Let $\mathrm{P} \subseteq \mathbf{Z}_{\geqslant 2}$ denote the set of prime numbers. Given $x \in \mathbf{R}_{\geqslant 2}$, let $\pi(x) := \#(\mathrm{P}_{\leqslant x})$.

We want to give an heuristic argument that $\pi(x) \approx \int_2^x \frac{\mathrm{d}t}{\ln t}$. This argument is an integral variant of Greg Martin's pretty argument; cf.
`www.maa.org/features/mathchat/mathchat_8_19_99.html`.

For other heuristical considerations, cf.
`mathoverflow.net/questions/16499/heuristic-argument-for-the-prime-number-theorem`
and
`terrytao.wordpress.com/2009/09/24/`
`the-prime-number-theorem-in-arithmetic-progressions-and-dueling-conspiracies`

Suppose, heuristically, that $\pi(x)$ is smooth. So $\pi'(x)$ is the prime number density at x.

By Eratosthenes, we ideally would have

$$\pi'(x) \;\approx\; \prod_{p\,\in\,\mathrm{P}_{<x}} \left(1 - \frac{1}{p}\right) .$$

Hence

$$\begin{aligned}\ln \pi'(x) \;&\approx\; \sum_{p\,\in\,\mathrm{P}_{<x}} \ln\left(1 - \frac{1}{p}\right) \\ &\approx\; \sum_{t\,\in\,\mathbf{Z},\; t\in[2,x]} \ln\left(1 - \frac{1}{t}\right) \pi'(t) \\ &\approx\; \int_2^x \ln\left(1 - \frac{1}{t}\right) \pi'(t)\,\mathrm{d}t \\ &\approx\; -\int_2^x \frac{1}{t}\cdot \pi'(t)\,\mathrm{d}t\end{aligned}$$

Differentiating yields

$$\frac{\pi''(x)}{\pi'(x)} \;\approx\; -\frac{\pi'(x)}{x} .$$

Hence

$$-\frac{\pi''(x)}{\pi'(x)^2} \;\approx\; \frac{1}{x} .$$

So

$$\frac{1}{\pi'(x)} \;\approx\; \ln(x) + c ,$$

i.e.

$$\pi'(x) \;\approx\; \frac{1}{\ln(x) + c}$$

for some $c \in \mathbf{R}$. For large x, this yields

$$\pi'(x) \;\approx\; \frac{1}{\ln(x)}$$

So

$$\pi(x) \;\approx\; \int_2^x \pi'(t)\,\mathrm{d}t \;\approx\; \int_2^x \frac{\mathrm{d}t}{\ln(t)} .$$