Dweak squares in Heller triangulated categories

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Abstract

We mention briefly an elementary compatibility of distinguished weak (dweak) squares with the shift functor, which holds in a closed Heller triangulated category.

In a Verdier triangulated category, one can still formulate this compatibility assertion, but I do not know whether it can be proven there.

To obtain a counterexample to this compatibility assertion in a Verdier triangulated category, however, one would need a Verdier triangulated category that is not a closed Heller triangulated category. I do not know an example of such a category.

Let $(\mathcal{C}, \mathsf{T}, \vartheta)$ be a closed Heller triangulated category; cf. [**2,** Def. 1.5.i], [**3,** Def. A.6]. Recall that closedness just means that each morphism can be completed to a 2-triangle. Heller triangulated categories in which idempotents split, are closed; cf. [**2,** Lem. 3.1]. The stable category of a Frobenius category is closed.

We use the conventions and notations of [**2**].

A closed Heller triangulated category is in particular Verdier triangulated, taking the set ([1]) of 2-triangles as set of distinguished triangles; cf. [**5,** Def. 1-1]; this is proven just as [**2,** Prop. 3.6].

Suppose given $n \geqslant 2$. Each chain of morphisms $X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n$ in $\mathcal{C}$ can be prolonged to an n-triangle, and this prolongation is unique up to isomorphism; cf. [**2,** Def. 1.5.ii.2, Lem. 3.1, Lem. 3.4.6].

A *distinguished weak square*, for short *dweak square* ([2]), is a weak square whose diagonal sequence fits into a 2-triangle. Dweak squares are indicated by the symbol Đ. Note that to define dweak squares, a Verdier triangulated category suffices.

Since a completion of a morphism to a 2-triangle is unique up to isomorphism, so is completion of an angle $\begin{array}{cc} \uparrow & \\ & \!\!\longrightarrow \end{array}$ to a dweak square $\begin{array}{ccc} & \longrightarrow & \\ \uparrow & \text{Đ} & \uparrow \\ & \longrightarrow & \end{array}$; and dually.

Using a 4-triangle, one sees that dweak squares compose. There is an elegant method, due to Neeman, to show this fact already in Verdier triangulated categories; cf. [**4,** Lem. 2.1].

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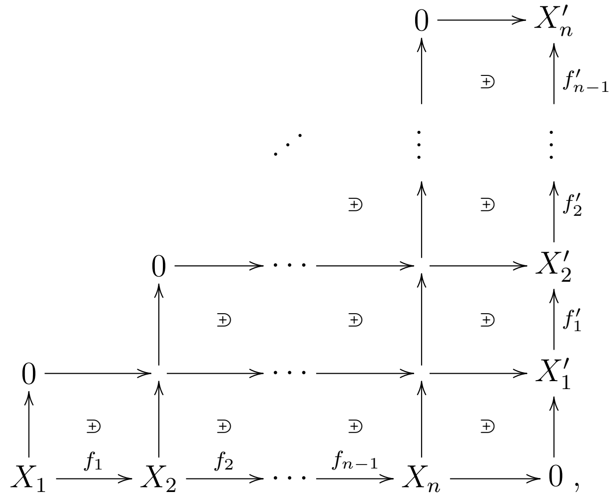
[1] In a sufficiently big universe.

[2] Also known as a *homotopy cartesian square*.

Completing iteratively to dweak squares, we may therefore form



and the resulting chain $X'_1 \xrightarrow{f'_1} \cdots \xrightarrow{f'_{n-1}} X'_n$ is unique up to isomorphism.

Remark 1. *The following compatibility of dweak squares and shift holds in our Heller triangulated category* $(\mathcal{C}, \mathsf{T}, \vartheta)$.

$$(X'_1 \xrightarrow{f'_1} \cdots \xrightarrow{f'_{n-1}} X'_n) \;\simeq\; (X_1^{+1} \xrightarrow{f_1^{+1}} \cdots \xrightarrow{f_{n-1}^{+1}} X_n^{+1})\,.$$

Proof. Complete the chain $X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n$ to an n-triangle. By [**2,** Lem. 3.4.1, Lem. 3.4.2], an n-triangle consists of dweak squares. The result follows from the uniqueness up to isomorphism stated just before Remark 1. ▫

In a Verdier triangulated category, in which the axiom [**1,** 1.1.13] holds, the assertion of Remark 1 holds if

$$n = 2$$

and, by the octahedral axiom in the form of loc. cit., if

$$n = 3\,.$$

But we may ask for this assertion for

$$n \geqslant 4$$

as well. In fact, both sides of the isomorphism in question are still welldefined up to isomorphism resp. welldefined.

If $n = 4$, the valid assertion in the cases $n = 2$ and $n = 3$, together with [**2,** Lem. 3.4.1], yields isomorphisms

$$\begin{array}{ccc} X'_1 & \xrightarrow{f'_1} & X'_2 \\ {\scriptstyle u_1^{1,2}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_2^{1,2}} \\ X_1^{+1} & \xrightarrow{f_1^{+1}} & X_2^{+1} \end{array} \qquad \begin{array}{ccc} X'_2 & \xrightarrow{f'_2} & X'_3 \\ {\scriptstyle u_2^{2,3}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_3^{2,3}} \\ X_2^{+1} & \xrightarrow{f_2^{+1}} & X_3^{+1} \end{array} \qquad \begin{array}{ccc} X'_1 & \xrightarrow{f'_1 f'_2} & X'_3 \\ {\scriptstyle u_1^{1,3}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_3^{1,3}} \\ X_1^{+1} & \xrightarrow{f_1^{+1} f_2^{+1}} & X_3^{+1} \end{array}$$

$$\begin{array}{ccc} X'_1 & \xrightarrow{f'_1 f'_2 f'_3} & X'_4 \\ {\scriptstyle u_1^{1,4}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_4^{1,4}} \\ X_1^{+1} & \xrightarrow{f_1^{+1} f_2^{+1} f_3^{+1}} & X_4^{+1} \end{array} \qquad \begin{array}{ccc} X'_2 & \xrightarrow{f'_2 f'_3} & X'_4 \\ {\scriptstyle u_2^{2,4}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_4^{2,4}} \\ X_2^{+1} & \xrightarrow{f_2^{+1} f_3^{+1}} & X_4^{+1} \end{array} \qquad \begin{array}{ccc} X'_3 & \xrightarrow{f'_3} & X'_4 \\ {\scriptstyle u_3^{3,4}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_4^{3,4}} \\ X_3^{+1} & \xrightarrow{f_3^{+1}} & X_4^{+1} \end{array}$$

$$
\begin{array}{ccccc}
X_1' & \xrightarrow{f_1'} & X_2' & \xrightarrow{f_2'} & X_3' \\
{\scriptstyle u_1^{1,2,3}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_2^{1,2,3}} & & \wr\downarrow{\scriptstyle u_3^{1,2,3}} \\
X_1^{+1} & \xrightarrow{f_1^{+1}} & X_2^{+1} & \xrightarrow{f_2^{+1}} & X_3^{+1}
\end{array}
\qquad
\begin{array}{ccccc}
X_1' & \xrightarrow{f_1'} & X_2' & \xrightarrow{f_2' f_3'} & X_4' \\
{\scriptstyle u_1^{1,2,4}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_2^{1,2,4}} & & \wr\downarrow{\scriptstyle u_4^{1,2,4}} \\
X_1^{+1} & \xrightarrow{f_1^{+1}} & X_2^{+1} & \xrightarrow{f_2^{+1} f_3^{+1}} & X_4^{+1}
\end{array}
$$

$$
\begin{array}{ccccc}
X_1' & \xrightarrow{f_1' f_2'} & X_3' & \xrightarrow{f_3'} & X_4' \\
{\scriptstyle u_1^{1,3,4}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_3^{1,3,4}} & & \wr\downarrow{\scriptstyle u_4^{1,3,4}} \\
X_1^{+1} & \xrightarrow{f_1^{+1} f_2^{+1}} & X_3^{+1} & \xrightarrow{f_3^{+1}} & X_4^{+1}
\end{array}
\qquad
\begin{array}{ccccc}
X_2' & \xrightarrow{f_2'} & X_3' & \xrightarrow{f_3'} & X_4' \\
{\scriptstyle u_2^{2,3,4}}\downarrow\wr & & \wr\downarrow{\scriptstyle u_3^{2,3,4}} & & \wr\downarrow{\scriptstyle u_4^{2,3,4}} \\
X_2^{+1} & \xrightarrow{f_2^{+1}} & X_3^{+1} & \xrightarrow{f_3^{+1}} & X_4^{+1}
\end{array}
$$

But in a Verdier triangulated category with [**1,** 1.1.13] added, I do not know how to prove that, say, $u_3^{1,2,3}$ may be chosen equal to $u_3^{3,4}$.

Neither do I know a Verdier triangulated category in which the assertion of Remark 1 fails, say, for $n = 4$. Worse still, I do not know an example of a Verdier triangulated category that is not closedly Heller triangulated.

References

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