nu}_0, n_1 \nu_1 \tilde{\nu}_1) \\
&\overset{\text{Lem. 66}}{=} (n_0 \nu_0, n_1 \nu_1)(\tilde{\nu} \operatorname{Rec}_1) \\
&\overset{\text{Lem. 66}}{=} (n_0, n_1)(\nu \operatorname{Rec}_1)(\tilde{\nu} \operatorname{Rec}_1) \\
&= (n_0, n_1)(\nu \operatorname{Rec}_1 \blacktriangle \tilde{\nu} \operatorname{Rec}_1).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
((n_0, n_1), (\check{n}_1, n_2))((\nu \blacktriangle \tilde{\nu}) \operatorname{Rec}_2) &\overset{\text{Lem. 70}}{=} ((n_0(\nu_0 \blacktriangle \tilde{\nu}_0), n_1(\nu_1 \blacktriangle \tilde{\nu}_1)), (\check{n}_1(\nu_1 \blacktriangle \tilde{\nu}_1), n_2(\nu_2 \blacktriangle \tilde{\nu}_2))) \\
&= ((n_0 \nu_0 \tilde{\nu}_0, n_1 \nu_1 \tilde{\nu}_1), (\check{n}_1 \nu_1 \tilde{\nu}_1, n_2 \nu_2 \tilde{\nu}_2)) \\
&\overset{\text{Lem. 70}}{=} ((n_0 \nu_0, n_1 \nu_1), (\check{n}_1 \nu_1, n_2 \nu_2))(\tilde{\nu} \operatorname{Rec}_2) \\
&\overset{\text{Lem. 70}}{=} ((n_0, n_1), (\check{n}_1, n_2))(\nu \operatorname{Rec}_2)(\tilde{\nu} \operatorname{Rec}_2) \\
&= ((n_0, n_1), (\check{n}_1, n_2))(\nu \operatorname{Rec}_2 \blacktriangle \tilde{\nu} \operatorname{Rec}_2).
\end{aligned}$$

□

Example 73 Suppose given $m \geqslant 1$. Let $N_2 := \langle d : d^m = 1 \rangle$. So $N_2 = \langle d \rangle \simeq \mathrm{C}_m$.

Suppose given a finite group N_0.

Suppose given $c \in \mathrm{Z}(N_0)$. Let $N_1 := \langle c \rangle \trianglelefteqslant N_0$. Write $k := |\langle c \rangle|$. So $\langle c \rangle \simeq \mathrm{C}_k$.

Suppose given $z \in \mathbb{Z}$ with $k \cdot z$ divisible by m.

We apply the functor Rec to the 2-crossed module of Example 38.

Recall that we have the group morphisms

$$\begin{array}{lclclcl}
N_2 & \xrightarrow{\partial_2 := !} & N_1 & : & n_2 & \longmapsto & 1 \\
N_1 & \xrightarrow{\partial_1} & N_0 & : & n_1 & \longmapsto & n_1 \\
N_0 & \xrightarrow{\beta_1} & \operatorname{Aut}(N_1) & : & n_0 & \longmapsto & (n_1 \mapsto n_1^{n_0} := n_1) \\
N_0 & \xrightarrow{\beta_2} & \operatorname{Aut}(N_2) & : & n_0 & \longmapsto & (n_2 \mapsto n_2^{n_0} := n_2)
\end{array}$$

and the map

$$\begin{array}{rcl} N_1 \times N_1 := \langle c \rangle \times \langle c \rangle & \xrightarrow{\zeta} & N_2 := \langle d \rangle \\ (c^i, c^j) & \longmapsto & \lfloor c^i, c^j \rfloor := d^{i \cdot j \cdot z}, \end{array}$$

yielding the 2-crossed module $(N_2,\, N_1,\, N_0,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta)$.

With the help of Lemma 55 we obtain the group morphism

$$\begin{array}{rcl} N_1 & \xrightarrow{\varepsilon_2} & \mathrm{Aut}(N_2) \\ n_1 & \longmapsto & \left(n_2 \mapsto \left(\begin{array}{rcl} n_2^{n_1} & := & n_2 \cdot \lfloor n_1, n_2\partial_2 \rfloor \\ & = & n_2 \cdot \lfloor n_1, 1 \rfloor \\ & = & n_2 \end{array} \right) \right). \end{array}$$

So

$$\begin{array}{rcl} N_0 \ltimes_{\beta_1} N_1 & = & N_0 \times N_1 \\ N_1 \ltimes_{\varepsilon_2} N_2 & = & N_1 \times N_2 \end{array}$$

are direct products.

With the help of Lemma 59 we obtain the group morphism

$$\begin{array}{rcl} N_0 \times N_1 & \xrightarrow{\varepsilon_{1,2}} & \mathrm{Aut}(N_1 \times N_2) \\ (n_0, n_1) & \longmapsto & \left((\check{n}_1, n_2) \mapsto \left(\begin{array}{rcl} (\check{n}_1, n_2)^{(n_0, n_1)} & := & ((\check{n}_1^{n_0})^{n_1}, \lfloor \check{n}_1^{n_0}, n_1 \rfloor \cdot n_2^{n_0 \cdot n_1 \partial_1}) \\ & = & (\check{n}_1^{n_1}, \lfloor \check{n}_1, n_1 \rfloor \cdot n_2) \\ & = & (\check{n}_1, \lfloor \check{n}_1, n_1 \rfloor \cdot n_2) \end{array} \right) \right). \end{array}$$

Suppose given d^i, $d^j \in N_2$, c^l, c^p, $c^{\tilde{l}}$, $c^{\tilde{p}} \in N_1$ and n_0, $n_0' \in N_0$ for suitably chosen i, j, l, p, $\tilde{l}$, $\tilde{p} \in \mathbb{Z}$.

Then we have

$$\begin{array}{rcl} ((n_0, c^l), (c^{\tilde{l}}, d^i)) \cdot ((n_0', c^p), (c^{\tilde{p}}, d^j)) & = & ((n_0, c^l) \cdot (n_0', c^p), (c^{\tilde{l}}, d^i)^{(n_0', c^p)} \cdot (c^{\tilde{p}}, d^j)) \\ & = & ((n_0, c^l) \cdot (n_0', c^p), (c^{\tilde{l}}, \lfloor c^{\tilde{l}}, c^p \rfloor \cdot d^i) \cdot (c^{\tilde{p}}, d^j)) \\ & = & ((n_0, c^l) \cdot (n_0', c^p), (c^{\tilde{l}}, d^{\tilde{l} \cdot p \cdot z + i}) \cdot (c^{\tilde{p}}, d^j)) \\ & = & ((n_0 \cdot n_0', c^l \cdot c^p), (c^{\tilde{l}} \cdot c^{\tilde{p}}, d^{\tilde{l} \cdot p \cdot z + i + j})). \end{array}$$

By Lemma 64, we obtain the group morphisms

$$N_0 \xrightarrow{\mathrm{s}_0^{N\,\mathrm{Rec},0}} N_0 \times N_1 \;:\; n_0 \longmapsto (n_0, 1),$$

and

$$\begin{array}{rcccl} N_0 \times N_1 & \xrightarrow{\mathrm{d}_0^{N\,\mathrm{Rec},1}} & N_0 & : & (n_0, n_1) \longmapsto n_0 \cdot n_1 \partial_1 \\ N_0 \times N_1 & \xrightarrow{\mathrm{d}_1^{N\,\mathrm{Rec},1}} & N_0 & : & (n_0, n_1) \longmapsto n_0. \end{array}$$

Furthermore, we get the group morphisms

$$\begin{array}{rcccl} N_0 \times N_1 & \xrightarrow{\mathrm{s}_0^{N\,\mathrm{Rec},1}} & (N_0 \times N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \times N_2) & : & (n_0, n_1) \longmapsto ((n_0, 1), (n_1, 1)) \\ N_0 \times N_1 & \xrightarrow{\mathrm{s}_1^{N\,\mathrm{Rec},1}} & (N_0 \times N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \times N_2) & : & (n_0, n_1) \longmapsto ((n_0, n_1), (1, 1)) \end{array}$$

and

$$\begin{array}{rcccl} (N_0 \times N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \times N_2) & \xrightarrow{\mathrm{d}_0^{N\,\mathrm{Rec},2}} & N_0 \times N_1 & : & ((n_0, n_1), (\check{n}_1, n_2)) \longmapsto (n_0 \cdot n_1 \partial_1, \check{n}_1) \\ (N_0 \times N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \times N_2) & \xrightarrow{\mathrm{d}_1^{N\,\mathrm{Rec},2}} & N_0 \times N_1 & : & ((n_0, n_1), (\check{n}_1, n_2)) \longmapsto (n_0, n_1 \cdot \check{n}_1) \\ (N_0 \times N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \times N_2) & \xrightarrow{\mathrm{d}_2^{N\,\mathrm{Rec},2}} & N_0 \times N_1 & : & ((n_0, n_1), (\check{n}_1, n_2)) \longmapsto (n_0, n_1). \end{array}$$

Recall that $\partial_1 : N_1 \to N_0$ is the inclusion morphism.

Then

$$((N_0 \times N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \times N_2),\ N_0 \times N_1,\ N_0,\ \mathrm{d}_0^{N\,\mathrm{Rec},2},\ \mathrm{d}_1^{N\,\mathrm{Rec},2},\ \mathrm{d}_2^{N\,\mathrm{Rec},2},\ \mathrm{s}_0^{N\,\mathrm{Rec},1},\ \mathrm{s}_1^{N\,\mathrm{Rec},1},\\ \mathrm{d}_0^{N\,\mathrm{Rec},1},\ \mathrm{d}_1^{N\,\mathrm{Rec},1},\ \mathrm{s}_0^{N\,\mathrm{Rec},0})$$

is a $\lfloor 2,0 \rfloor$-simplicial group.

$$\begin{tikzcd}
(N_0 \times N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \times N_2) \arrow[r, shift left=8, "\mathrm{d}_2^{N\,\mathrm{Rec},2}"] \arrow[r, "\mathrm{d}_1^{N\,\mathrm{Rec},2}"] \arrow[r, shift right=8, "\mathrm{d}_0^{N\,\mathrm{Rec},2}"] & N_0 \times N_1 \arrow[l, shift right=4, "\mathrm{s}_1^{N\,\mathrm{Rec},1}"'] \arrow[l, shift left=4, "\mathrm{s}_0^{N\,\mathrm{Rec},1}"'] \arrow[r, shift left=4, "\mathrm{d}_1^{N\,\mathrm{Rec},1}"] \arrow[r, shift right=4, "\mathrm{d}_0^{N\,\mathrm{Rec},1}"] & N_0 \arrow[l, "\mathrm{s}_0^{N\,\mathrm{Rec},0}"']
\end{tikzcd}$$

4.7 Equivalence of the categories $\lfloor 2,0 \rfloor$-$\mathcal{SimpGrp}$ and 2-$\mathcal{CrMod}$

4.7.1 The isotransformation $\zeta : \mathrm{Rec} \blacktriangle \hat{\mathrm{N}} \xrightarrow{\sim} \mathrm{Id}_{2\text{-}\mathcal{CrMod}}$

Lemma 74 Suppose given a 2-crossed module

$$N = (N_2,\ N_1,\ N_0,\ \partial_2,\ \partial_1,\ \beta_1,\ \beta_2,\ \zeta);$$

cf. Definition 36.

We have the following group isomorphisms.

$$\begin{array}{rcl} (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_2 & \xrightarrow{N\zeta_2} & N_2 \\ ((1,1),(1,n_2)) & \longmapsto & n_2 \end{array}$$

$$\begin{array}{rcl} (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_1 & \xrightarrow{N\zeta_1} & N_1 \\ (1,n_1) & \longmapsto & n_1 \end{array}$$

$$\begin{array}{rcl} (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_0 & \xrightarrow{N\zeta_0} & N_0 \\ n_0 & \longmapsto & n_0 \end{array}$$

Then

$$N\zeta := (N\zeta_2,\ N\zeta_1,\ N\zeta_0) : ((N\,\mathrm{Rec}\,\hat{\mathrm{N}})_2,\ (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_1,\ (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_0) \mapsto (N_2,\ N_1,\ N_0)$$

is an isomorphism of 2-crossed modules.

Proof. Recall that $N\,\mathrm{Rec}_2 = (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2)$, $N\,\mathrm{Rec}_1 = N_0 \ltimes_{\beta_1} N_1$ and $N\,\mathrm{Rec}_0 = N_0$; cf. Lemma 64.

Recall that

$$\begin{array}{rcl} ((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_1 & = & (n_0, n_1 \cdot \check{n}_1) \\ ((n_0,n_1),(\check{n}_1,n_2))\,\mathrm{d}_2 & = & (n_0,n_1) \end{array}$$

for $((n_0,\ n_1),\ (\check{n}_1,\ n_2)) \in N\,\mathrm{Rec}_2$; cf. Lemma 64.

Therefore

$$\begin{array}{rcl} (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_2 & = & \ker \mathrm{d}_1 \cap \ker \mathrm{d}_2 \\ & = & \{((1,1),(1,n_2)) : n_2 \in N_2\}. \end{array}$$

Recall that

$$(n_0, n_1)\,\mathrm{d}_1 \quad = \quad n_0$$

for $(n_0,\, n_1) \in N\,\mathrm{Rec}_1$; cf. Lemma 64.

Therefore

$$\begin{aligned} (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_1 \quad &= \quad \ker \mathrm{d}_1 \\ &= \quad \{(1, n_1) : n_1 \in N_1\}. \end{aligned}$$

Finally, $(N\,\mathrm{Rec}\,\hat{\mathrm{N}})_0 = N\,\mathrm{Rec}_0 = N_0$.

So we have the group isomorphisms $N\zeta_2$, $N\zeta_1$, $N\zeta_0$ as described above; cf. Remark 7.

We show that the tuple of group morphisms $(N\zeta_2,\, N\zeta_1,\, N\zeta_0)$ is a morphism of 2-crossed modules.

Recall that the 2-crossed module $N\,\mathrm{Rec}\,\hat{\mathrm{N}}$ has the differentials $\tilde{\partial}_2 := \mathrm{d}_0|_{(N\,\mathrm{Rec}\,\hat{\mathrm{N}})_2}^{(N\,\mathrm{Rec}\,\hat{\mathrm{N}})_1}$ and $\tilde{\partial}_1 := \mathrm{d}_0|_{(N\,\mathrm{Rec}\,\hat{\mathrm{N}})_1}^{(N\,\mathrm{Rec}\,\hat{\mathrm{N}})_0}$; cf. Lemma 51.

First, we have to show that the following diagram is commutative.

$$\begin{tikzcd}
(N\,\mathrm{Rec}\,\hat{\mathrm{N}})_2 \arrow[r, "\tilde{\partial}_2"] \arrow[d, "N\zeta_2"'] & (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_1 \arrow[r, "\tilde{\partial}_1"] \arrow[d, "N\zeta_1"'] & (N\,\mathrm{Rec}\,\hat{\mathrm{N}})_0 \arrow[d, "N\zeta_0"] \\
N_2 \arrow[r, "\partial_2"] & N_1 \arrow[r, "\partial_1"] & N_0
\end{tikzcd}$$

Suppose given $n_1 \in N_1$ and $n_2 \in N_2$.

Then we have

$$\begin{aligned} ((1,1),(1,n_2))(\tilde{\partial}_2 \blacktriangle N\zeta_1) \quad &= \quad (((1,1),(1,n_2))\tilde{\partial}_2)(N\zeta_1) \\ &= \quad (((1,1),(1,n_2))\,\mathrm{d}_0)(N\zeta_1) \\ &\overset{\text{Lem. 64}}{=} \quad (1, n_2\partial_2)(N\zeta_1) \\ &= \quad n_2\partial_2 \\ &= \quad (((1,1),(1,n_2))(N\zeta_2))\partial_2 \\ &= \quad ((1,1),(1,n_2))(N\zeta_2 \blacktriangle \partial_2) \end{aligned}$$

and

$$\begin{aligned} (1,n_1)(\tilde{\partial}_1 \blacktriangle N\zeta_0) \quad &= \quad ((1,n_1)\tilde{\partial}_1)(N\zeta_0) \\ &= \quad ((1,n_1)\,\mathrm{d}_0)(N\zeta_0) \\ &\overset{\text{Lem. 64}}{=} \quad (n_1\partial_1)(N\zeta_0) \\ &= \quad n_1\partial_1 \\ &= \quad ((1,n_1)(N\zeta_1))\partial_1 \\ &= \quad (1,n_1)(N\zeta_1 \blacktriangle \partial_1). \end{aligned}$$

So we have

$$\tilde{\partial}_2 \blacktriangle N\zeta_1 \quad = \quad N\zeta_2 \blacktriangle \partial_2$$

and

$$\tilde{\partial}_1 \blacktriangle N\zeta_0 \quad = \quad N\zeta_1 \blacktriangle \partial_1.$$

Second, we have to show (2CMM $1, 2$).

Ad (2CMM 1.1). Suppose given $n_0 \in N_0$ and $n_1 \in N_1$.

Then we have

$$\begin{aligned}
((1, n_1)^{n_0})(N\zeta_1) &\overset{\text{Lem. 51}}{=} ((1, n_1)^{n_0\,\mathrm{s}_0})(N\zeta_1) \\
&\overset{\text{Lem. 64}}{=} ((1, n_1)^{(n_0, 1)})(N\zeta_1) \\
&= ((n_0^-, 1) \cdot (1, n_1) \cdot (n_0, 1))(N\zeta_1) \\
&= ((n_0^-, 1) \cdot (n_0, n_1^{n_0}))(N\zeta_1) \\
&= (1, n_1^{n_0})(N\zeta_1) \\
&= n_1^{n_0} \\
&= ((1, n_1)(N\zeta_1))^{n_0(N\zeta_0)}.
\end{aligned}$$

Ad (2CMM 1.2). Suppose given $n_0 \in N_0$ and $n_2 \in N_2$.

Then we have

$$\begin{aligned}
(((1,1),(1,n_2))^{n_0})(N\zeta_2) &\overset{\text{Lem. 51}}{=} (((1,1),(1,n_2))^{n_0\,\mathrm{s}_0\,\mathrm{s}_0})(N\zeta_2) \\
&\overset{\text{Lem. 64}}{=} (((1,1),(1,n_2))^{(n_0,1)\,\mathrm{s}_0})(N\zeta_2) \\
&\overset{\text{Lem. 64}}{=} (((1,1),(1,n_2))^{((n_0,1),(1,1))})(N\zeta_2) \\
&= (((n_0^-,1),(1,1)) \cdot ((1,1),(1,n_2)) \cdot ((n_0,1),(1,1)))(N\zeta_2) \\
&= (((n_0^-,1),(1,1)) \cdot ((n_0,1),(1,n_2)^{(n_0,1)}))(N\zeta_2) \\
&= ((1,1),(1,n_2)^{(n_0,1)})(N\zeta_2) \\
&\overset{\text{Lem. 59}}{=} ((1,1),(1,\lfloor 1,1\rfloor \cdot n_2^{n_0}))(N\zeta_2) \\
&\overset{\text{Rem. 37}}{=} ((1,1),(1,n_2^{n_0}))(N\zeta_2) \\
&= n_2^{n_0} \\
&= (((1,1),(1,n_2))(N\zeta_2))^{n_0(N\zeta_0)}.
\end{aligned}$$

Ad (2CMM 2). Suppose given n_1, $n_1' \in N_1$.

Then we have

$$\begin{aligned}
&\lfloor (1,n_1),(1,n_1')\rfloor(N\zeta_2) \\
\overset{\text{Lem. 51}}{=}\ & (((1,n_1^-)\,\mathrm{s}_0)^{(1,n_1')\,\mathrm{s}_0} \cdot ((1,n_1)\,\mathrm{s}_0)^{(1,n_1')\,\mathrm{s}_1})(N\zeta_2) \\
\overset{\text{Lem. 64}}{=}\ & (((1,1),(n_1^-,1))^{((1,1),(n_1',1))} \cdot ((1,1),(n_1,1))^{((1,n_1'),(1,1))})(N\zeta_2) \\
\overset{\text{Rem. 7}}{=}\ & (((1,1),((n_1^-)^{n_1'},1)) \cdot ((1,1),(n_1,1))^{((1,n_1'),(1,1))})(N\zeta_2) \\
=\ & (((1,1),((n_1^-)^{n_1'},1)) \cdot ((1,n_1'^-),(1,1)) \cdot ((1,1),(n_1,1)) \cdot ((1,n_1'),(1,1)))(N\zeta_2) \\
=\ & (((1,1),((n_1^-)^{n_1'},1)) \cdot ((1,n_1'^-),(1,1)) \cdot ((1,n_1'),(n_1,1)^{(1,n_1')}))(N\zeta_2) \\
=\ & (((1,1),((n_1^-)^{n_1'},1)) \cdot ((1,1),(n_1,1)^{(1,n_1')}))(N\zeta_2) \\
\overset{\text{Lem. 59}}{=}\ & (((1,1),((n_1^-)^{n_1'},1)) \cdot ((1,1),(n_1^{n_1'},\lfloor n_1,n_1'\rfloor)))(N\zeta_2) \\
=\ & ((1,1),((n_1^-)^{n_1'},1) \cdot (n_1^{n_1'},\lfloor n_1,n_1'\rfloor))(N\zeta_2) \\
=\ & ((1,1),(1,\lfloor n_1,n_1'\rfloor))(N\zeta_2) \\
=\ & \lfloor n_1,n_1'\rfloor \\
=\ & \lfloor (1,n_1)(N\zeta_1),(1,n_1')(N\zeta_1)\rfloor.
\end{aligned}$$

Since $N\zeta_2$, $N\zeta_1$, $N\zeta_0$ are group isomorphisms, $N\zeta = (N\zeta_2,\ N\zeta_1,\ N\zeta_0)$ is an isomorphism of 2-crossed modules with inverse $N\zeta^- = (N\zeta_2^-,\ N\zeta_1^-,\ N\zeta_0^-)$; cf. Remark 43. $\square$

Lemma 75 We have the isotransformation

$$\zeta = (N\zeta)_{N\in\mathrm{Ob}(2\text{-}\mathscr{CrMod})} : \mathrm{Rec} \blacktriangle \hat{\mathrm{N}} \xrightarrow{\sim} \mathrm{Id}_{2\text{-}\mathscr{CrMod}} .$$

Cf. Lemma 74.

Proof. To show that ζ is an isotransformation, it remains to show that the quadrangle

$$\begin{tikzcd} N(\mathrm{Rec} \blacktriangle \hat{\mathrm{N}}) \arrow[r, "N\zeta"] \arrow[d, "\nu\,\mathrm{Rec}\,\hat{\mathrm{N}}"'] & N\,\mathrm{Id}_{2\text{-}\mathscr{CrMod}} \arrow[d, "\nu\,\mathrm{Id}_{2\text{-}\mathscr{CrMod}}"] \\ \tilde{N}(\mathrm{Rec} \blacktriangle \hat{\mathrm{N}}) \arrow[r, "\tilde{N}\zeta"'] & \tilde{N}\,\mathrm{Id}_{2\text{-}\mathscr{CrMod}} \end{tikzcd}$$

commutes for $N \xrightarrow{\nu} \tilde{N}$ in $2\text{-}\mathscr{CrMod}$.

To this end, we show that $N\zeta_k \blacktriangle \nu_k \overset{!}{=} (\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_k \blacktriangle \tilde{N}\zeta_k$ for $k \in \{0,\ 1,\ 2\}$; cf. Remark 40, Lemma 74.

Suppose given $n_0 \in N_0$, $n_1 \in N_1$ and $n_2 \in N_2$.

We have

$$\begin{aligned} ((1,1),(1,n_2))((\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_2 \blacktriangle \tilde{N}\zeta_2) &= (((1,1),(1,n_2))(\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_2)(\tilde{N}\zeta_2) \\ &\overset{\text{Def. 54}}{=} (((1,1),(1,n_2))(\nu\,\mathrm{Rec}_2))(\tilde{N}\zeta_2) \\ &\overset{\text{Lem. 70}}{=} ((1,1),(1,n_2\nu_2))(\tilde{N}\zeta_2) \\ &= n_2\nu_2 \\ &= (((1,1),(1,n_2))(N\zeta_2))\nu_2 \\ &= ((1,1),(1,n_2))(N\zeta_2 \blacktriangle \nu_2). \end{aligned}$$

We have

$$\begin{aligned} (1,n_1)((\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_1 \blacktriangle \tilde{N}\zeta_1) &= ((1,n_1)(\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_1)(\tilde{N}\zeta_1) \\ &\overset{\text{Def. 54}}{=} ((1,n_1)(\nu\,\mathrm{Rec}_1))(\tilde{N}\zeta_1) \\ &\overset{\text{Lem. 66}}{=} (1,n_1\nu_1)(\tilde{N}\zeta_1) \\ &= n_1\nu_1 \\ &= ((1,n_1)(N\zeta_1))\nu_1 \\ &= (1,n_1)(N\zeta_1 \blacktriangle \nu_1). \end{aligned}$$

Moreover, we have

$$\begin{aligned} n_0((\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_0 \blacktriangle \tilde{N}\zeta_0) &= (n_0(\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_0)(\tilde{N}\zeta_0) \\ &\overset{\text{Def. 54}}{=} (n_0(\nu\,\mathrm{Rec}_0))(\tilde{N}\zeta_0) \\ &\overset{\text{Rem. 65}}{=} n_0\nu_0(\tilde{N}\zeta_0) \\ &= n_0\nu_0 \\ &= (n_0(N\zeta_0))\nu_0 \\ &= n_0(N\zeta_0 \blacktriangle \nu_0). \end{aligned}$$

Altogether, we have

$$\begin{aligned} \nu\,\mathrm{Rec}\,\hat{\mathrm{N}} \blacktriangle \tilde{N}\zeta &= ((\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_2, (\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_1, (\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_0) \blacktriangle (\tilde{N}\zeta_2, \tilde{N}\zeta_1, \tilde{N}\zeta_0) \\ &\overset{\text{Rem. 40}}{=} ((\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_2 \blacktriangle \tilde{N}\zeta_2, (\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_1 \blacktriangle \tilde{N}\zeta_1, (\nu\,\mathrm{Rec}\,\hat{\mathrm{N}})_0 \blacktriangle \tilde{N}\zeta_0) \\ &= (N\zeta_2 \blacktriangle \nu_2, N\zeta_1 \blacktriangle \nu_1, N\zeta_0 \blacktriangle \nu_0) \\ &\overset{\text{Rem. 40}}{=} (N\zeta_2, N\zeta_1, N\zeta_0) \blacktriangle (\nu_2, \nu_1, \nu_0) \\ &= N\zeta \blacktriangle \nu. \end{aligned}$$

$\square$

4.7.2 The isotransformation $\vartheta : \hat{\mathrm{N}} \blacktriangle \mathrm{Rec} \xrightarrow{\sim} \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp}$

Lemma 76 Suppose given a $\lfloor 2,0\rfloor$-simplicial group G; cf. Definition 30.

We have the following group isomorphisms.

$$\begin{array}{rcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_2 & \xrightarrow{G\vartheta_2} & G_2 \\
((n_0, n_1), (\check{n}_1, n_2)) & \longmapsto & n_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot n_1\,\mathrm{s}_1 \cdot \check{n}_1\,\mathrm{s}_0 \cdot n_2 \\
\Big((g_2\,\mathrm{d}_2\,\mathrm{d}_1, g_2^-\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2), & & \\
(g_2^-\,\mathrm{d}_2 \cdot g_2\,\mathrm{d}_1, g_2^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{s}_1 \cdot g_2)\Big) & \longleftarrow\!\shortmid & g_2
\end{array}$$

$$\begin{array}{rcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_1 & \xrightarrow{G\vartheta_1} & G_1 \\
(n_0, n_1) & \longmapsto & n_0\,\mathrm{s}_0 \cdot n_1 \\
(g_1\,\mathrm{d}_1, g_1^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_1) & \longleftarrow\!\shortmid & g_1
\end{array}$$

$$\begin{array}{rcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_0 & \xrightarrow{G\vartheta_0} & G_0 \\
n_0 & \longmapsto & n_0 \\
g_0 & \longleftarrow\!\shortmid & g_0
\end{array}$$

Then

$$G\vartheta := (G\vartheta_2,\, G\vartheta_1,\, G\vartheta_0) : ((G\,\hat{\mathrm{N}}\,\mathrm{Rec})_2,\, (G\,\hat{\mathrm{N}}\,\mathrm{Rec})_1,\, (G\,\hat{\mathrm{N}}\,\mathrm{Rec})_0) \mapsto (G_2,\, G_1,\, G_0)$$

is an isomorphism of $\lfloor 2,0\rfloor$-simplicial groups.

Proof. Recall that given a 2-crossed module $N = (N_2,\, N_1,\, N_0)$, we have
$N\,\mathrm{Rec}_2 = (N_0 \ltimes_{\beta_1} N_1) \ltimes_{\varepsilon_{1,2}} (N_1 \ltimes_{\varepsilon_2} N_2)$, $N\,\mathrm{Rec}_1 = N_0 \ltimes_{\beta_1} N_1$ and $N\,\mathrm{Rec}_0 = N_0$; cf. Lemma 64.

Here, ε_2 is given as in Lemma 55 and $\varepsilon_{1,2}$ is given as in Lemma 59.

Recall the data of the 2-crossed module

$$G\,\hat{\mathrm{N}} = (G\mathrm{N}_2,\, G\mathrm{N}_1,\, G\mathrm{N}_0,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta)$$

from Lemma 51.

So

$$\begin{array}{rcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_2 & = & (G\mathrm{N}_0 \ltimes_{\beta_1} G\mathrm{N}_1) \ltimes_{\varepsilon_{1,2}} (G\mathrm{N}_1 \ltimes_{\varepsilon_2} G\mathrm{N}_2) \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_1 & = & G\mathrm{N}_0 \ltimes_{\beta_1} G\mathrm{N}_1 \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_0 & = & G\mathrm{N}_0\,.
\end{array}$$

We recall from [**1,** Lem. 55] with the help of [**1,** Rem. 56] the following group morphisms.

$$\begin{array}{rcl}
\gamma_1 : G\mathrm{N}_0 & \longrightarrow & \mathrm{Aut}(G\mathrm{N}_1) \\
n_0 & \longmapsto & (n_0\gamma_1 : n_1 \mapsto n_1^{n_0\,\mathrm{s}_0})
\end{array}$$

$$\begin{array}{rcl}
\gamma_2'' : G\mathrm{N}_1 & \longrightarrow & \mathrm{Aut}(G\mathrm{N}_2) \\
n_1 & \longmapsto & (n_1\gamma_2'' : n_2 \mapsto n_2^{n_1\,\mathrm{s}_0})
\end{array}$$

$$\begin{array}{rcl}
\gamma_2 : G\mathrm{N}_0 \ltimes_{\gamma_1} G\mathrm{N}_1 & \longrightarrow & \mathrm{Aut}(G\mathrm{N}_1 \ltimes_{\gamma_2''} G\mathrm{N}_2) \\
(n_0, n_1) & \longmapsto & \left((n_0, n_1)\gamma_2 : (\check{n}_1, n_2) \mapsto \left(\begin{array}{l} \check{n}_1^{\,n_0\,\mathrm{s}_0 \cdot n_1},\, (\check{n}_1^-\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot n_1\,\mathrm{s}_0} \\ \qquad \cdot\, (\check{n}_1\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot n_1\,\mathrm{s}_1} \cdot n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot n_1\,\mathrm{s}_1} \end{array} \right) \right).
\end{array}$$

Then we know that

$$\begin{array}{rcl}
(G\mathrm{N}_0 \ltimes_{\gamma_1} G\mathrm{N}_1) \ltimes_{\gamma_2} (G\mathrm{N}_1 \ltimes_{\gamma_2''} G\mathrm{N}_2) & \xrightarrow{G\vartheta_2} & G_2 \\
((n_0, n_1), (\check{n}_1, n_2)) & \longmapsto & n_0 \, \mathrm{s}_0 \, \mathrm{s}_1 \cdot n_1 \, \mathrm{s}_1 \cdot \check{n}_1 \, \mathrm{s}_0 \cdot n_2 \\
\begin{array}{r}\Big((g_2 \, \mathrm{d}_2 \, \mathrm{d}_1, g_2^- \, \mathrm{d}_2 \, \mathrm{d}_1 \, \mathrm{s}_0 \cdot g_2 \, \mathrm{d}_2), \\ (g_2^- \, \mathrm{d}_2 \cdot g_2 \, \mathrm{d}_1, g_2^- \, \mathrm{d}_1 \, \mathrm{s}_0 \cdot g_2 \, \mathrm{d}_2 \, \mathrm{s}_0 \cdot g_2^- \, \mathrm{d}_2 \, \mathrm{s}_1 \cdot g_2)\Big)\end{array} & \longleftarrow\!\shortmid & g_2
\end{array}$$

$$\begin{array}{rcl}
G\mathrm{N}_0 \ltimes_{\gamma_1} G\mathrm{N}_1 & \xrightarrow{G\vartheta_1} & G_1 \\
(n_0, n_1) & \longmapsto & n_0 \, \mathrm{s}_0 \cdot n_1 \\
(g_1 \, \mathrm{d}_1, g_1^- \, \mathrm{d}_1 \, \mathrm{s}_0 \cdot g_1) & \longleftarrow\!\shortmid & g_1
\end{array}$$

$$\begin{array}{rcl}
G\mathrm{N}_0 & \xrightarrow{G\vartheta_0} & G_0 \\
n_0 & \longmapsto & n_0 \\
g_0 & \longleftarrow\!\shortmid & g_0,
\end{array}$$

are group isomorphisms; cf. [**1,** Lem. 55].

Moreover, these group isomorphisms form an isomorphism of $\lfloor 2,0\rfloor$-simplicial groups in [**1,** Lem. 55].

Note that the maps d_i' and s_j' in [**1,** Lem. 55] are defined as the according morphisms in Lemma 64.

So it suffices to show that

$$\begin{array}{rcl}
\beta_1 & \overset{!}{=} & \gamma_1 \\
\varepsilon_2 & \overset{!}{=} & \gamma_2'' \\
\varepsilon_{1,2} & \overset{!}{=} & \gamma_2,
\end{array}$$

in order to show that $G\vartheta$ is an isomorphism of $\lfloor 2,0\rfloor$-simplicial groups.

Suppose given $n_0 \in G\mathrm{N}_0$ and $n_1 \in G\mathrm{N}_1$.

Then we have

$$\begin{array}{rcl}
(n_1)(n_0\beta_1) & \overset{\text{Lem. 51}}{=} & n_1^{n_0 \, \mathrm{s}_0} \\
& = & (n_1)(n_0\gamma_1).
\end{array}$$

Suppose given $n_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$.

Using the assertions (CC $2,4$) from Lemma 49, we obtain

$$\begin{array}{rcl}
(n_2)(n_1\varepsilon_2) & \overset{\text{Lem. 55}}{=} & n_2 \cdot \lfloor n_1, n_2 \partial_2 \rfloor \\
& = & n_2 \cdot \lfloor n_1, n_2 \, \mathrm{d}_0 \rfloor \\
& \overset{\text{Lem. 51}}{=} & n_2 \cdot (n_1^- \, \mathrm{s}_0)^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \cdot (n_1 \, \mathrm{s}_0)^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_1} \\
& \overset{\text{(CC 4)}}{=} & n_2 \cdot (n_1^- \, \mathrm{s}_0)^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \cdot (n_1 \, \mathrm{s}_0)^{n_2^- \cdot n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \\
& = & n_2 \cdot (n_1^- \, \mathrm{s}_0 \cdot (n_1 \, \mathrm{s}_0)^{n_2^-})^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \\
& = & n_2^{n_2} \cdot (n_1^- \, \mathrm{s}_0 \cdot n_2 \cdot n_1 \, \mathrm{s}_0 \cdot n_2^-)^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \\
& \overset{\text{(CC 2)}}{=} & n_2^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \cdot (n_1^- \, \mathrm{s}_0 \cdot n_2 \cdot n_1 \, \mathrm{s}_0 \cdot n_2^-)^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \\
& = & (n_2 \cdot n_1^- \, \mathrm{s}_0 \cdot n_2 \cdot n_1 \, \mathrm{s}_0 \cdot n_2^-)^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \\
& = & ((n_2^{n_1 \, \mathrm{s}_0})^{n_2^-})^{n_2 \, \mathrm{d}_0 \, \mathrm{s}_0} \\
& \overset{\text{(CC 2)}}{=} & ((n_2^{n_1 \, \mathrm{s}_0})^{n_2^-})^{n_2} \\
& = & n_2^{n_1 \, \mathrm{s}_0} \\
& = & (n_2)(n_1\gamma_2'').
\end{array}$$

Suppose given $n_0 \in G\mathrm{N}_0$, n_1, $\check{n}_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$.

Using the assertion (CC 1) from Lemma 49, we obtain

$$\begin{array}{rcl}
(\check{n}_1, n_2)((n_0,n_1)\varepsilon_{1,2}) & \overset{\text{Lem. 59}}{=} & ((\check{n}_1^{n_0})^{n_1}, \lfloor \check{n}_1^{n_0}, n_1\rfloor \cdot n_2^{n_0 \cdot n_1 \partial_1}) \\
& = & ((\check{n}_1^{n_0})^{n_1}, \lfloor \check{n}_1^{n_0}, n_1\rfloor \cdot n_2^{n_0 \cdot n_1 \,\mathrm{d}_0}) \\
& \overset{\text{Lem. 51}}{=} & ((\check{n}_1^{n_0})^{n_1}, ((\check{n}_1^{n_0})^- \,\mathrm{s}_0)^{n_1 \,\mathrm{s}_0} \cdot (\check{n}_1^{n_0} \,\mathrm{s}_0)^{n_1 \,\mathrm{s}_1} \cdot n_2^{n_0 \cdot n_1 \,\mathrm{d}_0}) \\
& \overset{\beta_1}{=} & ((\check{n}_1^{n_0 \,\mathrm{s}_0})^{n_1}, ((\check{n}_1^{n_0 \,\mathrm{s}_0})^- \,\mathrm{s}_0)^{n_1 \,\mathrm{s}_0} \cdot ((\check{n}_1^{n_0 \,\mathrm{s}_0}) \,\mathrm{s}_0)^{n_1 \,\mathrm{s}_1} \cdot n_2^{n_0 \cdot n_1 \,\mathrm{d}_0}) \\
& \overset{\beta_2}{=} & ((\check{n}_1^{n_0 \,\mathrm{s}_0})^{n_1}, ((\check{n}_1^{n_0 \,\mathrm{s}_0})^- \,\mathrm{s}_0)^{n_1 \,\mathrm{s}_0} \cdot ((\check{n}_1^{n_0 \,\mathrm{s}_0}) \,\mathrm{s}_0)^{n_1 \,\mathrm{s}_1} \cdot n_2^{(n_0 \cdot n_1 \,\mathrm{d}_0) \,\mathrm{s}_0 \,\mathrm{s}_0}) \\
& = & (\check{n}_1^{n_0 \,\mathrm{s}_0 \cdot n_1}, (\check{n}_1^- \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0} \cdot (\check{n}_1 \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_1} \cdot n_2^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0}) \\
& = & (\check{n}_1^{n_0 \,\mathrm{s}_0 \cdot n_1}, (\check{n}_1^- \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_0} \cdot (\check{n}_1 \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_1} \cdot (n_2^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1})^{n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0}) \\
& \overset{\text{Lem. 51}}{=} & (\check{n}_1^{n_0 \,\mathrm{s}_0 \cdot n_1}, (\check{n}_1^- \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_0} \cdot (\check{n}_1 \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_1} \cdot (n_2^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1})^{n_1 \,\mathrm{d}_0}) \\
& \overset{\text{(CC 1)}}{=} & (\check{n}_1^{n_0 \,\mathrm{s}_0 \cdot n_1}, (\check{n}_1^- \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_0} \cdot (\check{n}_1 \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_1} \cdot (n_2^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1})^{n_1 \,\mathrm{s}_1}) \\
& = & (\check{n}_1^{n_0 \,\mathrm{s}_0 \cdot n_1}, (\check{n}_1^- \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_0} \cdot (\check{n}_1 \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_1} \cdot n_2^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_1}) \\
& = & (\check{n}_1, n_2)((n_0,n_1)\gamma_2).
\end{array}$$

□

Lemma 77 We have the isotransformation

$$\vartheta = (G\vartheta)_{G \in \mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp)} : \hat{\mathrm{N}} \blacktriangle \mathrm{Rec} \xrightarrow{\sim} \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp}\,.$$

Cf. Lemma 76.

Proof. To show that ϑ is an isotransformation, it remains to show that the quadrangle

$$\begin{tikzcd}
G(\hat{\mathrm{N}} \blacktriangle \mathrm{Rec}) \arrow[r, "G\vartheta"] \arrow[d, "\varphi\, \hat{\mathrm{N}}\, \mathrm{Rec}"'] & G\, \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp} \arrow[d, "\varphi\, \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp}"] \\
\tilde{G}(\hat{\mathrm{N}} \blacktriangle \mathrm{Rec}) \arrow[r, "\tilde{G}\vartheta"'] & \tilde{G}\, \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp}
\end{tikzcd}$$

commutes for $G \xrightarrow{\varphi} \tilde{G}$ in $\lfloor 2,0\rfloor$-$\mathcal{S}imp\mathcal{G}rp$.

To this end, we show that $G\vartheta_k \blacktriangle \varphi_k \overset{!}{=} \varphi\, \hat{\mathrm{N}}\, \mathrm{Rec}_k \blacktriangle \tilde{G}\vartheta_k$ for $k \in \{0, 1, 2\}$; cf. Remark 32, Lemma 76.

Suppose given $n_0 \in G\mathrm{N}_0$, n_1, $\check{n}_1 \in G\mathrm{N}_1$ and $n_2 \in G\mathrm{N}_2$.

We have

$$\begin{array}{rcl}
((n_0,n_1),(\check{n}_1,n_2))((\varphi\, \hat{\mathrm{N}}\, \mathrm{Rec})_2 \blacktriangle \tilde{G}\vartheta_2) & = & (((n_0,n_1),(\check{n}_1,n_2))(\varphi\, \hat{\mathrm{N}}\, \mathrm{Rec})_2)(\tilde{G}\vartheta_2) \\
& \overset{\text{Def. 54.(2)}}{=} & ((n_0\varphi_0, n_1\varphi_1),(\check{n}_1\varphi_1, n_2\varphi_2))(\tilde{G}\vartheta_2) \\
& \overset{\text{Lem. 70}}{=} & n_0\varphi_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1\varphi_1 \,\mathrm{s}_1 \cdot \check{n}_1\varphi_1 \,\mathrm{s}_0 \cdot n_2\varphi_2 \\
& \overset{\text{Def. 31}}{=} & n_0 \,\mathrm{s}_0 \,\mathrm{s}_1\, \varphi_2 \cdot n_1 \,\mathrm{s}_1\, \varphi_2 \cdot \check{n}_1 \,\mathrm{s}_0\, \varphi_2 \cdot n_2\varphi_2 \\
& = & (n_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot n_1 \,\mathrm{s}_1 \cdot \check{n}_1 \,\mathrm{s}_0 \cdot n_2)\varphi_2 \\
& = & (((n_0,n_1),(\check{n}_1,n_2))(G\vartheta_2))\varphi_2 \\
& = & ((n_0,n_1),(\check{n}_1,n_2))(G\vartheta_2 \blacktriangle \varphi_2).
\end{array}$$

We have

$$\begin{aligned}
(n_0,n_1)((\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec})_1 \blacktriangle \tilde{G}\vartheta_1) &= ((n_0,n_1)(\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec})_1)(\tilde{G}\vartheta_1)\\
&\overset{\text{Def. 54.(2)}}{\underset{\text{Lem. 66}}{=}} (n_0\varphi_0, n_1\varphi_1)(\tilde{G}\vartheta_1)\\
&= n_0\varphi_0\,\mathrm{s}_0 \cdot n_1\varphi_1\\
&\overset{\text{Def. 31}}{=} n_0\,\mathrm{s}_0\,\varphi_1 \cdot n_1\varphi_1\\
&= (n_0\,\mathrm{s}_0 \cdot n_1)\varphi_1\\
&= ((n_0,n_1)(G\vartheta_1))\varphi_1\\
&= (n_0,n_1)(G\vartheta_1 \blacktriangle \varphi_1).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
n_0((\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec})_0 \blacktriangle \tilde{G}\vartheta_0) &= (n_0(\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec})_0)(\tilde{G}\vartheta_0)\\
&\overset{\text{Def. 54.(2)}}{\underset{\text{Rem. 65}}{=}} (n_0\varphi_0)(\tilde{G}\vartheta_0)\\
&= n_0\varphi_0\\
&= (n_0(G\vartheta_0))\varphi_0\\
&= n_0(G\vartheta_0 \blacktriangle \varphi_0).
\end{aligned}$$

Altogether, we have

$$\begin{aligned}
\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec} \blacktriangle \tilde{G}\vartheta &= ((\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec} \blacktriangle \tilde{G}\vartheta)_2, (\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec} \blacktriangle \tilde{G}\vartheta)_1, (\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec} \blacktriangle \tilde{G}\vartheta)_0)\\
&\overset{\text{Rem. 32}}{=} ((\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec})_2 \blacktriangle \tilde{G}\vartheta_2, (\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec})_1 \blacktriangle \tilde{G}\vartheta_1, (\varphi\,\hat{\mathrm{N}}\,\mathrm{Rec})_0 \blacktriangle \tilde{G}\vartheta_0)\\
&= (G\vartheta_2 \blacktriangle \varphi_2, G\vartheta_1 \blacktriangle \varphi_1, G\vartheta_0 \blacktriangle \varphi_0)\\
&\overset{\text{Rem. 32}}{=} ((G\vartheta \blacktriangle \varphi)_2, (G\vartheta \blacktriangle \varphi)_1, (G\vartheta \blacktriangle \varphi)_0)\\
&= G\vartheta \blacktriangle \varphi.
\end{aligned}$$

□

4.7.3 Rec and $\hat{\mathrm{N}}$ are equivalences

Proposition 78 We have the isotransformation

$$\zeta : \mathrm{Rec} \blacktriangle \hat{\mathrm{N}} \xrightarrow{\sim} \mathrm{Id}_{2\text{-}\mathcal{CrMod}}$$

from Lemma 75.

We have the isotransformation

$$\vartheta : \hat{\mathrm{N}} \blacktriangle \mathrm{Rec} \xrightarrow{\sim} \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp}}$$

from Lemma 77.

So we have the equivalences of categories

$$\begin{tikzcd}
\lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp} \arrow[rr, bend left, "\hat{\mathrm{N}}"] & & 2\text{-}\mathcal{CrMod}. \arrow[ll, bend left, "\mathrm{Rec}"]
\end{tikzcd}$$

5 From $\lfloor 2,0 \rfloor$-simplicial groups to crossed squares

5.1 The construction for the objects

Suppose given a $\lfloor 2,0 \rfloor$-simplicial group G; cf. Definition 30.

Lemma 79 Consider the groups

$$\begin{array}{lcl} L & := & G_{2;1,2} \\ M & := & G_{2;1}/G_{2;0,1} \\ M' & := & G_{2;2}/G_{2;0,2} \\ P & := & G_2/G_{2;0} \end{array}$$

and the group morphisms

$$\begin{array}{lclclcl} G_{2;1,2} & \xrightarrow{\lambda} & G_{2;1}/G_{2;0,1} & : & k & \longmapsto & kG_{2;0,1} \\ G_{2;1}/G_{2;0,1} & \xrightarrow{\mu} & G_2/G_{2;0} & : & gG_{2;0,1} & \longmapsto & gG_{2;0} \\ G_{2;1,2} & \xrightarrow{\lambda'} & G_{2;2}/G_{2;0,2} & : & k & \longmapsto & kG_{2;0,2} \\ G_{2;2}/G_{2;0,2} & \xrightarrow{\mu'} & G_2/G_{2;0} & : & hG_{2;0,2} & \longmapsto & hG_{2;0}\,. \end{array}$$

$$\begin{tikzcd} G_{2;1,2} \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & G_{2;2}/G_{2;0,2} \arrow[d, "\mu'"] \\ G_{2;1}/G_{2;0,1} \arrow[r, "\mu"'] & G_2/G_{2;0} \end{tikzcd}$$

We have the group morphisms

$$\begin{array}{lclclcl} G_{2;1}/G_{2;0,1} & \xrightarrow{\gamma_{M,L}} & \mathrm{Aut}(G_{2;1,2}) & : & gG_{2;0,1} & \longmapsto & (k \mapsto k^{gG_{2;0,1}} := k^g) \\ G_{2;2}/G_{2;0,2} & \xrightarrow{\gamma_{M',L}} & \mathrm{Aut}(G_{2;1,2}) & : & hG_{2;0,2} & \longmapsto & (k \mapsto k^{hG_{2;0,2}} := k^h) \\ G_2/G_{2;0} & \xrightarrow{\gamma_{P,L}} & \mathrm{Aut}(G_{2;1,2}) & : & fG_{2;0} & \longmapsto & (k \mapsto k^{fG_{2;0}} := k^f) \\ G_2/G_{2;0} & \xrightarrow{\gamma_{P,M}} & \mathrm{Aut}(G_{2;1}/G_{2;0,1}) & : & fG_{2;0} & \longmapsto & (gG_{2;0,1} \mapsto (gG_{2;0,1})^{fG_{2;0}} := g^f G_{2;0,1}) \\ G_2/G_{2;0} & \xrightarrow{\gamma_{P,M'}} & \mathrm{Aut}(G_{2;2}/G_{2;0,2}) & : & fG_{2;0} & \longmapsto & (hG_{2;0,2} \mapsto (hG_{2;0,2})^{fG_{2;0}} := h^f G_{2;0,2}). \end{array}$$

Here $g \in G_{2;1}$, $h \in G_{2;2}$, $f \in G_2$ and $k \in G_{2;1,2}$.

We have the map

$$\begin{array}{rcl} G_{2;1}/G_{2;0,1} \times G_{2;2}/G_{2;0,2} & \xrightarrow{\chi} & G_{2;1,2} \\ (gG_{2;0,1},\, hG_{2;0,2}) & \longmapsto & [g,h] =: \lceil gG_{2;0,1},\, hG_{2;0,2} \rceil, \end{array}$$

where $g \in G_{2;1}$ and $h \in G_{2;2}$.

Then

$$G\,\mathrm{Sq} := (G_{2;1,2},\, G_{2;1}/G_{2;0,1},\, G_{2;2}/G_{2;0,2},\, G_2/G_{2;0},\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

is a crossed square.

Cf. e.g. [**1,** Lem. 70], where a construction of PORTER has been slightly adapted; cf. [**8,** Prop. 7, Proof of Lem. A].

5.2 The construction for the morphisms

Suppose given a morphism $\varphi : G \to H$ of $\lfloor 2,0 \rfloor$-simplicial groups; cf. Definition 31.

Remark 80 We have the group morphisms

$$\begin{array}{rcl}
\varphi_{2;1,2} = \varphi_2|_{G_{2;1,2}}^{H_{2;1,2}} : G_{2;1,2} & \longrightarrow & H_{2;1,2} \\
g_2 & \longmapsto & g_2\varphi_2 \\
\\
\bar{\varphi}_{2;\emptyset} : G_2/G_{2;0} & \longrightarrow & H_2/H_{2;0} \\
g_2 G_{2;0} & \longmapsto & g_2\varphi_2 H_{2;0} \\
\\
\bar{\varphi}_{2;1} : G_{2;1}/G_{2;0,1} & \longrightarrow & H_{2;1}/H_{2;0,1} \\
g_2 G_{2;0,1} & \longmapsto & g_2\varphi_2 H_{2;0,1} \\
\\
\bar{\varphi}_{2;2} : G_{2;2}/G_{2;0,2} & \longrightarrow & H_{2;2}/H_{2;0,2} \\
g_2 G_{2;0,2} & \longmapsto & g_2\varphi_2 H_{2;0,2}\,.
\end{array}$$

Cf. Remark 52.(2), [**1,** Rem. 71], [**1,** Rem. 72] and [**1,** Rem. 73].

Lemma 81 Recall that we have the crossed squares

$$\begin{array}{rcl}
G\,\mathrm{Sq} & = & (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi) \\
& = & (G_{2;1,2},\, G_{2;1}/G_{2;0,1},\, G_{2;2}/G_{2;0,2},\, G_2/G_{2;0},\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)
\end{array}$$

and

$$\begin{array}{rcl}
H\,\mathrm{Sq} & = & (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}}\,,\, \gamma_{\tilde{M}',\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{M}}\,,\, \gamma_{\tilde{P},\tilde{M}'}\,,\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi}) \\
& = & (H_{2;1,2},\, H_{2;1}/H_{2;0,1},\, H_{2;2}/H_{2;0,2},\, H_2/H_{2;0},\, \gamma_{\tilde{M},\tilde{L}}\,,\, \gamma_{\tilde{M}',\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{M}}\,,\, \gamma_{\tilde{P},\tilde{M}'}\,,\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi}).
\end{array}$$

Cf. Lemma 79.

Let $\varphi\,\mathrm{Sq} := (\varphi_{2;1,2},\, \bar{\varphi}_{2;1},\, \bar{\varphi}_{2;2},\, \bar{\varphi}_{2;\emptyset})$; cf. Remark 80.

Then

$$\varphi\,\mathrm{Sq} : G\,\mathrm{Sq} \to H\,\mathrm{Sq}$$

is a morphism of crossed squares.

$$\begin{tikzcd}
 & G_{2;1,2} \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\varphi_{2;1,2}"' near end] & & G_{2;2}/G_{2;0,2} \arrow[dl, "\mu'"] \arrow[dd, "\bar{\varphi}_{2;2}"] \\
G_{2;1}/G_{2;0,1} \arrow[rr, "\mu" near end] \arrow[dd, "\bar{\varphi}_{2;1}"'] & & G_2/G_{2;0} \arrow[dd, "\bar{\varphi}_{2;\emptyset}" near start] & \\
 & H_{2;1,2} \arrow[rr, "\tilde{\lambda}'" near start] \arrow[dl, "\tilde{\lambda}"'] & & H_{2;2}/H_{2;0,2} \arrow[dl, "\tilde{\mu}'"] \\
H_{2;1}/H_{2;0,1} \arrow[rr, "\tilde{\mu}"] & & H_2/H_{2;0} &
\end{tikzcd} .$$

Cf. [**1,** Lem. 74].

5.3 The functor Sq

Definition 82 We shall define the following functor.

$$\begin{array}{rcl}
\mathrm{Sq} : \lfloor 2,0\rfloor\text{-}\mathit{SimpGrp} & \longrightarrow & \mathit{CrSq} \\
\left(\begin{tikzcd} G \arrow[d, "\varphi"] \\ H \end{tikzcd}\right) & \longmapsto & \left(\begin{tikzcd} G\,\mathrm{Sq} \arrow[d, "\varphi\,\mathrm{Sq}"] \\ H\,\mathrm{Sq} \end{tikzcd}\right)
\end{array}$$

(1) Suppose given a $\lfloor 2,0\rfloor$-simplicial group G.

The crossed square $G\,\mathrm{Sq}$ has been defined as follows in Lemma 79.

$$G\,\mathrm{Sq} = \left(\begin{tikzcd}
G_{2;1,2} \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & G_{2;2}/G_{2;0,2} \arrow[d, "\mu'"] \\
G_{2;1}/G_{2;0,1} \arrow[r, "\mu"'] & G_2/G_{2;0}
\end{tikzcd}\right)$$

In the notation of Definition 15 and Lemma 79, we have

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
(G\,\mathrm{Sq})_{1,1} & = & G_{2;1,2} & \gamma^{G\,\mathrm{Sq}}_{1,0} & = & \gamma_{M,L} & \lambda^{1,0}_{G\,\mathrm{Sq}} & = & \lambda \\
(G\,\mathrm{Sq})_{1,0} & = & G_{2;1}/G_{2;0,1} & \gamma^{G\,\mathrm{Sq}}_{0,1} & = & \gamma_{M',L} & \lambda^{0,1}_{G\,\mathrm{Sq}} & = & \lambda' \\
(G\,\mathrm{Sq})_{0,1} & = & G_{2;2}/G_{2;0,2} & \gamma^{1,1}_{G\,\mathrm{Sq}} & = & \gamma_{P,L} & \mu^{G\,\mathrm{Sq}}_{1,0} & = & \mu \\
(G\,\mathrm{Sq})_{0,0} & = & G_2/G_{2;0} & \gamma^{1,0}_{G\,\mathrm{Sq}} & = & \gamma_{P,M} & \mu^{G\,\mathrm{Sq}}_{0,1} & = & \mu' \\
 & & & \gamma^{0,1}_{G\,\mathrm{Sq}} & = & \gamma_{P,M'} & \chi_{G\,\mathrm{Sq}} & = & \chi\,.
\end{array}$$

(2) Suppose given a morphism of $\lfloor 2,0\rfloor$-simplicial groups $G \xrightarrow{\varphi} H$.

The following diagram morphism is a morphism of crossed squares, by Lemma 81.

$$\left(\begin{tikzcd} G \arrow[d, "\varphi"'] \\ H \end{tikzcd}\right)\mathrm{Sq} := \left(\begin{tikzcd}
& G_{2;1,2} \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\varphi_{2;1,2}"' near end] & & G_{2;2}/G_{2;0,2} \arrow[dl, "\mu'"] \arrow[dd, "\bar{\varphi}_{2;2}"] \\
G_{2;1}/G_{2;0,1} \arrow[rr, "\mu" near end] \arrow[dd, "\bar{\varphi}_{2;1}"'] & & G_2/G_{2;0} \arrow[dd, "\bar{\varphi}_{2;\emptyset}" near start] & \\
& H_{2;1,2} \arrow[rr, "\tilde{\lambda}'" near start] \arrow[dl, "\tilde{\lambda}"'] & & H_{2;2}/H_{2;0,2} \arrow[dl, "\tilde{\mu}'"] \\
H_{2;1}/H_{2;0,1} \arrow[rr, "\tilde{\mu}"] & & H_2/H_{2;0} &
\end{tikzcd}\right)$$

For short, $\varphi\,\mathrm{Sq} = (\varphi_{2;1,2},\ \bar{\varphi}_{2;1},\ \bar{\varphi}_{2;2},\ \bar{\varphi}_{2;\emptyset})$.

So we have

$$\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
(\varphi\,\mathrm{Sq})_{1,1} & = & \varphi_{2;1,2} & \lambda^{1,0}_{G\,\mathrm{Sq}} & = & \lambda & \lambda^{1,0}_{H\,\mathrm{Sq}} & = & \tilde{\lambda} \\
(\varphi\,\mathrm{Sq})_{1,0} & = & \bar{\varphi}_{2;1} & \lambda^{0,1}_{G\,\mathrm{Sq}} & = & \lambda' & \lambda^{0,1}_{H\,\mathrm{Sq}} & = & \tilde{\lambda}' \\
(\varphi\,\mathrm{Sq})_{0,1} & = & \bar{\varphi}_{2;2} & \mu^{G\,\mathrm{Sq}}_{1,0} & = & \mu & \mu^{H\,\mathrm{Sq}}_{1,0} & = & \tilde{\mu} \\
(\varphi\,\mathrm{Sq})_{0,0} & = & \bar{\varphi}_{2;\emptyset} & \mu^{G\,\mathrm{Sq}}_{0,1} & = & \mu' & \mu^{H\,\mathrm{Sq}}_{0,1} & = & \tilde{\mu}' \\
 & & & \chi_{G\,\mathrm{Sq}} & = & \chi & \chi_{H\,\mathrm{Sq}} & = & \tilde{\chi}\,.
\end{array}$$

(3) Suppose given morphisms of $\lfloor 2,0\rfloor$-simplicial groups $G \xrightarrow{\varphi} H \xrightarrow{\varphi'} K$.

Then we have

$$\begin{array}{ll}
\text{(a)} & (\mathrm{id}_G)\,\mathrm{Sq} = \mathrm{id}_{(G\,\mathrm{Sq})} \\
\text{(b)} & (\varphi \blacktriangle \varphi')\,\mathrm{Sq} = \varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq}\,.
\end{array}$$

In particular, $\mathrm{Sq} : \lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp \to \mathcal{C}r\mathcal{S}q$ is a functor.

Cf. [**1,** Def. 75].

Remark 83 The functor $\mathrm{Sq} : \lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp \to \mathcal{C}r\mathcal{S}q$ is not an equivalence.

More precisely Sq is not dense.

Cf. [**1,** Rem. 83].

5.4 The functor $\check{\mathrm{Sq}}$

Remark 84 Suppose given a $\lfloor 2,0\rfloor$-simplicial group G.

We have the following group isomorphisms.

(1)

$$\begin{array}{rrcl}
G\psi_{1,0} : & G_{2;1}/G_{2;0,1} & \xrightarrow{\sim} & G_{1;0} \\
 & g_2 G_{2;0,1} & \longmapsto & g_2\,\mathrm{d}_0 \\
 & (g_1\,\mathrm{s}_0 \cdot g_1^{-}\,\mathrm{s}_1)G_{2;0,1} & \longleftarrow\!\shortmid & g_1
\end{array}$$

(2)

$$\begin{array}{rrcl} G\psi_{0,1}: & G_{2;2}/G_{2;0,2} & \xrightarrow{\sim} & G_{1;1} \\ & h_2 G_{2;0,2} & \longmapsto & h_2\,\mathrm{d}_0 \\ & h_1\,\mathrm{s}_0\,G_{2;0,2} & \longleftarrow\!\!\shortmid & h_1 \end{array}$$

(3)

$$\begin{array}{rrcl} G\psi_{0,0}: & G_2/G_{2;0} & \xrightarrow{\sim} & G_1 \\ & k_2 G_{2;0} & \longmapsto & k_2\,\mathrm{d}_0 \\ & k_1\,\mathrm{s}_0\,G_{2;0} & \longleftarrow\!\!\shortmid & k_1 \end{array}$$

Cf. [**1,** Rem. 76].

Moreover, we have $G\psi_{1,1} = \mathrm{id}_{G_{2;1,2}} : G_{2;1,2} \xrightarrow{\sim} G_{2;1,2}$.

Definition 85 Suppose given a $\lfloor 2,0 \rfloor$-simplicial group G.

We have the crossed square

$$G\,\mathrm{Sq} \;=\; (G_{2;1,2},\, G_{2;1}/G_{2;0,1},\, G_{2;2}/G_{2;0,2},\, G_2/G_{2;0},\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi);$$

cf. Lemma 79, where we have written

$$\begin{array}{lcl} L & = & G_{2;1,2} \\ M & = & G_{2;1}/G_{2;0,1} \\ M' & = & G_{2;2}/G_{2;0,2} \\ P & = & G_2/G_{2;0}\,. \end{array}$$

Let

$$\begin{array}{lcl} \tilde{L} & := & G_{2;1,2} \\ \tilde{M} & := & G_{1;0} \\ \tilde{M}' & := & G_{1;1} \\ \tilde{P} & := & G_1\ . \end{array}$$

We have the group isomorphisms

$$\begin{array}{rcl} L = G_{2;1,2} & \xrightarrow[\sim]{G\psi_{1,1}} & G_{2;1,2} = \tilde{L} \\ M = G_{2;1}/G_{2;0,1} & \xrightarrow[\sim]{G\psi_{1,0}} & G_{1;0} = \tilde{M} \\ M' = G_{2;2}/G_{2;0,2} & \xrightarrow[\sim]{G\psi_{0,1}} & G_{1;1} = \tilde{M}' \\ P = G_2/G_{2;0} & \xrightarrow[\sim]{G\psi_{0,0}} & G_1 = \tilde{P}\ ; \end{array}$$

cf. Remark 84.

We abbreviate $G\psi_{1,1} =: \psi_{1,1}$, $G\psi_{1,0} =: \psi_{1,0}$, $G\psi_{0,1} =: \psi_{0,1}$, $G\psi_{0,0} =: \psi_{0,0}$.

Let $\tilde{\lambda} := \psi_{1,1}^{-} \blacktriangle \lambda \blacktriangle \psi_{1,0} = \lambda \blacktriangle \psi_{1,0} : G_{2;1,2} \to G_{1;0}$.

Let $\tilde{\lambda}' := \psi_{1,1}^{-} \blacktriangle \lambda' \blacktriangle \psi_{0,1} = \lambda' \blacktriangle \psi_{0,1} : G_{2;1,2} \to G_{1;1}$.

Let $\tilde{\mu} := \psi_{1,0}^{-} \blacktriangle \mu \blacktriangle \psi_{0,0} : G_{1;0} \to G_1$.

Let $\tilde{\mu}' := \psi_{0,1}^{-} \blacktriangle \mu' \blacktriangle \psi_{0,0} : G_{1;1} \to G_1$.

$$\begin{tikzcd}
& G_{2;1,2} \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\psi_{1,1}"', "\wr"] & & G_{2;2}/G_{2;0,2} \arrow[dl, "\mu'"] \arrow[dd, "\psi_{0,1}", "\wr"'] \\
G_{2;1}/G_{2;0,1} \arrow[rr, crossing over, "\mu"] \arrow[dd, "\psi_{1,0}"', "\wr"] & & G_2/G_{2;0} & \\
& G_{2;1,2} \arrow[rr, "\tilde{\lambda}'"] \arrow[dl, "\tilde{\lambda}"'] & & G_{1;1} \arrow[dl, "\tilde{\mu}'"] \\
G_{1;0} \arrow[rr, "\tilde{\mu}"] & & G_1 \arrow[from=uu, crossing over, "\psi_{0,0}", "\wr"'] &
\end{tikzcd}$$

Recall from Remark 9 that a group isomorphism $\mathfrak{q} : G \xrightarrow{\sim} H$ yields the group isomorphism

$$\begin{array}{rcl} \hat{\mathfrak{q}} : \mathrm{Aut}(G) & \xrightarrow{\sim} & \mathrm{Aut}(H) \\ \alpha & \longmapsto & \mathfrak{q}^{-} \blacktriangle \alpha \blacktriangle \mathfrak{q}. \end{array}$$

Let $\gamma_{\tilde{M},\tilde{L}} := \psi^{-}_{1,0} \blacktriangle \gamma_{M,L} \blacktriangle \hat{\psi}_{1,1} = \psi^{-}_{1,0} \blacktriangle \gamma_{M,L} : G_{1;0} \to \mathrm{Aut}(G_{2;1,2})$.

Let $\gamma_{\tilde{M}',\tilde{L}} := \psi^{-}_{0,1} \blacktriangle \gamma_{M',L} \blacktriangle \hat{\psi}_{1,1} = \psi^{-}_{0,1} \blacktriangle \gamma_{M',L} : G_{1;1} \to \mathrm{Aut}(G_{2;1,2})$.

Let $\gamma_{\tilde{P},\tilde{L}} := \psi^{-}_{0,0} \blacktriangle \gamma_{P,L} \blacktriangle \hat{\psi}_{1,1} = \psi^{-}_{0,0} \blacktriangle \gamma_{P,L} : G_1 \to \mathrm{Aut}(G_{2;1,2})$.

Let $\gamma_{\tilde{P},\tilde{M}} := \psi^{-}_{0,0} \blacktriangle \gamma_{P,M} \blacktriangle \hat{\psi}_{1,0} : G_1 \to \mathrm{Aut}(G_{1;0})$.

Let $\gamma_{\tilde{P},\tilde{M}'} := \psi^{-}_{0,0} \blacktriangle \gamma_{P,M'} \blacktriangle \hat{\psi}_{0,1} : G_1 \to \mathrm{Aut}(G_{1;1})$.

Let $\tilde{\chi} := (\psi^{-}_{1,0} \times \psi^{-}_{0,1}) \blacktriangle \chi \blacktriangle \psi_{1,1} = (\psi^{-}_{1,0} \times \psi^{-}_{0,1}) \blacktriangle \chi : G_{1;0} \times G_{1;1} \to G_{2;1,2}$

Then

$$G\check{\mathrm{S}}\mathrm{q} \quad := \quad (G_{2;1,2},\, G_{1;0},\, G_{1;1},\, G_1,\, \gamma_{\tilde{M},\tilde{L}},\, \gamma_{\tilde{M}',\tilde{L}},\, \gamma_{\tilde{P},\tilde{L}},\, \gamma_{\tilde{P},\tilde{M}},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi})$$

is a crossed square.

Cf. Remark 29.

In other words, we have

$$\begin{array}{rclrclrcl}
(G\check{\mathrm{S}}\mathrm{q})_{1,1} & = & G_{2;1,2} & \gamma^{G\check{\mathrm{S}}\mathrm{q}}_{1,0} & = & \gamma_{\tilde{M},\tilde{L}} = \psi^{-}_{1,0} \blacktriangle \gamma_{M,L} & \lambda^{1,0}_{G\check{\mathrm{S}}\mathrm{q}} & = & \tilde{\lambda} = \lambda \blacktriangle \psi_{1,0} \\
(G\check{\mathrm{S}}\mathrm{q})_{1,0} & = & G_{1;0} & \gamma^{G\check{\mathrm{S}}\mathrm{q}}_{0,1} & = & \gamma_{\tilde{M}',\tilde{L}} = \psi^{-}_{0,1} \blacktriangle \gamma_{M',L} & \lambda^{0,1}_{G\check{\mathrm{S}}\mathrm{q}} & = & \tilde{\lambda}' = \lambda' \blacktriangle \psi_{0,1} \\
(G\check{\mathrm{S}}\mathrm{q})_{0,1} & = & G_{1;1} & \gamma^{1,1}_{G\check{\mathrm{S}}\mathrm{q}} & = & \gamma_{\tilde{P},\tilde{L}} = \psi^{-}_{0,0} \blacktriangle \gamma_{P,L} & \mu^{G\check{\mathrm{S}}\mathrm{q}}_{1,0} & = & \tilde{\mu} = \psi^{-}_{1,0} \blacktriangle \mu \blacktriangle \psi_{0,0} \\
(G\check{\mathrm{S}}\mathrm{q})_{0,0} & = & G_1 & \gamma^{1,0}_{G\check{\mathrm{S}}\mathrm{q}} & = & \gamma_{\tilde{P},\tilde{M}} = \psi^{-}_{0,0} \blacktriangle \gamma_{P,M} \blacktriangle \hat{\psi}_{1,0} & \mu^{G\check{\mathrm{S}}\mathrm{q}}_{0,1} & = & \tilde{\mu}' = \psi^{-}_{0,1} \blacktriangle \mu' \blacktriangle \psi_{0,0} \\
 & & & \gamma^{0,1}_{G\check{\mathrm{S}}\mathrm{q}} & = & \gamma_{\tilde{P},\tilde{M}'} = \psi^{-}_{0,0} \blacktriangle \gamma_{P,M'} \blacktriangle \hat{\psi}_{0,1} & \chi_{G\check{\mathrm{S}}\mathrm{q}} & = & \tilde{\chi} = (\psi^{-}_{1,0} \times \psi^{-}_{0,1}) \blacktriangle \chi \\
 & & & & & & \kappa_{G\check{\mathrm{S}}\mathrm{q}} & = & \tilde{\kappa} = \tilde{\lambda} \blacktriangle \tilde{\mu} = \tilde{\lambda}' \blacktriangle \tilde{\mu}' \,.
\end{array}$$

Cf. Definition 15.

$$G\check{\mathrm{Sq}} = \left(\begin{tikzcd} G_{2;1,2} \arrow[r, "\lambda' \blacktriangle G\psi_{0,1}"] \arrow[d, "\lambda \blacktriangle G\psi_{1,0}"'] & G_{1;1} \arrow[d, "G\psi^{-}_{0,1} \blacktriangle \mu' \blacktriangle G\psi_{0,0}"] \\ G_{1;0} \arrow[r, "G\psi^{-}_{1,0} \blacktriangle \mu \blacktriangle G\psi_{0,0}"'] & G_1 \end{tikzcd}\right) = \left(\begin{tikzcd} G_{2;1,2} \arrow[r, "\lambda' \blacktriangle \psi_{0,1}"] \arrow[d, "\lambda \blacktriangle \psi_{1,0}"'] & G_{1;1} \arrow[d, "\psi^{-}_{0,1} \blacktriangle \mu' \blacktriangle \psi_{0,0}"] \\ G_{1;0} \arrow[r, "\psi^{-}_{1,0} \blacktriangle \mu \blacktriangle \psi_{0,0}"'] & G_1 \end{tikzcd}\right)$$

Note that the actions and morphisms in $G\check{\mathrm{Sq}}$ are given as follows.

Suppose given $g_2 \in G_{2;1,2}$, $h_2 \in G_{2;1}$, $k_2 \in G_{2;2}$, $g_1 \in G_{1;0}$, $h_1 \in G_{1;1}$ and $k_1 \in G_1$.

$$\begin{aligned}
g_2^{g_1} &= (g_2)(g_1\gamma_{1,0}^{G\check{\mathrm{Sq}}}) && \text{(action via } \gamma_{1,0}^{G\check{\mathrm{Sq}}}\text{)}\\
&= (g_2)(g_1(\psi^{-}_{1,0} \blacktriangle \gamma_{M,L}))\\
&= (g_2)((g_1\psi^{-}_{1,0})\gamma_{M,L})\\
&= (g_2)(((g_1\,\mathrm{s}_0 \cdot g_1^{-}\,\mathrm{s}_1)G_{2;0,1})\gamma_{M,L})\\
&= g_2^{g_1\,\mathrm{s}_0 \cdot g_1^{-}\,\mathrm{s}_1} && \text{(action via conjugation)}
\end{aligned}$$

$$\begin{aligned}
g_2^{h_1} &= (g_2)(h_1\gamma_{0,1}^{G\check{\mathrm{Sq}}}) && \text{(action via } \gamma_{0,1}^{G\check{\mathrm{Sq}}}\text{)}\\
&= (g_2)(h_1(\psi^{-}_{0,1} \blacktriangle \gamma_{M',L}))\\
&= (g_2)((h_1\psi^{-}_{0,1})\gamma_{M',L})\\
&= (g_2)((h_1\,\mathrm{s}_0\,G_{2;0,2})\gamma_{M',L})\\
&= g_2^{h_1\,\mathrm{s}_0} && \text{(action via conjugation)}
\end{aligned}$$

$$\begin{aligned}
g_2^{k_1} &= (g_2)(k_1\gamma^{1,1}_{G\check{\mathrm{Sq}}}) && \text{(action via } \gamma^{1,1}_{G\check{\mathrm{Sq}}}\text{)}\\
&= (g_2)(k_1(\psi^{-}_{0,0} \blacktriangle \gamma_{P,L}))\\
&= (g_2)((k_1\psi^{-}_{0,0})\gamma_{P,L})\\
&= (g_2)((k_1\,\mathrm{s}_0\,G_{2;0})\gamma_{P,L})\\
&= g_2^{k_1\,\mathrm{s}_0} && \text{(action via conjugation)}
\end{aligned}$$

$$\begin{aligned}
g_1^{k_1} &= (g_1)(k_1\gamma^{1,0}_{G\check{\mathrm{Sq}}}) && \text{(action via } \gamma^{1,0}_{G\check{\mathrm{Sq}}}\text{)}\\
&= (g_1)(k_1(\psi^{-}_{0,0} \blacktriangle \gamma_{P,M} \blacktriangle \hat{\psi}_{1,0}))\\
&= (g_1)(((k_1\psi^{-}_{0,0})\gamma_{P,M})\hat{\psi}_{1,0})\\
&= (g_1)(((k_1\,\mathrm{s}_0\,G_{2;0})\gamma_{P,M})\hat{\psi}_{1,0})\\
&= (g_1)(\psi^{-}_{1,0} \blacktriangle ((k_1\,\mathrm{s}_0\,G_{2;0})\gamma_{P,M}) \blacktriangle \psi_{1,0})\\
&= ((g_1\psi^{-}_{1,0})((k_1\,\mathrm{s}_0\,G_{2;0})\gamma_{P,M}))\psi_{1,0}\\
&= (((g_1\,\mathrm{s}_0 \cdot g_1^{-}\,\mathrm{s}_1)G_{2;0,1})((k_1\,\mathrm{s}_0\,G_{2;0})\gamma_{P,M}))\psi_{1,0}\\
&= ((g_1\,\mathrm{s}_0 \cdot g_1^{-}\,\mathrm{s}_1)^{k_1\,\mathrm{s}_0}G_{2;0,1})\psi_{1,0}\\
&= (g_1\,\mathrm{s}_0 \cdot g_1^{-}\,\mathrm{s}_1)^{k_1\,\mathrm{s}_0}\,\mathrm{d}_0\\
&= (g_1\,\mathrm{s}_0\,\mathrm{d}_0 \cdot g_1^{-}\,\mathrm{s}_1\,\mathrm{d}_0)^{k_1\,\mathrm{s}_0\,\mathrm{d}_0}\\
&= (g_1 \cdot g_1^{-}\,\mathrm{d}_0\,\mathrm{s}_0)^{k_1}\\
&= g_1^{k_1} && \text{(action via conjugation)}
\end{aligned}$$

$$
\begin{aligned}
h_1^{k_1} &= (h_1)(k_1 \gamma^{0,1}_{G\check{\mathrm{S}}\mathrm{q}}) && \text{(action via } \gamma^{0,1}_{G\check{\mathrm{S}}\mathrm{q}}\text{)} \\
&= (h_1)(k_1(\psi^{-}_{0,0} \blacktriangle \gamma_{P,M'} \blacktriangle \hat{\psi}_{0,1})) \\
&= (h_1)(((k_1 \psi^{-}_{0,0})\gamma_{P,M'})\hat{\psi}_{0,1}) \\
&= (h_1)(((k_1 \,\mathrm{s}_0\, G_{2;0})\gamma_{P,M'})\hat{\psi}_{0,1}) \\
&= (h_1)(\psi^{-}_{0,1} \blacktriangle ((k_1 \,\mathrm{s}_0\, G_{2;0})\gamma_{P,M'}) \blacktriangle \psi_{0,1}) \\
&= ((h_1 \psi^{-}_{0,1})((k_1 \,\mathrm{s}_0\, G_{2;0})\gamma_{P,M'}))\psi_{0,1} \\
&= ((h_1 \,\mathrm{s}_0\, G_{2;0,2})((k_1 \,\mathrm{s}_0\, G_{2;0})\gamma_{P,M'}))\psi_{0,1} \\
&= ((h_1 \,\mathrm{s}_0)^{k_1 \,\mathrm{s}_0} G_{2;0,2})\psi_{0,1} \\
&= (h_1 \,\mathrm{s}_0)^{k_1 \,\mathrm{s}_0}\, \mathrm{d}_0 \\
&= (h_1 \,\mathrm{s}_0\, \mathrm{d}_0)^{k_1 \,\mathrm{s}_0\, \mathrm{d}_0} \\
&= h_1^{k_1} && \text{(action via conjugation)}
\end{aligned}
$$

$$
\begin{aligned}
g_2 \lambda^{1,0}_{G\check{\mathrm{S}}\mathrm{q}} &= g_2(\lambda \blacktriangle \psi_{1,0}) \\
&= (g_2 \lambda)\psi_{1,0} \\
&= (g_2 G_{2;0,1})\psi_{1,0} \\
&= g_2 \,\mathrm{d}_0
\end{aligned}
$$

$$
\begin{aligned}
g_2 \lambda^{0,1}_{G\check{\mathrm{S}}\mathrm{q}} &= g_2(\lambda' \blacktriangle \psi_{0,1}) \\
&= (g_2 \lambda')\psi_{0,1} \\
&= (g_2 G_{2;0,2})\psi_{0,1} \\
&= g_2 \,\mathrm{d}_0
\end{aligned}
$$

$$
\begin{aligned}
g_1 \mu^{G\check{\mathrm{S}}\mathrm{q}}_{1,0} &= g_1(\psi^{-}_{1,0} \blacktriangle \mu \blacktriangle \psi_{0,0}) \\
&= ((g_1 \psi^{-}_{1,0})\mu)\psi_{0,0} \\
&= (((g_1 \,\mathrm{s}_0 \cdot g_1^{-} \,\mathrm{s}_1)G_{2;0,1})\mu)\psi_{0,0} \\
&= ((g_1 \,\mathrm{s}_0 \cdot g_1^{-} \,\mathrm{s}_1)G_{2;0})\psi_{0,0} \\
&= (g_1 \,\mathrm{s}_0 \cdot g_1^{-} \,\mathrm{s}_1)\,\mathrm{d}_0 \\
&= g_1 \,\mathrm{s}_0\, \mathrm{d}_0 \cdot g_1^{-} \,\mathrm{s}_1\, \mathrm{d}_0 \\
&= g_1 \cdot g_1^{-} \,\mathrm{d}_0\, \mathrm{s}_0 \\
&= g_1
\end{aligned}
$$

$$
\begin{aligned}
h_1 \mu^{G\check{\mathrm{S}}\mathrm{q}}_{0,1} &= h_1(\psi^{-}_{0,1} \blacktriangle \mu' \blacktriangle \psi_{0,0}) \\
&= ((h_1 \psi^{-}_{0,1})\mu')\psi_{0,0} \\
&= ((h_1 \,\mathrm{s}_0\, G_{2;0,2})\mu')\psi_{0,0} \\
&= (h_1 \,\mathrm{s}_0\, G_{2;0})\psi_{0,0} \\
&= h_1 \,\mathrm{s}_0\, \mathrm{d}_0 \\
&= h_1
\end{aligned}
$$

$$
\begin{aligned}
\lceil g_1, h_1 \rceil &= (g_1, h_1)\chi_{G\check{\mathrm{S}}\mathrm{q}} \\
&= (g_1, h_1)((\psi^{-}_{1,0} \times \psi^{-}_{0,1}) \blacktriangle \chi) \\
&= ((g_1, h_1)(\psi^{-}_{1,0} \times \psi^{-}_{0,1}))\chi \\
&= ((g_1 \,\mathrm{s}_0 \cdot g_1^{-} \,\mathrm{s}_1)G_{2;0,1}, h_1 \,\mathrm{s}_0\, G_{2;0,2})\chi \\
&= [g_1 \,\mathrm{s}_0 \cdot g_1^{-} \,\mathrm{s}_1, h_1 \,\mathrm{s}_0] && \text{(commutator bracket)}
\end{aligned}
$$

Moreover,

$$G\psi := (G\psi_{1,1},\, G\psi_{1,0},\, G\psi_{0,1},\, G\psi_{0,0}) = (\psi_{1,1},\, \psi_{1,0},\, \psi_{0,1},\, \psi_{0,0}) : G\,\mathrm{Sq} \to G\check{\mathrm{S}}\mathrm{q}$$

is an isomorphism of crossed squares.

Cf. Remark 29.

Remark 86 Suppose given a morphism $G \xrightarrow{\varphi} H$ of $\lfloor 2,0 \rfloor$-simplicial groups.

We have the following commutative quadrangles; cf. Remark 52.(1), Remark 80 and Remark 84.

(1)

$$\begin{tikzcd}
G_{2;1}/G_{2;0,1} \arrow[r, "G\psi_{1,0}", "\sim"'] \arrow[d, "\bar{\varphi}_{2;1}"'] \arrow[rd, phantom, "\circlearrowleft"] & G_{1;0} \arrow[d, "\varphi_{1;0}"] \\
H_{2;1}/H_{2;0,1} \arrow[r, "\sim", "H\psi_{1,0}"'] & H_{1;0}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
G_{2;2}/G_{2;0,2} \arrow[r, "G\psi_{0,1}", "\sim"'] \arrow[d, "\bar{\varphi}_{2;2}"'] \arrow[rd, phantom, "\circlearrowleft"] & G_{1;1} \arrow[d, "\varphi_{1;1}"] \\
H_{2;2}/H_{2;0,2} \arrow[r, "\sim", "H\psi_{0,1}"'] & H_{1;1}
\end{tikzcd}$$

(3)

$$\begin{tikzcd}
G_{2}/G_{2;0} \arrow[r, "G\psi_{0,0}", "\sim"'] \arrow[d, "\bar{\varphi}_{2;\emptyset}"'] \arrow[rd, phantom, "\circlearrowleft"] & G_{1} \arrow[d, "\varphi_{1}"] \\
H_{2}/H_{2;0} \arrow[r, "\sim", "H\psi_{0,0}"'] & H_{1}
\end{tikzcd}$$

Proof.

Ad (1). Suppose given $g_2 \in G_{2;1}$.

Then we have

$$\begin{aligned}
(g_2 G_{2;0,1})(\bar{\varphi}_{2;1} \blacktriangle H\psi_{1,0}) &= ((g_2 G_{2;0,1})\bar{\varphi}_{2;1})(H\psi_{1,0}) \\
&= (g_2 \varphi_2 H_{2;0,1})(H\psi_{1,0}) \\
&= g_2 \varphi_2 \, \mathrm{d}_0 \\
&= g_2 \, \mathrm{d}_0 \, \varphi_1 \\
&= (g_2 \, \mathrm{d}_0)\varphi_{1;0} \\
&= ((g_2 G_{2;0,1})(G\psi_{1,0}))\varphi_{1;0} \\
&= (g_2 G_{2;0,1})(G\psi_{1,0} \blacktriangle \varphi_{1;0}).
\end{aligned}$$

Ad (2). Suppose given $g_2 \in G_{2;2}$.

Then we have

$$\begin{aligned}
(g_2 G_{2;0,2})(\bar{\varphi}_{2;2} \blacktriangle H\psi_{0,1}) &= ((g_2 G_{2;0,2})\bar{\varphi}_{2;2})(H\psi_{0,1}) \\
&= (g_2 \varphi_2 H_{2;0,2})(H\psi_{0,1}) \\
&= g_2 \varphi_2 \, \mathrm{d}_0 \\
&= g_2 \, \mathrm{d}_0 \, \varphi_1 \\
&= (g_2 \, \mathrm{d}_0)\varphi_{1;1} \\
&= ((g_2 G_{2;0,2})(G\psi_{0,1}))\varphi_{1;1} \\
&= (g_2 G_{2;0,2})(G\psi_{0,1} \blacktriangle \varphi_{1;1}).
\end{aligned}$$

Ad (3). Suppose given $g_2 \in G_2$.

Then we have

$$\begin{array}{rcl}(g_2 G_{2;0})(\bar{\varphi}_{2;\emptyset} \blacktriangle H\psi_{0,0}) & = & ((g_2 G_{2;0})\bar{\varphi}_{2;\emptyset})(H\psi_{0,0}) \\ & = & (g_2 \varphi_2 H_{2;0})(H\psi_{0,0}) \\ & = & g_2 \varphi_2 \,\mathrm{d}_0 \\ & = & g_2 \,\mathrm{d}_0\, \varphi_1 \\ & = & (g_2 \,\mathrm{d}_0)\varphi_1 \\ & = & ((g_2 G_{2;0})(G\psi_{0,0}))\varphi_1 \\ & = & (g_2 G_{2;0})(G\psi_{0,0} \blacktriangle \varphi_1).\end{array}$$

□

Definition 87 Given a morphism $G \xrightarrow{\varphi} H$ of $\lfloor 2,0\rfloor$-simplicial groups, we let, following Remark 1,

$$\varphi\check{\mathrm{Sq}} := (G\psi)^- \blacktriangle \varphi \,\mathrm{Sq} \blacktriangle H\psi : G\check{\mathrm{Sq}} \to H\check{\mathrm{Sq}},$$

which is a morphism of crossed squares; cf. Lemma 81 and Definition 85.

By Remark 1, we have the following functor

$$\begin{array}{rcl}\check{\mathrm{Sq}} : \lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp} & \longrightarrow & \mathcal{CrSq} \\ \left(\begin{array}{c} G \\ \downarrow{\scriptstyle \varphi} \\ H \end{array}\right) & \longmapsto & \left(\begin{array}{c} G\check{\mathrm{Sq}} \\ \downarrow{\scriptstyle \varphi\check{\mathrm{Sq}}} \\ H\check{\mathrm{Sq}} \end{array}\right).\end{array}$$

In addition, Remark 1 gives the isotransformation

$$\psi = (G\psi)_{G\in\mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp})} : \mathrm{Sq} \xrightarrow{\sim} \check{\mathrm{Sq}}.$$

Recall that

$$G\psi = (G\psi_{1,1},\, G\psi_{1,0},\, G\psi_{0,1},\, G\psi_{0,0})$$

as defined in Remark 84.

Recall from Definition 85 that given the crossed square

$$G\,\mathrm{Sq} = \left(\begin{tikzcd} G_{2;1,2} \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & G_{2;2}/G_{2;0,2} \arrow[d, "\mu'"] \\ G_{2;1}/G_{2;0,1} \arrow[r, "\mu"'] & G_2/G_{2;0} \end{tikzcd}\right)$$

we have defined the crossed square

$$G\check{\mathrm{Sq}} = \left(\begin{tikzcd} G_{2;1,2} \arrow[r, "\lambda' \blacktriangle G\psi_{0,1}"] \arrow[d, "\lambda \blacktriangle G\psi_{1,0}"'] & G_{1;1} \arrow[d, "G\psi_{0,1}^- \blacktriangle \mu' \blacktriangle G\psi_{0,0}"] \\ G_{1;0} \arrow[r, "G\psi_{1,0}^- \blacktriangle \mu \blacktriangle G\psi_{0,0}"'] & G_1 \end{tikzcd}\right).$$

We write

$$G\check{\mathrm{Sq}} = (G_{2;1,2},\, G_{1;0},\, G_{1;1},\, G_1,\, \gamma_{1,0}^{G\check{\mathrm{Sq}}},\, \gamma_{0,1}^{G\check{\mathrm{Sq}}},\, \gamma^{1,1}_{G\check{\mathrm{Sq}}},\, \gamma^{1,0}_{G\check{\mathrm{Sq}}},\, \gamma^{0,1}_{G\check{\mathrm{Sq}}},\, \lambda^{1,0}_{G\check{\mathrm{Sq}}},\, \lambda^{0,1}_{G\check{\mathrm{Sq}}},\, \mu_{1,0}^{G\check{\mathrm{Sq}}},\, \mu_{0,1}^{G\check{\mathrm{Sq}}},\, \chi_{G\check{\mathrm{Sq}}})$$

and

$$H\check{\mathrm{Sq}} = (H_{2;1,2},\, H_{1;0},\, H_{1;1},\, H_1,\, \gamma_{1,0}^{H\check{\mathrm{Sq}}},\, \gamma_{0,1}^{H\check{\mathrm{Sq}}},\, \gamma^{1,1}_{H\check{\mathrm{Sq}}},\, \gamma^{1,0}_{H\check{\mathrm{Sq}}},\, \gamma^{0,1}_{H\check{\mathrm{Sq}}},\, \lambda^{1,0}_{H\check{\mathrm{Sq}}},\, \lambda^{0,1}_{H\check{\mathrm{Sq}}},\, \mu_{1,0}^{H\check{\mathrm{Sq}}},\, \mu_{0,1}^{H\check{\mathrm{Sq}}},\, \chi_{H\check{\mathrm{Sq}}}).$$

So we get

$$\begin{pmatrix} G \\ \varphi\big\downarrow \\ H \end{pmatrix}\check{\mathrm{Sq}} = \left(\begin{tikzcd}
& G_{2;1,2} \arrow[rr, "\lambda^{0,1}_{G\check{\mathrm{Sq}}}"] \arrow[dl, "\lambda^{1,0}_{G\check{\mathrm{Sq}}}"'] \arrow[dd, "\varphi_{2;1,2}"' near end] & & G_{1;1} \arrow[dl, "\mu_{0,1}^{G\check{\mathrm{Sq}}}"'] \arrow[dd, "G\psi^-_{0,1}\blacktriangle\bar{\varphi}_{2;2}\blacktriangle H\psi_{0,1}"] \\
G_{1;0} \arrow[rr, "\mu_{1,0}^{G\check{\mathrm{Sq}}}" near end] \arrow[dd, "G\psi^-_{1,0}\blacktriangle\bar{\varphi}_{2;1}\blacktriangle H\psi_{1,0}"'] & & G_1 \arrow[dd, "G\psi^-_{0,0}\blacktriangle\bar{\varphi}_{2;\emptyset}\blacktriangle H\psi_{0,0}" near start] & \\
& H_{2;1,2} \arrow[rr, "\lambda^{0,1}_{H\check{\mathrm{Sq}}}" near start] \arrow[dl, "\lambda^{1,0}_{H\check{\mathrm{Sq}}}"'] & & H_{1;1} \arrow[dl, "\mu_{0,1}^{H\check{\mathrm{Sq}}}"] \\
H_{1;0} \arrow[rr, "\mu_{1,0}^{H\check{\mathrm{Sq}}}"] & & H_1 &
\end{tikzcd}\right).$$

In other words,

$$\begin{aligned}
(\varphi\check{\mathrm{Sq}})_{1,1} &= \varphi_{2;1,2} \\
(\varphi\check{\mathrm{Sq}})_{1,0} &= G\psi^-_{1,0}\blacktriangle\bar{\varphi}_{2;1}\blacktriangle H\psi_{1,0} \overset{\text{Rem. 86.(1)}}{=} \varphi_{1;0} \\
(\varphi\,\mathrm{Sq})_{0,1} &= G\psi^-_{0,1}\blacktriangle\bar{\varphi}_{2;2}\blacktriangle H\psi_{0,1} \overset{\text{Rem. 86.(2)}}{=} \varphi_{1;1} \\
(\varphi\,\mathrm{Sq})_{0,0} &= G\psi^-_{0,0}\blacktriangle\bar{\varphi}_{2;\emptyset}\blacktriangle H\psi_{0,0} \overset{\text{Rem. 86.(3)}}{=} \varphi_1 \,.
\end{aligned}$$

So we get

$$\begin{pmatrix} G \\ \varphi\big\downarrow \\ H \end{pmatrix}\check{\mathrm{Sq}} = \left(\begin{tikzcd}
& G_{2;1,2} \arrow[rr, "\lambda^{0,1}_{G\check{\mathrm{Sq}}}"] \arrow[dl, "\lambda^{1,0}_{G\check{\mathrm{Sq}}}"'] \arrow[dd, "\varphi_{2;1,2}"' near end] & & G_{1;1} \arrow[dl, "\mu_{0,1}^{G\check{\mathrm{Sq}}}"'] \arrow[dd, "\varphi_{1;1}"] \\
G_{1;0} \arrow[rr, "\mu_{1,0}^{G\check{\mathrm{Sq}}}" near end] \arrow[dd, "\varphi_{1;0}"'] & & G_1 \arrow[dd, "\varphi_1" near start] & \\
& H_{2;1,2} \arrow[rr, "\lambda^{0,1}_{H\check{\mathrm{Sq}}}" near start] \arrow[dl, "\lambda^{1,0}_{H\check{\mathrm{Sq}}}"'] & & H_{1;1} \arrow[dl, "\mu_{0,1}^{H\check{\mathrm{Sq}}}"] \\
H_{1;0} \arrow[rr, "\mu_{1,0}^{H\check{\mathrm{Sq}}}"] & & H_1 &
\end{tikzcd}\right).$$

6 From crossed squares to 2-crossed modules

The following construction of the functor To : $\mathcal{CrSq} \to 2\text{-}\mathcal{CrMod}$ is due to Conduché [**3,** Cor. 3.5].

6.1 The construction for the objects

Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

$$C : \quad \begin{tikzcd} L \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & M' \arrow[d, "\mu'"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd}$$

Cf. Definition 15.

Remark 88 The following map is a group morphism as a composite of group morphisms.

$$\begin{array}{rcl} \alpha := \mu \blacktriangle \gamma_{P,M'} : M & \longrightarrow & \mathrm{Aut}(M') \\ m & \longmapsto & (m' \mapsto (m')(m\alpha) = (m')(m(\mu \blacktriangle \gamma_{P,M'})) = m'^{\,m\mu} =: m'^{\,m}) \end{array}$$

In particular, we may form the semidirect product $M \ltimes_\alpha M'$, in which

$$\begin{array}{rcl} (m, m') \cdot (\tilde{m}, \tilde{m}') & = & (m \cdot \tilde{m}, m'^{\,\tilde{m}} \cdot \tilde{m}') \\ & = & (m \cdot \tilde{m}, m'^{\,\tilde{m}\mu} \cdot \tilde{m}'), \end{array}$$

for (m, m'), $(\tilde{m}, \tilde{m}') \in M \ltimes_\alpha M'$.

Remark 89 Suppose given (m, m'), $(\tilde{m}, \tilde{m}') \in M \ltimes_\alpha M'$.
Then we have

$$(m, m')^{(\tilde{m}, \tilde{m}')} = (m^{\tilde{m}\mu}, (\tilde{m}'^{-})^{(m\mu)^{\tilde{m}\mu}} \cdot m'^{\,\tilde{m}\mu} \cdot \tilde{m}').$$

Proof. We have

$$\begin{array}{rcl} (m, m')^{(\tilde{m}, \tilde{m}')} & \overset{\text{Rem. 8.(2)}}{=} & (m^{\tilde{m}}, (\tilde{m}'^{-})^{m^{\tilde{m}}} \cdot m'^{\,\tilde{m}} \cdot \tilde{m}') \\ & \overset{\text{Rem. 88}}{=} & (m^{\tilde{m}}, (\tilde{m}'^{-})^{(m^{\tilde{m}})\mu} \cdot m'^{\,\tilde{m}\mu} \cdot \tilde{m}') \\ & = & (m^{\tilde{m}}, (\tilde{m}'^{-})^{(m\mu)^{\tilde{m}\mu}} \cdot m'^{\,\tilde{m}\mu} \cdot \tilde{m}') \\ & \underset{\text{for } \mu}{\overset{\text{(CM 2)}}{=}} & (m^{\tilde{m}\mu}, (\tilde{m}'^{-})^{(m\mu)^{\tilde{m}\mu}} \cdot m'^{\,\tilde{m}\mu} \cdot \tilde{m}'). \end{array}$$

□

Remark 90 We have the following group morphism.

$$\begin{array}{rcl} L & \xrightarrow{\partial_2} & M \ltimes_\alpha M' \\ l & \longmapsto & (l\lambda, l^-\lambda') \end{array}$$

Proof. Suppose given l, $\tilde{l} \in L$.

Then we have

$$\begin{array}{rcl} (l \cdot \tilde{l})\partial_2 & = & ((l \cdot \tilde{l})\lambda, (l \cdot \tilde{l})^-\lambda') \\ & = & (l\lambda \cdot \tilde{l}\lambda, \tilde{l}^-\lambda' \cdot l^-\lambda') \\ & = & (l\lambda \cdot \tilde{l}\lambda, \tilde{l}^-\lambda' \cdot l^-\lambda' \cdot \tilde{l}\lambda' \cdot \tilde{l}^-\lambda') \\ & = & (l\lambda \cdot \tilde{l}\lambda, (l^-\lambda')^{\tilde{l}\lambda'} \cdot \tilde{l}^-\lambda') \\ & \underset{\text{for } \mu'}{\overset{\text{(CM 2)}}{=}} & (l\lambda \cdot \tilde{l}\lambda, (l^-\lambda')^{\tilde{l}\lambda'\mu'} \cdot \tilde{l}^-\lambda') \\ & = & (l\lambda \cdot \tilde{l}\lambda, (l^-\lambda')^{\tilde{l}\kappa} \cdot \tilde{l}^-\lambda') \\ & = & (l\lambda \cdot \tilde{l}\lambda, (l^-\lambda')^{(\tilde{l}\lambda)\mu} \cdot \tilde{l}^-\lambda') \\ & = & (l\lambda, l^-\lambda') \cdot (\tilde{l}\lambda, \tilde{l}^-\lambda') \\ & = & l\partial_2 \cdot \tilde{l}\partial_2 . \end{array}$$

☐

Remark 91 We have the following group morphism.

$$\begin{array}{rcl} M \ltimes_\alpha M' & \xrightarrow{\partial_1} & P \\ (m, m') & \longmapsto & m\mu \cdot m'\mu' \end{array}$$

Proof. Suppose given (m, m'), $(\tilde{m}, \tilde{m}') \in M \ltimes_\alpha M'$.

Then we have

$$\begin{array}{rcl} ((m, m') \cdot (\tilde{m}, \tilde{m}'))\partial_1 & = & (m \cdot \tilde{m}, m'^{\tilde{m}\mu} \cdot \tilde{m}')\partial_1 \\ & = & (m \cdot \tilde{m})\mu \cdot (m'^{\tilde{m}\mu} \cdot \tilde{m}')\mu' \\ & = & m\mu \cdot \tilde{m}\mu \cdot m'^{\tilde{m}\mu}\mu' \cdot \tilde{m}'\mu' \\ & \underset{\text{for } \mu'}{\overset{\text{(CM 1)}}{=}} & m\mu \cdot \tilde{m}\mu \cdot (m'\mu')^{\tilde{m}\mu} \cdot \tilde{m}'\mu' \\ & = & m\mu \cdot \tilde{m}\mu \cdot \tilde{m}^-\mu \cdot m'\mu' \cdot \tilde{m}\mu \cdot \tilde{m}'\mu' \\ & = & m\mu \cdot m'\mu' \cdot \tilde{m}\mu \cdot \tilde{m}'\mu' \\ & = & (m, m')\partial_1 \cdot (\tilde{m}, \tilde{m}')\partial_1 . \end{array}$$

☐

Lemma 92 We have the following group morphism.

$$\begin{array}{rcl} P & \xrightarrow{\beta_1} & \operatorname{Aut}(M \ltimes_\alpha M') \\ p & \longmapsto & ((m, m') \mapsto (m, m')^p := (m^p, m'^p)) \end{array}$$

Proof. We recall that

$$\begin{array}{rcl} (m, m') \cdot (\tilde{m}, \tilde{m}') & = & (m \cdot \tilde{m}, m'^{\tilde{m}} \cdot \tilde{m}') \\ & = & (m \cdot \tilde{m}, m'^{\tilde{m}\mu} \cdot \tilde{m}') \end{array}$$

for m, $\tilde{m} \in M$ and m', $\tilde{m}' \in M'$.

We use Remark 6.

Suppose given p, $\tilde{p} \in P$ and (m, m'), $(\tilde{m}, \tilde{m}') \in M \ltimes_\alpha M'$.

Then we have

$$\begin{array}{rcl} (m, m')^1 & = & (m^1, m'^1) \\ & = & (m, m'). \end{array}$$

Moreover, we obtain

$$\begin{aligned} ((m,m')^p)^{\tilde{p}} &= (m^p, m'^{\,p})^{\tilde{p}} \\ &= (m^{p\cdot\tilde{p}}, m'^{\,p\cdot\tilde{p}}) \\ &= (m,m')^{p\cdot\tilde{p}}. \end{aligned}$$

Finally, we calculate

$$\begin{aligned} ((m,m')\cdot(\tilde{m},\tilde{m}'))^p &= (m\cdot\tilde{m}, m'^{\,\tilde{m}\mu}\cdot\tilde{m}')^p \\ &= ((m\cdot\tilde{m})^p, (m'^{\,\tilde{m}\mu}\cdot\tilde{m}')^p) \\ &= (m^p\cdot\tilde{m}^p, (m'^{\,\tilde{m}\mu})^p\cdot\tilde{m}'^{\,p}) \\ &= (m^p\cdot\tilde{m}^p, m'^{\,\tilde{m}\mu\cdot p}\cdot\tilde{m}'^{\,p}) \\ &= (m^p\cdot\tilde{m}^p, m'^{\,p\cdot p^-\cdot\tilde{m}\mu\cdot p}\cdot\tilde{m}'^{\,p}) \\ &= (m^p\cdot\tilde{m}^p, (m'^{\,p})^{p^-\cdot\tilde{m}\mu\cdot p}\cdot\tilde{m}'^{\,p}) \\ &= (m^p\cdot\tilde{m}^p, (m'^{\,p})^{(\tilde{m}\mu)^p}\cdot\tilde{m}'^{\,p}) \\ &\underset{\text{for }\mu}{\overset{\text{(CM 1)}}{=}} (m^p\cdot\tilde{m}^p, (m'^{\,p})^{\tilde{m}^p\mu}\cdot\tilde{m}'^{\,p}) \\ &= (m^p, m'^{\,p})\cdot(\tilde{m}^p, \tilde{m}'^{\,p}) \\ &= (m,m')^p\cdot(\tilde{m},\tilde{m}')^p. \end{aligned}$$

□

The following construction of the total 2-crossed module of a crossed square is due to CONDUCHÉ [**3,** Cor. 3.5], refining a construction of LODAY [**7,** Def. and Lem. 3.1, Def. 5.1].

Lemma 93 Consider the groups L, $M \ltimes_\alpha M'$ and P; cf. Remark 88.

With the help of Remark 90, Remark 91, Lemma 92 and Definition 15 we have the group morphisms

$$\begin{array}{llllll} L & \xrightarrow{\partial_2} & M \ltimes_\alpha M' & : & l & \longmapsto & (l\lambda, l^-\lambda') \\ M \ltimes_\alpha M' & \xrightarrow{\partial_1} & P & : & (m,m') & \longmapsto & m\mu\cdot m'\mu' \\ P & \xrightarrow{\beta_1} & \operatorname{Aut}(M \ltimes_\alpha M') & : & p & \longmapsto & ((m,m')\mapsto (m,m')^p = (m^p, m'^{\,p})) \\ P & \xrightarrow{\gamma_{P,L}} & \operatorname{Aut}(L) & : & p & \longmapsto & (l\mapsto l^p). \end{array}$$

Let

$$\begin{array}{rcl} (M \ltimes_\alpha M')\times(M \ltimes_\alpha M') & \xrightarrow{\zeta} & L \\ ((m,m'),(\tilde{m},\tilde{m}')) & \longmapsto & \lfloor (m,m'),(\tilde{m},\tilde{m}')\rfloor := \lceil m^{\tilde{m}}, \tilde{m}'\rceil. \end{array}$$

In other words, $((m,m'),(\tilde{m},\tilde{m}'))\zeta = (m^{\tilde{m}}, \tilde{m}')\chi$.

Then

$$C\,\mathrm{To} := (L,\ M \ltimes_\alpha M',\ P,\ \partial_2,\ \partial_1,\ \beta_1,\ \gamma_{P,L},\ \zeta)$$

is a 2-crossed module, called the *total* 2-*crossed module* of C.

$$L \xrightarrow{\partial_2} M \ltimes_\alpha M' \xrightarrow{\partial_1} P$$

Proof.

Ad (2CM 1). Suppose given l, $\tilde{l}\in L$.

Then we have

$$\begin{aligned} \lfloor \tilde{l}\partial_2, l\partial_2\rfloor &= \lfloor (\tilde{l}\lambda, \tilde{l}^-\lambda'), (l\lambda, l^-\lambda')\rfloor \\ &= \lceil (\tilde{l}\lambda)^{l\lambda}, l^-\lambda'\rceil \\ &= \lceil (\tilde{l}^l)\lambda, l^-\lambda'\rceil \\ &\overset{\text{(CS 4.3)}}{=} (\tilde{l}^l)^-\cdot(\tilde{l}^l)^{l^-\lambda'} \end{aligned}$$

$$\begin{aligned}
&\underset{\text{for }\lambda'}{\overset{\text{(CM 2)}}{=}} && (\tilde{l}^{l})^{-} \cdot (\tilde{l}^{l})^{l^{-}} \\
&= && (\tilde{l}^{l})^{-} \cdot \tilde{l} \\
&= && l^{-} \cdot \tilde{l}^{-} \cdot l \cdot \tilde{l} \\
&= && [l, \tilde{l}].
\end{aligned}$$

Ad (2CM 2). Suppose given $(m, m') \in M \ltimes_{\alpha} M'$ and $l \in L$.

Then we have

$$\begin{aligned}
\lfloor (m,m'), l\partial_2 \rfloor \cdot \lfloor l\partial_2, (m,m') \rfloor
&= && \lfloor (m,m'), (l\lambda, l^{-}\lambda') \rfloor \cdot \lfloor (l\lambda, l^{-}\lambda'), (m,m') \rfloor \\
&= && \lceil m^{l\lambda}, l^{-}\lambda' \rceil \cdot \lceil (l\lambda)^{m}, m' \rceil \\
&\underset{\text{for }\lambda}{\overset{\text{(CM 1)}}{=}} && \lceil m^{l\lambda}, l^{-}\lambda' \rceil \cdot \lceil (l^{m})\lambda, m' \rceil \\
&\overset{\text{(CS 4.4)}}{=} && l^{(m^{l\lambda})} \cdot l^{-} \cdot \lceil (l^{m})\lambda, m' \rceil \\
&\overset{\text{(CS 4.3)}}{=} && l^{(m^{l\lambda})} \cdot l^{-} \cdot (l^{m})^{-} \cdot (l^{m})^{m'} \\
&= && l^{l^{-}\lambda \cdot m \cdot l\lambda} \cdot l^{-} \cdot (l^{m})^{-} \cdot (l^{m})^{m'} \\
&= && ((l^{l^{-}\lambda})^{m})^{l\lambda} \cdot l^{-} \cdot (l^{m})^{-} \cdot (l^{m})^{m'} \\
&\underset{\text{for }\lambda}{\overset{\text{(CM 2)}}{=}} && ((l^{l^{-}})^{m})^{l\lambda} \cdot l^{-} \cdot (l^{m})^{-} \cdot (l^{m})^{m'} \\
&= && (l^{m})^{l\lambda} \cdot l^{-} \cdot (l^{m})^{-} \cdot (l^{m})^{m'} \\
&\underset{\text{for }\lambda}{\overset{\text{(CM 2)}}{=}} && (l^{m})^{l} \cdot l^{-} \cdot (l^{m})^{-} \cdot (l^{m})^{m'} \\
&= && l^{-} \cdot l^{m} \cdot l \cdot l^{-} \cdot (l^{m})^{-} \cdot (l^{m})^{m'} \\
&= && l^{-} \cdot (l^{m})^{m'} \\
&\overset{\text{Rem. 16}}{=} && l^{-} \cdot (l^{m\mu})^{m'\mu'} \\
&= && l^{-} \cdot l^{m\mu \cdot m'\mu'} \\
&= && l^{-} \cdot l^{(m,m')\partial_1}.
\end{aligned}$$

Ad (2CM 3). Suppose given (m, m'), $(\tilde{m}, \tilde{m}') \in M \ltimes_{\alpha} M'$.

We have to show that

$$\lfloor (m,m'), (\tilde{m},\tilde{m}') \rfloor \partial_2 \overset{!}{=} ((m,m')^{-})^{(\tilde{m},\tilde{m}')} \cdot (m,m')^{(\tilde{m},\tilde{m}')\partial_1}.$$

In order to calculate, we write $a := m$, $b := \tilde{m}$, $c := m'$ and $d := \tilde{m}'$.

Then a, $b \in M$ and c, $d \in M'$.

We have to show that

$$(a,c)^{(b.d)} \cdot \lfloor (a,c), (b,d) \rfloor \partial_2 \overset{!}{=} (a,c)^{(b,d)\partial_1}.$$

We obtain

$$\begin{aligned}
& && (a,c)^{(b,d)} \cdot \lfloor (a,c), (b,d) \rfloor \partial_2 \\
&\overset{\text{Rem. 89}}{=} && (a^{b\mu}, (d^{-})^{(a\mu)^{b\mu}} \cdot c^{b\mu} \cdot d) \cdot \lfloor (a,c), (b,d) \rfloor \partial_2 \\
&= && (a^{b\mu}, (d^{-})^{(a^{b})\mu} \cdot c^{b\mu} \cdot d) \cdot \lfloor (a,c), (b,d) \rfloor \partial_2 \\
&\underset{\text{for }\mu}{\overset{\text{(CM 2)}}{=}} && (a^{b}, (d^{-})^{(a^{b})\mu} \cdot c^{b\mu} \cdot d) \cdot \lfloor (a,c), (b,d) \rfloor \partial_2 \\
&= && (a^{b}, (d^{-})^{(a^{b})\mu} \cdot c^{b\mu} \cdot d) \cdot \lceil a^{b}, d \rceil \partial_2 \\
&= && (a^{b}, (d^{-})^{(a^{b})\mu} \cdot c^{b\mu} \cdot d) \cdot (\lceil a^{b}, d \rceil \lambda, (\lceil a^{b}, d \rceil \lambda')^{-}) \\
&\overset{\text{(CS 4.1)}}{=} && (a^{b}, (d^{-})^{(a^{b})\mu} \cdot c^{b\mu} \cdot d) \cdot ((a^{b})^{-} \cdot (a^{b})^{d\mu'}, (\lceil a^{b}, d \rceil \lambda')^{-}) \\
&\overset{\text{(CS 4.2)}}{=} && (a^{b}, (d^{-})^{(a^{b})\mu} \cdot c^{b\mu} \cdot d) \cdot ((a^{b})^{-} \cdot (a^{b})^{d\mu'}, d^{-} \cdot d^{(a^{b})\mu}) \\
&= && (a^{b} \cdot (a^{b})^{-} \cdot (a^{b})^{d\mu'}, (d^{-})^{(a^{b})\mu \cdot ((a^{b})^{-} \cdot (a^{b})^{d\mu'})\mu} \cdot c^{b\mu \cdot ((a^{b})^{-} \cdot (a^{b})^{d\mu'})\mu} \cdot d^{((a^{b})^{-} \cdot (a^{b})^{d\mu'})\mu} \cdot d^{-} \cdot d^{(a^{b})\mu})
\end{aligned}$$

$$
\begin{aligned}
&= ((a^b)^{d\mu'}, (d^-)^{(a^b)\mu \cdot ((a^b)\mu)^- \cdot ((a^b)^{d\mu'})\mu} \cdot c^{b\mu \cdot ((a^b)\mu)^- \cdot ((a^b)^{d\mu'})\mu} \cdot d^{((a^b)\mu)^- \cdot ((a^b)^{d\mu'})\mu} \cdot d^- \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, (d^-)^{((a^b)^{d\mu'})\mu} \cdot c^{b\mu \cdot ((a^b)\mu)^- \cdot ((a^b)^{d\mu'})\mu} \cdot d^{((a^b)\mu)^- \cdot ((a^b)^{d\mu'})\mu} \cdot d^- \cdot d^{(a^b)\mu}) \\
&\overset{\text{(CM 1)}}{\underset{\text{for } \mu}{=}} ((a^b)^{d\mu'}, (d^-)^{((a^b)\mu)^{d\mu'}} \cdot c^{b\mu \cdot ((a\mu)^{b\mu})^- \cdot ((a^b)\mu)^{d\mu'}} \cdot d^{((a^b)\mu)^- \cdot ((a^b)\mu)^{d\mu'}} \cdot d^- \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, (d^-)^{(d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot c^{b\mu \cdot (b\mu)^- \cdot (a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^- \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, (d^-)^{(d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^- \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, ((d^-)^{(d\mu')^-})^{(a^b)\mu \cdot d\mu'} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^- \cdot d^{(a^b)\mu}) \\
&\overset{\text{(CM 2)}}{\underset{\text{for } \mu'}{=}} ((a^b)^{d\mu'}, ((d^-)^{d^-})^{(a^b)\mu \cdot d\mu'} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^- \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, (d^-)^{(a^b)\mu \cdot d\mu'} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^- \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, d^- \cdot d \cdot (d^-)^{(a^b)\mu \cdot d\mu'} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^- \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, d^- \cdot ((d^-)^{(a^b)\mu \cdot d\mu'} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'})^{d^-} \cdot d^{(a^b)\mu}) \\
&\overset{\text{(CM 2)}}{\underset{\text{for } \mu'}{=}} ((a^b)^{d\mu'}, d^- \cdot ((d^-)^{(a^b)\mu \cdot d\mu'} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu'})^{d^-\mu'} \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, d^- \cdot (d^-)^{(a^b)\mu \cdot d\mu' \cdot d^-\mu'} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu' \cdot d^-\mu'} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot d\mu' \cdot d^-\mu'} \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, d^- \cdot (d^-)^{(a^b)\mu} \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu} \cdot d^{(a^b)\mu}) \\
&= ((a^b)^{d\mu'}, d^- \cdot (c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu})^{d^{(a^b)\mu}}) \\
&\overset{\text{(CM 2)}}{\underset{\text{for } \mu'}{=}} ((a^b)^{d\mu'}, d^- \cdot (c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu})^{(d^{(a^b)\mu})\mu'}) \\
&\overset{\text{(CM 1)}}{\underset{\text{for } \mu'}{=}} ((a^b)^{d\mu'}, d^- \cdot (c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu})^{(d\mu')^{(a^b)\mu}}) \\
&= ((a^b)^{d\mu'}, d^- \cdot c^{(a\mu)^- \cdot b\mu \cdot (d\mu')^- \cdot (a^b)\mu \cdot (a^b)^-\mu \cdot d\mu' \cdot (a^b)\mu} \cdot d^{((a^b)\mu)^- \cdot (d\mu')^- \cdot (a^b)\mu \cdot (a^b)^-\mu \cdot d\mu' \cdot (a^b)\mu}) \\
&= ((a^b)^{d\mu'}, d^- \cdot c^{(a\mu)^- \cdot b\mu \cdot (a^b)\mu} \cdot d) \\
&= ((a^b)^{d\mu'}, d^- \cdot c^{(a\mu)^- \cdot b\mu \cdot (a\mu)^{b\mu}} \cdot d) \\
&= ((a^b)^{d\mu'}, d^- \cdot c^{(a\mu)^- \cdot b\mu \cdot (b\mu)^- \cdot a\mu \cdot b\mu} \cdot d) \\
&= ((a^b)^{d\mu'}, d^- \cdot c^{b\mu} \cdot d) \\
&= ((a^b)^{d\mu'}, (c^{b\mu})^d) \\
&\overset{\text{(CM 2)}}{\underset{\text{for } \mu,\ \mu'}{=}} ((a^{b\mu})^{d\mu'}, (c^{b\mu})^{d\mu'}) \\
&= (a, c)^{b\mu \cdot d\mu'} \\
&= (a, c)^{(b,d)\partial_1}.
\end{aligned}
$$

Ad (2CM 4). Suppose given $p \in P$ and (m, m'), $(\tilde{m}, \tilde{m}') \in M \ltimes_\alpha M'$.

Then we have

$$
\begin{aligned}
\lfloor (m, m'), (\tilde{m}, \tilde{m}') \rfloor^p &= \lceil m^{\tilde{m}}, \tilde{m}' \rceil^p \\
&\overset{\text{(CS 4.7)}}{=} \lceil (m^{\tilde{m}})^p, (\tilde{m}')^p \rceil \\
&= \lceil (\tilde{m}^- \cdot m \cdot \tilde{m})^p, (\tilde{m}')^p \rceil \\
&= \lceil (\tilde{m}^p)^- \cdot m^p \cdot \tilde{m}^p, (\tilde{m}')^p \rceil \\
&= \lceil (m^p)^{(\tilde{m}^p)}, (\tilde{m}')^p \rceil \\
&= \lfloor (m^p, (m')^p), (\tilde{m}^p, (\tilde{m}')^p) \rfloor \\
&\overset{\text{Lem. 92}}{=} \lfloor (m, m')^p, (\tilde{m}, \tilde{m}')^p \rfloor .
\end{aligned}
$$

Ad (2CM 5). Suppose given $l \in L$.

Then we have

$$
\begin{aligned}
l\partial_2\partial_1 &= (l\lambda, l^-\lambda')\partial_1 \\
&= l\lambda\mu \cdot l^-\lambda'\mu' \\
&= l\kappa \cdot l^-\kappa \\
&= l\kappa \cdot (l\kappa)^- \\
&= 1.
\end{aligned}
$$

Ad (2CM 6). Suppose given $p \in P$ and $(m, m') \in M \ltimes_\alpha M'$.

Then we have

$$\begin{aligned} ((m,m')^p)\partial_1 &\overset{\text{Lem. 92}}{=} (m^p, (m')^p)\partial_1 \\ &= m^p\mu \cdot (m')^p\mu' \\ &\underset{\text{for } \mu,\, \mu'}{\overset{\text{(CM 1)}}{=}} (m\mu)^p \cdot (m'\mu')^p \\ &= (m\mu \cdot m'\mu')^p \\ &= ((m,m')\partial_1)^p. \end{aligned}$$

Ad (2CM 7). Suppose given $p \in P$ and $l \in L$.

Then we have

$$\begin{aligned} (l^p)\partial_2 &= (l^p\lambda, (l^p)^-\lambda') \\ &= (l^p\lambda, (l^-)^p\lambda') \\ &\overset{\text{Rem. 16}}{=} ((l\lambda)^p, (l^-\lambda')^p) \\ &\overset{\text{Lem. 92}}{=} (l\lambda, l^-\lambda')^p \\ &= (l\partial_2)^p. \end{aligned}$$

Ad (2CM 8). Suppose given (m, m'), $(\tilde{m}, \tilde{m}')$, $(\tilde{\tilde{m}}, \tilde{\tilde{m}}') \in M \ltimes_\alpha M'$.

We have to show that

$$\lfloor (m,m'), (\tilde{m},\tilde{m}') \cdot (\tilde{\tilde{m}},\tilde{\tilde{m}}') \rfloor \overset{!}{=} \lfloor (m,m')^{(\tilde{m},\tilde{m}')}, (\tilde{\tilde{m}},\tilde{\tilde{m}}') \rfloor \cdot \lfloor (m,m'), (\tilde{m},\tilde{m}') \rfloor^{(\tilde{\tilde{m}},\tilde{\tilde{m}}')\partial_1}.$$

In order to calculate, we write $a := m$, $b := \tilde{m}$, $c := \tilde{\tilde{m}}$, $d := m'$, $e := \tilde{m}'$ and $f := \tilde{\tilde{m}}'$.

Then a, b, $c \in M$ and d, e, $f \in M'$.

We have to show that

$$\lfloor (a,d), (b,e) \cdot (c,f) \rfloor \overset{!}{=} \lfloor (a,d)^{(b,e)}, (c,f) \rfloor \cdot \lfloor (a,d), (b,e) \rfloor^{(c,f)\partial_1}.$$

We obtain

$$\begin{aligned} \lfloor (a,d), (b,e) \cdot (c,f) \rfloor &= \lfloor (a,d), (b \cdot c, e^{c\mu} \cdot f) \rfloor \\ &= \lceil a^{b\cdot c}, e^{c\mu} \cdot f \rceil \\ &\overset{\text{(CS 4.6)}}{=} \lceil a^{b\cdot c}, f \rceil \cdot \lceil a^{b\cdot c}, e^{c\mu} \rceil^{f} \\ &\overset{\text{Rem. 16}}{=} \lceil a^{b\cdot c}, f \rceil \cdot \lceil a^{b\cdot c}, e^{c\mu} \rceil^{f\mu'} \\ &= \lceil a^{b\cdot c}, f \rceil \cdot \lceil (a^b)^c, e^{c\mu} \rceil^{f\mu'} \\ &\underset{\text{for } \mu}{\overset{\text{(CM 2)}}{=}} \lceil a^{b\cdot c}, f \rceil \cdot \lceil (a^b)^{c\mu}, e^{c\mu} \rceil^{f\mu'} \\ &\overset{\text{(CS 4.7)}}{=} \lceil a^{b\cdot c}, f \rceil \cdot \lceil a^b, e \rceil^{c\mu\cdot f\mu'} \\ &= \lceil (a^b)^c, f \rceil \cdot \lceil a^b, e \rceil^{c\mu\cdot f\mu'} \\ &\underset{\text{for } \mu}{\overset{\text{(CM 2)}}{=}} \lceil (a^{b\mu})^c, f \rceil \cdot \lceil a^b, e \rceil^{c\mu\cdot f\mu'} \\ &= \lfloor (a^{b\mu}, (e^-)^{(a\mu)^{b\mu}} \cdot d^{b\mu} \cdot e), (c,f) \rfloor \cdot \lfloor (a,d), (b,e) \rfloor^{c\mu\cdot f\mu'} \\ &\overset{\text{Rem. 89}}{=} \lfloor (a,d)^{(b,e)}, (c,f) \rfloor \cdot \lfloor (a,d), (b,e) \rfloor^{c\mu\cdot f\mu'} \\ &= \lfloor (a,d)^{(b,e)}, (c,f) \rfloor \cdot \lfloor (a,d), (b,e) \rfloor^{(c,f)\partial_1}. \end{aligned}$$

Ad (2CM 9). Suppose given (m, m'), $(\tilde{m}, \tilde{m}')$, $(\tilde{\tilde{m}}, \tilde{\tilde{m}}') \in M \ltimes_\alpha M'$.

We have to show that

$$\begin{aligned} & \lfloor (m,m') \cdot (\tilde{m},\tilde{m}'), (\tilde{\tilde{m}},\tilde{\tilde{m}}') \rfloor \\ \overset{!}{=}\; & \lfloor (m,m'), (\tilde{\tilde{m}},\tilde{\tilde{m}}') \rfloor \cdot \lfloor (\tilde{m},\tilde{m}')^{(\tilde{\tilde{m}},\tilde{\tilde{m}}')}, \lfloor (m,m'), (\tilde{\tilde{m}},\tilde{\tilde{m}}') \rfloor\partial_2 \rfloor \cdot \lfloor (\tilde{m},\tilde{m}'), (\tilde{\tilde{m}},\tilde{\tilde{m}}') \rfloor. \end{aligned}$$

In order to calculate, we write $a := m$, $b := \tilde{m}$, $c := \tilde{\tilde{m}}$, $d := m'$, $e := \tilde{m}'$ and $f := \tilde{\tilde{m}}'$.

Then $a,\ b,\ c \in M$ and $d,\ e,\ f \in M'$.

We have to show that

$$\lfloor (a,d)\cdot(b,e),(c,f)\rfloor \overset{!}{=} \lfloor (a,d),(c,f)\rfloor \cdot \lfloor (b,e)^{(c,f)}, \lfloor (a,d),(c,f)\rfloor\partial_2\rfloor \cdot \lfloor (b,e),(c,f)\rfloor .$$

We obtain

$$\begin{aligned}
\lfloor (a,d)\cdot(b,e),(c,f)\rfloor &= \lfloor (a\cdot b, d^{b\mu}\cdot e),(c,f)\rfloor \\
&= \lceil (a\cdot b)^c, f\rceil \\
&= \lceil a^c\cdot b^c, f\rceil \\
&\overset{\text{(CS 4.5)}}{=} \lceil a^c, f\rceil^{b^c}\cdot\lceil b^c, f\rceil \\
&= (\lceil a^c, f\rceil^{b^c})^{\lceil a^c,f\rceil\cdot\lceil a^c,f\rceil^-}\cdot\lceil b^c, f\rceil \\
&= ((\lceil a^c, f\rceil^{b^c})^{\lceil a^c,f\rceil})^{\lceil a^c,f\rceil^-}\cdot\lceil b^c, f\rceil \\
&= \lceil a^c, f\rceil\cdot(\lceil a^c, f\rceil^{b^c})^{\lceil a^c,f\rceil}\cdot\lceil a^c, f\rceil^-\cdot\lceil b^c, f\rceil \\
&= \lceil a^c, f\rceil\cdot((\lceil a^c, f\rceil^{\lceil a^c,f\rceil^-})^{b^c})^{\lceil a^c,f\rceil}\cdot\lceil a^c, f\rceil^-\cdot\lceil b^c, f\rceil \\
&\underset{\text{for }\lambda}{\overset{\text{(CM 2)}}{=}} \lceil a^c, f\rceil\cdot((\lceil a^c, f\rceil^{\lceil a^c,f\rceil^-\lambda})^{b^c})^{\lceil a^c,f\rceil\lambda}\cdot\lceil a^c, f\rceil^-\cdot\lceil b^c, f\rceil \\
&= \lceil a^c, f\rceil\cdot\lceil a^c, f\rceil^{\lceil a^c,f\rceil^-\lambda\cdot b^c\cdot\lceil a^c,f\rceil\lambda}\cdot\lceil a^c, f\rceil^-\cdot\lceil b^c, f\rceil \\
&= \lceil a^c, f\rceil\cdot\lceil a^c, f\rceil^{(b^c)^{\lceil a^c,f\rceil\lambda}}\cdot\lceil a^c, f\rceil^-\cdot\lceil b^c, f\rceil \\
&\overset{\text{(CS 4.4)}}{=} \lceil a^c, f\rceil\cdot\lceil (b^c)^{\lceil a^c,f\rceil\lambda}, \lceil a^c, f\rceil^-\lambda'\rceil\cdot\lceil b^c, f\rceil \\
&= \lceil a^c, f\rceil\cdot\lfloor (b^c,(f^-)^{(b\mu)^{c\mu}}\cdot e^{c\mu}\cdot f),(\lceil a^c, f\rceil\lambda, \lceil a^c, f\rceil^-\lambda')\rfloor\cdot\lceil b^c, f\rceil \\
&\underset{\text{for }\mu}{\overset{\text{(CM 2)}}{=}} \lceil a^c, f\rceil\cdot\lfloor (b^{c\mu},(f^-)^{(b\mu)^{c\mu}}\cdot e^{c\mu}\cdot f),(\lceil a^c, f\rceil\lambda, \lceil a^c, f\rceil^-\lambda')\rfloor\cdot\lceil b^c, f\rceil \\
&\overset{\text{Rem. 89}}{=} \lceil a^c, f\rceil\cdot\lfloor (b,e)^{(c,f)},(\lceil a^c, f\rceil\lambda, \lceil a^c, f\rceil^-\lambda')\rfloor\cdot\lceil b^c, f\rceil \\
&= \lceil a^c, f\rceil\cdot\lfloor (b,e)^{(c,f)},\lceil a^c, f\rceil\partial_2\rfloor\cdot\lceil b^c, f\rceil \\
&= \lfloor (a,d),(c,f)\rfloor \cdot \lfloor (b,e)^{(c,f)}, \lfloor (a,d),(c,f)\rfloor\partial_2\rfloor \cdot \lfloor (b,e),(c,f)\rfloor .
\end{aligned}$$

□

6.2 The construction for the morphisms

Suppose given crossed squares

$$C = (L,\ M,\ M',\ P) = (L,\ M,\ M',\ P,\ \gamma_{M,L}\ ,\ \gamma_{M',L}\ ,\ \gamma_{P,L}\ ,\ \gamma_{P,M}\ ,\ \gamma_{P,M'}\ ,\ \lambda,\ \lambda',\ \mu,\ \mu',\ \chi)$$

and

$$\tilde{C} = (\tilde{L},\ \tilde{M},\ \tilde{M}',\ \tilde{P}) = (\tilde{L},\ \tilde{M},\ \tilde{M}',\ \tilde{P},\ \gamma_{\tilde{M},\tilde{L}},\ \gamma_{\tilde{M}',\tilde{L}},\ \gamma_{\tilde{P},\tilde{L}},\ \gamma_{\tilde{P},\tilde{M}},\ \gamma_{\tilde{P},\tilde{M}'},\ \tilde{\lambda},\ \tilde{\lambda}',\ \tilde{\mu},\ \tilde{\mu}',\ \tilde{\chi}).$$

Suppose given a morphism of crossed squares

$$\mathfrak{c} = (\mathfrak{l},\ \mathfrak{m},\ \mathfrak{m}',\ \mathfrak{p}) : (L,\ M,\ M',\ P) \to (\tilde{L},\ \tilde{M},\ \tilde{M}',\ \tilde{P}).$$

$$\begin{tikzcd}
 & & L \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\mathfrak{l}"] & & M' \arrow[dl, "\mu'"] \arrow[dd, "\mathfrak{m}'"] \\
 & M \arrow[rr, "\mu"] \arrow[dd, "\mathfrak{m}"'] & & P \arrow[dd, "\mathfrak{p}"] & \\
\mathfrak{c} : & & \tilde{L} \arrow[rr, "\tilde{\lambda}'"] \arrow[dl, "\tilde{\lambda}"'] & & \tilde{M}' \arrow[dl, "\tilde{\mu}'"] \\
 & \tilde{M} \arrow[rr, "\tilde{\mu}"] & & \tilde{P} &
\end{tikzcd}$$

Let $\alpha := \mu \blacktriangle \gamma_{P,M'} : M \to \operatorname{Aut}(M')$.

Let $\tilde{\alpha} := \tilde{\mu} \blacktriangle \gamma_{\tilde{P},\tilde{M}'} : \tilde{M} \to \operatorname{Aut}(\tilde{M}')$.

Remark 94 We have the following group morphism.

$$\begin{array}{rcl} M \ltimes_\alpha M' & \xrightarrow{\mathfrak{m}\ltimes\mathfrak{m}'} & \tilde{M} \ltimes_{\tilde{\alpha}} \tilde{M}' \\ (m, m') & \longmapsto & (m, m')(\mathfrak{m} \ltimes \mathfrak{m}') := (m\mathfrak{m}, m'\mathfrak{m}') \end{array}$$

Proof. Suppose given (m, m'), $(\check{m}, \check{m}') \in M \ltimes_\alpha M'$.

We recall that

$$\begin{array}{rcl} (m, m') \cdot (\check{m}, \check{m}') & = & (m \cdot \check{m}, m'^{\check{m}} \cdot \check{m}') \\ & = & (m \cdot \check{m}, m'^{\check{m}\mu} \cdot \check{m}'). \end{array}$$

We have

$$\begin{array}{rcl} (m, m')(\mathfrak{m} \ltimes \mathfrak{m}') \cdot (\check{m}, \check{m}')(\mathfrak{m} \ltimes \mathfrak{m}') & = & (m\mathfrak{m}, m'\mathfrak{m}') \cdot (\check{m}\mathfrak{m}, \check{m}'\mathfrak{m}') \\ & = & (m\mathfrak{m} \cdot \check{m}\mathfrak{m}, (m'\mathfrak{m}')^{\check{m}\mathfrak{m}\tilde{\mu}} \cdot \check{m}'\mathfrak{m}') \\ & \overset{\text{(CSM 1.3)}}{=} & (m\mathfrak{m} \cdot \check{m}\mathfrak{m}, (m'\mathfrak{m}')^{\check{m}\mu\mathfrak{p}} \cdot \check{m}'\mathfrak{m}') \\ & \overset{\text{(CSM 1.4)}}{=} & (m\mathfrak{m} \cdot \check{m}\mathfrak{m}, (m'^{\check{m}\mu})\mathfrak{m}' \cdot \check{m}'\mathfrak{m}') \\ & = & ((m \cdot \check{m})\mathfrak{m}, (m'^{\check{m}\mu} \cdot \check{m}')\mathfrak{m}') \\ & = & (m \cdot \check{m}, m'^{\check{m}\mu} \cdot \check{m}')(\mathfrak{m} \ltimes \mathfrak{m}') \\ & = & ((m, m') \cdot (\check{m}, \check{m}'))(\mathfrak{m} \ltimes \mathfrak{m}'). \end{array}$$

□

Lemma 95 Recall that we have the 2-crossed modules

$$C\,\mathrm{To} =: (L,\ M \ltimes_\alpha M',\ P,\ \partial_2,\ \partial_1,\ \beta_1,\ \gamma_{P,L},\ \zeta)$$

and

$$\tilde{C}\,\mathrm{To} =: (\tilde{L},\ \tilde{M} \ltimes_{\tilde{\alpha}} \tilde{M}',\ \tilde{P},\ \tilde{\partial}_2,\ \tilde{\partial}_1,\ \tilde{\beta}_1,\ \gamma_{\tilde{P},\tilde{L}},\ \tilde{\zeta}).$$

Cf. Lemma 93.

Recall the group morphisms $\mathfrak{l}$, $\mathfrak{m} \ltimes \mathfrak{m}'$, $\mathfrak{p}$; cf. Definition 19 and Remark 94.

Then

$$\mathfrak{c}\,\mathrm{To} := (\mathfrak{l},\ \mathfrak{m} \ltimes \mathfrak{m}',\ \mathfrak{p}) : (L,\ M \ltimes_\alpha M',\ P) \mapsto (\tilde{L},\ \tilde{M} \ltimes_{\tilde{\alpha}} \tilde{M}',\ \tilde{P})$$

is a morphism of 2-crossed modules.

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "\mathfrak{l}"] & M \ltimes_\alpha M' \arrow[r, "\partial_1"] \arrow[d, "\mathfrak{m}\ltimes\mathfrak{m}'"] & P \arrow[d, "\mathfrak{p}"] \\ \tilde{L} \arrow[r, "\tilde{\partial}_2"] & \tilde{M} \ltimes_{\tilde{\alpha}} \tilde{M}' \arrow[r, "\tilde{\partial}_1"] & \tilde{P} \end{tikzcd}$$

Proof. First, we have to show that the diagram is commutative.

Suppose given $l \in L$ and $(m, m') \in M \ltimes_\alpha M'$.

Then we have

$$\begin{aligned} l(\partial_2 \blacktriangle (\mathfrak{m} \ltimes \mathfrak{m}')) &= (l\partial_2)(\mathfrak{m} \ltimes \mathfrak{m}') \\ &= (l\lambda, l^- \lambda')(\mathfrak{m} \ltimes \mathfrak{m}') \\ &= (l\lambda\mathfrak{m}, l^- \lambda'\mathfrak{m}') \\ &\overset{\text{(CSM 1.1)}}{=} (l\mathfrak{l}\tilde{\lambda}, l^- \lambda'\mathfrak{m}') \\ &\overset{\text{(CSM 1.2)}}{=} (l\mathfrak{l}\tilde{\lambda}, l^- \mathfrak{l}\tilde{\lambda}') \\ &= (l\mathfrak{l}\tilde{\lambda}, (l\mathfrak{l})^- \tilde{\lambda}') \\ &= (l\mathfrak{l})\tilde{\partial}_2 \\ &= l(\mathfrak{l} \blacktriangle \tilde{\partial}_2) \end{aligned}$$

and

$$\begin{aligned} (m, m')(\partial_1 \blacktriangle \mathfrak{p}) &= ((m, m')\partial_1)\mathfrak{p} \\ &= (m\mu \cdot m'\mu')\mathfrak{p} \\ &= m\mu\mathfrak{p} \cdot m'\mu'\mathfrak{p} \\ &\overset{\text{(CSM 1.3)}}{=} m\mathfrak{m}\tilde{\mu} \cdot m'\mu'\mathfrak{p} \\ &\overset{\text{(CSM 1.4)}}{=} m\mathfrak{m}\tilde{\mu} \cdot m'\mathfrak{m}'\tilde{\mu}' \\ &= (m\mathfrak{m}, m'\mathfrak{m}')\tilde{\partial}_1 \\ &= ((m, m')(\mathfrak{m} \ltimes \mathfrak{m}'))\tilde{\partial}_1 \\ &= (m, m')((\mathfrak{m} \ltimes \mathfrak{m}') \blacktriangle \tilde{\partial}_1). \end{aligned}$$

So we have

$$\partial_2 \blacktriangle (\mathfrak{m} \ltimes \mathfrak{m}') = \mathfrak{l} \blacktriangle \tilde{\partial}_2$$

and

$$\partial_1 \blacktriangle \mathfrak{p} = (\mathfrak{m} \ltimes \mathfrak{m}') \blacktriangle \tilde{\partial}_1.$$

Second, we have to show (2CMM 1, 2).

Ad (2CMM 1.1). Suppose given $(m, m') \in M \ltimes_\alpha M'$ and $p \in P$.

Then we have

$$\begin{aligned} ((m, m')^p)(\mathfrak{m} \ltimes \mathfrak{m}') &\overset{\text{Lem. 93}}{=} (m^p, m'^p)(\mathfrak{m} \ltimes \mathfrak{m}') \\ &= (m^p\mathfrak{m}, m'^p\mathfrak{m}') \\ &\overset{\text{(CSM 1.3)}}{=} ((m\mathfrak{m})^{p\mathfrak{p}}, m'^p\mathfrak{m}') \\ &\overset{\text{(CSM 1.4)}}{=} ((m\mathfrak{m})^{p\mathfrak{p}}, (m'\mathfrak{m}')^{p\mathfrak{p}}) \\ &\overset{\text{Lem. 93}}{=} (m\mathfrak{m}, m'\mathfrak{m}')^{p\mathfrak{p}} \\ &= ((m, m')(\mathfrak{m} \ltimes \mathfrak{m}'))^{p\mathfrak{p}}. \end{aligned}$$

Ad (2CMM 1.2). Suppose given $l \in L$ and $p \in P$.

Then we have

$$(l^p)\mathfrak{l} \overset{\text{(CSM 1.5)}}{=} (l\mathfrak{l})^{p\mathfrak{p}}.$$

Ad (2CMM 2). Suppose given (m, m'), $(\check{m}, \check{m}') \in M \ltimes_\alpha M'$.

Then we have

$$\begin{aligned} \lfloor (m, m'), (\check{m}, \check{m}') \rfloor \mathfrak{l} &= \lceil m^{\check{m}}, \check{m}' \rceil \mathfrak{l} \\ &\overset{\text{(CSM 2)}}{=} \lceil m^{\check{m}}\mathfrak{m}, \check{m}'\mathfrak{m}' \rceil \\ &= \lceil m\mathfrak{m}^{\check{m}\mathfrak{m}}, \check{m}'\mathfrak{m}' \rceil \end{aligned}$$

$$\begin{aligned} &= \lfloor (m\mathfrak{m}, m'\mathfrak{m}'), (\check{m}\mathfrak{m}, \check{m}'\mathfrak{m}') \rfloor \\ &= \lfloor (m, m')(\mathfrak{m} \ltimes \mathfrak{m}'), (\check{m}, \check{m}')(\mathfrak{m} \ltimes \mathfrak{m}') \rfloor . \end{aligned}$$

□

6.3 The functor To

Definition 96 We shall define the following *total* 2-*crossed module functor*.

$$\mathrm{To} : \mathscr{CrSq} \longrightarrow 2\text{-}\mathscr{CrMod}$$

$$\left(\begin{tikzcd} C \arrow[d, "\mathfrak{c}"] \\ \tilde{C} \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} C\,\mathrm{To} \arrow[d, "\mathfrak{c}\,\mathrm{To}"] \\ \tilde{C}\,\mathrm{To} \end{tikzcd}\right)$$

(1) Suppose given a crossed square $C = (L,\, M,\, M',\, P)$.

The 2-crossed module $C\,\mathrm{To}$ has been defined as follows in Lemma 93.

$$C\,\mathrm{To} = \left(L \xrightarrow{\partial_2} M \ltimes_\alpha M' \xrightarrow{\partial_1} P \right)$$

(2) Suppose given a morphism of crossed squares

$$\mathfrak{c} = (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) : (L,\, M,\, M',\, P) \to (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P}).$$

The following diagram morphism is a morphism of 2-crossed modules, by Lemma 95.

$$\left(\begin{tikzcd} C \arrow[d, "\mathfrak{c}"'] \\ \tilde{C} \end{tikzcd}\right) \mathrm{To} = \left(\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "\mathfrak{l}"] & M \ltimes_\alpha M' \arrow[r, "\partial_1"] \arrow[d, "\mathfrak{m} \ltimes \mathfrak{m}'"] & P \arrow[d, "\mathfrak{p}"] \\ \tilde{L} \arrow[r, "\tilde{\partial}_2"] & \tilde{M} \ltimes_{\tilde{\alpha}} \tilde{M}' \arrow[r, "\tilde{\partial}_1"] & \tilde{P} \end{tikzcd}\right)$$

For short, $\mathfrak{c}\,\mathrm{To} = (\mathfrak{l}, \mathfrak{m} \ltimes \mathfrak{m}', \mathfrak{p})$.

(3) Suppose given morphisms of crossed squares $C \xrightarrow{\mathfrak{c}} \tilde{C} \xrightarrow{\tilde{\mathfrak{c}}} \tilde{\tilde{C}}$.

Then we have

(a) $(\mathrm{id}_C)\,\mathrm{To} = \mathrm{id}_{C\,\mathrm{To}}$

(b) $(\mathfrak{c} \blacktriangle \tilde{\mathfrak{c}})\,\mathrm{To} = \mathfrak{c}\,\mathrm{To} \blacktriangle \tilde{\mathfrak{c}}\,\mathrm{To}$.

In particular, $\mathrm{To} : \mathscr{CrSq} \to 2\text{-}\mathscr{CrMod}$ is a functor.

Cf. [**2,** Cor. 3.5].

Proof.

Ad (3.a). Write $C =: (L,\, M,\, M',\, P)$.

We have $\mathrm{id}_C = (\mathrm{id}_L,\, \mathrm{id}_M,\, \mathrm{id}_{M'},\, \mathrm{id}_P)$; cf. Remark 21.

We have $(\mathrm{id}_C)\,\mathrm{To} = (\mathrm{id}_L,\, \mathrm{id}_M \ltimes \mathrm{id}_{M'},\, \mathrm{id}_P)$; cf. Lemma 95.

On the other hand, we have $C\,\mathrm{To} = (L,\, M \ltimes_\alpha M',\, P)$; cf. Lemma 93.

So we have $\mathrm{id}_{C\,\mathrm{To}} = (\mathrm{id}_L,\, \mathrm{id}_{M \ltimes_\alpha M'},\, \mathrm{id}_P)$; cf. Remark 41.

So it suffices to show that $\mathrm{id}_M \ltimes \mathrm{id}_{M'} \overset{!}{=} \mathrm{id}_{M \ltimes_\alpha M'}$.

In fact, suppose given $(m, m') \in M \ltimes_\alpha M'$.

We obtain

$$\begin{aligned} (m, m')(\mathrm{id}_M \ltimes \mathrm{id}_{M'}) &= (m\,\mathrm{id}_M, m'\,\mathrm{id}_{M'}) \\ &= (m, m') \\ &= (m, m')(\mathrm{id}_{M \ltimes_\alpha M'}). \end{aligned}$$

Ad (3.b). Write $C =: (L,\, M,\, M',\, P)$, $\tilde{C} =: (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P})$, $\tilde{\tilde{C}} =: (\tilde{\tilde{L}},\, \tilde{\tilde{M}},\, \tilde{\tilde{M}}',\, \tilde{\tilde{P}})$, $\mathfrak{c} =: (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p})$ and $\tilde{\mathfrak{c}} =: (\tilde{\mathfrak{l}},\, \tilde{\mathfrak{m}},\, \tilde{\mathfrak{m}}',\, \tilde{\mathfrak{p}})$.

So

$$(C \xrightarrow{\mathfrak{c}} \tilde{C} \xrightarrow{\tilde{\mathfrak{c}}} \tilde{\tilde{C}}) = ((L,\, M,\, M',\, P) \xrightarrow{(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p})} (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P}) \xrightarrow{(\tilde{\mathfrak{l}},\, \tilde{\mathfrak{m}},\, \tilde{\mathfrak{m}}',\, \tilde{\mathfrak{p}})} (\tilde{\tilde{L}},\, \tilde{\tilde{M}},\, \tilde{\tilde{M}}',\, \tilde{\tilde{P}})).$$

We have

$$\begin{aligned} &(C\,\mathrm{To} \xrightarrow{\mathfrak{c}\,\mathrm{To}} \tilde{C}\,\mathrm{To} \xrightarrow{\tilde{\mathfrak{c}}\,\mathrm{To}} \tilde{\tilde{C}}\,\mathrm{To}) \\ = \;&((L,\, M \ltimes_\alpha M',\, P) \xrightarrow{(\mathfrak{l},\, \mathfrak{m} \ltimes \mathfrak{m}',\, \mathfrak{p})} (\tilde{L},\, \tilde{M} \ltimes_{\tilde{\alpha}} \tilde{M}',\, \tilde{P}) \xrightarrow{(\tilde{\mathfrak{l}},\, \tilde{\mathfrak{m}} \ltimes \tilde{\mathfrak{m}}',\, \tilde{\mathfrak{p}})} (\tilde{\tilde{L}},\, \tilde{\tilde{M}} \ltimes_{\tilde{\tilde{\alpha}}} \tilde{\tilde{M}}',\, \tilde{\tilde{P}})); \end{aligned}$$

cf. Lemma 93 and Lemma 95.

So we have

$$(C\,\mathrm{To} \xrightarrow{\mathfrak{c}\,\mathrm{To} \,\blacktriangle\, \tilde{\mathfrak{c}}\,\mathrm{To}} \tilde{\tilde{C}}\,\mathrm{To}) = ((L,\, M \ltimes_\alpha M',\, P) \xrightarrow{(\mathfrak{l} \,\blacktriangle\, \tilde{\mathfrak{l}},\, (\mathfrak{m} \ltimes \mathfrak{m}') \,\blacktriangle\, (\tilde{\mathfrak{m}} \ltimes \tilde{\mathfrak{m}}'),\, \mathfrak{p} \,\blacktriangle\, \tilde{\mathfrak{p}})} (\tilde{\tilde{L}},\, \tilde{\tilde{M}} \ltimes_{\tilde{\tilde{\alpha}}} \tilde{\tilde{M}}',\, \tilde{\tilde{P}}));$$

cf. Remark 40.

On the other hand, we have

$$(C \xrightarrow{\mathfrak{c} \,\blacktriangle\, \tilde{\mathfrak{c}}} \tilde{\tilde{C}}) = ((L,\, M,\, M',\, P) \xrightarrow{(\mathfrak{l} \,\blacktriangle\, \tilde{\mathfrak{l}},\, \mathfrak{m} \,\blacktriangle\, \tilde{\mathfrak{m}},\, \mathfrak{m}' \,\blacktriangle\, \tilde{\mathfrak{m}}',\, \mathfrak{p} \,\blacktriangle\, \tilde{\mathfrak{p}})} (\tilde{\tilde{L}},\, \tilde{\tilde{M}},\, \tilde{\tilde{M}}',\, \tilde{\tilde{P}}));$$

cf. Remark 20.

Hence

$$(C\,\mathrm{To} \xrightarrow{(\mathfrak{c} \,\blacktriangle\, \tilde{\mathfrak{c}})\,\mathrm{To}} \tilde{\tilde{C}}\,\mathrm{To}) = ((L,\, M \ltimes_\alpha M',\, P) \xrightarrow{(\mathfrak{l} \,\blacktriangle\, \tilde{\mathfrak{l}},\, (\mathfrak{m} \,\blacktriangle\, \tilde{\mathfrak{m}}) \ltimes (\mathfrak{m}' \,\blacktriangle\, \tilde{\mathfrak{m}}'),\, \mathfrak{p} \,\blacktriangle\, \tilde{\mathfrak{p}})} (\tilde{\tilde{L}},\, \tilde{\tilde{M}} \ltimes_{\tilde{\tilde{\alpha}}} \tilde{\tilde{M}}',\, \tilde{\tilde{P}}));$$

cf. Lemma 95.

In order to show that $\mathfrak{c}\,\mathrm{To} \blacktriangle \tilde{\mathfrak{c}}\,\mathrm{To} \overset{!}{=} (\mathfrak{c} \blacktriangle \tilde{\mathfrak{c}})\,\mathrm{To}$, it suffices to show that $(\mathfrak{m} \ltimes \mathfrak{m}') \blacktriangle (\tilde{\mathfrak{m}} \ltimes \tilde{\mathfrak{m}}') \overset{!}{=} (\mathfrak{m} \blacktriangle \tilde{\mathfrak{m}}) \ltimes (\mathfrak{m}' \blacktriangle \tilde{\mathfrak{m}}')$, as morphisms from $M \ltimes_\alpha M'$ to $\tilde{\tilde{M}} \ltimes_{\tilde{\tilde{\alpha}}} \tilde{\tilde{M}}'$.

Suppose given $(m, m') \in M \ltimes_\alpha M'$.

We obtain

$$\begin{aligned} (m, m')((\mathfrak{m} \ltimes \mathfrak{m}') \blacktriangle (\tilde{\mathfrak{m}} \ltimes \tilde{\mathfrak{m}}')) &= ((m, m')(\mathfrak{m} \ltimes \mathfrak{m}'))(\tilde{\mathfrak{m}} \ltimes \tilde{\mathfrak{m}}') \\ &= (m\mathfrak{m}, m'\mathfrak{m}')(\tilde{\mathfrak{m}} \ltimes \tilde{\mathfrak{m}}') \\ &= (m\mathfrak{m}\tilde{\mathfrak{m}}, m'\mathfrak{m}'\tilde{\mathfrak{m}}') \\ &= (m(\mathfrak{m} \blacktriangle \tilde{\mathfrak{m}}), m'(\mathfrak{m}' \blacktriangle \tilde{\mathfrak{m}}')) \\ &= (m, m')((\mathfrak{m} \blacktriangle \tilde{\mathfrak{m}}) \ltimes (\mathfrak{m}' \blacktriangle \tilde{\mathfrak{m}}')). \end{aligned}$$

□

Example 97 Suppose given a group P and normal subgroups $M,\, M' \trianglelefteqslant P$.

Let $L \trianglelefteqslant P$ with $[M, M'] \leqslant L \leqslant M \cap M'$.

We apply the functor To to the crossed square of Example 23, in order to obtain an example of a 2-crossed module.

We obtain the group morphisms

$$\begin{array}{lcllcl} L & \xrightarrow{\partial_2} & M \ltimes_\alpha M' & : \; l & \longmapsto & (l, l^-) \\ M \ltimes_\alpha M' & \xrightarrow{\partial_1} & P & : \; (m, m') & \longmapsto & m \cdot m' \\ P & \xrightarrow{\beta_1} & \mathrm{Aut}(M \ltimes_\alpha M') & : \; p & \longmapsto & ((m, m') \mapsto (m^p, m'^p)) \\ P & \xrightarrow{\beta_2} & \mathrm{Aut}(L) & : \; p & \longmapsto & (l \mapsto l^p). \end{array}$$

We obtain the map

$$\begin{array}{rcl} (M \ltimes_\alpha M') \times (\tilde{M} \ltimes_\alpha \tilde{M}') & \xrightarrow{\zeta} & L \\ ((m, m'), (\tilde{m}, \tilde{m}')) & \longmapsto & \left(\begin{array}{rcl} \lfloor (m, m'), (\tilde{m}, \tilde{m}') \rfloor & := & [m^{\tilde{m}}, \tilde{m}'] \\ & = & (\tilde{m}'^-)^{m^{\tilde{m}}} \cdot \tilde{m}' \\ & = & (m^{\tilde{m}})^- \cdot \tilde{m}'^- \cdot m^{\tilde{m}} \cdot \tilde{m}' \\ & = & (m^{\tilde{m}})^- \cdot (m^{\tilde{m}})^{\tilde{m}'} \end{array} \right). \end{array}$$

Then

$$(L,\, M \ltimes_\alpha M',\, P,\, \partial_2,\, \partial_1,\, \beta_1,\, \beta_2,\, \zeta)$$

is a 2-crossed module.

For instance, we see that we have

$$\begin{array}{rcl} l\partial_2\partial_1 & = & (l, l^-)\partial_1 \\ & = & l \cdot l^- \\ & = & 1. \end{array}$$

as required by (2CM 5).

$$L \xrightarrow{\partial_2} M \ltimes_\alpha M' \xrightarrow{\partial_1} P$$

Remark 98 The functor $\mathrm{To} : \mathcal{CrSq} \to 2\text{-}\mathcal{CrMod}$ is not full.

Proof. Let A be an abelian group with $A \neq 1$.

Consider the following crossed square as a particular case of the construction in [**1,** Ex. 65].

$$\begin{tikzcd} & A \arrow[r, "\lambda' = \mathrm{id}_A"] \arrow[d, "\lambda = \mathrm{id}_A"'] & A \arrow[d, "\mu'"] \\ C : & A \arrow[r, "\mu"'] & 1\,. \end{tikzcd}$$

Let

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\lambda}} & A \\ a & \longmapsto & a^- \end{array}$$

and

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\lambda}'} & A \\ a & \longmapsto & a^-. \end{array}$$

Consider the following crossed square as a particular case of the construction in [**1,** Ex. 65].

$$\begin{tikzcd}
\tilde{C}: & A \arrow[r, "\tilde{\lambda}'"] \arrow[d, "\tilde{\lambda}"'] & A \arrow[d, "\mu'"] \\
& A \arrow[r, "\mu"'] & 1\,.
\end{tikzcd}$$

Consider the 2-crossed modules

$$C\,\mathrm{To} =: (A,\, A \times A,\, 1,\, \partial_2,\, \partial_1,\, \beta_1,\, \gamma_{P,L},\, \zeta)$$

and

$$\tilde{C}\,\mathrm{To} =: (A,\, A \times A,\, 1,\, \tilde{\partial}_2,\, \tilde{\partial}_1,\, \tilde{\beta}_1,\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\zeta}).$$

Cf. Lemma 93.

We have

$$\begin{array}{ccccccc}
A & \xrightarrow{\partial_2} & A \times A & : & a & \longmapsto & (a\lambda, a^-\lambda') = (a, a^-) \\
A & \xrightarrow{\tilde{\partial}_2} & A \times A & : & a & \longmapsto & (a\tilde{\lambda}, a^-\tilde{\lambda}') = (a^-, a)
\end{array}$$

and

$$\begin{array}{ccccccc}
(A \times A) \times (A \times A) & \xrightarrow{\zeta} & A & : & ((a,b),(a',b')) & \longmapsto & \lfloor (a,b),(a',b') \rfloor := \lceil a^{a'}, b' \rceil = \lceil a, b' \rceil = 1 \\
(A \times A) \times (A \times A) & \xrightarrow{\tilde{\zeta}} & A & : & ((a,b),(a',b')) & \longmapsto & \lfloor (a,b),(a',b') \rfloor := \lceil a^{a'}, b' \rceil = \lceil a, b' \rceil = 1.
\end{array}$$

Let

$$\begin{array}{rcl}
A & \xrightarrow{\nu_2} & A \\
a & \longmapsto & a
\end{array}$$

$$\begin{array}{rcl}
A \times A & \xrightarrow{\nu_1} & A \times A \\
(a,b) & \longmapsto & (b,a)
\end{array}$$

$$\begin{array}{rcl}
1 & \xrightarrow{\nu_0} & 1 \\
1 & \longmapsto & 1.
\end{array}$$

We show that we have the following morphism of 2-crossed modules.

$$\nu := (\nu_2,\, \nu_1,\, \nu_0) : (A,\, A \times A,\, 1) \to (A,\, A \times A,\, 1)$$

$$\begin{tikzcd}
A \arrow[r, "\partial_2"] \arrow[d, "\nu_2"'] & A \times A \arrow[r, "\partial_1"] \arrow[d, "\nu_1"] & 1 \arrow[d, "\nu_0"] \\
A \arrow[r, "\tilde{\partial}_2"] & A \times A \arrow[r, "\tilde{\partial}_1"] & 1
\end{tikzcd}$$

First, we have to show that the diagram is commutative.

Suppose given $a \in A$.

Then we have

$$\begin{array}{rcl}
a(\partial_2 \blacktriangle \nu_1) & = & (a\partial_2)\nu_1 \\
& = & (a, a^-)\nu_1 \\
& = & (a^-, a) \\
& = & a\tilde{\partial}_2 \\
& = & (a\nu_2)\tilde{\partial}_2 \\
& = & a(\nu_2 \blacktriangle \tilde{\partial}_2).
\end{array}$$

Second, we have to show (2CMM $1, 2$).

Suppose given a, a', b, $b' \in A$.

Ad (2CMM 1.1). We have

$$((a,b)^1)\nu_1 = ((a,b)\nu_1)^{(1\nu_0)}.$$

Ad (2CMM 1.2). We have

$$(a^1)\nu_2 = (a\nu_2)^{(1\nu_0)}.$$

Ad (2CMM 2). Furthermore, we have

$$\begin{array}{rcl} \lfloor (a,b),(a',b')\rfloor \nu_2 & = & 1\nu_2 \\ & = & 1 \\ & = & \lfloor (b,a),(b',a')\rfloor \\ & = & \lfloor (a,b)\nu_1,(a',b')\nu_1\rfloor . \end{array}$$

Suppose given a morphism of crossed squares

$$\mathfrak{c} := (\mathfrak{l},\ \mathfrak{m},\ \mathfrak{m}',\ \mathfrak{p}) : (A,\ A,\ A,\ 1) \to (A,\ A,\ A,\ 1).$$

$$\begin{tikzcd}
& A \arrow[rr, "\lambda'"] \arrow[dl, "\lambda"'] \arrow[dd, "\mathfrak{l}", near end] & & A \arrow[dl, "\mu'"] \arrow[dd, "\mathfrak{m}'"] \\
A \arrow[rr, "\mu", near end, crossing over] \arrow[dd, "\mathfrak{m}"'] & & 1 & \\
& A \arrow[rr, "\tilde{\lambda}'", near start] \arrow[dl, "\tilde{\lambda}"] & & A \arrow[dl, "\mu'"] \\
A \arrow[rr, "\mu"] & & 1 \arrow[from=uu, "\mathfrak{p}", near start, crossing over] &
\end{tikzcd}$$

Consider the following morphism of 2-crossed modules.

$$\mathfrak{c}\,\mathrm{To} := (\mathfrak{l},\ \mathfrak{m}\times\mathfrak{m}',\ \mathfrak{p}) : (A,\ A\times A,\ 1) \to (A,\ A\times A,\ 1)$$

$$\begin{tikzcd}
A \arrow[r, "\partial_2"] \arrow[d, "\mathfrak{l}"'] & A\times A \arrow[r, "\partial_1"] \arrow[d, "\mathfrak{m}\times\mathfrak{m}'"] & 1 \arrow[d, "\mathfrak{p}"] \\
A \arrow[r, "\tilde{\partial}_2"] & A\times A \arrow[r, "\tilde{\partial}_1"] & 1
\end{tikzcd}$$

Cf. Lemma 95.

It suffices to show that ν and $\mathfrak{c}\,\mathrm{To}$ are not the same.

Assume that $\nu = \mathfrak{c}\,\mathrm{To}$.

Then $\mathfrak{l} = \nu_2 = \mathrm{id}_A$.

For $a \in A$, we obtain

$$\begin{array}{rcl} a(\partial_2 \blacktriangle (\mathfrak{m}\times\mathfrak{m}')) & = & (a\partial_2)(\mathfrak{m}\times\mathfrak{m}') \\ & = & (a, a^-)(\mathfrak{m}\times\mathfrak{m}') \\ & = & (a\mathfrak{m}, a^-\mathfrak{m}'). \end{array}$$

Furthermore, we obtain

$$\begin{array}{rcl} a(\mathfrak{l} \blacktriangle \tilde{\partial}_2) & = & (a\mathfrak{l})\tilde{\partial}_2 \\ & = & a\tilde{\partial}_2 \\ & = & (a^-, a). \end{array}$$

Hence $a\mathfrak{m} = a^-$ and $a^-\mathfrak{m}' = a$.

Then we have

$$(a, 1)\nu_1 \quad = \quad (1, a)$$

and

$$\begin{array}{rcl} (a, 1)(\mathfrak{m} \times \mathfrak{m}') & = & (a\mathfrak{m}, 1\mathfrak{m}') \\ & = & (a^-, 1). \end{array}$$

So $a = 1$ for $a \in A$. Therefore, $A = 1$.

This yields a *contradiction* to $A \neq 1$. $\square$

7 The adjoint functors $\mathrm{Sq} \dashv (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})$

7.1 The transformation $\varepsilon : \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp} \to \mathrm{Sq} \blacktriangle (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})$

Remark 99 Suppose given $\lfloor 2,0\rfloor$-simplicial groups G and H and a morphism of $\lfloor 2,0\rfloor$-simplicial groups $\varphi : G \to H$.

Cf. Definition 30 and Definition 31.

We recall that

$$G\,\mathrm{Sq} = (G_{2;1,2},\ G_{2;1}/G_{2;0,1},\ G_{2;2}/G_{2;0,2},\ G_2/G_{2;0},\ \gamma_{M,L}\,,\ \gamma_{M',L}\,,\ \gamma_{P,L}\,,\ \gamma_{P,M}\,,\ \gamma_{P,M'}\,,\ \lambda,\ \lambda',\ \mu,\ \mu',\ \chi)$$

is a crossed square and

$$\varphi\,\mathrm{Sq} = (\varphi_{2;1,2},\ \bar{\varphi}_{2;1},\ \bar{\varphi}_{2;2},\ \bar{\varphi}_{2;\emptyset}) : G\,\mathrm{Sq} \to H\,\mathrm{Sq}$$

is a morphism of crossed squares; cf. Definition 82.

Using the functor Tr from Definition 27 we have that

$$G\,\mathrm{Sq}\,\mathrm{Tr} = (G_{2;1,2},\ G_{2;2}/G_{2;0,2},\ G_{2;1}/G_{2;0,1},\ G_2/G_{2;0},\ \gamma_{M',L}\,,\ \gamma_{M,L}\,,\ \gamma_{P,L}\,,\ \gamma_{P,M'}\,,\ \gamma_{P,M}\,,\ \lambda',\ \lambda,\ \mu',\ \mu,\ \chi^{\mathrm{tr}})$$

with the map

$$\begin{array}{rcl} G_{2;2}/G_{2;0,2} \times G_{2;1}/G_{2;0,1} & \xrightarrow{\chi^{\mathrm{tr}}} & G_{2;1,2} \\ (n_2 G_{2;0,2}, \check{n}_2 G_{2;0,1}) & \longmapsto & (n_2 G_{2;0,2}, \check{n}_2 G_{2;0,1})\chi^{\mathrm{tr}} = \lceil n_2 G_{2;0,2}, \check{n}_2 G_{2;0,1} \rceil^{\mathrm{tr}} = \lceil \check{n}_2 G_{2;0,1}, n_2 G_{2;0,2} \rceil^{-} \end{array}$$

is also a crossed square; cf. Remark 24.

Moreover,

$$\varphi\,\mathrm{Sq}\,\mathrm{Tr} = (\varphi_{2;1,2},\ \bar{\varphi}_{2;2},\ \bar{\varphi}_{2;1},\ \bar{\varphi}_{2;\emptyset}) : G\,\mathrm{Sq}\,\mathrm{Tr} \to H\,\mathrm{Sq}\,\mathrm{Tr}$$

is also a morphism of crossed squares; cf. Remark 25.

The following map is a group morphism as a composite of group morphisms; cf. Remark 88.

$$\begin{array}{rcl} \alpha = \mu' \blacktriangle \gamma_{P,M} : G_{2;2}/G_{2;0,2} & \longrightarrow & \mathrm{Aut}(G_{2;1}/G_{2;0,1}) \\ g_2 G_{2;0,2} & \longmapsto & \left(g'_2 G_{2;0,1} \ \mapsto \ \left(\begin{array}{rl} & (g'_2 G_{2;0,1})((g_2 G_{2;0,2})\alpha) \\ = & (g'_2 G_{2;0,1})(g_2 G_{2;0,2}(\mu' \blacktriangle \gamma_{P,M})) \\ = & (g'_2 G_{2;0,1})^{(g_2 G_{2;0,2})\mu'} \\ = & (g'_2 G_{2;0,1})^{g_2 G_{2;0}} \\ := & {g'_2}^{g_2} G_{2;0,1} \end{array} \right) \right) \end{array}$$

In particular, we may form the semidirect product $G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1}$, in which

$$(g_2 G_{2;0,2}, g'_2 G_{2;0,1}) \cdot (\tilde{g}_2 G_{2;0,2}, \tilde{g}'_2 G_{2;0,1}) \quad = \quad ((g_2 \cdot \tilde{g}_2) G_{2;0,2}, ({g'_2}^{\tilde{g}_2} \cdot \tilde{g}'_2) G_{2;0,1}),$$

for $(g_2 G_{2;0,2}, g'_2 G_{2;0,1}),\ (\tilde{g}_2 G_{2;0,2}, \tilde{g}'_2 G_{2;0,1}) \in G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1}$.

So we have the 2-crossed module

$$G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To} =: \left(G_{2;1,2} \xrightarrow{\partial_2'} G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1} \xrightarrow{\partial_1'} G_2/G_{2;0} \right)$$

with the following group morphisms

$$\begin{array}{rcl} G_{2;1,2} & \xrightarrow{\partial_2'} & G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1} \\ n_2 & \longmapsto & (n_2 G_{2;0,2}, n_2^- G_{2;0,1}) \end{array}$$

$$\begin{array}{rcl} G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1} & \xrightarrow{\partial_1'} & G_2/G_{2;0} \\ (n_2 G_{2;0,2}, \check{n}_2 G_{2;0,1}) & \longmapsto & n_2 G_{2;0} \cdot \check{n}_2 G_{2;0} \end{array}$$

$$\begin{array}{rcl} G_2/G_{2;0} & \xrightarrow{\beta_1'} & \mathrm{Aut}(G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1}) \\ n_2 G_{2;0} & \longmapsto & \left((\check{n}_2 G_{2;0,2}, \check{n}_2' G_{2;0,1}) \mapsto \left(\begin{array}{cl} & (\check{n}_2 G_{2;0,2}, \check{n}_2' G_{2;0,1})^{n_2 G_{2;0}} \\ = & ((\check{n}_2 G_{2;0,2})^{n_2 G_{2;0}}, (\check{n}_2' G_{2;0,1})^{n_2 G_{2;0}}) \\ \overset{\text{Lem. 79}}{=} & (\check{n}_2^{n_2} G_{2;0,2}, \check{n}_2^{\prime\, n_2} G_{2;0,1})) \end{array} \right) \right) \\ G_2/G_{2;0} & \xrightarrow{\gamma_{P,L}'} & \mathrm{Aut}(G_{2;1,2}) \\ n_2 G_{2,0} & \longmapsto & \left(\check{n}_2 \mapsto \left(\begin{array}{cl} & \check{n}_2^{n_2 G_{2;0}} \\ \overset{\text{Lem. 79}}{=} & \check{n}_2^{n_2} \end{array} \right) \right) \end{array}$$

and the map

$$\begin{array}{rcl} (G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1}) \times (G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1}) & \xrightarrow{\zeta'} & G_{2;1,2} \\ ((n_2 G_{2;0,2}, \check{n}_2 G_{2;0,1}), (n_2' G_{2;0,2}, \check{n}_2' G_{2;0,1})) & \longmapsto & \left(\begin{array}{cl} & \lfloor (n_2 G_{2;0,2}, \check{n}_2 G_{2;0,1}), (n_2' G_{2;0,2}, \check{n}_2' G_{2;0,1}) \rfloor \\ = & \lceil (n_2 G_{2;0,2})^{n_2' G_{2;0,2}}, \check{n}_2' G_{2;0,1} \rceil^{\mathrm{tr}} \\ = & \lceil n_2^{n_2'} G_{2;0,2}, \check{n}_2' G_{2;0,1} \rceil^{\mathrm{tr}} \\ = & \lceil \check{n}_2' G_{2;0,1}, n_2^{n_2'} G_{2;0,2} \rceil^{-} \\ \overset{\text{Lem. 79}}{=} & [\check{n}_2', n_2^{n_2'}]^{-} \\ = & [n_2^{n_2'}, \check{n}_2'] \end{array} \right) \end{array} .$$

Moreover,

$$\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To} := (\varphi_{2;1,2},\ \bar{\varphi}_{2;2} \ltimes \bar{\varphi}_{2;1},\ \bar{\varphi}_{2;\emptyset})$$

is a morphism of 2-crossed modules.

Cf. Definition 96.

Lemma 100 Suppose given a $\lfloor 2, 0 \rfloor$-simplicial group G.

We have the following group morphisms.

$$\begin{array}{rcl} G_{2;1,2} & \xrightarrow{G\mathrm{v}_2} & G_{2;1,2} \\ n_2 & \longmapsto & n_2 \end{array}$$

$$\begin{array}{rcl} G_{1;1} & \xrightarrow{G\mathrm{v}_1} & G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1} \\ n_1 & \longmapsto & (n_1\,\mathrm{s}_0\, G_{2;0,2}, (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1) G_{2;0,1}) \end{array}$$

$$\begin{array}{rcl} G_0 & \xrightarrow{G\mathrm{v}_0} & G_2/G_{2;0} \\ n_0 & \longmapsto & n_0\,\mathrm{s}_0\,\mathrm{s}_0\, G_{2;0} \end{array}$$

Proof. We consider $G\nu_1$.

Suppose given n_1, $n'_1 \in G_{1;1}$.

We have

$$\begin{aligned} (n_1\,\mathrm{s}_0)\,\mathrm{d}_2 &= n_1\,\mathrm{s}_0\,\mathrm{d}_2 \\ &= n_1\,\mathrm{d}_1\,\mathrm{s}_0 \\ &= 1, \end{aligned}$$

$$\begin{aligned} (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)\,\mathrm{d}_1 &= n_1^-\,\mathrm{s}_0\,\mathrm{d}_1 \cdot n_1\,\mathrm{s}_1\,\mathrm{d}_1 \\ &= n_1^- \cdot n_1 \\ &= 1. \end{aligned}$$

So we have $n_1\,\mathrm{s}_0 \in G_{2;2}$ and $n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1 \in G_{2;1}$.

We have

$$\begin{aligned} (n_1 \cdot n'_1)(G\nu_1) &= ((n_1 \cdot n'_1)\,\mathrm{s}_0\,G_{2;0,2}, ((n_1 \cdot n'_1)^-\,\mathrm{s}_0 \cdot (n_1 \cdot n'_1)\,\mathrm{s}_1)G_{2;0,1}) \\ &= ((n_1\,\mathrm{s}_0 \cdot n'_1\,\mathrm{s}_0)G_{2;0,2}, (n'^-_1\,\mathrm{s}_0 \cdot n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1 \cdot n'_1\,\mathrm{s}_1)G_{2;0,1}) \\ &= ((n_1\,\mathrm{s}_0 \cdot n'_1\,\mathrm{s}_0)G_{2;0,2}, ((n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)^{n'_1\,\mathrm{s}_0} \cdot (n'^-_1\,\mathrm{s}_0 \cdot n'_1\,\mathrm{s}_1))G_{2;0,1}) \\ &= (n_1\,\mathrm{s}_0\,G_{2;0,2} \cdot n'_1\,\mathrm{s}_0\,G_{2;0,2}, (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)^{n'_1\,\mathrm{s}_0}G_{2;0,1} \cdot (n'^-_1\,\mathrm{s}_0 \cdot n'_1\,\mathrm{s}_1)G_{2;0,1}) \\ &= (n_1\,\mathrm{s}_0\,G_{2;0,2} \cdot n'_1\,\mathrm{s}_0\,G_{2;0,2}, ((n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1})^{n'_1\,\mathrm{s}_0\,G_{2;0,2}} \cdot (n'^-_1\,\mathrm{s}_0 \cdot n'_1\,\mathrm{s}_1)G_{2;0,1}) \\ &= (n_1\,\mathrm{s}_0\,G_{2;0,2}, (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1}) \cdot (n'_1\,\mathrm{s}_0\,G_{2;0,2}, (n'^-_1\,\mathrm{s}_0 \cdot n'_1\,\mathrm{s}_1)G_{2;0,1}) \\ &= n_1(G\nu_1) \cdot n'_1(G\nu_1). \end{aligned}$$

We consider $G\nu_0$.

We have that $G\nu_0$ is a group morphism as a composite of group morphisms. □

Lemma 101 Suppose given a $\lfloor 2, 0\rfloor$-simplicial group G.

We have the morphism

$$G\nu := (G\nu_2,\, G\nu_1,\, G\nu_0) : G\,\hat{\mathrm{N}} \to G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}$$

of 2-crossed modules.

Cf. Lemma 51, Remark 99 and Lemma 100.

$$\begin{tikzcd} G_{2;1,2} \arrow[r, "\partial_2"] \arrow[d, "G\nu_2"] & G_{1;1} \arrow[r, "\partial_1"] \arrow[d, "G\nu_1"] & G_0 \arrow[d, "G\nu_0"] \\ G_{2;1,2} \arrow[r, "\partial'_2"] & G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1} \arrow[r, "\partial'_1"] & G_2/G_{2;0} \end{tikzcd}$$

Proof. We abbreviate $\nu_i := G\nu_i$ for $i \in [0,\, 2]$.

First, we have to show that the diagram is commutative.

Suppose given $n_1 \in G_{1;1}$ and $n_2 \in G_{2;1,2}$.

We have

$$\begin{aligned} (n_2^- \cdot n_2\,\mathrm{d}_0\,\mathrm{s}_0)\,\mathrm{d}_0 &= n_2^-\,\mathrm{d}_0 \cdot n_2\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{d}_0 \\ &= n_2^-\,\mathrm{d}_0 \cdot n_2\,\mathrm{d}_0 \\ &= 1, \end{aligned}$$

$$\begin{aligned}
(n_2^- \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_0) \,\mathrm{d}_2 &= n_2^- \,\mathrm{d}_2 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{d}_2 \\
&= n_2^- \,\mathrm{d}_2 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{d}_1 \,\mathrm{s}_0 \\
&= n_2^- \,\mathrm{d}_2 \cdot n_2 \,\mathrm{d}_2 \,\mathrm{d}_0 \,\mathrm{s}_0 \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
(n_2 \cdot n_2^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_1) \,\mathrm{d}_0 &= n_2 \,\mathrm{d}_0 \cdot n_2^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{d}_0 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_1 \,\mathrm{d}_0 \\
&= n_2 \,\mathrm{d}_0 \cdot n_2^- \,\mathrm{d}_0 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_1 \,\mathrm{d}_0 \\
&= n_2 \,\mathrm{d}_0 \,\mathrm{s}_1 \,\mathrm{d}_0 \\
&= n_2 \,\mathrm{d}_0 \,\mathrm{d}_0 \,\mathrm{s}_0 \\
&= n_2 \,\mathrm{d}_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
(n_2 \cdot n_2^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_1) \,\mathrm{d}_1 &= n_2 \,\mathrm{d}_1 \cdot n_2^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_1 \,\mathrm{d}_1 \\
&= n_2^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_1 \,\mathrm{d}_1 \\
&= n_2^- \,\mathrm{d}_0 \cdot n_2 \,\mathrm{d}_0 \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
(n_1^- \,\mathrm{s}_1 \cdot n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0) \,\mathrm{d}_0 &= n_1^- \,\mathrm{s}_1 \,\mathrm{d}_0 \cdot n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \,\mathrm{d}_0 \\
&= n_1^- \,\mathrm{s}_1 \,\mathrm{d}_0 \cdot n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \\
&= n_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \\
&= 1.
\end{aligned}$$

So we have $n_2^- \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_0 \in G_{2;0,2}$, $n_2 \cdot n_2^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_1 \in G_{2;0,1}$ and $n_1^- \,\mathrm{s}_1 \cdot n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \in G_{2;0}$.

Then we have

$$\begin{aligned}
n_2(\partial_2 \blacktriangle \nu_1) &= (n_2\partial_2)\nu_1 \\
&= (n_2 \,\mathrm{d}_0)\nu_1 \\
&= (n_2 \,\mathrm{d}_0 \,\mathrm{s}_0 \, G_{2;0,2}, (n_2^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_2 \,\mathrm{d}_0 \,\mathrm{s}_1) G_{2;0,1}) \\
&= (n_2 G_{2;0,2}, n_2^- G_{2;0,1}) \\
&= n_2 \partial'_2 \\
&= (n_2 \nu_2)\partial'_2 \\
&= n_2(\nu_2 \blacktriangle \partial'_2)
\end{aligned}$$

and

$$\begin{aligned}
n_1(\partial_1 \blacktriangle \nu_0) &= (n_1\partial_1)\nu_0 \\
&= (n_1 \,\mathrm{d}_0)\nu_0 \\
&= n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \, G_{2;0} \\
&= n_1 \,\mathrm{s}_1 \, G_{2;0} \\
&= n_1 \,\mathrm{s}_0 \, G_{2;0} \cdot (n_1^- \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_1) G_{2;0} \\
&= (n_1 \,\mathrm{s}_0 \, G_{2;0,2}, (n_1^- \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_1) G_{2;0,1})\partial'_1 \\
&\overset{\text{Lem. 100}}{=} (n_1 \nu_1)\partial'_1 \\
&= n_1(\nu_1 \blacktriangle \partial'_1).
\end{aligned}$$

Second, we have to show (2CMM 1, 2).

Ad (2CMM 1.1). Suppose given $n_0 \in G_0$ and $n_1 \in G_{1;1}$.

Then we have

$$\begin{aligned}
(n_1^{n_0})\nu_1 &\overset{\text{Lem. 51}}{=} (n_1^{n_0 \,\mathrm{s}_0})\nu_1 \\
&\overset{\text{Lem. 100}}{=} (n_1^{n_0 \,\mathrm{s}_0} \,\mathrm{s}_0 \, G_{2;0,2}, ((n_1^-)^{n_0 \,\mathrm{s}_0} \,\mathrm{s}_0 \cdot n_1^{n_0 \,\mathrm{s}_0} \,\mathrm{s}_1) G_{2;0,1}) \\
&= ((n_1 \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0} G_{2;0,2}, ((n_1^- \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_1)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_1}) G_{2;0,1}) \\
&= ((n_1 \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0} G_{2;0,2}, ((n_1^- \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0} \cdot (n_1 \,\mathrm{s}_1)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0}) G_{2;0,1})
\end{aligned}$$

$$\begin{aligned} &= ((n_1\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0}G_{2;0,2},\, (n_1^-\,\mathrm{s}_0\cdot n_1\,\mathrm{s}_1)^{n_0\,\mathrm{s}_0\,\mathrm{s}_0}G_{2;0,1}) \\ &\overset{\text{Rem. 99}}{=} (n_1\,\mathrm{s}_0\,G_{2;0,2},\, (n_1^-\,\mathrm{s}_0\cdot n_1\,\mathrm{s}_1)G_{2;0,1})^{n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}} \\ &\overset{\text{Lem. 100}}{=} (n_1\nu_1)^{n_0\nu_0}. \end{aligned}$$

Ad (2CMM 1.2). Suppose given $n_0 \in G_0$ and $n_2 \in G_{2;1,2}$.

Then we have

$$\begin{aligned} (n_2^{n_0})\nu_2 &= n_2^{n_0} \\ &\overset{\text{Lem. 51}}{=} n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0} \\ &\overset{\text{Rem. 99}}{=} n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}} \\ &= (n_2\nu_2)^{n_0\nu_0}. \end{aligned}$$

Ad (2CMM 2). Suppose given n_1, $n_1' \in G_{1;1}$.

Then we have

$$\begin{aligned} \lfloor n_1, n_1'\rfloor\nu_2 &\overset{\text{Lem. 51}}{=} ((n_1^-\,\mathrm{s}_0)^{n_1'\,\mathrm{s}_0}\cdot(n_1\,\mathrm{s}_0)^{n_1'\,\mathrm{s}_1})\nu_2 \\ &= (n_1^-\,\mathrm{s}_0)^{n_1'\,\mathrm{s}_0}\cdot(n_1\,\mathrm{s}_0)^{n_1'\,\mathrm{s}_1} \\ &= (n_1^-\,\mathrm{s}_0)^{n_1'\,\mathrm{s}_0}\cdot n_1'^-\,\mathrm{s}_1\cdot n_1\,\mathrm{s}_0\cdot n_1'\,\mathrm{s}_1 \\ &= (n_1^-\,\mathrm{s}_0)^{n_1'\,\mathrm{s}_0}\cdot n_1'^-\,\mathrm{s}_1\cdot n_1'\,\mathrm{s}_0\cdot(n_1\,\mathrm{s}_0)^{n_1'\,\mathrm{s}_0}\cdot\ n_1'^-\,\mathrm{s}_0\cdot n_1'\,\mathrm{s}_1 \\ &= [(n_1\,\mathrm{s}_0)^{n_1'\,\mathrm{s}_0}, n_1'^-\,\mathrm{s}_0\cdot n_1'\,\mathrm{s}_1] \\ &\overset{\text{Rem. 99}}{=} \lfloor(n_1\,\mathrm{s}_0\,G_{2;0,2}, (n_1^-\,\mathrm{s}_0\cdot n_1\,\mathrm{s}_1)G_{2;0,1}), (n_1'\,\mathrm{s}_0\,G_{2;0,2}, (n_1'^-\,\mathrm{s}_0\cdot n_1'\,\mathrm{s}_1)G_{2;0,1})\rfloor \\ &\overset{\text{Lem. 100}}{=} \lfloor n_1\nu_1, n_1'\nu_1\rfloor. \end{aligned}$$

□

Lemma 102 We have the transformation

$$\nu = (G\nu)_{G\in\mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp})} : \hat{\mathrm{N}} \to \mathrm{Sq} \blacktriangle \mathrm{Tr} \blacktriangle \mathrm{To}\,.$$

Cf. Lemma 100 and Lemma 101.

Proof. To show that ν is a transformation, we have to show that the quadrangle

$$\begin{tikzcd} G\,\hat{\mathrm{N}} \arrow[r, "G\nu"] \arrow[d, "\varphi\,\hat{\mathrm{N}}"'] & G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To} \arrow[d, "\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}"] \\ H\,\hat{\mathrm{N}} \arrow[r, "H\nu"'] & H\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To} \end{tikzcd}$$

commutes for $G \xrightarrow{\varphi} H$ in $\lfloor 2,0\rfloor$-$\mathcal{SimpGrp}$.

To this end, we show that $G\nu_k \blacktriangle (\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_k \overset{!}{=} (\varphi\,\hat{\mathrm{N}})_k \blacktriangle H\nu_k$ for $k \in \{0,\ 1,\ 2\}$; cf. Remark 40, Lemma 101.

Suppose given $n_0 \in G_0$, $n_1 \in G_{1;1}$ and $n_2 \in G_{2;1,2}$.

We have

$$\begin{aligned} n_2((\varphi\,\hat{\mathrm{N}})_2 \blacktriangle H\nu_2) &= (n_2(\varphi\,\hat{\mathrm{N}})_2)(H\nu_2) \\ &\overset{\text{Def. 54}}{=} n_2\varphi_2(H\nu_2) \\ &\overset{\text{Lem. 100}}{=} n_2\varphi_2 \\ &= n_2(\varphi_{2;1,2}) \\ &\overset{\text{Rem. 99}}{=} n_2(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_2 \\ &\overset{\text{Lem. 100}}{=} (n_2(G\nu_2))(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_2 \\ &= n_2(G\nu_2 \blacktriangle (\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_2). \end{aligned}$$

We have

$$\begin{array}{rcl}
n_1((\varphi\,\hat{\mathrm{N}})_1 \blacktriangle H\nu_1) & = & (n_1(\varphi\,\hat{\mathrm{N}})_1)(H\nu_1) \\
& \overset{\text{Def. 54}}{=} & n_1\varphi_1(H\nu_1) \\
& \overset{\text{Lem. 100}}{=} & (n_1\varphi_1\,\mathrm{s}_0\,H_{2;0,2},(n_1^-\varphi_1\,\mathrm{s}_0\cdot n_1\varphi_1\,\mathrm{s}_1)H_{2;0,1}) \\
& = & (n_1\,\mathrm{s}_0\,\varphi_2 H_{2;0,2},(n_1^-\,\mathrm{s}_0\,\varphi_2\cdot\ n_1\,\mathrm{s}_1\,\varphi_2)H_{2;0,1}) \\
& = & (n_1\,\mathrm{s}_0\,\varphi_2 H_{2;0,2},(n_1^-\,\mathrm{s}_0\cdot n_1\,\mathrm{s}_1)\varphi_2 H_{2;0,1}) \\
& \overset{\text{Rem. 80}}{=} & ((n_1\,\mathrm{s}_0\,G_{2;0,2})\bar{\varphi}_{2;2},((n_1^-\,\mathrm{s}_0\cdot n_1\,\mathrm{s}_1)G_{2;0,1})\bar{\varphi}_{2;1}) \\
& = & (n_1\,\mathrm{s}_0\,G_{2;0,2},(n_1^-\,\mathrm{s}_0\cdot n_1\,\mathrm{s}_1)G_{2;0,1})(\bar{\varphi}_{2;2}\ltimes\bar{\varphi}_{2;1}) \\
& \overset{\text{Rem. 99}}{=} & (n_1\,\mathrm{s}_0\,G_{2;0,2},(n_1^-\,\mathrm{s}_0\cdot n_1\,\mathrm{s}_1)G_{2;0,1})(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_1 \\
& \overset{\text{Lem. 100}}{=} & (n_1(G\nu_1))(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_1 \\
& = & n_1(G\nu_1\blacktriangle(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_1).
\end{array}$$

Moreover, we have

$$\begin{array}{rcl}
n_0((\varphi\,\hat{\mathrm{N}})_0 \blacktriangle H\nu_0) & = & (n_0(\varphi\,\hat{\mathrm{N}})_0)(H\nu_0) \\
& \overset{\text{Def. 54}}{=} & n_0\varphi_0(H\nu_0) \\
& \overset{\text{Lem. 100}}{=} & n_0\varphi_0\,\mathrm{s}_0\,\mathrm{s}_0\,H_{2;0} \\
& = & n_0\,\mathrm{s}_0\,\varphi_1\,\mathrm{s}_0\,H_{2;0} \\
& = & n_0\,\mathrm{s}_0\,\mathrm{s}_0\,\varphi_2 H_{2;0} \\
& \overset{\text{Rem. 80}}{=} & (n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0})(\bar{\varphi}_{2;\emptyset}) \\
& \overset{\text{Rem. 99}}{=} & (n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0})(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_0 \\
& \overset{\text{Lem. 100}}{=} & (n_0(G\nu_0))(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_0 \\
& = & n_0(G\nu_0\blacktriangle(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_0).
\end{array}$$

Altogether, we have

$$\begin{array}{rcl}
\varphi\,\hat{\mathrm{N}}\blacktriangle H\nu & = & ((\varphi\,\hat{\mathrm{N}})_2,(\varphi\,\hat{\mathrm{N}})_1,(\varphi\,\hat{\mathrm{N}})_0)\blacktriangle(H\nu_2,H\nu_1,H\nu_0) \\
& \overset{\text{Rem. 40}}{=} & ((\varphi\,\hat{\mathrm{N}})_2\blacktriangle H\nu_2,(\varphi\,\hat{\mathrm{N}})_1\blacktriangle H\nu_1,(\varphi\,\hat{\mathrm{N}})_0\blacktriangle H\nu_0) \\
& = & (G\nu_2\blacktriangle(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_2,G\nu_1\blacktriangle(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_1,G\nu_0\blacktriangle(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_0) \\
& \overset{\text{Rem. 40}}{=} & (G\nu_2,G\nu_1,G\nu_0)\blacktriangle((\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_2,(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_1,(\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To})_0) \\
& = & G\nu\blacktriangle\varphi\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,.
\end{array}$$

□

Lemma 103 We have the transformation

$$\left(\begin{array}{rcl}
\nu\,\mathrm{Rec} & = & (G(\nu\,\mathrm{Rec}))_{G\in\mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp)} \\
& := & ((G\nu)\,\mathrm{Rec})_{G\in\mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp)} \\
& = & (G\nu\,\mathrm{Rec})_{G\in\mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathcal{S}imp\mathcal{G}rp)}
\end{array}\right) : \hat{\mathrm{N}}\blacktriangle\mathrm{Rec}\to\mathrm{Sq}\blacktriangle\mathrm{Tr}\blacktriangle\mathrm{To}\blacktriangle\mathrm{Rec}\,.$$

Cf. Definition 72 and Lemma 102.

We have, by Lemma 64, Definition 54, Remark 99,

$$\begin{array}{rcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_2 & = & (G_0\ltimes_{\beta_1}G_{1;1})\ltimes_{\varepsilon_{1,2}}(G_{1;1}\ltimes_{\varepsilon_2}G_{2;1,2}) \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_1 & = & G_0\ltimes_{\beta_1}G_{1;1} \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_0 & = & G_0 \\
(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_2 & = & (G_2/G_{2;0}\ltimes_{\beta_1}(G_{2;2}/G_{2;0,2}\ltimes_\alpha G_{2;1}/G_{2;0,1}))\ltimes_{\varepsilon_{1,2}}((G_{2;2}/G_{2;0,2}\ltimes_\alpha G_{2;1}/G_{2;0,1})\ltimes_{\varepsilon_2}G_{2;1,2}) \\
(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_1 & = & G_2/G_{2;0}\ltimes_{\beta_1}(G_{2;2}/G_{2;0,2}\ltimes_\alpha G_{2;1}/G_{2;0,1}) \\
(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_0 & = & G_2/G_{2;0}\,.
\end{array}$$

Moreover, by Lemma 70, Lemma 66, Remark 65, Lemma 100, we have

$$\begin{array}{rcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_2 & \xrightarrow{G\nu\,\mathrm{Rec}_2} & (G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_2 \\
((n_0, n_1), (\check{n}_1, n_2)) & \longmapsto & ((n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, (n_1\,\mathrm{s}_0\,G_{2;0,2}, (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1})), ((\check{n}_1\,\mathrm{s}_0\,G_{2;0,2}, (\check{n}_1^-\,\mathrm{s}_0 \cdot \check{n}_1\,\mathrm{s}_1)G_{2;0,1}), n_2))
\end{array}$$

$$\begin{array}{rcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_1 & \xrightarrow{G\nu\,\mathrm{Rec}_1} & (G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_1 \\
(n_0, n_1) & \longmapsto & (n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, (n_1\,\mathrm{s}_0\,G_{2;0,2}, (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1}))
\end{array}$$

$$\begin{array}{rcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_0 & \xrightarrow{G\nu\,\mathrm{Rec}_0} & (G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_0 \\
n_0 & \longmapsto & n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}\,.
\end{array}$$

Lemma 104 We have the transformation

$$\left(\begin{array}{rcl}
\varepsilon & := & \vartheta^- \blacktriangle \nu\,\mathrm{Rec} \\
& = & (G(\vartheta^- \blacktriangle \nu\,\mathrm{Rec}))_{G \in \mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathit{SimpGrp})} \\
& = & (G\vartheta^- \blacktriangle G\nu\,\mathrm{Rec})_{G \in \mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathit{SimpGrp})} \\
& = & (G\vartheta^-)_{G \in \mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathit{SimpGrp})} \blacktriangle (G\nu\,\mathrm{Rec})_{G \in \mathrm{Ob}(\lfloor 2,0\rfloor\text{-}\mathit{SimpGrp})}
\end{array}\right) : \mathrm{Id}_{\lfloor 2,0\rfloor\text{-}\mathit{SimpGrp}} \to \mathrm{Sq} \blacktriangle (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})$$

as a composite of transformations; cf. Remark 2, Lemma 77 and Lemma 103.

So $G\varepsilon = G\vartheta^- \blacktriangle G\nu\,\mathrm{Rec}$.

In particular,

$$\begin{array}{rcl}
G_2 & \xrightarrow{G\varepsilon_2} & (G_2/G_{2;0} \ltimes_{\beta_1} (G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1})) \ltimes_{\varepsilon_{1,2}} ((G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1}) \ltimes_{\varepsilon_2} G_{2;1,2}) \\
g_2 & \longmapsto & \left(\begin{array}{rl}
& ((g_2\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, \\
& ((g_2^-\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0)G_{2;0,2}, (g_2^-\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_1 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_1)G_{2;0,1})), \\
& ((g_2^-\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_1\,\mathrm{s}_0)G_{2;0,2}, (g_2^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{s}_1 \cdot g_2\,\mathrm{d}_1\,\mathrm{s}_1)G_{2;0,1}), \\
& g_2^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{s}_1 \cdot g_2)) \\
= & ((g_2\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, \\
& ((g_2^-\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0)G_{2;0,2}, (g_2^-\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_1)G_{2;0,1})), \\
& ((g_2^-\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_1\,\mathrm{s}_0)G_{2;0,2}, (g_2^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{s}_1 \cdot g_2\,\mathrm{d}_1\,\mathrm{s}_1)G_{2;0,1}), \\
& g_2^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{s}_1 \cdot g_2)) \\
= & ((g_2\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, \\
& ((g_2^-\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0)G_{2;0,2}, (g_2^-\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_1)G_{2;0,1})), \\
& ((g_2^-\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_1\,\mathrm{s}_0)G_{2;0,2}, (g_2^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{s}_1 \cdot g_2\,\mathrm{d}_1\,\mathrm{s}_1)G_{2;0,1}), \\
& g_2^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_2\,\mathrm{d}_2\,\mathrm{s}_0 \cdot g_2^-\,\mathrm{d}_2\,\mathrm{s}_1 \cdot g_2))
\end{array}\right)
\end{array}$$

$$\begin{array}{rcl}
G_1 & \xrightarrow{G\varepsilon_1} & G_2/G_{2;0} \ltimes_{\beta_1} (G_{2;2}/G_{2;0,2} \ltimes_\alpha G_{2;1}/G_{2;0,1}) \\
g_1 & \longmapsto & \left(\begin{array}{rl}
& (g_1\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, ((g_1^-\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_1\,\mathrm{s}_0)G_{2;0,2}, (g_1^-\,\mathrm{s}_0 \cdot g_1\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_1^-\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_1 \cdot g_1\,\mathrm{s}_1)G_{2;0,1})) \\
= & (g_1\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, ((g_1^-\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_1\,\mathrm{s}_0)G_{2;0,2}, (g_1^-\,\mathrm{s}_0 \cdot g_1\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_1^-\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_1\,\mathrm{s}_1)G_{2;0,1})) \\
= & (g_1\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, ((g_1^-\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{s}_0 \cdot g_1\,\mathrm{s}_0)G_{2;0,2}, (g_1^-\,\mathrm{s}_0 \cdot g_1\,\mathrm{s}_1)G_{2;0,1}))
\end{array}\right)
\end{array}$$

$$\begin{array}{rcl}
G_0 & \xrightarrow{G\varepsilon_0} & G_2/G_{2;0} \\
g_0 & \longmapsto & g_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}\,.
\end{array}$$

Cf. Lemma 76 and Lemma 103.

7.2 The transformation $\eta : (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \mathrm{Sq} \to \mathrm{Id}_{CrSq}$

Remark 105 Suppose given crossed squares

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi),$$

$$\tilde{C} = (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}}\,,\, \gamma_{\tilde{M}',\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{M}}\,,\, \gamma_{\tilde{P},\tilde{M}'}\,,\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi})$$

and a morphism of crossed squares

$$\mathfrak{c} = (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) \quad : \quad C = (L,\, M,\, M',\, P) \quad \to \quad \tilde{C} = (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P}).$$

Cf. Definition 15 and Definition 19.

(1) *Application of* Tr .

We recall that

$$C\,\mathrm{Tr} = (L,\, M',\, M,\, P,\, \gamma_{M',L}\,,\, \gamma_{M,L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M'}\,,\, \gamma_{P,M}\,,\, \lambda',\, \lambda,\, \mu',\, \mu,\, \chi^{\mathrm{tr}})$$

is a crossed square and

$$\mathfrak{c}\,\mathrm{Tr} = (\mathfrak{l},\, \mathfrak{m}',\, \mathfrak{m},\, \mathfrak{p}) : C\,\mathrm{Tr} \to \tilde{C}\,\mathrm{Tr}$$

is a morphism of crossed squares; cf. Definition 27.

The following map is a group morphism as a composite of group morphisms; cf. Remark 88.

$$\begin{array}{rcl} \alpha = \mu' \blacktriangle \gamma_{P,M} : M' & \longrightarrow & \mathrm{Aut}(M) \\ m' & \longmapsto & (m \mapsto (m)(m'\alpha) = (m)(m'(\mu' \blacktriangle \gamma_{P,M})) = m^{m'\mu'} =: m^{m'}) \end{array}$$

In particular, we may form the semidirect product $M' \ltimes_\alpha M$, in which

$$\begin{array}{rcl} (m', m) \cdot (\tilde{m}', \tilde{m}) & = & (m' \cdot \tilde{m}', m^{\tilde{m}'} \cdot \tilde{m}) \\ & = & (m' \cdot \tilde{m}', m^{\tilde{m}'\mu'} \cdot \tilde{m}), \end{array}$$

for (m', m), $(\tilde{m}', \tilde{m}) \in M' \ltimes_\alpha M$.

(2) *Application of* To .

Using the functor To from Definition 96 we have the 2-crossed module

$$C\,\mathrm{Tr}\,\mathrm{To} = \Big(\; L \xrightarrow{\partial_2} M' \ltimes_\alpha M \xrightarrow{\partial_1} P \;\Big),$$

with the following group morphisms

$$\begin{array}{llllll} L & \xrightarrow{\partial_2} & M' \ltimes_\alpha M & : & l & \longmapsto & (l\lambda', l^- \lambda) \\ M' \ltimes_\alpha M & \xrightarrow{\partial_1} & P & : & (m', m) & \longmapsto & m'\mu' \cdot m\mu \\ P & \xrightarrow{\beta_1} & \mathrm{Aut}(M' \ltimes_\alpha M) & : & p & \longmapsto & ((m', m) \mapsto (m', m)^p = (m'^p, m^p)) \\ P & \xrightarrow{\gamma_{P,L}} & \mathrm{Aut}(L) & : & p & \longmapsto & (l \mapsto l^p). \end{array}$$

and the map

$$\begin{array}{rcl} (M' \ltimes_\alpha M) \times (M' \ltimes_\alpha M) & \xrightarrow{\zeta} & L \\ ((m', m), (\tilde{m}', \tilde{m})) & \longmapsto & \lfloor (m', m), (\tilde{m}', \tilde{m}) \rfloor := \lceil m'^{\tilde{m}'}, \tilde{m} \rceil^{\mathrm{tr}} = \lceil \tilde{m}, m'^{\tilde{m}'} \rceil^-; \end{array}$$

cf. Lemma 93 and Remark 24.

Moreover,

$$\mathfrak{c}\,\mathrm{Tr}\,\mathrm{To} = (\mathfrak{l},\ \mathfrak{m}' \ltimes \mathfrak{m},\ \mathfrak{p}) : C\,\mathrm{Tr}\,\mathrm{To} \to \tilde{C}\,\mathrm{Tr}\,\mathrm{To}$$

is a morphism of 2-crossed modules, where $(m',m)(\mathfrak{m}'\ltimes\mathfrak{m}) = (m'\mathfrak{m}', m\mathfrak{m})$ for $(m',m) \in M'\ltimes_\alpha M$; cf. Lemma 95.

(3) *Application of* Rec .

We have the following group morphism; cf. Lemma 55.

$$\begin{array}{rcl} M' \ltimes_\alpha M & \xrightarrow{\varepsilon_2} & \operatorname{Aut}(L) \\ (m',m) & \longmapsto & \left(l \mapsto \left(\begin{array}{rcl} l^{(m',m)} & = & l \cdot \lfloor (m',m), l\partial_2 \rfloor \\ & = & l \cdot \lfloor (m',m), (l\lambda', l^-\lambda) \rfloor \\ & = & l \cdot \lceil l^-\lambda, m'^{\,l\lambda'} \rceil^- \\ & \overset{\text{(CS 4.3)}}{=} & l \cdot l^{m'^{\,l\lambda'}} \cdot l^- \\ & = & l \cdot l^{l^-\lambda' \cdot m' \cdot l\lambda'} \cdot l^- \\ & = & l \cdot ((l^{l^-\lambda'})^{m'})^{l\lambda'} \cdot l^- \\ & \overset{\text{(CM 2)}}{=} & l \cdot ((l^{l^-})^{m'})^{l} \cdot l^- \\ & = & l \cdot l^- \cdot (l \cdot l \cdot l^-)^{m'} \cdot l \cdot l^- \\ & = & l^{m'} \end{array} \right) \right) \end{array}$$

We have the following group morphism; cf. Lemma 59.

$$\begin{array}{rcl} P \ltimes_{\beta_1} (M' \ltimes_\alpha M) & \xrightarrow{\varepsilon_{1,2}} & \operatorname{Aut}((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) \\ (p,(m',m)) & \longmapsto & \left(((\check{m}',\check{m}),l) \mapsto \left(\begin{array}{rl} & ((\check{m}',\check{m}),l)^{(p,(m',m))} \\ = & (((\check{m}',\check{m})^p)^{(m',m)}, \lfloor (\check{m}',\check{m})^p, (m',m) \rfloor \cdot l^{p\,\cdot\,(m',m)\partial_1}) \\ = & ((\check{m}'^{\,p}, \check{m}^p)^{(m',m)}, \lfloor (\check{m}'^{\,p}, \check{m}^p), (m',m) \rfloor \cdot l^{p\,\cdot\, m'\mu' \cdot\, m\mu}) \\ = & ((\check{m}'^{\,p}, \check{m}^p)^{(m',m)}, \lceil (\check{m}'^{\,p})^{m'}, m \rceil^{\mathrm{tr}} \cdot l^{p\,\cdot\, m'\mu' \cdot\, m\mu}) \\ \overset{\text{Rem. 8.(2)}}{=} & (((\check{m}'^{\,p})^{m'}, (m^-)^{(\check{m}'^{\,p})^{m'}} \cdot (\check{m}^p)^{m'} \cdot m), \\ & \lceil (\check{m}'^{\,p})^{m'}, m \rceil^{\mathrm{tr}} \cdot l^{p\,\cdot\, m'\mu' \cdot\, m\mu}) \\ \overset{\text{(CM2)}}{=} & ((\check{m}'^{\,p\,\cdot\, m'\mu'}, (m^-)^{\check{m}'^{\,p\,\cdot\, m'\mu'}} \cdot \check{m}^{p\,\cdot\, m'\mu'} \cdot m), \\ & \lceil \check{m}'^{\,p\,\cdot\, m'\mu'}, m \rceil^{\mathrm{tr}} \cdot l^{p\,\cdot\, m'\mu' \cdot\, m\mu}) \end{array} \right) \right) \end{array}$$

We abbreviate

$$\mathrm{G}^C := C\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec}\,.$$

Using the functor Rec from Definition 72, we have the $\lfloor 2,0 \rfloor$-simplicial group

$$\mathrm{G}^C = \left(\begin{tikzcd}[column sep=huge] (P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) \arrow[r, shift left=8, "\mathrm{d}_2^{\mathrm{G}^C,2}"] \arrow[r, "\mathrm{d}_1^{\mathrm{G}^C,2}"] \arrow[r, shift right=8, "\mathrm{d}_0^{\mathrm{G}^C,2}"] & P \ltimes_{\beta_1} (M' \ltimes_\alpha M) \arrow[l, shift right=4, "\mathrm{s}_1^{\mathrm{G}^C,1}"'] \arrow[l, shift left=4, "\mathrm{s}_0^{\mathrm{G}^C,1}"'] \arrow[r, shift left=4, "\mathrm{d}_1^{\mathrm{G}^C,1}"] \arrow[r, shift right=4, "\mathrm{d}_0^{\mathrm{G}^C,1}"'] & P \arrow[l, "\mathrm{s}_0^{\mathrm{G}^C,0}"'] \end{tikzcd} \right)$$

with the following group morphisms; cf. Lemma 64.

$$\begin{array}{rcl}
P & \xrightarrow{\mathrm{s}_0^{\mathrm{G}^C,0}} & P \ltimes_{\varepsilon_2} (M' \ltimes_\alpha M) \\
p & \longmapsto & (p,(1,1))
\end{array}$$

$$\begin{array}{rcl}
P \ltimes_{\beta_1} (M' \ltimes_\alpha M) & \xrightarrow{\mathrm{d}_0^{\mathrm{G}^C,1}} & P \\
(p,(m',m)) & \longmapsto & \left(\begin{array}{rl} & p \cdot (m',m)\partial_1 \\ = & p \cdot m'\mu' \cdot m\mu \end{array}\right)
\end{array}$$

$$\begin{array}{rcl}
P \ltimes_{\beta_1} (M' \ltimes_\alpha M) & \xrightarrow{\mathrm{d}_1^{\mathrm{G}^C,1}} & P \\
(p,(m',m)) & \longmapsto & p
\end{array}$$

$$\begin{array}{rcl}
P \ltimes_{\beta_1} (M' \ltimes_\alpha M) & \xrightarrow{\mathrm{s}_0^{\mathrm{G}^C,1}} & (P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) \\
(p,(m',m)) & \longmapsto & ((p,(1,1)),((m',m),1))
\end{array}$$

$$\begin{array}{rcl}
P \ltimes_{\beta_1} (M' \ltimes_\alpha M) & \xrightarrow{\mathrm{s}_1^{\mathrm{G}^C,1}} & (P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) \\
(p,(m',m)) & \longmapsto & ((p,(m',m)),((1,1),1))
\end{array}$$

$$\begin{array}{rcl}
(P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) & \xrightarrow{\mathrm{d}_0^{\mathrm{G}^C,2}} & P \ltimes_{\beta_1} (M' \ltimes_\alpha M) \\
((p,(m',m)),((\check{m}',\check{m}),l)) & \longmapsto & \left(\begin{array}{rl}
 & (p \cdot (m',m)\partial_1, (\check{m}',\check{m}) \cdot l\partial_2) \\
= & (p \cdot m'\mu' \cdot m\mu, (\check{m}',\check{m}) \cdot (l\lambda', l^-\lambda)) \\
= & (p \cdot m'\mu' \cdot m\mu, (\check{m}' \cdot l\lambda', \check{m}^{l\lambda'} \cdot l^-\lambda)) \\
= & (p \cdot m'\mu' \cdot m\mu, (\check{m}' \cdot l\lambda', \check{m}^{l\lambda'\mu'} \cdot l^-\lambda)) \\
= & (p \cdot m'\mu' \cdot m\mu, (\check{m}' \cdot l\lambda', \check{m}^{l\lambda\mu} \cdot l^-\lambda)) \\
\overset{(\mathrm{CM}\ 2)}{=} & (p \cdot m'\mu' \cdot m\mu, (\check{m}' \cdot l\lambda', \check{m}^{l\lambda} \cdot l^-\lambda)) \\
= & (p \cdot m'\mu' \cdot m\mu, (\check{m}' \cdot l\lambda', l^-\lambda \cdot \check{m} \cdot l\lambda \cdot l^-\lambda)) \\
= & (p \cdot m'\mu' \cdot m\mu, (\check{m}' \cdot l\lambda', l^-\lambda \cdot \check{m}))
\end{array}\right)
\end{array}$$

$$\begin{array}{rcl}
(P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) & \xrightarrow{\mathrm{d}_1^{\mathrm{G}^C,2}} & P \ltimes_{\beta_1} (M' \ltimes_\alpha M) \\
((p,(m',m)),((\check{m}',\check{m}),l)) & \longmapsto & \left(\begin{array}{rl}
 & (p,(m',m) \cdot (\check{m}',\check{m})) \\
= & (p,(m' \cdot \check{m}', m^{\check{m}'} \cdot \check{m})) \\
= & (p,(m' \cdot \check{m}', m^{\check{m}'\mu'} \cdot \check{m}))
\end{array}\right)
\end{array}$$

$$\begin{array}{rcl}
(P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) & \xrightarrow{\mathrm{d}_2^{\mathrm{G}^C,2}} & P \ltimes_{\beta_1} (M' \ltimes_\alpha M) \\
((p,(m',m)),((\check{m}',\check{m}),l)) & \longmapsto & (p,(m',m))
\end{array}$$

Recall that we have

$$\begin{array}{rcl}
\mathrm{G}_2^C & := & (P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) \\
\mathrm{G}_1^C & := & P \ltimes_{\beta_1} (M' \ltimes_\alpha M) \\
\mathrm{G}_0^C & := & P.
\end{array}$$

We abbreviate

$$\varphi^{\mathfrak{c}} := \mathfrak{c}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec} : \mathrm{G}^C \to \mathrm{G}^{\tilde{C}}.$$

We have the following group morphism; cf. Remark 65.

$$\begin{array}{rcl}
P & \xrightarrow{\varphi_0^{\mathfrak{c}}} & \tilde{P} \\
p & \longmapsto & p\mathfrak{p}
\end{array}$$

So $\varphi_0^{\mathfrak{c}} := \mathfrak{p}$.

We have the following group morphism; cf. Lemma 66.

$$\begin{array}{rcl} P \ltimes_{\beta_1} (M' \ltimes_\alpha M) & \xrightarrow{\varphi_1^{\mathfrak{c}}} & \tilde{P} \ltimes_{\tilde{\beta}_1} (\tilde{M}' \ltimes_{\tilde{\alpha}} \tilde{M}) \\ (p,(m',m)) & \longmapsto & \left(\begin{array}{rl} & (p\mathfrak{p}, (m',m)(\mathfrak{m}' \ltimes \mathfrak{m})) \\ = & (p\mathfrak{p}, (m'\mathfrak{m}', m\mathfrak{m})) \end{array}\right) \end{array}$$

We have the following group morphism; cf. Lemma 70.

$$\begin{array}{rcl} (P \ltimes_{\beta_1} (M' \ltimes_\alpha M)) \ltimes_{\varepsilon_{1,2}} ((M' \ltimes_\alpha M) \ltimes_{\varepsilon_2} L) & \xrightarrow{\varphi_2^{\mathfrak{c}}} & (\tilde{P} \ltimes_{\tilde{\beta}_1} (\tilde{M}' \ltimes_{\tilde{\alpha}} \tilde{M})) \ltimes_{\tilde{\varepsilon}_{1,2}} ((\tilde{M}' \ltimes_{\tilde{\alpha}} \tilde{M}) \ltimes_{\tilde{\varepsilon}_2} \tilde{L}) \\ ((p,(m',m)),((\check{m}',\check{m}),l)) & \longmapsto & \left(\begin{array}{rl} & ((p\mathfrak{p}, (m',m)(\mathfrak{m}' \ltimes \mathfrak{m})), ((\check{m}',\check{m})(\mathfrak{m}' \ltimes \mathfrak{m}), l\mathfrak{l})) \\ = & ((p\mathfrak{p}, (m'\mathfrak{m}', m\mathfrak{m})), ((\check{m}'\mathfrak{m}', \check{m}\mathfrak{m}), l\mathfrak{l})) \end{array}\right) \end{array}$$

Moreover,

$$\mathfrak{c}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec} = \varphi^{\mathfrak{c}} = (\varphi_2^{\mathfrak{c}},\ \varphi_1^{\mathfrak{c}},\ \varphi_0^{\mathfrak{c}}) : \mathrm{G}^C \to \mathrm{G}^{\tilde{C}}$$

is a morphism of $\lfloor 2, 0 \rfloor$-simplicial groups; cf. Lemma 71.

Remark 106 Suppose given a crossed square

$$C = (L,\ M,\ M',\ P,\ \gamma_{M,L}\ ,\ \gamma_{M',L}\ ,\ \gamma_{P,L}\ ,\ \gamma_{P,M}\ ,\ \gamma_{P,M'}\ ,\ \lambda,\ \lambda',\ \mu,\ \mu',\ \chi).$$

Cf. Definition 15.

We shall calculate the data appearing in the crossed square

$$\mathrm{G}^C\check{\mathrm{Sq}} = C\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec}\,\check{\mathrm{Sq}}\ ,$$

namely groups, group morphisms, actions and the Loday bracket; cf. Definition 87.

(1) *Groups*

We recall from Remark 105.(3) that we have

$$\mathrm{G}_1^C = P \ltimes_{\beta_1} (M' \ltimes_\alpha M) = \{(p,(m',m)) : p \in P,\ (m',m) \in M' \ltimes_\alpha M\}$$

and we also have

$$\begin{array}{rcll} \mathrm{G}_{1;0}^C & = & \ker \mathrm{d}_0^{\mathrm{G}^C,1} & \overset{\text{Rem. 105.(3)}}{=} \{(m^-\mu \cdot m'^-\mu', (m',m)) : (m',m) \in M' \ltimes_\alpha M\} \\ \mathrm{G}_{1;1}^C & = & \ker \mathrm{d}_1^{\mathrm{G}^C,1} & \overset{\text{Rem. 105.(3)}}{=} \{(1,(m',m)) : (m',m) \in M' \ltimes_\alpha M\} \\ \mathrm{G}_{2;1,2}^C & = & \ker \mathrm{d}_1^{\mathrm{G}^C,2} \cap \ker \mathrm{d}_2^{\mathrm{G}^C,2} & \overset{\text{Rem. 105.(3)}}{=} \{((1,(1,1)),((1,1),l)) : l \in L\}. \end{array}$$

We often abbreviate $\boldsymbol{(}l\boldsymbol{)} := ((1,(1,1)),((1,1),l))$ for $l \in L$.

So $\mathrm{G}_{2;1,2}^C = \{\boldsymbol{(}l\boldsymbol{)} : l \in L\}$.

Note that for $l,\ \tilde{l} \in L$, we have

$$\begin{array}{rcl} \boldsymbol{(}l\boldsymbol{)} \cdot \boldsymbol{(}\tilde{l}\boldsymbol{)} & = & ((1,(1,1)),((1,1),l)) \cdot ((1,(1,1)),((1,1),\tilde{l})) \\ & = & ((1,(1,1)),((1,1),l) \cdot ((1,1),\tilde{l})) \\ & = & ((1,(1,1)),((1,1),l \cdot \tilde{l})) \\ & = & \boldsymbol{(}l \cdot \tilde{l}\boldsymbol{)}. \end{array}$$

(2) *Group morphisms*

We have the group morphisms

$$\begin{array}{rcl}
\mathrm{G}^C_{2;1,2} & \xrightarrow{\lambda^{1,0}_{\mathrm{G}^C\check{\mathrm{Sq}}} =: \check{\lambda}} & \mathrm{G}^C_{1;0} \\
(l) & \longmapsto & (l)\,\mathrm{d}_0 = ((1,(1,1)),((1,1),l))\,\mathrm{d}_0 = (1,(l\lambda', l^-\lambda)) \\[2ex]
\mathrm{G}^C_{1;0} & \xrightarrow{\mu^{\mathrm{G}^C\check{\mathrm{Sq}}}_{1,0} =: \check{\mu}} & \mathrm{G}^C_{1} \\
(m^-\mu \cdot m'^-\mu', (m', m)) & \longmapsto & (m^-\mu \cdot m'^-\mu', (m', m)) \\[2ex]
\mathrm{G}^C_{2;1,2} & \xrightarrow{\lambda^{0,1}_{\mathrm{G}^C\check{\mathrm{Sq}}} =: \check{\lambda}'} & \mathrm{G}^C_{1;1} \\
(l) & \longmapsto & (l)\,\mathrm{d}_0 = ((1,(1,1)),((1,1),l))\,\mathrm{d}_0 = (1,(l\lambda', l^-\lambda)) \\[2ex]
\mathrm{G}^C_{1;1} & \xrightarrow{\mu^{\mathrm{G}^C\check{\mathrm{Sq}}}_{0,1} =: \check{\mu}'} & \mathrm{G}^C_{1} \\
(1,(m', m)) & \longmapsto & (1,(m', m))\ .
\end{array}$$

Cf. Definition 85 and Remark 105.(3).

Using the functor $\check{\mathrm{Sq}}$ from Definition 87 we have the crossed square

$$\mathrm{G}^C\check{\mathrm{Sq}} = \left(\begin{tikzcd}
\mathrm{G}^C_{2;1,2} \arrow[r, "\lambda^{0,1}_{\mathrm{G}^C\check{\mathrm{Sq}}}"] \arrow[d, "\lambda^{1,0}_{\mathrm{G}^C\check{\mathrm{Sq}}}"'] & \mathrm{G}^C_{1;1} \arrow[d, "\mu^{\mathrm{G}^C\check{\mathrm{Sq}}}_{0,1}"] \\
\mathrm{G}^C_{1;0} \arrow[r, "\mu^{\mathrm{G}^C\check{\mathrm{Sq}}}_{1,0}"'] & \mathrm{G}^C_{1}
\end{tikzcd}\right) =: \left(\begin{tikzcd}
\check{L} \arrow[r, "\check{\lambda}'"] \arrow[d, "\check{\lambda}"'] & \check{M}' \arrow[d, "\check{\mu}'"] \\
\check{M} \arrow[r, "\check{\mu}"'] & \check{P}
\end{tikzcd}\right).$$

Due to (CS 1) we have $\check{\lambda} \blacktriangle \check{\mu} = \check{\lambda}' \blacktriangle \check{\mu}'$.

We write $\check{\kappa} := \kappa_{\mathrm{G}^C\check{\mathrm{Sq}}} = \check{\lambda} \blacktriangle \check{\mu} = \check{\lambda}' \blacktriangle \check{\mu}'$.

(3) *Actions*

In order to simplify our computations, we need some preparations.

(3.1) For $(m^-\mu \cdot m'^-\mu', (m', m)) \in \mathrm{G}^C_{1;0}$ we have

$$\begin{array}{rcl}
(m^-\mu \cdot m'^-\mu', (m', m))^- & \overset{\text{Rem. 8.(1)}}{=} & ((m^-\mu \cdot m'^-\mu')^-, ((m', m)^-)^{(m^-\mu \cdot m'^-\mu')^-}) \\
& \overset{\text{Rem. 8.(1)}}{=} & (m'\mu' \cdot m\mu, (m'^-, (m^-)^{m'^-})^{m'\mu' \cdot m\mu}) \\
& \overset{\text{Rem. 105.(1)}}{=} & (m'\mu' \cdot m\mu, (m'^-, (m^-)^{m'^-\mu'})^{m'\mu' \cdot m\mu}) \\
& \overset{\text{Rem. 105.(2)}}{=} & (m'\mu' \cdot m\mu, ((m'^-)^{m'\mu' \cdot m\mu}, (m^-)^{m'^-\mu' \cdot m'\mu' \cdot m\mu})) \\
& = & (m'\mu' \cdot m\mu, (((m'^-)^{m'\mu'})^{m\mu}, (m^-)^{m\mu})) \\
& \overset{\text{(CM 2)}}{=} & (m'\mu' \cdot m\mu, (((m'^-)^{m'})^{m\mu}, (m^-)^{m})) \\
& = & (m'\mu' \cdot m\mu, ((m'^-)^{m\mu}, m^-)).
\end{array}$$

(3.2) Suppose given $l \in L$, $m \in M$, $m' \in M'$ and $p \in P$.

Then we have

$$
\begin{aligned}
& (l)^{((p,(1,1)),((m',m),1))} \\
= \;& ((1,(1,1)),((1,1),l))^{((p,(1,1)),((m',m),1))} \\
\overset{\text{Rem. 8.(2)}}{=} \;& ((1,(1,1))^{(p,(1,1))}, (((m',m),1)^-)^{(1,(1,1))^{(p,(1,1))}} \cdot ((1,1),l)^{(p,(1,1))} \cdot ((m',m),1)) \\
= \;& ((1,(1,1)),((m',m),1)^- \cdot ((1,1),l)^{(p,(1,1))} \cdot ((m',m),1)) \\
= \;& ((1,(1,1)),((m',m)^-,1) \cdot ((1,1),l)^{(p,(1,1))} \cdot ((m',m),1)) \\
\underset{\varepsilon_{1,2}}{\overset{\text{Rem. 105.(3)}}{=}} \;& ((1,(1,1)),((m',m)^-,1) \cdot ((1,1),l^p) \cdot ((m',m),1)) \\
= \;& ((1,(1,1)),((m',m)^-,l^p) \cdot ((m',m),1)) \\
= \;& ((1,(1,1)),((m',m)^- \cdot (m',m),(l^p)^{(m',m)})) \\
= \;& ((1,(1,1)),((1,1),(l^p)^{(m',m)})) \\
\underset{\varepsilon_2}{\overset{\text{Rem. 105.(3)}}{=}} \;& ((1,(1,1)),((1,1),(l^p)^{m'})) \\
\overset{\text{Rem. 16}}{=} \;& ((1,(1,1)),((1,1),(l^p)^{m'\mu'})) \\
= \;& ((1,(1,1)),((1,1),l^{p \cdot m'\mu'})) \\
\overset{\text{short}}{=} \;& (l^{p \cdot m'\mu'}).
\end{aligned}
$$

(3.3) Suppose given m, $\tilde{m} \in M$, m', $\tilde{m}' \in M'$ and p, $\tilde{p} \in P$.

Then we have

$$
\begin{aligned}
& (\tilde{p},(\tilde{m}',\tilde{m}))^{(p,(m',m))} \\
\overset{\text{Rem. 8.(2)}}{=} \;& (\tilde{p}^p,((m',m)^-)^{\tilde{p}^p} \cdot (\tilde{m}',\tilde{m})^p \cdot (m',m)) \\
\overset{\text{Rem. 8.(1)}}{=} \;& (\tilde{p}^p,(m'^-,(m^-)^{m'^-})^{\tilde{p}^p} \cdot (\tilde{m}',\tilde{m})^p \cdot (m',m)) \\
\overset{\text{Rem. 105.(1)}}{=} \;& (\tilde{p}^p,(m'^-,(m^-)^{m'^-\mu'})^{\tilde{p}^p} \cdot (\tilde{m}',\tilde{m})^p \cdot (m',m)) \\
\overset{\text{Rem. 105.(2)}}{=} \;& (\tilde{p}^p,((m'^-)^{\tilde{p}^p},(m^-)^{m'^-\mu' \cdot \tilde{p}^p}) \cdot (\tilde{m}'^p,\tilde{m}^p) \cdot (m',m)) \\
= \;& (\tilde{p}^p,((m'^-)^{\tilde{p}^p} \cdot \tilde{m}'^p,((m^-)^{m'^-\mu' \cdot \tilde{p}^p})^{\tilde{m}'^p} \cdot \tilde{m}^p) \cdot (m',m)) \\
= \;& (\tilde{p}^p,((m'^-)^{\tilde{p}^p} \cdot \tilde{m}'^p \cdot m',(((m^-)^{m'^-\mu' \cdot \tilde{p}^p})^{\tilde{m}'^p} \cdot \tilde{m}^p)^{m'} \cdot m)) \\
= \;& (\tilde{p}^p,((m'^-)^{\tilde{p}^p} \cdot \tilde{m}'^p \cdot m',(((m^-)^{m'^-\mu' \cdot \tilde{p}^p})^{\tilde{m}'^p})^{m'} \cdot (\tilde{m}^p)^{m'} \cdot m)) \\
\overset{\text{Rem. 105.(1)}}{=} \;& (\tilde{p}^p,((m'^-)^{\tilde{p}^p} \cdot \tilde{m}'^p \cdot m',(((m^-)^{m'^-\mu' \cdot \tilde{p}^p})^{\tilde{m}'^p\mu'})^{m'\mu'} \cdot (\tilde{m}^p)^{m'\mu'} \cdot m)) \\
\overset{\text{(CM 1)}}{=} \;& (\tilde{p}^p,((m'^-)^{\tilde{p}^p} \cdot \tilde{m}'^p \cdot m',(((m^-)^{m'^-\mu' \cdot \tilde{p}^p})^{(\tilde{m}'\mu')^p})^{m'\mu'} \cdot (\tilde{m}^p)^{m'\mu'} \cdot m)) \\
= \;& (\tilde{p}^p,((m'^-)^{\tilde{p}^p} \cdot \tilde{m}'^p \cdot m',(m^-)^{m'^-\mu' \cdot \tilde{p}^p \cdot (\tilde{m}'\mu')^p \cdot m'\mu'} \cdot \tilde{m}^{p \cdot m'\mu'} \cdot m)) \\
= \;& (\tilde{p}^p,((m'^-)^{\tilde{p}^p} \cdot \tilde{m}'^p \cdot m',(m^-)^{(\tilde{p} \cdot \tilde{m}'\mu')^{p \cdot m'\mu'}} \cdot \tilde{m}^{p \cdot m'\mu'} \cdot m)).
\end{aligned}
$$

We have the following group morphisms.

$$\begin{array}{ccc}
\mathrm{G}^{C}_{1;0} & \xrightarrow{\gamma_{1,0}^{\mathrm{G}^{C}\check{\mathrm{Sq}}} = \gamma_{\check{M},\check{L}}} & \mathrm{Aut}(\mathrm{G}^{C}_{2;1,2}) \\
(m^{-}\mu\cdot\ m'^{-}\mu', (m',m)) & \longmapsto & \left(\begin{array}{cl}
(l) & \\
\mapsto & (l)^{(m^{-}\mu\cdot m'^{-}\mu',(m',m))} \\
\overset{\text{Def. 85}}{=} & (l)^{(m^{-}\mu\cdot m'^{-}\mu',(m',m))\,\mathrm{s}_0\,\cdot\,(m^{-}\mu\cdot m'^{-}\mu',(m',m))^{-}\,\mathrm{s}_1} \\
\overset{(3.1)}{=} & (l)^{(m^{-}\mu\cdot m'^{-}\mu',(m',m))\,\mathrm{s}_0\,\cdot\,(m'\mu'\cdot m\mu,((m'^{-})^{m\mu},m^{-}))\,\mathrm{s}_1} \\
\overset{\text{Rem. 105.(3)}}{=} & (l)^{((m^{-}\mu\cdot m'^{-}\mu',(1,1)),((m',m),1))\cdot((m'\mu'\cdot m\mu,((m'^{-})^{m\mu},m^{-})),((1,1),1))} \\
= & ((l)^{((m^{-}\mu\cdot m'^{-}\mu',(1,1)),((m',m),1))})^{((m'\mu'\cdot m\mu,((m'^{-})^{m\mu},m^{-})),((1,1),1))} \\
\overset{(3.2)}{=} & (l^{m^{-}\mu\cdot m'^{-}\mu'\cdot m'\mu'})^{((m'\mu'\cdot m\mu,((m'^{-})^{m\mu},m^{-})),((1,1),1))} \\
= & ((1,(1,1)),((1,1),l^{m^{-}\mu}))^{((m'\mu'\cdot m\mu,((m'^{-})^{m\mu},m^{-})),((1,1),1))} \\
\overset{\text{Rem. 8.(2)}}{=} & ((1,(1,1)),((1,1),l^{m^{-}\mu})^{(m'\mu'\cdot m\mu,((m'^{-})^{m\mu},m^{-}))}) \\
\underset{\varepsilon_{1,2}}{\overset{\text{Rem. 105.(3)}}{=}} & ((1,(1,1)),((1,m\cdot m^{-}),\lceil 1,m^{-}\rceil^{\mathrm{tr}}\cdot(l^{m^{-}\mu})^{m'\mu'\cdot m\mu\,\cdot\,(m'^{-})^{m\mu}\mu'\cdot m^{-}\mu})) \\
\overset{\text{Rem. 24}}{=} & ((1,(1,1)),((1,1),\lceil m^{-},1\rceil^{-}\cdot l^{m^{-}\mu\cdot m'\mu'\cdot m\mu\,\cdot\,(m'^{-})^{m\mu}\mu'\cdot m^{-}\mu})) \\
\overset{\text{Rem. 18}}{=} & ((1,(1,1)),((1,1),l^{m^{-}\mu\cdot m'\mu'\cdot m\mu\,\cdot\,(m'^{-})^{m\mu}\mu'\cdot m^{-}\mu})) \\
\overset{\text{(CM 1)}}{=} & ((1,(1,1)),((1,1),l^{m^{-}\mu\cdot m'\mu'\cdot m\mu\,\cdot\,(m'^{-}\mu')^{m\mu}\cdot m^{-}\mu})) \\
= & ((1,(1,1)),((1,1),l^{m^{-}\mu\cdot m'\mu'\cdot m\mu\cdot m^{-}\mu\cdot m'^{-}\mu'\cdot m\mu\cdot m^{-}\mu})) \\
= & ((1,(1,1)),((1,1),l^{m^{-}\mu})) \\
\overset{\text{Rem. 16}}{=} & ((1,(1,1)),((1,1),l^{m^{-}})) \\
\overset{\text{short}}{=} & (l^{m^{-}})
\end{array}\right)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{G}^{C}_{1;1} & \xrightarrow{\gamma_{0,1}^{\mathrm{G}^{C}\check{\mathrm{Sq}}} = \gamma_{\check{M}',\check{L}}} & \mathrm{Aut}(\mathrm{G}^{C}_{2;1,2}) \\
(1,(m',m)) & \longmapsto & \left((l) \mapsto \left(\begin{array}{ccl}
(l)^{(1,(m',m))} & \overset{\text{Def. 85}}{=} & (l)^{(1,(m',m))\,\mathrm{s}_0} \\
 & \overset{\text{Rem. 105.(3)}}{=} & (l)^{((1,(1,1)),((m',m),1))} \\
 & \overset{(3.2)}{=} & (l^{m'\mu'}) \\
 & \overset{\text{Rem. 16}}{=} & (l^{m'})
\end{array}\right)\right)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{G}^{C}_{1} & \xrightarrow{\gamma^{1,1}_{\mathrm{G}^{C}\check{\mathrm{Sq}}} = \gamma_{\check{P},\check{L}}} & \mathrm{Aut}(\mathrm{G}^{C}_{2;1,2}) \\
(p,(m',m)) & \longmapsto & \left((l) \mapsto \left(\begin{array}{ccl}
(l)^{(p,(m',m))} & \overset{\text{Def. 85}}{=} & (l)^{(p,(m',m))\,\mathrm{s}_0} \\
 & \overset{\text{Rem. 105.(3)}}{=} & (l)^{((p,(1,1)),((m',m),1))} \\
 & \overset{(3.2)}{=} & (l^{p\,\cdot\,m'\mu'})
\end{array}\right)\right)
\end{array}$$

$$\mathrm{G}^C_1 \xrightarrow{\gamma^{1,0}_{\mathrm{G}^C\check{\mathrm{Sq}}}=\gamma_{\check{P},\check{M}}} \mathrm{Aut}(\mathrm{G}^C_{1;0})$$

$$(p,(m',m)) \longmapsto \left(\begin{array}{cl}
 & (\tilde{m}^-\mu\cdot\tilde{m}'^-\mu',(\tilde{m}',\tilde{m})) \\
\mapsto & (\tilde{m}^-\mu\cdot\tilde{m}'^-\mu',(\tilde{m}',\tilde{m}))^{(p,(m',m))} \\
\overset{\text{Def. 85}}{\underset{(3.3)}{=}} & ((\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p,((m'^-)^{(\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p}\cdot\tilde{m}'^p\cdot m', \\
 & (m^-)^{(\tilde{m}^-\mu\cdot\tilde{m}'^-\mu'\cdot\tilde{m}'\mu')^{p\,\cdot\,m'\mu'}}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m)) \\
= & ((\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p,((m'^-)^{(\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p}\cdot\tilde{m}'^p\cdot m', \\
 & (m^-)^{(\tilde{m}^-\mu)^{p\,\cdot\,m'\mu'}}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m)) \\
\overset{\text{(CM 1)}}{=} & ((\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p,((m'^-)^{(\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p}\cdot\tilde{m}'^p\cdot m', \\
 & (m^-)^{((\tilde{m}^-)^{p\,\cdot\,m'\mu'})\mu}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m)) \\
\overset{\text{(CM 2)}}{=} & ((\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p,((m'^-)^{(\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p}\cdot\tilde{m}'^p\cdot m', \\
 & (m^-)^{(\tilde{m}^-)^{p\,\cdot\,m'\mu'}}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m)) \\
= & ((\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p,((m'^-)^{(\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p}\cdot\tilde{m}'^p\cdot m', \\
 & \tilde{m}^{p\,\cdot\,m'\mu'}\cdot m^-\cdot(\tilde{m}^-)^{p\,\cdot\,m'\mu'}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m)) \\
= & ((\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p,((m'^-)^{(\tilde{m}^-\mu\cdot\tilde{m}'^-\mu')^p}\cdot\tilde{m}'^p\cdot m',\tilde{m}^{p\,\cdot\,m'\mu'}))
\end{array}\right)$$

$$\mathrm{G}^C_1 \xrightarrow{\gamma^{0,1}_{\mathrm{G}^C\check{\mathrm{Sq}}}=\gamma_{\check{P},\check{M}'}} \mathrm{Aut}(\mathrm{G}^C_{1;1})$$

$$(p,(m',m)) \longmapsto \left(\,(1,(\tilde{m}',\tilde{m}))\mapsto\left(\begin{array}{cl}
 & (1,(\tilde{m}',\tilde{m}))^{(p,(m',m))} \\
\overset{\text{Def. 85}}{\underset{(3.3)}{=}} & (1,(m'^-\cdot\tilde{m}'^p\cdot m',(m^-)^{(\tilde{m}'\mu')^{p\,\cdot\,m'\mu'}}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m)) \\
= & (1,((\tilde{m}'^p)^{m'},(m^-)^{(\tilde{m}'\mu')^{p\,\cdot\,m'\mu'}}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m)) \\
\overset{\text{(CM 2)}}{=} & (1,((\tilde{m}'^p)^{m'\mu'},(m^-)^{(\tilde{m}'\mu')^{p\,\cdot\,m'\mu'}}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m)) \\
= & (1,(\tilde{m}'^{p\,\cdot\,m'\mu'},(m^-)^{(\tilde{m}'\mu')^{p\,\cdot\,m'\mu'}}\cdot\tilde{m}^{p\,\cdot\,m'\mu'}\cdot m))
\end{array}\right)\right)$$

Cf. Definition 85.

$$\mathrm{G}^C\check{\mathrm{Sq}} = \left(\begin{tikzcd}
\mathrm{G}^C_{2;1,2} \arrow[r, "\lambda^{0,1}_{\mathrm{G}^C\check{\mathrm{Sq}}}"] \arrow[d, "\lambda^{1,0}_{\mathrm{G}^C\check{\mathrm{Sq}}}"'] & \mathrm{G}^C_{1;1} \arrow[d, "\mu^{\mathrm{G}^C\check{\mathrm{Sq}}}_{0,1}"] \\
\mathrm{G}^C_{1;0} \arrow[r, "\mu^{\mathrm{G}^C\check{\mathrm{Sq}}}_{1,0}"'] & \mathrm{G}^C_1
\end{tikzcd}\right) = \left(\begin{tikzcd}
\check{L} \arrow[r, "\check{\lambda}'"] \arrow[d, "\check{\lambda}"'] & \check{M}' \arrow[d, "\check{\mu}'"] \\
\check{M} \arrow[r, "\check{\mu}"'] & \check{P}
\end{tikzcd}\right)$$

Altogether, we have

$$\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
(\mathrm{G}^C\check{\mathrm{Sq}})_{1,1} & = & \mathrm{G}^C_{2;1,2} & \gamma^{\mathrm{G}^C\check{\mathrm{Sq}}}_{1,0} & = & \gamma_{\check{M},\check{L}} & \lambda^{1,0}_{\mathrm{G}^C\check{\mathrm{Sq}}} & = & \check{\lambda} \\
(\mathrm{G}^C\check{\mathrm{Sq}})_{1,0} & = & \mathrm{G}^C_{1;0} & \gamma^{\mathrm{G}^C\check{\mathrm{Sq}}}_{0,1} & = & \gamma_{\check{M}',\check{L}} & \lambda^{0,1}_{\mathrm{G}^C\check{\mathrm{Sq}}} & = & \check{\lambda}' \\
(\mathrm{G}^C\check{\mathrm{Sq}})_{0,1} & = & \mathrm{G}^C_{1;1} & \gamma^{1,1}_{\mathrm{G}^C\check{\mathrm{Sq}}} & = & \gamma_{\check{P},\check{L}} & \mu^{\mathrm{G}^C\check{\mathrm{Sq}}}_{1,0} & = & \check{\mu} \\
(\mathrm{G}^C\check{\mathrm{Sq}})_{0,0} & = & \mathrm{G}^C_1 & \gamma^{1,0}_{\mathrm{G}^C\check{\mathrm{Sq}}} & = & \gamma_{\check{P},\check{M}} & \mu^{\mathrm{G}^C\check{\mathrm{Sq}}}_{0,1} & = & \check{\mu}' \\
 & & & \gamma^{0,1}_{\mathrm{G}^C\check{\mathrm{Sq}}} & = & \gamma_{\check{P},\check{M}'} & \chi_{\mathrm{G}^C\check{\mathrm{Sq}}} & = & \check{\chi} \qquad\qquad \text{(see (4) below)} \\
 & & & & & & \kappa_{\mathrm{G}^C\check{\mathrm{Sq}}} & = & \check{\kappa}=\check{\lambda}\blacktriangle\check{\mu}=\check{\lambda}'\blacktriangle\check{\mu}'.
\end{array}$$

(4) *Loday bracket*

Suppose given $m,\ \tilde{m} \in M$ and $m',\ \tilde{m}' \in M'$.

We *claim* that in $\mathrm{G}^C\check{\mathrm{Sq}}$ we have

$$\begin{aligned} \lceil (m^-\mu \cdot m'^-\mu', (m',m)),\ (1,(\tilde{m}',\tilde{m}))\rceil &= ((1,(1,1)),((1,1),\lceil m^-,\tilde{m}'\rceil)) \\ &\overset{\text{short}}{=} (\lceil m^-,\tilde{m}'\rceil). \end{aligned}$$

We calculate

$$\begin{aligned}
& \lceil (m^-\mu \cdot m'^-\mu', (m',m)),\ (1,(\tilde{m}',\tilde{m}))\rceil \\
\overset{\text{Def. 85}}{=}\ & [(m^-\mu \cdot m'^-\mu', (m',m))\,\mathrm{s}_0 \cdot (m^-\mu \cdot m'^-\mu', (m',m))^-\,\mathrm{s}_1, (1,(\tilde{m}',\tilde{m}))\,\mathrm{s}_0] \\
\overset{(3.1)}{=}\ & [(m^-\mu \cdot m'^-\mu', (m',m))\,\mathrm{s}_0 \cdot (m'\mu' \cdot m\mu, ((m'^-)^{m\mu}, m^-))\,\mathrm{s}_1, (1,(\tilde{m}',\tilde{m}))\,\mathrm{s}_0] \\
\overset{\text{Rem. 105.(3)}}{=}\ & [((m^-\mu \cdot m'^-\mu', (1,1)),((m',m),1)) \cdot ((m'\mu' \cdot m\mu, ((m'^-)^{m\mu}, m^-)),((1,1),1)),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((m^-\mu \cdot m'^-\mu', (1,1)) \cdot\ (m'\mu' \cdot m\mu, ((m'^-)^{m\mu}, m^-)), \\
& ((m',m),1)^{(m'\mu'\cdot m\mu,((m'^-)^{m\mu},m^-))} \cdot ((1,1),1)),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((m^-\mu \cdot m'^-\mu' \cdot\ m'\mu' \cdot m\mu, (1,1)^{m'\mu'\cdot m\mu} \cdot ((m'^-)^{m\mu}, m^-)), \\
& ((m',m),1)^{(m'\mu'\cdot m\mu,((m'^-)^{m\mu},m^-))}),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((1,((m'^-)^{m\mu},m^-)),((m',m),1)^{(m'\mu'\cdot m\mu,((m'^-)^{m\mu},m^-))}),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
\underset{\varepsilon_{1,2}}{\overset{\text{Rem. 105.(3)}}{=}}\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m'\mu'\cdot m\mu\cdot((m'^-)^{m\mu})\mu'}, m^{({m'}^{m'\mu'\cdot m\mu\cdot((m'^-)^{m\mu})\mu'})} \cdot m^{m'\mu'\cdot m\mu\cdot((m'^-)^{m\mu})\mu'} \cdot m^-), \\
& \lceil {m'}^{m'\mu'\cdot m\mu\cdot((m'^-)^{m\mu})\mu'}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
\overset{\text{(CM 1)}}{=}\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m'\mu'\cdot m\mu\cdot(m'^-\mu')^{m\mu}}, m^{({m'}^{m'\mu'\cdot m\mu\cdot(m'^-\mu')^{m\mu}})} \cdot m^{m'\mu'\cdot m\mu\cdot(m'^-\mu')^{m\mu}} \cdot m^-), \\
& \lceil {m'}^{m'\mu'\cdot m\mu\cdot(m'^-\mu')^{m\mu}}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m'\mu'\cdot m\mu\cdot m^-\mu\cdot m'^-\mu'\cdot m\mu}, m^{({m'}^{m'\mu'\cdot m\mu\cdot m^-\mu\cdot m'^-\mu'\cdot m\mu})} \cdot m^{m'\mu'\cdot m\mu\cdot m^-\mu\cdot m'^-\mu'\cdot m\mu} \cdot m^-), \\
& \lceil {m'}^{m'\mu'\cdot m\mu\cdot m^-\mu\cdot m'^-\mu'\cdot m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, m^{({m'}^{m\mu})} \cdot m^{m\mu} \cdot m^-), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
\overset{\text{(CM 2)}}{=}\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, m^{({m'}^{m\mu})} \cdot m^{m} \cdot m^-), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, m^{({m'}^{m\mu})}), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
\overset{\text{Rem. 105.(1)}}{=}\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, m^{({m'}^{m\mu})\mu'}), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
\overset{\text{(CM 1)}}{=}\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, m^{(m'\mu')^{m\mu}}), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, m^{m^-\mu\cdot m'\mu'\cdot m\mu}), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, (m^{m^-\mu})^{m'\mu'\cdot m\mu}), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
\overset{\text{(CM 2)}}{=}\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, (m^{m^-})^{m'\mu'\cdot m\mu}), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
=\ & [((1,((m'^-)^{m\mu},m^-)),(({m'}^{m\mu}, m^{m'\mu'\cdot m\mu}), \lceil {m'}^{m\mu}, m^-\rceil^{\mathrm{tr}})),((1,(1,1)),((\tilde{m}',\tilde{m}),1))] \\
\overset{\text{Rem. 8.(3)}}{=}\ & ((1,(1,1)),(((\tilde{m}',\tilde{m})^-,1)^{(1,((m'^-)^{m\mu},m^-))((({m'}^{m\mu},m^{m'\mu'\cdot m\mu}),\lceil {m'}^{m\mu},m^-\rceil^{\mathrm{tr}})} \cdot ((\tilde{m}',\tilde{m}),1)) \\
\overset{\text{Rem. 8.(1)}}{=}\ & ((1,(1,1)),(((\tilde{m}'^-,(\tilde{m}^-)^{\tilde{m}'^-}),1)^{(1,((m'^-)^{m\mu},m^-))(({m'}^{m\mu},m^{m'\mu'\cdot m\mu}),\lceil {m'}^{m\mu},m^-\rceil^{\mathrm{tr}})} \cdot ((\tilde{m}',\tilde{m}),1)) \\
\overset{\text{Rem. 105.(1)}}{=}\ & ((1,(1,1)),(((\tilde{m}'^-,(\tilde{m}^-)^{\tilde{m}'^-\mu'}),1)^{(1,((m'^-)^{m\mu},m^-))(({m'}^{m\mu},m^{m'\mu'\cdot m\mu}),\lceil {m'}^{m\mu},m^-\rceil^{\mathrm{tr}})} \cdot ((\tilde{m}',\tilde{m}),1)) \\
\overset{\text{Rem. 8.(1)}}{=}\ & ((1,(1,1)),(((\tilde{m}'^-,(\tilde{m}^-)^{\tilde{m}'^-\mu'}),1)^{(1,((m'^-)^{m\mu},m^-))(({m'}^{m\mu},m^{m'\mu'\cdot m\mu}),\lceil {m'}^{m\mu},m^-\rceil^{\mathrm{tr}})} \cdot ((\tilde{m}'^-,(\tilde{m}^-)^{\tilde{m}'^-})^-,1)) \\
\overset{\text{Rem. 105.(1)}}{=}\ & ((1,(1,1)),(((\tilde{m}'^-,(\tilde{m}^-)^{\tilde{m}'^-\mu'}),1)^{(1,((m'^-)^{m\mu},m^-))(({m'}^{m\mu},m^{m'\mu'\cdot m\mu}),\lceil {m'}^{m\mu},m^-\rceil^{\mathrm{tr}})} \cdot ((\tilde{m}'^-,(\tilde{m}^-)^{\tilde{m}'^-\mu'})^-,1)).
\end{aligned}$$

In order to facilitate calculations, we write $a := m$, $b := m'$, $c := (\tilde{m}^-)^{\tilde{m}'^-\mu'}$ and $d := \tilde{m}'^-$.

Then $a,\ c \in M$ and $b,\ d \in M'$.

We have to show that

$$\begin{aligned} & (((\tilde{m}'^-,(\tilde{m}^-)^{\tilde{m}'^-\mu'}),1)^{(1,((m'^-)^{m\mu},m^-))(({m'}^{m\mu},m^{m'\mu'\cdot m\mu}),\lceil {m'}^{m\mu},m^-\rceil^{\mathrm{tr}})} \cdot ((\tilde{m}'^-,(\tilde{m}^-)^{\tilde{m}'^-\mu'})^-,1) \\ \overset{!}{=}\ & ((1,1),\lceil m^-,\tilde{m}'\rceil). \end{aligned}$$

So we have to show that

$$(((d,c),1)^{(1,((b^-)^{a\mu},a^-))})^{((b^{a\mu},a^{b\mu'\cdot a\mu}),\lceil b^{a\mu},a^-\rceil^{\mathrm{tr}})}\cdot((d,c)^-,1)\overset{!}{=}((1,1),\lceil a^-,d^-\rceil).$$

We obtain

$$\begin{aligned}
& (((d,c),1)^{(1,((b^-)^{a\mu},a^-))})^{((b^{a\mu},a^{b\mu'\cdot a\mu}),\lceil b^{a\mu},a^-\rceil^{\mathrm{tr}})}\cdot((d,c)^-,1)\\
\overset{\text{Rem. 105.(3)}}{\underset{\varepsilon_{1,2}}{=}}\ & ((d^{((b^-)^{a\mu})\mu'},a^{d^{((b^-)^{a\mu})\mu'}}\cdot c^{((b^-)^{a\mu})\mu'}\cdot a^-),\lceil d^{((b^-)^{a\mu})\mu'},a^-\rceil^{\mathrm{tr}})^{((b^{a\mu},a^{b\mu'\cdot a\mu}),\lceil b^{a\mu},a^-\rceil^{\mathrm{tr}})}\cdot((d,c)^-,1)\\
\overset{\text{Rem. 105.(1)}}{=}\ & ((d^{((b^-)^{a\mu})\mu'},a^{d^{((b^-)^{a\mu})\mu'}}\cdot c^{(b^-)^{a\mu}}\cdot a^-),\lceil d^{((b^-)^{a\mu})\mu'},a^-\rceil^{\mathrm{tr}})^{((b^{a\mu},a^{b\mu'\cdot a\mu}),\lceil b^{a\mu},a^-\rceil^{\mathrm{tr}})}\cdot((d,c)^-,1)\\
\overset{\text{(CM 2)}}{=}\ & ((d^{(b^-)^{a\mu}},a^{d^{(b^-)^{a\mu}}}\cdot c^{(b^-)^{a\mu}}\cdot a^-),\lceil d^{(b^-)^{a\mu}},a^-\rceil^{\mathrm{tr}})^{((b^{a\mu},a^{b\mu'\cdot a\mu}),\lceil b^{a\mu},a^-\rceil^{\mathrm{tr}})}\cdot((d,c)^-,1)\\
\overset{\text{Rem. 24}}{=}\ & ((d^{(b^-)^{a\mu}},a^{d^{(b^-)^{a\mu}}}\cdot c^{(b^-)^{a\mu}}\cdot a^-),\lceil a^-,d^{(b^-)^{a\mu}}\rceil^-)^{((b^{a\mu},a^{b\mu'\cdot a\mu}),\lceil a^-,b^{a\mu}\rceil^-)}\cdot((d,c)^-,1)\\
\overset{\text{Rem. 8.(2)}}{=}\ & ((d^{(b^-)^{a\mu}},a^{d^{(b^-)^{a\mu}}}\cdot c^{(b^-)^{a\mu}}\cdot a^-)^{(b^{a\mu},a^{b\mu'\cdot a\mu})},\\
& \lceil a^-,b^{a\mu}\rceil^{(d^{(b^-)^{a\mu}},a^{d^{(b^-)^{a\mu}}}\cdot c^{(b^-)^{a\mu}}\cdot a^-)^{(b^{a\mu},a^{b\mu'\cdot a\mu})}}\cdot(\lceil a^-,d^{(b^-)^{a\mu}}\rceil^-)^{(b^{a\mu},a^{b\mu'\cdot a\mu})}\cdot\lceil a^-,b^{a\mu}\rceil^-)\cdot((d,c)^-,1).
\end{aligned}$$

We consider the first factor and calculate

$$\begin{aligned}
& (d^{(b^-)^{a\mu}},a^{d^{(b^-)^{a\mu}}}\cdot c^{(b^-)^{a\mu}}\cdot a^-)^{(b^{a\mu},a^{b\mu'\cdot a\mu})}\\
\overset{\text{Rem. 8.(2)}}{=}\ & ((d^{(b^-)^{a\mu}})^{b^{a\mu}},((a^-)^{b\mu'\cdot a\mu})^{(d^{(b^-)^{a\mu}})^{b^{a\mu}}}\cdot(a^{d^{(b^-)^{a\mu}}}\cdot c^{(b^-)^{a\mu}}\cdot a^-)^{b^{a\mu}}\cdot a^{b\mu'\cdot a\mu})\\
=\ & (d,((a^-)^{b\mu'\cdot a\mu})^d\cdot(a^{d^{(b^-)^{a\mu}}})^{b^{a\mu}}\cdot(c^{(b^-)^{a\mu}})^{b^{a\mu}}\cdot(a^-)^{b^{a\mu}}\cdot a^{b\mu'\cdot a\mu})\\
=\ & (d,((a^-)^{b\mu'\cdot a\mu})^d\cdot a^{d^{(b^-)^{a\mu}}\cdot b^{a\mu}}\cdot c\cdot(a^-)^{b^{a\mu}}\cdot a^{b\mu'\cdot a\mu})\\
\overset{\text{(CM 2)}}{=}\ & (d,((a^-)^{b\mu'\cdot a\mu})^d\cdot a^{d^{((b^-)^{a\mu})\mu'}\cdot b^{a\mu}}\cdot c\cdot(a^-)^{b^{a\mu}}\cdot a^{b\mu'\cdot a\mu})\\
\overset{\text{Rem. 105.(1)}}{=}\ & (d,((a^-)^{b\mu'\cdot a\mu})^{d\mu'}\cdot a^{(d^{((b^-)^{a\mu})\mu'}\cdot b^{a\mu})\mu'}\cdot c\cdot(a^-)^{(b^{a\mu})\mu'}\cdot a^{b\mu'\cdot a\mu})\\
\overset{\text{(CM 1)}}{=}\ & (d,(a^-)^{b\mu'\cdot a\mu\cdot d\mu'}\cdot a^{(d\mu')^{(b^-\mu')^{a\mu}}\cdot(b\mu')^{a\mu}}\cdot c\cdot(a^-)^{(b\mu')^{a\mu}}\cdot a^{b\mu'\cdot a\mu})\\
=\ & (d,(a^-)^{b\mu'\cdot a\mu\cdot d\mu'}\cdot a^{(b\mu')^{a\mu}\cdot d\mu'\cdot(b^-\mu')^{a\mu}\cdot(b\mu')^{a\mu}}\cdot c\cdot(a^-)^{(b\mu')^{a\mu}}\cdot a^{b\mu'\cdot a\mu})\\
=\ & (d,(a^-)^{b\mu'\cdot a\mu\cdot d\mu'}\cdot a^{a^-\mu\cdot b\mu'\cdot a\mu\cdot d\mu'}\cdot c\cdot(a^-)^{a^-\mu\cdot b\mu'\cdot a\mu}\cdot a^{b\mu'\cdot a\mu})\\
=\ & (d,(a^-)^{b\mu'\cdot a\mu\cdot d\mu'}\cdot(a^{a^-\mu})^{b\mu'\cdot a\mu\cdot d\mu'}\cdot c\cdot((a^-)^{a^-\mu})^{b\mu'\cdot a\mu}\cdot a^{b\mu'\cdot a\mu})\\
\overset{\text{(CM 2)}}{=}\ & (d,(a^-)^{b\mu'\cdot a\mu\cdot d\mu'}\cdot(a^{a^-})^{b\mu'\cdot a\mu\cdot d\mu'}\cdot c\cdot((a^-)^{a^-})^{b\mu'\cdot a\mu}\cdot a^{b\mu'\cdot a\mu})\\
=\ & (d,(a^-)^{b\mu'\cdot a\mu\cdot d\mu'}\cdot a^{b\mu'\cdot a\mu\cdot d\mu'}\cdot c\cdot(a^-)^{b\mu'\cdot a\mu}\cdot a^{b\mu'\cdot a\mu})\\
=\ & (d,c).
\end{aligned}$$

So we can continue

$$\begin{aligned}
& ((d^{(b^-)^{a\mu}},a^{d^{(b^-)^{a\mu}}}\cdot c^{(b^-)^{a\mu}}\cdot a^-)^{(b^{a\mu},a^{b\mu'\cdot a\mu})},\\
& \lceil a^-,b^{a\mu}\rceil^{(d^{(b^-)^{a\mu}},a^{d^{(b^-)^{a\mu}}}\cdot c^{(b^-)^{a\mu}}\cdot a^-)^{(b^{a\mu},a^{b\mu'\cdot a\mu})}}\cdot(\lceil a^-,d^{(b^-)^{a\mu}}\rceil^-)^{(b^{a\mu},a^{b\mu'\cdot a\mu})}\cdot\lceil a^-,b^{a\mu}\rceil^-)\cdot((d,c)^-,1)\\
=\ & ((d,c),\lceil a^-,b^{a\mu}\rceil^{(d,c)}\cdot(\lceil a^-,d^{(b^-)^{a\mu}}\rceil^-)^{(b^{a\mu},a^{b\mu'\cdot a\mu})}\cdot\lceil a^-,b^{a\mu}\rceil^-)\cdot((d,c)^-,1)\\
\overset{\text{Rem. 105.(3)}}{\underset{\varepsilon_2}{=}}\ & ((d,c),\lceil a^-,b^{a\mu}\rceil^d\cdot(\lceil a^-,d^{(b^-)^{a\mu}}\rceil^-)^{b^{a\mu}}\cdot\lceil a^-,b^{a\mu}\rceil^-)\cdot((d,c)^-,1)\\
=\ & ((d,c),\lceil a^-,b^{a\mu}\rceil^d\cdot(\lceil a^-,d^{(b^-)^{a\mu}}\rceil^{b^{a\mu}})^-\cdot\lceil a^-,b^{a\mu}\rceil^-)\cdot((d,c)^-,1)\\
\overset{\text{(CS 4.6)}}{=}\ & ((d,c),\lceil a^-,d\rceil^-\cdot\lceil a^-,b^{a\mu}\cdot d\rceil\cdot(\lceil a^-,b^{a\mu}\rceil^-\cdot\lceil a^-,d^{(b^-)^{a\mu}}\cdot b^{a\mu}\rceil)^-\cdot\lceil a^-,b^{a\mu}\rceil^-)\cdot((d,c)^-,1)\\
=\ & ((d,c),\lceil a^-,d\rceil^-\cdot\lceil a^-,b^{a\mu}\cdot d\rceil\cdot(\lceil a^-,b^{a\mu}\rceil^-\cdot\lceil a^-,b^{a\mu}\cdot d\cdot(b^-)^{a\mu}\cdot b^{a\mu}\rceil)^-\cdot\lceil a^-,b^{a\mu}\rceil^-)\cdot((d,c)^-,1)\\
=\ & ((d,c),\lceil a^-,d\rceil^-\cdot\lceil a^-,b^{a\mu}\cdot d\rceil\cdot\lceil a^-,b^{a\mu}\cdot d\rceil^-\cdot\lceil a^-,b^{a\mu}\rceil\cdot\lceil a^-,b^{a\mu}\rceil^-)\cdot((d,c)^-,1)\\
=\ & ((d,c),\lceil a^-,d\rceil^-)\cdot((d,c)^-,1)\\
=\ & ((d,c)\cdot(d,c)^-,(\lceil a^-,d\rceil^-)^{(d,c)^-}\cdot 1)\\
=\ & ((1,1),(\lceil a^-,d\rceil^-)^{(d,c)^-})\\
\overset{\text{Rem. 8.(1)}}{=}\ & ((1,1),(\lceil a^-,d\rceil^-)^{(d^-,(c^-)^{d^-})})\\
\overset{\text{Rem. 105.(3)}}{\underset{\varepsilon_2}{=}}\ & ((1,1),(\lceil a^-,d\rceil^-)^{d^-})
\end{aligned}$$

$$\begin{array}{cl}
= & ((1,1),(\lceil a^-,d\rceil^{d^-})^-) \\
\overset{\text{(CS 4.6)}}{=} & ((1,1),(\lceil a^-,d^-\rceil^- \cdot \lceil a^-,d\cdot d^-\rceil)^-) \\
= & ((1,1),(\lceil a^-,d^-\rceil^- \cdot \lceil a^-,1\rceil)^-) \\
\overset{\text{Rem. 18}}{=} & ((1,1),\lceil a^-,d^-\rceil).
\end{array}$$

This shows the *claim.*

We have the following map.

$$\begin{array}{rcl}
\mathrm{G}^C_{1;0}\times\mathrm{G}^C_{1;1} & \xrightarrow{\chi_{\mathrm{G}^C\check{\mathrm{Sq}}}=\check{\chi}} & \mathrm{G}^C_{2;1,2} \\
((m^-\mu\cdot m'^-\mu',(m',m)),\,(1,(\tilde{m}',\tilde{m}))) & \longmapsto & \left(\begin{array}{cl} & \lceil(m^-\mu\cdot m'^-\mu',(m',m)),\,(1,(\tilde{m}',\tilde{m}))\rceil \\ = & (\lceil m^-,\tilde{m}'\rceil)\end{array}\right)
\end{array}$$

Cf. Definition 85.

Remark 107 Suppose given crossed squares

$$C=(L,\,M,\,M',\,P,\,\gamma_{M,L}\,,\,\gamma_{M',L}\,,\,\gamma_{P,L}\,,\,\gamma_{P,M}\,,\,\gamma_{P,M'}\,,\,\lambda,\,\lambda',\,\mu,\,\mu',\,\chi)$$

and

$$\tilde{C}=(\tilde{L},\,\tilde{M},\,\tilde{M}',\,\tilde{P},\,\gamma_{\tilde{M},\tilde{L}}\,,\,\gamma_{\tilde{M}',\tilde{L}}\,,\,\gamma_{\tilde{P},\tilde{L}}\,,\,\gamma_{\tilde{P},\tilde{M}}\,,\,\gamma_{\tilde{P},\tilde{M}'}\,,\,\tilde{\lambda},\,\tilde{\lambda}',\,\tilde{\mu},\,\tilde{\mu}',\,\tilde{\chi}).$$

Suppose given a morphism of crossed squares

$$\mathfrak{c}=(\mathfrak{l},\,\mathfrak{m},\,\mathfrak{m}',\,\mathfrak{p})\ :\ C=(L,\,M,\,M',\,P)\ \to\ \tilde{C}=(\tilde{L},\,\tilde{M},\,\tilde{M}',\,\tilde{P}).$$

Cf. Definition 15 and Definition 19.

We have the following group morphisms; cf. Remark 105.(3) and Remark 106.(1).

$$\begin{array}{rcl}
\mathrm{G}^C_{2;1,2} & \xrightarrow{\varphi^{\mathfrak{c}}_{2;1,2}} & \mathrm{G}^{\tilde{C}}_{2;1,2} \\
(l) & \longmapsto & \left(\begin{array}{cl} & (l)\varphi^{\mathfrak{c}}_2 \\ = & (l\mathfrak{l})\end{array}\right)
\end{array}$$

$$\begin{array}{rcl}
\mathrm{G}^C_{1;0} & \xrightarrow{\varphi^{\mathfrak{c}}_{1;0}} & \mathrm{G}^{\tilde{C}}_{1;0} \\
(m^-\mu\cdot m'^-\mu',(m',m)) & \longmapsto & \left(\begin{array}{cl} & (m^-\mu\cdot m'^-\mu',(m',m))\varphi^{\mathfrak{c}}_1 \\ = & ((m^-\mu\cdot m'^-\mu')\mathfrak{p},(m'\mathfrak{m}',m\mathfrak{m})) \\ = & ((m^-\mu)\mathfrak{p}\cdot(m'^-\mu')\mathfrak{p},(m'\mathfrak{m}',m\mathfrak{m})) \\ \overset{\text{(CSM 1.3)}}{\underset{\text{(CSM 1.4)}}{=}} & (m^-\mathfrak{m}\tilde{\mu}\cdot m'^-\mathfrak{m}'\tilde{\mu}',(m'\mathfrak{m}',m\mathfrak{m}))\end{array}\right)
\end{array}$$

$$\begin{array}{rcl}
\mathrm{G}^C_{1;1} & \xrightarrow{\varphi^{\mathfrak{c}}_{1;1}} & \mathrm{G}^{\tilde{C}}_{1;1} \\
(1,(m',m)) & \longmapsto & \left(\begin{array}{cl} & (1,(m',m))\varphi^{\mathfrak{c}}_1 \\ = & (1,(m'\mathfrak{m}',m\mathfrak{m}))\end{array}\right)
\end{array}$$

$$\begin{array}{rcl}
\mathrm{G}^C_1 & \xrightarrow{\varphi^{\mathfrak{c}}_1} & \mathrm{G}^{\tilde{C}}_1 \\
(p,(m',m)) & \longmapsto & (p\mathfrak{p},(m'\mathfrak{m}',m\mathfrak{m}))
\end{array}$$

Moreover,

$$\begin{array}{rcl}
\varphi^{\mathfrak{c}}\check{\mathrm{Sq}} & := & ((\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{1,1},\,(\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{1,0},\,(\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{0,1},\,(\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{0,0}) \\
& = & (\varphi^{\mathfrak{c}}_{2;1,2},\,\varphi^{\mathfrak{c}}_{1;0},\,\varphi^{\mathfrak{c}}_{1;1},\,\varphi^{\mathfrak{c}}_1)
\end{array}$$

is a morphism of crossed squares; cf. Definition 87.

Lemma 108 Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

We recall that we often abbreviate $\boldsymbol{(}l\boldsymbol{)} := ((1,(1,1)),((1,1),l))$ for $l \in L$.

We have the following group morphisms.

$$\begin{array}{rcl} \mathrm{G}^{C}_{2;1,2} & \xrightarrow{C\xi_{1,1}} & L \\ \boldsymbol{(}l\boldsymbol{)} & \longmapsto & l \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^{C}_{1;0} & \xrightarrow{C\xi_{1,0}} & M \\ (m^{-}\mu \cdot m'^{-}\mu', (m',m)) & \longmapsto & m^{-} \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^{C}_{1;1} & \xrightarrow{C\xi_{0,1}} & M' \\ (1,(m',m)) & \longmapsto & m' \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^{C}_{1} & \xrightarrow{C\xi_{0,0}} & P \\ (p,(m',m)) & \longmapsto & p \cdot m'\mu' \end{array}$$

Cf. Remark 106.(1).

Proof.

Suppose given $l,\, \tilde{l} \in L$, $m,\, \tilde{m} \in M$, $m',\, \tilde{m}' \in M'$ and $p,\, \tilde{p} \in P$.

Then we obtain

$$\begin{array}{rcl} (\boldsymbol{(}l\boldsymbol{)} \cdot \boldsymbol{(}\tilde{l}\boldsymbol{)})(C\xi_{1,1}) & \overset{\text{Rem. 106.(1)}}{=} & \boldsymbol{(}l \cdot \tilde{l}\boldsymbol{)}(C\xi_{1,1}) \\ & = & l \cdot \tilde{l} \\ & = & \boldsymbol{(}l\boldsymbol{)}(C\xi_{1,1}) \cdot \boldsymbol{(}\tilde{l}\boldsymbol{)}(C\xi_{1,1}) \end{array}$$

and

$$\begin{array}{rl} & ((m^{-}\mu \cdot m'^{-}\mu', (m',m)) \cdot (\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (\tilde{m}',\tilde{m})))(C\xi_{1,0}) \\ = & (m^{-}\mu \cdot m'^{-}\mu' \cdot \tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (m',m)^{\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu'} \cdot (\tilde{m}',\tilde{m}))(C\xi_{1,0}) \\ \overset{\text{Rem. 105.(2)}}{=} & (m^{-}\mu \cdot m'^{-}\mu' \cdot \tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (m'^{\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu'}, m^{\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu'}) \cdot (\tilde{m}',\tilde{m}))(C\xi_{1,0}) \\ \overset{\text{Rem. 105.(1)}}{=} & (m^{-}\mu \cdot m'^{-}\mu' \cdot \tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (m'^{\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu'}, (m^{\tilde{m}^{-}\mu})^{\tilde{m}'^{-}} \cdot (\tilde{m}',\tilde{m}))(C\xi_{1,0}) \\ = & (m^{-}\mu \cdot m'^{-}\mu' \cdot \tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (m'^{\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu'} \cdot \tilde{m}', (m^{\tilde{m}^{-}\mu})^{\tilde{m}'^{-} \cdot \tilde{m}'} \cdot \tilde{m}))(C\xi_{1,0}) \\ = & (m^{-}\mu \cdot m'^{-}\mu' \cdot \tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (m'^{\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu'} \cdot \tilde{m}', m^{\tilde{m}^{-}\mu} \cdot \tilde{m}))(C\xi_{1,0}) \\ = & \tilde{m}^{-} \cdot (m^{-})^{\tilde{m}^{-}\mu} \\ \overset{\text{(CM 2)}}{=} & \tilde{m}^{-} \cdot (m^{-})^{\tilde{m}^{-}} \\ = & \tilde{m}^{-} \cdot \tilde{m} \cdot m^{-} \cdot \tilde{m}^{-} \\ = & m^{-} \cdot \tilde{m}^{-} \\ = & (m^{-}\mu \cdot m'^{-}\mu', (m',m))(C\xi_{1,0}) \cdot (\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (\tilde{m}',\tilde{m}))(C\xi_{1,0}). \end{array}$$

We obtain

$$\begin{array}{rcl} ((1,(m',m)) \cdot (1,(\tilde{m}',\tilde{m})))(C\xi_{0,1}) & = & (1,(m',m) \cdot (\tilde{m}',\tilde{m}))(C\xi_{0,1}) \\ & = & (1,(m' \cdot \tilde{m}', m^{\tilde{m}'} \cdot \tilde{m}))(C\xi_{0,1}) \\ & = & m' \cdot \tilde{m}' \\ & = & (1,(m',m))(C\xi_{0,1}) \cdot (1,(\tilde{m}',\tilde{m}))(C\xi_{0,1}). \end{array}$$

Moreover, we obtain

$$\begin{array}{rcl}
((p,(m',m))\cdot(\tilde{p},(\tilde{m}',\tilde{m})))(C\xi_{0,0}) & = & (p\cdot\tilde{p},(m',m)^{\tilde{p}}\cdot(\tilde{m}',\tilde{m}))(C\xi_{0,0}) \\
& \overset{\text{Rem. 105.(2)}}{=} & (p\cdot\tilde{p},({m'}^{\tilde{p}},m^{\tilde{p}})\cdot(\tilde{m}',\tilde{m}))(C\xi_{0,0}) \\
& = & (p\cdot\tilde{p},({m'}^{\tilde{p}}\cdot\tilde{m}',(m^{\tilde{p}})^{\tilde{m}'}\cdot\tilde{m}))(C\xi_{0,0}) \\
& = & p\cdot\tilde{p}\cdot({m'}^{\tilde{p}}\cdot\tilde{m}')\mu' \\
& = & p\cdot\tilde{p}\cdot({m'}^{\tilde{p}})\mu'\cdot\tilde{m}'\mu' \\
& \overset{\text{(CM 1)}}{=} & p\cdot\tilde{p}\cdot(m'\mu')^{\tilde{p}}\cdot\tilde{m}'\mu' \\
& = & p\cdot\tilde{p}\cdot\tilde{p}^{-}\cdot m'\mu'\cdot\tilde{p}\cdot\tilde{m}'\mu' \\
& = & p\cdot m'\mu'\cdot\tilde{p}\cdot\tilde{m}'\mu' \\
& = & (p,(m',m))(C\xi_{0,0})\cdot(\tilde{p},(\tilde{m}',\tilde{m}))(C\xi_{0,0}).
\end{array}$$

□

Lemma 109 Suppose given a crossed square

$$C = (L,\ M,\ M',\ P,\ \gamma_{M,L}\,,\ \gamma_{M',L}\,,\ \gamma_{P,L}\,,\ \gamma_{P,M}\,,\ \gamma_{P,M'}\,,\ \lambda,\ \lambda',\ \mu,\ \mu',\ \chi).$$

We have the morphism

$$C\xi := (C\xi_{1,1},\ C\xi_{1,0},\ C\xi_{0,1},\ C\xi_{0,0}):\ \ \mathrm{G}^C\check{\mathrm{Sq}} = (\mathrm{G}^C_{2;1,2},\ \mathrm{G}^C_{1;0},\ \mathrm{G}^C_{1;1},\ \mathrm{G}^C_{1})\ \ \mapsto\ \ C = (L,\ M,\ M',\ P)$$

of crossed squares.

Cf. Lemma 108 and Definition 85.

$$\begin{tikzcd}
& \mathrm{G}^C_{2;1,2} \arrow[rr, "\check{\lambda}'"] \arrow[dl, "\check{\lambda}"'] \arrow[dd, "C\xi_{1,1}"' near end] & & \mathrm{G}^C_{1;1} \arrow[dl, "\check{\mu}'"] \arrow[dd, "C\xi_{0,1}"] \\
\mathrm{G}^C_{1;0} \arrow[rr, "\check{\mu}" near end] \arrow[dd, "C\xi_{1,0}"'] & & \mathrm{G}^C_{1} \arrow[dd, "C\xi_{0,0}" near start] & \\
& L \arrow[rr, "\lambda'" near start] \arrow[dl, "\lambda"'] & & M' \arrow[dl, "\mu'"] \\
M \arrow[rr, "\mu"] & & P &
\end{tikzcd}$$

Proof.

We recall that we often abbreviate $\mathbf{(}l\mathbf{)} := ((1,(1,1)),((1,1),l))$ for $l \in L$.

Ad (CSM 1.1). Suppose given $l \in L$, $m \in M$ and $m' \in M'$.

Then we have

$$\begin{array}{rcl}
\mathbf{(}l\mathbf{)}(\check{\lambda}\blacktriangle C\xi_{1,0}) & = & (\mathbf{(}l\mathbf{)}\check{\lambda})(C\xi_{1,0}) \\
& \overset{\text{Rem. 106.(2)}}{=} & (1,(l\lambda',l^{-}\lambda))(C\xi_{1,0}) \\
& = & l\lambda \\
& = & (\mathbf{(}l\mathbf{)}(C\xi_{1,1}))\lambda \\
& = & \mathbf{(}l\mathbf{)}(C\xi_{1,1}\blacktriangle\lambda)
\end{array}$$

and

$$\begin{array}{rcl}
\mathbf{(}l\mathbf{)}^{((m^{-}\mu\,,\,m'^{-}\mu'),(m',m))}(C\xi_{1,1}) & \overset{\text{Rem. 106.(3)}}{=} & \mathbf{(}l^{m^{-}}\mathbf{)}(C\xi_{1,1}) \\
& = & l^{m^{-}} \\
& = & (\mathbf{(}l\mathbf{)}(C\xi_{1,1}))^{((m^{-}\mu\,,\,m'^{-}\mu'),(m',m))(C\xi_{1,0})}.
\end{array}$$

So $(C\xi_{1,1},\ C\xi_{1,0})$ is a morphism of crossed modules.

Ad (CSM 1.2). Suppose given $l \in L$, $m \in M$ and $m' \in M'$.

Then we have

$$\begin{aligned} (l)(\check{\lambda}' \blacktriangle C\xi_{0,1}) &= ((l)\check{\lambda}')(C\xi_{0,1}) \\ &\overset{\text{Rem. 106.(2)}}{=} (1, (l\lambda', l^{-}\lambda))(C\xi_{0,1}) \\ &= l\lambda' \\ &= ((l)(C\xi_{1,1}))\lambda' \\ &= (l)(C\xi_{1,1} \blacktriangle \lambda') \end{aligned}$$

and

$$\begin{aligned} (l)^{(1,(m',m))}(C\xi_{1,1}) &\overset{\text{Rem. 106.(3)}}{=} (l^{m'})(C\xi_{1,1}) \\ &= l^{m'} \\ &= ((l)(C\xi_{1,1}))^{(1,(m',m))(C\xi_{0,1})}. \end{aligned}$$

So $(C\xi_{1,1},\, C\xi_{0,1})$ is a morphism of crossed modules.

Ad (CSM 1.3). Suppose given $m,\, \tilde{m} \in M$, $m',\, \tilde{m}' \in M'$ and $p \in P$.

Then we have

$$\begin{aligned} (m^{-}\mu \cdot m'^{-}\mu', (m',m))(\check{\mu} \blacktriangle C\xi_{0,0}) &= ((m^{-}\mu \cdot m'^{-}\mu', (m',m))\check{\mu})(C\xi_{0,0}) \\ &\overset{\text{Rem. 106.(2)}}{=} (m^{-}\mu \cdot m'^{-}\mu', (m',m))(C\xi_{0,0}) \\ &= m^{-}\mu \cdot m'^{-}\mu' \cdot m'\mu' \\ &= m^{-}\mu \\ &= ((m^{-}\mu \cdot m'^{-}\mu', (m',m))(C\xi_{1,0}))\mu \\ &= (m^{-}\mu \cdot m'^{-}\mu', (m',m))(C\xi_{1,0} \blacktriangle \mu) \end{aligned}$$

and

$$\begin{aligned} & (\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (\tilde{m}',\tilde{m}))^{(p,(m',m))}(C\xi_{1,0}) \\ \overset{\text{Rem. 106.(3)}}{=} \;& ((\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu')^{p}, ((m'^{-})^{(\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu')^{p}} \cdot \tilde{m}'^{p} \cdot m', \tilde{m}^{p \cdot m'\mu'}))(C\xi_{1,0}) \\ = \;& (\tilde{m}^{-})^{p \cdot m'\mu'} \\ = \;& ((\tilde{m}^{-}\mu \cdot \tilde{m}'^{-}\mu', (\tilde{m}',\tilde{m}))(C\xi_{1,0}))^{(p,(m',m))(C\xi_{0,0})}. \end{aligned}$$

So $(C\xi_{1,0},\, C\xi_{0,0})$ is a morphism of crossed modules.

Ad (CSM 1.4). Suppose given $m,\, \tilde{m} \in M$, $m',\, \tilde{m}' \in M'$ and $p \in P$.

Then we have

$$\begin{aligned} (1, (m',m))(\check{\mu}' \blacktriangle C\xi_{0,0}) &= ((1, (m',m))\check{\mu}')(C\xi_{0,0}) \\ &\overset{\text{Rem. 106.(2)}}{=} (1, (m',m))(C\xi_{0,0}) \\ &= m'\mu' \\ &= ((1, (m',m))(C\xi_{0,1}))\mu' \\ &= (1, (m',m))(C\xi_{0,1} \blacktriangle \mu') \end{aligned}$$

and

$$\begin{aligned} (1, (\tilde{m}',\tilde{m}))^{(p,(m',m))}(C\xi_{0,1}) &\overset{\text{Rem. 106.(3)}}{=} (1, (\tilde{m}'^{p \cdot m'\mu'}, (m^{-})^{(\tilde{m}'\mu')^{p \cdot m'\mu'}} \cdot \tilde{m}^{p \cdot m'\mu'} \cdot m))(C\xi_{0,1}) \\ &= \tilde{m}'^{p \cdot m'\mu'} \\ &= ((1, (\tilde{m}',\tilde{m}))(C\xi_{0,1}))^{(p,(m',m))(C\xi_{0,0})}. \end{aligned}$$

So $(C\xi_{0,1},\, C\xi_{0,0})$ is a morphism of crossed modules.

Ad (CSM 1.5). Suppose given $l \in L$, $m \in M$, $m' \in M'$ and $p \in P$.

Then we have

$$\begin{aligned}
(l)(\check{\kappa} \blacktriangle C\xi_{0,0}) &\overset{\text{Rem. 106.(2)}}{=} (l)((\check{\lambda} \blacktriangle \check{\mu}) \blacktriangle C\xi_{0,0}) \\
&= (((l)\check{\lambda})\check{\mu})(C\xi_{0,0}) \\
&\overset{\text{Rem. 106.(2)}}{=} ((1,(l\lambda', l^-\lambda))\check{\mu})(C\xi_{0,0}) \\
&\overset{\text{Rem. 106.(2)}}{=} (1,(l\lambda', l^-\lambda))(C\xi_{0,0}) \\
&= (l\lambda')\mu' \\
&= l(\lambda' \blacktriangle \mu') \\
&\overset{\text{Def. 15}}{=} l\kappa \\
&= ((l)(C\xi_{1,1}))\kappa \\
&= (l)(C\xi_{1,1} \blacktriangle \kappa)
\end{aligned}$$

and

$$\begin{aligned}
(l)^{(p,(m',m))}(C\xi_{1,1}) &\overset{\text{Rem. 106.(3)}}{=} (l^{p\,\cdot\, m'\mu'})(C\xi_{1,1}) \\
&= l^{p\,\cdot\, m'\mu'} \\
&= ((l)(C\xi_{1,1}))^{(p,(m',m))(C\xi_{0,0})}.
\end{aligned}$$

So $(C\xi_{1,1},\, C\xi_{0,0})$ is a morphism of crossed modules.

The result of the first calculation also follows from the composite of two commutative quadrangles being commutative.

Ad (CSM 2). Suppose given m, $\tilde{m} \in M$ and m', $\tilde{m}' \in M'$.

Then we have

$$\begin{aligned}
&\lceil (m^-\mu \cdot m'^-\mu', (m',m))(C\xi_{1,0}), (1,(\tilde{m}',\tilde{m}))(C\xi_{0,1}) \rceil \\
=\ &\lceil m^-, \tilde{m}' \rceil \\
=\ &(\lceil m^-, \tilde{m}' \rceil)(C\xi_{1,1}) \\
\overset{\text{Rem. 106.(3)}}{=}\ &\lceil (m^-\mu \cdot m'^-\mu', (m',m)), (1,(\tilde{m}',\tilde{m})) \rceil (C\xi_{1,1}).
\end{aligned}$$

□

Lemma 110 We have the transformation

$$\xi = (C\xi)_{C \in \mathrm{Ob}(\mathcal{C}r\mathcal{S}q)} : (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \check{\mathrm{Sq}} \to \mathrm{Id}_{\mathcal{C}r\mathcal{S}q}\,.$$

Cf. Lemma 108 and Lemma 109.

Proof. To show that ξ is a transformation, we have to show that the quadrangle

$$\begin{tikzcd}
\mathrm{G}^{C}\check{\mathrm{Sq}} \arrow[r, "C\xi"] \arrow[d, "\varphi^{\mathfrak{c}}\check{\mathrm{Sq}}"'] & C \arrow[d, "\mathfrak{c}"] \\
\mathrm{G}^{\tilde{C}}\check{\mathrm{Sq}} \arrow[r, "\tilde{C}\xi"'] & \tilde{C}
\end{tikzcd}$$

commutes for $C \xrightarrow{\mathfrak{c}} \tilde{C}$ in $\mathcal{C}r\mathcal{S}q$; cf. Remark 105.(3). Concerning $\mathrm{G}^{C}\check{\mathrm{Sq}}$; cf. Remark 106.

We write

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi),$$

$$\tilde{C} = (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}}\,,\, \gamma_{\tilde{M}',\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{L}}\,,\, \gamma_{\tilde{P},\tilde{M}}\,,\, \gamma_{\tilde{P},\tilde{M}'}\,,\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi})$$

and

$$\mathfrak{c} = (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) \quad : \quad C = (L,\, M,\, M',\, P) \quad \to \quad \tilde{C} = (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P}).$$

Suppose given $l \in L$, $m \in M$, $m' \in M'$ and $p \in P$.

We recall that we often abbreviate $(l) := ((1,(1,1)),((1,1),l))$ for $l \in L$.

We have

$$\begin{aligned} (l)((\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{1,1} \mathbin{\blacktriangle} \tilde{C}\xi_{1,1}) &= ((l)(\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{1,1})(\tilde{C}\xi_{1,1}) \\ &\overset{\text{Rem. 107}}{=} (l\mathfrak{l})(\tilde{C}\xi_{1,1}) \\ &\overset{\text{Lem. 108}}{=} l\mathfrak{l} \\ &\overset{\text{Lem. 108}}{=} ((l)(C\xi_{1,1}))\mathfrak{l} \\ &= (l)(C\xi_{1,1} \mathbin{\blacktriangle} \mathfrak{l}). \end{aligned}$$

We have

$$\begin{aligned} (m^{-}\mu \cdot m'^{-}\mu', (m',m))((\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{1,0} \mathbin{\blacktriangle} \tilde{C}\xi_{1,0}) &= ((m^{-}\mu \cdot m'^{-}\mu', (m',m))(\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{1,0})(\tilde{C}\xi_{1,0}) \\ &\overset{\text{Rem. 107}}{=} (m^{-}\mathfrak{m}\tilde{\mu} \cdot m'^{-}\mathfrak{m}'\tilde{\mu}', (m'\mathfrak{m}', m\mathfrak{m}))(\tilde{C}\xi_{1,0}) \\ &\overset{\text{Lem. 108}}{=} m^{-}\mathfrak{m} \\ &\overset{\text{Lem. 108}}{=} ((m^{-}\mu \cdot m'^{-}\mu', (m',m))(C\xi_{1,0}))\mathfrak{m} \\ &= (m^{-}\mu \cdot m'^{-}\mu', (m',m))(C\xi_{1,0} \mathbin{\blacktriangle} \mathfrak{m}). \end{aligned}$$

We have

$$\begin{aligned} (1,(m',m))((\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{0,1} \mathbin{\blacktriangle} \tilde{C}\xi_{0,1}) &= ((1,(m',m))(\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{0,1})(\tilde{C}\xi_{0,1}) \\ &\overset{\text{Rem. 107}}{=} (1,(m'\mathfrak{m}', m\mathfrak{m}))(\tilde{C}\xi_{0,1}) \\ &\overset{\text{Lem. 108}}{=} m'\mathfrak{m}' \\ &\overset{\text{Lem. 108}}{=} ((1,(m',m))(C\xi_{0,1}))\mathfrak{m}' \\ &= (1,(m',m))(C\xi_{0,1} \mathbin{\blacktriangle} \mathfrak{m}'). \end{aligned}$$

Moreover, we have

$$\begin{aligned} (p,(m',m))((\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{0,0} \mathbin{\blacktriangle} \tilde{C}\xi_{0,0}) &= ((p,(m',m))(\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{0,0})(\tilde{C}\xi_{0,0}) \\ &\overset{\text{Rem. 107}}{=} (p\mathfrak{p}, (m'\mathfrak{m}', m\mathfrak{m}))(\tilde{C}\xi_{0,0}) \\ &\overset{\text{Lem. 108}}{=} p\mathfrak{p} \cdot m'\mathfrak{m}'\tilde{\mu}' \\ &\overset{\text{(CSM 1.4)}}{=} p\mathfrak{p} \cdot m'\mu'\mathfrak{p} \\ &= (p \cdot m'\mu')\mathfrak{p} \\ &\overset{\text{Lem. 108}}{=} ((p,(m',m))(C\xi_{0,0}))\mathfrak{p} \\ &= (p,(m',m))(C\xi_{0,0} \mathbin{\blacktriangle} \mathfrak{p}). \end{aligned}$$

Altogether, we have

$$\begin{aligned} & \varphi^{\mathfrak{c}}\check{\mathrm{Sq}} \mathbin{\blacktriangle} \tilde{C}\xi \\ =\; & ((\varphi^{\mathfrak{c}}\check{\mathrm{Sq}} \mathbin{\blacktriangle} \tilde{C}\xi)_{1,1},\, (\varphi^{\mathfrak{c}}\check{\mathrm{Sq}} \mathbin{\blacktriangle} \tilde{C}\xi)_{1,0},\, (\varphi^{\mathfrak{c}}\check{\mathrm{Sq}} \mathbin{\blacktriangle} \tilde{C}\xi)_{0,1},\, (\varphi^{\mathfrak{c}}\check{\mathrm{Sq}} \mathbin{\blacktriangle} \tilde{C}\xi)_{0,0}) \\ \overset{\text{Rem. 20}}{=}\; & ((\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{1,1} \mathbin{\blacktriangle} \tilde{C}\xi_{1,1},\, (\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{1,0} \mathbin{\blacktriangle} \tilde{C}\xi_{1,0},\, (\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{0,1} \mathbin{\blacktriangle} \tilde{C}\xi_{0,1},\, (\varphi^{\mathfrak{c}}\check{\mathrm{Sq}})_{0,0} \mathbin{\blacktriangle} \tilde{C}\xi_{0,0}) \\ =\; & (C\xi_{1,1} \mathbin{\blacktriangle} \mathfrak{l},\, C\xi_{1,0} \mathbin{\blacktriangle} \mathfrak{m},\, C\xi_{0,1} \mathbin{\blacktriangle} \mathfrak{m}',\, C\xi_{0,0} \mathbin{\blacktriangle} \mathfrak{p}) \\ \overset{\text{Def. 19}}{=}\; & (C\xi_{1,1} \mathbin{\blacktriangle} \mathfrak{c}_{1,1},\, C\xi_{1,0} \mathbin{\blacktriangle} \mathfrak{c}_{1,0},\, C\xi_{0,1} \mathbin{\blacktriangle} \mathfrak{c}_{0,1},\, C\xi_{0,0} \mathbin{\blacktriangle} \mathfrak{c}_{0,0}) \\ \overset{\text{Rem. 20}}{=}\; & ((C\xi \mathbin{\blacktriangle} \mathfrak{c})_{1,1},\, (C\xi \mathbin{\blacktriangle} \mathfrak{c})_{1,0},\, (C\xi \mathbin{\blacktriangle} \mathfrak{c})_{0,1},\, (C\xi \mathbin{\blacktriangle} \mathfrak{c})_{0,0}) \\ =\; & C\xi \mathbin{\blacktriangle} \mathfrak{c}. \end{aligned}$$

□

Lemma 111 We have the transformation

$$\left(\begin{array}{rcl} \eta & := & (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})\psi \blacktriangle \xi \\ & = & (C((\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})\psi \blacktriangle \xi))_{C \in \mathrm{Ob}(\mathcal{CrSq})} \\ & = & (\mathrm{G}^C \psi \blacktriangle C\xi)_{C \in \mathrm{Ob}(\mathcal{CrSq})} \\ & = & (\mathrm{G}^C \psi)_{C \in \mathrm{Ob}(\mathcal{CrSq})} \blacktriangle (C\xi)_{C \in \mathrm{Ob}(\mathcal{CrSq})} \end{array}\right) : (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \mathrm{Sq} \to \mathrm{Id}_{\mathcal{CrSq}} \,.$$

Cf. Definition 87 and Lemma 110.

So $C\eta = \mathrm{G}^C \psi \blacktriangle C\xi$.

In particular, using Definition 82, Remark 105.(3), Remark 84 and Lemma 108,

$$\begin{array}{rcl} \mathrm{G}^C_{2;1,2} & \xrightarrow{C\eta_{1,1}} & L \\ (l) & \longmapsto & l \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^C_{2;1}/\mathrm{G}^C_{2;0,1} & \xrightarrow{C\eta_{1,0}} & M \\ ((1,(m',m)),((m'^-,(m^-)^{m'^-}),l))\mathrm{G}^C_{2;0,1} & \longmapsto & \left(\begin{array}{cl} & (((1,(m',m)),((m'^-,(m^-)^{m'^-}),l))\,\mathrm{d}_0)(C\xi_{1,0}) \\ \overset{\text{Rem. 105.(3)}}{=} & (m'\mu' \cdot m\mu, (m'^- \cdot l\lambda', l^-\lambda \cdot (m^-)^{m'^-})(C\xi_{1,0}) \\ \overset{\text{Lem. 108}}{=} & m^{m'^-} \cdot l\lambda \\ \overset{\text{Rem. 105.(1)}}{=} & m^{m'^-\mu'} \cdot l\lambda \end{array}\right) \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^C_{2;2}/\mathrm{G}^C_{2;0,2} & \xrightarrow{C\eta_{0,1}} & M' \\ ((1,(1,1)),((m',m),l))\mathrm{G}^C_{2;0,2} & \longmapsto & \left(\begin{array}{cl} & (((1,(1,1)),((m',m),l))\,\mathrm{d}_0)(C\xi_{0,1}) \\ \overset{\text{Rem. 105.(3)}}{=} & (1,(m' \cdot l\lambda', l^-\lambda \cdot m))(C\xi_{0,1}) \\ \overset{\text{Lem. 108}}{=} & m' \cdot l\lambda' \end{array}\right) \end{array}$$

$$\begin{array}{rcl} \mathrm{G}^C_2/\mathrm{G}^C_{2;0} & \xrightarrow{C\eta_{0,0}} & P \\ ((p,(m',m)),((\check{m}',\check{m}),l))\mathrm{G}^C_{2;0} & \longmapsto & \left(\begin{array}{cl} & (((p,(m',m)),((\check{m}',\check{m}),l))\,\mathrm{d}_0)(C\xi_{0,0}) \\ \overset{\text{Rem. 105.(3)}}{=} & (p \cdot m'\mu' \cdot m\mu, (\check{m}' \cdot l\lambda', l^-\lambda \cdot \check{m}))(C\xi_{0,0}) \\ \overset{\text{Lem. 108}}{=} & p \cdot m'\mu' \cdot m\mu \cdot \check{m}'\mu' \cdot l\lambda'\mu' \end{array}\right) . \end{array}$$

Proof. The quadrangle

$$\begin{tikzcd}
\mathrm{G}^C \mathrm{Sq} \arrow[r, "\mathrm{G}^C\psi", "\sim"'] \arrow[d, "\varphi^{\mathfrak{c}}\,\mathrm{Sq}"'] & \mathrm{G}^C \check{\mathrm{Sq}} \arrow[r, "C\xi"] \arrow[d, "\varphi^{\mathfrak{c}}\check{\mathrm{Sq}}"] & C \arrow[d, "\mathfrak{c}"] \\
\mathrm{G}^{\tilde{C}} \mathrm{Sq} \arrow[r, "\sim", "\mathrm{G}^{\tilde{C}}\psi"'] & \mathrm{G}^{\tilde{C}} \check{\mathrm{Sq}} \arrow[r, "\tilde{C}\xi"'] & \tilde{C}
\end{tikzcd}$$

commutes for $C \xrightarrow{\mathfrak{c}} \tilde{C}$ in $\mathcal{CrSq}$ as a composite of two commutative quadrangles; cf. Definition 85, Definition 87 and Lemma 110.

So η is a transformation. $\square$

7.3 Adjunction triangles

Lemma 112 We have the following commutative diagram; cf. Lemma 104 and Lemma 111.

$$\begin{tikzcd}
\mathrm{Sq} \arrow[r, "\varepsilon\,\mathrm{Sq}"] \arrow[dr, "\mathrm{id}_{\mathrm{Sq}}"'] & \mathrm{Sq} \blacktriangle (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \mathrm{Sq} \arrow[d, "\mathrm{Sq}\,\eta"] \\
& \mathrm{Sq}
\end{tikzcd}$$

Proof. Suppose given a $\lfloor 2, 0 \rfloor$-simplicial group G.

We have to show that

$$\begin{array}{rcl}
(G\varepsilon)\,\mathrm{Sq} \blacktriangle (G\,\mathrm{Sq})\eta & \overset{\text{Lem. 104}}{=} & (G(\vartheta^- \blacktriangle \nu\,\mathrm{Rec}))\,\mathrm{Sq} \blacktriangle (G\,\mathrm{Sq})\eta \\
& = & (G\vartheta^- \blacktriangle G\nu\,\mathrm{Rec})\,\mathrm{Sq} \blacktriangle (G\,\mathrm{Sq})\eta \\
& = & G\vartheta^-\,\mathrm{Sq} \blacktriangle G\nu\,\mathrm{Rec}\,\mathrm{Sq} \blacktriangle (G\,\mathrm{Sq})\eta \\
& \overset{!}{=} & \mathrm{id}_{G\,\mathrm{Sq}}\,.
\end{array}$$

It suffices to show that

$$G\nu\,\mathrm{Rec}\,\mathrm{Sq} \blacktriangle G\,\mathrm{Sq}\,\eta \overset{!}{=} G\vartheta\,\mathrm{Sq},$$

as morphisms from $G\,\hat{\mathrm{N}}\,\mathrm{Rec}\,\mathrm{Sq}$ to $G\,\mathrm{Sq}$.

By Definition 54 and Lemma 64, we have

$$\begin{array}{rclcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;1} & = & \ker \mathrm{d}_1^{G\,\hat{\mathrm{N}}\,\mathrm{Rec},2} & = & \{((1, n_1), (n_1^-, n_2)) : n_1 \in G_{1;1}, n_2 \in G_{2;1,2}\} \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;2} & = & \ker \mathrm{d}_2^{G\,\hat{\mathrm{N}}\,\mathrm{Rec},2} & = & \{((1, 1), (n_1, n_2)) : n_1 \in G_{1;1}, n_2 \in G_{2;1,2}\} \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;1,2} & = & \ker \mathrm{d}_1^{G\,\hat{\mathrm{N}}\,\mathrm{Rec},2} \cap \ker \mathrm{d}_2^{G\,\hat{\mathrm{N}}\,\mathrm{Rec},2} & = & \{((1, 1), (1, n_2)) : n_2 \in G_{2;1,2}\}.
\end{array}$$

So we have; cf. Definition 82

$$\begin{array}{rclcl}
(G\,\hat{\mathrm{N}}\,\mathrm{Rec}\,\mathrm{Sq})_{1,1} & = & (G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;1,2} & = & \{((1, 1), (1, n_2)) : n_2 \in G_{2;1,2}\} \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec}\,\mathrm{Sq})_{1,0} & = & (G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;1}/(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,1} & = & \{((1, n_1), (n_1^-, n_2)) : n_1 \in G_{1;1}, n_2 \in G_{2;1,2}\}/(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,1} \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec}\,\mathrm{Sq})_{0,1} & = & (G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;2}/(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,2} & = & \{((1, 1), (n_1, n_2)) : n_1 \in G_{1;1}, n_2 \in G_{2;1,2}\}/(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,2} \\
(G\,\hat{\mathrm{N}}\,\mathrm{Rec}\,\mathrm{Sq})_{0,0} & = & (G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2}/(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0} & = & ((G_0 \ltimes_{\beta_1} G_{1;1}) \ltimes_{\varepsilon_{1,2}} (G_{1;1} \ltimes_{\varepsilon_2} G_{2;1,2}))/(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0}\,.
\end{array}$$

Suppose given $n_0 \in G_0$, n_1, $\check{n}_1 \in G_{1;1}$ and $n_2 \in G_{2;1,2}$.

Then we have

$$\begin{array}{cl}
& ((1, 1), (1, n_2))((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{1,1} \blacktriangle (G\,\mathrm{Sq}\,\eta)_{1,1}) \\
= & (((1, 1), (1, n_2))((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{1,1}))((G\,\mathrm{Sq}\,\eta)_{1,1}) \\
\underset{\text{Def. 82}}{\overset{\text{Rem. 80}}{=}} & (((1, 1), (1, n_2))(G\nu\,\mathrm{Rec}_2))((G\,\mathrm{Sq}\,\eta)_{1,1}) \\
\overset{\text{Lem. 103}}{=} & ((1, (1, 1)), ((1, 1), n_2))((G\,\mathrm{Sq}\,\eta)_{1,1}) \\
\overset{\text{Lem. 111}}{=} & n_2 \\
\overset{\text{Lem. 76}}{=} & ((1, 1), (1, n_2))(G\vartheta_2) \\
\underset{\text{Def. 82}}{\overset{\text{Rem. 80}}{=}} & ((1, 1), (1, n_2))(G\vartheta\,\mathrm{Sq})_{1,1}\,.
\end{array}$$

We have

$$\begin{aligned}
& (((1, n_1), (n_1^-, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,1})((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{1,0} \blacktriangle (G\,\mathrm{Sq}\,\eta)_{1,0}) \\
= {} & ((((1, n_1), (n_1^-, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,1})((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{1,0}))((G\,\mathrm{Sq}\,\eta)_{1,0}) \\
\overset{\text{Rem. 80}}{\underset{\text{Def. 82}}{=}} {} & (((1, n_1), (n_1^-, n_2))(G\nu\,\mathrm{Rec}_2)(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_{2;0,1})((G\,\mathrm{Sq}\,\eta)_{1,0}) \\
\overset{\text{Lem. 103}}{=} {} & (((1, (n_1\,\mathrm{s}_0\,G_{2;0,2}, (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1})), \\
& ((n_1^-\,\mathrm{s}_0\,G_{2;0,2}, (n_1\,\mathrm{s}_0 \cdot n_1^-\,\mathrm{s}_1)G_{2;0,1}), n_2))(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_{2;0,1})((G\,\mathrm{Sq}\,\eta)_{1,0}) \\
\overset{\text{Lem. 111}}{=} {} & ((n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1})^{(n_1^-\,\mathrm{s}_0\,G_{2;0,2})\mu'} \cdot n_2\lambda \\
\overset{\text{Lem. 79}}{=} {} & ((n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1})^{n_1^-\,\mathrm{s}_0\,G_{2;0}} \cdot n_2 G_{2;0,1} \\
\overset{\text{Lem. 79}}{=} {} & ((n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)^{n_1^-\,\mathrm{s}_0})G_{2;0,1} \cdot n_2 G_{2;0,1} \\
= {} & (n_1\,\mathrm{s}_0 \cdot n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1 \cdot n_1^-\,\mathrm{s}_0 \cdot n_2)G_{2;0,1} \\
= {} & (n_1\,\mathrm{s}_1 \cdot n_1^-\,\mathrm{s}_0 \cdot n_2)G_{2;0,1} \\
\overset{\text{Lem. 76}}{=} {} & ((1, n_1), (n_1^-, n_2))(G\vartheta_2)G_{2;0,1} \\
\overset{\text{Rem. 80}}{\underset{\text{Def. 82}}{=}} {} & (((1, n_1), (n_1^-, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,1})(G\vartheta\,\mathrm{Sq})_{1,0}\,.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
& (((1, 1), (n_1, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,2})((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{0,1} \blacktriangle (G\,\mathrm{Sq}\,\eta)_{0,1}) \\
= {} & ((((1, 1), (n_1, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,2})((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{0,1}))((G\,\mathrm{Sq}\,\eta)_{0,1}) \\
\overset{\text{Rem. 80}}{\underset{\text{Def. 82}}{=}} {} & (((1, 1), (n_1, n_2))(G\nu\,\mathrm{Rec}_2)(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_{2;0,2})((G\,\mathrm{Sq}\,\eta)_{0,1}) \\
\overset{\text{Lem. 103}}{=} {} & (((1, (1, 1)), ((n_1\,\mathrm{s}_0\,G_{2;0,2}, (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1}), n_2))(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_{2;0,2})((G\,\mathrm{Sq}\,\eta)_{0,1}) \\
\overset{\text{Lem. 111}}{=} {} & n_1\,\mathrm{s}_0\,G_{2;0,2} \cdot\ n_2\lambda' \\
\overset{\text{Lem. 79}}{=} {} & (n_1\,\mathrm{s}_0 \cdot n_2)G_{2;0,2} \\
\overset{\text{Lem. 76}}{=} {} & ((1, 1), (n_1, n_2))(G\vartheta_2)G_{2;0,2} \\
\overset{\text{Rem. 80}}{\underset{\text{Def. 82}}{=}} {} & (((1, 1), (n_1, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0,2})(G\vartheta\,\mathrm{Sq})_{0,1}
\end{aligned}$$

and

$$\begin{aligned}
& (((n_0, n_1), (\check{n}_1, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0})((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{0,0} \blacktriangle (G\,\mathrm{Sq}\,\eta)_{0,0}) \\
= {} & ((((n_0, n_1), (\check{n}_1, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0})((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{0,0}))((G\,\mathrm{Sq}\,\eta)_{0,0}) \\
\overset{\text{Rem. 80}}{\underset{\text{Def. 82}}{=}} {} & (((n_0, n_1), (\check{n}_1, n_2))(G\nu\,\mathrm{Rec}_2)(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_{2;0})((G\,\mathrm{Sq}\,\eta)_{0,0}) \\
\overset{\text{Lem. 103}}{=} {} & (((n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0}, (n_1\,\mathrm{s}_0\,G_{2;0,2}, (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1})), \\
& ((\check{n}_1\,\mathrm{s}_0\,G_{2;0,2}, (\check{n}_1^-\,\mathrm{s}_0 \cdot \check{n}_1\,\mathrm{s}_1)G_{2;0,1}), n_2))(G\,\mathrm{Sq}\,\mathrm{Tr}\,\mathrm{To}\,\mathrm{Rec})_{2;0})((G\,\mathrm{Sq}\,\eta)_{0,0}) \\
\overset{\text{Lem. 111}}{=} {} & n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0} \cdot\ (n_1\,\mathrm{s}_0\,G_{2;0,2})\mu' \cdot ((n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0,1})\mu \cdot\ (\check{n}_1\,\mathrm{s}_0\,G_{2;0,2})\mu' \cdot\ n_2\lambda'\mu' \\
\overset{\text{Lem. 79}}{=} {} & n_0\,\mathrm{s}_0\,\mathrm{s}_0\,G_{2;0} \cdot\ n_1\,\mathrm{s}_0\,G_{2;0} \cdot (n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1)G_{2;0} \cdot\ \check{n}_1\,\mathrm{s}_0\,G_{2;0} \cdot\ n_2 G_{2;0} \\
= {} & (n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_0 \cdot n_1^-\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_1 \cdot \check{n}_1\,\mathrm{s}_0 \cdot n_2)G_{2;0} \\
= {} & (n_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot n_1\,\mathrm{s}_1 \cdot \check{n}_1\,\mathrm{s}_0 \cdot n_2)G_{2;0} \\
\overset{\text{Lem. 76}}{=} {} & ((n_0, n_1), (\check{n}_1, n_2))(G\vartheta_2)G_{2;0} \\
\overset{\text{Rem. 80}}{\underset{\text{Def. 82}}{=}} {} & (((n_0, n_1), (\check{n}_1, n_2))(G\,\hat{\mathrm{N}}\,\mathrm{Rec})_{2;0})(G\vartheta\,\mathrm{Sq})_{0,0}\,.
\end{aligned}$$

Altogether we have

$$\begin{aligned}
& G\nu\,\mathrm{Rec}\,\mathrm{Sq} \blacktriangle G\,\mathrm{Sq}\,\eta \\
= {} & ((G\nu\,\mathrm{Rec}\,\mathrm{Sq} \blacktriangle G\,\mathrm{Sq}\,\eta)_{1,1},\ (G\nu\,\mathrm{Rec}\,\mathrm{Sq} \blacktriangle G\,\mathrm{Sq}\,\eta)_{1,0},\ (G\nu\,\mathrm{Rec}\,\mathrm{Sq} \blacktriangle G\,\mathrm{Sq}\,\eta)_{0,1},\ (G\nu\,\mathrm{Rec}\,\mathrm{Sq} \blacktriangle G\,\mathrm{Sq}\,\eta)_{0,0})
\end{aligned}$$

$$\begin{array}{cl}
\overset{\text{Rem. 20}}{=} & ((G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{1,1} \blacktriangle (G\,\mathrm{Sq}\,\eta)_{1,1},\ (G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{1,0} \blacktriangle (G\,\mathrm{Sq}\,\eta)_{1,0}, \\
& (G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{0,1} \blacktriangle (G\,\mathrm{Sq}\,\eta)_{0,1},\ (G\nu\,\mathrm{Rec}\,\mathrm{Sq})_{0,0} \blacktriangle (G\,\mathrm{Sq}\,\eta)_{0,0}) \\
= & ((G\vartheta\,\mathrm{Sq})_{1,1},\ (G\vartheta\,\mathrm{Sq})_{1,0},\ (G\vartheta\,\mathrm{Sq})_{0,1},\ (G\vartheta\,\mathrm{Sq})_{0,0}) \\
= & G\vartheta\,\mathrm{Sq}\,.
\end{array}$$

□

Lemma 113 We have the following commutative diagram; cf. Lemma 104 and Lemma 111.

$$\begin{tikzcd}
\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec} \arrow[r, "(\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})\varepsilon"] \arrow[dr, "\mathrm{id}_{\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}}"'] & (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \mathrm{Sq} \blacktriangle (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \arrow[d, "\eta(\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})"] \\
& \mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}
\end{tikzcd}$$

Proof. Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L}\,,\, \gamma_{M',L}\,,\, \gamma_{P,L}\,,\, \gamma_{P,M}\,,\, \gamma_{P,M'}\,,\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

We have to show that

$$\begin{array}{rcl}
\mathrm{G}^C\varepsilon \blacktriangle \varphi^{C\eta} & = & \mathrm{G}^C(\vartheta^- \blacktriangle \nu\,\mathrm{Rec}) \blacktriangle \varphi^{C\eta} \\
& = & \mathrm{G}^C\vartheta^- \blacktriangle \mathrm{G}^C\nu\,\mathrm{Rec} \blacktriangle \varphi^{C\eta} \\
& \overset{!}{=} & \mathrm{id}_{\mathrm{G}^C}\,.
\end{array}$$

Cf. Lemma 104 and Remark 105.$(2,3)$.

It suffices to show that

$$\mathrm{G}^C\nu\,\mathrm{Rec} \blacktriangle \varphi^{C\eta} \overset{!}{=} \mathrm{G}^C\vartheta,$$

as morphisms from $\mathrm{G}^C\,\hat{\mathrm{N}}\,\mathrm{Rec}$ to G^C.

By Lemma 64, we have

$$\begin{array}{rclcl}
(\mathrm{G}^C\,\hat{\mathrm{N}}\,\mathrm{Rec})_2 & = & (\mathrm{G}^C\,\hat{\mathrm{N}}_0 \ltimes_{\beta_1} \mathrm{G}^C\,\hat{\mathrm{N}}_1) \ltimes_{\varepsilon_{1,2}} (\mathrm{G}^C\,\hat{\mathrm{N}}_1 \ltimes_{\varepsilon_2} \mathrm{G}^C\,\hat{\mathrm{N}}_2) & = & (\mathrm{G}^C_0 \ltimes_{\beta_1} \mathrm{G}^C_{1;1}) \ltimes_{\varepsilon_{1,2}} (\mathrm{G}^C_{1;1} \ltimes_{\varepsilon_2} \mathrm{G}^C_{2;1,2}) \\
(\mathrm{G}^C\,\hat{\mathrm{N}}\,\mathrm{Rec})_1 & = & \mathrm{G}^C\,\hat{\mathrm{N}}_0 \ltimes_{\beta_1} \mathrm{G}^C\,\hat{\mathrm{N}}_1 & = & \mathrm{G}^C_0 \ltimes_{\beta_1} \mathrm{G}^C_{1;1} \\
(\mathrm{G}^C\,\hat{\mathrm{N}}\,\mathrm{Rec})_0 & = & \mathrm{G}^C\,\hat{\mathrm{N}}_0 & = & \mathrm{G}^C_0\,.
\end{array}$$

We recall that we often abbreviate $(\!(l)\!) := ((1,(1,1)),((1,1),l))$ for $l \in L$.

We recall from Remark 105.(3) that we have

$$\mathrm{G}^C_0 = P,$$

and we also have

$$\begin{array}{rcl}
\mathrm{G}^C_{1;1} & \overset{\text{Rem. 106.(1)}}{=} & \{(1,(m',m)) : (m',m) \in M' \ltimes_\alpha M\} \\
\mathrm{G}^C_{2;1,2} & \overset{\text{Rem. 106.(1)}}{=} & \{(\!(l)\!) : l \in L\}.
\end{array}$$

So we have

$$\begin{array}{rcl}
(\mathrm{G}^C\,\hat{\mathrm{N}}\,\mathrm{Rec})_2 & = & (P \ltimes_{\beta_1} \{(1,(m',m)) : (m',m) \in M' \ltimes_\alpha M\}) \ltimes_{\varepsilon_{1,2}} (\{(1,(\check{m}',\check{m})) : (\check{m}',\check{m}) \in M' \ltimes_\alpha M\} \ltimes_{\varepsilon_2} \{(\!(l)\!) : l \in L\}) \\
(\mathrm{G}^C\,\hat{\mathrm{N}}\,\mathrm{Rec})_1 & = & P \ltimes_{\beta_1} \{(1,(m',m)) : (m',m) \in M' \ltimes_\alpha M\} \\
(\mathrm{G}^C\,\hat{\mathrm{N}}\,\mathrm{Rec})_0 & = & P.
\end{array}$$

Suppose given $p \in P$, (m, m'), $(\check{m}, \check{m}') \in M \ltimes_\alpha M'$ and $l \in L$.

We have

$$\begin{aligned}
p\,\mathrm{s}_0^{\mathrm{G}^C,0}\,\mathrm{s}_0^{\mathrm{G}^C,1} &\overset{\text{Rem. 105.(3)}}{=} (p,(1,1))\,\mathrm{s}_0^{\mathrm{G}^C,1} \\
&\overset{\text{Rem. 105.(3)}}{=} ((p,(1,1)),((1,1),1))
\end{aligned}$$

$$(1,(m',m))\,\mathrm{s}_0^{\mathrm{G}^C,1} \overset{\text{Rem. 105.(3)}}{=} ((1,(1,1)),((m',m),1))$$

and

$$\begin{aligned}
& (1,(m',m))^-\,\mathrm{s}_0^{\mathrm{G}^C,1} \cdot (1,(m',m))\,\mathrm{s}_1^{\mathrm{G}^C,1} \\
= \; & (1,(m',m)^-)\,\mathrm{s}_0^{\mathrm{G}^C,1} \cdot (1,(m',m))\,\mathrm{s}_1^{\mathrm{G}^C,1} \\
\overset{\text{Rem. 8.(1)}}{=} \; & (1,(m'^-,(m^-)^{m'^-}))\,\mathrm{s}_0^{\mathrm{G}^C,1} \cdot (1,(m',m))\,\mathrm{s}_1^{\mathrm{G}^C,1} \\
\overset{\text{Rem. 105.(3)}}{=} \; & ((1,(1,1)),((m'^-,(m^-)^{m'^-}),1)) \cdot ((1,(m',m)),((1,1),1)) \\
= \; & ((1,(1,1)) \cdot (1,(m',m)),((m'^-,(m^-)^{m'^-}),1)^{(1,(m',m))} \cdot ((1,1),1)) \\
= \; & ((1,(m',m)),((m'^-,(m^-)^{m'^-}),1)^{(1,(m',m))}) \\
\underset{\varepsilon_{1,2}}{\overset{\text{Rem. 105.(3)}}{=}} \; & ((1,(m',m)),(((m'^-)^{m'\mu'},(m^-)^{(m'^-)^{m'\mu'}} \cdot ((m^-)^{m'^-})^{m'\mu'} \cdot m),\lceil (m'^-)^{m'\mu'},m\rceil^{\mathrm{tr}})) \\
\overset{\text{(CM 2)}}{=} \; & ((1,(m',m)),(((m'^-)^{m'},(m^-)^{(m'^-)^{m'}} \cdot ((m^-)^{m'^-})^{m'\mu'} \cdot m),\lceil (m'^-)^{m'},m\rceil^{\mathrm{tr}})) \\
= \; & ((1,(m',m)),((m'^-,(m^-)^{m'^-} \cdot ((m^-)^{m'^-})^{m'\mu'} \cdot m),\lceil m'^-,m\rceil^{\mathrm{tr}})) \\
\overset{\text{Rem. 105.(1)}}{=} \; & ((1,(m',m)),((m'^-,(m^-)^{m'^-} \cdot ((m^-)^{m'^-\mu'})^{m'\mu'} \cdot m),\lceil m'^-,m\rceil^{\mathrm{tr}})) \\
= \; & ((1,(m',m)),((m'^-,(m^-)^{m'^-}),\lceil m'^-,m\rceil^{\mathrm{tr}})) \\
\overset{\text{Rem. 105.(2)}}{=} \; & ((1,(m',m)),((m'^-,(m^-)^{m'^-}),\lceil m,m'^-\rceil^-)).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
& ((p,(1,(m',m))),((1,(\check{m}',\check{m})),(l)))(\mathrm{G}^C\nu\,\mathrm{Rec}_2 \blacktriangle \varphi_2^{C\eta}) \\
= \; & (((p,(1,(m',m))),((1,(\check{m}',\check{m})),(l)))(\mathrm{G}^C\nu\,\mathrm{Rec}_2))\varphi_2^{C\eta} \\
\overset{\text{Lem. 103}}{=} \; & ((p\,\mathrm{s}_0^{\mathrm{G}^C,0}\,\mathrm{s}_0^{\mathrm{G}^C,1}\,\mathrm{G}^C_{2;0},((1,(m',m))\,\mathrm{s}_0^{\mathrm{G}^C,1}\,\mathrm{G}^C_{2;0,2},((1,(m',m))^-\,\mathrm{s}_0^{\mathrm{G}^C,1} \cdot (1,(m',m))\,\mathrm{s}_1^{\mathrm{G}^C,1})\mathrm{G}^C_{2;0,1})), \\
& (((1,(\check{m}',\check{m}))\,\mathrm{s}_0^{\mathrm{G}^C,1}\,\mathrm{G}^C_{2;0,2},((1,(\check{m}',\check{m}))^-\,\mathrm{s}_0^{\mathrm{G}^C,1} \cdot (1,(\check{m}',\check{m}))\,\mathrm{s}_1^{\mathrm{G}^C,1})\mathrm{G}^C_{2;0,1}),(l)))\varphi_2^{C\eta} \\
\overset{\text{see above}}{=} \; & ((((p,(1,1)),((1,1),1))\mathrm{G}^C_{2;0}, \\
& (((1,(1,1)),((m',m),1))\mathrm{G}^C_{2;0,2},((1,(m',m)),((m'^-,(m^-)^{m'^-}),\lceil m,m'^-\rceil^-))\mathrm{G}^C_{2;0,1})), \\
& ((((1,(1,1)),((\check{m}',\check{m}),1))\mathrm{G}^C_{2;0,2},((1,(\check{m}',\check{m})),((\check{m}'^-,(\check{m}^-)^{\check{m}'^-}),\lceil \check{m},\check{m}'^-\rceil^-))\mathrm{G}^C_{2;0,1}),(l)))\varphi_2^{C\eta} \\
\overset{\text{Rem. 105.(3)}}{=} \; & (((((p,(1,1)),((1,1),1))\mathrm{G}^C_{2;0})(C\eta_{0,0}), \\
& ((((1,(1,1)),((m',m),1))\mathrm{G}^C_{2;0,2})(C\eta_{0,1}),(((1,(m',m)),((m'^-,(m^-)^{m'^-}),\lceil m,m'^-\rceil^-))\mathrm{G}^C_{2;0,1})(C\eta_{1,0}))), \\
& (((((1,(1,1)),((\check{m}',\check{m}),1))\mathrm{G}^C_{2;0,2})(C\eta_{0,1}),(((1,(\check{m}',\check{m})),((\check{m}'^-,(\check{m}^-)^{\check{m}'^-}),\lceil \check{m},\check{m}'^-\rceil^-))\mathrm{G}^C_{2;0,1})(C\eta_{1,0})), \\
& (l)(C\eta_{1,1}))) \\
\overset{\text{Lem. 111}}{=} \; & ((p,(m',m^{m'^-\mu'} \cdot \lceil m,m'^-\rceil^-\lambda)),((\check{m}',\check{m}^{\check{m}'^-\mu'} \cdot \lceil \check{m},\check{m}'^-\rceil^-\lambda),l)) \\
\overset{\text{(CS 4.1)}}{=} \; & ((p,(m',m \cdot \lceil m,m'^-\rceil\lambda \cdot \lceil m,m'^-\rceil^-\lambda)),((\check{m}',\check{m} \cdot \lceil \check{m},\check{m}'^-\rceil\lambda \cdot \lceil \check{m},\check{m}'^-\rceil^-\lambda),l)) \\
= \; & ((p,(m',m)),((\check{m}',\check{m}),l)) \\
= \; & ((p,(m',m)),((1,1),1)) \cdot ((1,(1,1)),((\check{m}',\check{m}),l)) \\
= \; & ((p,(1,1)) \cdot (1,(m',m)),((1,1),1)) \cdot ((1,(1,1)),((\check{m}',\check{m}),1) \cdot ((1,1),l)) \\
= \; & ((p,(1,1)),((1,1),1)) \cdot ((1,(m',m)),((1,1),1)) \cdot ((1,(1,1)),((\check{m}',\check{m}),1)) \cdot ((1,(1,1)),((1,1),l)) \\
\overset{\text{Rem. 105.(3)}}{=} \; & p\,\mathrm{s}_0^{\mathrm{G}^C,0}\,\mathrm{s}_1^{\mathrm{G}^C,1} \cdot (1,(m',m))\,\mathrm{s}_1^{\mathrm{G}^C,1} \cdot (1,(\check{m}',\check{m}))\,\mathrm{s}_0^{\mathrm{G}^C,1} \cdot (l) \\
\overset{\text{Lem. 76}}{=} \; & ((p,(1,(m',m))),((1,(\check{m}',\check{m})),(l)))(\mathrm{G}^C\vartheta_2).
\end{aligned}$$

We obtain

$$
\begin{aligned}
& (p,(1,(m',m)))(\mathrm{G}^C\nu\,\mathrm{Rec}_1 \blacktriangle \varphi_1^{C\eta}) \\
= \; & ((p,(1,(m',m)))(\mathrm{G}^C\nu\,\mathrm{Rec}_1))\varphi_1^{C\eta} \\
\overset{\text{Lem. 103}}{=} \; & (p\,\mathrm{s}_0^{\mathrm{G}^C,0}\,\mathrm{s}_0^{\mathrm{G}^C,1}\,\mathrm{G}^C_{2;0}, ((1,(m',m))\,\mathrm{s}_0^{\mathrm{G}^C,1}\,\mathrm{G}^C_{2;0,2}, ((1,(m',m))^-\,\mathrm{s}_0^{\mathrm{G}^C,1}\cdot(1,(m',m))\,\mathrm{s}_1^{\mathrm{G}^C,1})\mathrm{G}^C_{2;0,1}))\varphi_1^{C\eta} \\
\overset{\text{see above}}{=} \; & (((p,(1,1)),((1,1),1))\mathrm{G}^C_{2;0}, \\
& (((1,(1,1)),((m',m),1))\mathrm{G}^C_{2;0,2}, ((1,(m',m)),((m'^-,(m^-)^{m'^-}),\lceil m,m'^-\rceil^-))\mathrm{G}^C_{2;0,1}))\varphi_1^{C\eta} \\
\overset{\text{Rem. 105.(3)}}{=} \; & ((((p,(1,1)),((1,1),1))\mathrm{G}^C_{2;0})(C\eta_{0,0}), \\
& ((((1,(1,1)),((m',m),1))\mathrm{G}^C_{2;0,2})(C\eta_{0,1}), (((1,(m',m)),((m'^-,(m^-)^{m'^-}),\lceil m,m'^-\rceil^-))\mathrm{G}^C_{2;0,1})(C\eta_{1,0}))) \\
\overset{\text{Lem. 111}}{=} \; & (p,(m', m^{m'^-\mu'}\cdot\lceil m,m'^-\rceil^-\lambda)) \\
\overset{\text{(CS 4.1)}}{=} \; & (p,(m', m\cdot\lceil m,m'^-\rceil\lambda\cdot\lceil m,m'^-\rceil^-\lambda)) \\
= \; & (p,(m',m)) \\
= \; & (p,(1,1))\cdot(1,(m',m)) \\
\overset{\text{Rem. 105.(3)}}{=} \; & p\,\mathrm{s}_0^{\mathrm{G}^C,0}\cdot(1,(m',m)) \\
\overset{\text{Lem. 76}}{=} \; & (p,(1,(m',m)))(\mathrm{G}^C\vartheta_1).
\end{aligned}
$$

Moreover, we obtain

$$
\begin{aligned}
p(\mathrm{G}^C\nu\,\mathrm{Rec}_0 \blacktriangle \varphi_0^{C\eta}) \; = \; & (p(\mathrm{G}^C\nu\,\mathrm{Rec}_0))\varphi_0^{C\eta} \\
\overset{\text{Lem. 103}}{=} \; & (p\,\mathrm{s}_0^{\mathrm{G}^C,0}\,\mathrm{s}_0^{\mathrm{G}^C,1}\,\mathrm{G}^C_{2;0})\varphi_0^{C\eta} \\
\overset{\text{see above}}{=} \; & (((p,(1,1)),((1,1),1))\mathrm{G}^C_{2;0})\varphi_0^{C\eta} \\
\overset{\text{Rem. 105.(3)}}{=} \; & (((p,(1,1)),((1,1),1))\mathrm{G}^C_{2;0})(C\eta_{0,0}) \\
\overset{\text{Lem. 111}}{=} \; & p \\
\overset{\text{Lem. 76}}{=} \; & p(\mathrm{G}^C\vartheta_0).
\end{aligned}
$$

Altogether we have

$$
\begin{aligned}
& \mathrm{G}^C\nu\,\mathrm{Rec} \blacktriangle \varphi^{C\eta} \\
= \; & ((\mathrm{G}^C\nu\,\mathrm{Rec} \blacktriangle \varphi^{C\eta})_2,\ (\mathrm{G}^C\nu\,\mathrm{Rec} \blacktriangle \varphi^{C\eta})_1,\ (\mathrm{G}^C\nu\,\mathrm{Rec} \blacktriangle \varphi^{C\eta})_0) \\
\overset{\text{Rem. 32}}{=} \; & (\mathrm{G}^C\nu\,\mathrm{Rec}_2 \blacktriangle \varphi_2^{C\eta},\ \mathrm{G}^C\nu\,\mathrm{Rec}_1 \blacktriangle \varphi_1^{C\eta},\ \mathrm{G}^C\nu\,\mathrm{Rec}_0 \blacktriangle \varphi_0^{C\eta}) \\
= \; & (\mathrm{G}^C\vartheta_2,\ \mathrm{G}^C\vartheta_1,\ \mathrm{G}^C\vartheta_0) \\
= \; & \mathrm{G}^C\vartheta.
\end{aligned}
$$

□

Theorem 114 Recall that we have the functors

$$\begin{tikzcd}
\lfloor 2,0 \rfloor\text{-}\mathit{SimpGrp} \arrow[r, bend left, "\mathrm{Sq}"] & \mathit{CrSq} \arrow[l, bend left, "\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}"] .
\end{tikzcd}$$

Cf. Definition 82, Definition 27, Definition 96, Definition 72 and Remark 105.

Then

$$(\mathrm{Sq},\ \mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec},\ \varepsilon,\ \eta)$$

is an adjunction; cf. Lemma 104 and Lemma 111.

In particular,

$$\mathrm{Sq} \dashv (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}).$$

Proof. By Lemma 112, we have the following commutative triangle.

$$\begin{tikzcd}
\mathrm{Sq} \arrow[r, "\varepsilon\,\mathrm{Sq}"] \arrow[dr, "\mathrm{id}_{\mathrm{Sq}}"'] & \mathrm{Sq} \blacktriangle (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \mathrm{Sq} \arrow[d, "\mathrm{Sq}\,\eta"] \\
 & \mathrm{Sq}
\end{tikzcd}$$

By Lemma 113, we have the following commutative triangle.

$$\begin{tikzcd}
\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec} \arrow[r, "(\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})\varepsilon"] \arrow[dr, "\mathrm{id}_{\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}}"'] & (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \blacktriangle \mathrm{Sq} \blacktriangle (\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}) \arrow[d, "\eta(\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec})"] \\
 & \mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}
\end{tikzcd}$$

Hence

$$(\mathrm{Sq},\ \mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec},\ \varepsilon,\ \eta)$$

is an adjunction; cf. Definition 4.

□

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Zusammenfassung

Die Simplex-Kategorie $\triangle$ ist die Kategorie der totalgeordneten Mengen der Form $[0, n]$ und ihrer monotonen Abbildungen. Eine simpliziale Gruppe ist ein Funktor von $\triangle^{\mathrm{op}}$ in die Kategorie der Gruppen. Somit ist eine simpliziale Gruppe G gegeben durch eine Folge von Gruppen

$$\ldots, G_3, G_2, G_1, G_0,$$

zusammen mit Randmorphismen $\mathrm{d}_i^{G,n}$ und Ausartungsmorphismen $\mathrm{s}_j^{G,n}$, die bestimmte Relationen erfüllen. Simpliziale Gruppen modellieren topologische Räume auf algebraische Weise.

Eine $\lfloor 2, 0\rfloor$-simpliziale Gruppe G besteht aus den Gruppen G_2, G_1, G_0 sowie Randmorphismen $\mathrm{d}_i^{G,n}$ und Ausartungsmorphismen $\mathrm{s}_j^{G,n}$, die bestimmte Relationen erfüllen. Dabei soll zusätzlich die Conduché-Bedingung gelten, die besagt, dass bestimmte Untergruppen von G_2 kommutieren.

Mit Hilfe der Abschneideoperation $\mathrm{Trunc}_{\lfloor 2,0\rfloor}$ erhält man aus einer simplizialen Gruppe G eine $\lfloor 2, 0\rfloor$-simpliziale Gruppe $G\,\mathrm{Trunc}$. Wenn man also den $\lfloor 2, 0\rfloor$-Teil einer simplizialen Gruppe G untersuchen möchte, kann man $G\,\mathrm{Trunc}_{\lfloor 2,0\rfloor}$ betrachten.

Der Begriff des verschränkten Quadrats ist eine Verallgemeinerung des Begriffs des verschränkten Moduls. Es handelt sich um ein kommutatives Viereck von Gruppen mit Zusatzdaten, die bestimmte Voraussetzungen erfüllen.

Wir konstruieren den Funktor Sq von der Kategorie der $\lfloor 2, 0\rfloor$-simplizialen Gruppen in die Kategorie der verschränkten Quadrate.

Sei G eine $\lfloor 2, 0\rfloor$-simpliziale Gruppe. Dann hat $G\,\mathrm{Sq}$ die Gestalt

$$G\,\mathrm{Sq} = \left(\begin{tikzcd} G_{2;1,2} \arrow[r] \arrow[d] & G_{2;2}/G_{2;0,2} \arrow[d] \\ G_{2;1}/G_{2;0,1} \arrow[r] & G_2/G_{2;0} \end{tikzcd}\right),$$

wobei $G_{2;1} = \ker(\mathrm{d}_1^{G,2})$, $G_{2;1,2} = \ker(\mathrm{d}_1^{G,2}) \cap \ker(\mathrm{d}_2^{G,2})$, usw.

Der Funktor Sq ist nicht dicht, also keine Äquivalenz.

Sei C ein verschränktes Quadrat:

$$C: \quad \begin{tikzcd} L \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & M' \arrow[d, "\mu'"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd}.$$

Dann hat das transponierte verschränkte Quadrat die Form

$$C^{\mathrm{tr}}: \quad \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ M' \arrow[r, "\mu'"'] & P \end{tikzcd} .$$

Damit erhalten wir den Transpositionsfunktor

$$\mathit{CrSq} \xrightarrow{\mathrm{Tr}} \mathit{CrSq} .$$

Laut Conduché ist ein 2-verschränkter Modul ein Diagramm von Gruppen mit Zusatzdaten, die bestimmte Voraussetzungen erfüllen:

$$N_2 \xrightarrow{\partial_2} N_1 \xrightarrow{\partial_1} N_0 .$$

Unter Verwendung eines semidirektes Produkts erhalten wir ausgehend von einem verschränkten Quadrat C, Conduché und Loday folgend, den totalen 2-verschränkten Modul

$$C\,\mathrm{To} := \left(L \xrightarrow{\partial_2} M \ltimes_\alpha M' \xrightarrow{\partial_1} P \right).$$

Damit erhalten wir den totalen-2-verschränkten-Modul-Funktor

$$\mathit{CrSq} \xrightarrow{\mathrm{To}} 2\text{-}\mathit{CrMod} .$$

Der Funktor To ist nicht voll und somit keine Äquivalenz.

Wir konstruieren die zueinander inversen Äquivalenzen

$$\begin{tikzcd} \lfloor 2,0 \rfloor\text{-}\mathit{SimpGrp} \arrow[r, bend left, "\hat{\mathrm{N}}"] \arrow[r, phantom, "\sim"] & 2\text{-}\mathit{CrMod} \arrow[l, bend left, "\mathrm{Rec}"] \end{tikzcd} .$$

Insgesamt erhalten wir die folgenden Funktoren:

$$\begin{tikzcd} \lfloor 2,0 \rfloor\text{-}\mathit{SimpGrp} \arrow[r, bend left, "\mathrm{Sq}"] & \mathit{CrSq} \arrow[l, bend left, "\mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}"] \end{tikzcd} .$$

Wir konstruieren eine Adjunktion

$$(\mathrm{Sq}, \mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec}, \varepsilon, \eta)$$

Insbesondere gilt:

$$\mathrm{Sq} \dashv \mathrm{Tr} \blacktriangle \mathrm{To} \blacktriangle \mathrm{Rec} .$$

Erklärung

Hiermit versichere ich, Natalia-Maria Asiki, dass ich meine Arbeit selbständig verfasst habe, dass ich keine anderen als die angegebenen Quellen benutzt und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe, dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist, dass ich die Arbeit weder vollständig noch in Teilen bereits veröffentlicht habe, es sei denn, der Prüfer/die Prüferin hat die Veröffentlichung zuvor genehmigt, und dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

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