Remarks on the axioms for exact categories:

(Ex 2) is redundant

If idempotents split, then first to pure mono is pure mono

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Abstract

We will show that the axioms (Ex 2) and (Ex 2°) in [**2,** §A.2.1] are redundant. Moreover, we will show that if idempotents split the first morphism in a composition yielding a pure monomorphism is purely monomorphic; and dually.

Contents

1 Introduction

Exact categories have been defined by QUILLEN [**3,** §2]. In [**1,** App. A], KELLER has cut down redundancies in QUILLEN's set of axioms.

We have given a slightly different set of axioms in [**2,** §A.2.1]. We will show that in this set of axioms, (Ex 2) and, dually, (Ex 2°) are redundant; cf. Lemma 4.1.

If idempotents are stipulated to be pure in axiom (Ex $1'$) below, then (Ex $1, 1°$) follow; cf. Remark 5.1. Moreover, in this case idempotents split. And, which is convenient, the first morphism in a composition yielding a pure monomorphism is purely monomorphic; and dually; cf. Lemma 5.3. We do not know whether this holds in an exact category with not necessarily pure idempotents.

2 Pure short exact sequences

Suppose given an additive category $\mathcal{E}$. A *split monomorphism* in $\mathcal{E}$ is a morphism isomorphic to a morphism of the form $X \xrightarrow{(1\,0)} X \oplus Y$. A *split epimorphism* in $\mathcal{E}$ is a morphism isomorphic to a morphism of the form $X \oplus Y \xrightarrow{\binom{0}{1}} Y$.

The sequence $X' \xrightarrow{i} X \xrightarrow{p} X''$ in $\mathcal{E}$ is called *short exact* if i is a kernel of p and p is a cokernel of i.

Suppose given an isomorphism closed set of short exact sequences in $\mathcal{E}$, called the set of *pure short exact sequences.*

A *pure monomorphism* is a morphism appearing as a kernel in a pure short exact sequence; denoted by $\rightarrowtail$.

A *pure epimorphism* is a morphism appearing as a cokernel in a pure short exact sequence; denoted $\twoheadrightarrow$.

A *pure morphism* is a morphism that can be written as a composite of a pure epimorphism followed by a pure monomorphism.

Note that any kernel of a pure epimorphism is a pure monomorphism since the set of pure short exact sequences is supposed to be closed under isomorphisms. Dually, any cokernel of a pure monomorphism is a pure epimorphism.

Note that a morphism that is pure and monomorphic is purely monomorphic. In fact, its kernel is 0, whence it factors as a pure epimorphism with kernel 0, followed by a pure monomorphism, so that it suffices to show that the former is an isomorphism. However, in a short exact sequence with kernel term 0, the cokernel morphism is an isomorphism.

Dually, a morphism that is pure and epimorphic is purely epimorphic.

Remark 2.1 *Let $X \xrightarrow{f} Y$ be a pure morphism in $\mathcal{E}$. Suppose given factorisations*

$$(X \xrightarrow{f} Y) = (X \overset{\bar{f}}{\twoheadrightarrow} I \overset{\dot{f}}{\rightarrowtail} Y) = (X \xrightarrow{a} J \xrightarrow{b} Y),$$

where a is epimorphic and b is monomorphic.

Then there exists an isomorphism $I \xrightarrow[\sim]{u} J$ such that $\bar{f}u = a$ and $ub = \dot{f}$.

In particular, a is purely epimorphic and b is purely monomorphic.

Proof. By the universal property of the cokernel $\bar{f}$, there exists $I \xrightarrow{u} J$ such that $\bar{f}u = a$. Since $\bar{f}$ is epimorphic, we have $ub = \dot{f}$. The dual argument yields a morphism $J \xrightarrow{v} I$ such that $v\dot{f} = b$ and $av = \bar{f}$. Since $\bar{f}$ is epimorphic and $\dot{f}$ is monomorphic, we conclude that $uv = 1_I$. Since a is epimorphic and b is monomorphic, we conclude that $vu = 1_J$. $\square$

3 Some axioms

Consider the following axioms.

(Ex 1) All split monomorphisms in $\mathcal{E}$ are pure monomorphisms.

(Ex 1°) All split epimorphisms in $\mathcal{E}$ are pure epimorphisms.

(Ex 3) Given a commutative diagram

$$\begin{tikzcd} & Y \arrow[dr, "|" marking] & \\ X \arrow[ur] \arrow[rr, "\bullet" marking] & & Z \end{tikzcd},$$

we may insert it into a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "\bullet" marking] \arrow[dr, "\bullet" marking] & & B \\ & Y \arrow[ur, "|" marking] \arrow[dr, "|" marking] & \\ X \arrow[ur, "\bullet" marking] \arrow[rr, "\bullet" marking] & & Z \end{tikzcd}$$

with (X, Y, B) and (A, Y, Z) pure short exact sequences.

(Ex 3°) Given a commutative diagram

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur, "\bullet" marking] \arrow[rr, "|" marking] & & Z \end{tikzcd},$$

we may insert it into a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "|" marking] \arrow[dr, "\bullet" marking] & & B \\ & Y \arrow[ur, "|" marking] \arrow[dr, "|" marking] & \\ X \arrow[ur, "\bullet" marking] \arrow[rr, "|" marking] & & Z \end{tikzcd}$$

with (X, Y, B) and (A, Y, Z) pure short exact sequences.

4 (Ex 2) follows

Suppose given an additive category $\mathcal{E}$ and an isomorphism closed set of pure short exact sequences in $\mathcal{E}$. Suppose that (Ex $1, 1°, 3, 3°$) hold.

We shall show in Lemma 4.1 that (Ex 2) of [**2,** §A.2.1] holds. By duality, we then can conclude that (Ex 2°) of loc. cit. holds as well. In particular, the category $\mathcal{E}$, together with this set of pure short exact sequences, is an exact category in the sense of loc. cit., and therefore, which amounts to the same, in the sense of QUILLEN [**3,** §2]; cf. [**1,** App. A]. Conversely, (Ex $1, 1^\circ, 3, 3^\circ$) hold in the case of an extension closed additive full subcategory of an abelian category, where the pure short exact sequences are stipulated to be the short exact sequences of this abelian category with all three objects in that subcategory; cf. [**2,** §A.2.1].

Lemma 4.1 *Suppose given pure monomorphisms $X \xrightarrow{i} Y$ and $Y \xrightarrow{j} Z$. Then their composite $X \xrightarrow{ij} Z$ is a pure monomorphism.*

Proof. Let $X \xrightarrow{i} Y \xrightarrow{p} \mathrm{C}_i$ be a pure short exact sequence. By (Ex 1°), the morphism

$$\mathrm{C}_i \oplus Z \xrightarrow{\binom{0}{1}} Z$$

is purely epimorphic.

Applying (Ex 3) to the commutative triangle

$$\begin{tikzcd}
 & \mathrm{C}_i \oplus Z \arrow[dr, "\binom{0}{1}"] & \\
Y \arrow[ur, "(p\ j)"] \arrow[rr, "j"] & & Z ,
\end{tikzcd}$$

we obtain a diagram as follows.

$$\begin{tikzcd}
\mathrm{C}_i \arrow[rr, "j'"] \arrow[dr, "(1\ 0)"'] & & W \\
 & \mathrm{C}_i \oplus Z \arrow[ur, "\binom{j'}{p'}"'] \arrow[dr, "\binom{0}{1}"] & \\
Y \arrow[ur, "(p\ j)"] \arrow[rr, "j"] & & Z
\end{tikzcd}$$

Applying (Ex 3°) to the commutative triangle

$$\begin{tikzcd}
 & \mathrm{C}_i \oplus Z \arrow[dr, "\binom{1}{0}"] & \\
Y \arrow[ur, "(p\ j)"] \arrow[rr, "p"] & & \mathrm{C}_i ,
\end{tikzcd}$$

we obtain the diagram

$$\begin{tikzcd}
Z \arrow[rr, "p'"] \arrow[dr, "(0\ 1)"'] & & W \\
 & \mathrm{C}_i \oplus Z \arrow[ur, "\binom{j'}{p'}"'] \arrow[dr, "\binom{1}{0}"] & \\
Y \arrow[ur, "(p\ j)"] \arrow[rr, "p"] & & \mathrm{C}_i .
\end{tikzcd}$$

By the Changing Lemma (also known as X-Lemma), we may complete this diagram to a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[rr, "p'"] \arrow[dr, "(0\ 1)"'] & & W \\
X \arrow[ur, "u"] \arrow[dr, "i"'] & & \mathrm{C}_i \oplus Z \arrow[ur, "\binom{j'}{p'}"'] \arrow[dr, "\binom{1}{0}"] & \\
 & Y \arrow[rr, "p"] \arrow[ur, "(p\ j)"] & & \mathrm{C}_i\ ,
\end{tikzcd}$$

in which u is the kernel of p' and therefore purely monomorphic. By composition, we see that $u = ij$. $\square$

5 If idempotents split

Suppose given an additive category $\mathcal{E}$ and an isomorphism closed set of pure short exact sequences in $\mathcal{E}$.

Recall that an idempotent in $\mathcal{E}$ is an endomorphism $X \xrightarrow{e} X$ such that $e = e^2$. Consider the following axiom.

(Ex 1′) All idempotents in $\mathcal{E}$ are pure.

Suppose that (Ex 1′) and (Ex 3, 3°) hold. We shall show that then (Ex 1) and, dually, (Ex 1°) hold. Moreover, we shall show that idempotents actually split, cf. Lemma 5.2; and that the first morphism in a composition yielding a pure monomorphism is purely monomorphic; and dually, cf. Lemma 5.3.

To summarise roughly, (Ex 1′) is a stronger replacement for (Ex 1, 1°) in the set of axioms in §3.

Remark 5.1 *Coretractions in $\mathcal{E}$ are purely monomorphic. Retractions in $\mathcal{E}$ are purely epimorphic. In particular,* (Ex 1, 1°) *hold.*

Proof. Assume $ab = 1$. Then ba is idempotent, hence pure by (Ex 1′). Since b is epimorphic and a is monomorphic, this implies b purely epimorphic and a purely monomorphic; cf. Remark 2.1. $\square$

Lemma 5.2

(1) *Suppose given an idempotent $X \xrightarrow{e} X$ in $\mathcal{E}$. There exists an isomorphism $X \xrightarrow[\sim]{u} I \oplus J$ for some I, $J \in \operatorname{Ob} \mathcal{E}$ such that $u^- e u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.*

(2) *Given $(X \xrightarrow{a} Y \xrightarrow{b} X) = (X \xrightarrow{1_X} X)$ in $\mathcal{E}$, there exists an isomorphism $Y \xrightarrow[\sim]{v} X \oplus Z$ for some $Z \in \operatorname{Ob} \mathcal{E}$ such that $av = (1\ 0)$ and $v^- b = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.*

Proof. Ad (1). Since e and $f := 1 - e$ are pure by (Ex 1′), we may factor

$$\begin{aligned}
(X \xrightarrow{\;e\;} X) \;&=\; (X \overset{\bar e}{\twoheadrightarrow} I \overset{\dot e}{\rightarrowtail} X)\\
(X \xrightarrow{\;f\;} X) \;&=\; (X \overset{\bar f}{\twoheadrightarrow} J \overset{\dot f}{\rightarrowtail} X)
\end{aligned}$$

Note that $(\bar e\ \bar f)\begin{pmatrix}\dot e\\ \dot f\end{pmatrix} = 1$ and $\begin{pmatrix}\dot e\\ \dot f\end{pmatrix}(\bar e\ \bar f) = \begin{pmatrix}1&0\\0&1\end{pmatrix}$. In fact, for the latter equation we note that $\bar e\dot e\bar e\dot e = \bar e\dot e$ implies that $\dot e\bar e = 1$, and that $\bar e\dot e\bar f\dot f = 0$ implies that $\dot e\bar f = 0$. So we may take $u = (\bar e\ \bar f)$. We obtain

$$\begin{pmatrix}\dot e\\ \dot f\end{pmatrix} \; e \; (\bar e\ \bar f) \;=\; \begin{pmatrix}\dot e e\bar e & \dot f e\bar e\\ \dot e e\bar f & \dot f e\bar f\end{pmatrix} \;=\; \begin{pmatrix}1&0\\0&0\end{pmatrix} .$$

Ad (2). Since ba is idempotent, by Remark 5.1 we may assume that $b = \bar e$ and $a = \dot e$ in the situation of the proof of (1). Now $\dot e\,(\bar e\ \bar f) = (1\ 0)$ and $\begin{pmatrix}\dot e\\ \dot f\end{pmatrix}\bar e = \begin{pmatrix}1\\0\end{pmatrix}$. ▫

Lemma 5.3 *Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$.*

(1) *If fg is purely monomorphic, then f is purely monomorphic.*

(1°) *If fg is purely epimorphic, then g is purely epimorphic.*

Proof. Ad (1). Applying (Ex 3) to the commutative triangle

$$\begin{tikzcd}
 & Y \oplus Z \arrow[dr, "\begin{pmatrix}0\\1\end{pmatrix}", "|" marking] & \\
X \arrow[ur, "(f\ \ fg)"] \arrow[rr, "fg", "\bullet" marking] & & Z ,
\end{tikzcd}$$

we obtain

$$\begin{tikzcd}
Y \arrow[rr, "s", "\bullet" marking] \arrow[dr, "(1\,0)"', "\bullet" marking] & & Z' \\
 & Y \oplus Z \arrow[ur, "\begin{pmatrix}s\\t\end{pmatrix}"', "|" marking] \arrow[dr, "\begin{pmatrix}0\\1\end{pmatrix}", "|" marking] & \\
X \arrow[ur, "(f\ \ fg)", "\bullet" marking] \arrow[rr, "fg", "\bullet" marking] & & Z .
\end{tikzcd}$$

By the universal property of the cokernel $\begin{pmatrix}s\\t\end{pmatrix}$, there exists x such that $\begin{pmatrix}g\\-1\end{pmatrix} = \begin{pmatrix}s\\t\end{pmatrix}x$. Hence t is a coretraction, and thus purely monomorphic; cf. Remark 5.1.

Applying (Ex 3) to the commutative triangle

$$\begin{tikzcd}
 & Y \oplus Z \arrow[dr, "\begin{pmatrix}s\\t\end{pmatrix}", "|" marking] & \\
Z \arrow[ur, "(0\,1)"] \arrow[rr, "t", "\bullet" marking] & & Z' ,
\end{tikzcd}$$

we obtain the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "(f\ fg)"'] & & Y \\
& Y \oplus Z \arrow[ur, "\binom{1}{0}"'] \arrow[dr, "\binom{s}{t}"] & \\
Z \arrow[ur, "(0\,1)"] \arrow[rr, "t"] & & Z' \,.
\end{tikzcd}$$

In particular, f is purely monomorphic. □

Does there exist an exact category and a pure monomorphism i in it that factors $i = fg$ with f not being a pure monomorphism? By Lemma 5.3, in such an exact category there have to exist non-pure idempotents. Cf. also [**1,** A.1].

If $\mathcal{E}$ is an extension closed and summand closed additive full subcategory of an abelian category $\mathcal{A}$, then we get, forming Cokernels in $\mathcal{A}$, a bicartesian square $(Y, Z, \mathrm{C}_f, \mathrm{C}_{fg})$ in $\mathcal{A}$. Its short exact diagonal sequence $Y \longrightarrow Z \oplus \mathrm{C}_f \longrightarrow \mathrm{C}_{fg}$ has Y and C_{fg} in $\mathrm{Ob}\,\mathcal{E}$, and thus $Z \oplus \mathrm{C}_f \in \mathrm{Ob}\,\mathcal{E}$. Since $\mathcal{E}$ is summand closed in $\mathcal{A}$, we conclude that $\mathrm{C}_f \in \mathrm{Ob}\,\mathcal{E}$, whence f is purely monomorphic. Note that we needed summand closedness only in the case of a direct sum one of whose summands is already in $\mathrm{Ob}\,\mathcal{E}$, to conclude that so is the other.

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