On the center of the derived category (manuscript)

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Abstract

We give an example of a ring A whose center is strictly contained in the center of its bounded derived category, which is defined to be the ring of shift-compatible endomorphisms of the identity functor of $\mathrm{D}^{\mathrm{b}}(A)$.

1 Preliminaries

We denote composition of morphisms on the right, i.e. $\xrightarrow{a}\xrightarrow{b} = \xrightarrow{ab}$. Complexes are denoted cohomologically, i.e. with upper indices. Let

$$\mathrm{Z}(\mathcal{C}) \;:=\; \{\eta \in \mathrm{End}\, 1_{\mathcal{C}} \mid \eta(X[1]) = (\eta X)[1]\}$$

denote the center of a triangulated category $\mathcal{C}$ in the sense of Verdier [**V**].

Let A be a noetherian ring. Let $\mathrm{D}^{\mathrm{b}}(A\text{-mod})$ denote the derived category of bounded complexes of finitely generated left A-modules. Note that $\mathrm{D}^{\mathrm{b}}(A\text{-mod}) \simeq \mathrm{K}^{-,\mathrm{b}}(A\text{-proj})$, the latter denoting the homotopy category of right bounded complexes of finitely generated projective left A-modules with bounded homology.

Lemma 1.1 *The ring morphism*

$$\begin{array}{rcl} \mathrm{Z}(A) & \xrightarrow{\iota} & \mathrm{Z}(\mathrm{D}^{\mathrm{b}}(A\text{-mod})) \\ z & \longmapsto & \left(\begin{array}{ccc} X & \longrightarrow & X \\ X^i & \longrightarrow & X^i \\ x & \longmapsto & xz \end{array}\right) \end{array}$$

is injective.

Proof. Suppose that z is mapped to zero. In particular, $z\iota$ vanishes on A, considered as a complex concentrated in degree 0. Since $A\text{-mod} \longrightarrow \mathrm{D}^{\mathrm{b}}(A\text{-mod})$ is full and faithful, we conclude that $z = 0$. $\square$

The aim of this note is to show that ι is not necessarily surjective. See Lemma 3.2.

2 Rickard's construction principle

Definition 2.1 (Happel, [**H**, I.4.1]) A morphism $X \xrightarrow{\partial} Y$ in a triangulated category $\mathcal{C}$ is called *almost vanishing* if (i), (ii) and (iii) hold.

(i) We have $\partial \neq 0$.

(ii) If $W \xrightarrow{\varphi} X$ is not a split epimorphism, then $\varphi\partial = 0$.

(iii) If $Y \xrightarrow{\psi} Z$ is not a split monomorphism, then $\partial\psi = 0$.

Note that if $X \xrightarrow{\partial} Y$ is almost vanishing, then X and Y are indecomposable; cf. [**H**, p. 35]. Moreover, given another almost vanishing morphism $X \xrightarrow{\partial'} Y'$, there is an isomorphism $Y \xrightarrow{\alpha} Y'$ such that $\partial\alpha = \partial'$. Likewise dually. Concerning existence of almost vanishing morphisms, see [**H**, I.4.6]. Under certain circumstances, the conditions (ii) and (iii) are equivalent; cf. [**H**, I.4.1].

Lemma 2.2 (Rickard, [**R**]) *Suppose given a triangulated category $\mathcal{C}$ satisfying Krull-Schmidt. Suppose given an almost vanishing* endo*morphism $X \xrightarrow{\partial} X$ in $\mathcal{C}$ in $\mathrm{Z}(\mathrm{End}_{\mathcal{C}}(X))$. There is an element $\eta \in \mathrm{Z}(\mathcal{C})$ such that $\eta(X[i]) = \partial[i]$ for $i \in \mathbf{Z}$, and such that η vanishes on the indecomposable objects not isomorphic to any $X[i]$.*

Proof. By direct sums and a skeleton argument, using Krull-Schmidt, we may extend a family of endomorphisms on isorepresentatives of the indecomposable objects of $\mathcal{C}$ to a family of endomorphisms on each object of $\mathcal{C}$. The resulting family is an endomorphism of the identity functor if and only if at each isorepresentative of the indecomposable objects, the member of this family is central in the endomorphism ring of this indecomposable object.

Since $\partial[i]$ is almost vanishing for $i \in \mathbf{Z}$, it remains to use the assumption that ∂ commutes with all endomorphisms of X to prove that η furnishes an endomorphism of the identity functor. □

Remark 2.3 Let A be a finite dimensional algebra over an algebraically closed field k of finite projective dimension. Let $\mathcal{C} = \mathrm{D}^{\mathrm{b}}(A\text{-mod})$. We claim that $\mathcal{C}$ is Krull-Schmidt; cf. [**H**, p. 42]. It suffices to show that all Hom-spaces are finite dimensional and that the endomorphism ring of an indecomposable object is local. Since $\mathrm{D}^{\mathrm{b}}(A\text{-mod}) \simeq \mathrm{K}^{\mathrm{b}}(A\text{-proj})$, all Hom-spaces are finite dimensional. In an abelian category that is noetherian and artinian with respect to subobjects, an application of the kernel-cokernel-sequence to a commutative triangle $X \xrightarrow{u^m} X \xrightarrow{u^m} X$ shows that for $m \geq 1$ big enough, u^m is split, i.e. it factors over a split epimorphism, followed by a split monomorphism (Fitting lemma). Suppose given $X \in \mathrm{Ob}\, \mathrm{C}^{\mathrm{b}}(A\text{-proj})$ that is indecomposable in $\mathrm{K}^{\mathrm{b}}(A\text{-proj})$ and an endomorphism $X \xrightarrow{u} X$. An application of the Fitting lemma in the abelian category $\mathrm{C}^{\mathrm{b}}(A\text{-mod})$ shows that there is an $m \geq 1$ such that u^m is split in $\mathrm{C}^{\mathrm{b}}(A\text{-mod})$, hence in $\mathrm{C}^{\mathrm{b}}(A\text{-proj})$, hence in

$\mathrm{K}^{\mathrm{b}}(A\text{-proj})$. Since X is indecomposable in the latter category, we infer that either u^m and thus also u is an isomorphism in $\mathrm{K}^{\mathrm{b}}(A\text{-proj})$, or that $u^m = 0$ in $\mathrm{K}^{\mathrm{b}}(A\text{-proj})$. Therefore, $\mathrm{End}_{\mathcal{C}} X$ is local ([1]).

Since k is algebraically closed, we may write any endomorphism of X as a sum of a scalar plus an element of the Jacobson radical of $\mathrm{End}_{\mathcal{C}}(X)$. Hence, any almost vanishing endomorphism $X \xrightarrow{\partial} X$ is central in $\mathrm{End}_{\mathcal{C}} X$, for it commutes with scalars and annihilates elements of the Jacobson radical by composition from the left or from the right.

3 An example

Let R be a discrete valuation ring with maximal ideal generated by π, let $A = R/\pi^2$. Given integers $a \leq b$, we denote

$$X^{[a,b]} \;=\; (\cdots \to 0 \to \underbrace{A}_{a} \xrightarrow{\pi} A \xrightarrow{\pi} A \xrightarrow{\pi} \cdots \xrightarrow{\pi} \underbrace{A}_{b} \to 0 \to \cdots)\,,$$

considered as an object of $\mathrm{K}^{-,\mathrm{b}}(\mathrm{proj}\, A)$. Moreover, for $b \in \mathbf{Z}$ we denote

$$X^{[-\infty,b]} \;:=\; (\cdots \xrightarrow{\pi} A \xrightarrow{\pi} \underbrace{A}_{b} \to 0 \to \cdots)$$

(nonetheless, the position $-\infty$ itself does not exist).

Lemma 3.1 *Each indecomposable object of* $\mathrm{K}^{-,\mathrm{b}}(\mathrm{proj}\, A)$ $(\simeq \mathrm{D}^{\mathrm{b}}(A))$ *is isomorphic to an object of the form* $X^{[a,b]}$*, where* $a \in \mathbf{Z} \sqcup \{-\infty\}$*,* $b \in \mathbf{Z}$ *and* $a \leq b$*. Different pairs of parameters* (a,b) *yield nonisomorphic indecomposable objects* $X^{[a,b]}$*.*

This is shown at the end of §4.

Lemma 3.2 *Let the endomorphism* η *of the identity functor on* $\mathrm{D}^{\mathrm{b}}(A)$ *be defined by* $(X^{[i,i]} \xrightarrow{\eta} X^{[i,i]}) := (A \xrightarrow{\pi} A)[i]$ *for* $i \in \mathbf{Z}$*, and by zero on* $X^{[a,b]}$ *for* $a \in \mathbf{Z} \sqcup \{-\infty\}$*,* $b \in \mathbf{Z}$ *such that* $a < b$*. Then* $\eta \in \mathrm{Z}(\mathrm{D}^{\mathrm{b}}(A\text{-mod})) \smallsetminus \mathrm{Z}(A)\iota$*.*

Proof. In order for η to be welldefined, by Lemma 2.2 it suffices to show that the endomorphism $(X^{[0,0]} \xrightarrow{\eta} X^{[0,0]}) = (A \xrightarrow{\pi} A)$ is almost vanishing.

Given $X^{[a,b]} \xrightarrow{\varphi} X^{[0,0]}$, the composition $(X^{[a,b]} \xrightarrow{\varphi} X^{[0,0]} \xrightarrow{\eta} X^{[0,0]})$ vanishes as morphisms of complexes if $a \neq 0$; if $a = 0 < b$, it vanishes modulo homotopy; if $a = 0 = b$, it vanishes if φ is not an isomorphism.

[1]Does there exist an abelian category $\mathcal{A}$ in which Krull-Schmidt holds, but for which Krull-Schmidt fails for $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$?

Given $X^{[0,0]} \xrightarrow{\psi} X^{[0,0]}$, the composition $(X^{[0,0]} \xrightarrow{\eta} X^{[0,0]} \xrightarrow{\psi} X^{[a,b]})$ vanishes as morphism of complexes if $b \neq 0$; if $a < 0 = b$, it vanishes modulo homotopy; if $a = 0 = b$, it vanishes if ψ is not an isomorphism.

For $a \in A \smallsetminus \{0\}$, we claim that $(X^{[0,2]} \xrightarrow{a\iota} X^{[0,2]}) \neq 0$. In fact, it vanishes if and only if there are elements $u,\, v \in A$ such that $\pi u = a$, $u\pi + \pi v = a$ and $v\pi = a$. The latter two equations imply that $u\pi = 0$, which is impossible.

On the other hand, we have $(X^{[0,2]} \xrightarrow{\eta} X^{[0,2]}) = 0$, but $\eta \neq 0$. Hence $\eta \neq a\iota$ for all $a \in A = \mathrm{Z}(A)$. ▫

4 Indecomposables

The aim of this section is to provide a proof of Lemma 3.1 via linear algebra. Quite probably, there is a more conceptional method to derive this classification of indecomposables.

We denote the unit matrix by E_n if it is in $A^{n\times n}$ for some $n \geq 0$, or by E if we do not want to specify n.

Lemma 4.1 *Suppose given $k \geq 1$ and $n_1 \geq n_2 \geq \cdots \geq n_k$. Let $n_{k+1} := 0$. For $i \in [1, k-1]$, we denote*

$$D^i \;:=\; \begin{pmatrix} 0 \\ \pi\mathrm{E}_{n_{i+1}} \end{pmatrix} \;\in\; A^{n_i \times n_{i+1}}$$

Denote the diagram

$$(*) \qquad Y \;:=\; \left(A^{n_1} \xrightarrow{D^1} A^{n_2} \xrightarrow{D^2} A^{n_3} \xrightarrow{D^3} \cdots \xrightarrow{D^{k-2}} A^{n_{k-1}} \xrightarrow{D^{k-1}} A^{n_k}\right)$$

Suppose given $m \geq 0$ and $D' \in A^{m\times n_1}$ such that $D'D^1 = 0$.

There is an automorphism $(S^1, \ldots, S^k)$ of Y, where $S^i \in \mathrm{GL}_{n_i}(A)$, and an automorphism $S' \in \mathrm{GL}_m(A)$ of A^m such that the following holds. We have

$$S'D'S^1 \;=\; \begin{pmatrix} 0 \\ D'' \end{pmatrix} \;\in\; A^{m\times n_1}\,,$$

where

$$D'' \;=\; \mathrm{diag}(\mathrm{E}_\ell,\, C',\, C_1,\, \ldots,\, C_{k-1})\,,$$

where C' and C_i are of the form $(\pi\mathrm{E}\ 0)$, where C' has $n_1 - n_2 - \ell$ columns, and where C_i has $n_{i+1} - n_{i+2}$ columns for $i \in [1, k-1]$.

Note that we specified neither the number of rows of C', nor of any of the C_i, nor of D''.

Proof. An automorphism $S^1 \in \mathrm{GL}_{n_1}(A)$ can be extended to an automorphism $(S^1, \ldots, S^k)$ of Y if and only if it is of the block form

$$S_1 \;=\; \left(\begin{array}{c|c|c|c|c|c}
* & \pi* & \cdots & & \cdots & \pi* \\ \hline
* & * & \pi* & \cdots & \cdots & \pi* \\ \hline
\vdots & & \ddots & \ddots & & \vdots \\ \hline
* & \cdots & \cdots & * & \pi* & \pi* \\ \hline
* & \cdots & & \cdots & * & \pi* \\ \hline
* & \cdots & & & \cdots & *
\end{array}\right) ,$$

where the blocks are of sizes $(n_1 - n_2,\ n_2 - n_3,\ \ldots,\ n_k - n_{k+1})$.

Note that the last n_2 columns of D' are divisible by π since $D'D^1 = 0$.

In the following calculations, if no matrix entry is specified, then it is supposed to be zero.

Step 1. By the elementary divisor theorem for R, we may assume that

$$D' \;=\; \left(\begin{array}{c|c|c|c}
\mathrm{E} & & & \pi* \\ \hline
 & \pi\mathrm{E} & & \pi* \\ \hline
 & & & \pi*
\end{array}\right) \;\in\; A^{m\times n_1} ,$$

the last block column consisting of n_2 columns.

Step 2. By multiplication from the right, we may assume that

$$D' \;=\; \left(\begin{array}{c|c|c|c}
\mathrm{E} & & & \\ \hline
 & \pi\mathrm{E} & & \pi* \\ \hline
 & & & \pi*
\end{array}\right) .$$

Step 3. If a row in the batch in block position $(2, 4)$ is nonzero, then we use a nonzero entry to annihilate the entry π in the same row in the batch at block position $(2, 2)$. Then we move this row to the third block row. Changing the numbers of rows and of columns of the batches, we may therefore assume

$$D' \;=\; \left(\begin{array}{c|c|c|c}
\mathrm{E} & & & \\ \hline
 & \pi\mathrm{E} & & \\ \hline
 & & & \pi*
\end{array}\right) .$$

Step 4. Row echelonising the batch in block position $(3, 4)$ by multiplication from the left, with pivotal elements π on the rightmost position, cleaning rows with these pivotal elements by multiplication from the right, and finally sorting columns by multiplication from the right and sorting rows by multiplication from the left, we obtain

$$D' = \left(\begin{array}{c|c|c|c} \mathrm{E} & & & \\ \hline & \pi\mathrm{E} & & \\ \hline & & & \left(\begin{array}{c|c|c} C_1 & & \\ \hline & \ddots & \\ \hline & & C_{k-1} \\ \hline & & \end{array}\right) \end{array}\right),$$

with C_i as introduced above. Sorting the zero rows to the top, we obtain the claimed form. $\square$

Proof of Lemma 3.1.

Suppose given an indecomposable object $P \in \operatorname{Ob} \mathrm{K}^{-,\mathrm{b}}(\operatorname{proj} A)$, and assume it not to be of the form $X^{[a,b]}$ for some $a \in \mathbf{Z} \sqcup \{-\infty\}$ and some $b \in \mathbf{Z}$ such that $a \leq b$. By induction, starting on the right, we may assume that P is isomorphic to a complex of the form $(*)$ as in Lemma 4.1 at its last k nonzero positions. By loc. cit., and using the notation introduced there, we may bring the next differential D' into the form $\begin{pmatrix} 0 \\ \operatorname{diag}(\mathrm{E}_\ell,\ C',\ C_1,\ \ldots,\ C_{k-1}) \end{pmatrix}$ by base change. Zero columns yield direct summands of the form $X^{[a,b]}$ for finite a, b. So D' does not contain zero columns. If $\ell > 0$, then composition of D' with the next differential to the left shows that the first ℓ columns of the latter vanish. Thus we may split off a split acyclic summand, which is isomorphic to 0. Hence we may assume that $\ell = 0$. Thus D' is of the form $\binom{0}{\pi\mathrm{E}}$.

So for arbitrarily small $k \in \mathbf{Z}$, the complex P is isomorphic to a complex $P^{(k)}$ whose differentials to the right of position k are of the form $\binom{0}{\pi E}$. Moreover, if $k \leq \ell$, then to the right of position ℓ, the isomorphisms $P \xrightarrow{\sim} P^{(k)}$ and $P \xrightarrow{\sim} P^{(\ell)}$ coincide as morphisms in $\mathrm{C}(A\text{-proj})$. So we can glue these complex morphisms to obtain a complex morphism from P to a complex $P^{(-\infty)}$ that has all differentials of the form $\binom{0}{\pi\mathrm{E}}$. Since a morphism is an isomorphism in $\mathrm{K}^{-,\mathrm{b}}(A\text{-proj})$ if and only if its cone is split acyclic, and since at a given position (and a finite neighbourhood of this position) the cone of $P \longrightarrow P^{(-\infty)}$ coincides with the cone of $P \longrightarrow P^{(k)}$ for k small enough, we conclude that $P \longrightarrow P^{(-\infty)}$ is an isomorphism in $\mathrm{K}^{-,\mathrm{b}}(A\text{-proj})$.

If we consider a position at which, and to the left of which $P^{(-\infty)}$ is acyclic, then $D' = \pi\mathrm{E}$, for an element which is not a multiple of π cannot lie in the image of the differential. Hence $P^{(-\infty)}$, and thus also P is a direct sum of complexes of the form $X^{[-\infty,b]}$ with $b \in \mathbf{Z}$, yielding a contradiction.

That different pairs of parameters (a, b) yield nonisomorphic indecomposable objects $X^{[a,b]}$ follows by considering the homology. $\square$

5 References

[**H**] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras,* Lond. Math. Soc. LN 119, 1988.

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