Let \( P \subseteq \mathbb{Z}_{\geq 2} \) denote the set of prime numbers. Given \( x \in \mathbb{R}_{\geq 2} \), let \( \pi(x) := \#(P_{\leq x}) \).

We want to give an heuristic argument that \( \pi(x) \approx \int_2^x \frac{dt}{\ln t} \). This argument is an integral variant of Greg Martin’s pretty argument; cf. www.maa.org/features/mathchat/mathchat_8_19_99.html.


Suppose, heuristically, that \( \pi(x) \) is smooth. So \( \pi'(x) \) is the prime number density at \( x \).

By Eratosthenes, we ideally would have

\[
\pi'(x) \approx \prod_{p \in P_{\leq x}} \left( 1 - \frac{1}{p} \right).
\]

Hence

\[
\ln \pi'(x) \approx \sum_{p \in P_{\leq x}} \ln \left( 1 - \frac{1}{p} \right)
\approx \sum_{t \in \mathbb{Z}, t \in [2,x]} \ln \left( 1 - \frac{1}{t} \right) \pi'(t)
\approx \int_2^x \ln \left( 1 - \frac{1}{t} \right) \pi'(t) \, dt
\approx -\int_2^x \frac{1}{t} \cdot \pi'(t) \, dt.
\]

Differentiating yields

\[
\frac{\pi''(x)}{\pi'(x)} \approx -\frac{\pi'(x)}{x}.
\]

Hence

\[
-\frac{\pi''(x)}{\pi'(x)^2} \approx \frac{1}{x}.
\]

So

\[
\frac{1}{\pi'(x)} \approx \ln(x) + c,
\]

i.e.

\[
\pi'(x) \approx \frac{1}{\ln(x) + c}
\]

for some \( c \in \mathbb{R} \). For large \( x \), this yields

\[
\pi'(x) \approx \frac{1}{\ln(x)}
\]

So

\[
\pi(x) \approx \int_2^x \pi'(t) \, dt \approx \int_2^x \frac{dt}{\ln(t)}.
\]