On exact functors for Heller triangulated categories

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Abstract

We show certain standard constructions of the theory of Verdier triangulated categories to be valid in the Heller triangulated framework as well; viz. Karoubi hull, exactness of adjoints, localisation.

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0 Introduction

0.1 Extending from Verdier to Heller

The following facts are part of the classical theory that Verdier triangulated categories.

- Verdier triangulated categories are stable under formation of the Karoubi hull [**1**].
- The Karoubi hull construction is functorial within Verdier triangulated categories and exact functors [**1**].
- Verdier triangulated categories are stable under localisation at a thick subcategory [**13**].
- Such a localisation has a universal property within Verdier triangulated categories and exact functors [**13**].
- An adjoint functor of an exact functor is exact [**11,** App. 2, Prop. 11], [**5,** 1.6].

We extend these assertions somewhat to fit into the Heller triangulated setting.

- Heller triangulated categories are stable under formation of the Karoubi hull; cf. Proposition 12.
- The Karoubi hull construction is functorial within Heller triangulated categories and exact functors; cf. Proposition 13.
- Closed Heller triangulated categories are stable under localisation at a thick subcategory; cf. Proposition 36. (Concerning closedness, see remark below.)
- Such a localisation has a universal property within closed Heller triangulated categories and exact functors; cf. Proposition 38.
- An adjoint functor of an exact functor is exact; cf. Proposition 28.

In a general Heller triangulated category, it is unknown whether there exists a cone on a given morphism. This however is true if all idempotents split [**8,** Lem. 3.1]. It is technically convenient to extend this assertion in the following manner. Define a Heller triangulated category to be closed if this property holds; cf. [**9,** Def. 13], Definition 14, Remark 15, Lemma 20. Prove that certain constructions yield closed Heller triangulated categories or preserve closedness; cf. [**9,** Cor. 21], Proposition 36.

An exact functor between Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$ is a pair (F, a) consisting of a subexact functor F and an isotransformation $a : \mathsf{T}\, F \longrightarrow F\, \mathsf{T}'$ such that ϑ, ϑ' and a are compatible; cf. [**5,** Def. 1.4], Definition 1. Exactness of such a pair can also be characterised via n-triangles; cf. Proposition 25. The deeper reason behind that fact is that closed Heller triangulated categories can, alternatively, be defined via sets of n-triangles for $n \geqslant 0$ with suitable properties with respect to quasicyclic and folding operations, as S. Thomas informed me.

The proof of the exactness of an adjoint of an exact functor does not have to make recourse to n-triangles. Neither does the construction of the Heller triangulation on the Karoubi hull. This shows the convenience of the definition of a triangulation via a tuple $\vartheta = (\vartheta_n)_{n\geqslant 0}$ of isomorphisms between certain shift functors, and to view the n-triangles as accessory, if useful; which is no longer the point of view taken in [**7**].

0.2 Desirables

Still missing is a precise formulation in which sense the dual of a Heller triangulated category is again a Heller triangulated category, and also in which sense the constructions above are compatible with duality. Moreover, we do not treat exactness of derived functors, except implicitly, in those cases where a derived functor can be written as a composite of an adjoint of a localisation functor, an exact functor and another localisation functor. Still missing, in the Heller triangulated context, is furthermore the exactness of the lift of the inclusion of the heart to a functor on the bounded derived category [**2,** Prop. 3.1.10], or more generally, the functor Z appearing in the construction of [**5,** Ex. 2.3] ; cf. [**5,** Th. 3.2].

0.3 Notations and conventions

We use the notations and conventions from [**8**]. In particular, we write composition of morphisms and functors in the natural order; viz. morphisms as $\xrightarrow{f}\xrightarrow{g} = \xrightarrow{fg} = \xrightarrow{f\cdot g}$ and functors as $\xrightarrow{F}\xrightarrow{G} = \xrightarrow{FG} = \xrightarrow{F\star G}$. Similarly for transformations.

Epic and *epimorphic* are synonymous, and so are *monic* and *monomorphic.*

1 Exact functors

Let $(\mathcal{C}, \mathsf{T}, \vartheta)$, $(\mathcal{C}', \mathsf{T}', \vartheta')$ and $(\mathcal{C}'', \mathsf{T}'', \vartheta'')$ be Heller triangulated categories; cf. [**8,** Def. 1.5.(ii)].

> In [**8,** Def. 1.5.(ii)], we required a strictly exact functor $\mathcal{C} \longrightarrow \mathcal{C}'$ to satisfy $F\,\mathsf{T}' = \mathsf{T}\,F$. The adjoint functor of a strictly exact functor does not always seem to be strictly exact. Following Keller and Vossieck, we shall prove below that if we call a functor *exact,* if it satisfies the weakened condition $F\,\mathsf{T}' \simeq \mathsf{T}\,F$ instead (and an accordingly modified compatibility with the Heller triangulations), then an adjoint of an exact functor is exact; cf. [**5,** 1.4].
>
> Nonetheless, generally speaking, usually one deals with strictly exact functors. Hence we shall also state an extra condition of shiftcompatibility on the adjunction that ensures a shiftcompatibly adjoint functor of a strictly exact functor to be strictly exact.

Given $n \geqslant 0$ and a transformation $G \xrightarrow{a} G'$ between subexact additive functors $\mathcal{C} \underset{G'}{\overset{G}{\rightrightarrows}} \mathcal{C}'$, we denote by $G^+(\bar{\Delta}_n^\#) \xrightarrow{a^+(\bar{\Delta}_n^\#)} G'^+(\bar{\Delta}_n^\#)$ the transformation given by

$$\big(X(a^+(\bar{\Delta}_n^\#))\big)_{\beta/\alpha} \;:=\; X_{\beta/\alpha}a \;:\; X_{\beta/\alpha}G \longrightarrow X_{\beta/\alpha}G'$$

for $X \in \mathrm{Ob}\,\mathcal{C}(\bar{\Delta}_n^{\#})$, and for $\beta/\alpha \in \bar{\Delta}_n^{\#}$, i.e. for $\alpha,\ \beta \in \bar{\Delta}_n$ with $\beta^{-1} \leqslant \alpha \leqslant \beta \leqslant \alpha^{+1}$. Moreover, we denote by $\underline{G^+(\bar{\Delta}_n^{\#})} \xrightarrow{\underline{a^+(\bar{\Delta}_n^{\#})}} \underline{G'^+(\bar{\Delta}_n^{\#})}$ the induced transformation between the induced functors on the stable categories.

Sometimes, we abbreviate $(\underline{G} \xrightarrow{\underline{a}} \underline{G'}) := \big(\underline{G^+(\bar{\Delta}_n^{\#})} \xrightarrow{\underline{a^+(\bar{\Delta}_n^{\#})}} \underline{G'^+(\bar{\Delta}_n^{\#})}\big)$.

Definition 1

A pair (F, a), consisting of an additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ and a transformation $\mathsf{T}\,F \xrightarrow{a} F\ \mathsf{T}'$, is called an *exact pair,* or an *exact functor,* if the following conditions hold.

(1) a is an isotransformation.

(2) F is subexact, i.e. its induced functor $\hat{\mathcal{C}} \xrightarrow{\hat{F}} \hat{\mathcal{C}}'$ on the Freyd categories is exact.

(3) We have

$$(\vartheta_n \star \underline{F^+(\bar{\Delta}_n^{\#})}) \cdot \underline{a^+(\bar{\Delta}_n^{\#})} \ = \ \underline{F^+(\bar{\Delta}_n^{\#})} \star \vartheta'_n$$

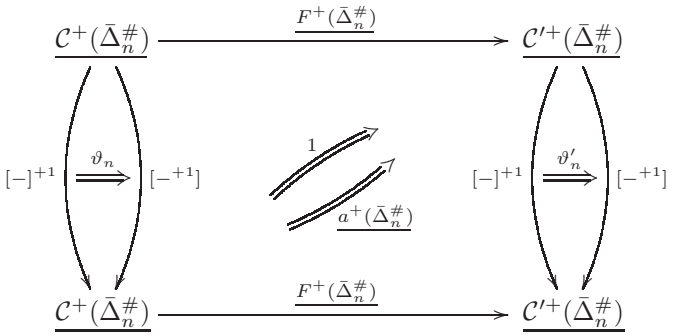
for all $n \geqslant 0$.

In particular, provided $\mathsf{T}\,F = F\ \mathsf{T}'$, then $(F, 1)$ is exact if and only if F is strictly exact; cf. [**8,** Def. 1.5.(iii)]. In this case, we sometimes identify F and $(F, 1)$.

Calling a pair (F, a) an exact functor instead of an exact pair is an abuse of notation.

We shall not discuss whether condition (1) is redundant; we need it for the construction of $\tilde{Y}$ in §4, but that may be due to the order of our arguments.

Condition (3) asserts that the following cylindrical diagram commutes for all $n \geqslant 0$.



I.e., using the abbreviation just introduced, we require $X\vartheta_n\underline{F} \cdot X\underline{a} = X\underline{F}\vartheta'_n$ to hold in $\underline{\mathcal{C}'^+(\bar{\Delta}_n^{\#})}$ for all $X \in \mathrm{Ob}\,\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})} = \mathrm{Ob}\,\mathcal{C}^+(\bar{\Delta}_n^{\#})$.

Definition 2 Suppose given exact functors (F, a) from $\mathcal{C}$ to $\mathcal{C}'$, and (F', a') from $\mathcal{C}'$ to $\mathcal{C}''$. The composite of (F, a) and (F', a') is defined to be

$$(F, a) \star (F', a') \ = \ (F, a)(F', a') \ := \ (FF',\, (aF')(Fa')) \ = \ (F \star F',\, (a \star F') \cdot (F \star a'))\ .$$

Composition is associative.

Remark 3 *If (F, a) and (F', a') are exact, then so is their composite $(F, a)(F', a')$.*

Proof. To be able to distinguish more easily, we shall make use, from the second to the last but first step, of the notation $a \star F = aF$, $F \star F' = FF'$ etc. Given $n \geqslant 0$, we obtain

$$\begin{aligned}
& \Big(\vartheta_n \star \underline{(F \star F')^+(\bar{\Delta}_n^{\#})}\Big) \cdot \underline{\big((a \star F') \cdot (F \star a')\big)^+(\bar{\Delta}_n^{\#})} \\
= \; & \Big(\vartheta_n \star \underline{F^+(\bar{\Delta}_n^{\#})} \star \underline{F'^+(\bar{\Delta}_n^{\#})}\Big) \cdot \Big(\underline{a^+(\bar{\Delta}_n^{\#})} \star \underline{F'^+(\bar{\Delta}_n^{\#})}\Big) \cdot \Big(\underline{F^+(\bar{\Delta}_n^{\#})} \star \underline{a'^+(\bar{\Delta}_n^{\#})}\Big) \\
= \; & \Big(\underline{F^+(\bar{\Delta}_n^{\#})} \star \vartheta'_n \star \underline{F'^+(\bar{\Delta}_n^{\#})}\Big) \cdot \Big(\underline{F^+(\bar{\Delta}_n^{\#})} \star \underline{a'^+(\bar{\Delta}_n^{\#})}\Big) \\
= \; & \underline{F^+(\bar{\Delta}_n^{\#})} \star \underline{F'^+(\bar{\Delta}_n^{\#})} \star \vartheta''_n \\
= \; & \underline{(F \star F')^+(\bar{\Delta}_n^{\#})} \star \vartheta''_n \; .
\end{aligned}$$

□

Definition 4 Suppose given exact functors (F, a) and (G, b) from $(\mathcal{C}, \mathsf{T}, \vartheta)$ to $(\mathcal{C}', \mathsf{T}', \vartheta')$.
A transformation $F \xrightarrow{s} G$ such that $(\mathsf{T} \star s) \cdot b = a \cdot (s \star \mathsf{T}')$ holds, is called *periodic.*

The periodicity condition requires that

$$\begin{array}{ccc}
X^{+1}F & \xrightarrow{X^{+1}s} & X^{+1}G \\
{\scriptstyle Xa}\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\downarrow{\scriptstyle Xb} \\
(XF)^{+1} & \xrightarrow{(Xs)^{+1}} & (XG)^{+1}
\end{array}$$

commute for all $X \in \mathrm{Ob}\,\mathcal{C}$.

Remark 5 Suppose given exact functors (F, a), (G, b) and (H, c) from $\mathcal{C}$ to $\mathcal{C}'$, and periodic transformations $F \underset{s'}{\overset{s}{\rightrightarrows}} G \xrightarrow{t} H$.

(1) The composite $F \xrightarrow{s \cdot t} H$ is periodic.

(2) The identity $F \xrightarrow{1} F$ is periodic.

(3) If s is a periodic isotransformation from (F, a) to (G, b), then s^- is a periodic isotransformation from (G, b) to (F, a).

(4) The difference $F \xrightarrow{s - s'} G$ of two periodic transformations is periodic.

(5) The direct sum $(F, a) \oplus (G, b) := (F \oplus G, a \oplus b) = (F \oplus G, \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right))$ is exact, with periodic inclusions from and periodic projections to (F, a) and (G, b).

Definition 6 Write $\mathbb{I}\mathcal{C}, \mathcal{C}'\mathbb{I}_{\mathrm{ex}}$ for the category of the exact functors and periodic transformations from $\mathcal{C}$ to $\mathcal{C}'$; cf. Definitions 1, 4, Remark 5.

Write $\mathbb{I}\mathcal{C}, \mathcal{C}'\mathbb{I}_{\mathrm{st\,ex}}$ for the full subcategory of $\mathbb{I}\mathcal{C}, \mathcal{C}'\mathbb{I}_{\mathrm{ex}}$ of the strictly exact functors and periodic transformations from $\mathcal{C}$ to $\mathcal{C}'$.

2 Idempotents and cones

Let $(\mathcal{C}, \mathsf{T}, \vartheta)$ be a Heller triangulated category; cf. [**8,** Def. 1.5].

2.1 A general remark on residue classes

Concerning Frobenius categories, cf. e.g. [**8,** Sec. A.2.3].

Remark 7 *Given a full and faithful exact functor $G : \mathcal{F} \longrightarrow \mathcal{F}'$ of Frobenius categories that sends all bijective objects to bijective objects. Then the induced functor $\underline{G} : \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{F}}'$ on the classical stable categories is full and faithful.*

Proof. By construction, it is full. We claim that it is faithful. Suppose given $X \longrightarrow Y$ in $\mathcal{F}$ such that

$$(XG \longrightarrow YG) \;=\; (XG \longrightarrow B' \longrightarrow YG)$$

in $\mathcal{C}'$ for some bijective object B' of $\mathcal{C}'$. Choose $X \rightarrowtail B$ in $\mathcal{C}$ with B bijective in $\mathcal{C}$. Since G preserves pure monomorphy, $XG \longrightarrow B'$ factors over $XG \rightarrowtail BG$, whence $XG \longrightarrow YG$ factors over $XG \rightarrowtail BG$, whence $X \longrightarrow Y$ factors over $X \rightarrowtail B$. ▫

Suppose given weakly abelian categories $\mathcal{A}$ and $\mathcal{A}'$. Suppose given a subexact functor $\mathcal{A} \xrightarrow{F} \mathcal{A}'$. Suppose given $n \geqslant 0$. We obtain an induced functor

$$\underline{F^+(\bar{\Delta}_n^{\#})} \;:\; \underline{\mathcal{A}^+(\bar{\Delta}_n^{\#})} \longrightarrow \underline{\mathcal{A}'^+(\bar{\Delta}_n^{\#})}$$

on the respective stable categories of n-pretriangles. Cf. [**8,** §1.2.1.3, §A.6.3].

Remark 8 *If F is full and faithful, so is $\underline{F^+(\bar{\Delta}_n^{\#})}$.*

In particular, if F is the embedding of a full subcategory, we may and will also consider $\underline{F^+(\bar{\Delta}_n^{\#})}$ to be the embedding of a full subcategory.

Proof. By [**8,** Prop. 5.5], both $\mathcal{A}^+(\bar{\Delta}_n^{\#})$ and $\mathcal{A}'^+(\bar{\Delta}_n^{\#})$ are Frobenius categories; and the full and faithful functor $F^+(\bar{\Delta}_n^{\#}) : \mathcal{A}^+(\bar{\Delta}_n^{\#}) \longrightarrow \mathcal{A}'^+(\bar{\Delta}_n^{\#})$ induced by F preserves bijective objects, viz. split objects, and pure short exact sequences, viz. pointwise split short exact sequences. So by Remark 7, the assertion follows. ▫

2.2 A Heller triangulation on the Karoubi hull

Let $\hat{\mathcal{C}}$ denote the Freyd category of $\mathcal{C}$; cf. e.g. [**8,** A.6.3]. We consider the full and faithful functor $\mathcal{C} \longrightarrow \hat{\mathcal{C}}$ as an embedding of a full subcategory. Let $\tilde{\mathcal{C}}$ denote the full subcategory of bijectives in the abelian Frobenius category $\hat{\mathcal{C}}$. So we have full subcategories

$$\mathcal{C} \;\subseteq\; \tilde{\mathcal{C}} \;\subseteq\; \hat{\mathcal{C}}$$

Since the image of $\mathcal{C}$ in $\hat{\mathcal{C}}$ is a big enough subcategory of bijectives, the embedding $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ is a Karoubi hull of $\mathcal{C}$; cf. [**6,** III.II]. Cf. also Remark 43, Lemma 44 – which we will not use and argue directly instead.

We shall give a Heller triangulation on this Karoubi hull $\tilde{\mathcal{C}}$ of $\mathcal{C}$. The Verdier triangulated version of this construction is due to Balmer and Schlichting; cf. [**1,** Th. 1.12].

As a full subcategory of bijective objects in abelian Frobenius category, the category $\tilde{\mathcal{C}}$ is weakly abelian.

The shift T on $\mathcal{C}$ induces a shift $\hat{\mathsf{T}}$ on $\hat{\mathcal{C}}$, which restricts to a shift $\tilde{\mathsf{T}}$ on $\tilde{\mathcal{C}}$.

Remark 9 *Suppose given $n \geqslant 0$ and $X \in \mathrm{Ob}\, \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$. There exists $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ such that X is isomorphic to a direct summand of Z in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$. In other words, there exists $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ and a split monomorphism $X \xrightarrow{i} Z$ in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$.*

Proof. By [**8,** Prop. 2.6], it suffices to prove that given $X \in \mathrm{Ob}\, \underline{\tilde{\mathcal{C}}(\dot{\Delta}_n)}$, there exists $Z \in \mathrm{Ob}\, \underline{\mathcal{C}(\dot{\Delta}_n)}$ such that X is isomorphic, in $\underline{\tilde{\mathcal{C}}(\dot{\Delta}_n)}$, to a direct summand of Z.

It suffices to prove the existence of a split monomorphism $X \longrightarrow Z$ in $\tilde{\mathcal{C}}(\dot{\Delta}_n)$ with $Z \in \mathrm{Ob}\, \mathcal{C}(\dot{\Delta}_n)$.

For $i \in [1, n]$, let $Y_i \in \mathrm{Ob}\, \tilde{\mathcal{C}}$ be such that $X_i \oplus Y_i$ is isomorphic to an object in $\mathcal{C}$. Let $Y \in \mathrm{Ob}\, \tilde{\mathcal{C}}(\dot{\Delta}_n)$ have entry Y_i at position i for $1 \leqslant i \leqslant n$ and the morphism from position i to position j be zero for $1 \leqslant i < j \leqslant n$. The diagram $X \oplus Y$ has X as a summand and is isomorphic to an object in $\mathcal{C}(\dot{\Delta}_n)$. ▫

Remark 10 *Given $n \geqslant 0$, a diagram $X \in \mathrm{Ob}\, \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$, a split monomorphism $X \longrightarrow Z$ with $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ and a morphism $X \xrightarrow{x} X'$, then there exists a commutative quadrangle*

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z' \end{array}$$

in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$ with $Z' \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

Moreover, if $X \longrightarrow X'$ is a split monomorphism, we may choose $Z \longrightarrow Z'$ to be a split monomorphism.

Proof. We form

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ {\scriptstyle (1\,0)}\downarrow & & \downarrow{\scriptstyle (1\,0)} \\ X \oplus Y & \xrightarrow[\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)]{} & X' \oplus Y\ , \end{array}$$

where $X \oplus Y \simeq Z$. By Remark 9, there is a split monomorphism from $X' \oplus Y$ to an object Z' of $\mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

Moreover, if $X \xrightarrow{x} X'$ is split monic, so is the composite $(X \oplus Y \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)} X' \oplus Y \longrightarrow Z')$. ▫

Construction 11 Given $n \geqslant 0$, we define $[-]^{+1} \xrightarrow{\tilde{\vartheta}_n} [-^{+1}]$ on $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$ as follows.

Given $X \in \mathrm{Ob}\, \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$, choose a split monomorphism $X \xrightarrow{i} Z$ with $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$, existent by Remark 9, and choose a retraction p to i. Define

$$([X]^{+1} \xrightarrow{X\tilde{\vartheta}_n} [X^{+1}]) \;:=\; ([X]^{+1} \xrightarrow{[i]^{+1}} [Z]^{+1} \xrightarrow{Z\vartheta_n} [Z^{+1}] \xrightarrow{[p^{+1}]} [X^{+1}])\,.$$

To prove that $X\tilde{\vartheta}_n$ is welldefined, we shall first show that it is independent of the choice of the retraction p. Given $d : Z \longrightarrow X$ with $id = 0$, we have to show that $[i]^{+1}Z\vartheta_n[d^{+1}] = 0$. Since $[i^{+1}]$ is monic, it suffices to show that $[i]^{+1}Z\vartheta_n[d^{+1}][i^{+1}] = 0$. In fact,

$$[i]^{+1}Z\vartheta_n[d^{+1}][i^{+1}] \;=\; [i]^{+1}Z\vartheta_n[(di)^{+1}] \;=\; [i]^{+1}[di]^{+1}Z\vartheta_n \;=\; [id]^{+1}[i^{+1}]Z\vartheta_n \;=\; 0\,,$$

since di is in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

Now assume given another split monomorphism $X \longrightarrow Z'$ with $Z' \in \mathrm{Ob}\,\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$. By Remark 10, we may assume that this split monomorphism factors into two split monomorphisms $X \xrightarrow{i} Z \xrightarrow{i'} Z'$. Let $ip = 1$ and $i'p' = 1$. Then $(ii')(p'p) = 1$, and we may conclude

$$[ii']^{+1}(Z'\vartheta_n)[(p'p)^{+1}] \;=\; [i]^{+1}[i']^{+1}\Big(Z'\vartheta_n[p'^{+1}]\Big)[p^{+1}] \;=\; [i]^{+1}[i']^{+1}\Big([p']^{+1}Z\vartheta_n\Big)[p^{+1}] \;=\; [i]^{+1}Z\vartheta_n[p^{+1}]\,,$$

since p' is in $\underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

To show that ϑ_n is a transformation, we suppose given a morphism $X \xrightarrow{f} X'$ in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$ and have to show that $X\tilde{\vartheta}_n[f^{+1}] \stackrel{!}{=} [f]^{+1}X'\tilde{\vartheta}_n$. By Remarks 9 and 10, we find a commutative quadrangle

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ Z & \xrightarrow{g} & Z' \end{array}$$

in $\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$ with $Z,\, Z' \in \mathrm{Ob}\, \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^\#)}$. Choose p and p' such that $ip = 1$ and $i'p' = 1$. It suffices to show that $X\tilde{\vartheta}_n[f^{+1}][i'^{+1}] \stackrel{!}{=} [f]^{+1}X'\tilde{\vartheta}_n[i'^{+1}]$ by monomorphy of $[i'^{+1}]$. In fact,

$$\begin{array}{lcllcl} & X\tilde{\vartheta}_n[f^{+1}][i'^{+1}] & = & X\tilde{\vartheta}_n[i^{+1}][g^{+1}] & = & [i]^{+1}Z\vartheta_n[p^{+1}][i^{+1}][g^{+1}] \\ = & [i]^{+1}Z\vartheta_n[(pig)^{+1}] & = & [i]^{+1}[pig]^{+1}Z'\vartheta_n & = & [ig]^{+1}Z'\vartheta_n \\ = & [fi']^{+1}Z'\vartheta_n & = & [fi']^{+1}[p'i']^{+1}Z'\vartheta_n & = & [fi']^{+1}Z'\vartheta_n[(p'i')^{+1}] \\ = & [f]^{+1}[i']^{+1}Z'\vartheta_n[p'^{+1}][i'^{+1}] & = & [f]^{+1}X'\tilde{\vartheta}_n[i'^{+1}]\,. & & \end{array}$$

Note that $Z\tilde{\vartheta}_n = Z\vartheta_n$ for $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$.

End of construction.

Proposition 12

(1) *The tuple $\tilde{\vartheta} := (\tilde{\vartheta}_n)_{n\geqslant 0}$ is the unique Heller triangulation on $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}})$ such that the full and faithful inclusion functor $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ is strictly exact; cf.* [**8,** Def. 1.5.(i, ii)].

(2) *An n-pretriangle $U \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^\#)}$ is an n-triangle with respect to $(\mathcal{C}, \mathsf{T}, \vartheta)$ if and only if it is an n-triangle with respect to $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}}, \tilde{\vartheta})$.*

Proof. Ad (1). We have to show that given $m,\ n \geqslant 0$ and a periodic monotone map $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$, we have $\underline{p}^{\#} \star \tilde{\vartheta}_m \stackrel{!}{=} \tilde{\vartheta}_n \star \underline{p}^{\#}$. Let us verify this at $X \in \mathrm{Ob}\, \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^{\#})}$. Choose a split monomorphism $X \xrightarrow{i} Z$ with $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ by Remark 9. It suffices to show that $(X\underline{p}^{\#})\tilde{\vartheta}_m[(i\underline{p}^{\#})^{+1}] \stackrel{!}{=} (X\tilde{\vartheta}_n)\underline{p}^{\#}[(i\underline{p}^{\#})^{+1}]$. In fact, we obtain

$$\begin{aligned}(X\underline{p}^{\#})\tilde{\vartheta}_m[(i\underline{p}^{\#})^{+1}] &= [i\underline{p}^{\#}]^{+1}(Z\underline{p}^{\#})\vartheta_m = [i\underline{p}^{\#}]^{+1}(Z\vartheta_n)\underline{p}^{\#} \\ &= ([i]^{+1}Z\vartheta_n)\underline{p}^{\#} = (X\tilde{\vartheta}_n[i^{+1}])\underline{p}^{\#} = (X\tilde{\vartheta}_n)\underline{p}^{\#}[(i\underline{p}^{\#})^{+1}]\,.\end{aligned}$$

We have to show that given $n \geqslant 0$, we have $\underline{\mathsf{f}}_n \star \tilde{\vartheta}_{n+1} \stackrel{!}{=} \tilde{\vartheta}_{2n+1} \star \underline{\mathsf{f}}_n$. Let us verify this at $X \in \mathrm{Ob}\, \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_{2n+1}^{\#})}$. Choose a split monomorphism $X \xrightarrow{i} Z$ with $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_{2n+1}^{\#})}$ by Remark 9. It suffices to show that $(X\underline{\mathsf{f}}_n)\tilde{\vartheta}_{n+1}[(i\underline{\mathsf{f}}_n)^{+1}] \stackrel{!}{=} (X\tilde{\vartheta}_{2n+1})\underline{\mathsf{f}}_n[(i\underline{\mathsf{f}}_n)^{+1}]$. In fact, we obtain

$$\begin{aligned}(X\underline{\mathsf{f}}_n)\tilde{\vartheta}_{n+1}[(i\underline{\mathsf{f}}_n)^{+1}] &= [i\underline{\mathsf{f}}_n]^{+1}(Z\underline{\mathsf{f}}_n)\vartheta_{n+1} = [i\underline{\mathsf{f}}_n]^{+1}(Z\vartheta_{2n+1})\underline{\mathsf{f}}_n \\ &= ([i]^{+1}Z\vartheta_{2n+1})\underline{\mathsf{f}}_n = (X\tilde{\vartheta}_{2n+1}[i^{+1}])\underline{\mathsf{f}}_n = (X\tilde{\vartheta}_{2n+1})\underline{\mathsf{f}}_n[(i\underline{\mathsf{f}}_n)^{+1}]\,.\end{aligned}$$

The inclusion functor $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ is strictly exact since it strictly commutes with shift by construction, since it is subexact because the induced functor on the Freyd categories is an equivalence, and since $Z\tilde{\vartheta}_n = Z\vartheta_n$ for $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$.

Now suppose that both $\tilde{\vartheta}$ and $\tilde{\vartheta}'$ are Heller triangulations on $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}})$ such that $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}}$ is strictly exact. Suppose given $n \geqslant 0$ and $X \in \mathrm{Ob}\, \underline{\tilde{\mathcal{C}}^+(\bar{\Delta}_n^{\#})}$. We have to show that $X\tilde{\vartheta}_n \stackrel{!}{=} X\tilde{\vartheta}'_n$. Choose a split monomorphism $X \xrightarrow{i} Z$ with $Z \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$; cf. Remark 9. It suffices to show that $X\tilde{\vartheta}_n[i^{+1}] \stackrel{!}{=} X\tilde{\vartheta}'_n[i^{+1}]$. In fact,

$$X\tilde{\vartheta}_n[i^{+1}] = [i]^{+1}Z\tilde{\vartheta}_n = [i]^{+1}Z\vartheta_n = [i]^{+1}Z\tilde{\vartheta}'_n = X\tilde{\vartheta}'_n[i^{+1}]\,.$$

Ad (2). Suppose given an n-pretriangle $U \in \mathrm{Ob}\, \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$. Now U is an n-triangle with respect to $(\mathcal{C}, \mathsf{T}, \vartheta)$ if and only if $U\vartheta_n = 1$, and with respect to $(\tilde{\mathcal{C}}, \tilde{\mathsf{T}}, \tilde{\vartheta})$ if and only if $U\tilde{\vartheta}_n = 1$; cf. [**8,** Def. 1.5.(ii)]. Since $U\vartheta_n = U\tilde{\vartheta}_n$, these assertions are equivalent. Cf. also [**8,** Lem. 3.8]. ▫

2.3 Functoriality of the Karoubi hull

We shall prove the universal property of the Karoubi hull directly, without making recourse to Remark 43 and Lemma 44. We will make use of the universal property and the abelianness of the Freyd category, however.

Proposition 13 *Suppose given Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$, $(\mathcal{C}', \mathsf{T}', \vartheta')$. Call the strictly exact inclusion functors $\mathsf{K} : \mathcal{C} \longrightarrow \tilde{\mathcal{C}}$ and $\mathsf{K}' : \mathcal{C}' \longrightarrow \tilde{\mathcal{C}}'$.*

(1) *Suppose given an exact functor (F, a) from $\mathcal{C}$ to $\mathcal{C}'$.*

We may construct an exact functor $(\tilde{F}, \tilde{a})$ from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}'$ such that

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{(F,a)} & \mathcal{C}' \\ {\scriptstyle \mathsf{K}}\downarrow & & \downarrow{\scriptstyle \mathsf{K}'} \\ \tilde{\mathcal{C}} & \xrightarrow{(\tilde{F},\tilde{a})} & \tilde{\mathcal{C}}' \end{array}$$

commutes, i.e. such that $(F,a)(\mathsf{K}',1) = (\mathsf{K},1)(\tilde{F},\tilde{a})$*, i.e. such that* $F \star \mathsf{K}' = \mathsf{K} \star \tilde{F}$ *and* $a \star \mathsf{K}' = \mathsf{K} \star \tilde{a}$*, i.e. such that* $u\tilde{F} = uF$ *for* $u \in \operatorname{Mor}\mathcal{C}$ *and* $Z\tilde{a} = Za$ *for* $Z \in \operatorname{Ob}\mathcal{C}$*.*

The functor $\tilde{F}$ *and the condition* $a \star \mathsf{K}' = \mathsf{K} \star \tilde{a}$ *uniquely determines* $\tilde{a}$*. If* $a = 1$*, then* $\tilde{a} = 1$*.*

(2) *Given two exact functors* $(\tilde{F}_1\,,\,\tilde{a}_1)$ *and* $(\tilde{F}_2\,,\,\tilde{a}_2)$ *such that* $(F,a)(\mathsf{K}',1) = (\mathsf{K},1)(\tilde{F}_1\,,\,\tilde{a}_1) = (\mathsf{K},1)(\tilde{F}_2\,,\,\tilde{a}_2)$*, there exists a unique isotransformation* $\tilde{F}_1 \xrightarrow{\varphi}_{\sim} \tilde{F}_2$ *such that* $\mathsf{K} \star \varphi = 1$*, i.e. such that* $Z\varphi = 1$ *for* $Z \in \operatorname{Ob}\mathcal{C}$*. This isotransformation* φ *is periodic.*

(3) *Suppose given exact functors* (F,a) *and* (G,b) *from* $\mathcal{C}$ *to* $\mathcal{C}'$*. Suppose given a periodic transformation* s *from* F *to* G*.*

Construct $(\tilde{F},\tilde{a})$ *and* $(\tilde{G},\tilde{b})$ *as in* (1)*.*

There exists a unique periodic transformation $\tilde{s}$ *from* $\tilde{F}$ *to* $\tilde{G}$ *such that* $\mathsf{K} \star \tilde{s} = s \star \mathsf{K}'$*, i.e. such that* $Z\tilde{s} = Zs$ *for* $Z \in \operatorname{Ob}\mathcal{C}$*.*

Proof. Given $X \in \operatorname{Ob}\tilde{\mathcal{C}}$, we choose $X \xrightarrow{i_X} Z_X \xrightarrow{p_X} X$ in $\tilde{\mathcal{C}}$ such that $i_X \cdot p_X = 1_X$ and such that $Z_X \in \operatorname{Ob}\mathcal{C}$.

Moreover, choose these objects and morphisms in such a way that $Z_{X\tilde{\mathsf{T}}} = Z_X\,\mathsf{T}$, $i_{X\tilde{\mathsf{T}}} = i_X\tilde{\mathsf{T}}$ and $p_{X\tilde{\mathsf{T}}} = p_X\tilde{\mathsf{T}}$ for $X \in \operatorname{Ob}\tilde{\mathcal{C}}$.

Furthermore, if $X \in \operatorname{Ob}\mathcal{C}$, then choose $Z_X = X$ and $i_X = 1_X$ and $p_X = 1_X$.

Given $X \xrightarrow{u} Y$ in $\tilde{\mathcal{C}}$, we let $Z_X \xrightarrow{z_u} Z_Y$ be defined by $z_u := p_X \cdot u \cdot i_Y$; cf. Remark 39.

Ad (1). Since F is subexact, $\hat{F}$ is exact. Since W is a summand of an object in $\mathcal{C}$, also $W\hat{F}$ is a summand of an object in $\mathcal{C}'$, hence bijective. So $\tilde{F} := \hat{F}|^{\tilde{\mathcal{C}}'}_{\tilde{\mathcal{C}}}$ is welldefined.

We want to show that the functor $\tilde{F}$ preserves weak kernels and is therefore subexact; cf. Lemma 41. In fact, given $W \xrightarrow{w} B \xrightarrow{f} C$ in $\tilde{\mathcal{C}}$ such that w is a weak kernel of f, we get a factorisation $w = w'i$, where $K \xrightarrow{i} B$ is a kernel of f in $\hat{\mathcal{C}}$. Considering an epimorphism $P \xrightarrow{p} K$ in $\hat{\mathcal{C}}$ with $P \in \operatorname{Ob}\tilde{\mathcal{C}}$, we obtain a factorisation $pi = p'w = p'w'i$, whence $p = p'w'$, whence w' is epic. Since $w'\hat{F}$ is epic and $i\hat{F}$ is a kernel of $f\hat{F}$, we obtain that $w\hat{F} = w\tilde{F}$ is a weak kernel of $f\hat{F} = f\tilde{F}$.

The universal property of the Freyd construction yields a transformation $\hat{a} : \hat{\mathsf{T}}\hat{F} \longrightarrow \hat{F}\,\hat{\mathsf{T}}'$. We let the transformation $\tilde{a} : \tilde{\mathsf{T}}\tilde{F} \longrightarrow \tilde{F}\,\tilde{\mathsf{T}}'$ be defined on $X \in \operatorname{Ob}\tilde{\mathcal{C}} \subseteq \operatorname{Ob}\hat{\mathcal{C}}$ as $X\tilde{a} := X\hat{a}$. In particular, $Z\tilde{a} = Za$ for $Z \in \operatorname{Ob}\mathcal{C}$.

Given $n \geqslant 0$, it remains to be shown that $\underline{\tilde{F}^+(\bar{\Delta}^{\#}_n)} \star \tilde{\vartheta}'_n \stackrel{!}{=} (\tilde{\vartheta}_n \star \underline{\tilde{F}^+(\bar{\Delta}^{\#}_n)}) \cdot \underline{\tilde{a}^+(\bar{\Delta}^{\#}_n)}$; cf. Definition 1. Let us verify this at $X \in \operatorname{Ob}\underline{\tilde{\mathcal{C}}^+(\bar{\Delta}^{\#}_n)}$. Let $X \xrightarrow{i} Z$ be a split monomorphism with $Z \in \operatorname{Ob}\underline{\mathcal{C}^+(\bar{\Delta}^{\#}_n)}$, existent by Remark 9. It suffices to show that

$$(X\underline{\tilde{F}^+(\bar{\Delta}^{\#}_n)})\tilde{\vartheta}'_n \cdot [(i\underline{\tilde{F}^+(\bar{\Delta}^{\#}_n)})^{+1}] \stackrel{!}{=} (X\tilde{\vartheta}_n)\underline{\tilde{F}^+(\bar{\Delta}^{\#}_n)} \cdot X\underline{\tilde{a}^+(\bar{\Delta}^{\#}_n)} \cdot [(i\underline{\tilde{F}^+(\bar{\Delta}^{\#}_n)})^{+1}]\,.$$

In fact, we obtain

$$\begin{aligned}
(X\underline{\tilde{F}^+(\bar{\Delta}_n^\#)})\tilde{\vartheta}'_n \cdot [(i\underline{\tilde{F}^+(\bar{\Delta}_n^\#)})^{+1}] &= [i\underline{\tilde{F}^+(\bar{\Delta}_n^\#)}]^{+1} \cdot (Z\underline{\tilde{F}^+(\bar{\Delta}_n^\#)})\tilde{\vartheta}'_n \\
&= [i\underline{\tilde{F}^+(\bar{\Delta}_n^\#)}]^{+1} \cdot (Z\underline{F^+(\bar{\Delta}_n^\#)})\vartheta'_n \\
&\overset{(F,a)\text{ ex.}}{=} [i\underline{\tilde{F}^+(\bar{\Delta}_n^\#)}]^{+1} \cdot (Z\vartheta_n)\underline{F^+(\bar{\Delta}_n^\#)} \cdot Z\underline{a^+(\bar{\Delta}_n^\#)} \\
&= [i\underline{\tilde{F}^+(\bar{\Delta}_n^\#)}]^{+1} \cdot (Z\tilde{\vartheta}_n)\underline{\tilde{F}^+(\bar{\Delta}_n^\#)} \cdot Z\underline{\tilde{a}^+(\bar{\Delta}_n^\#)} \\
&= [i]^{+1}\underline{\tilde{F}^+(\bar{\Delta}_n^\#)} \cdot (Z\tilde{\vartheta}_n)\underline{\tilde{F}^+(\bar{\Delta}_n^\#)} \cdot Z\underline{\tilde{a}^+(\bar{\Delta}_n^\#)} \\
&= (X\tilde{\vartheta}_n)\underline{\tilde{F}^+(\bar{\Delta}_n^\#)} \cdot [i^{+1}]\underline{\tilde{F}^+(\bar{\Delta}_n^\#)} \cdot Z\underline{\tilde{a}^+(\bar{\Delta}_n^\#)} \\
&= (X\tilde{\vartheta}_n)\underline{\tilde{F}^+(\bar{\Delta}_n^\#)} \cdot X\underline{\tilde{a}^+(\bar{\Delta}_n^\#)} \cdot [(i\underline{\tilde{F}^+(\bar{\Delta}_n^\#)})^{+1}] \,.
\end{aligned}$$

If $a = 1$, then $\hat{a} = 1$, so $\tilde{a} = 1$.

It remains to show that $\tilde{a}$ is uniquely determined by $\tilde{F}$ and the condition $a \star \mathsf{K}' = \mathsf{K} \star \tilde{a}$. In fact, given $X \in \mathrm{Ob}\,\tilde{\mathcal{C}}$, we have

$$X\tilde{a} \cdot i_X\tilde{F}\tilde{\mathsf{T}}' \;=\; i_X\tilde{\mathsf{T}}\tilde{F} \cdot Z_X\tilde{a} \;=\; i_X\tilde{\mathsf{T}}\tilde{F} \cdot Z_Xa \;,$$

and $i_X\tilde{F}\tilde{\mathsf{T}}'$ is monic.

Ad (2). Define $\varphi : \tilde{F}_1 \xrightarrow{\sim} \tilde{F}_2$ at $X \in \mathrm{Ob}\,\tilde{\mathcal{C}}$ by

$$\begin{array}{ccccc}
Z_XF & \xrightarrow{p_X\tilde{F}_1} & X\tilde{F}_1 & \xrightarrow{i_X\tilde{F}_1} & Z_XF \\
{\scriptstyle 1}\downarrow & & {\scriptstyle X\varphi}\downarrow\wr & & \downarrow{\scriptstyle 1} \\
Z_XF & \xrightarrow{p_X\tilde{F}_2} & X\tilde{F}_2 & \xrightarrow{i_X\tilde{F}_2} & Z_XF \,;
\end{array}$$

cf. Remark 40.

The tuple $\varphi = (X\varphi)_{X \in \mathrm{Ob}\,\tilde{\mathcal{C}}}$ is actually a transformation from $\tilde{F}_1$ to $\tilde{F}_2$, for given $X \xrightarrow{u} Y$ in $\tilde{\mathcal{C}}$, we obtain

$$\begin{aligned}
p_X\tilde{F}_1 \cdot u\tilde{F}_1 \cdot Y\varphi \cdot i_Y\tilde{F}_2 &= p_X\tilde{F}_1 \cdot u\tilde{F}_1 \cdot i_Y\tilde{F}_1 \\
= \; p_X\tilde{F}_1 \cdot (u \cdot i_Y)\tilde{F}_1 &= p_X\tilde{F}_1 \cdot (i_X \cdot z_u)\tilde{F}_1 \\
= \; p_X\tilde{F}_1 \cdot i_X\tilde{F}_1 \cdot z_u\tilde{F}_1 &= ((p_X \cdot i_X) \cdot z_u)F \\
= \; (z_u \cdot (p_Y \cdot i_Y))F &= z_u\tilde{F}_2 \cdot p_Y\tilde{F}_2 \cdot i_Y\tilde{F}_2 \\
= \; (z_u \cdot p_Y)\tilde{F}_2 \cdot i_Y\tilde{F}_2 &= (p_X \cdot u)\tilde{F}_2 \cdot i_Y\tilde{F}_2 \\
= \; p_X\tilde{F}_2 \cdot u\tilde{F}_2 \cdot i_Y\tilde{F}_2 &= p_X\tilde{F}_1 \cdot X\varphi \cdot u\tilde{F}_2 \cdot i_Y\tilde{F}_2 \,,
\end{aligned}$$

and $p_X\tilde{F}_1$ is epic and $i_Y\tilde{F}_2$ is monic.

Note that commutativity of the diagram above is also necessary, for we require $\mathsf{K} \star \varphi = 1$. This ensures uniqueness of φ.

It remains to show that φ is a periodic transformation from $(\tilde{F}_1\,,\,\tilde{a}_1)$ to $(\tilde{F}_2\,,\,\tilde{a}_2)$. In fact, given

$X \in \mathrm{Ob}\,\tilde{\mathcal{C}}$, we get

$$\begin{array}{rcl@{\qquad}rcl}
 & & X\tilde{a}_1 \cdot X\varphi\tilde{\mathsf{T}}' \cdot i_X\tilde{F}_2\tilde{\mathsf{T}}' & = & X\tilde{a}_1 \cdot i_X\tilde{F}_1\tilde{\mathsf{T}}' \\
= & & i_X\tilde{\mathsf{T}}\tilde{F}_1 \cdot Z_X\tilde{a}_1 & = & i_{X\tilde{\mathsf{T}}}\tilde{F}_1 \cdot Z_X\tilde{a}_1 \\
= & & i_{X\tilde{\mathsf{T}}}\tilde{F}_1 \cdot Z_X a & = & i_{X\tilde{\mathsf{T}}}\tilde{F}_1 \cdot Z_X\tilde{a}_2 \\
= & & X\tilde{\mathsf{T}}\varphi \cdot i_{X\tilde{\mathsf{T}}}\tilde{F}_2 \cdot Z_X\tilde{a}_2 & = & X\tilde{\mathsf{T}}\varphi \cdot i_X\tilde{\mathsf{T}}\tilde{F}_2 \cdot Z_X\tilde{a}_2 \\
= & & X\tilde{\mathsf{T}}\varphi \cdot X\tilde{a}_2 \cdot i_X\tilde{F}_2\tilde{\mathsf{T}}' , & & \\
\end{array}$$

and $i_X\tilde{F}_2\tilde{\mathsf{T}}'$ is monic.

Ad (3). Define $\tilde{s} : \tilde{F} \longrightarrow \tilde{G}$ at $X \in \mathrm{Ob}\,\tilde{\mathcal{C}}$ by

$$\begin{array}{ccccc}
Z_XF & \xrightarrow{p_X\tilde{F}} & X\tilde{F} & \xrightarrow{i_X\tilde{F}} & Z_XF \\
{\scriptstyle Z_Xs}\downarrow & & {\scriptstyle X\tilde{s}}\downarrow\wr & & \downarrow{\scriptstyle Z_Xs} \\
Z_XG & \xrightarrow{p_X\tilde{G}} & X\tilde{G} & \xrightarrow{i_X\tilde{G}} & Z_XG\ ;
\end{array}$$

cf. Remark 40.

The tuple $s = (Xs)_{X\in\mathrm{Ob}\,\tilde{\mathcal{C}}}$ is actually a transformation from $\tilde{F}$ to $\tilde{G}$, for given $X \xrightarrow{u} Y$ in $\tilde{\mathcal{C}}$, we obtain

$$\begin{array}{rcl@{\qquad}rcl}
 & & p_X\tilde{F} \cdot u\tilde{F} \cdot Y\tilde{s} \cdot i_Y\tilde{G} & = & p_X\tilde{F} \cdot u\tilde{F} \cdot i_Y\tilde{F} \cdot Z_Ys \\
= & & p_X\tilde{F} \cdot (u \cdot i_Y)\tilde{F} \cdot Z_Ys & = & p_X\tilde{F} \cdot (i_X \cdot z_u)\tilde{F} \cdot Z_Ys \\
= & & p_X\tilde{F} \cdot i_X\tilde{F} \cdot z_u\tilde{F} \cdot Z_Ys & = & ((p_X \cdot i_X) \cdot z_u)F \cdot Z_Ys \\
= & & (z_u \cdot (p_Y \cdot i_Y))F \cdot Z_Ys & = & Z_Xs \cdot (z_u \cdot (p_Y \cdot i_Y))G \\
= & & Z_Xs \cdot z_u\tilde{G} \cdot p_Y\tilde{G} \cdot i_Y\tilde{G} & = & Z_Xs \cdot (z_u \cdot p_Y)\tilde{G} \cdot i_Y\tilde{G} \\
= & & Z_Xs \cdot (p_X \cdot u)\tilde{G} \cdot i_Y\tilde{G} & = & Z_Xs \cdot p_X\tilde{G} \cdot u\tilde{G} \cdot i_Y\tilde{G} \\
= & & p_X\tilde{F} \cdot X\tilde{s} \cdot u\tilde{G} \cdot i_Y\tilde{G} , & & \\
\end{array}$$

and $p_X\tilde{F}$ is epic and $i_Y\tilde{G}$ is monic.

Note that commutativity of the diagram above is also necessary, for we require $\mathsf{K} \star \tilde{s} = s \star \mathsf{K}'$. This ensures uniqueness of s.

It remains to show that $\tilde{s}$ is a periodic transformation from $(\tilde{F}, \tilde{a})$ to $(\tilde{G}, \tilde{b})$. In fact, given $X \in \mathrm{Ob}\,\tilde{\mathcal{C}}$, we get

$$\begin{array}{rcl@{\qquad}rcl}
 & & X\tilde{a} \cdot X\tilde{s}\tilde{\mathsf{T}}' \cdot i_X\tilde{G}\,\tilde{\mathsf{T}}' & = & X\tilde{a} \cdot i_X\tilde{F}\,\tilde{\mathsf{T}}' \cdot Z_Xs\tilde{\mathsf{T}}' \\
= & & i_X\tilde{\mathsf{T}}\tilde{F} \cdot Z_X\tilde{a} \cdot Z_Xs\tilde{\mathsf{T}}' & = & i_{X\tilde{\mathsf{T}}}\tilde{F} \cdot Z_X\tilde{a} \cdot Z_Xs\tilde{\mathsf{T}}' \\
= & & i_{X\tilde{\mathsf{T}}}\tilde{F} \cdot Z_Xa \cdot Z_Xs\,\mathsf{T}' & = & i_{X\tilde{\mathsf{T}}}\tilde{F} \cdot Z_X\,\mathsf{T}\,s \cdot Z_Xb \\
= & & i_{X\tilde{\mathsf{T}}}\tilde{F} \cdot Z_X\tilde{\mathsf{T}}s \cdot Z_X\tilde{b} & = & i_{X\tilde{\mathsf{T}}}\tilde{F} \cdot Z_{X\tilde{\mathsf{T}}}s \cdot Z_X\tilde{b} \\
= & & X\tilde{\mathsf{T}}\tilde{s} \cdot i_{X\tilde{\mathsf{T}}}\tilde{G} \cdot Z_X\tilde{b} & = & X\tilde{\mathsf{T}}\tilde{s} \cdot i_X\tilde{\mathsf{T}}\tilde{G} \cdot Z_X\tilde{b} \\
= & & X\tilde{\mathsf{T}}\tilde{s} \cdot X\tilde{b} \cdot i_X\tilde{G}\,\tilde{\mathsf{T}}' , & & \\
\end{array}$$

and $i_X\tilde{G}\,\tilde{\mathsf{T}}'$ is monic. ∎

2.4 Closed Heller triangulated categories

Recall that given a Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$, its Karoubi hull $\tilde{\mathcal{C}}$ is Heller triangulated, too; cf. Proposition 12.(1). More precisely, $(\tilde{\mathcal{C}}, \tilde{T}, \tilde{\vartheta})$ is Heller triangulated, where $\tilde{T}$ and $\tilde{\vartheta}$ are as in §2.2.

Definition 14

A Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$ is called *closed* if whenever $(X, Y, \tilde{Z})$ is a 2-triangle in $\tilde{\mathcal{C}}$ and $X, Y \in \mathrm{Ob}\,\mathcal{C}$, then $\tilde{Z}$ is isomorphic to an object of $\mathcal{C}$.

Cf. [**8,** Def. 1.5.(i, iii)].

I do not know an example of a non-closed Heller triangulated category.

As usual, we will call $\tilde{Z}$ the *cone* of $X \longrightarrow Y$, being unique up to isomorphism. Thus we may rephrase that by definition, $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed if it is closed under taking cones in the Karoubi hull $\tilde{\mathcal{C}}$.

Remark 15 *The Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed if and only if given $X \xrightarrow{f} Y$ in $\mathcal{C}$, there exists a 2-triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X^{+1}$ in $\mathcal{C}$.*

Cf. [**9,** Def. A.6].

Proof. If $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed, then given $X \xrightarrow{f} Y$ in $\mathcal{C}$, there exists a 2-triangle $X \xrightarrow{f} Y \longrightarrow \tilde{Z} \longrightarrow X^{+1}$ in $\tilde{\mathcal{C}}$ by [**8,** Lem. 3.1], and we may substitute $\tilde{Z}$ isomorphically by an object Z in $\mathrm{Ob}\,\mathcal{C}$, so we are done by [**8,** Lem. 3.4.(4)].

Conversely, if we dispose of this existence property, and if we are given a 2-triangle $(X, Y, \tilde{Z})$ in $\tilde{\mathcal{C}}$ with $X, Y \in \mathrm{Ob}\,\mathcal{C}$, then there exists a 2-triangle (X, Y, Z) with $Z \in \mathrm{Ob}\,\mathcal{C}$, too, and we may apply [**8,** Lem. 3.4.(6)] to conclude that $Z \simeq \tilde{Z}$. So $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed. ▫

Remark 16 *If idempotents split in $\mathcal{C}$, then $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed.*

Proof. If idempotents split in $\mathcal{C}$, then $\mathcal{C} = \tilde{\mathcal{C}}$. ▫

Remark 17 *Suppose given Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$, $(\mathcal{C}', \mathsf{T}', \vartheta')$ and a full and faithful strictly exact functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$. Furthermore, suppose that whenever given a 2-triangle (XF, YF, Z') in $\mathcal{C}'$, where $X, Y \in \mathrm{Ob}\,\mathcal{C}$, then there exists $Z \in \mathrm{Ob}\,\mathcal{C}$ such that $Z' \simeq ZF$.*

Suppose that $\mathcal{C}'$ is closed. Then $\mathcal{C}$ is closed.

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{C}$. There exists a 2-triangle $XF \xrightarrow{fF} YF \longrightarrow Z' \longrightarrow XF^{+1}$ in $\mathcal{C}'$. By assumption, there exists $Z \in \mathrm{Ob}\,\mathcal{C}$ such that $ZF \simeq Z'$. By isomorphic substitution and

fullness of F, we obtain a 2-triangle $XF \xrightarrow{fF} YF \xrightarrow{gF} ZF \xrightarrow{hF} XF^{+1}$ in $\mathcal{C}'$; cf. [**8,** Lem. 3.4.(4)]. Since

$$(X,Y,Z)\vartheta_2\underline{F^+(\bar{\Delta}_2^{\#})} \;=\; (X,Y,Z)\underline{F^+(\bar{\Delta}_2^{\#})}\vartheta_2' \;=\; (XF,YF,ZF)\vartheta_2' \;=\; 1\,,$$

we conclude by faithfulness of $\underline{F^+(\bar{\Delta}_2^{\#})}$ that $(X,Y,Z)\vartheta_2 = 1$; cf. Remark 8, [**8,** Def. 1.5.(ii)]. So we are done by Remark 15. ▫

Remark 18 *A closed Heller triangulated category is Verdier triangulated.*

Proof. Its Karoubian hull is Verdier triangulated [**8,** Prop. 3.6]. An additive shift-closed subcategory of a Verdier triangulated category that is closed under forming cones is Verdier triangulated. ▫

Definition 19 Suppose given a closed Heller triangulated category $(\mathcal{C},\mathsf{T},\vartheta)$.

Suppose given $n \geqslant 0$ and $Y \in \mathrm{Ob}\,\mathcal{C}(\dot{\Delta}_n)$ and $X \in \mathrm{Ob}\,\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ such that $X|_{\dot{\Delta}_n} = Y$.

Then Y is called the *base* of the n-triangle X.

Lemma 20 *Suppose given a closed Heller triangulated category $(\mathcal{C},\mathsf{T},\vartheta)$ and $n \geqslant 0$. The restriction functor $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#}) \xrightarrow{(-)|_{\dot{\Delta}_n}} \mathcal{C}(\dot{\Delta}_n)$ is strictly dense, i.e. surjective on objects. In other words, each object $Y \in \mathrm{Ob}\,\mathcal{C}(\dot{\Delta}_n)$ is the base of an n-triangle.*

> So weakening the assumption in [**8,** Lem. 3.1] that idempotents be split in $\mathcal{C}$ to the assumption that $\mathcal{C}$ be closed, we nonetheless obtain the conclusion of loc. cit.

Proof. Suppose given $Y \in \mathrm{Ob}\,\mathcal{C}(\dot{\Delta}_n)$. By [**8,** Lem. 3.1], we obtain an n-triangle $\tilde{X} \in \mathrm{Ob}\,\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ such that $\tilde{X}|_{\dot{\Delta}_n} = Y$.

By [**8,** Lem. 3.4.(1)], we have a triangle $(\tilde{X}_{\alpha/0}\,,\,\tilde{X}_{\beta/0}\,,\,\tilde{X}_{\beta/\alpha})$ whenever $0 < \alpha < \beta < 0^{+1}$. Since $\mathcal{C}$ is closed, $\tilde{X}_{\beta/\alpha}$ is isomorphic to an object of $\mathcal{C}$. Isomorphic substitution, which is permitted without leaving $\tilde{\mathcal{C}}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ by [**8,** Lem. 3.4.(4)], yields an n-triangle in $\mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}_n^{\#})$ that restricts to Y on $\dot{\Delta}_n$; cf. Proposition 12.(2). ▫

3 Heller triangulated subcategories

Definition 21 Given a Heller triangulated category $(\mathcal{C}',\mathsf{T}',\vartheta')$, a full subcategory $\mathcal{C} \subseteq \mathcal{C}'$ is called a full *Heller triangulated subcategory* of $\mathcal{C}'$ if there exist T and ϑ such that $(\mathcal{C},\mathsf{T},\vartheta)$ is a Heller triangulated category and such that the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{C}'$ is strictly exact.

We remark that in this case, the automorphism T and the tuple of transformations ϑ are uniquely determined by $(\mathcal{C}',\mathsf{T}',\vartheta')$ as respective restrictions; cf. [**8,** Def. 1.5.(iii)], Remark 8.

Example 22 Let $(\mathcal{C},\mathsf{T},\vartheta)$ be a Heller triangulated category. Let $\tilde{\mathcal{C}}$ be the Karoubi hull of $\mathcal{C}$, and let $(\tilde{\mathcal{C}},\tilde{\mathsf{T}},\tilde{\vartheta}_n)$ be the Heller triangulated category from Construction 11. By Proposition 12.(1), $\mathcal{C}$ is a Heller triangulated subcategory of $\tilde{\mathcal{C}}$.

Lemma 23 *Suppose given a closed Heller triangulated category $(\mathcal{C}', \mathsf{T}', \vartheta')$, and a full subcategory $\mathcal{C} \subseteq \mathcal{C}'$ such that the following conditions $(1, 2)$ hold.*

(1) $\mathcal{C}\,\mathsf{T}' = \mathcal{C}$.

(2) *Given a 2-triangle (X, Y, Z') in $\mathcal{C}'$ with X, $Y \in \mathrm{Ob}\,\mathcal{C}$, then Z' is isomorphic to an object of $\mathcal{C}$.*

Then $\mathcal{C}$, equipped with the shift T and the tuple ϑ obtained by restriction from T' and ϑ', respectively, is a Heller triangulated subcategory of $\mathcal{C}'$. Moreover, $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed.

Proof. Let T denote the restriction of T' to an automorphism of $\mathcal{C}$, which exists by assumption (1).

Write $\mathcal{C} \overset{i}{\hookrightarrow} \mathcal{C}'$ for the inclusion functor.

Since $\mathcal{C}'$ is closed, assumption (2) allows to conclude that $\mathcal{C}$ is a full additive subcategory of $\mathcal{C}'$, and moreover, that $\mathcal{C}$ is weakly abelian such that i is subexact; cf. Lemma 41.

Given $n \geqslant 0$ and $X \in \mathrm{Ob}\,\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$, we define, by restriction, $([X]^{+1} \xrightarrow{X\vartheta_n} [X^{+1}]) := ([X]^{+1} \xrightarrow{X\vartheta'_n} [X^{+1}])$. Since $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})} \xrightarrow{\underline{i}} \underline{\mathcal{C}'^+(\bar{\Delta}_n^{\#})}$ is full and faithful by Remark 8, this is a welldefined transformation satisfying $\vartheta_n \star \underline{i} = \underline{i} \star \vartheta'_n$.

Given m, $n \geqslant 0$ and a periodic monotone map $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$, we have $\underline{p}^{\#} \star \underline{i} = \underline{i} \star \underline{p}^{\#}$, whence

$$\underline{p}^{\#} \star \vartheta_m \star \underline{i} \;=\; \underline{p}^{\#} \star \underline{i} \star \vartheta'_m \;=\; \underline{i} \star \underline{p}^{\#} \star \vartheta'_m \;=\; \underline{i} \star \vartheta'_n \star \underline{p}^{\#} \;=\; \vartheta_n \star \underline{i} \star \underline{p}^{\#} \;=\; \vartheta_n \star \underline{p}^{\#} \star \underline{i}\,,$$

so that we may conclude that $\underline{p}^{\#} \star \vartheta_m = \vartheta_n \star \underline{p}^{\#}$, for $\underline{i}$ is faithful.

Given $n \geqslant 0$, we have $\underline{\mathsf{f}}_n \star \underline{i} = \underline{i} \star \underline{\mathsf{f}}_n$, whence

$$\underline{\mathsf{f}}_n \star \vartheta_{n+1} \star \underline{i} \;=\; \underline{\mathsf{f}}_n \star \underline{i} \star \vartheta'_{n+1} \;=\; \underline{i} \star \underline{\mathsf{f}}_n \star \vartheta'_{n+1} \;=\; \underline{i} \star \vartheta'_{2n+1} \star \underline{\mathsf{f}}_n \;=\; \vartheta_{2n+1} \star \underline{i} \star \underline{\mathsf{f}}_n \;=\; \vartheta_{2n+1} \star \underline{\mathsf{f}}_n \star \underline{i}\,,$$

so that we may conclude that $\underline{\mathsf{f}}_n \star \vartheta_{n+1} = \vartheta_{2n+1} \star \underline{\mathsf{f}}_n$, for $\underline{i}$ is faithful.

Hence ϑ is a Heller triangulation on $(\mathcal{C}, \mathsf{T})$; cf. [**8,** Def. 1.5.(i)]. By construction, $\mathcal{C} \overset{i}{\hookrightarrow} \mathcal{C}'$ is strictly exact.

By (2) and Remark 17, the Heller triangulated category $(\mathcal{C}, \mathsf{T}, \vartheta)$ is closed. ▫

4 Functors are exact if and only if they are compatible with n-triangles

Suppose given Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$.

Concerning the notion of n-triangles in a Heller triangulated category, cf. [**8,** Def. 1.5.(ii)].

For $n \geqslant 0$, an object Y in $\mathcal{C}(\bar{\Delta}_n^{\#})$ is called *periodic* if $[Y]^{+1} = [Y^{+1}]$.

Suppose given an additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ and an isomorphism $\mathsf{T}\,F \xrightarrow{a}_{\sim} F\,\mathsf{T}'$.

For $z \in \mathbf{Z}$, we let $\mathsf{T}^z\, F \xrightarrow[\sim]{a^{(z)}} F\, \mathsf{T}'^z$ be defined by

$$\begin{array}{lcll} a^{(0)} & := & 1_F & \\ a^{(z+1)} & := & (\mathsf{T} \star a^{(z)}) \cdot (a \star \mathsf{T}'^z) & \text{for } z \geqslant 0 \\ a^{(z-1)} & := & (\mathsf{T}^- \star a^{(z)}) \cdot (\mathsf{T}^- \star a^- \star \mathsf{T}'^{z-1}) & \text{for } z \leqslant 0 \end{array}$$

Then $(\mathsf{T}^z \star a^{(w)}) \cdot (a^{(z)} \star \mathsf{T}'^w) = a^{(z+w)} : \mathsf{T}^{z+w}\, F \xrightarrow{\sim} F\, \mathsf{T}'^{z+w}$ for $z,\, w \in \mathbf{Z}$.

Given a periodic n-pretriangle $X \in \mathrm{Ob}\, \mathcal{C}^{+,\,\mathrm{periodic}}(\bar{\Delta}_n^{\#})$, for sake of brevity we denote in this section by

$$Y \;:=\; X(F(\bar{\Delta}_n^{\#})) \;\in\; \mathrm{Ob}\, \mathcal{C}'(\bar{\Delta}_n^{\#})$$

the diagram obtained by pointwise application of F to X. We have

$$Y|_{\dot{\Delta}_n^{+1}} \;=\; X|_{\dot{\Delta}_n}((\mathsf{T}\, F)(\dot{\Delta}_n)) \;\xrightarrow[\sim]{X|_{\dot{\Delta}_n}(a(\dot{\Delta}_n))}\; X|_{\dot{\Delta}_n}((F\, \mathsf{T}')(\dot{\Delta}_n)) \;=\; (Y|_{\dot{\Delta}_n})^{+1}\,.$$

Isomorphic substitution along this isomorphism turns $Y|_{\bar{\Delta}_n^{\Delta\nabla}}$ into a diagram $\breve{Y}|_{\bar{\Delta}_n^{\Delta\nabla}}$ for a periodic object $\breve{Y} \in \mathrm{Ob}\, \mathcal{C}(\bar{\Delta}_n^{\#})$ thus defined. We have an isomorphism $Y \xrightarrow[\sim]{\breve{a}} \breve{Y}$ in $\mathcal{C}'(\bar{\Delta}_n^{\#})$ that at $(\beta/\alpha)^{+z}$ for $0 \leqslant \alpha \leqslant \beta \leqslant n$ and $z \in \mathbf{Z}$ is given by

$$\left(Y_{(\beta/\alpha)^{+z}} \xrightarrow[\sim]{\breve{a}_{(\beta/\alpha)^{+z}}} \breve{Y}_{(\beta/\alpha)^{+z}}\right) \;:=\; \left(X_{\beta/\alpha}\, \mathsf{T}^z\, F \xrightarrow[\sim]{X_{\beta/\alpha}\, a^{(z)}} X_{\beta/\alpha} F\, \mathsf{T}'^z\right).$$

In fact, given $0 \leqslant \alpha \leqslant n$ and $z \in \mathbf{Z}$, we obtain a commutative quadrangle

$$\begin{array}{ccc} X_{n/\alpha}\, \mathsf{T}^z\, F & \xrightarrow{x\, \mathsf{T}^z\, F} & X_{\alpha/0}\, \mathsf{T}^{z+1}\, F \\ {\scriptstyle X_{n/\alpha}\, a^{(z)}}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle X_{\alpha/0}\, a^{(z+1)}} \\ X_{n/\alpha} F\, \mathsf{T}'^z & \xrightarrow{(xF\, \mathsf{T}'^z)(X_{\alpha/0}\, a\, \mathsf{T}'^z)} & X_{\alpha/0} F\, \mathsf{T}'^{z+1}\,, \end{array}$$

for

$$(X_{n/\alpha}\, a^{(z)})(xF\, \mathsf{T}'^z)(X_{\alpha/0}\, a\, \mathsf{T}'^z) \;=\; (x\, \mathsf{T}^z\, F)(X_{\alpha/0}\, \mathsf{T}\, a^{(z)})(X_{\alpha/0}\, a\, \mathsf{T}'^z) \;=\; (x\, \mathsf{T}^z\, F)(X_{\alpha/0}\, a^{(z+1)})\,.$$

The remaining commutativities required for the naturality of $Y \xrightarrow[\sim]{\breve{a}} \breve{Y}$ follow by naturality of $a^{(z)}$.

We remark that $\breve{a}|_{\dot{\Delta}_n} = 1_{F(\dot{\Delta}_n)}$.

If F is subexact, then Y is an n-pretriangle and $\breve{Y}$ is a periodic n-pretriangle.

Lemma 24 *Suppose given an exact functor (F, a).*

Then for each n-triangle X of $\mathcal{C}$, i.e. $X \in \mathrm{Ob}\, \mathcal{C}^{\vartheta=1,+}(\bar{\Delta}_n^{\#})$, the object $\breve{Y}$ of $\mathcal{C}'(\bar{\Delta}_n^{\#})$ defined by (1) and (2) is an n-triangle of $\mathcal{C}'$, i.e. $\breve{Y} \in \mathrm{Ob}\, \mathcal{C}'^{\vartheta=1,+}(\bar{\Delta}_n^{\#})$.

(1) *We have $[\breve{Y}]^{+1} = [\breve{Y}^{+1}]$.*

(2) *On $\bar{\Delta}_n^{\Delta\nabla}$, the object $\breve{Y}|_{\bar{\Delta}_n^{\Delta\nabla}}$ arises from $Y := X(F(\bar{\Delta}_n^{\#}))|_{\bar{\Delta}_n^{\Delta\nabla}}$ by isomorphic substitution along $Y|_{\dot{\Delta}_n^{+1}} = X|_{\dot{\Delta}_n}(\mathsf{T}(\dot{\Delta}_n))(F(\dot{\Delta}_n)) \xrightarrow[\sim]{a(\dot{\Delta}_n)} X|_{\dot{\Delta}_n}(F(\dot{\Delta}_n))(\mathsf{T}'(\dot{\Delta}_n)) = (Y|_{\dot{\Delta}_n})^{+1}$.*

Cf. [**8,** Lem. 3.8] for the case of a strictly exact functor.

Proof. Suppose given $n \geqslant 0$ and an n-triangle $X \in \mathrm{Ob}\, \mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^{\#})$. By construction, $\breve{Y}$ is periodic. We have to show that $\breve{Y}\vartheta'_n \stackrel{!}{=} 1_{[\breve{Y}]^{+1}}$. We obtain

$$\begin{aligned} Y\vartheta'_n &= X\left(\underline{F^+(\bar{\Delta}_n^{\#})} \star \vartheta'_n\right) \\ &= X\left(\left(\vartheta_n \star \underline{F^+(\bar{\Delta}_n^{\#})}\right) \cdot \underline{a^+(\bar{\Delta}_n^{\#})}\right) \\ &= X\left(\vartheta_n \star \underline{F^+(\bar{\Delta}_n^{\#})}\right) \cdot X\underline{a^+(\bar{\Delta}_n^{\#})} \\ &= X\underline{a^+(\bar{\Delta}_n^{\#})}\,. \end{aligned}$$

In particular, $Y\vartheta'|_{\dot{\Delta}_n} = X|_{\dot{\Delta}_n}\underline{a(\dot{\Delta}_n)}$. Hence, restricting the stably commutative quadrangle

$$\begin{array}{ccc} [Y]^{+1} & \xrightarrow{[\breve{a}]^{+1}} & [\breve{Y}]^{+1} \\ {\scriptstyle Y\vartheta'_n}\downarrow & & \downarrow{\scriptstyle \breve{Y}\vartheta'_n} \\ [Y^{+1}] & \xrightarrow{[\breve{a}^{+1}]} & [\breve{Y}^{+1}] \end{array}$$

to $\dot{\Delta}_n$, we obtain the stably commutative quadrangle

$$\begin{array}{ccc} Y|_{\dot{\Delta}_n^{+1}} & \xrightarrow{X|_{\dot{\Delta}_n}\underline{a(\dot{\Delta}_n)}} & (Y|_{\dot{\Delta}_n})^{+1} \\ {\scriptstyle X|_{\dot{\Delta}_n}\underline{a(\dot{\Delta}_n)}}\downarrow & & \downarrow{\scriptstyle \breve{Y}\vartheta'_n|_{\dot{\Delta}_n}} \\ (Y|_{\dot{\Delta}_n})^{+1} & \xrightarrow{1} & (Y|_{\dot{\Delta}_n})^{+1}\,. \end{array}$$

whence $\breve{Y}\vartheta'_n|_{\dot{\Delta}_n} = 1_{(Y|_{\dot{\Delta}_n})^{+1}}$. Since the functor from $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ to $\underline{\mathcal{C}(\dot{\Delta}_n)}$ induced by restriction is an equivalence by [**8,** Prop. 2.6], this implies that $\breve{Y}\vartheta'_n = 1_{[\breve{Y}]^{+1}}$. ▫

Proposition 25 *Suppose $\mathcal{C}$ to be closed.*

The pair (F, a) is an exact functor if and only if for each n-triangle X of $\mathcal{C}$, the object $\breve{Y}$ of $\mathcal{C}'(\bar{\Delta}_n^{\#})$ defined by $(1,2)$ is an n-triangle of $\mathcal{C}'$.

(1) *We have $[\breve{Y}]^{+1} = [\breve{Y}^{+1}]$.*

(2) *On $\bar{\Delta}_n^{\Delta\nabla}$, the object $\breve{Y}|_{\bar{\Delta}_n^{\Delta\nabla}}$ arises from $Y := X(F(\bar{\Delta}_n^{\#}))|_{\bar{\Delta}_n^{\Delta\nabla}}$ by isomorphic substitution along $Y|_{\dot{\Delta}_n^{+1}} = X|_{\dot{\Delta}_n}(\mathsf{T}(\dot{\Delta}_n))(F(\dot{\Delta}_n)) \stackrel{a(\dot{\Delta}_n)}{\xrightarrow{\sim}} X|_{\dot{\Delta}_n}(F(\dot{\Delta}_n))(\mathsf{T}'(\dot{\Delta}_n)) = (Y|_{\dot{\Delta}_n})^{+1}$.*

Proof. In view of Lemma 24, it suffices to show that if each n-triangle X in $\mathcal{C}$ yields an n-triangle $\breve{Y}$ in $\mathcal{C}'$ by $(1,2)$, then (F, a) is exact.

We *claim* that F is subexact. By Lemma 41, it suffices to show that given a morphism $S \xrightarrow{p} T$ in $\mathcal{C}$, there exists a weak cokernel of p that is mapped by F to a weak cokernel. Since $\mathcal{C}$ is a closed Heller triangulated category, a weak cokernel of p is contained in the the completion of $S \xrightarrow{p} T$ to a 2-triangle X by Lemma 20. We form the corresponding 2-triangle $\breve{Y}$ defined

by $(1,2)$. Since it contains a weak cokernel of $SF \xrightarrow{pF} TF$, and since $\breve{Y}$ is isomorphic, in $\mathcal{C}'^{+}(\bar{\Delta}_n^{\#})$, to $X(F^{+}(\bar{\Delta}_2^{\#}))$, the image under F of the weak cokernel of p that is contained in the 2-triangle X is in fact a weak cokernel of pF. This proves the *claim.*

We *claim* that

$$(\vartheta_n \star \underline{F^{+}(\bar{\Delta}_n^{\#})}) \cdot \underline{a^{+}(\bar{\Delta}_n^{\#})} \;=\; \underline{F^{+}(\bar{\Delta}_n^{\#})} \star \vartheta'_n$$

for all $n \geqslant 0$. Suppose given $X \in \mathrm{Ob}\, \mathcal{C}^{+}(\bar{\Delta}_n^{\#})$. Since $\mathcal{C}$ is a closed Heller triangulated category, there exists an n-triangle X' such that $X'|_{\dot{\Delta}_n} = X|_{\dot{\Delta}_n}$; cf. Lemma 20. By [**8,** Prop. 2.6], we have an isomorphism $X \xrightarrow{f}_{\sim} X'$ in $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$ that restricts to the identity on $\dot{\Delta}_n$. We dispose of a commutative diagram

$$\begin{array}{ccc} [X]^{+1} & \xrightarrow{X\vartheta_n} & [X^{+1}] \\ {\scriptstyle [f]^{+1}}\downarrow & & \downarrow{\scriptstyle [f^{+1}]} \\ [X']^{+1} & \xrightarrow{X'\vartheta_n} & [X'^{+1}] \end{array}$$

in $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$. Since, by construction, $X'\vartheta_n = 1$, we have $X\vartheta_n = [f]^{+1} \cdot [f^{+1}]^{-}$ in $\underline{\mathcal{C}^{+}(\bar{\Delta}_n^{\#})}$.

Likewise, we have a commutative quadrangle

$$\begin{array}{ccc} [X\underline{F^{+}(\bar{\Delta}_n)}]^{+1} & \xrightarrow{X\underline{F^{+}(\bar{\Delta}_n)}\vartheta'_n} & [(X\underline{F^{+}(\bar{\Delta}_n)})^{+1}] \\ {\scriptstyle [f\underline{F^{+}(\bar{\Delta}_n)}]^{+1}}\downarrow & & \downarrow{\scriptstyle [(f\underline{F^{+}(\bar{\Delta}_n)})^{+1}]} \\ [X'\underline{F^{+}(\bar{\Delta}_n)}]^{+1} & \xrightarrow{X'\underline{F^{+}(\bar{\Delta}_n)}\vartheta'_n} & [(X'\underline{F^{+}(\bar{\Delta}_n)})^{+1}]\,, \end{array}$$

in $\underline{\mathcal{C}'^{+}(\bar{\Delta}_n^{\#})}$. We want to calculate its lower arrow. Since X' is an n-triangle, we have an isomorphism $Y' \xrightarrow{\breve{a}'}_{\sim} \breve{Y}'$ formed as above, where $\breve{Y}'\vartheta'_n = 1$. The stably commutative quadrangle

$$\begin{array}{ccc} [Y']^{+1} & \xrightarrow{[\breve{a}']^{+1}} & [\breve{Y}']^{+1} \\ {\scriptstyle Y'\vartheta'_n}\downarrow & & \downarrow{\scriptstyle \breve{Y}'\vartheta'_n = 1} \\ [Y'^{+1}] & \xrightarrow{[\breve{a}'^{+1}]} & [\breve{Y}'^{+1}] \end{array}$$

yields by restriction to $\dot{\Delta}_n$ the commutative diagram

$$\begin{array}{ccc} X'|_{\dot{\Delta}_n}\,\underline{(\mathsf{T}\,F)(\dot{\Delta}_n)} & \xrightarrow{X'|_{\dot{\Delta}_n}\,\underline{a(\dot{\Delta}_n)}} & X'|_{\dot{\Delta}_n}\,\underline{(F\;\mathsf{T}')(\dot{\Delta}_n)} \\ \| & & \| \\ [Y']^{+1}|_{\dot{\Delta}_n} & \xrightarrow{[\breve{a}]^{+1}|_{\dot{\Delta}_n}} & [\breve{Y}']^{+1}|_{\dot{\Delta}_n} \\ {\scriptstyle Y'\vartheta'_n|_{\dot{\Delta}_n}}\downarrow & & \downarrow{\scriptstyle 1} \\ [Y'^{+1}]|_{\dot{\Delta}_n} & \xrightarrow{1} & [\breve{Y}'^{+1}]|_{\dot{\Delta}_n}\,, \end{array}$$

whence $Y'\vartheta'_n|_{\dot{\Delta}_n} = X'|_{\dot{\Delta}_n}\,\underline{a(\dot{\Delta}_n)} = X'\underline{a^{+}(\bar{\Delta}_n^{\#})}|_{\dot{\Delta}_n}$. Since the functor from $\underline{\mathcal{C}'^{+}(\bar{\Delta}_n^{\#})}$ to $\underline{\mathcal{C}'(\dot{\Delta}_n)}$ induced by restriction is an equivalence by [**8,** Prop. 2.6], this implies that

$$X'\underline{F^{+}(\bar{\Delta}_n)}\vartheta'_n \;=\; Y'\vartheta'_n \;=\; X'\underline{a^{+}(\bar{\Delta}_n^{\#})}\,.$$

So we can conclude that

$$\begin{aligned}
X\underline{F^+(\bar{\Delta}_n)}\vartheta'_n &= [f\underline{F^+(\bar{\Delta}_n)}]^{+1} \cdot X'\underline{F^+(\bar{\Delta}_n)}\vartheta'_n \cdot [(f\underline{F^+(\bar{\Delta}_n)})^{+1}]^- \\
&= [f\underline{F^+(\bar{\Delta}_n)}]^{+1} \cdot X'\underline{a^+(\bar{\Delta}^\#_n)} \cdot [(f\underline{F^+(\bar{\Delta}_n)})^{+1}]^- \\
&= [f\underline{F^+(\bar{\Delta}^\#_n)}]^{+1} \cdot X'\underline{a^+(\bar{\Delta}^\#_n)} \cdot (f\underline{(F\,\mathsf{T}')^+(\bar{\Delta}^\#_n)})^- \\
&= [f\underline{F^+(\bar{\Delta}^\#_n)}]^{+1} \cdot (f\underline{(\mathsf{T}\,F)^+(\bar{\Delta}^\#_n)})^- \cdot X\underline{a^+(\bar{\Delta}^\#_n)} \\
&= ([f]^{+1} \cdot [f^{+1}]^-)\underline{F^+(\bar{\Delta}^\#_n)} \cdot X\underline{a^+(\bar{\Delta}^\#_n)} \\
&= X(\vartheta_n \star \underline{F^+(\bar{\Delta}^\#_n)}) \cdot X\underline{a^+(\bar{\Delta}^\#_n)} \ .
\end{aligned}$$

This proves the *claim.* □

Corollary 26 *Suppose* $(\mathcal{C}, \mathsf{T}, \vartheta)$ *to be closed.*

Suppose given an additive functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ *such that* $\mathsf{T}\,F = F\ \mathsf{T}'$.

Then F *is strictly exact if and only if for each* $n \geqslant 0$ *and each* n*-triangle* $X \in \mathrm{Ob}\ \mathcal{C}^{+,\,\vartheta=1}(\bar{\Delta}^\#_n)$*, the diagram* $X(F(\bar{\Delta}^\#_n)) \in \mathrm{Ob}\ \mathcal{C}'(\bar{\Delta}^\#_n)$*, obtained by pointwise application of* F*, is an* n*-triangle.*

Proof. In this case, we have $a = 1_{\mathsf{T}\,F} = 1_{F\,\mathsf{T}'}$ and $\breve{Y} = Y = X(F(\bar{\Delta}^\#_n))$. Since $(F, 1)$ is exact if and only if F is strictly exact, the assertion follows by Proposition 25. □

5 Adjoints

5.1 Adjoints and shifts

Suppose given categories $\mathcal{A}$ and $\mathcal{A}'$. Suppose given an endofunctor T of $\mathcal{A}$. Suppose given an endofunctor T' of $\mathcal{A}'$.

Suppose given functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{A}'$ such that $F \dashv G$ via unit $1 \xrightarrow{\varepsilon} FG$ and counit $GF \xrightarrow{\eta} 1$, i.e. $(G\varepsilon)(\eta G) = 1_G$ and $(\varepsilon F)(F\eta) = 1_F$.

Suppose given $TF \xrightarrow{\alpha} FT'$.

Let

$$(GT \xrightarrow{\beta} T'G) \ := \ \big(GT \xrightarrow{GT\varepsilon} GTFG \xrightarrow{G\alpha G} GFT'G \xrightarrow{\eta T'G} T'G\big) \ .$$

So we have the commutative diagram

$$\begin{array}{ccc}
GT & \xrightarrow{\ \beta\ } & T'G \\
{\scriptstyle GT\varepsilon}\downarrow & & \uparrow{\scriptstyle \eta T'G} \\
GTFG & \xrightarrow{\ G\alpha G\ } & GFT'G \ .
\end{array}$$

Lemma 27

(1) *We have the commutative diagram*

$$\begin{array}{ccc} TF & \xrightarrow{\alpha} & FT' \\ {\scriptstyle \varepsilon TF}\downarrow & & \uparrow{\scriptstyle FT'\eta} \\ FGTF & \xrightarrow{F\beta F} & FT'GF\ . \end{array}$$

(2) *We have the commutative quadrangle*

$$\begin{array}{ccc} T & \xrightarrow{\varepsilon T} & FGT \\ {\scriptstyle T\varepsilon}\downarrow & & \downarrow{\scriptstyle F\beta} \\ TFG & \xrightarrow[\alpha G]{} & FT'G\ . \end{array}$$

(2°) *We have the commutative quadrangle*

$$\begin{array}{ccc} GTF & \xrightarrow{G\alpha} & GFT' \\ {\scriptstyle \beta F}\downarrow & & \downarrow{\scriptstyle \eta T'} \\ T'GF & \xrightarrow[T'\eta]{} & T'\ . \end{array}$$

(3) *Suppose that T and T' are autofunctors. Write $G' = T'GT^-$. If α is an isotransformation, then so is β, where*

$$\begin{aligned} &(T'G \xrightarrow{\beta^-} GT) \ = \\ &\big(T'G = T'GT^-T \xrightarrow{T'GT^-\varepsilon T} T'GT^-FGT = \\ &T'GT^-FT'T'^-GT \xrightarrow{T'GT^-\alpha^-T'^-GT} T'GT^-TFT'^-GT = \\ &T'GFT'^-GT \xrightarrow{T'\eta T'^-GT} T'T'^-GT = GT\big)\ . \end{aligned}$$

(3°) *Suppose that T and T' are autofunctors. If β is an isotransformation, then so is α, where*

$$\begin{aligned} &(FT' \xrightarrow{\alpha^-} TF) \ = \\ &\big(FT' = TT^-FT' \xrightarrow{T\varepsilon T^-FT'} TFGT^-FT' = \\ &TFT'^-T'GT^-FT' \xrightarrow{TFT'^-\beta^-T^-FT'} TFT'^-GTT^-FT' = \\ &TFT'^-GFT' \xrightarrow{TFT'^-\eta T'} TFT'^-T' = TF\big)\ . \end{aligned}$$

Proof. Ad (2). We have

$$\begin{aligned} (\varepsilon T)(F\beta) &= (\varepsilon T)(FGT\varepsilon)(FG\alpha G)(F\eta T'G) \\ &= (T\varepsilon)(\varepsilon TFG)(FG\alpha G)(F\eta T'G) \\ &= (T\varepsilon)(\alpha G)(\varepsilon FT'G)(F\eta T'G) \\ &= (T\varepsilon)(\alpha G)\ . \end{aligned}$$

Ad (1). We have

$$\begin{aligned}(\varepsilon TF)(F\beta F)(FT'\eta) &\overset{(2)}{=} (T\varepsilon F)(\alpha GF)(FT'\eta)\\ &= (T\varepsilon F)(TF\eta)\alpha\\ &= \alpha\ .\end{aligned}$$

□

Ad (3). We have

$$\begin{aligned}&\beta\cdot(T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT)\\ =\ &(\beta T^-T)(T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT)\\ =\ &(GTT^-\varepsilon T)(\beta T^-FGT)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT)\\ =\ &(G\varepsilon T)(\beta T^-FT'T'^-GT)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT)\\ =\ &(G\varepsilon T)(GTT^-\alpha^-T'^-GT)(\beta T^-TFT'^-GT)(T'\eta T'^-GT)\\ =\ &(G\varepsilon T)(G\alpha^-T'^-GT)(\beta FT'^-GT)(T'\eta T'^-GT)\\ \overset{(2^\circ)}{=}\ &(G\varepsilon T)(G\alpha^-T'^-GT)(G\alpha T'^-GT)(\eta T'T'^-GT)\\ =\ &(G\varepsilon T)(\eta GT)\\ =\ &1\end{aligned}$$

and

$$\begin{aligned}&(T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT)\cdot\beta\\ =\ &(T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'\eta T'^-GT)(T'T'^-\beta)\\ =\ &(T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'GFT'^-\beta)(T'\eta T'^-T'G)\\ =\ &(T'GT^-\varepsilon T)(T'GT^-\alpha^-T'^-GT)(T'GT^-TFT'^-\beta)(T'\eta G)\\ =\ &(T'GT^-\varepsilon T)(T'GT^-FT'T'^-\beta)(T'GT^-\alpha^-T'^-T'G)(T'\eta G)\\ =\ &(T'GT^-\varepsilon T)(T'GT^-F\beta)(T'GT^-\alpha^-G)(T'\eta G)\\ \overset{(2)}{=}\ &(T'GT^-T\varepsilon)(T'GT^-\alpha G)(T'GT^-\alpha^-G)(T'\eta G)\\ =\ &(T'G\varepsilon)(T'\eta G)\\ =\ &1\ .\end{aligned}$$

5.2 An adjoint of an exact functor is exact

The Verdier triangulated version of the following proposition is due to Margolis [**11,** App. 2, Prop. 11], and, in a more general form, to Keller and Vossieck [**5,** 1.6].

Proposition 28 *Suppose given Heller triangulated categories* $(\mathcal{C},\mathsf{T},\vartheta)$ *and* $(\mathcal{C}',\mathsf{T}',\vartheta')$.

Suppose given an exact functor (F,a) *from* $\mathcal{C}$ *to* $\mathcal{C}'$*; cf. Definition* 1.

Suppose given a functor $\mathcal{C}\xleftarrow{G}\mathcal{C}'$.

So $\mathcal{C}\underset{G}{\overset{F}{\rightleftarrows}}\mathcal{C}'$ *and* $\mathsf{T}F\xrightarrow[\sim]{a}F\mathsf{T}'$.

(1) *If* $F\dashv G$*, then there exists an isomorphism* $\mathsf{T}'G\xrightarrow[\sim]{b}G\mathsf{T}$ *such that* (G,b) *is an exact functor from* $\mathcal{C}'$ *to* $\mathcal{C}$.

Choose a unit $1_{\mathcal{C}'} \xrightarrow{\varepsilon} FG$ *and a counit* $GF \xrightarrow{\eta} 1_{\mathcal{C}}$. *Then, more precisely, we may choose*

$$(\mathsf{T}'G \xrightarrow{b} G\mathsf{T}) \;:=\; (G\mathsf{T} \xrightarrow{G\mathsf{T}\varepsilon} G\mathsf{T}FG \xrightarrow{GaG} GF\mathsf{T}'G \xrightarrow{\eta\mathsf{T}'G} \mathsf{T}'G)^{-} .$$

(1°) *If* $G \dashv F$, *then there exists an isomorphism* $\mathsf{T}'G \xrightarrow[\sim]{b} G\mathsf{T}$ *such that* (G, b) *is an exact functor from* $\mathcal{C}'$ *to* $\mathcal{C}$.

Choose a unit $1_{\mathcal{C}'} \xrightarrow{\varepsilon} GF$ *and a counit* $FG \xrightarrow{\eta} 1_{\mathcal{C}}$. *Then, more precisely, we may choose*

$$(\mathsf{T}'G \xrightarrow{b} G\mathsf{T}) \;:=\; (\mathsf{T}'G \xrightarrow{\varepsilon\mathsf{T}'G} GF\mathsf{T}'G \xrightarrow{Ga^{-}G} G\mathsf{T}FG \xrightarrow{G\mathsf{T}\eta} G\mathsf{T}) .$$

Proof. Ad (1). By Lemma 42.(1°), G is subexact.

Lemma 27.(3) yields the isotransformation $b^{-} := (G\mathsf{T}\varepsilon)(GaG)(\eta\mathsf{T}'G)$.

Suppose given $n \geqslant 0$. We shall make use of the abbreviation $\underline{G} = \underline{G^{+}(\bar{\Delta}^{\#}_{n})}$, etc. We have to show that

$$(\vartheta'_{n} \star \underline{G}) \cdot \underline{b} \;\overset{!}{=}\; \underline{G} \star \vartheta_{n} ,$$

i.e. that

$$(\underline{G} \star \vartheta_{n}) \cdot \underline{b}^{-} \;\overset{!}{=}\; \vartheta'_{n} \star \underline{G} ,$$

i.e. that

$$(\underline{G} \star \vartheta_{n}) \cdot (\underline{G} \star \underline{\mathsf{T}} \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{a} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}'} \star \underline{G}) \;\overset{!}{=}\; \vartheta'_{n} \star \underline{G}$$

Recall that $[-]^{+1}$ denotes the outer shift, that $[-^{+1}]$ denotes the inner shift and that $\vartheta_{n} : [-]^{+1} \xrightarrow{\sim} [-^{+1}]$ on $\underline{\mathcal{C}^{+}(\bar{\Delta}^{\#}_{n})}$; similarly on $\underline{\mathcal{C}'^{+}(\bar{\Delta}^{\#}_{n})}$.

We obtain

$$\begin{aligned}
& (\underline{G} \star \vartheta_{n}) \cdot (\underline{G} \star \underline{\mathsf{T}} \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{a} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}'} \star \underline{G}) \\
= \;& (\underline{G} \star \vartheta_{n}) \cdot (\underline{G} \star [-^{+1}] \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{a} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}'} \star \underline{G}) \\
= \;& (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{G} \star \vartheta_{n} \star \underline{F} \star \underline{G}) \cdot (\underline{G} \star \underline{a} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}'} \star \underline{G}) \\
= \;& (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{G} \star ((\vartheta_{n} \star \underline{F}) \cdot \underline{a}) \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}'} \star \underline{G}) \\
\overset{(F,a)\text{ ex.}}{=} \;& (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{F} \star \vartheta'_{n} \star \underline{G}) \cdot (\underline{\eta} \star \underline{\mathsf{T}'} \star \underline{G}) \\
= \;& (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{G} \star \underline{F} \star \vartheta'_{n} \star \underline{G}) \cdot (\underline{\eta} \star [-^{+1}] \star \underline{G}) \\
= \;& (\underline{G} \star [-]^{+1} \star \underline{\varepsilon}) \cdot (\underline{\eta} \star [-]^{+1} \star \underline{G}) \cdot (\vartheta'_{n} \star \underline{G}) \\
= \;& (\underline{G} \star \underline{\varepsilon} \star [-]^{+1}) \cdot (\underline{\eta} \star \underline{G} \star [-]^{+1}) \cdot (\vartheta'_{n} \star \underline{G}) \\
= \;& \vartheta'_{n} \star \underline{G} .
\end{aligned}$$

Ad (1°). Cf. Lemma 27.(1). ▫

Example 29 Suppose we are in the situation of Proposition 28.(1). Then ε and η are periodic; cf. Definition 4.

Ad $\varepsilon : 1 \longrightarrow FG$. The functor $(F, a)(G, b) = (FG\,,\, aG \cdot Fb)$ is exact; cf. Remark 3. The functor $(1_{\mathcal{C}}\,,\, 1)$ is exact. The quadrangle

$$\begin{array}{ccc}
\mathsf{T} & \xrightarrow{T\varepsilon} & \mathsf{T}FG \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle aG\cdot Fb} \\
\mathsf{T} & \xrightarrow{\varepsilon T} & FG\mathsf{T}
\end{array}$$

commutes by Lemma 27.(2).

Ad $\eta : GF \longrightarrow 1$. The functor $(G, b)(F, a) = (GF\,,\, bF \cdot Ga)$ is exact; cf. Remark 3. The functor $(1_{\mathcal{C}'}\,,\, 1)$ is exact. The quadrangle

$$\begin{array}{ccc} \mathsf{T}'\,GF & \xrightarrow{\mathsf{T}'\,\eta} & \mathsf{T}' \\ {\scriptstyle bF\cdot Ga}\downarrow & & \downarrow{\scriptstyle 1} \\ GF\,\mathsf{T}' & \xrightarrow{\eta\,\mathsf{T}'} & \mathsf{T}' \end{array}$$

commutes by Lemma 27.(2°).

5.3 A functor shiftcompatibly adjoint to a strictly exact functor is strictly exact

Suppose given closed Heller triangulated categories $(\mathcal{C}, \mathsf{T}, \vartheta)$ and $(\mathcal{C}', \mathsf{T}', \vartheta')$

Recall that an additive functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is strictly exact if and only if $(F, 1)$ is exact; cf. [**8,** Def. 1.5.(iii)], Definition 1.

Corollary 30

Suppose given a strictly exact functor $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ *.*

Suppose given a functor $\mathcal{C} \xleftarrow{G} \mathcal{C}'$ *.*

(1) *If* $F \dashv G$*, with unit* $\varepsilon : 1 \longrightarrow FG$ *and counit* $\eta : GF \longrightarrow 1$ *such that* $(G\,\mathsf{T}\,\varepsilon)(\eta\,\mathsf{T}'\,G) = 1$*, then* G *is strictly exact.*

(1°) *If* $G \dashv F$*, with unit* $\varepsilon : 1 \longrightarrow FG$ *and counit* $\eta : GF \longrightarrow 1$ *such that* $(\varepsilon\,\mathsf{T}'\,G)(G\,\mathsf{T}\,\eta) = 1$*, then* G *is strictly exact.*

Proof. Ad (1). In the notation of Proposition 28.(1), we have $a = 1$, and, consequently, $b = (G\,\mathsf{T}\,\varepsilon)(\eta\,\mathsf{T}'\,G) = 1$. Hence by loc. cit., $(G, 1)$ is exact, i.e. G is strictly exact. □

6 Localisation

We prove that the localisation $\mathcal{C}/\!\!/\mathcal{N}$ of a Heller triangulated category $\mathcal{C}$ at a thick subcategory $\mathcal{N}$ is Heller triangulated in such a way that the localisation functor $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$ is strictly exact; cf. [**8,** Def. 1.5]. There is considerable overlap with the classical localisation theory of Verdier triangulated categories, due to Verdier [**13**], which we include for sake of self-containedness.

Let $(\mathcal{C}, \mathsf{T}, \vartheta)$ be a closed Heller triangulated category; cf. Definition 14.

Definition 31 A full additive subcategory $\mathcal{N} \subseteq \mathcal{C}$ is called *thick* if the conditions $(1, 2, 3)$ are satisfied; cf. [**12,** Prop. 1.3]

(1) We have $\mathcal{N}^{+1} = \mathcal{N}$ (closed under shift).

(2) Given a 2-triangle (X, Y, Z) in $\mathcal{C}$ with X and Y in $\mathrm{Ob}\,\mathcal{N}$, then $Z \in \mathrm{Ob}\,\mathcal{N}$ (closed under taking cones).

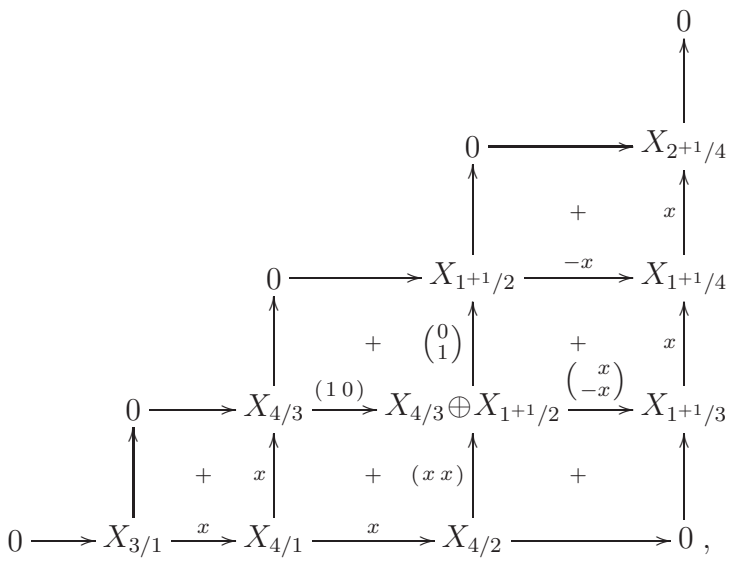
(3) Given X, $Y \in \mathrm{Ob}\,\mathcal{C}$ with $X \oplus Y$ in $\mathrm{Ob}\,\mathcal{N}$, then $X \in \mathrm{Ob}\,\mathcal{N}$ (closed under taking summands).

Let $\mathcal{N}$ be a thick subcategory of $\mathcal{C}$. By Lemma 23, conditions (1) and (2) of Definition 31 yield that $\mathcal{N}$ is a Heller triangulated subcategory of $\mathcal{C}$.

Let $M(\mathcal{N}) := \{(X \xrightarrow{f} Y) \in \mathcal{C} \ : \ \text{the cone of } f \text{ is in } \mathrm{Ob}\,\mathcal{N}\}$. An element of $M(\mathcal{N})$ is called an $M(\mathcal{N})$-*isomorphism* or often just an $\mathcal{N}$-*isomorphism* (not to be confused with "an isomorphism in $\mathcal{N}$"). If $\mathcal{N}$ is unambiguous, then an $\mathcal{N}$-isomorphism is denoted by $X \Longrightarrow Y$. For instance, $X \Longrightarrow 0$ if and only if $0 \Longrightarrow X$ if and only if $X \in \mathrm{Ob}\,\mathcal{N}$.

Lemma 32 *The subset $M(\mathcal{N})$ of $\mathcal{N}$-isomorphisms in $\mathcal{C}$ is a multiplicative system in $\mathcal{C}$ in the sense of Definition* 45.

Proof. Ad (Fr 2). Suppose given $X_{1/0} \xrightarrow{x} X_{2/0} \xrightarrow{x} X_{3/0} \xrightarrow{x} X_{4/0}$ such that $X_{1/0} \xrightarrow{x} X_{3/0}$ and $X_{2/0} \xrightarrow{x} X_{4/0}$ are M-isomorphisms. We complete to a 4-triangle $X \in \mathrm{Ob}\,\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_n^{\#})$ using closedness of $\mathcal{C}$; cf. Lemma 20. By [**8,** Lem. 3.4.(1, 6)], we have $X_{3/1}$, $X_{4/2} \in \mathrm{Ob}\,\mathcal{N}$. We have to show that $X_{2/1}$, $X_{3/2}$, $X_{4/3}$, $X_{4/1} \in \mathrm{Ob}\,\mathcal{N}$. Let the periodic monotone map $\bar{\Delta}_5 \xrightarrow{p} \bar{\Delta}_4$ be defined by $0p := 1$, $1p := 1$, $2p := 2$, $3p := 3$, $4p := 4$ and $5p := 4$. The 2-triangle $Xp^{\#}\mathfrak{f}_2 \in \mathrm{Ob}\,\mathcal{C}^{+,\vartheta=1}(\bar{\Delta}_2^{\#})$ is given by



cf. [**8,** Lem. 3.4.(1, 2), §1.2.1.2, §1.2.2.2.]. Since $X_{3/1}$, $X_{4/2} \in \mathrm{Ob}\,\mathcal{N}$, and since $\mathcal{N}$ is closed under cones, we have $X_{4/3} \oplus X_{1+1/2} \in \mathrm{Ob}\,\mathcal{N}$. Since $\mathcal{N}$ is closed under summands and under shift, we obtain $X_{4/3}$, $X_{2/1} \in \mathrm{Ob}\,\mathcal{N}$. Since $\mathcal{N}$ is closed under cones and under shift, $X_{4/1} \in \mathrm{Ob}\,\mathcal{N}$ ensues. Considering X again, since $\mathcal{N}$ is closed under cones, we finally obtain $X_{3/2} \in \mathrm{Ob}\,\mathcal{N}$.

Ad (Fr 3). Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}$ such that there exists $Y \xRightarrow{s} Z$ with $fs = 0$. We obtain a factorisation $(X \xrightarrow{f} Y) = (X \xrightarrow{u} N \xrightarrow{v} Y)$ with $N \in \mathrm{Ob}\,\mathcal{N}$. Completing $(T_{1/0} \xrightarrow{t} T_{2/0} \xrightarrow{t} T_{3/0}) := (X \xrightarrow{u} N \xrightarrow{v} Y)$ to a 3-triangle by Lemma 20, we obtain $T_{1/2^{-1}} \xRightarrow{t} T_{1/0}$, which composes to zero with $(T_{1/0} \xrightarrow{t} T_{3/0}) = (X \xrightarrow{f} Y)$.

Ad (Fr 4). Suppose given

$$\begin{array}{ccc} & & X' \\ & & \uparrow \\ X & \Longrightarrow & Y \end{array}$$

in $\mathcal{C}$. Prolonging $X \longrightarrow X'$ to a 2-triangle (X'', X, X'), then completing $X'' \longrightarrow X \Longrightarrow Y$ to a 3-triangle using Lemma 20, we obtain, by [**8,** Lem. 3.4.(6)], a 3-triangle T with $(T_{2/0} \xrightarrow{t} T_{3/0}) = (X \Longrightarrow Y)$ and $(T_{2/0} \xrightarrow{t} T_{2/1}) = (X \longrightarrow X')$. Then $T_{3/2} \in \mathrm{Ob}\,\mathcal{N}$, whence $T_{2/1} \Longrightarrow T_{3/1}$. The weak square $(T_{2/0}, T_{3/0}, T_{2/1}, T_{3/1})$ is a completion as sought. ▫

> Note that if $X \in (\mathrm{Ob}\,\mathcal{C}) \setminus (\mathrm{Ob}\,\mathcal{N})$, then $(0, 0, 0, X)$ is a weak square in which $0 \longrightarrow 0$ is an $\mathcal{N}$-isomorphism, but $0 \longrightarrow X$ is not.

The localisation of $\mathcal{C}$ at $M(\mathcal{N})$, defined as in §A.4, is also called the *localisation of* $\mathcal{C}$ *at* $\mathcal{N}$, and also written $\mathcal{C}/\!\!/\mathcal{N} := \mathcal{C}_{M(\mathcal{N})}$. Concerning the *localisation functor* $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$, we refer to §A.4.

Recall that an additive functor between weakly abelian categories is called subexact if it induces an exact functor on the Freyd categories; cf. [**8,** §1.2.1.3]; cf. also Lemma 41.

Lemma 33 *The category $\mathcal{C}/\!\!/\mathcal{N}$ is weakly abelian. The functor $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$ is subexact.*

Proof. By Remark 51, the category $\mathcal{C}/\!\!/\mathcal{N}$ is additive, and the localisation functor $\mathsf{L} : \mathcal{C} \longrightarrow \mathcal{C}/\!\!/\mathcal{N}$ is additive. We claim that L maps weak kernels to weak kernels. Let $X \xrightarrow{f} Y$ be a weak kernel of $Y \xrightarrow{g} Z$ in $\mathcal{C}$. We claim that it remains a weak kernel in $\mathcal{C}/\!\!/\mathcal{N}$. Suppose given a morphism $T \xrightarrow{t} Y$ in $\mathcal{C}$ such that $tg = 0$ in $\mathcal{C}/\!\!/\mathcal{N}$, which we, by isomorphic replacement, may assume given. Let $T' \xRightarrow{s} T$ be such that $stg = 0$ in $\mathcal{C}$; cf. Remark 47. Since f is a weak kernel of g in $\mathcal{C}$, we have a factorisation $st = uf$. Hence $t = (s^{-1}u)f$ is a factorisation of t over f in $\mathcal{C}/\!\!/\mathcal{N}$.

Substituting isomorphically in $\mathcal{C}/\!\!/\mathcal{N}$ and using duality, for $\mathcal{C}/\!\!/\mathcal{N}$ to be weakly abelian, it suffices to show that each morphism $X \xrightarrow{f} Y$ has a weak kernel resp. is a weak kernel in $\mathcal{C}/\!\!/\mathcal{N}$. But by the property of L just shown, we may use a weak kernel of f in $\mathcal{C}$ resp. a morphism f is a weak kernel of in $\mathcal{C}$. ▫

Remark 34 *The category $\mathcal{C}/\!\!/\mathcal{N}$ carries a shift automorphism $\mathcal{C}/\!\!/\mathcal{N} \xrightarrow{\mathsf{T}} \mathcal{C}/\!\!/\mathcal{N}$, $f/t \longmapsto f^{+1}/t^{+1}$. We have $\mathsf{L}\,\mathsf{T} = \mathsf{T}\,\mathsf{L}$.*

Proof. This functor is welldefined since $\mathcal{N}$, and hence $M(\mathcal{N})$, is closed under shift in $\mathcal{C}$. Likewise, its inverse $f/t \longmapsto f^{-1}/t^{-1}$ is welldefined. ▫

Lemma 35 *Suppose given a Heller triangulated category $(\mathcal{D}, \mathsf{T}, \theta)$.*

Suppose given a weakly abelian category $\mathcal{D}'$ and an automorphism $\mathcal{D}' \xrightarrow{\mathsf{T}'} \mathcal{D}'$. Suppose given a subexact additive functor $\mathcal{D} \xrightarrow{G} \mathcal{D}'$ strictly compatible with shift, i.e. $G\,\mathsf{T}' = \mathsf{T}\,G$. Suppose that $\mathcal{D}(\dot{\Delta}_n) \xrightarrow{G(\dot{\Delta}_n)} \mathcal{D}'(\dot{\Delta}_n)$ is 1-epimorphic for $n \geqslant 0$.

Then the functor $\underline{\mathcal{D}^+(\bar{\Delta}_n^{\#})} \xrightarrow{G^+(\bar{\Delta}_n^{\#})} \underline{\mathcal{D}'^+(\bar{\Delta}_n^{\#})}$ is 1-epimorphic.

Moreover, there exists a unique Heller triangulation θ' on $(\mathcal{D}', \mathsf{T}')$ such that $\mathcal{D} \xrightarrow{G} \mathcal{D}'$ is strictly exact; cf. [**8,** Def. 1.5].

Proof. Given $n \geqslant 0$. Since the residue class functors $\mathcal{D}(\dot{\Delta}_n) \longrightarrow \underline{\mathcal{D}(\dot{\Delta}_n)}$ and $\mathcal{D}'(\dot{\Delta}_n) \longrightarrow \underline{\mathcal{D}'(\dot{\Delta}_n)}$ are full and dense, they are 1-epimorphic by [**8,** Cor. A.37] ; concerning notation, cf. [**8,** §2.4]. The commutative quadrangle

$$\begin{array}{ccc} \mathcal{D}(\dot{\Delta}_n) & \xrightarrow{G(\dot{\Delta}_n)} & \mathcal{D}'(\dot{\Delta}_n) \\ \downarrow & & \downarrow \\ \underline{\mathcal{D}(\dot{\Delta}_n)} & \xrightarrow{\underline{G(\dot{\Delta}_n)}} & \underline{\mathcal{D}'(\dot{\Delta}_n)} \end{array}$$

shows that $\underline{G(\dot{\Delta}_n)}$ is 1-epimorphic. Restriction induces equivalences $\underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} \xrightarrow{(-)|_{\dot{\Delta}n}} \underline{\mathcal{D}(\dot{\Delta}_n)}$ and $\underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)} \xrightarrow{(-)|_{\dot{\Delta}n}} \underline{\mathcal{D}'(\dot{\Delta}_n)}$ by [**8,** Prop. 2.6]. Therefore, the commutative quadrangle

$$\begin{array}{ccc} \underline{\mathcal{D}(\dot{\Delta}_n)} & \xrightarrow{\underline{G(\dot{\Delta}_n)}} & \underline{\mathcal{D}'(\dot{\Delta}_n)} \\ {\scriptstyle (-)|_{\dot{\Delta}n}}\uparrow & & \uparrow{\scriptstyle (-)|_{\dot{\Delta}n}} \\ \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} & \xrightarrow{\underline{G^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)} \end{array}$$

shows that $\underline{G^+(\bar{\Delta}_n^\#)}$ is 1-epimorphic; concerning notation, cf. [**8,** §1.2.1.1, §1.2.1.3]. Therefore, we may define a transformation θ'_n for $\mathcal{D}'$ by the requirement that

$$\begin{array}{ccc} \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} & \xrightarrow{\underline{G^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)} \\ {\scriptstyle [-]^{+1}}\Big(\Rightarrow{\scriptstyle \theta_n}\Big){\scriptstyle [-^{+1}]} & & {\scriptstyle [-]^{+1}}\Big(\Rightarrow{\scriptstyle \theta'_n}\Big){\scriptstyle [-^{+1}]} \\ \underline{\mathcal{D}^+(\bar{\Delta}_n^\#)} & \xrightarrow{\underline{G^+(\bar{\Delta}_n^\#)}} & \underline{\mathcal{D}'^+(\bar{\Delta}_n^\#)} \end{array}$$

be commutative, i.e. that $\theta_n \star \underline{G^+(\bar{\Delta}_n^\#)} = \underline{G^+(\bar{\Delta}_n^\#)} \star \theta'_n$. In other words, there exists a unique θ'_n making this diagram commutative.

Let $\theta' := (\theta'_n)_{n\geqslant 0}$, where for $n = 0$, we make use of $\underline{\mathcal{D}'^+(\bar{\Delta}_0^\#)} = 0$. We claim that θ' is a Heller triangulation on $(\mathcal{D}', \mathsf{T}')$, i.e. that $(\mathcal{D}', \mathsf{T}', \theta')$ is a Heller triangulated category. Once this is proven, we see that by construction, $\mathcal{D} \xrightarrow{G} \mathcal{D}'$ is strictly exact; cf. [**8,** Def. 1.5.(iii)].

Suppose given $m, n \geqslant 0$ and a periodic monotone map $\bar{\Delta}_n \xleftarrow{p} \bar{\Delta}_m$. To prove that $\underline{p}^\# \star \theta'_m \overset{!}{=} \theta'_n \star \underline{p}^\#$, we may precompose with the 1-epimorphic functor $\underline{G^+(\bar{\Delta}_n^\#)}$ to obtain

$$\begin{array}{rcccccl} \underline{G^+(\bar{\Delta}_n^\#)} \star \underline{p}^\# \star \theta'_m & = & \underline{p}^\# \star \underline{G^+(\bar{\Delta}_m^\#)} \star \theta'_m & = & \underline{p}^\# \star \theta_m \star \underline{G^+(\bar{\Delta}_m^\#)} & & \\ & \underset{\text{Heller triangulated}}{\overset{(\mathcal{D},\mathsf{T},\theta)}{=}} & \theta_n \star \underline{p}^\# \star \underline{G^+(\bar{\Delta}_m^\#)} & = & \theta_n \star \underline{G^+(\bar{\Delta}_n^\#)} \star \underline{p}^\# & = & \underline{G^+(\bar{\Delta}_n^\#)} \star \theta'_n \star \underline{p}^\# \ . \end{array}$$

Suppose given $n \geqslant 0$. To prove that $\underline{\mathfrak{f}}_n \star \theta'_{n+1} \overset{!}{=} \theta'_{2n+1} \star \underline{\mathfrak{f}}_n$, we may precompose with the

1-epimorphic functor $\underline{G^+(\bar{\Delta}^{\#}_{2n+1})}$ to obtain

$$\begin{aligned}
\underline{G^+(\bar{\Delta}^{\#}_{2n+1})} \star \underline{\mathfrak{f}}_n \star \theta'_{n+1} &= \underline{\mathfrak{f}}_n \star \underline{G^+(\bar{\Delta}^{\#}_{n+1})} \star \theta'_{n+1} \\
&= \underline{\mathfrak{f}}_n \star \theta_{n+1} \star \underline{G^+(\bar{\Delta}^{\#}_{n+1})} \\
&\overset{(\mathcal{D},\mathsf{T},\theta)}{\underset{\text{Heller triangulated}}{=}} \theta_{2n+1} \star \underline{\mathfrak{f}}_n \star \underline{G^+(\bar{\Delta}^{\#}_{n+1})} \\
&= \theta_{2n+1} \star \underline{G^+(\bar{\Delta}^{\#}_{2n+1})} \star \underline{\mathfrak{f}}_n \;=\; \underline{G^+(\bar{\Delta}^{\#}_{2n+1})} \star \theta'_{2n+1} \star \underline{\mathfrak{f}}_n \,.
\end{aligned}$$

□

Proposition 36 *Recall that $(\mathcal{C}, \mathsf{T}, \vartheta)$ is a closed Heller triangulated category, and that $\mathcal{N}$ is a thick subcategory of $\mathcal{C}$.*

There exists a unique Heller triangulation θ on $(\mathcal{C}/\!\!/\mathcal{N}, \mathsf{T})$ such that $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$ is strictly exact; cf. [**8,** Def. 1.5].

Then $(\mathcal{C}/\!\!/\mathcal{N}, \mathsf{T}, \theta)$ is a closed Heller triangulated category; cf. Definition 14.

Proof. By Lemma 33, the category $\mathcal{C}/\!\!/\mathcal{N}$ is weakly abelian, and $\mathsf{L} : \mathcal{C} \longrightarrow \mathcal{C}/\!\!/\mathcal{N}$ is subexact. By Remark 34, $\mathcal{C}/\!\!/\mathcal{N}$ carries a shift automorphism, and L is compatible with the shift automorphisms on $\mathcal{C}$ and on $\mathcal{C}/\!\!/\mathcal{N}$. By Lemma 50, the functor $\mathcal{C}(\dot{\Delta}_n) \xrightarrow{\mathsf{L}(\dot{\Delta}_n)} (\mathcal{C}/\!\!/\mathcal{N})(\dot{\Delta}_n)$ is 1-epimorphic for $n \geqslant 0$. Therefore, existence and uniqueness of θ follow by Lemma 35.

It remains to be shown that $\mathcal{C}/\!\!/\mathcal{N}$ is closed. By isomorphic substitution, it suffices to show that each morphism in the image of L has a cone in $\mathcal{C}/\!\!/\mathcal{N}$; cf. [**8,** Lem. 3.4.(6)]. But this follows from $\mathcal{C}$ being closed and from L being strictly exact. □

An object $(X \xrightarrow{x} X')$ of the Freyd category $\hat{\mathcal{C}}$ is called *$\mathcal{N}$-zero* if x factors over an object of $\mathcal{N}$; concerning $\hat{\mathcal{C}}$, cf. [**8,** §A.6.3]. Note that an object of $\hat{\mathcal{C}}$ that is isomorphic to a summand of an $\mathcal{N}$-zero object is itself $\mathcal{N}$-zero.

Remark 37 *A morphism in $\mathcal{C}$ is an $\mathcal{N}$-isomorphism if and only if its kernel and its cokernel, taken in $\hat{\mathcal{C}}$, are $\mathcal{N}$-zero.*

Note that this criterion does not make reference to the Heller triangulated structure on $\mathcal{C}$, but only to the fact that $\mathcal{C}$ is weakly abelian. One might ask for conditions on $\mathcal{N}$ that only use weak abelianess of $\mathcal{C}$, and that nonetheless suffice to turn $\mathcal{C}_{M(\mathcal{N})}$ into a weakly abelian category – where now $M(\mathcal{N})$ is the subset of morphisms of $\mathcal{C}$ defined by the criterion given in Remark 37.

Proof of Remark 37. Suppose that $X \xrightarrow{f} Y$ is an $\mathcal{N}$-isomorphism in $\mathcal{C}$. Then it has a weak kernel N and a weak cokernel M in $\mathrm{Ob}\,\mathcal{N}$. By construction of the kernel in $\hat{\mathcal{C}}$, it is of the form $(N \longrightarrow X)$. Dually, the cokernel is of the form $(Y \longrightarrow M)$; cf. [**8,** §A.6.3].

Conversely, suppose that the kernel and the cokernel of the morphism $X \xrightarrow{f} Y$, taken in $\hat{\mathcal{C}}$, are $\mathcal{N}$-zero. Consider the exact functor $\hat{\mathcal{C}} \xrightarrow{\hat{\mathsf{L}}} (\mathcal{C}/\!\!/\mathcal{N})\hat{}$ that prolongs L on the level of Freyd categories. It maps f to an isomorphism, since in the abelian category $(\mathcal{C}/\!\!/\mathcal{N})\hat{}$, the image of f has zero kernel and zero cokernel. Since $\mathcal{C}/\!\!/\mathcal{N} \longrightarrow (\mathcal{C}/\!\!/\mathcal{N})\hat{}$ is full and faithful, the image of f under L in $\mathcal{C}/\!\!/\mathcal{N}$ is an isomorphism, too. Hence f is an $\mathcal{N}$-isomorphism in $\mathcal{C}$; cf. Remark 46. □

Proposition 38 (universal property) *Recall that $(\mathcal{C}, \mathsf{T}, \vartheta)$ is a closed Heller triangulated category, and that $\mathcal{N}$ is a thick subcategory of $\mathcal{C}$.*

Let θ be the unique Heller triangulation on $(\mathcal{C}/\!\!/\mathcal{N}, \mathsf{T})$ such that the localisation functor $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}/\!\!/\mathcal{N}$ is strictly exact; cf. Proposition 36*. Suppose given a Heller triangulated category $(\mathcal{C}', \mathsf{T}', \vartheta')$.*

Recall that we write $[\![\mathcal{C}, \mathcal{C}']\!]_{\mathrm{ex}}$ for the category of exact functors and periodic transformations from $\mathcal{C}$ to $\mathcal{C}'$; cf. Definition 6*.*

Write $[\![\mathcal{C}, \mathcal{C}']\!]_{\mathrm{ex},\,\mathcal{N}} \subseteq [\![\mathcal{C}, \mathcal{C}']\!]_{\mathrm{ex}}$ for the full subcategory consisting of exact functors (F, a) such that $NF \simeq 0$ for all $N \in \mathrm{Ob}\,\mathcal{N}$.

Recall that we write $[\![\mathcal{C}, \mathcal{C}']\!]_{\mathrm{st\,ex}}$ for the category of strictly exact functors and periodic transformations from $\mathcal{C}$ to $\mathcal{C}'$; cf. Definition 6*.*

Write $[\![\mathcal{C}, \mathcal{C}']\!]_{\mathrm{st\,ex},\,\mathcal{N}} \subseteq [\![\mathcal{C}, \mathcal{C}']\!]_{\mathrm{st\,ex}}$ for the full subcategory consisting of strictly exact functors F such that $NF \simeq 0$ for all $N \in \mathrm{Ob}\,\mathcal{N}$.

(1) *We have a strictly dense equivalence*

$$\begin{array}{rcl} [\![\mathcal{C}, \mathcal{C}']\!]_{\mathrm{ex},\,\mathcal{N}} & \xleftarrow{\mathsf{L}\star(-)} & [\![\mathcal{C}/\!\!/\mathcal{N},\, \mathcal{C}']\!]_{\mathrm{ex}} \\ (\mathsf{L}\star G\,,\, \mathsf{L}\star b) \;=\; (\mathsf{L}, 1)\star(G, b) & \longmapsto & (G, b)\,. \end{array}$$

(2) *We have a strictly dense equivalence*

$$\begin{array}{rcl} [\![\mathcal{C}, \mathcal{C}']\!]_{\mathrm{st\,ex},\,\mathcal{N}} & \xleftarrow{\mathsf{L}\star(-)} & [\![\mathcal{C}/\!\!/\mathcal{N},\, \mathcal{C}']\!]_{\mathrm{st\,ex}} \\ \mathsf{L}\star G & \longmapsto & G\,. \end{array}$$

Proof.

Ad (1). Welldefinedness of the functor $\mathsf{L}\star(-)$ follows from L being strictly exact and exact functors being stable under composition; cf. Proposition 36, Remark 3.

We make use of the universal property of the localisation to the extent stated in Remark 51.

Suppose given exact functors $\mathcal{C} \underset{(G,b)}{\overset{(F,a)}{\rightrightarrows}} \mathcal{C}'$ and a periodic transformation $F \xrightarrow{u} G$.

Let $\breve{F} : \mathcal{C}/\!\!/\mathcal{N} \longrightarrow \mathcal{C}'$ be defined by $\mathsf{L}\star\breve{F} := F$. Let $\breve{G} : \mathcal{C}/\!\!/\mathcal{N} \longrightarrow \mathcal{C}'$ be defined by $\mathsf{L}\star\breve{G} := G$.

Recall that the shift on $\mathcal{C}/\!\!/\mathcal{N}$ is, abusively, also denoted by T, so that $\mathsf{T}\star\mathsf{L} = \mathsf{L}\star\mathsf{T}$. Let the transformations $\breve{a}$ and $\breve{b}$ be defined by

$$\begin{array}{rcl} \mathsf{L}\star(\mathsf{T}\star\breve{F} \xrightarrow{\breve{a}} \breve{F}\star\mathsf{T}') & := & (\mathsf{T}\star F \xrightarrow{a} F\star\mathsf{T}') \\ \mathsf{L}\star(\mathsf{T}\star\breve{G} \xrightarrow{\breve{b}} \breve{G}\star\mathsf{T}') & := & (\mathsf{T}\star G \xrightarrow{b} G\star\mathsf{T}')\,. \end{array}$$

Let the transformation $\breve{F} \xrightarrow{\breve{u}} \breve{G}$ be defined by

$$\mathsf{L}\star(\breve{F} \xrightarrow{\breve{u}} \breve{G}) \;:=\; (F \xrightarrow{u} G)\,.$$

We have to show that $(\breve{F}, \breve{a})$ is exact and that $\breve{u}$ is periodic.

Ad $\breve{F}$ *exact.* Since $X\breve{a} = X\,\mathsf{L}\,\breve{a} = Xa$ is an isomorphism for $X \in \mathrm{Ob}\,\mathcal{C}/\!\!/\mathcal{N} = \mathrm{Ob}\,\mathcal{C}$, the transformation a is an isotransformation.

To show that $\breve{F}$ is subexact, by Lemma 41, it suffices to show that given a morphism f in $\mathcal{C}/\!\!/\mathcal{N}$, it has a weak cokernel that is preserved by $\breve{F}$. By isomorphic substitution, we may assume that $f = f'\,\mathsf{L}$ for some morphism f' in $\mathcal{C}$. Let (f', g', h') be a 2-triangle in $\mathcal{C}$; cf. Lemma 20. Since L is strictly exact, the 2-triangle $(f, g'\,\mathsf{L}, h'\,\mathsf{L})$ results. In particular, $g'\,\mathsf{L}$ is a weak cokernel of f. Since F is subexact, $g'F = g'\,\mathsf{L}\,\breve{F}$ is a weak cokernel of $f'F = f'\,\mathsf{L}\,\breve{F} = f\breve{F}$.

Suppose given $n \geqslant 0$. We shall make use of the abbreviation $\underline{F} = \underline{F^{+}(\bar{\Delta}_n^{\#})}$, etc. It remains to show that

$$(\theta_n \star \underline{\breve{F}}) \cdot \underline{\breve{a}} \stackrel{!}{=} \underline{\breve{F}} \star \vartheta'_n \,.$$

Since $\underline{\mathsf{L}} = \underline{\mathsf{L}^{+}(\bar{\Delta}_n^{\#})}$ is 1-epimorphic by Lemmata 50 and 35, it suffices to show that

$$\underline{\mathsf{L}} \star ((\theta_n \star \underline{\breve{F}}) \cdot \underline{\breve{a}}) \stackrel{!}{=} \underline{\mathsf{L}} \star \underline{\breve{F}} \star \vartheta'_n \,.$$

In fact,

$$\begin{aligned}
\underline{\mathsf{L}} \star ((\theta_n \star \underline{\breve{F}}) \cdot \underline{\breve{a}}) &= (\underline{\mathsf{L}} \star \theta_n \star \underline{\breve{F}}) \cdot (\underline{\mathsf{L}} \star \underline{\breve{a}}) \\
&\stackrel{\mathsf{L}\text{ ex.}}{=} (\vartheta_n \star \underline{\mathsf{L}} \star \underline{\breve{F}}) \cdot (\underline{\mathsf{L}} \star \underline{\breve{a}}) \\
&= (\vartheta_n \star \underline{F}) \cdot \underline{a} \\
&\stackrel{(F,a)\text{ ex.}}{=} \underline{F} \star \vartheta'_n \\
&= \underline{\mathsf{L}} \star \underline{\breve{F}} \star \vartheta'_n \,.
\end{aligned}$$

Ad $\breve{u}$ *periodic.* We have to show that

$$(\mathsf{T} \star \breve{u}) \cdot \breve{b} \stackrel{!}{=} \breve{a} \cdot (\breve{u} \star \mathsf{T}')$$

as transformations from $\mathsf{T} \star \breve{F}$ to $\breve{G} \star \mathsf{T}'$. By Remark 51, it suffices to show that

$$\mathsf{L} \star ((\mathsf{T} \star \breve{u}) \cdot \breve{b}) \stackrel{!}{=} \mathsf{L} \star (\breve{a} \cdot (\breve{u} \star \mathsf{T}')) \,.$$

In fact,

$$\begin{aligned}
\mathsf{L} \star ((\mathsf{T} \star \breve{u}) \cdot \breve{b}) &= (\mathsf{L} \star \mathsf{T} \star \breve{u}) \cdot (\mathsf{L} \star \breve{b}) \\
&= (\mathsf{T} \star \mathsf{L} \star \breve{u}) \cdot (\mathsf{L} \star \breve{b}) \\
&= (\mathsf{T} \star u) \cdot b \\
&\stackrel{u\text{ per.}}{=} a \cdot (u \star \mathsf{T}') \\
&= (\mathsf{L} \star \breve{a}) \cdot (\mathsf{L} \star \breve{u} \star \mathsf{T}') \\
&= \mathsf{L} \star (\breve{a} \cdot (\breve{u} \star \mathsf{T}')) \,.
\end{aligned}$$

Ad (2). Welldefinedness of the functor $\mathsf{L} \star (-)$ follows from L being strictly exact and strictly exact functors being stable under composition; cf. Proposition 36, Remark 3.

Keep the notation of the proof of (1). Given an exact functor (F, a) from $\mathcal{C}$ to $\mathcal{C}'$, we infer from $a = 1$, using $\mathsf{L} \star \breve{a} = a$, that $\breve{a} = 1$. ∎

A Some general assertions

This appendix serves as a tool kit consisting of known results and folklore lemmata. We do not claim originality.

A.1 Remarks on coretractions and retractions

Remark 39 *Let $\mathcal{A}$ be a category.*

Suppose given X, Z in $\mathrm{Ob}\,\mathcal{A}$, and morphisms $X \xrightarrow{i} Z \xrightarrow{p} X$ such that $ip = 1_X$.

Suppose given Y, W in $\mathrm{Ob}\,\mathcal{A}$, and morphisms $Y \xrightarrow{j} W \xrightarrow{q} Y$ such that $jq = 1_Y$.

Suppose given $X \xrightarrow{u} Y$ in $\mathcal{A}$. Let $Z \xrightarrow{v} W$ be defined by $v := puj$. Then $vq = pu$ and $iv = uj$.

$$\begin{array}{ccccc} Z & \xrightarrow{p} & X & \xrightarrow{i} & Z \\ {\scriptstyle v}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ W & \xrightarrow{q} & Y & \xrightarrow{j} & W \end{array}$$

Proof. We have $vq = pujq = pu$ and $iv = ipuj = uj$. ∎

Remark 40 *Let $\mathcal{A}$ be a category. Suppose given Z, X, Z', W, Y, $W' \in \mathrm{Ob}\,\mathcal{A}$.*

Suppose given morphisms $X \xrightarrow{i} Z \xrightarrow{p} X$ such that $ip = 1_X$.

Suppose given morphisms $X \xrightarrow{i'} Z' \xrightarrow{p'} X$ such that $i'p' = 1_X$.

Suppose given morphisms $Y \xrightarrow{j} W \xrightarrow{q} Y$ such that $jq = 1_Y$.

Suppose given morphisms $Y \xrightarrow{j'} W' \xrightarrow{q'} Y$ such that $j'q' = 1_Y$.

Suppose given $Z \xrightarrow{v} W$ and $Z' \xrightarrow{v'} W'$ such that $pi'v' = vqj'$.

Then there exists a unique morphism $X \xrightarrow{u} Y$ in $\mathcal{A}$ such that $vq = pu$ and $i'v' = uj'$.

$$\begin{array}{ccccc} Z & \xrightarrow{p} & X & \xrightarrow{i'} & Z' \\ {\scriptstyle v}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v'} \\ W & \xrightarrow{q} & Y & \xrightarrow{j'} & W' \end{array}$$

If v and v' are isomorphisms, so is u.

Proof. Uniqueness follows from p being epic and j' being monic.

For existence, we let $u := ivq = i'v'q'$, the latter equality holding because of $pivqj' = pipi'v' = pi'v' = vqj' = vqj'q'j' = pi'v'q'j'$, using p epic and j' monic. Then $pu = pi'v'q' = vqj'q' = vq$ and $uj' = ivqj' = ipi'v' = i'v'$.

If v and v' are isomorphisms, then let $u' := jv^-p = j'v'^-p'$ to get $uu' = ivqj'v'^-p' = ipi'v'v'^-p' = 1$ and $u'u = jv^-pi'v'q' = jv^-vqj'q' = 1$, so that $u' = u^-$. In particular, u is an isomorphism. ∎

A.2 Two lemmata on subexact functors

Suppose given weakly abelian categories $\mathcal{A}$ and $\mathcal{A}'$; cf. e.g. [**8,** Def. A.26.(3)]. Suppose given an additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$. Recall that F is called subexact if the induced functor $\hat{F} : \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{A}}'$ on the Freyd categories is exact; cf. [**8,** §1.2.1.3].

Lemma 41 *The following assertions* $(1, 2, 3, 3^\circ, 4, 4^\circ)$ *are equivalent.*

(1) *The functor* F *is subexact.*

(2) *The functor* F *preserves weak kernels and weak cokernels.*

(3) *The functor* F *preserves weak kernels.*

(3°) *The functor* F *preserves weak cokernels.*

(4) *For each morphism* $X \xrightarrow{t} Y$ *in* $\mathcal{A}$*, there exists a weak kernel* $W \xrightarrow{w} X$ *such that* wF *is a weak kernel of* tF*.*

(4°) *For each morphism* $X \xrightarrow{t} Y$ *in* $\mathcal{A}$*, there exists a weak cokernel* $Y \xrightarrow{w'} W'$ *such that* $w'F$ *is a weak cokernel of* tF*.*

Proof. Ad (1) ⇒ (4). Suppose given a morphism $X \xrightarrow{t} Y$ in $\mathcal{A}$. Let $K \overset{i}{\rightarrowtail} X$ be a kernel of t in $\hat{\mathcal{A}}$. Choose $A \overset{b}{\twoheadrightarrow} K$ with $A \in \operatorname{Ob}\mathcal{A}$. Since $\hat{F}$ is exact, $A\hat{F} \xrightarrow{(bi)\hat{F}} X\hat{F} \xrightarrow{t\hat{F}} Y\hat{F}$ is exact at $X\hat{F}$. So $(bi)\hat{F} = (bi)F$ is a weak kernel of $t\hat{F} = tF$ in $\mathcal{A}'$.

Ad (4) ⇒ (3). Given a morphism $X \xrightarrow{t} Y$ in $\mathcal{A}$ and a weak kernel $W \xrightarrow{w} X$, a morphism $V \xrightarrow{v} X$ is a weak kernel of t if and only if both (w factors over v) and (v factors over w). So if wF is a weak kernel of tF, so is vF. Consequently, if F preserves a single weak kernel of t, it preserves all of them.

Ad (3) ⇒ (2). This follows by [**8,** Rem. A.27].

Ad (2) ⇒ (1). Using duality and uniqueness of the kernel up to isomorphism, it suffices to show that $\hat{F}$ maps a chosen kernel of a given morphism to a kernel of its image under $\hat{F}$. Since F preserves weak kernels, this follows by construction of a kernel; cf. e.g. [**8,** §A.6.3, item (1) before Rem. A.27]. □

Lemma 42 *Suppose that* $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ *is subexact. Suppose given a functor* $\mathcal{C} \xleftarrow{G} \mathcal{C}'$*.*

(1) *If* $G \dashv F$*, then* G *is subexact.*

(1°) *If* $F \dashv G$*, then* G *is subexact.*

Proof. Ad (1). As an adjoint functor between additive categories, G is additive.

Let $1 \xrightarrow{\varepsilon} GF$ be a unit and $FG \xrightarrow{\eta} 1$ a counit of the adjunction $G \dashv F$.

By Lemma 41, it suffices to show that G preserves weak cokernels. Suppose given $X' \xrightarrow{u} X \xrightarrow{v} X''$ such that v is a weak cokernel of u. We have to show that Gv is a weak cokernel of Gu. Suppose given $t : XG \longrightarrow T$ such that $uG \cdot t = 0$. Then

$$u \cdot X\varepsilon \cdot tF \;=\; X'\varepsilon \cdot uGF \cdot tF \;=\; X'\varepsilon \cdot (uG \cdot t)F \;=\; 0\,.$$

Since v is a weak cokernel of u, we obtain a morphism $s : X'' \longrightarrow TF$ such that $v \cdot s = X\varepsilon \cdot tF$. Then

$$vG \cdot (sG \cdot T\eta) \;=\; X\varepsilon G \cdot tFG \cdot T\eta \;=\; X\varepsilon G \cdot XG\eta \cdot t \;=\; t\,.$$

□

A.3 Karoubi hull

The construction of the Karoubi hull is due to Karoubi; cf. [**6,** III.II].

Suppose given an additive category $\mathcal{A}$. The *Karoubi hull* $\tilde{\mathcal{A}}$ has

$$\operatorname{Ob}\tilde{\mathcal{A}} \;:=\; \{\,(A, e) \,:\, A \in \operatorname{Ob}\mathcal{A},\, e \in {}_{\mathcal{A}}(A, A) \text{ with } e^2 = e\,\}$$

and, given (A, e), $(B, f) \in \operatorname{Ob} \tilde{\mathcal{A}}$,

$$\mathcal{A}\big((A, e), (B, f)\big) \;:=\; \{\, u \in \mathcal{A}(A, B) \,:\, e \cdot u \cdot f = u \,\} \,.$$

Then $\tilde{\mathcal{A}}$ is an additive category, in which all idempotents are split.

Composition is inherited from $\mathcal{A}$. We have a full and faithful additive functor

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathsf{K}} & \tilde{\mathcal{A}} \\ (X \xrightarrow{u} Y) & \longmapsto & \big((X,1) \xrightarrow{u} (Y,1)\big) \,, \end{array}$$

which we often consider as an inclusion of a full subcategory.

Suppose given an additive category $\mathcal{B}$ in which all idempotents are split.

Remark 43 Write $[\![\mathcal{A}, \mathcal{B}]\!]_{\mathrm{add}}$ for the category of additive functors and transformations between such from $\mathcal{A}$ to $\mathcal{B}$. The induced functor $[\![\mathcal{A}, \mathcal{B}]\!] \xleftarrow{\mathsf{K}\star(-)} [\![\tilde{\mathcal{A}}, \mathcal{B}]\!]$ restricts to a strictly dense equivalence

$$[\![\mathcal{A}, \mathcal{B}]\!]_{\mathrm{add}} \xleftarrow{\mathsf{K}\star(-)} [\![\tilde{\mathcal{A}}, \mathcal{B}]\!]_{\mathrm{add}} \,.$$

Lemma 44 Suppose given an additive functor $\mathcal{A} \xrightarrow{I} \mathcal{A}'$ to an additive category $\mathcal{A}'$ in which all idempotents split. By Remark 43, we obtain a functor $J : \tilde{\mathcal{A}} \longrightarrow \mathcal{A}'$, unique up to isomorphism, such that the following triangle of functors commutes.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{I} & \mathcal{A}' \\ {\scriptstyle \mathsf{K}}\downarrow & \nearrow_{J} & \\ \tilde{\mathcal{A}} & & \end{array}$$

If I is full and faithful, and if every object of $\mathcal{A}'$ is a direct summand of an object in the image of I, then J is an equivalence.

By abuse of notation, in the situation of Lemma 44, we also write $\tilde{\mathcal{A}} = \mathcal{A}'$ and consider I to be an inclusion of a full subcategory.

A.4 Multiplicative systems

The construction of the quotient category of a Verdier triangulated category is due to Verdier; cf. [**13**].

Suppose given a category $\mathcal{C}$.

Definition 45 A set M of morphisms of $\mathcal{C}$ is called a *multiplicative system* in $\mathcal{C}$ if (Fr 1-4) are satisfied. An element of M is called an M-isomorphism and denoted by $X \Longrightarrow Y$.

(Fr 1) Each identity in $\mathcal{C}$ is an M-isomorphism.

(Fr 2) Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{C}$ such that fg and gh are M-isomorphisms.
Then f, g, h and $f \cdot g \cdot h$ are M-isomorphisms.

(Fr 3) Suppose given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathcal{C}$. There exists an M-isomorphism s such that $sf = sg$ if and only if there exists an M-isomorphism t such that $ft = gt$.

(Fr 4) Given $\begin{array}{ccc} X & \Longrightarrow & Y \\ \downarrow & & \\ X' & & \end{array}$ in $\mathcal{C}$, there exists a completion to a commutative quadrangle $\begin{array}{ccc} X & \Longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \Longrightarrow & Y' \end{array}$.

Dually, given $\begin{array}{ccc} & & Y \\ & & \downarrow \\ X' & \Longrightarrow & Y' \end{array}$ in $\mathcal{C}$, there exists a completion to a commutative quadrangle $\begin{array}{ccc} X & \Longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \Longrightarrow & Y' \end{array}$.

Cf. [**13,** §2, no. 1].

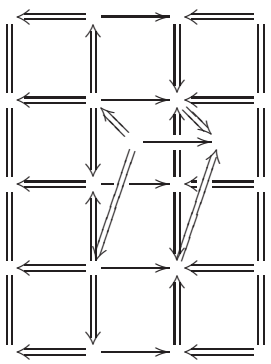
Suppose given a multiplicative system M in $\mathcal{C}$. Using (Fr 2), we note that in the first assertion of (Fr 4), if $X \longrightarrow X'$ is an M-isomorphism, then there exists a commutative completion with $Y \longrightarrow Y'$ being an M-isomorphism. And dually.

The category $\mathcal{C}_M$, called *localisation of* $\mathcal{C}$ *at* M, is defined as follows. Let $\mathrm{Ob}\,\mathcal{C}_M := \mathrm{Ob}\,\mathcal{C}$. A morphism from X to Y is a *double fraction,* which is an equivalence class of diagrams of the following form.

$$X \xleftarrow{s} X' \xrightarrow{f} Y' \xleftarrow{t} Y$$

The diagrams (s, f, t) and $(s's, s'ft', tt')$ are declared to be *elementarily equivalent*, provided s' and t' are M-isomorphisms. To form double fractions, we take the equivalence relation generated by elementary equivalence.

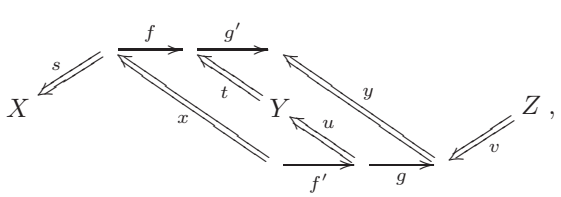
The equivalence class of the diagram (s, f, t) is written $s\backslash f/t$. So $s\backslash f/t = \tilde{s}\backslash \tilde{f}/\tilde{t}$ if and only if there exist M-isomorphisms u, $\tilde{u}$, v, $\tilde{v}$ such that $us = \tilde{u}\tilde{s}$ and $tv = \tilde{t}\tilde{v}$ and $ufv = \tilde{u}\tilde{f}\tilde{v}$.



Write $f/t := 1\backslash f/t$, called a *right fraction*, and $s\backslash f := s\backslash f/1$, called a *left fraction*. Using (Fr 4), each morphism in $\mathcal{C}_M$ can be represented both by a left fraction and by a right fraction. Given right fractions f/t and $\tilde{f}/\tilde{t}$, they are equal if there exist M-isomorphisms u, v and $\tilde{v}$ such that $ufv = u\tilde{f}\tilde{v}$ and $tv = \tilde{t}\tilde{v}$. By (Fr 3), this implies the existence of M-isomorphisms v, $\tilde{v}$ and u' such that $f(vu') = \tilde{f}(\tilde{v}u')$ and $t(vu') = \tilde{t}(\tilde{v}u')$. Dually for left fractions.

> So double fractions are a self-dual way to represent morphisms in $\mathcal{C}_M$. Right or left fractions are more efficient in many arguments.

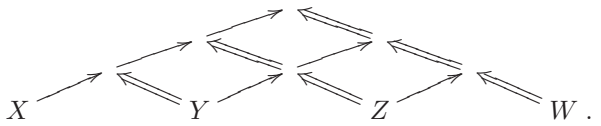
The composite of two double fractions $s\backslash f/t$ and $u\backslash g/v$ is defined, using (Fr 4) for the commutative diagram



to be equivalently $s\backslash fg'/vy$ or $xs\backslash f'g/v$. By (Fr 4, 2, 3), this definition is independent of the chosen completion with g' and y, and, likewise, of the chosen completion with x and f'.

Independence of the choice of the representative $s\backslash f/t$ is seen considering an elementary equivalence and using (Fr 4, 2), thus obtaining an elementary equivalence of the two possible representatives of the composite. Likewise independence of the representative $u\backslash g/v$.

Associativity follows using right fractions and a commutative diagram constructed by means of (Fr 4),



Given $f \in \operatorname{Mor}\mathcal{C}$, we also write $1\backslash f/1 =: f$ in $\mathcal{C}_M$, by abuse of notation. Note that in $\mathcal{C}_M$, we have $s\backslash f/t = s^- f\, t^-$.

Remark 46 *A double fraction $s\backslash f/t$ represents an isomorphism in $\mathcal{C}_M$ if and only if f is an M-isomorphism.*

Sketch. First, using (Fr 2), we reduce to the case of a right fraction g/u. For a right fraction in turn, the assertion follows applying (Fr 2) to an associativity diagram as above. □

Remark 47 Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathcal{C}$, we have $f = g$ in $\mathcal{C}_M$ if and only if there exists an M-isomorphism t such that $ft = gt$ in $\mathcal{C}$, or, equivalently, if and only if there exists an M-isomorphism s such that $sf = sg$ in $\mathcal{C}$.

Remark 48 *We have a functor $\mathcal{C} \xrightarrow{\mathsf{L}} \mathcal{C}_M$, $f \longmapsto 1\backslash f/1 = f$, called localisation functor.*

Given a category $\mathcal{T}$, we let $[\![\mathcal{C}, \mathcal{T}]\!]_M$ be the full subcategory of $[\![\mathcal{C}, \mathcal{T}]\!]$ consisting of functors that send all M-isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{T}$. The induced functor

$$[\![\mathcal{C}, \mathcal{T}]\!]_M \xleftarrow{\mathsf{L}\star(-)} [\![\mathcal{C}_M, \mathcal{T}]\!]$$

is a strictly dense equivalence, i.e. it is surjective on objects, full and faithful.

Sketch. Given a functor $F \in \operatorname{Ob}[\![\mathcal{C}, \mathcal{T}]\!]_M$, we may define $\breve{F}$ on $\mathcal{C}_M$ by letting $X\breve{F} := XF$ for $X \in \operatorname{Ob}\mathcal{C}_M = \operatorname{Ob}\mathcal{C}$ and by $(s\backslash f/t)\breve{F} := (sF)^- \cdot (fF) \cdot (tF)^-$. Then $\mathsf{L}\star\breve{F} = F$.

Given a transformation $(F \xrightarrow{u} G) \in \operatorname{Mor}[\![\mathcal{C}, \mathcal{T}]\!]_M$, we may define $\breve{F} \xrightarrow{\breve{u}} \breve{G}$ by setting $X\breve{u} := Xu$ for $X \in \operatorname{Ob}\mathcal{C}_M = \operatorname{Ob}\mathcal{C}$. Then $\mathsf{L}\star\breve{u} = u$. □

Lemma 49 *Given $n \geqslant 0$, the functor*

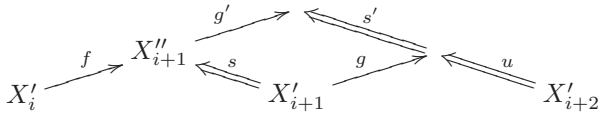
$$\mathcal{C}(\dot{\Delta}_n) \xrightarrow{\mathsf{L}(\dot{\Delta}_n)} \mathcal{C}_M(\dot{\Delta}_n)\,,$$

given by pointwise application of L, is dense.

Proof. We may assume $n \geqslant 1$.

Suppose given $X \in \operatorname{Ob}\mathcal{C}_M(\dot{\Delta}_n)$. To prove that for $i \in [1, n-1]$ there exists an $X' \in \operatorname{Ob}\mathcal{C}_M(\dot{\Delta}_n)$ isomorphic to X such that $X'_j \xrightarrow{x'} X'_{j+1}$ is in the image of L for $j \in [1, i-1]$, we proceed by induction on $i \geqslant 1$. Suppose the assertion to be true for i. Let us prove the assertion for $i+1$. Write $X'_i \xrightarrow{x'} X'_{i+1}$ as a right fraction f/s. If

$i = n-1$, we replace X'_{i+1} by the target of f, and f/s by f. If $i \leqslant n-2$, we write $X'_{i+1} \xrightarrow{x'} X'_{i+2}$ as a right fraction g/u and construct the following commutative diagram using (Fr 4).



We replacing the object X'_{i+1} by X''_{i+1}, the morphism f/s by f and the morphism g/u by g'/us'.

In both cases, we obtain a diagram isomorphic to X' that conincides with X' on $[1,i]$ and whose morphism from i to $i+1$ is in the image of L. ▫

Lemma 50 *Given $n \geqslant 0$, the functor*

$$\mathcal{C}(\dot{\Delta}_n) \xrightarrow{\mathsf{L}(\dot{\Delta}_n)} \mathcal{C}_M(\dot{\Delta}_n)$$

is 1*-epimorphic.*

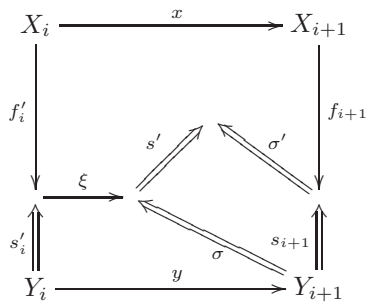
Proof. We shall apply [**8,** Lem. A.35]. By Lemma 49, $\mathsf{L}(\dot{\Delta}_n)$ is dense.

Suppose given X, $Y \in \mathrm{Ob}\,\mathcal{C}(\dot{\Delta}_n)$ and a morphism $X\,\mathsf{L}(\dot{\Delta}_n) \xrightarrow{g} Y\,\mathsf{L}(\dot{\Delta}_n)$ in $\mathcal{C}_M(\dot{\Delta}_n)$. Let g_i be represented by a right fraction f_i/s_i for $i \in [1,n]$.

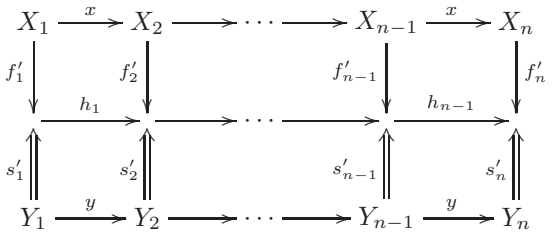
We *claim* that for $i \in [1,n]$, we can find representatives f'_j/s'_j for $j \in [1,i]$ such that there exist h_j with $s'_j h_j = y s'_{j+1}$ and $f'_j h_j = x f'_{j+1}$ in $\mathcal{C}$ for $j \in [1,i-1]$. Let $f'_1 := f_1$ and $s'_1 := s_1$. Proceeding by induction on i, we have to write the right fraction f_{i+1}/s_{i+1} suitably as f'_{i+1}/s'_{i+1}. First of all, by (Fr 4), we find an M-isomorphism σ and a morphism ξ such that $y\sigma = s'_i \xi$ in $\mathcal{C}$. We have $f'_i \xi \sigma^- = x f_{i+1} s_{i+1}^-$ in $\mathcal{C}_M$. Using (Fr 4) and (Fr 2), we find M-isomorphisms s' and σ' such that $\sigma s' = s_{i+1}\sigma'$ in $\mathcal{C}$. Hence

$$f'_i \xi s' \;=\; f'_i \xi s' s'^- \sigma^- s_{i+1}\sigma' \;=\; f'_i \xi \sigma^- s_{i+1}\sigma' \;=\; x f_{i+1} s_{i+1}^- s_{i+1}\sigma' \;=\; x f_{i+1}\sigma'$$

in $\mathcal{C}_M$. Composing with a further M-isomorphism, we may assume that $f'_i \xi s' = x f_{i+1}\sigma'$ in $\mathcal{C}$; cf. Remark 47. We take $h_i := \xi s'$ and $s'_{i+1} := s_{i+1}\sigma'$ and $f'_{i+1} := f_{i+1}\sigma'$.



This proves the *claim,* in particular for $i = n$.



Condition (C) of loc. cit. is satisfied letting the epizigzag have length 0, letting the monozigzag be the single backwards diagram morphism consisting of the morphisms s'_i, and letting the required diagram morphism in the image of $L(\dot{\Delta}_n)$ consist of the morphisms f'_i. ▫

Remark 51 *Suppose the category $\mathcal{C}$ to be additive.*

(1) *An object X is isomorphic to 0 in $\mathcal{C}_M$ if and only if $X \Longrightarrow 0$, or, equivalently, if and only if $0 \Longrightarrow X$.*

(2) *The category $\mathcal{C}_M$ is additive, and the functor $\mathsf{L} : \mathcal{C} \longrightarrow \mathcal{C}_M$ is additive.*

(3) *Given an additive category $\mathcal{T}$, the strictly dense equivalence*

$$[\![\, \mathcal{C}\,,\, \mathcal{T}\,]\!]_M \xleftarrow{\ \mathsf{L}\star(-)\ } [\![\, \mathcal{C}_M\,,\, \mathcal{T}\,]\!]$$

restricts to a strictly dense equivalence from the category of additive functors from $\mathcal{C}_M$ to $\mathcal{T}$ to the category of additive functors from $\mathcal{C}$ to $\mathcal{T}$ that sends all M-isomorphisms to isomorphisms, written

$$[\![\, \mathcal{C}\,,\, \mathcal{T}\,]\!]_{\mathrm{add},\,M} \xleftarrow{\ \mathsf{L}\star(-)\ } [\![\, \mathcal{C}_M\,,\, \mathcal{T}\,]\!]_{\mathrm{add}}\ .$$

Sketch.

Ad (1). If X is isomorphic to 0, then $X \Longrightarrow X' \Longleftarrow 0$; cf. Remark 46. By (Fr 4), we conclude that $0 \Longrightarrow X$.

Ad (2). Given X, $Y \in \mathrm{Ob}\,\mathcal{C}$, the direct sum $X \oplus Y$, together with $X \xrightarrow{(1\,0)} X \oplus Y$ and $Y \xrightarrow{(0\,1)} X \oplus Y$, remains a coproduct in $\mathcal{C}_M$.

For existence of an induced morphism from the coproduct, we use (Fr 4, 2) to produce a common denominator of two right fractions.

To prove uniqueness of the induced morphism, we suppose given $\binom{f}{g}/s$ and $\binom{f'}{g'}/s$, without loss of generality with common denominator, such that $f/s = (1\,0)\cdot\binom{f}{g}/s = (1\,0)\cdot\binom{f'}{g'}/s = f'/s$ and $g/s = (0\,1)\cdot\binom{f}{g}/s = (0\,1)\cdot\binom{f'}{g'}/s = g'/s$ in $\mathcal{C}_M$. So there exists an M-isomorphism u such that $fu = f'u$, and an M-isomorphism v such that $gv = g'v$, both in $\mathcal{C}$. By (Fr 4, 2), we obtain a common M-isomorphism w such that $fw = f'w$ and $gw = g'w$ in $\mathcal{C}$. Hence $\binom{f}{g}\,w = \binom{f'}{g'}\,w$ in $\mathcal{C}$. Therefore $\binom{f}{g}/s = \binom{f'}{g'}/s$ in $\mathcal{C}_M$.

Moreover, the automorphism $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ of $X \oplus X$ remains an automorphism in $\mathcal{C}_M$.

Ad (3). Since L is additive, $\mathsf{L}\star(-)$ sends additive functors to additive functors. Conversely, given an additive functor $F : \mathcal{C} \longrightarrow \mathcal{T}$, the functor $\breve{F}$ as constructed in the proof of Remark 48 is additive. ▫

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