A construction principle for Frobenius categories

Manuscript

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Abstract

This note is a supplement to [**3**]. Let $\mathcal{C}$ be a weakly abelian category. Let $n \geq 0$. Let $\mathcal{C}(\dot{\Delta}_n)$ be the category of diagrams of shape $\dot{\Delta}_n = [1,n]$ with values in $\mathcal{C}$. Let $\underline{\mathcal{C}(\dot{\Delta}_n)}$ be its quotient modulo split such diagrams. We know by [**3,** Prop. 5.5.(1), Prop. 2.6] that there is a Frobenius category $\mathcal{C}(\bar{\Delta}_n^{\#})$ whose classical stable category $\underline{\mathcal{C}(\bar{\Delta}_n^{\#})}$ is equivalent to $\underline{\mathcal{C}(\dot{\Delta}_n)}$. In particular, $\underline{\mathcal{C}(\dot{\Delta}_n)}$ is weakly abelian. We give a direct proof of this fact, exhibiting a structure of a Frobenius category on $\mathcal{C}(\dot{\Delta}_n)$ such that $\underline{\mathcal{C}(\dot{\Delta}_n)}$ is its classical stable category.

Contents

0 Introduction

0.1 A construction principle for Frobenius categories

Given an exact category $\mathcal{E}$ and a full subcategory $\mathcal{N} \subseteq \mathcal{E}$, we ask for a modification of the exact structure on $\mathcal{E}$ in such a way that the result is a Frobenius category with $\mathcal{N}$ as a sufficiently big subcategory of bijective objects.

Declaring a pure short exact sequence in $\mathcal{E}$ to be $\mathcal{N}$-pure if each object of $\mathcal{N}$ is bijective with respect to it, we verify that $\mathcal{E}$, equipped with the set of $\mathcal{N}$-pure short exact sequences,

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actually is an exact category. For it to be Frobenius, $\mathcal{N}$ only has to be big enough; see Remark 4.

0.2 Application to $\mathcal{C}(\dot{\Delta}_n)$

Let $\mathcal{C}$ be a weakly abelian category; cf. e.g. [**3,** Def. A.26]. Let $n \geq 0$.

Let $\mathcal{C}(\dot{\Delta}_n)$ be the category of diagrams of shape $\dot{\Delta}_n = [1, n]$ with values in $\mathcal{C}$. Let $\underline{\mathcal{C}(\dot{\Delta}_n)}$ be its quotient modulo split such diagrams.

For the definition of the poset $\bar{\Delta}_n^{\#}$, see [**3,** §1.1]. For the definition of the category $\mathcal{C}^+(\bar{\Delta}_n^{\#})$, see [**3,** §1.2.1.1]. Roughly, it is the category of diagrams on $\bar{\Delta}_n^{\#}$ that have zeroes on the boundaries and weak squares wherever possible. The category $\mathcal{C}^+(\bar{\Delta}_n^{\#})$ is Frobenius by [**3,** Prop. 5.5.(1)].

Its classical stable category $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ is equivalent to $\underline{\mathcal{C}(\dot{\Delta}_n)}$ by [**3,** Prop. 2.6]. In particular, since $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ is weakly abelian, so is $\underline{\mathcal{C}(\dot{\Delta}_n)}$. Cf. also [**1,** Prop. 8.4].

We find a structure of an exact category on $\mathcal{C}(\dot{\Delta}_n)$ such that it is a Frobenius category with $\underline{\mathcal{C}(\dot{\Delta}_n)}$ as its classical stable category. This reproves the fact that $\underline{\mathcal{C}(\dot{\Delta}_n)}$ is weakly abelian.

Whereas the category $\mathcal{C}(\dot{\Delta}_n)$ looks smaller and simpler than $\mathcal{C}^+(\bar{\Delta}_n^{\#})$, it behaves worse. Firstly, while $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$ carries a shift functor by diagram shift, the category $\underline{\mathcal{C}(\dot{\Delta}_n)}$ does not allow such a diagram shift, and can only artificially be given a shift functor via the equivalence $\underline{\mathcal{C}(\dot{\Delta}_n)} \simeq \underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$. Therefore, in the definition of a Heller triangulated category [**3,** Def. 1.5.(i)], we rather use $\underline{\mathcal{C}^+(\bar{\Delta}_n^{\#})}$. Secondly, and of relevance here, the exact structure on $\mathcal{C}^+(\bar{\Delta}_n^{\#})$ is the obvious one that declares pointwise split short exact sequences to be pure. The exact structure on $\mathcal{C}(\dot{\Delta}_n)$ has to be constructed; see Proposition 6 below.

0.3 Notation and conventions

(i) Given elements x, y of some set X, we let $\partial_{x,y} = 1$ in case $x = y$ and $\partial_{x,y} = 0$ in case $x \neq y$.

(ii) For an assertion X, which might be true or not, we let $\{X\}$ equal 1 if X is true, and equal 0 if X is false. So for instance, $\{x = y\} = \partial_{x,y}$.

(iii) For $a,\, b \in \mathbf{Z}$, we denote by $[a, b] := \{z \in \mathbf{Z} \,:\, a \leq z \leq b\}$ the integral interval.

(iv) Given $n \geq 0$, we denote by $\Delta_n := [0, n]$ the linearly ordered set with ordering induced by standard ordering on $\mathbf{Z}$. Let $\dot{\Delta}_n := \Delta_n \smallsetminus \{0\} = [1, n]$, considered as a linearly ordered set.

(v) Maps act on the right. Composition of maps, and of more general morphisms, is written on the right, i.e. $\xrightarrow{a}\xrightarrow{b} = \xrightarrow{ab}$.

(vi) Functors act on the right. Composition of functors is written on the right, i.e. $\xrightarrow{F}\xrightarrow{G} = \xrightarrow{FG}$. Accordingly, the entry of a transformation a between functors at an object X will be written Xa.

(vii) All categories are supposed to be small with respect to a sufficiently big universe.

(viii) Given a category $\mathcal{C}$, and objects X, Y in $\mathcal{C}$, we denote the set of morphisms from X to Y by ${}_{\mathcal{C}}(X, Y)$, or simply by (X, Y), if unambiguous.

(ix) Pure monomorphy in an exact category is indicated by $X \rightarrowtail Y$, pure epimorphy by $X \twoheadrightarrow Y$. Concerning exact categories in the sense of Quillen, cf. [**3,** §A.2].

(x) A morphism in an additive category $\mathcal{A}$ is *split* if it is isomorphic, in $\mathcal{A}(\Delta_1)$, to a morphism of the form $X \oplus Y \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} Y \oplus Z$. A morphism being split is indicated by $X \rightarrowtail\!\!\!\!\rightarrow Y$ (not to be confused with monomorphy). Accordingly, a morphism being a split monomorphism is indicated by $X \rightarrowtail\!\!\bullet\!\!\rightarrow Y$, a morphism being a split epimorphism by $X \rightarrowtail\!\!+\!\!\rightarrow Y$.

(xi) A sequence $X' \longrightarrow X \longrightarrow X''$ in an additive category $\mathcal{A}$ is *split short exact* if it is isomorphic, in $\mathcal{A}(\Delta_2)$, to the sequence $X' \xrightarrow{(1\,0)} X' \oplus X'' \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} X''$.

(xii) For the definition of a weakly abelian category, see e.g. [**3,** Def. A.26]; cf. [**2,** §3, l. 1–2], [**1,** Def. 8.6].

(xiii) Given a weakly abelian category $\mathcal{C}$ and $n \geq 1$, the category $\underline{\mathcal{C}(\dot{\Delta}_n)}$ is defined as $\mathcal{C}(\dot{\Delta}_n)$ modulo the subcategory of split diagrams; cf. [**3,** §2.4].

(xiv) Concerning the Freyd category $\hat{\mathcal{C}}$ of a weakly abelian category $\mathcal{C}$, we refer to [**3,** §A.6.3]. The Freyd category $\hat{\mathcal{C}}$ is an abelian Frobenius category that contains $\mathcal{C}$ as a sufficiently big subcategory of bijectives.

1 Construction of exact categories

Remark 1 *If $(\mathcal{E}, \mathcal{S}_i)$ are exact categories for i in some index set I, where $\mathcal{S}_i$ denotes the respective set of pure short exact sequences, then also $(\mathcal{E}, \bigcap_{i\in I} \mathcal{S}_i)$ is an exact category.*

A sequence $X' \longrightarrow X \longrightarrow X''$ in an exact category $(\mathcal{E}, \mathcal{S})$ is called *left exact* if $X' \longrightarrow X$ is purely monomorphic and a kernel of $X \longrightarrow X''$.

A sequence $X' \longrightarrow X \longrightarrow X''$ in an exact category $(\mathcal{E}, \mathcal{S})$ is called *right exact* if $X \longrightarrow X''$ is purely epimorphic and a cokernel of $X' \longrightarrow X$.

Let $(\mathcal{E}, \mathcal{S})$ and $(\mathcal{E}', \mathcal{S}')$ be exact categories, and let $\mathcal{E} \xrightarrow{F} \mathcal{E}'$ be an additive functor. Let $\mathcal{S}_F$ denote the set of short exact sequences in $\mathcal{S}$ whose image under F, applied pointwise, is in $\mathcal{S}'$.

The short exact sequences in $\mathcal{S}$ will also be called $\mathcal{S}$-pure; etc. The short exact sequences in $\mathcal{S}_F$ will also be called $\mathcal{S}_F$-pure; etc. We will continue to denote an $\mathcal{S}$-pure monomorphism in $\mathcal{E}$ by $\rightarrowtail$, and an $\mathcal{S}$-pure epimorphism by $\twoheadrightarrow$.

The functor F is called *left exact* if for any pure short exact sequence (X, Y, Z) in $\mathcal{E}$, the sequence (XF, YF, ZF) is left exact.

The functor F is called *right exact* if for any pure short exact sequence (X, Y, Z) in $\mathcal{E}$, the sequence (XF, YF, ZF) is right exact.

Lemma 2

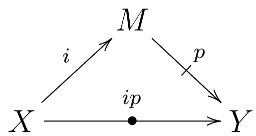
(1) *If $\mathcal{E} \xrightarrow{F} \mathcal{E}'$ is left exact, then $(\mathcal{E}, \mathcal{S}_F)$ is an exact category.*

(2) *If $\mathcal{E} \xrightarrow{F} \mathcal{E}'$ is right exact, then $(\mathcal{E}, \mathcal{S}_F)$ is an exact category.*

Proof. Ad (1). Consider a left exact functor $\mathcal{E} \xrightarrow{F} \mathcal{E}'$.

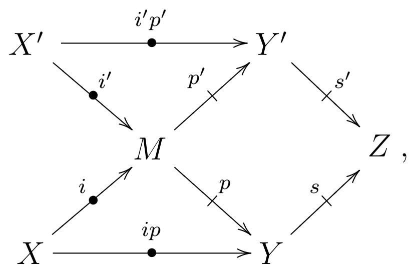
We use the axioms from [**3,** §A.2.1]. The axiom (Ex 2) is redundant; cf. [**4**].

Verification of (Ex 3). Suppose given a commutative triangle

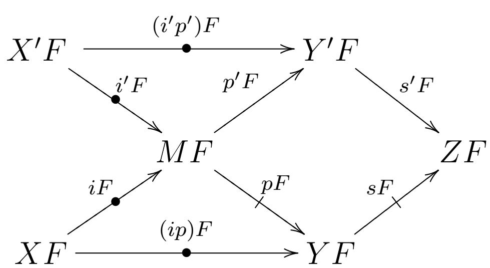


in $\mathcal{E}$ in which, moreover, ip is $\mathcal{S}_F$-purely monomorphic and p is $\mathcal{S}_F$-purely epimorphic.

By exactness of $(\mathcal{E}, \mathcal{S})$, we can complete it to a diagram

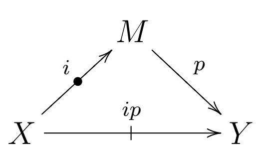


in $\mathcal{E}$ with $\mathcal{S}$-pure short exact sequences (X, M, Y'), (X', M, Y), (X, Y, Z) and (X', Y', Z). Moreover, (X', M, Y) and (X, Y, Z) are $\mathcal{S}_F$-purely short exact, i.e. $(X'F, MF, YF)$ and (XF, YF, ZF) are pure short exact sequences in $\mathcal{E}'$. Hence, application of the left exact functor F yields a diagram



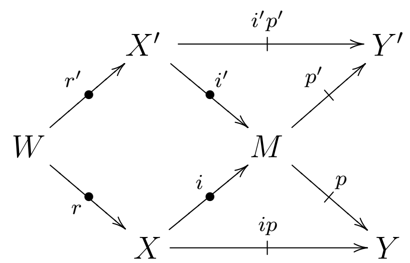
in $\mathcal{E}'$ with $(XF, MF, Y'F)$ and $(X'F, Y'F, ZF)$ left exact. By composition, $s'F$ is purely epimorphic, and hence $(X'F, Y'F, ZF)$ is a pure short exact sequence. The quadrangle $(MF, YF, Y'F, ZF)$ is a pure square, for on the kernels, we have the identity on $X'F$ as induced morphism, and the cokernels are zero; cf. [**3,** §A.4; §A.2.2; Lem. A.11]. In particular, it is a pullback, and so $p'F$ is purely epimorphic. We conclude that $(XF, MF, Y'F)$ is a pure short exact sequence.

Verification of (Ex 3°). Suppose given a commutative triangle

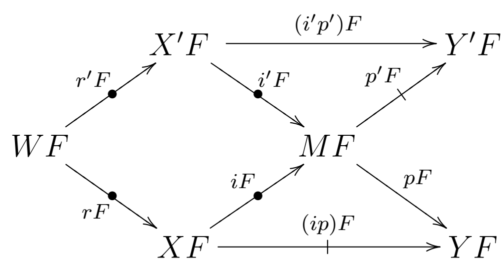


in which, moreover, ip is $\mathcal{S}_F$-purely epimorphic and i is $\mathcal{S}_F$-purely monomorphic.

By exactness of $(\mathcal{E}, \mathcal{S})$, we can complete it to a diagram



in $\mathcal{E}$ with pure short exact sequences (X, M, Y'), (X', M, Y), (W, X, Y) and (W, X', Y'). Moreover, (X, M, Y') and (W, X, Y) are $\mathcal{S}_F$-purely short exact, i.e. $(XF, MF, Y'F)$ and (WF, XF, YF) are pure short exact sequences in $\mathcal{E}'$. Hence, application of F yields a diagram



in $\mathcal{E}'$ with $(WF, X'F, Y'F)$ and $(X'F, MF, YF)$ left exact. By composition, the morphism pF is purely epimorphic, and thus $(X'F, MF, YF)$ is a pure short exact sequence. By (Ex 3°) in $\mathcal{E}'$, the morphism $(i'p')F$ is purely epimorphic, and thus $(WF, X'F, Y'F)$ is a pure short exact sequence. $\square$

Remark 3 A possible source of mistakes. Given an $\mathcal{S}$-pure monomorphism $X \rightarrowtail Y$ in $\mathcal{E}$ such that its image $FX \rightarrowtail FY$ is purely monomorphic, we cannot conclude that $X \rightarrowtail Y$ is $\mathcal{S}_F$-purely monomorphic. In fact, the image of *every* $\mathcal{S}$-pure monomorphism under F is purely monomorphic.

2 The construction principle

Let $(\mathcal{E}, \mathcal{S})$ be an exact category, where $\mathcal{S}$ denotes the set of pure short exact sequences, and let $\mathcal{N} \subseteq \mathcal{E}$ be a full additive subcategory.

Consider the following set of pure short exact sequences.

$$\mathcal{S}_{\mathcal{N}} := \left(\bigcap_{N \in \operatorname{Ob} \mathcal{N}} \mathcal{S}_{\mathcal{E}(N,-)} \right) \cap \left(\bigcap_{N \in \operatorname{Ob} \mathcal{N}} \mathcal{S}_{\mathcal{E}(-,N)} \right) .$$

Then $(\mathcal{E}, \mathcal{S}_{\mathcal{N}})$ is an exact category by Lemma 2 and Remark 1. The short exact sequences in $\mathcal{S}_{\mathcal{N}}$ are called *$\mathcal{N}$-pure short exact sequences.* The pure monomorphisms in this exact category are called *$\mathcal{N}$-pure monomorphisms,* and the pure epimorphisms therein are called *$\mathcal{N}$-pure epimorphisms.*

By construction, the subcategory $\mathcal{N} \subseteq \mathcal{E}$ consists of bijective objects in $(\mathcal{E}, \mathcal{S}_{\mathcal{N}})$; that is, each $N \in \operatorname{Ob} \mathcal{N}$ is bijective with respect to the $\mathcal{N}$-pure short exact sequences.

Written out, an $\mathcal{N}$-pure short exact sequence in $\mathcal{E}$ is a pure short exact sequence $X' \rightarrowtail X \twoheadrightarrow X''$ such that for any $N \in \operatorname{Ob}\mathcal{N}$ and any morphism $N \longrightarrow X''$, there exists a factorisation $(N \longrightarrow X'') = (N \longrightarrow X \twoheadrightarrow X'')$; and, dually, such that for any $N \in \operatorname{Ob}\mathcal{N}$ and any morphism $X' \longrightarrow N$, there exists a factorisation $(X' \longrightarrow N) = (X' \rightarrowtail X \longrightarrow N)$.

An $\mathcal{N}$-pure short exact sequence $X' \rightarrowtail N \twoheadrightarrow X''$ in $\mathcal{E}$ is called *$\mathcal{N}$-resolving* if $N \in \operatorname{Ob}\mathcal{N}$.

Remark 4 *The category $(\mathcal{E}, \mathcal{S}_{\mathcal{N}})$, i.e. the given exact category $\mathcal{E}$ together with the set of $\mathcal{N}$-pure short exact sequences $\mathcal{S}_{\mathcal{N}}$, is a Frobenius category if the following conditions* (1) *and* (2) *are fulfilled. In this case, $\mathcal{N}$ is a sufficiently big subcategory of bijectives.*

(1) *For all $X'' \in \operatorname{Ob}\mathcal{E}$, there exists a $\mathcal{N}$-resolving pure short exact sequence with cokernel term X''.*

(2) *For all $X' \in \operatorname{Ob}\mathcal{E}$, there exists a $\mathcal{N}$-resolving pure short exact sequence with kernel term X'.*

3 Application to $\mathcal{C}(\dot{\Delta}_n)$

Suppose given $n \geq 1$. Recall that $\dot{\Delta}_n = \Delta_n \smallsetminus \{0\} = [1, n]$.

Let $\mathcal{C}$ be a weakly abelian category. We shall consider the category $\mathcal{C}(\dot{\Delta}_n)$. For ease of notation, we formally put $X_{n+1} := 0$ for $X \in \operatorname{Ob}\mathcal{C}(\dot{\Delta}_n)$.

A sequence $X' \xrightarrow{i} X \xrightarrow{p} X''$ in $\mathcal{C}(\dot{\Delta}_n)$ is called *pointwise split short exact,* if the sequence $X'_k \xrightarrow{i_k} X_k \xrightarrow{p_k} X''_k$ is split short exact for all $k \in [1, n]$. The kernel in a pointwise split short exact sequence is *pointwise split monomorphic,* the cokernel *pointwise split epimorphic.* The additive category $\mathcal{C}(\dot{\Delta}_n)$, equipped with the set of pointwise split short exact sequences as pure short exact sequences, is an exact category; cf. e.g. [**3,** Ex. A.3, Ex. A.4].

Consider the full subcategory $\mathcal{C}^{\text{split}}(\dot{\Delta}_n) \subseteq \mathcal{C}(\dot{\Delta}_n)$ whose objects are diagrams $X \in \operatorname{Ob}\mathcal{C}(\dot{\Delta}_n)$ such that $X_k \xrightarrow{x} X_l$ is split for all $k, l \in [1, n]$ with $k \leq l$.

Let $\mathcal{S}$ denote the set of pointwise split short exact sequences in $\mathcal{C}(\dot{\Delta}_n)$.

Lemma 5 *Suppose given a pointwise split short exact sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ in $\mathcal{C}(\dot{\Delta}_n)$ such that, for all $l, m \in [1, n]$ with $l \leq m$, the quadrangle (X_l, X_m, X''_l, X''_m) has the following property* $(*)$.

$(*)$ *The morphism induced from the kernel of $X_l \xrightarrow{x} X_m$ in $\hat{\mathcal{C}}$ to the kernel of $X''_l \xrightarrow{x} X''_m$ in $\hat{\mathcal{C}}$ is epimorphic.*

Suppose given $Z \in \operatorname{Ob}\mathcal{C}^{\text{split}}(\dot{\Delta}_n)$. Then application of the functors $(Z, -) = {}_{\mathcal{C}(\dot{\Delta}_n)}(Z, -)$ and $(-, Z) = {}_{\mathcal{C}(\dot{\Delta}_n)}(-, Z)$ yields short exact sequences

$$\begin{array}{ccccc} (Z, X') & \xrightarrow{(Z,f)} & (Z, X) & \xrightarrow{(Z,g)} & (Z, X'') \\ (X', Z) & \xleftarrow{(f,Z)} & (X, Z) & \xleftarrow{(g,Z)} & (X'', Z) \end{array}$$

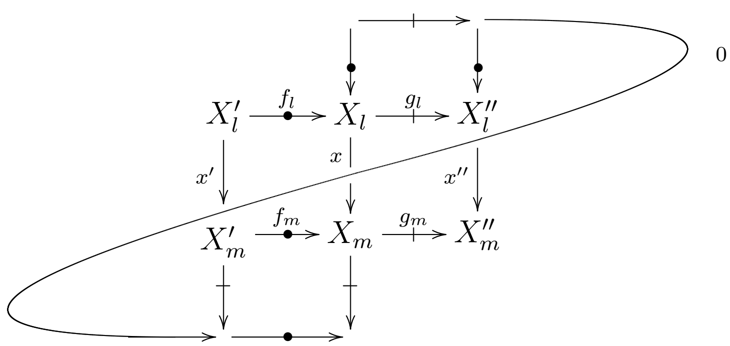
of abelian groups. In other words, the sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ *is* $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$*-purely short exact; still in other words, it is contained in* $\mathcal{S}_{\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)}$.

Proof. We claim that $(Z, X) \xrightarrow{(Z,g)} (Z, X'')$ is surjective. By Lemma [**3,** A.25], applied to the abelian Frobenius category $\hat{\mathcal{C}}$, we may assume that Z is an interval, say $Z = C_{[l,m]}$ with $C \in \operatorname{Ob}\mathcal{C}$ and $l,\, m \in [1, n]$ with $l \le m$; cf. [**3,** §A.6.2].

A morphism $C_{[l,m]} \longrightarrow X''$ is determined by a morphism $C \xrightarrow{t} X''_l$ such that the composite $(C \xrightarrow{t} X''_l \xrightarrow{x''} X''_{m+1})$ vanishes. To prove the asserted surjectivity, we have to find a morphism $C \xrightarrow{t'} X_l$ such that the composite $(C \xrightarrow{t'} X_l \xrightarrow{x} X_{m+1})$ vanishes and such that $(C \xrightarrow{t'} X_l \xrightarrow{g_l} X''_l) = (C \xrightarrow{t} X''_l)$.

To do so, we may assume that $m < n$. Let $K_x \xrightarrow{s} X_l$ denote the kernel of $X_l \xrightarrow{x} X_{m+1}$ in $\hat{\mathcal{C}}$, and let $K_{x''} \xrightarrow{s''} X''_l$ denote the kernel of $X''_l \xrightarrow{x''} X''_{m+1}$ in $\hat{\mathcal{C}}$. By $(*)$, we obtain an induced epimorphism $K_x \xrightarrow{\tilde{g}} K_{x''}$, characterized by $\tilde{g}s'' = sg_l$. We factor $(C \xrightarrow{t} X''_l) = (C \xrightarrow{t_1} K_{x''} \xrightarrow{s''} X''_l)$ by the universal property of s''. Then we factor $(C \xrightarrow{t_1} K_{x''}) = (C \xrightarrow{t_2} K_x \xrightarrow{\tilde{g}} K_{\tilde{x}})$ by epimorphy of $\tilde{g}$ and by bijectivity of C in $\hat{\mathcal{C}}$. We may use $t' := t_2 s$. This proves the claim.

We claim that $(X', Z) \xrightarrow{(f,Z)} (X, Z)$ is surjective. By duality, it suffices to show that, given $l,\, m \in [1, n]$ with $l \le m$, the morphism induced from the cokernel of $X'_l \longrightarrow X'_m$ in $\hat{\mathcal{C}}$ to the cokernel of $X_l \longrightarrow X_m$ in $\hat{\mathcal{C}}$, is monomorphic. This in turn follows by an application of the snake lemma in $\hat{\mathcal{C}}$ to the morphism $(X'_l, X_l, X''_l) \xrightarrow{(x', x, x'')} (X'_m, X_m, X''_m)$ of short exact sequences.



This proves the second claim. □

Proposition 6 *The category* $\mathcal{C}(\dot{\Delta}_n)$*, equipped with the set* $\mathcal{S}_{\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)}$ *of* $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$*-pure short exact sequences, is a Frobenius category with* $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$ *as a sufficiently big subcategory of bijectives.*

Proof. By Remark 4 and by duality, it suffices to show that for each object $X \in \operatorname{Ob}\mathcal{C}(\dot{\Delta}_n)$ there exists a $\mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$-resolving pure exact sequence with cokernel term X.

Write $K_{i,n+1} := X_i$ for $i \in [1, n]$. For the notion of a weak square, we refer to [**3,** Def. A.9].

Choose a diagram

$$
\begin{array}{ccccccccccccc}
K_{1,2} & \longrightarrow & 0 & & & & & & & & & & \\
{\scriptstyle k}\downarrow & + & \downarrow & & & & & & & & & & \\
K_{1,3} & \xrightarrow{k} & K_{2,3} & \longrightarrow & 0 & & & & & & & & \\
{\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & \downarrow & & & & & & & & \\
K_{1,4} & \xrightarrow{k} & K_{2,4} & \xrightarrow{k} & K_{3,4} & \longrightarrow & 0 & & & & & & \\
{\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & \downarrow & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \cdots & & \vdots & & \\
{\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & & + & {\scriptstyle k}\downarrow & & \\
K_{1,n} & \xrightarrow{k} & K_{2,n} & \xrightarrow{k} & K_{3,n} & \xrightarrow{k} & K_{4,n} & \xrightarrow{k} & \cdots & \xrightarrow{k} & K_{n-1,n} & \longrightarrow & 0 \\
{\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & {\scriptstyle k}\downarrow & + & & + & {\scriptstyle k}\downarrow & + & \downarrow \\
K_{1,n+1} & \xrightarrow{k} & K_{2,n+1} & \xrightarrow{k} & K_{3,n+1} & \xrightarrow{k} & K_{4,n+1} & \xrightarrow{k} & \cdots & \xrightarrow{k} & K_{n-1,n+1} & \xrightarrow{k} & K_{n,n+1} \longrightarrow 0
\end{array}
$$

in $\mathcal{C}$, and where $(K_{l,n+1} \xrightarrow{k} K_{m,n+1}) = (X_l \xrightarrow{x} X_m)$ for $l,\, m \in [1,n]$ with $l \leq m$. This is possible since $[1,n]$ is linearly ordered, proceeding from right to left and from the bottom to the top.

We write also $K_{i,j} \xrightarrow{k} K_{i',j'}$ whenever $i,\, j,\, i',\, j' \in [1, n+1]$ with $i < j$, with $i' < j'$, with $i \leq i'$ and $j \leq j'$. In particular, for $i = i'$ and $j = j'$, the morphism $K_{i,j} \xrightarrow{k} K_{i,j}$ is an identity. Note that $K_{i,j} \xrightarrow{k} K_{i',j'}$ is zero unless $i' < j$.

The morphism $(K_{i,j} \xrightarrow{k} K_{i,n+1})$ is a weak kernel of $(K_{i,n+1} \xrightarrow{k} K_{j,n+1}) = (X_i \xrightarrow{x} X_j)$ for $i,\, j \in [1,n]$ with $i \leq j$; cf. [**3,** Lem. A.14, Rem. A.27].

We shall define an object $P \in \operatorname{Ob} \mathcal{C}^{\mathrm{split}}(\dot{\Delta}_n)$. Given $l \in [1,n]$, we let

$$P_l \;:=\; \bigoplus_{i \in [1,l]} \; \bigoplus_{j \in [l+1,n+1]} K_{i,j} \; .$$

Given $l,\, m \in [1,n]$ with $l < m$, we let the morphism $P_l \xrightarrow{p} P_m$ be defined by the matrix $p = (p_{(i,j),(i',j')})_{(i,j),(i',j')}$, where

$$p_{(i,j),(i',j')} \;:=\; \partial_{j,j'}(\partial_{i,i'} + k\partial_{i,l}\{i' \in [l+1,m]\}) \; .$$

First, let us verify that $(P_l \xrightarrow{p} P_m \xrightarrow{p} P_r) = (P_l \xrightarrow{p} P_r)$ for $l < m < r$ in $[1,n]$. In fact, at $i \in [1,l]$, $j \in [l+1, n+1]$, $i'' \in [1,r]$, $j'' \in [r+1, n+1]$, we obtain

$$
\begin{array}{rl}
 & \sum_{i' \in [1,m]} \sum_{j' \in [m+1,n+1]} \partial_{j,j'}\, (\partial_{i,i'} + k\partial_{i,l}\{i' \in [l+1,m]\})\, \partial_{j',j''}\, (\partial_{i',i''} + k\partial_{i',m}\{i'' \in [m+1,r]\}) \\
= & \partial_{j,j''}\Big(\partial_{i,i''} + k\partial_{i,m}\{i'' \in [m+1,r]\} + k\partial_{i,l}\{i'' \in [l+1,m]\} + k\partial_{i,l}\{i'' \in [m+1,r]\}\Big) \\
= & \partial_{j,j''}\Big(\partial_{i,i''} + k\partial_{i,l}\{i'' \in [l+1,r]\}\Big) \; .
\end{array}
$$

Given $l,\, m \in [1,n]$ with $l < m$, we let $P_{l,m} := \bigoplus_{i \in [1,l]} \bigoplus_{j \in [m+1,n+1]} K_{i,j}$. The projection $P_l \longrightarrow P_{l,m}$ is split epimorphic. The morphism $P_{l,m} \longrightarrow P_m$ given by the matrix $p|_{P_{l,m}} = (p_{(i,j),(i',j')})_{(i,j),(i',j')}$ is split monomorphic, for it has the projection $P_m \longrightarrow P_{l,m}$ as a retraction. Now since our morphism factors as $(P_l \xrightarrow{p} P_m) = (P_l \longrightarrow P_{l,m} \longrightarrow P_m)$, it is split. We conclude that $P \in \operatorname{Ob} \mathcal{C}^{\text{split}}(\dot{\Delta}_n)$.

Given $l \in [1, n]$, we let $P_l \xrightarrow{\pi} K_{l,n+1} = X_l$ be the morphism given by the column vector $\pi = (\pi_{(i,j)})_{(i,j)}$ with

$$\pi_{(i,j)} \;=\; \partial_{i,l} k \;.$$

So $P_l \xrightarrow{\pi} K_{l,n+1} = X_l$ is split epimorphic, for it has the inclusion of $K_{l,n+1}$ into P_l as a coretraction.

We claim that these morphisms furnish a pointwise split epimorphism $P \xrightarrow{\pi} X$. Suppose given $l,\, m \in [1, n]$ with $l < m$. We have to show that

$$(P_l \xrightarrow{\pi} K_{l,n+1} \xrightarrow{k} K_{m,n+1}) \stackrel{!}{=} (P_l \xrightarrow{p} P_m \xrightarrow{\pi} K_{m,n+1}) \;.$$

Suppose given $i \in [1, l]$ and $j \in [l{+}1, n]$. At position (i, j), the right hand side composition has the entry

$$\begin{aligned} & \textstyle\sum_{i' \in [1,m]} \sum_{j' \in [m+1,n+1]} \partial_{j,j'} \, (\partial_{i,i'} + k \partial_{i,l} \{ i' \in [l+1, m] \}) \, \partial_{i',m} k \\ = \;& \{ j \in [m+1, n+1] \} \partial_{i,l} k \\ = \;& \pi_{(i,j)} k \;, \end{aligned}$$

being the entry and so does the left hand side composition. We conclude that $\pi k = p\pi$.

We claim that $P \xrightarrow{\pi} X$ is $\mathcal{C}^{\text{split}}(\dot{\Delta}_n)$-purely epimorphic. By Lemma 5, it suffices to show that for $l,\, m \in [1, n]$ with $l < m$, for the quadrangle (P_l, P_m, X_l, X_m), the induced morphism from the kernel of $P_l \xrightarrow{p} P_m$ in $\hat{\mathcal{C}}$ to the kernel of $X_l \xrightarrow{x} X_m$ in $\hat{\mathcal{C}}$ is epimorphic. Since by [**3,** Rem. A.27], the induced map from the weak kernel $K_{l,m}$ to the kernel of $X_l \xrightarrow{x} X_m$ is epimorphic, it suffices to find an epimorphic induced morphism from the kernel of $P_l \xrightarrow{p} P_m$ to $K_{l,m}$.

The kernel of $P_l \xrightarrow{p} P_m$ is given by $\bigoplus_{i \in [1,l]} \; \bigoplus_{j \in [l+1,m]} K_{i,j}$, together with the inclusion into P_l.

As induced morphism $\bigoplus_{i \in [1,l]} \; \bigoplus_{j \in [l+1,m]} K_{i,j} \;\longrightarrow\; K_{l,m}$, we take the column vector $(\partial_{i,l} k)_{(i,j)}$.

This induced morphism is split epimorphic, for it has the inclusion of $K_{l,m}$ into that kernel as a coretraction. This proves the claim on $P \xrightarrow{\pi} X$. ▫

Example 7 We display the matrix of the morphism $P_3 \xrightarrow{p} P_5$ in the case $n = 7$ (in the notation of the proof of Proposition 6). We have

$$\begin{aligned} P_3 &= (K_{1,4} \oplus K_{1,5} \oplus K_{1,6} \oplus K_{1,7} \oplus K_{1,8}) \oplus (K_{2,4} \oplus K_{2,5} \oplus K_{2,6} \oplus K_{2,7} \oplus K_{2,8}) \oplus (K_{3,4} \oplus K_{3,5} \oplus K_{3,6} \oplus K_{3,7} \oplus K_{3,8}) \\ P_5 &= (K_{1,6} \oplus K_{1,7} \oplus K_{1,8}) \oplus (K_{2,6} \oplus K_{2,7} \oplus K_{2,8}) \oplus (K_{3,6} \oplus K_{3,7} \oplus K_{3,8}) \oplus (K_{4,6} \oplus K_{4,7} \oplus K_{4,8}) \oplus (K_{5,6} \oplus K_{5,7} \oplus K_{5,8}) \;, \end{aligned}$$

and the morphism $P_3 \xrightarrow{p} P_5$ is given by the matrix

$$
\left(\begin{array}{ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&k&0&0&k&0&0\\
0&0&0&0&0&0&0&1&0&0&k&0&0&k&0\\
0&0&0&0&0&0&0&0&1&0&0&k&0&0&k
\end{array}\right) .
$$

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