1 What is a Heller triangulated category?

A Heller triangulated category is a triple $\mathcal{C} = (\mathcal{C}, \mathsf{T}, \vartheta)$, where $\mathcal{C}$ is a weakly abelian category, T a shift functor and ϑ a tuple of isotransformations.

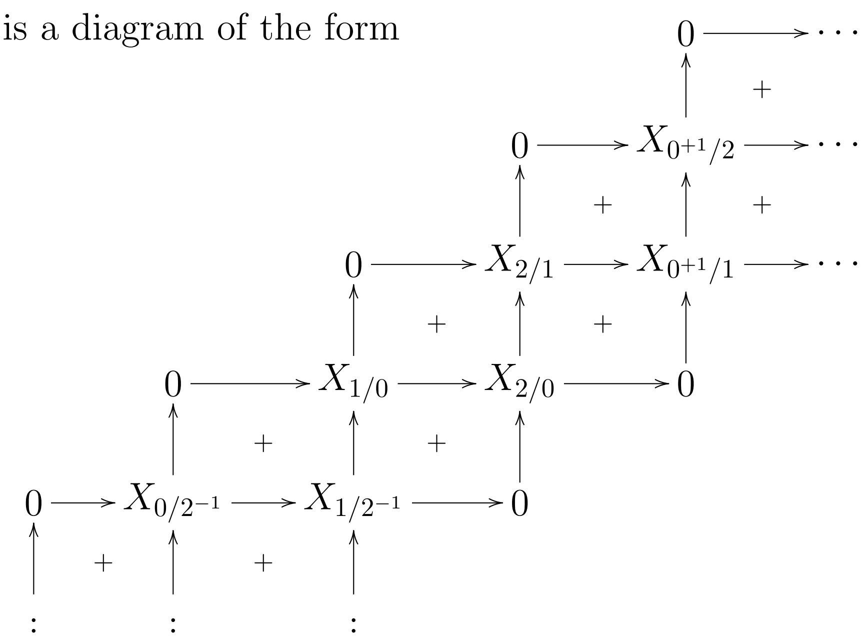
1.1 What is a weakly abelian category?

A weakly abelian category is an additive category, with split idempotents to simplify, in which each morphism is and has a weak kernel and a weak cokernel. It is the category of bijective objects of an abelian category, viz. its Freyd category.

1.2 What is theta?

1.2.1 It lives on n-pretriangles...

A 2-*pretriangle* is a diagram of the form



All quadrangles marked + are weak squares, i.e. their respective diagonal sequence is exact in the middle.

In other words, a 2-pretriangle is an acyclic complex, indexed in a convenient manner.

Modulo split 2-pretriangles, we obtain the homotopy category of acyclic complexes.

Analogously *n-pretriangles* and split n-pretriangles.

1.2.2 ... as follows

Given an n-pretriangle X.

We can apply T pointwise to obtain the n-pretriangle $[X^{+1}]$:

$$[X^{+1}]_{\beta/\alpha} \;=\; X_{\beta/\alpha}\,\mathsf{T}\;.$$

This operation is called the *inner shift.*

We can apply a diagram shift to obtain the n-pretriangle $[X]^{+1}$:

$$([X]^{+1})_{\alpha/\beta} \;=\; X_{\beta^{+1}/\alpha}\;.$$

This operation is called the *outer shift.*

$$\text{Now, } \vartheta \;=\; \left(\frac{n\text{-pretriangles}}{\text{split } n\text{-pretriangles}} \underset{[-^{+1}]}{\overset{[-]^{+1}}{\underset{\sim\Downarrow\,\vartheta_n}{\rightrightarrows}}} \frac{n\text{-pretriangles}}{\text{split } n\text{-pretriangles}} \right)_{n\,\geqslant\,0}\;.$$

1.3 Any axioms?

The tuple ϑ should be compatible with

- generalised simplicial operations, and with
- folding.

1.3.1 What is folding?

Roughly the following.

Given a $(2n+1)$-pretriangle X. Consider a sequence of morphisms lying diagonally in X. Embed this sequence canonically into an $(n+1)$-pretriangle $X\mathfrak{f}_n$, made out of sums of entries of X. The operation $\mathfrak{f}_n$ is called folding.

More details in §A.

1.4 Where are the distinguished triangles of Verdier?

An n-pretriangle X is an *n-triangle* if $X\vartheta_n = \mathrm{id}$.

Then a 2-triangle is a distinguished triangle in the sense of Verdier.

Every 3-triangle is a Verdier octahedron, but not conversely.

Now, n-triangles are stable under generalised simplicial operations and under folding.

2 What is an exact functor?

Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be an additive functor between Heller triangulated categories that respects weak kernels and weak cokernels, and for which $\mathsf{T}F = F\mathsf{T}'$ holds in this strict manner, to simplify.

We call F *strictly exact* if for an n-pretriangle X we have $X\vartheta_n F = XF\vartheta'_n$, where F is to be read as applied pointwise.

3 Why Heller triangulated categories?

As S. Thomas has recently shown, one can start with “n-triangles plus axioms” and recover ϑ. So why not work with the conventional approach “n-triangles plus axioms”?

To have n-triangles at one’s disposal is surely useful.

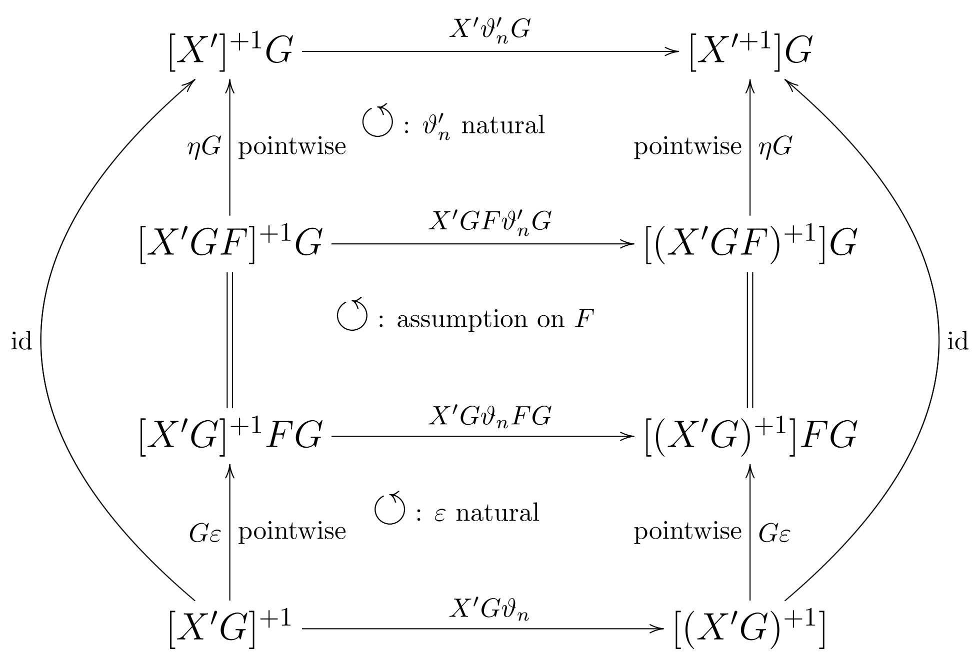
Having to check compatibility with n-triangles can be clumsy, though.

So ϑ simplifies.

3.1 Where does theta simplify something?

- Suppose given a strictly exact functor F. Suppose G is right adjoint to F in a shift-compatible manner. Then G is strictly exact.

  Proof of compatibility with ϑ, modulo introduction of the obvious notation :



- Dropping our assumption that idempotents split in $\mathcal{C}$, one can use ϑ for a simple proof that the Karoubi hull of a Heller triangulated category is Heller triangulated.

- Not only linear bases are allowed to build n-triangles on and to prolong morphisms, but also zigzag bases.

4 Is there a connection to derivators?

4.1 From triangulated derivators to n-triangles

As G. Maltsiniotis has shown, the base category of a triangulated derivator has n-triangles [people.math.jussieu.fr/~maltsin/ps/triansup.ps].

Folding remains to be discussed. First step : does [BBD, Astérisque 100, 1.1.13] hold in such a base category?

4.2 Difference

There is a fundamental difference between triangulated derivators and Heller triangulated categories, though :

A "morphism of derivators" is a compatible family of functors.

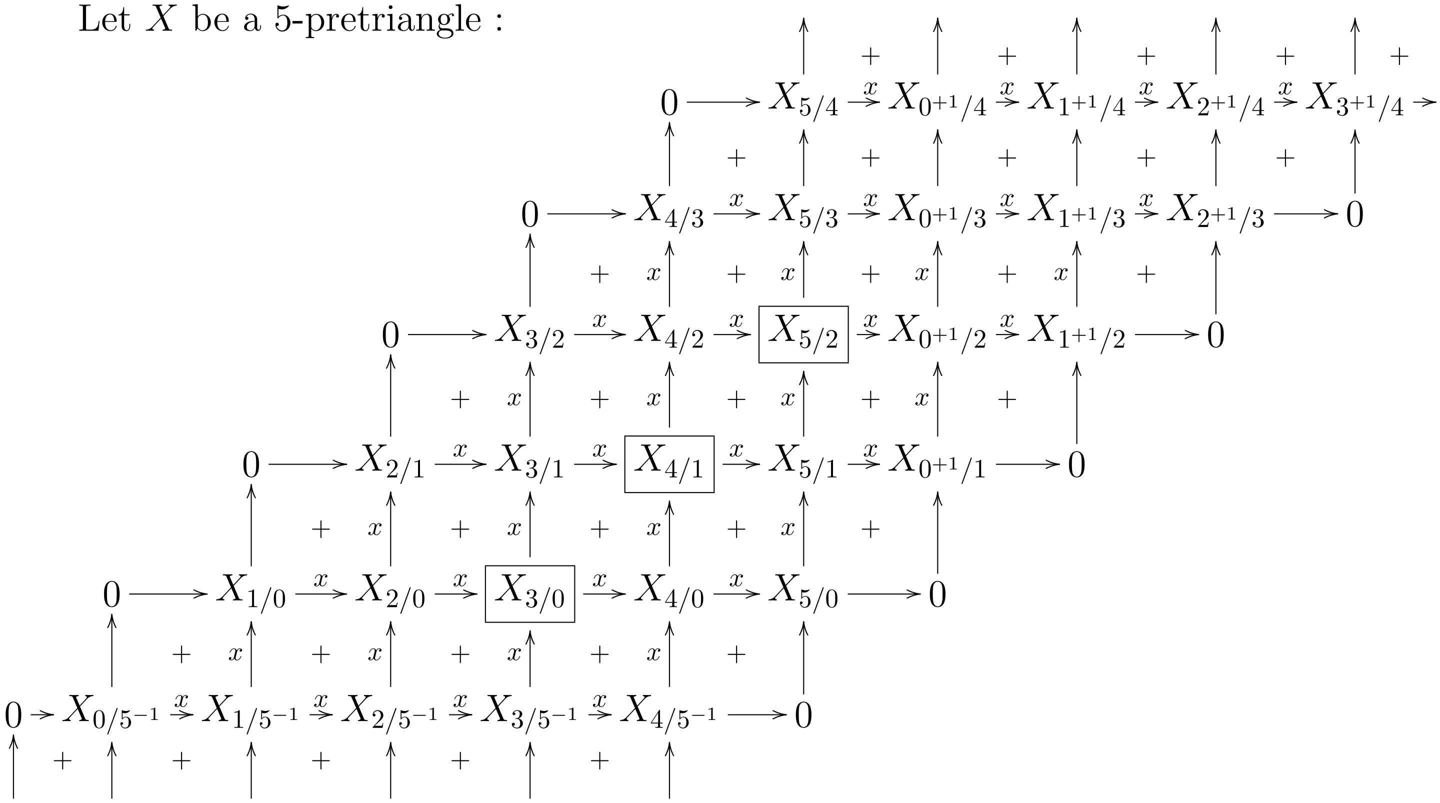
A "morphism of Heller triangulated categories" is a single additive functor that respects shift and ϑ.

A Heller triangulation is a step into the direction of a wished-for "*maximal* exactness structure" on $\mathcal{C}$. Its purpose is to help to clarify the question how much information the single category $\mathcal{C}$ can carry.

Imagine that recently we would have discovered the passage from $\mathbf{Z}$ to $\mathbf{Z}/p\mathbf{Z}$. Now I want to know *all* properties of $\mathbf{Z}/p\mathbf{Z}$. Whether $\varprojlim_n \mathbf{Z}/p^n\mathbf{Z}$ is a complete discrete valuation ring is a related, useful, but different question.

A More on the folding operation

Let X be a 5-pretriangle :



Folding turns X into the 3-pretriangle $X\mathfrak{f}_2$: