Classifying group extensions with not necessarily abelian kernel

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Chapter 0

Introduction

Suppose given groups A and G, not necessarily abelian.

0.1 A problem and its history

If A is abelian and we are given an action of G on A, the equivalence classes of group extensions of A by G that induce this action are in bijection to the second cohomology group $\mathrm{H}^2(G, A)$; cf. e.g. [**2,** IV, Th. 3.12].

For general A, we want to describe the set $\mathrm{Ext}(G, A)$ of equivalence classes of group extensions of A by G.

In 1926, Schreier introduced what we call normalized generalized 2-cocycles to describe the set $\mathrm{Ext}(G, A)$; cf. [**8,** Satz I], Lemma 2.7.

We write $\mathrm{Out}(A) = \mathrm{Aut}(A)/\,\mathrm{Int}(A)$ for the outer automorphism group of A. A group extension $1 \to A \to E \to G \to 1$ induces a group morphism $G \to \mathrm{Out}(A)$.

In 1947, Eilenberg and Mac Lane conversely supposed given a group morphism

$$\varpi \;:\; G \;\to\; \mathrm{Out}(A)$$

and found an obstruction in $\mathrm{H}^3(G, \mathrm{Z}(A))$ to the existence of a group extension of A by G that induces ϖ; cf. [**6**].

They proceeded as follows. Let $\xi_g \in \mathrm{Aut}(A)$ be a lift of $\varpi_g \in \mathrm{Out}(A)$ for $g \in G$. Note that $G \to \mathrm{Aut}(A)$, $g \mapsto \xi_g$ is not a group morphism in general. But we may choose a map $f : G \times G \to A$ such that

$$\xi_g \circ \xi_h \;=\; \mathrm{Int}(f(g,h)) \circ \xi_{gh}$$

for g, $h \in G$. It turns out that there is a 3-cocycle $c \in \mathrm{Z}^3(G, \mathrm{Z}(A))$ such that

$$f(g,h) \cdot f(gh,k) = c(g,h,k) \cdot \xi_g(f(h,k)) \cdot f(g,hk)$$

for g, h, $k \in G$. Then a group extension of A by G that induces ϖ exists if and only if

$$c \cdot \mathrm{B}^3(G, \mathrm{Z}(A)) \;=\; 1_{\mathrm{H}^3(G,\mathrm{Z}(A))}\,.$$

Cf. Remark 3.2, Lemma 3.5 and Theorem 3.7.

More generally, EILENBERG and MAC LANE start with an abelian group Z and consider the set of pairs $(\tilde{A}, G \to \mathrm{Out}(\tilde{A}))$, where $\tilde{A}$ is a group with center isomorphic to Z and $G \to \mathrm{Out}(\tilde{A})$ is a group morphism. On this set, they define an equivalence relation. On the set of equivalence classes, they define a group structure; cf. [**6,** §6]. They show that this group is isomorphic to $\mathrm{H}^3(G, Z)$; cf. [**6,** Th. 10.1]. An equivalence class of a pair $(\tilde{A}, G \to \mathrm{Out}(\tilde{A}))$ is mapped to the trivial element of $\mathrm{H}^3(G, Z)$ if and only if there exists a group extension $1 \to \tilde{A} \to E \to G \to 1$ inducing $G \to \mathrm{Out}(\tilde{A})$.

In addition, they show that if a group extension $1 \to A \to E \to G \to 1$ inducing ϖ exists, then the set of equivalence classes of group extensions of A by G inducing ϖ is in bijection to $\mathrm{H}^2(G, \mathrm{Z}(A))$; cf. [**6,** Th. 11.1].

Already in 1934, BAER has shown the obstruction part and the H^2-part of this theory in the particular case $\mathrm{Z}(A) = 1$; cf. [**1,** p. 375].

MAC LANE gave an account of this theory in [**5,** Ch. IV, §8 and §9]. BROWN gives a sketch in [**2,** Ch. IV, §6].

In 2000, MORANDI introduced an equivalence relation on the set $\mathrm{z}^2(G, A)$ of normalized generalized 2-cocycles. The set of equivalence classes is written $\mathrm{h}^2(G, A)$. He showed a bijection between $\mathrm{h}^2(G, A)$ and $\mathrm{Ext}(G, A)$. Cf. [**7**].

We give an account of the result of SCHREIER and MORANDI. Moreover, we give an account of the obstruction part and the H^2-part of the theory of EILENBERG and MAC LANE. We shall summarize the results in the following §0.2.

0.2 Results

The following results are a reformulation of results of SCHREIER, MORANDI, EILENBERG and MAC LANE; cf. [**8**], [**7**], [**6**]. For the history of these results, see §0.1.

Recall that A and G are groups, not necessarily abelian.

The set $\mathrm{h}^2(G, A)$ of equivalence classes of normalized 2-cocycles is defined in Lemma 2.7. We write $\mathrm{Ext}(G, A)$ for the set of equivalence classes of group extensions of A by G ; cf. Definition 2.3.

Theorem 2.17. We construct mutually inverse bijections

$$\mathrm{Ext}(G, A) \underset{\beta}{\overset{\alpha}{\underset{\sim}{\rightleftarrows}}} \mathrm{h}^2(G, A) \; ;$$

cf. Propositions 2.9 and 2.14.

Suppose given a group morphism $\varpi : G \to \mathrm{Out}(A)$. Then ϖ induces a group morphism $\varpi^{\mathrm{Z}} : G \to \mathrm{Aut}(\mathrm{Z}(A))$; cf. Remark 3.2. Cohomology groups of G with values in $\mathrm{Z}(A)$ are formed with respect to ϖ^{Z}.

To ϖ we attach an element $\zeta_\varpi \in \mathrm{H}^3(G, \mathrm{Z}(A))$; cf. Lemma 3.5.(3).

Theorem 3.7. There is a group extension of A by G inducing the group morphism

$$\varpi \;:\; G \;\to\; \mathrm{Out}(A)$$

if and only if

$$\zeta_\varpi = 1$$

in $\mathrm{H}^3(G, \mathrm{Z}(A))$.

We give an example in which the map ζ is trivial in spite of $|\,\mathrm{Mor}(G, \mathrm{Out}(A))| \neq 1$ and $|\,\mathrm{H}^3(G, \mathrm{Z}(A))| \neq 1$ and $1 < \mathrm{Z}(A) < A$; cf. Example 3.17.

Let $\mathrm{Ext}_\varpi(G, A)$ be the subset of $\mathrm{Ext}(G, A)$ consisting of those equivalence classes of group extensions of A by G that induce ϖ in the sense of Remark 3.1.

Let $\mathrm{h}^2_\varpi(G, A)$ be the subset of $\mathrm{h}^2(G, A)$ consisting of those equivalence classes of normalized generalized 2-cocycles (ω, f) such that ω lifts ϖ ; cf. Lemmata 2.7 and 3.9.

Then the bijections of Theorem 2.17 restrict to the subsets $\mathrm{Ext}_\varpi(G, A)$ and $\mathrm{h}^2_\varpi(G, A)$. The restrictions are written α_ϖ and β_ϖ, respectively.

We use a normalized variant $\mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$ of $\mathrm{H}^2(G, \mathrm{Z}(A))$, in which the cohomology classes are represented by normalized 2-cocycles; cf. Remark 3.3, part 2. By Lemma 1.13, we have

$$\mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) \;\overset{\varphi}{\underset{\sim}{\to}}\; \mathrm{H}^2(G, \mathrm{Z}(A)) \;.$$

Theorem 3.13. Suppose that there exists a group extension of A by G inducing the given group morphism $\varpi : G \to \mathrm{Out}(A)$.

Therefore $\mathrm{h}^2_\varpi(G, A) \neq \emptyset$; cf. Lemma 3.9. So we may choose an element (ξ, f_0) in $\mathrm{z}^2(G, A)$ such that $[\xi, f_0] \in \mathrm{h}^2_\varpi(G, A)$.

We construct mutually inverse bijections $\vartheta^{\,\xi, f_0}$ and $\eta^{\,\xi, f_0}$ so that altogether we obtain the following diagram.

$$\begin{tikzcd} \mathrm{Ext}_\varpi(G, A) \arrow[r, shift left, "\alpha_\varpi"] & \mathrm{h}^2_\varpi(G, A) \arrow[l, shift left, "\beta_\varpi"] \arrow[r, shift left, "\vartheta^{\,\xi, f_0}"] & \mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) \arrow[l, shift left, "\eta^{\,\xi, f_0}"] \arrow[r, "\varphi", "\sim"'] & \mathrm{H}^2(G, \mathrm{Z}(A)) \end{tikzcd}$$

In particular, if A is abelian, we recover the theory of group extensions with abelian kernel; cf. Remarks 3.8 and 3.16.

Conventions

(1) Given $a,\, b \in \mathbf{Z}$, we write $[a,b] := \{\, z \in \mathbf{Z} \,:\, a \le z \le b \,\}$.

(2) Suppose given a set X. Suppose given $a,\, b \in \mathbf{Z}$. Suppose given $x_i \in X$ for $i \in [a,b]$. We write

$$x_{[a,b]} := (x_a, x_{a+1}, \ldots, x_b)\ .$$

In particular, if $a > b$, then $x_{[a,b]}$ is the empty tuple.

(3) Suppose given a set X. Suppose given $k \ge 1$. Suppose given $a_i\,,\, b_i \in \mathbf{Z}$ for $i \in [1,k]$. Suppose given a tuple $x_{[a_i,b_i]}$ with entries in X for $i \in [1,k]$. Then, by abuse of notation, we write

$$(x_{[a_1,b_1]}, x_{[a_2,b_2]}, \ldots, x_{[a_k,b_k]}) \;:=\; (x_{a_1}, x_{a_1+1}, \ldots, x_{b_1},\ x_{a_2}, x_{a_2+1}, \ldots, x_{b_2},\quad \ldots\quad,\ x_{a_k}, x_{a_k+1}, \ldots, x_{b_k})$$

for the concatenation of the tuples $x_{[a_i,b_i]}$ for $i \in [1,k]$.

Single elements of X are viewed as tuples with one element in this context.

Empty tuples vanish when concatenated.

For example,

$$(x_{[1,3]}, y, x_{[7,10]}, x_{[13,12]}, z) \;=\; (x_1, x_2, x_3, y, x_7, x_8, x_9, x_{10}, z)\ ,$$

where all entries are in X.

(4) Given sets X and Y, we write $\mathrm{Map}(X,Y)$ for the set of maps from X to Y.

Given groups G and H, we write $\mathrm{Mor}(G,H)$ for the set of group morphisms from G to H.

The group morphism from G to H that maps g to 1 for $g \in G$ is denoted by $!$.

Cf. also Definition 1.1.

(5) For a set X and $k \ge 0$, we let $X^k := \prod_{i\in[1,k]} X$.

In particular, $X^0 = \{(\ )\}$, containing only the empty tuple.

Chapter 1

Preliminaries on cohomology groups

Let G be a group acting on an abelian group M via a group morphism $\varphi : G \to \mathrm{Aut}(M)$.

We often write $(\varphi(g))(m) = {}^g m$ for $g \in G$ and $m \in M$.

1.1 The cohomology groups

Definition 1.1. Let X be a set. Let Y be a group. We write $\mathrm{Map}(X, Y)$ for the set of maps from X to Y. Suppose given $f, f' \in \mathrm{Map}(X, Y)$. We define their product via

$$(f \cdot f')(x) \;=\; (f \cdot_{\mathrm{Map}(X,Y)} f')(x) \;:=\; f(x) \cdot_Y f'(x)$$

for $x \in X$. Then $(\mathrm{Map}(X, Y), \cdot)$ is a group, often denoted by $\mathrm{Map}(X, Y)$.

If Y is abelian, then $\mathrm{Map}(X, Y)$ is abelian.

Its identity element is given by the map

$$\begin{array}{rcrcl} ! & : & X & \to & Y \\ & & x & \mapsto & !(x) := 1 \,. \end{array}$$

Lemma 1.2. Suppose given $n \geq 0$. We define the map $\partial_n : \mathrm{Map}(G^n, M) \to \mathrm{Map}(G^{n+1}, M)$ by

$$\begin{array}{rcl} (\partial_n(f))(g_1, g_2, \ldots, g_{n+1}) & := & {}^{g_1} f(g_2, \ldots, g_{n+1}) \cdot f(g_1 g_2, g_3, \ldots, g_{n+1})^{-1} \cdot f(g_1, g_2 g_3, g_4, \ldots, g_{n+1}) \\ & & \cdots \\ & & \cdot\, f(g_1, g_2, \ldots, g_{n-1}, g_n g_{n+1})^{(-1)^n} \cdot f(g_1, g_2, \ldots, g_n)^{(-1)^{n+1}} \end{array}$$

for $f \in \mathrm{Map}(G^n, M)$ and $g_1, \ldots, g_{n+1} \in G$.

We often write just $\partial := \partial_n$ and $\partial f := \partial_n(f)$.

Then $\partial : \mathrm{Map}(G^n, M) \to \mathrm{Map}(G^{n+1}, M)$ is a group morphism between abelian groups.

Proof. Suppose given $f, f' \in \mathrm{Map}(G^n, M)$. Because M is abelian, we have

$$\begin{aligned}
& (\partial(f\cdot f'))(g_1, g_2, \ldots, g_{n+1}) \\
= \; & {}^{g_1}(f\cdot f')(g_2, \ldots, g_{n+1}) \cdot (f\cdot f')(g_1g_2, g_3, \ldots, g_{n+1})^{-1} \\
& \ldots \\
& \cdot (f\cdot f')(g_1, g_2, \ldots, g_{n-1}, g_ng_{n+1})^{(-1)^n} \cdot (f\cdot f')(g_1, g_2, \ldots, g_n)^{(-1)^{n+1}} \\
= \; & {}^{g_1}f(g_2, \ldots, g_{n+1}) \cdot f(g_1g_2, g_3, \ldots, g_{n+1})^{-1} \\
& \ldots \\
& \cdot f(g_1, g_2, \ldots, g_{n-1}, g_ng_{n+1})^{(-1)^n} \cdot f(g_1, g_2, \ldots, g_n)^{(-1)^{n+1}} \\
& \cdot {}^{g_1}f'(g_2, \ldots, g_{n+1}) \cdot f'(g_1g_2, g_3, \ldots, g_{n+1})^{-1} \cdot \\
& \ldots \\
& \cdot f'(g_1, g_2, \ldots, g_{n-1}, g_ng_{n+1})^{(-1)^n} \cdot f'(g_1, g_2, \ldots, g_n)^{(-1)^{n+1}} \\
= \; & \partial f(g_1, \ldots, g_{n+1}) \cdot \partial f'(g_1, \ldots, g_{n+1}) \\
= \; & (\partial f \cdot \partial f')(g_1, \ldots, g_{n+1})
\end{aligned}$$

So ∂ is a group morphism. □

Lemma 1.3. Suppose given $g_{[1,n+1]} := (g_1, g_2, \ldots, g_{n+1}) \in G^{n+1}$. Suppose given $i \in [1, n+1]$. We write

$$g_{[1,n+1]} * i \;:=\; \begin{cases}
(g_2, \ldots, g_{n+1}) & = \; g_{[2,n+1]} & \text{if } i = 0 \\
(g_1g_2, g_3 \ldots, g_{n+1}) & = \; (g_1g_2, g_{[3,n+1]}) & \text{if } i = 1 \\
(g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_{n+1}) & = \; (g_{[1,i-1]}, g_ig_{i+1}, g_{[i+2,n+1]}) & \text{if } i \in [2, n-1] \\
(g_1, \ldots, g_{n-1}, g_ng_{n+1}) & = \; (g_{[1,n-1]}, g_ng_{n+1}) & \text{if } i = n \\
(g_1, \ldots, g_n) & = \; g_{[1,n]} & \text{if } i = n+1
\end{cases}$$

Then we have

$$g_{[1,n+2]} * j * i \;=\; g_{[1,n+2]} * i * (j-1)$$

for $0 \le i < j \le n+2$.

Proof. Suppose that $i = 0$ and $j = 1$. Then

$$\begin{aligned}
g_{[1,n+2]} * i * (j-1) \;&=\; g_{[1,n+2]} * 0 * 0 \\
&=\; g_{[2,n+2]} * 0 \\
&=\; g_{[3,n+2]} \\
&=\; (g_1g_2, g_{[3,n+2]}) * 0 \\
&=\; g_{[1,n+2]} * 1 * 0 \\
&=\; g_{[1,n+2]} * j * i\,.
\end{aligned}$$

Suppose that $i = 0$ and $j \in [2, n+1]$. Then

$$\begin{aligned} g_{[1,n+2]} * i * (j-1) &= g_{[1,n+2]} * 0 * (j-1) \\ &= g_{[2,n+2]} * (j-1) \\ &= (g_{[2,j-1]}, g_j g_{j+1}, g_{[j+2,n+2]}) \\ &= (g_{[1,j-1]}, g_j g_{j+1}, g_{[j+2,n+2]}) * 0 \\ &= g_{[1,n+2]} * j * 0 \\ &= g_{[1,n+2]} * j * i \;. \end{aligned}$$

Suppose that $i = 0$ and $j = n+2$. Then

$$\begin{aligned} g_{[1,n+2]} * i * (j-1) &= g_{[1,n+2]} * 0 * (n+1) \\ &= g_{[2,n+2]} * (n+1) \\ &= g_{[2,n+1]} \\ &= g_{[1,n+1]} * 0 \\ &= g_{[1,n+2]} * (n+2) * 0 \\ &= g_{[1,n+2]} * j * i \;. \end{aligned}$$

Suppose that $i \in [1, n]$ and $j = i+1$. Then

$$\begin{aligned} g_{[1,n+2]} * i * (j-1) &= g_{[1,n+2]} * i * i \\ &= (g_{[1,i-1]}, g_i g_{i+1}, g_{[i+2,n+2]}) * i \\ &= (g_{[1,i-1]}, g_i g_{i+1} g_{i+2}, g_{[i+3,n+2]}) \\ &= (g_{[1,i]}, g_{i+1} g_{i+2}, g_{[i+3,n+2]}) * i \\ &= g_{[1,n+2]} * (i+1) * i \\ &= g_{[1,n+2]} * j * i \;. \end{aligned}$$

Suppose that $i \in [1, n-1]$ and $j \in [i+2, n+1]$. Then

$$\begin{aligned} g_{[1,n+2]} * i * (j-1) &= (g_{[1,i-1]}, g_i g_{i+1}, g_{[i+2,n+2]}) * (j-1) \\ &= (g_{[1,i-1]}, g_i g_{i+1}, g_{[i+2,j-1]}, g_j g_{j+1}, g_{[j+2,n+2]}) \\ &= (g_{[1,j-1]}, g_j g_{j+1}, g_{[j+2,n+2]}) * i \\ &= g_{[1,n+2]} * j * i \;. \end{aligned}$$

Suppose that $i \in [1, n]$ and $j = n+2$. Then

$$\begin{aligned} g_{[1,n+2]} * i * (j-1) &= g_{[1,n+2]} * i * (n+1) \\ &= (g_{[1,i-1]}, g_i g_{i+1}, g_{[i+2,n+2]}) * (n+1) \\ &= (g_{[1,i-1]}, g_i g_{i+1}, g_{[i+2,n+1]}) \\ &= g_{[1,n+1]} * i \\ &= g_{[1,n+2]} * (n+2) * i \\ &= g_{[1,n+2]} * j * i \;. \end{aligned}$$

Suppose that $i = n+1$ and $j = n+2$. Then

$$\begin{aligned} g_{[1,n+2]} * i * (j-1) &= g_{[1,n+2]} * (n+1) * (n+1) \\ &= (g_{[1,n]}, g_{n+1}g_{n+2}) * (n+1) \\ &= g_{[1,n]} \\ &= g_{[1,n+1]} * (n+1) \\ &= g_{[1,n+2]} * (n+2) * (n+1) \\ &= g_{[1,n+2]} * j * i\ . \end{aligned}$$

Then we have proved that

$$g_{[1,n+2]} * j * i = g_{[1,n+2]} * i * (j-1)$$

for $0 \le i < j \le n+2$. $\square$

Example 1.4. Suppose given $n \ge 0$. Suppose given $f \in \mathrm{Map}(G^n, M)$.

For $g := g_{[1,n+1]} \in G^{n+1}$, we have

$$(\partial f)(g) \;=\; {}^{g_1}f(g*0) \cdot \big(\prod_{i\in[1,n+1]} f(g*i)^{(-1)^i} \big)\ .$$

Proposition 1.5. Suppose given $n \ge 0$. We have

$$(\partial\partial f)(g_1, \ldots, g_{n+2}) = 1$$

for $f \in \mathrm{Map}(G^n, M)$ and for $g_1, \ldots, g_{n+2} \in G$.

I.e. we have $\partial\partial f = !$ for $f \in \mathrm{Map}(G^n, M)$.

I.e. we have $\partial\partial = ! : \mathrm{Map}(G^n, M) \to \mathrm{Map}(G^{n+2}, M)$.

Proof. Let $g := g_{[1,n+2]} \in G^{n+2}$. Using Lemma 1.3 repeatedly, we obtain

$$\begin{aligned} \partial\partial f(g) &= {}^{g_1}(\partial f)(g*0) \cdot \big(\prod_{j\in[1,n+2]} (\partial f)(g*j)^{(-1)^j} \big) \\ &= {}^{g_1}\Big({}^{g_2}(f(g*0*0) \cdot \prod_{i\in[1,n+1]} f(g*0*i)^{(-1)^i} \Big) \\ &\quad \cdot {}^{g_1g_2}(f(g*1*0)^{-1}) \cdot \prod_{i\in[1,n+1]} f(g*1*i)^{(-1)^{1+i}} \\ &\quad \cdot \prod_{j\in[2,n+2]} \Big({}^{g_1}(f(g*j*0)^{(-1)^j}) \cdot \prod_{i\in[1,n+1]} f(g*j*i)^{(-1)^{i+j}} \Big) \\ &= {}^{g_1}\Big(\prod_{i\in[1,n+1]} f(g*0*i)^{(-1)^i} \Big) \cdot \big(\prod_{j\in[2,n+2]} {}^{g_1}f(g*j*0)^{(-1)^j} \big) \\ &\quad \cdot \big(\prod_{i\in[1,n+1]} f(g*1*i)^{(-1)^{1+i}} \big) \cdot \big(\prod_{j\in[2,n+2]} \prod_{i\in[1,n+1]} f(g*j*i)^{(-1)^{i+j}} \big) \\ &= {}^{g_1}\Big(\prod_{j\in[2,n+2]} (f(g*0*(j-1)))^{(-1)^{j-1}} \cdot f(g*j*0)^{(-1)^j}) \Big) \\ &\quad \cdot \big(\prod_{i\in[1,n+1]} f(g*1*i)^{(-1)^{1+i}} \big) \cdot \big(\prod_{j\in[2,n+2]} \prod_{i\in[1,n+1]} f(g*j*i)^{(-1)^{i+j}} \big) \\ &= \prod_{j\in[1,n+2]} \prod_{i\in[1,n+1]} f(g*j*i)^{(-1)^{i+j}} \\ &= \prod_{(i,j)\in[1,n+1]\times[1,n+2]} f(g*j*i)^{(-1)^{i+j}}\ . \end{aligned}$$

Let

$$\begin{array}{rcl} X & := & \{\,(i,j) \in [1,n+1] \times [1,n+2] \,:\, i < j\} \\ Y & := & \{\,(i,j) \in [1,n+1] \times [1,n+2] \,:\, i \geq j\} \end{array}$$

We have $X \cap Y = \emptyset$ and $X \cup Y = [1,n+1] \times [1,n+2]$.

We have mutually inverse bijections

$$\begin{array}{rcl} X & \leftrightarrow & Y \\ (i,j) & \overset{q}{\mapsto} & (j-1,i) \\ (j,i+1) & \overset{q^{-1}}{\leftarrow\!\shortmid} & (i,j)\ . \end{array}$$

Therefore, using Lemma 1.3, we obtain

$$\begin{array}{rcl} \partial\partial f(g) & = & \prod\limits_{i \in [1,n+1]\times[1,n+2]} f(g * j * i)^{(-1)^{i+j}} \\ & = & \big(\prod\limits_{(i,j)\in X} f(g*j*i)^{(-1)^{i+j}}\big) \cdot \big(\prod\limits_{(i,j)\in Y} f(g*j*i)^{(-1)^{i+j}}\big) \\ & = & \big(\prod\limits_{(i,j)\in X} f(g*j*i)^{(-1)^{i+j}}\big) \cdot \big(\prod\limits_{(i,j)\in q(X)} f(g*j*i)^{(-1)^{i+j}}\big) \\ & = & \big(\prod\limits_{(i,j)\in X} f(g*j*i)^{(-1)^{i+j}}\big) \cdot \big(\prod\limits_{(i,j)\in X} f(g*i*(j-1))^{(-1)^{j-1+i}}\big) \\ & = & \big(\prod\limits_{(i,j)\in X} f(g*j*i)^{(-1)^{i+j}}\big) \cdot \big(\prod\limits_{(i,j)\in X} f(g*j*i)^{(-1)^{j-1+i}}\big) \\ & = & 1\ . \end{array}$$

□

Definition 1.6. Suppose given $n \geq 1$. We define subgroups

$$\mathrm{Z}^n(G,M) \;:=\; \ker(\mathrm{Map}(G^n,M) \xrightarrow{\partial} \mathrm{Map}(G^{n+1},M)) \;=\; \{\, f \in \mathrm{Map}(G^n,M) \,:\, \partial f = !\,\}.$$

and

$$\mathrm{B}^n(G,M) \;:=\; \mathrm{im}(\mathrm{Map}(G^{n-1},M) \xrightarrow{\partial} \mathrm{Map}(G^n,M)) \;=\; \{\, \partial f \,:\, f \in \mathrm{Map}(G^{n-1},M)\,\}$$

of the abelian group $\mathrm{Map}(G^n,M)$; cf. Lemma 1.2.

An element of $\mathrm{Z}^n(G,M)$ is called an *n-cocycle* of G with values in M.

An element of $\mathrm{B}^n(G,M)$ is called an *n-coboundary* of G with values in M.

Lemma 1.7. The group $\mathrm{B}^n(G,M)$ is a subgroup of $\mathrm{Z}^n(G,M)$.

Proof. Note that $\partial\partial f = !$; cf. Proposition 1.5. That means

$$\partial f \in \ker(\mathrm{Map}(G^n,M) \xrightarrow{\partial} \mathrm{Map}(G^{n+1},M))$$

for $f \in \mathrm{Map}(G^n,M)$. Hence $\mathrm{B}^n(G,M) \leq \mathrm{Z}^n(G,M)$. □

Definition 1.8. Suppose given $n \geq 1$. The nth cohomology group of G with values in M is defined via

$$\mathrm{H}^n(G,M) \;:=\; \mathrm{Z}^n(G,M)/\,\mathrm{B}^n(G,M)$$

Example 1.9. We consider the particular case $n = 2$; cf. Definitions 1.6 and 1.8, Lemma 1.2.

(a) The group of 2-cocycles of G with values in M is given by

$$\mathrm{Z}^2(G,M) \;=\; \left\{ G\times G \xrightarrow{f} M \;:\; \begin{array}{l} f \text{ is a map such that} \\ {}^{g}f(h,k)\cdot f(gh,k)^{-1}\cdot f(g,hk)\cdot f(g,h)^{-1} = 1 \\ \text{for } g,h,k\in G \end{array} \right\}.$$

(b) The group of 2-coboundaries of G with values in M is given by

$$\mathrm{B}^2(G,M) \;=\; \left\{ G\times G \xrightarrow{\partial v} M \;:\; \begin{array}{l} v : G\to M \text{ is a map, and} \\ (\partial v)(g,h) = {}^{g}v(h)\cdot v(gh)^{-1}\cdot v(g) \\ \text{for } g,h\in G \end{array} \right\}.$$

(c) The second cohomology group of G with values in M is the factor group

$$\mathrm{H}^2(G,M) \;=\; \mathrm{Z}^2(G,M)/\,\mathrm{B}^2(G,M)\,.$$

Example 1.10. We consider the particular case $n = 3$; cf. Definitions 1.6 and 1.8, Lemma 1.2.

(a) The group of 3-cocycles of G with values in M is given by

$$\mathrm{Z}^3(G,M) \;=\; \left\{ G\times G\times G \xrightarrow{f} M \;:\; \begin{array}{l} f \text{ is a map such that} \\ {}^{g}f(h,k,l)\cdot f(gh,k,l)^{-1}\cdot f(g,hk,l) \\ \cdot\, f(g,h,kl)^{-1}\cdot f(g,h,k) = 1 \\ \text{for } g,h,k,l\in G \end{array} \right\}.$$

(b) The group of 3-coboundaries of G with values in M is given by

$$\mathrm{B}^3(G,M) \;=\; \left\{ G\times G\times G \xrightarrow{\partial v} M \;:\; \begin{array}{l} v : G\times G\to M \text{ is a map, and} \\ (\partial v)(g,h,k) = {}^{g}v(h,k)\cdot v(gh,k)^{-1}\cdot v(g,hk)\cdot v(g,h)^{-1} \\ \text{for } g,h,k\in G \end{array} \right\}.$$

(c) The third cohomology group of G with values in M is the factor group

$$\mathrm{H}^3(G,M) \;=\; \mathrm{Z}^3(G,M)\;/\,\mathrm{B}^3(G,M)\,.$$

1.2 The groups $\mathrm{Z}^2_{\mathrm{norm}}$, $\mathrm{B}^2_{\mathrm{norm}}$ and $\mathrm{H}^2_{\mathrm{norm}}$

Definition 1.11. The group of *normalized* 2*-cocycles* of G with values in M is defined as

$$\mathrm{Z}^2_{\mathrm{norm}}(G,M) \;:=\; \big\{ f\in \mathrm{Z}^2(G,M) : f(g,1) = 1 \text{ and } f(1,g) = 1 \text{ for } g\in G \big\}\,.$$

The group of *normalized* 2*-coboundaries* of G with values in M is defined as

$$\mathrm{B}^2_{\mathrm{norm}}(G,M) \;:=\; \mathrm{Z}^2_{\mathrm{norm}}(G,M)\;\cap\;\mathrm{B}^2(G,M)\,.$$

The *normalized second cohomology group* of G with values in M is defined as

$$\mathrm{H}^2_{\mathrm{norm}}(G,M) \;:=\; \mathrm{Z}^2_{\mathrm{norm}}(G,M)/\,\mathrm{B}^2_{\mathrm{norm}}(G,M)$$

Cf. Example 1.9.

Remark 1.12. We have

$$\mathrm{B}^2_{\mathrm{norm}}(G,M) \;=\; \left\{ G\times G \xrightarrow{\partial v} M \;:\; \begin{array}{l} v: G\to M \text{ is a map with } v(1)=1, \text{ and} \\ (\partial v)(g,h) = {}^{g}v(h)\cdot v(gh)^{-1}\cdot v(g) \\ \text{for } g,h\in G \end{array} \right\}.$$

Proof.

Ad $\supseteq$. Suppose given a map $v : G \to M$ with $v(1) = 1$. We have to show that ∂v is contained in $\mathrm{B}^2_{\mathrm{norm}}(G,M) = \mathrm{Z}^2_{\mathrm{norm}}(G,M)\cap \mathrm{B}^2(G,M)$. It suffices to show that $(\partial v)(g,1) \overset{!}{=} 1$ and that $(\partial v)(1,g) \overset{!}{=} 1$ for $g\in G$.

We obtain $(\partial v)(g,1) = {}^{g}v(1)\cdot v(g\cdot 1)^{-1}\cdot v(g) = {}^{g}1 = 1$.

Moreover, we obtain $(\partial v)(1,g) = {}^{1}v(g)\cdot v(1\cdot g)^{-1}\cdot v(1) = v(1) = 1$.

Ad $\subseteq$. Suppose given a map $v : G \to M$ such that ∂v is contained in the subgroup $\mathrm{B}^2_{\mathrm{norm}}(G,M) = \mathrm{Z}^2_{\mathrm{norm}}(G,M)\cap \mathrm{B}^2(G,M)$. We have to show that $v(1) \overset{!}{=} 1$.

Since $\partial v \in \mathrm{Z}^2_{\mathrm{norm}}(G,M)$, we have in fact $1 = (\partial v)(1,1) = {}^{1}v(1)\cdot v(1\cdot 1)^{-1}\cdot v(1) = v(1)$. □

Lemma 1.13. Given a map $f : G\times G\to M$, we define the constant map

$$\begin{array}{rcl} G & \xrightarrow{\dot f} & M \\ g & \mapsto & \dot f(g) := f(1,1) \end{array}$$

We have the isomorphism of abelian groups

$$\begin{array}{rcl} \mathrm{H}^2_{\mathrm{norm}}(G,M) & \xrightarrow{\sim} & \mathrm{H}^2(G,M) \\ f\cdot \mathrm{B}^2_{\mathrm{norm}}(G,M) & \overset{\varphi}{\mapsto} & f\cdot \mathrm{B}^2(G,M) \\ f\cdot(\partial\dot f)^{-1}\cdot \mathrm{B}^2_{\mathrm{norm}}(G,M) & \overset{\varphi^{-1}}{\leftarrow\!\shortmid} & f\cdot \mathrm{B}^2(G,M) \end{array}$$

Proof. The map $\varphi : \mathrm{H}^2_{\mathrm{norm}}(G,M) \to \mathrm{H}^2(G,M)$ is a well-defined group morphism because $\mathrm{Z}^2_{\mathrm{norm}}(G,M)$ is a subgroup of $\mathrm{Z}^2(G,M)$ and because $\mathrm{B}^2_{\mathrm{norm}}(G,M)$ is a subgroup of $\mathrm{B}^2(G,M)$.

We show that $\varphi^{-1} : \mathrm{H}^2(G,M) \to \mathrm{H}^2_{\mathrm{norm}}(G,M)$ is well-defined.

First, we show that $f\cdot(\partial\dot f)^{-1} \overset{!}{\in} \mathrm{Z}^2_{\mathrm{norm}}(G,M)$ for $f\in \mathrm{Z}^2(G,M)$.

Note that $1 = {}^{g}f(1,1)\cdot f(g\cdot 1,1)^{-1}\cdot f(g,1\cdot 1)\cdot f(g,1)^{-1}$, whence ${}^{g}f(1,1) = f(g,1)$ for $g\in G$.

Note that $1 = {}^{1}f(1,g)\cdot f(1\cdot 1,g)^{-1}\cdot f(1,1\cdot g)\cdot f(1,1)^{-1}$, whence $f(1,g) = f(1,1)$ for $g\in G$.

Suppose given $g\in G$. We get

$$\begin{array}{rcl} (f\cdot(\partial\dot f)^{-1})(g,1) & = & f(g,1)\cdot(\partial\dot f)(g,1)^{-1} \\ & = & f(g,1)\cdot {}^{g}\dot f(1)^{-1}\cdot \dot f(g\cdot 1)\cdot \dot f(g)^{-1} \\ & = & f(g,1)\cdot {}^{g}f(1,1)^{-1} \\ & = & 1\,. \end{array}$$

Moreover, we get

$$\begin{aligned}(f \cdot (\partial \dot{f})^{-1})(1,g) &= f(1,g) \cdot (\partial \dot{f})(1,g)^{-1} \\ &= f(1,g) \cdot {}^{1}\dot{f}(g)^{-1} \cdot \dot{f}(1 \cdot g) \cdot \dot{f}(1)^{-1} \\ &= f(1,g) \cdot f(1,1)^{-1} \\ &= 1 \,.\end{aligned}$$

Second, given $f,\, \tilde{f} \in \mathrm{Z}^2(G,M)$ such that $f \cdot \mathrm{B}^2(G,M) = \tilde{f} \cdot \mathrm{B}^2(G,M)$, we show that

$$f \cdot (\partial \dot{f})^{-1} \cdot \mathrm{B}^2_{\mathrm{norm}}(G,M) \overset{!}{=} \tilde{f} \cdot (\partial \dot{\tilde{f}})^{-1} \cdot \mathrm{B}^2_{\mathrm{norm}}(G,M) \,.$$

We have $b \in \mathrm{B}^2(G,M)$ with $\tilde{f} = f \cdot b$. Then $\dot{\tilde{f}} = \dot{f} \cdot \dot{b}$, because $\dot{\tilde{f}}(g) = \tilde{f}(1,1) = f(1,1) \cdot b(1,1) = \dot{f}(g) \cdot \dot{b}(g)$ for $g \in G$. So

$$\tilde{f} \cdot (\partial \dot{\tilde{f}})^{-1} \;=\; f \cdot b \cdot (\partial(\dot{f} \cdot \dot{b}))^{-1} \;=\; \big(f \cdot (\partial \dot{f})^{-1}\big) \cdot \big(b \cdot (\partial \dot{b})^{-1}\big)$$

Now $b \cdot (\partial \dot{b})^{-1} \in \mathrm{Z}^2_{\mathrm{norm}}(G,M)$, as seen in the first step. Moreover, $b \cdot (\partial \dot{b})^{-1} \in \mathrm{B}^2(G,M)$, as $b \in \mathrm{B}^2(G,M)$ and $\partial \dot{b} \in \mathrm{B}^2(G,M)$. Hence $b \cdot (\partial \dot{b})^{-1} \in \mathrm{B}^2_{\mathrm{norm}}(G,M)$, as required.

We have to show that the claimed inverse φ^{-1} actually is a both-sided inverse of φ.

For $f \in \mathrm{Z}^2_{\mathrm{norm}}(G,M)$, we get

$$f \cdot \mathrm{B}^2_{\mathrm{norm}}(G,M) \overset{\varphi}{\mapsto} f \cdot \mathrm{B}^2(G,M) \overset{\varphi^{-1}}{\mapsto} f \cdot (\partial \dot{f})^{-1} \cdot \mathrm{B}^2_{\mathrm{norm}}(G,M) \;=\; f \cdot \mathrm{B}^2_{\mathrm{norm}}(G,M) \,,$$

since $\dot{f}(g) = f(1,1) = 1$ for $g \in G$, whence $\dot{f} = !\,$, whence $\partial \dot{f} = !\,$.

For $f \in \mathrm{Z}^2(G,M)$, we get

$$f \cdot \mathrm{B}^2(G,M) \overset{\varphi^{-1}}{\mapsto} f \cdot (\partial \dot{f})^{-1} \cdot \mathrm{B}^2_{\mathrm{norm}}(G,M) \overset{\varphi}{\mapsto} f \cdot (\partial \dot{f})^{-1} \cdot \mathrm{B}^2(G,M) \;=\; f \cdot \mathrm{B}^2(G,M) \,,$$

since $(\partial \dot{f})^{-1} \in \mathrm{B}^2(G,M)$. $\square$

Chapter 2

Group extensions with not necessarily abelian kernel

Let A and G be groups, both not necessarily abelian.

2.1 The sets $\mathrm{Ext}(G, A)$ and $\mathrm{h}^2(G, A)$

Definition 2.1.

(1) We denote by $\mathrm{Aut}(A)$ the group of automorphisms of A, carrying the multiplication given by composition $(\circ)$.

(2) Given $a \in A$, we write $\mathrm{Int}(a) \in \mathrm{Aut}(A)$ for the inner automorphism $t \mapsto ata^{-1}$ of A.

(3) We write $\mathrm{Int}(A) := \{\mathrm{Int}(a) : a \in A\}$ for the set of inner automorphisms of A. We have $\mathrm{Int}(A) \trianglelefteq \mathrm{Aut}(A)$.

(4) We write $\mathrm{Out}(A) := \mathrm{Aut}(A)/\,\mathrm{Int}(A)$ for the group of outer automorphisms of A. Its multiplication is again written $(\circ)$.

(5) We write $\rho : \mathrm{Aut}(A) \to \mathrm{Out}(A)$, $\sigma \mapsto \sigma \circ \mathrm{Int}(A)$ for the residue class morphism.

Remark 2.2. Recall that in a short exact sequence $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ of groups and group morphisms, ι is injective, π is surjective and $\iota(A) = \ker(\pi)$.

Definition 2.3. A *group extension* of A by G is a short exact sequence $1 \to A \to E \to G \to 1$.

We say that two group extensions

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

and

$$1 \to A \xrightarrow{\iota'} E' \xrightarrow{\pi'} G \to 1$$

of A by G are *equivalent*, if there exists a group isomorphism $\varphi : E \to E'$ such that the diagram

$$\begin{tikzcd}[column sep=small] & E \arrow[dr, "\pi"] \arrow[dd, "\wr", "\varphi"'] & \\ A \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & & G \\ & E' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

is commutative, which means that $\iota' = \varphi \circ \iota$ and $\pi = \pi' \circ \varphi$. The set of equivalence classes of group extensions of A by G is denoted by

$$\mathrm{Ext}(G, A)\ .$$

Remark 2.4. Consider a group extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$.

There exists a map s such that $\pi \circ s = \mathrm{id}_G$ and such that $s(1_G) = 1_E$. In fact, we may choose an image $s(g) \in \pi^{-1}(g)$ for each element $g \in G$. Since $1_E \in \pi^{-1}(1_G)$, we may in particular choose $s(1_G) = 1_E$. Note that such an element $s(g)$ is determined only up to multiplication by an element of $\ker(\pi) = \iota(A)$.

If we find a group morphism s such that $\pi \circ s = \mathrm{id}_G$, then the group extension (ι, π) is said to be *split*. In this case E is a semidirect product of A and G.

Definition 2.5. Suppose given a group extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$.

Suppose given a map $s : G \to E$ such that $\pi \circ s = \mathrm{id}_G$ and that $s(1_G) = 1_E$; cf. Remark 2.4.

(1) We have a group isomorphism $\bar\iota := \iota|^{\iota(A)} : A \xrightarrow{\sim} \iota(A)$. So we get a group isomorphism

$$\begin{array}{rccl} \tilde\iota & : & \mathrm{Aut}(\iota(A)) & \xrightarrow{\sim} & \mathrm{Aut}(A) \\ & & \sigma & \mapsto & \bar\iota^{-1} \circ \sigma \circ \bar\iota \end{array}$$

In fact, given σ, $\sigma' \in \mathrm{Aut}(\iota(A))$, we get

$$\tilde\iota(\sigma) \circ \tilde\iota(\sigma') \;=\; \bar\iota^{-1} \circ \sigma \circ \bar\iota \circ \bar\iota^{-1} \circ \sigma' \circ \bar\iota \;=\; \bar\iota^{-1} \circ \sigma \circ \sigma' \circ \bar\iota \;=\; \tilde\iota(\sigma \circ \sigma')\ .$$

The inverse to $\tilde\iota$ is given by $\tilde\iota^{-1} : \mathrm{Aut}(A) \to \mathrm{Aut}(\iota(A))$, $\sigma \mapsto \bar\iota \circ \sigma \circ \bar\iota^{-1}$.

(2) Define the map

$$\begin{array}{rcccl} \bar\omega & : & G & \to & \mathrm{Aut}(\iota(A)) \\ & & g & \mapsto & (\bar\omega_g : \iota(a) \mapsto s(g) \cdot \iota(a) \cdot s(g)^{-1})\ \text{, where } a \in A. \end{array}$$

Then $\bar\omega_g$ is an automorphism of $\iota(A)$ since $\iota(A) \trianglelefteq E$.

Define the map

$$\begin{array}{l} \omega^{(\iota,\pi),s} \;=\; \omega \;:=\; \tilde\iota \circ \bar\omega \;\;:\;\; G \;\to\; \mathrm{Aut}(A) \\ \qquad\qquad\qquad\qquad\qquad\quad\;\; g \;\mapsto\; \omega_g \;=\; \tilde\iota(\bar\omega_g) \;=\; \bar\iota^{-1} \circ \bar\omega_g \circ \bar\iota\ . \end{array}$$

So for $g \in G$ and $a \in A$, we have

$$\iota(\omega_g(a)) \;=\; \bar\iota(\omega_g(a)) \;=\; \bar\iota(\bar\iota^{-1}(\bar\omega_g(\bar\iota(a)))) \;=\; \bar\omega_g(\iota(a)) \;=\; s(g) \cdot \iota(a) \cdot s(g)^{-1}\ .$$

(3) Given $g,\, h \in G$, we have

$$\pi(s(g)\cdot s(h)\cdot s(gh)^{-1}) \;=\; \pi(s(g))\cdot\pi(s(h))\cdot\pi(s(gh))^{-1} \;=\; g\cdot h\cdot (gh)^{-1} \;=\; 1$$

and thus $s(g)\cdot s(h)\cdot s(gh)^{-1} \in \iota(A)$.

So we may define

$$\begin{array}{rcl} \mathrm{f}^{(\iota,\pi),s} = f \;:\; G\times G & \to & A \\ (g,h) & \mapsto & f(g,h) \;:=\; \bar{\iota}^{-1}(s(g)\cdot s(h)\cdot s(gh)^{-1})\,. \end{array}$$

Note that $\iota(f(g,h)) = \bar{\iota}(f(g,h)) = s(g)\cdot s(h)\cdot s(gh)^{-1}$ for $g,\, h \in G$.

Lemma 2.6. Suppose given a group extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$.
Suppose given a map $s : G \to E$ such that $\pi\circ s = \mathrm{id}_G$ and that $s(1_G) = 1_E$.
Abbreviate $\omega := \boldsymbol{\omega}^{(\iota,\pi),s}$ and $f := \mathrm{f}^{(\iota,\pi),s}$; cf. Definition 2.5.(1, 2).
The following assertions (1, 2, 3, 4) hold.

(1) We have $\omega_g\circ\omega_h \;=\; \mathrm{Int}(f(g,h))\circ\omega_{gh}$ for $g,\, h \in G$.

(2) We have $\omega_{1_G} = \mathrm{id}_A$.

(3) We have $f(g,h)\cdot f(gh,k) \;=\; \omega_g(f(h,k))\cdot f(g,hk)$ for $g,\, h,\, k \in G$.

(4) We have $f(1_G, g) = f(g, 1_G) = 1_A$ for $g \in G$.

Proof. Ad (1). Suppose given $g,\, h \in G$. We need to show that

$$(\omega_g\circ\omega_h)(a) \overset{!}{=} (\mathrm{Int}(f(g,h))\circ\omega_{gh})(a)$$

for $a \in A$. Since ι is injective, it suffices to show that

$$\iota\big((\omega_g\circ\omega_h)(a)\big) \overset{!}{=} \iota\big((\mathrm{Int}(f(g,h))\circ\omega_{gh})(a)\big)$$

The left side is calculated as follows.

$$\begin{array}{rcl} \iota\big((\omega_g\circ\omega_h)(a)\big) & = & \iota\big(\omega_g(\omega_h(a))\big) \\ & = & s(g)\cdot\iota(\omega_h(a))\cdot s(g)^{-1} \\ & = & s(g)\cdot s(h)\cdot\iota(a)\cdot s(h)^{-1}\cdot s(g)^{-1} \end{array}$$

The right side is calculated as follows.

$$\begin{array}{rcl} \iota\big((\mathrm{Int}(f(g,h))\circ\omega_{gh})(a)\big) & = & \iota\big({}^{f(g,h)}\omega_{gh}(a)\big) \\ & = & {}^{\iota(f(g,h))}\iota(\omega_{gh}(a)) \\ & = & {}^{s(g)\cdot s(h)\cdot s(gh)^{-1}}(s(gh)\cdot\iota(a)\cdot s(gh)^{-1}) \\ & = & s(g)\cdot s(h)\cdot\iota(a)\cdot s(h)^{-1}\cdot s(g)^{-1} \end{array}$$

So both sides are equal.

Ad (2). We know that $s(1_G) = 1_E$. So for $a \in A$, we get

$$\iota(\omega_{1_G}(a)) = s(1_G) \cdot \iota(a) \cdot s(1_G)^{-1} = \iota(a)\ .$$

Hence $\omega_{1_G}(a) = a$ for any $a \in A$. Therefore we have $\omega_{1_G} = \mathrm{id}_A$.

Ad (3). Suppose given $g,\ h,\ k\ \in G$. Since ι is injective, it suffices to show that

$$\iota\big(f(g,h) \cdot f(gh,k)\big) \stackrel{!}{=} \iota\big(\omega_g(f(h,k)) \cdot f(g,hk)\big)\ .$$

Since ι is a group morphism, the left side is calculated as follows.

$$\begin{array}{rcl} \iota\big(f(g,h) \cdot f(gh,k)\big) & = & \iota(f(g,h)) \cdot \iota(f(gh,k)) \\ & = & s(g) \cdot s(h) \cdot s(gh)^{-1} \cdot s(gh) \cdot s(k) \cdot s(ghk)^{-1} \\ & = & s(g) \cdot s(h) \cdot s(k) \cdot s(ghk)^{-1} \end{array}$$

The right side is calculated as follows.

$$\begin{array}{rcl} \iota\big(\omega_g(f(h,k)) \cdot f(g,hk)\big) & = & \iota(\omega_g(f(h,k))) \cdot \iota(f(g,hk)) \\ & = & s(g) \cdot \iota(f(h,k)) \cdot s(g)^{-1} \cdot \iota(f(g,hk)) \\ & = & s(g) \cdot s(h) \cdot s(k) \cdot s(hk)^{-1} \cdot s(g)^{-1} \cdot s(g) \cdot s(hk) \cdot s(ghk)^{-1} \\ & = & s(g) \cdot s(h) \cdot s(k) \cdot s(ghk)^{-1} \end{array}$$

So both sides are equal.

Ad (4). For $g \in G$, we have

$$\iota\big(f(1_G,g)\big) \ = \ s(1_G) \cdot s(g) \cdot s(1_G \cdot g)^{-1} \ = \ 1_E \ = \ \iota(1_A)$$

and

$$\iota\big(f(g,1_G)\big) \ = \ s(g) \cdot s(1_G) \cdot s(g \cdot 1_G)^{-1} \ = \ 1_E \ = \ \iota(1_A)\ .$$

Hence $f(1_G,g) = f(g,1_G) = 1_A$. □

Lemma 2.7 (and definition)**.** A *normalized generalized* 2*-cocycle* of G with coefficients in A is a pair (ω, f), where $\omega : G \to \mathrm{Aut}(A)$ and $f : G \times G \to A$ are maps satisfying the following conditions $(1, 2, 3, 4)$.

(1) We have $\omega_g \circ \omega_h = \mathrm{Int}(f(g,h)) \circ \omega_{gh}$ for $g,\ h\ \in G$.

(2) We have $\omega_{1_G} = \mathrm{id}_A$.

(3) We have $f(g,h) \cdot f(gh,k) = \omega_g(f(h,k)) \cdot f(g,hk)$ for $g,\ h,\ k\ \in G$.

(4) We have $f(g,1_G) = 1_A = f(1_G,g)$ for $g \in G$.

The set of normalized generalized 2-cocycles of G with coefficients in A is called

$$\mathrm{z}^2(G,A)\ .$$

Note that ω and f here are arbitrary maps that do not necessarily stem from a group extension.

Now we define a relation $(\sim)$ on the set $\mathrm{z}^2(G,A)$. For $(\omega,f),(\omega',f')\in\mathrm{z}^2(G,A)$, we write

$$(\omega,f)\sim(\omega',f')$$

and say that (ω,f) and (ω',f') are *cohomologous*, if there is a map $t:G\to A$ such that

$$t(1)\;=\;1\,,$$

such that

$$\omega'_g\;=\;\mathrm{Int}(t(g))\circ\omega_g$$

for $g\in G$ and such that

$$f'(g,h)\;=\;t(g)\cdot\omega_g(t(h))\cdot f(g,h)\cdot t(gh)^{-1}$$

for $g,h\in G$.

Then $(\sim)$ is a equivalence relation on $\mathrm{z}^2(G,A)$.

The equivalence class of $(\omega,f)\in\mathrm{z}^2(G,A)$ is denoted by $[\omega,f]$. The set of equivalence classes is denoted by

$$\mathrm{h}^2(G,A)\;:=\;\mathrm{z}^2(G,A)/(\sim)\;=\;\{\,[\omega,f]\,:\,(\omega,f)\in\mathrm{z}^2(G,A)\,\}\,.$$

Proof. We need to prove that $(\sim)$ is reflexive, symmetric and transitive.

Reflexivity.

Suppose given $(\omega,f)\in\mathrm{z}^2(G,A)$. We have $(\omega,f)\sim(\omega,f)$ by choosing the trivial function $t:G\to A$, $g\mapsto 1$.

Symmetry.

Suppose given (ω,f), $(\omega',f')\in\mathrm{z}^2(G,A)$. Suppose that $(\omega,f)\sim(\omega',f')$ via $t:G\to A$.

Define $t':G\to A$, $g\mapsto t'(g):=t(g)^{-1}$. Then

$$\begin{aligned}\omega_g &= \mathrm{Int}(t(g))^{-1}\circ\omega'_g\\ &= \mathrm{Int}(t'(g))\circ\omega'_g\end{aligned}$$

for $g\in G$ and

$$\begin{aligned}f(g,h) &= \omega_g(t(h))^{-1}\cdot t(g)^{-1}\cdot f'(g,h)\cdot t(gh)\\ &= \omega_g(t'(h))\cdot t'(g)\cdot f'(g,h)\cdot t'(gh)^{-1}\\ &= (\mathrm{Int}(t'(g))\circ\omega'_g)(t'(h))\cdot t'(g)\cdot f'(g,h)\cdot t'(gh)^{-1}\\ &= t'(g)\cdot\omega'_g(t'(h))\cdot t'(g)^{-1}\cdot t'(g)\cdot f'(g,h)\cdot t'(gh)^{-1}\\ &= t'(g)\cdot\omega'_g(t'(h))\cdot f'(g,h)\cdot t'(gh)^{-1}\end{aligned}$$

for $g,h\in G$. Therefore $(\omega',f')\sim(\omega,f)$.

Transitivity.

Suppose given (ω,f), (ω',f'), $(\omega'',f'')\in\mathrm{z}^2(G,A)$ such that $(\omega,f)\sim(\omega',f')$ via $t:G\to A$ and $(\omega',f')\sim(\omega'',f'')$ via $t':G\to A$. So for $g,h\in G$ we have

$$\begin{aligned}\omega'_g &= \mathrm{Int}(t(g))\circ\omega_g\\ f'(g,h) &= t(g)\cdot\omega_g(t(h))\cdot f(g,h)\cdot t(gh)^{-1}\\ \omega''_g &= \mathrm{Int}(t'(g))\circ\omega'_g\\ f''(g,h) &= t'(g)\cdot\omega'_g(t'(h))\cdot f'(g,h)\cdot t'(gh)^{-1}\,.\end{aligned}$$

Define $\hat{t} : G \to A$, $g \mapsto \hat{t}(g) := t'(g) \cdot t(g)$. Then

$$\begin{aligned} \omega''_g &= \mathrm{Int}(t'(g)) \circ \omega'_g \\ &= \mathrm{Int}(t'(g)) \circ \mathrm{Int}(t(g)) \circ \omega_g \\ &= \mathrm{Int}(t'(g)t(g)) \circ \omega_g \\ &= \hat{t}(g) \circ \omega_g \end{aligned}$$

for $g \in G$ and

$$\begin{aligned} f''(g,h) &= t'(g) \cdot \omega'_g(t'(h)) \cdot f'(g,h) \cdot t'(gh)^{-1} \\ &= t'(g) \cdot t(g) \cdot \omega_g(t'(h)) \cdot t(g)^{-1} \cdot t(g) \cdot \omega_g(t(h)) \cdot f(g,h) \cdot t(gh)^{-1} \cdot t'(gh)^{-1} \\ &= \hat{t}(g) \cdot \omega_g(t'(h)t(h)) \cdot f(g,h) \cdot \hat{t}(gh)^{-1} \\ &= \hat{t}(g) \cdot \omega_g(\hat{t}(h)) \cdot f(g,h) \cdot \hat{t}(gh)^{-1} \end{aligned}$$

for $g, h \in G$. Therefore $(\omega, f) \sim (\omega'', f'')$. $\square$

2.2 The bijection between $\mathrm{Ext}(G, A)$ and $\mathrm{h}^2(G, A)$

Remark 2.8. Suppose given a group extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$.

Suppose given a map $s : G \to E$ such that $\pi \circ s = \mathrm{id}_G$ and that $s(1_G) = 1_E$.

Then $(\omega^{(\iota,\pi),s}, \mathrm{f}^{(\iota,\pi),s}) \in \mathrm{z}^2(G, A)$; cf. Lemmata 2.6 and 2.7.

Proposition 2.9. The map

$$\begin{array}{rcl} \mathrm{Ext}(G, A) & \xrightarrow{\alpha} & \mathrm{h}^2(G, A) \\ [1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1] & \mapsto & [\omega^{(\iota,\pi),s}, \mathrm{f}^{(\iota,\pi),s}]. \end{array}$$

is well-defined, where $s : G \to E$ is an arbitrary map such that $\pi \circ s = \mathrm{id}_G$ and that $s(1_G) = 1_E$. Cf. Definition 2.3, Lemma 2.7.

Proof. Given two equivalent group extensions

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

and

$$1 \to A \xrightarrow{\iota'} E' \xrightarrow{\pi'} G \to 1$$

and maps $s, s' : G \to E$ such that $\pi \circ s = \pi \circ s' = \mathrm{id}_G$ and that $s(1) = s'(1) = 1$, we need to prove that the representing normalized generalized 2-cocycles $(\omega^{(\iota,\pi),s}, \mathrm{f}^{(\iota,\pi),s})$ and $(\omega^{(\iota',\pi'),s'}, \mathrm{f}^{(\iota',\pi'),s'})$ are cohomologous; cf. Lemma 2.7.

Due to the equivalence of the given group extensions we have a commutative diagram as follows, cf. Definition 2.3.

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "\varphi", "\wr"'] & G \arrow[d, equal] \\ A \arrow[r, "\iota'"] & E' \arrow[r, "\pi'"] & G \end{tikzcd}$$

We write $\omega := \boldsymbol{\omega}^{(\iota,\pi),s}$, $f := \mathrm{f}^{(\iota,\pi),s}$ and $\omega' := \boldsymbol{\omega}^{(\iota',\pi'),s'}$, $f' := \mathrm{f}^{(\iota',\pi'),s'}$ like in Definition 2.5.

We have to show $(\omega, f) \overset{!}{\sim} (\omega', f')$.

Write $\bar{\iota} := \iota|^{\iota(A)} : A \xrightarrow{\sim} \iota(A)$.

Note that for $g \in G$, we have

$$\pi(\varphi^{-1}(s'(g)) \cdot s(g)^{-1}) \;=\; \pi(\varphi^{-1}(s'(g))) \cdot \pi(s(g))^{-1} \;=\; \pi'(s'(g)) \cdot \pi(s(g))^{-1} \;=\; g \cdot g^{-1} \;=\; 1$$

and thus $\varphi^{-1}(s'(g)) \cdot s(g)^{-1} \in \iota(A)$.

Define

$$\begin{array}{rcl} t : G & \to & A \\ g & \mapsto & \bar{\iota}^{-1}(\varphi^{-1}(s'(g)) \cdot s(g)^{-1})\,. \end{array}$$

Then

$$\iota(t(g)) \;=\; \varphi^{-1}(s'(g)) \cdot s(g)^{-1}$$

for $g \in G$.

We aim to show that $(\omega, f) \overset{!}{\sim} (\omega', f')$ via t; cf. Lemma 2.7.

We have $\iota(t(1)) = \varphi^{-1}(s'(1)) \cdot s(1)^{-1} = 1 \cdot 1 = 1$, and thus $t(1) = 1$.

We have to show that $\omega'_g \overset{!}{=} \mathrm{Int}(t(g)) \circ \omega_g$ for $g \in G$.

Suppose given $a \in A$. It suffices to show $\iota(\omega'_g(a)) \overset{!}{=} \iota((\mathrm{Int}(t(g)) \circ \omega_g)(a))$.

On the one hand, we get the following.

$$\begin{array}{rcl} \iota(\omega'_g(a)) & = & \varphi^{-1}(\iota'(\omega'_g(a))) \\ & = & \varphi^{-1}(s'(g) \cdot \iota'(a) \cdot s'(g)^{-1}) \\ & = & \varphi^{-1}(s'(g) \cdot \varphi(\iota(a)) \cdot s'(g)^{-1}) \\ & = & \varphi^{-1}(s'(g)) \cdot \iota(a) \cdot \varphi^{-1}(s'(g)^{-1}) \end{array}$$

On the other hand, we get the following.

$$\begin{array}{rcl} \iota((\mathrm{Int}(t(g)) \circ \omega_g)(a)) & = & \iota(\mathrm{Int}(t(g))(\omega_g(a))) \\ & = & \iota(t(g) \cdot \omega_g(a) \cdot t(g)^{-1}) \\ & = & \iota(t(g)) \cdot \iota(\omega_g(a)) \cdot \iota(t(g))^{-1} \\ & = & \varphi^{-1}(s'(g)) \cdot s(g)^{-1} \cdot \iota(\omega_g(a)) \cdot s(g) \cdot \varphi^{-1}(s'(g)^{-1}) \\ & = & \varphi^{-1}(s'(g)) \cdot s(g)^{-1} \cdot s(g) \cdot \iota(a) \cdot s(g)^{-1} \cdot s(g) \cdot \varphi^{-1}(s'(g)^{-1}) \\ & = & \varphi^{-1}(s'(g)) \cdot \iota(a) \cdot \varphi^{-1}(s'(g)^{-1}) \end{array}$$

We have to show that $f'(g,h) \overset{!}{=} t(g) \cdot \omega_g(t(h)) \cdot f(g,h) \cdot t(gh)^{-1}$ for $g,\, h \in G$.

It suffices to show that $\iota(f'(g,h)) \overset{!}{=} \iota(t(g) \cdot \omega_g(t(h)) \cdot f(g,h) \cdot t(gh)^{-1})$. We calculate.

On the one hand, we get the following.

$$\begin{array}{rl} & \iota(f'(g,h)) \\ = & \varphi^{-1}(\iota'(f'(g,h))) \\ = & \varphi^{-1}(s'(g) \cdot s'(h) \cdot s'(gh)^{-1}) \end{array}$$

On the other hand, we get the following.

$$
\begin{aligned}
& \iota(t(g)\cdot\omega_g(t(h))\cdot f(g,h)\cdot t(gh)^{-1}) \\
= \ & \iota(t(g))\cdot\iota(\omega_g(t(h)))\cdot\iota(f(g,h))\cdot\iota(t(gh)^{-1}) \\
= \ & \iota(t(g))\cdot\iota(\omega_g(t(h)))\cdot\iota(f(g,h))\cdot\iota(t(gh)^{-1}) \\
= \ & \varphi^{-1}(s'(g))\cdot s(g)^{-1}\cdot\iota(\omega_g(t(h)))\cdot s(g)\cdot s(h)\cdot s(gh)^{-1}\cdot s(gh)\cdot\varphi^{-1}(s'(gh)^{-1}) \\
= \ & \varphi^{-1}(s'(g))\cdot s(g)^{-1}\cdot\iota(\omega_g(t(h)))\cdot s(g)\cdot s(h)\cdot\varphi^{-1}(s'(gh)^{-1}) \\
= \ & \varphi^{-1}(s'(g))\cdot s(g)^{-1}\cdot s(g)\cdot\iota(t(h))\cdot s(g)^{-1}\cdot s(g)\cdot s(h)\cdot\varphi^{-1}(s'(gh)^{-1}) \\
= \ & \varphi^{-1}(s'(g))\cdot\iota(t(h))\cdot s(h)\cdot\varphi^{-1}(s'(gh)^{-1}) \\
= \ & \varphi^{-1}(s'(g))\cdot\varphi^{-1}(s'(h))\cdot s(h)^{-1}\cdot s(h)\cdot\varphi^{-1}(s'(gh)^{-1}) \\
= \ & \varphi^{-1}(s'(g))\cdot\varphi^{-1}(s'(h))\cdot\varphi^{-1}(s'(gh)^{-1}) \\
= \ & \varphi^{-1}(s'(g)\cdot s'(h)\cdot s'(gh)^{-1})
\end{aligned}
$$

Thus, we have $(\omega, f)\sim(\omega', f')$. So the map $\boldsymbol{\alpha}$ is well-defined. $\square$

Example 2.10. Suppose given a group extension $1\to A\xrightarrow{\iota}E\xrightarrow{\pi}G\to 1$ with maps $s: G\to E$ and $s': G\to E$ such that $\pi\circ s=\pi\circ s'=\mathrm{id}_G$ and that $s(1)=s'(1)=1$.

Then $(\boldsymbol{\omega}^{(\iota,\pi),s},\mathrm{f}^{(\iota,\pi),s})$ and $(\boldsymbol{\omega}^{(\iota,\pi),s'},\mathrm{f}^{(\iota,\pi),s'})$ are cohomologous.

In fact, by Proposition 2.9 the image of $[1\to A\xrightarrow{\iota}E\xrightarrow{\pi}G\to 1]$ under the map $\boldsymbol{\alpha}$ is independent of the choice of the map s.

Lemma 2.11. Suppose given $(\omega,\ f)\in\mathrm{z}^2(G,\ A)$, we define an operation $(\underset{\omega,f}{\bullet})$ on $A\times G$ by

$$
\begin{aligned}
(\underset{\omega,f}{\bullet})\colon (A\times G)\times(A\times G) &\ \to\ A\times G \\
((a,g),(b,h)) &\ \mapsto\ (a,g)\underset{\omega,f}{\bullet}(b,h) := (a\cdot\omega_g(b)\cdot f(g,h),\ g\cdot h)
\end{aligned}
$$

Then $(A\times G,\ \underset{\omega,f}{\bullet})$ is a group, denoted by $A\underset{\omega,f}{\rtimes}G$.

Specifically, for $(a,\ g)\in A\underset{\omega,f}{\rtimes}G$, we have $(a,\ g)^{-1}=(\omega_g^{-1}(a^{-1}\cdot f(g,\ g^{-1})^{-1}),\ g^{-1})$.

Note that $(a,1)\underset{\omega,f}{\bullet}(1,g)=(\,a\cdot\omega_1(1)\cdot f(1,g),g\,)=(a,g)$ for $(a,g)\in A\underset{\omega,f}{\rtimes}G$.

We simply write $(\bullet)$ instead of $(\underset{\omega,f}{\bullet})$ if unambiguous.

Proof. First of all, we have $a\cdot\omega_g(b)\cdot f(g,h)\in A$ and $g\cdot h\in G$ for $a,\ b\ \in\ A$ and $g,\ h\ \in\ G$.

We show associativity. Suppose given (a,g), (b,h), $(c,k)\in A\times G$:

$$
\begin{aligned}
((a,g)\bullet(b,h))\bullet(c,k) &= (a\cdot\omega_g(b)\cdot f(g,h),\ gh)\bullet(c,k) \\
&= ((a\cdot\omega_g(b)\cdot f(g,h))\cdot\omega_{gh}(c)\cdot f(gh,k),\ ghk) \\
(a,g)\bullet((b,h)\bullet(c,k)) &= (a,g)\bullet(b\cdot\omega_h(c)\cdot f(h,k),\ hk) \\
&= (a\cdot\omega_g(b\cdot\omega_h(c)\cdot f(h,k))\cdot f(g,hk),\ ghk)
\end{aligned}
$$

We need to prove that

$$
a\cdot\omega_g(b)\cdot f(g,h)\cdot\omega_{gh}(c)\cdot f(gh,k)\ \overset{!}{=}\ a\cdot\omega_g(b\cdot\omega_h(c)\cdot f(h,k))\cdot f(g,hk)\ .
$$

So we need to prove that

$$f(g,h)\cdot\omega_{gh}(c)\cdot f(gh,k) \stackrel{!}{=} \omega_g(\omega_h(c))\cdot\omega_g(f(h,k))\cdot f(g,hk)\ .$$

Using $(\omega,\ f) \in \mathrm{z}^2(G,\ A)$, cf. Lemma 2.7, we obtain in fact

$$\begin{aligned}(\omega_g\circ\omega_h)(c)\cdot\omega_g(f(h,k))\cdot f(g,hk) &= f(g,h)\cdot\omega_{gh}(c)\cdot f(g,h)^{-1}\cdot f(g,h)\cdot f(gh,k)\\ &= f(g,h)\cdot\omega_{gh}(c)\cdot f(gh,k)\ .\end{aligned}$$

So we have the associativity.

We need to show that $(1_A,\ 1_G)$ is the identity element. Suppose given $(a,g)\in A\times G$. We get

$$\begin{aligned}(1_A,1_G)\bullet(a,g) &= (1_A\cdot\omega_{1_G}(a)\cdot f(1_G,g),\ 1_G\cdot g)\\ &= (1_A\cdot\mathrm{id}_A(a)\cdot 1_A,\ g)\\ &= (a,\ g)\end{aligned}$$

and

$$\begin{aligned}(a,g)\bullet(1_A,1_G) &= (a\cdot\omega_g(1_A)\cdot f(g,1_G),\ g\cdot 1_G)\\ &= (a,\ g)\ .\end{aligned}$$

Suppose given $(a,\ g)\in A\times G$. We need to find the inverse element $(b,\ h)\in A\times G$ to $(a,\ g)$. So we need

$$(1_A,\ 1_G) \stackrel{!}{=} (a,\ g)\bullet(b,\ h) = (a\cdot\omega_g(b)\cdot f(g,h),\ gh)\ ,$$

which is equivalent to $h=g^{-1}$ and $b=\omega_g^{-1}(a^{-1}\cdot f(g,\ g^{-1})^{-1})$.

Therefore $(\omega_g^{-1}(a^{-1}\cdot f(g,\ g^{-1})^{-1}),\ g^{-1})$ is a right inverse of $(a,\ g)$. Now the right inverse is also a left inverse, so that we have found an inverse element of $(a,\ g)$. $\square$

Remark 2.12. If $f(g,h)=1$ for $g,\ h\in G$, then $\omega : G\to\mathrm{Aut}(A)$, $g\mapsto\omega_g$ is a group morphism, i.e. G acts on A via ω. In this case, $A\underset{\omega,f}{\rtimes}G$ is the semidirect product of A by G via ω.

In general, despite the similarity, the symbol $A\underset{\omega,f}{\rtimes}G$ defined here does not necessarily denote a semidirect product of A by G.

Lemma 2.13. Suppose given $(\omega,\ f)\in\mathrm{z}^2(G,\ A)$. Then we have group morphisms

$$\begin{array}{ccccc} A & \xrightarrow{\iota^{(\omega,f)}} & A\underset{\omega,f}{\rtimes}G & \xrightarrow{\pi^{(\omega,f)}} & G\\ a & \mapsto & (a,\ 1) & & \\ & & (a,\ g) & \mapsto & g\end{array}$$

and $(1\to A\xrightarrow{\iota^{(\omega,f)}} A\underset{\omega,f}{\rtimes}G \xrightarrow{\pi^{(\omega,f)}} G\to 1)$ is a group extension of A by G.

Proof. Write $\iota := \iota^{(\omega,f)}$ and $\pi := \pi^{(\omega,f)}$.

We show that ι is a group morphism. For $a,\, b \in A$, we have

$$\begin{array}{rcl} \iota(a)\cdot\iota(b) & = & (a,\,1)\bullet(b,\,1) \\ & = & (a\cdot\omega_1(b)\cdot f(1,1),\,1\cdot 1) \\ & = & (a\cdot \mathrm{id}_A(b),\,1) \\ & = & (a\cdot b,\,1) \\ & = & \iota(a\cdot b)\,. \end{array}$$

We have $\ker(\iota) = \{1\}$, so that ι is a injective group morphism.

We show that π is a group morphism. For $(a,\,g),\,(b,\,h) \in A \underset{\omega,f}{\rtimes} G$, we have

$$\begin{array}{rcl} \pi(\,(a,\,g)\bullet(b,\,h)\,) & = & \pi(\,(a\cdot\omega_g(b)\cdot f(g,h),\,g\cdot h)\,) \\ & = & g\cdot h \\ & = & \pi(\,(a,\,g)\,)\cdot\pi(\,(b,\,h)\,)\,. \end{array}$$

For $g \in G$, we have $\pi((1,g)) = g$, so that π is a surjective group morphism.

We need to prove that $(1 \to A \overset{\iota}{\to} A \underset{\omega,f}{\rtimes} G \overset{\pi}{\to} G \to 1)$ is a short exact sequence. To that end, it remains to show that $\iota(A) \overset{!}{=} \ker(\pi)$. In fact, we have $\iota(A) = \{(a,\,1) \,:\, a \in A\} = \ker(\pi)$. $\square$

Proposition 2.14. We have a well-defined map

$$\begin{array}{rcl} \mathrm{h}^2(G,A) & \overset{\beta}{\to} & \mathrm{Ext}(G,A) \\ [\omega,\,f] & \mapsto & [1 \to A \xrightarrow{\iota^{(\omega,f)}} A \underset{\omega,f}{\rtimes} G \xrightarrow{\pi^{(\omega,f)}} G \to 1]\,. \end{array}$$

Concerning $A \underset{\omega,f}{\rtimes} G$, cf. Lemma 2.11. Concerning $\iota^{(\omega,f)}$ and $\pi^{(\omega,f)}$, cf. Lemma 2.13.

Proof. Suppose given cohomologous pairs $(\omega,\,f) \sim (\omega',\,f')$ in $\mathrm{z}^2(G,A)$; cf. Lemma 2.7.

Write $\iota := \iota^{(\omega,f)}$ and $\pi := \pi^{(\omega,f)}$. Write $\iota' := \iota^{(\omega',f')}$ and $\pi' := \pi^{(\omega',f')}$.

We need to prove that there is a group isomorphism φ between $A \underset{\omega,f}{\rtimes} G$ and $A \underset{\omega',f'}{\rtimes} G$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, equal] & A \underset{\omega,f}{\rtimes} G \arrow[r, "\pi"] \arrow[d, "\varphi", "\wr"'] & G \arrow[d, equal] \\ A \arrow[r, "\iota'"] & A \underset{\omega',f'}{\rtimes} G \arrow[r, "\pi'"] & G \end{tikzcd}$$

is commutative.

Since (ω, f) and $(\omega',\,f')$ are cohomologous, there is a map $t : A \to G$ such that

$$\begin{array}{rcl} t(1) & = & 1 \\ \omega'_g & = & \mathrm{Int}(t(g))\circ\omega_g \\ f'(g,h) & = & t(g)\cdot\omega_g(t(h))\cdot f(g,h)\cdot t(gh)^{-1} \end{array}$$

for $g, h \in G$.

Let

$$
\begin{array}{rcl}
\underset{\omega,f}{A \rtimes G} & \xrightarrow{\varphi} & \underset{\omega',f'}{A \rtimes G} \\
(a,1) & \mapsto & (a,1) \\
(1,g) & \mapsto & (t(g)^{-1}, g) \\
(a,g) = (a,1) \underset{\omega,f}{\bullet} (1,g) & \mapsto & (a,1) \underset{\omega',f'}{\bullet} (t(g)^{-1}, g) = (a \cdot t(g)^{-1},\, g)\ .
\end{array}
$$

The map φ is bijective, for its inverse is given by $\underset{\omega',f'}{A \rtimes G} \to \underset{\omega,f}{A \rtimes G}$, $(a,g) \mapsto (a \cdot t(g), g)$.

We show that φ is a group morphism. Suppose given (a,g), $(b,h) \in \underset{\omega,f}{A \rtimes G}$. Then

$$
\begin{array}{rcl}
\varphi(\,(a,g) \underset{\omega,f}{\bullet} (b,h)\,) & = & \varphi(\,a \cdot \omega_g(b) \cdot f(g,h),\, gh\,) \\
& = & (\,a \cdot \omega_g(b) \cdot f(g,h) \cdot t(gh)^{-1},\, gh\,)
\end{array}
$$

and

$$
\begin{array}{rcl}
\varphi(\,(a,g)\,) \underset{\omega',f'}{\bullet} \varphi(\,(b,h)\,) & = & (\,a \cdot t(g)^{-1}, g\,) \underset{\omega',f'}{\bullet} (b \cdot t(h)^{-1}, h\,) \\
& = & (\,a \cdot t(g)^{-1} \cdot \omega'_g(b \cdot t(h)^{-1}) \cdot f'(g,h),\, gh\,) \\
& = & (\,a \cdot t(g)^{-1} \cdot t(g) \cdot \omega_g(b) \cdot \omega_g(t(h)^{-1}) \\
& & \cdot\, t(g)^{-1} \cdot t(g) \cdot \omega_g(t(h)) \cdot f(g,h\,) \cdot t(gh)^{-1},\, gh) \\
& = & (\,a \cdot \omega_g(b) \cdot f(g,h) \cdot t(gh)^{-1},\, gh\,)
\end{array}
$$

Hence $\varphi(\,(a,g) \underset{\omega,f}{\bullet} (b,h)\,) = \varphi(\,(a,g)\,) \underset{\omega',f'}{\bullet} \varphi(\,(b,h)\,)$.

We get

$$
\begin{array}{rcl}
\varphi(\iota(a)) & = & \varphi(\,(a,1)\,) \\
& = & (a \cdot t(1)^{-1},\, 1) \\
& = & (a,\, 1) \\
& = & \iota'(a)
\end{array}
$$

for $a \in A$. Hence $\varphi \circ \iota = \iota'$.

We get

$$
\begin{array}{rcl}
\pi'(\varphi(\,(a,g)\,)) & = & \pi'(\,(a \cdot t(g)^{-1},\, g)\,) \\
& = & g \\
& = & \pi(\,(a,g)\,)
\end{array}
$$

for $(a,g) \in \underset{\omega,f}{A \rtimes G}$. Hence $\pi' \circ \varphi = \pi$.

$\square$

Proposition 2.15. The composite $\beta \circ \alpha$ is the identity on $\mathrm{Ext}(G,A)$.

Cf. Propositions 2.9 and 2.14.

Proof. Given a group extension $(1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1)$, we choose a map $s : G \to E$ such that $\pi \circ s = \mathrm{id}_G$ and such that $s(1) = 1$.

Write $\omega := \boldsymbol{\omega}^{(\iota,\pi),s}$ and $f := \mathrm{f}^{(\iota,\pi),s}$.

Write $\sigma := \iota^{(\omega,f)}$ and $\tau := \pi^{(\omega,f)}$. Cf. Lemmata 2.11 and 2.13.

We have to show that the group extension

$$(1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1)$$

is equivalent to the group extension

$$(1 \to A \xrightarrow{\sigma} A \underset{\omega,f}{\rtimes} G \xrightarrow{\tau} G \to 1)\ .$$

Recall that we have the isomorphism $\bar\iota := \iota|^{\iota(A)} : A \to \iota(A)$. We remark that for $x \in E$, the element $x \cdot s(\pi(x))^{-1}$ satisfies $\pi(x \cdot s(\pi(x))^{-1}) = \pi(x) \cdot ((\pi \circ s)(\pi(x)))^{-1} = 1$, whence it lies in $\ker(\pi) = \iota(A)$. So we may define

$$\begin{array}{rcl} E & \xrightarrow{\varphi} & A \underset{\omega,f}{\rtimes} G \\ x & \mapsto & (\,\bar\iota^{\,-1}(x \cdot s(\pi(x))^{-1}), \pi(x)\,)\ . \end{array}$$

We show that φ is a group morphism. For $x,\ y\ \in\ E$, we get

$$\begin{array}{rcl} \varphi(x) \bullet \varphi(y) & = & (\,\bar\iota^{\,-1}(x \cdot s(\pi(x))^{-1}), \pi(x)\,) \bullet (\,\bar\iota^{\,-1}(y \cdot s(\pi(y))^{-1}), \pi(y)\,) \\ & = & (\,\bar\iota^{\,-1}(x \cdot s(\pi(x))^{-1}) \cdot \omega_{\pi(x)}(\bar\iota^{\,-1}(y \cdot s(\pi(y))^{-1})) \cdot f(\pi(x), \pi(y)),\ \pi(xy)\,) \\ & = & (\,\bar\iota^{\,-1}(x \cdot s(\pi(x))^{-1}) \cdot \bar\iota^{\,-1}(s(\pi(x)) \cdot y \cdot s(\pi(y))^{-1} \cdot s(\pi(x))^{-1}) \\ & & \quad \cdot\, \bar\iota^{\,-1}(s(\pi(x)) \cdot s(\pi(y)) \cdot s(\pi(xy))^{-1}),\ \pi(xy)) \\ & = & (\,\bar\iota^{\,-1}(xy \cdot s(\pi(xy))^{-1}), \pi(xy)\,) \\ & = & \varphi(xy)\ . \end{array}$$

We define $\theta : A \underset{\omega,f}{\rtimes} G \to E$, $(a,g) \mapsto \iota(a) \cdot s(g)$. For $x \in E$, we have

$$(\theta \circ \varphi)(x) = \iota(\,\bar\iota^{\,-1}(x \cdot s(\pi(x))^{-1})) \cdot s(\pi(x)) = x \cdot s(\pi(x))^{-1} \cdot s(\pi(x)) = x\ .$$

Thus $\theta \circ \varphi = \mathrm{id}_E$. For $(a,g) \in A \underset{\omega,f}{\rtimes} G$, we have $\pi(\iota(a) \cdot s(g)) = g$ and thus

$$(\varphi \circ \theta)(a,g) = \varphi(\iota(a) \cdot s(g)) = (\,\bar\iota^{\,-1}(\iota(a) \cdot s(g) \cdot s(\pi(\iota(a) \cdot s(g)))^{-1}),\ \pi(\iota(a) \cdot s(g))\,) = (a,g)\ .$$

Thus $\varphi \circ \theta = \mathrm{id}_{A \underset{\omega,f}{\rtimes} G}$. Altogether, φ is bijective.

We have

$$\varphi(\iota(a)) = (\,\bar\iota^{\,-1}(\iota(a) \cdot s(\pi(\iota(a)))^{-1}),\ \pi(\iota(a))\,) = (a,1) = \sigma(a)$$

for $a \in A$ and

$$\tau(\varphi(x)) = \tau(\bar\iota^{\,-1}(x \cdot s(\pi(x))^{-1}),\ \pi(x)) = \pi(x)$$

for $x \in E$. Therefore we have $\varphi \circ \iota = \sigma$ and $\tau \circ \varphi = \pi$. Hence

$$[1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1]\ =\ [1 \to A \xrightarrow{\sigma} A \underset{\omega,f}{\rtimes} G \xrightarrow{\tau} G \to 1]\ .$$

$\square$

Proposition 2.16. The composite $\alpha \circ \beta$ is the identity on $\mathrm{h}^2(G, A)$.

Cf. Propositions 2.9 and 2.14.

Proof. Suppose given $(\omega, f) \in \mathrm{z}^2(G, A)$. Then

$$\beta([\omega, f]) \;=\; [1 \to A \xrightarrow{\iota^{(\omega,f)}} A \underset{\omega,f}{\rtimes} G \xrightarrow{\pi^{(\omega,f)}} G \to 1]\,.$$

We write $\iota := \iota^{(\omega,f)}$ and $\pi := \pi^{(\omega,f)}$; cf. Lemma 2.13.

We define a map $s : G \to A \underset{\omega,f}{\rtimes} G$ by

$$\begin{array}{rcl} G & \xrightarrow{s} & A \underset{\omega,f}{\rtimes} G \\ g & \mapsto & (1, g) \end{array}$$

Note that $(\pi \circ s)(g) = \pi(\,(1, g)\,) = g$ for $g \in G$, hence $\pi \circ s = \mathrm{id}_G$. Furthermore, note that $s(1_G) = (1, 1) = 1_{A \underset{\omega,f}{\rtimes} G}$.

So

$$(\alpha \circ \beta)([\omega, f]) \;=\; \alpha([1 \to A \xrightarrow{\iota^{(\omega,f)}} A \underset{\omega,f}{\rtimes} G \xrightarrow{\pi^{(\omega,f)}} G \to 1]) \;=\; [\omega^{(\iota,\pi),s}, \mathrm{f}^{(\iota,\pi),s}]\,.$$

Write $\hat{\omega} := \omega^{(\iota,\pi),s}$ and $\hat{f} := \mathrm{f}^{(\iota,\pi),s}$.

We have to show that $[\omega, f] \overset{!}{=} [\hat{\omega}, \hat{f}]$, i.e. that $(\omega, f) \overset{!}{\sim} (\hat{\omega}, \hat{f})$.

For $a \in A$ and $g \in G$ we have

$$\begin{array}{rcl} (\hat{\omega}_g(a), 1) & = & \iota(\hat{\omega}_g(a)) \\ & = & s(g) \underset{\omega,f}{\bullet} \iota(a) \underset{\omega,f}{\bullet} s(g)^{-1} \\ & = & (1, g) \underset{\omega,f}{\bullet} (a, 1) \underset{\omega,f}{\bullet} (1, g)^{-1} \\ & = & (1 \cdot \omega_g(a) \cdot f(g, 1),\; g) \underset{\omega,f}{\bullet} (1, g)^{-1} \\ & = & (\omega_g(a), g) \underset{\omega,f}{\bullet} (1, g)^{-1} \\ & = & (\omega_g(a), 1) \underset{\omega,f}{\bullet} (1, g) \underset{\omega,f}{\bullet} (1, g)^{-1} \\ & = & (\omega_g(a), 1)\,. \end{array}$$

Thus $\hat{\omega}_g(a) = \omega_g(a)$. We conclude that $\hat{\omega} = \omega$.

For $g, h \in G$ we have

$$\begin{array}{rcl} (\hat{f}(g, h), 1) & = & \iota(\hat{f}(g, h)) \\ & = & s(g) \underset{\omega,f}{\bullet} s(h) \underset{\omega,f}{\bullet} s(gh)^{-1} \\ & = & (1, g) \underset{\omega,f}{\bullet} (1, h) \underset{\omega,f}{\bullet} (1, gh)^{-1} \\ & = & (f(g, h), gh) \underset{\omega,f}{\bullet} (1, gh)^{-1} \\ & = & (f(g, h), 1) \underset{\omega,f}{\bullet} (1, gh) \underset{\omega,f}{\bullet} (1, gh)^{-1} \\ & = & (f(g, h), 1)\,. \end{array}$$

Thus $\hat{f}(g,h) = f(g,h)$. We conclude that $\hat{f} = f$.

Altogether, we have $(\omega, f) = (\hat{\omega}, \hat{f})$. Hence $(\omega, f) \sim (\hat{\omega}, \hat{f})$.

$\square$

Theorem 2.17. Consider the maps

$$\mathrm{Ext}(G,A) \underset{\beta}{\overset{\alpha}{\underset{\sim}{\rightleftarrows}}} \mathrm{h}^2(G,A)$$

defined in Propositions 2.9 and 2.14, running between the sets defined in Definition 2.3 and Lemma 2.7.

Then α and β are mutually inverse bijections.

Proof. By Propositions 2.15 and 2.16, the maps α and β are mutually inverse. $\square$

Chapter 3

Existence and classification of group extensions

Let A and G be groups, both not necessarily abelian.

3.1 The problem

A group extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ induces a group morphism $G \;\to\; \mathrm{Out}(A)$; cf. Remark 3.1 below.

Conversely, suppose given a group morphism

$$\varpi \;:\; G \;\to\; \mathrm{Out}(A)\;.$$

There are two questions. First, when does there exist a group extension of A by G inducing ϖ? Second, if such a group extension exists, can we classify all group extensions of A by G inducing ϖ?

Remark 3.1. Suppose given a group extension

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1\;.$$

Choose a map $s : G \to E$ such that $\pi \circ s = \mathrm{id}_G$. Then we have a map

$$\begin{array}{rcrcl} \omega^{(\iota,\pi),s} & : & G & \to & \mathrm{Aut}(A) \\ & & g & \mapsto & \omega_g\;; \end{array}$$

cf. Definition 2.5.(2). This map satisfies

$$\omega_g^{(\iota,\pi),s} \circ \omega_h^{(\iota,\pi),s} \;=\; \mathrm{Int}(f(g,h)) \circ \omega_{gh}^{(\iota,\pi),s}$$

for $g,\, h \,\in\, G$; cf. Lemma 2.6. Therefore,

$$\rho(\omega_g^{(\iota,\pi),s}) \circ \rho(\omega_h^{(\iota,\pi),s}) \;=\; \rho(\omega_{gh}^{(\iota,\pi),s})$$

in $\mathrm{Out}(A)$; cf. Definition 2.1. So the composite

$$\rho \circ \omega^{(\iota,\pi),s} \;:\; G \;\to\; \mathrm{Out}(A)$$

is a group morphism, *induced* by the group extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$.

More formally, recall that we have the bijection

$$\mathrm{Ext}(G,A) \ \xrightarrow{\alpha}\ \mathrm{h}^2(G,A)\ .$$

We shall show well-definedness of the map

$$\begin{array}{rcl} \mathrm{h}^2(G,A) & \xrightarrow{\gamma} & \mathrm{Mor}(G,\mathrm{Out}(A)) \\ [\omega,f] & \mapsto & \rho\circ\omega \end{array}$$

The group morphism induced by $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ is then

$$(\gamma\circ\alpha)([1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1])\ .$$

To show well-definedness of γ, suppose given (ω,f), $(\omega',f') \in \mathrm{z}^2(G,A)$ such that

$$(\omega,f)\ \sim\ (\omega',f')\ .$$

Then there is a map $t : G \to A$ such that $\omega'_g = \mathrm{Int}(t(g))\circ\omega_g$ for $g \in G$. So

$$(\rho\circ\omega')(g) \ =\ \omega'_g\circ\mathrm{Int}(A) \ =\ \mathrm{Int}(t(g))\circ\omega_g\circ\mathrm{Int}(A) \ =\ \omega_g\circ\mathrm{Int}(A) \ =\ (\rho\circ\omega)(g)\ .$$

for $g \in G$. Hence $\rho\circ\omega' = \rho\circ\omega$.

3.2 Action of G on $\mathrm{Z}(A)$ induced by a group morphism from G to $\mathrm{Out}(A)$

Suppose given a group morphism $\varpi : G \to \mathrm{Out}(A)$.

Remark 3.2. We choose a map $\xi : G \to \mathrm{Aut}(A)$ such that $\rho\circ\xi = \varpi$. We write ξ_g for the image of $g \in G$ under ξ. So $\xi_g\circ\mathrm{Int}(A) = \rho(\xi_g) = \varpi_g$.

Note that $\mathrm{Z}(A)$ is a characteristic subgroup of A and ξ_g is a group automorphism for $g \in G$. So we have

$$\xi_g(\mathrm{Z}(A)) \subseteq \mathrm{Z}(A)$$

Therefore we can define a map ϖ^{Z} via

$$\begin{array}{rcl} G & \xrightarrow{\varpi^{\mathrm{Z}}} & \mathrm{Aut}(\mathrm{Z}(A)) \\ g & \mapsto & \varpi^{\mathrm{Z}}_g \ :=\ \xi_g|^{\mathrm{Z}(A)}_{\mathrm{Z}(A)} \end{array}$$

Then ϖ^{Z} is a group morphism. So G acts on $\mathrm{Z}(A)$ via ϖ^{Z}.

Moreover, ϖ^{Z} does not depend on the choice of ξ.

Proof. For $g,\ h \in G$, we have

$$\rho(\xi_{gh}^{-1}\circ\xi_g\circ\xi_h) \ =\ \varpi_{gh}^{-1}\circ\varpi_g\circ\varpi_h \ =\ 1_{\mathrm{Out}(A)}$$

and thus

$$\xi_{gh}^{-1} \circ \xi_g \circ \xi_h = \mathrm{Int}(a)$$

for some element $a \in A$. So, for $z \in \mathrm{Z}(A)$ we have

$$(\xi_{gh}^{-1} \circ \xi_g \circ \xi_h)(z) \;=\; a \cdot z \cdot a^{-1} \;=\; z\,.$$

Hence

$$\xi_g|_{\mathrm{Z}(A)}^{\mathrm{Z}(A)} \circ \xi_h|_{\mathrm{Z}(A)}^{\mathrm{Z}(A)} \;=\; \xi_{gh}|_{\mathrm{Z}(A)}^{\mathrm{Z}(A)}\,.$$

Therefore ϖ^{Z} is a group morphism.

Suppose given a map $\xi' : G \to \mathrm{Aut}(A)$ such that $\rho \circ \xi' = \varpi$. We write ξ'_g for the image of $g \in G$ under ξ'.

Suppose given $g \in G$ and $z \in \mathrm{Z}(A)$. We have to show that $\xi'_g(z) \stackrel{!}{=} \xi_g(z)$.

In fact, we have

$$\rho(\xi_g) \;=\; \varpi_g \;=\; \rho(\xi'_g)$$

and thus

$$\xi_g = \xi'_g \circ \mathrm{Int}(a)$$

for some $a \in A$. So

$$\xi_g(z) \;=\; (\xi'_g \circ \mathrm{Int}(a))(z) \;=\; \xi'_g(aza^{-1}) \;=\; \xi'_g(z)\,.$$

So the induced group morphism ϖ^{Z} does not depend on the choice of ξ. □

Remark 3.3. Recall that the group morphism $\varpi : G \to \mathrm{Out}(A)$ induces a group morphism $\varpi^{\mathrm{Z}} : G \to \mathrm{Aut}(\mathrm{Z}(A))$, by means of which G acts on $\mathrm{Z}(A)$; cf. Remark 3.2.

We want to specialize Examples 1.9 and 1.10 and Definition 1.11 to the present case, i.e. we let $\varpi^{\mathrm{Z}} : G \to \mathrm{Aut}(\mathrm{Z}(A))$ play the role of $\varphi : G \to \mathrm{Aut}(M)$ there.

Note that $\varpi^{\mathrm{Z}}_g(z) = \xi_g(z)$ for $g \in G$ and $z \in \mathrm{Z}(A)$ by construction.

1. *The second cohomology group.*

   (a) The group of 2-cocycles of G with values in $\mathrm{Z}(A)$ is given by

   $$\mathrm{Z}^2(G, \mathrm{Z}(A)) \;=\; \left\{ G \times G \xrightarrow{f} \mathrm{Z}(A) : \begin{array}{l} f \text{ is a map such that} \\ \xi_g(f(h,k)) \cdot f(gh,k)^{-1} \cdot f(g,hk) \cdot f(g,h)^{-1} = 1 \\ \text{for } g,h,k \in G \end{array} \right\}.$$

   Recall that $(f_1 \cdot f_2)(g,h) = f_1(g,h) \cdot f_2(g,h)$ for $f_1,\ f_2 \in \mathrm{Z}^2(G, \mathrm{Z}(A))$ and $g,h \in G$.

   (b) The group of 2-coboundaries of G with values in $\mathrm{Z}(A)$ is given by

   $$\mathrm{B}^2(G, \mathrm{Z}(A)) \;=\; \left\{ G \times G \xrightarrow{\partial v} \mathrm{Z}(A) : \begin{array}{l} v : G \to \mathrm{Z}(A) \text{ is a map, and} \\ (\partial v)(g,h) = \xi_g(v(h)) \cdot v(gh)^{-1} \cdot v(g) \\ \text{for } g,h \in G \end{array} \right\}.$$

(c) The second cohomology group of G with values in $\mathrm{Z}(A)$ is given by

$$\mathrm{H}^2(G,\mathrm{Z}(A)) \;=\; \mathrm{Z}^2(G,\mathrm{Z}(A))/\,\mathrm{B}^2(G,\mathrm{Z}(A))\,.$$

2. *The normalized second cohomology group.*

(a) The group of normalized 2-cocycles of G with values in $\mathrm{Z}(A)$ is given by

$$\mathrm{Z}^2_{\mathrm{norm}}(G,\mathrm{Z}(A)) \;=\; \big\{ f\in \mathrm{Z}^2(G,\mathrm{Z}(A)) : f(g,1)=1 \text{ and } f(1,g)=1 \text{ for } g\in G\big\}$$

(b) The group of normalized 2-coboundaries of G with values in $\mathrm{Z}(A)$ is given by

$$\begin{array}{rcl} \mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A)) & = & \mathrm{Z}^2_{\mathrm{norm}}(G,\mathrm{Z}(A)) \,\cap\, \mathrm{B}^2(G,\mathrm{Z}(A)) \\ & \overset{\text{R. 1.12}}{=} & \left\{ G\times G \xrightarrow{\partial v} \mathrm{Z}(A) \;:\; \begin{array}{l} v: G\to \mathrm{Z}(A) \text{ is a map with } v(1)=1, \text{ and} \\ (\partial v)(g,h) = \xi_g(v(h))\cdot v(gh)^{-1}\cdot v(g) \\ \text{for } g,h\in G \end{array} \right\}. \end{array}$$

(c) The normalized second cohomology group of G with values in $\mathrm{Z}(A)$ is given by

$$\mathrm{H}^2_{\mathrm{norm}}(G,\mathrm{Z}(A)) \;=\; \mathrm{Z}^2_{\mathrm{norm}}(G,\mathrm{Z}(A))/\,\mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A))\,.$$

By Lemma 1.13, we have

$$\begin{array}{rcl} \mathrm{H}^2_{\mathrm{norm}}(G,\mathrm{Z}(A)) & \overset{\varphi}{\underset{\sim}{\to}} & \mathrm{H}^2(G,\mathrm{Z}(A)) \\ f\cdot \mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A)) & \mapsto & f\cdot \mathrm{B}^2(G,\mathrm{Z}(A))\,. \end{array}$$

3. *The third cohomology group.*

(a) The group of 3-cocycles of G with values in $\mathrm{Z}(A)$ is given by

$$\mathrm{Z}^3(G,\mathrm{Z}(A)) \;=\; \left\{ G\times G\times G \xrightarrow{f} \mathrm{Z}(A) : \begin{array}{l} f \text{ is a map such that} \\ \xi_g(f(h,k,l))\cdot f(gh,k,l)^{-1}\cdot f(g,hk,l) \\ \cdot f(g,h,kl)^{-1}\cdot f(g,h,k) = 1 \\ \text{for } g,h,k,l\in G \end{array} \right\}.$$

Recall that $(f_1\cdot f_2)(g,h,k) = f_1(g,h,k)\cdot f_2(g,h,k)$ for $f_1\,,\; f_2 \in \mathrm{Z}^3(G,\mathrm{Z}(A))$ and $g,\; h,\; k\;\in G$.

(b) The group of 3-coboundaries of G with values in $\mathrm{Z}(A)$ is given by

$$\mathrm{B}^3(G,\mathrm{Z}(A)) \;=\; \left\{ G\times G\times G \xrightarrow{\partial v} \mathrm{Z}(A) : \begin{array}{l} v: G\times G\to \mathrm{Z}(A) \text{ is a map, and} \\ (\partial v)(g,h,k) \\ = \xi_g(v(h,k))\cdot v(gh,k)^{-1}\cdot v(g,hk)\cdot v(g,h)^{-1} \\ \text{for } g,h,k\in G \end{array} \right\}.$$

(c) The third cohomology group of G with values in $\mathrm{Z}(A)$ is given by

$$\mathrm{H}^3(G,\mathrm{Z}(A)) \;=\; \mathrm{Z}^3(G,\mathrm{Z}(A))\;/\;\mathrm{B}^3(G,\mathrm{Z}(A))\,.$$

3.3 Obstruction in $\mathrm{H}^3(G, \mathrm{Z}(A))$ against the existence of a group extension inducing $\varpi : G \to \mathrm{Out}(A)$

Suppose given a group morphism $\varpi : G \to \mathrm{Out}(A)$. We write $\varpi_g \in \mathrm{Out}(A)$ for the image of $g \in G$ under ϖ. So we have an induced group morphism $\varpi^{\mathrm{Z}} : G \to \mathrm{Aut}(\mathrm{Z}(A))$ by means of which $\mathrm{Z}(A)$ becomes a G-module; cf. Remark 3.2. Using ϖ^{Z}, we form $\mathrm{H}^3(G, \mathrm{Z}(A))$, cf. Example 1.10 and Remark 3.3.(3.c).

Remark 3.4. Suppose given $\gamma \in \mathrm{Aut}(A)$ and $a \in A$. We have

$$\gamma \circ \mathrm{Int}(a) = \mathrm{Int}(\gamma(a)) \circ \gamma$$

Proof. We have

$$\begin{aligned} (\gamma \circ \mathrm{Int}(a))(b) &= \gamma(a \cdot b \cdot a^{-1}) \\ &= \gamma(a) \cdot \gamma(b) \cdot \gamma(a)^{-1} \\ &= (\mathrm{Int}(\gamma(a)) \circ \gamma)(b) \end{aligned}$$

for $b \in A$. □

Lemma 3.5 (and definition)**.** Recall that $\rho : \mathrm{Aut}(A) \to \mathrm{Out}(A)$ is the residue class morphism.

We choose a map $\xi : G \to \mathrm{Aut}(A)$ such that $\rho \circ \xi = \varpi$. We write ξ_g for the image of $g \in G$ under ξ. So $\xi_g \circ \mathrm{Int}(A) = \rho(\xi_g) = \varpi_g$.

In particular, we may choose $\xi_1 := \mathrm{id}_A$.

Consequently, for $g,\ h\ \in\ G$ we get $\rho(\xi_g \circ \xi_h \circ \xi_{gh}^{-1}) = \varpi_g \circ \varpi_h \circ \varpi_{gh}^{-1} = 1$, i.e. $\xi_g \circ \xi_h \circ \xi_{gh}^{-1} \in \mathrm{Int}(A)$.

We choose a map $f : G \times G \to A$ such that $\xi_g \circ \xi_h = \mathrm{Int}(f(g,h)) \circ \xi_{gh}$ for $g,\ h\ \in\ G$.

In particular, we may choose $f(g,1) := 1$ and $f(1,h) := 1$ for $g,\ h\ \in\ G$.

Let

$$\begin{array}{rcl} G \times G \times G & \xrightarrow{c} & A \\ (g,h,k) & \mapsto & c(g,h,k) \;:=\; f(g,h) \cdot f(gh,k) \cdot f(g,hk)^{-1} \cdot \xi_g(f(h,k))^{-1} \;. \end{array}$$

So

$$f(g,h) \cdot f(gh,k) = c(g,h,k) \cdot \xi_g(f(h,k)) \cdot f(g,hk) \;.$$

(1) We have $c(g,h,k) \in \mathrm{Z}(A)$ for $g,\ h,\ k\ \in\ G$.

We write again $c := c|^{\mathrm{Z}(A)} : G \times G \times G \to \mathrm{Z}(A)$ by abuse of notation.

(2) The map $c : G \times G \times G \to \mathrm{Z}(A)$ is a 3-cocycle, i.e. $c \in \mathrm{Z}^3(G, \mathrm{Z}(A))$.

(3) The cohomology class

$$\zeta_\varpi \;:=\; c \cdot \mathrm{B}^3(G, \mathrm{Z}(A)) \;\in\; \mathrm{H}^3(G, \mathrm{Z}(A))$$

is uniquely determined by ϖ.

I.e. $c \cdot \mathrm{B}^3(G, \mathrm{Z}(A))$ is independent of the choice of ξ and of the choice of f.

Proof. *Ad* (1). For $a \in A$ and $g \in G$, we get, for $x \in A$,

$$\begin{array}{rcl} (\mathrm{Int}(\xi_g(a)))(x) & = & \xi_g(a) \cdot x \cdot \xi_g(a)^{-1} \\ & = & \xi_g(a \cdot \xi_g^{-1}(x) \cdot a^{-1}) \\ & = & (\xi_g \circ \mathrm{Int}(a) \circ \xi_g^{-1})(x) \; . \end{array}$$

So

$$\mathrm{Int}(\xi_g(a)) \;=\; \xi_g \circ \mathrm{Int}(a) \circ \xi_g^{-1} \; .$$

For $g,\, h,\, k \in G$, we have

$$\mathrm{Int}(f(g,h) \cdot f(gh,k)) \;=\; (\xi_g \circ \xi_h \circ \xi_{gh}^{-1}) \circ (\xi_{gh} \circ \xi_k \circ \xi_{ghk}^{-1}) \;=\; \xi_g \circ \xi_h \circ \xi_k \circ \xi_{ghk}^{-1} \; .$$

Moreover,

$$\mathrm{Int}(\xi_g(f(h,k)) \cdot f(g,hk)) \;=\; \xi_g \circ (\xi_h \circ \xi_k \circ \xi_{hk}^{-1}) \circ \xi_g^{-1} \circ (\xi_g \circ \xi_{hk} \circ \xi_{ghk}^{-1}) \;=\; \xi_g \circ \xi_h \circ \xi_k \circ \xi_{ghk}^{-1} \; .$$

Therefore we have

$$\mathrm{Int}(f(g,h) \cdot f(gh,k)) = \mathrm{Int}(\xi_g(f(h,k)) \cdot f(g,hk)) \; .$$

So

$$c(g,h,k) \;=\; (f(g,h) \cdot f(gh,k)) \cdot (\xi_g(f(h,k)) \cdot f(g,hk))^{-1} \;\in\; \mathrm{Z}(A) \; .$$

Ad (2). Suppose given $g,\, h,\, k \in G$. Since $c(g,h,k) \in \mathrm{Z}(A)$ by (1), we may conjugate $c(g,h,k)$ in A without changing it. In particular, we get

$$\begin{array}{rcl} c(g,h,k) & = & f(g,h) \cdot f(gh,k) \cdot f(g,hk)^{-1} \cdot \xi_g(f(h,k))^{-1} \\ & = & f(gh,k) \cdot f(g,hk)^{-1} \cdot \xi_g(f(h,k))^{-1} \cdot f(g,h) \\ & = & f(g,hk)^{-1} \cdot \xi_g(f(h,k))^{-1} \cdot f(g,h) \cdot f(gh,k) \\ & = & \xi_g(f(h,k))^{-1} \cdot f(g,h) \cdot f(gh,k) \cdot f(g,hk)^{-1} \; . \end{array}$$

We need to prove that

$$\xi_g(c(h,k,l)) \cdot c(gh,k,l)^{-1} \cdot c(g,hk,l) \cdot c(g,h,kl)^{-1} \cdot c(g,h,k) \overset{!}{=} 1$$

for $g, h, k, l \in G$; cf. Remark 3.3.(3.a). Because c has images in $\mathrm{Z}(A)$, it suffices to show that

$$\xi_g(c(h,k,l)) \cdot c(g,h,k) \cdot c(g,hk,l) \;\overset{!}{=}\; c(g,h,kl) \cdot c(gh,k,l) \; .$$

We have

$$\begin{array}{rcl} \xi_g(c(h,k,l)) & = & \xi_g(f(h,k)) \cdot \xi_g(f(hk,l)) \cdot \xi_g(f(h,kl))^{-1} \cdot \xi_g(\xi_h(f(k,l)))^{-1} \\ & = & \xi_g(f(h,k)) \cdot \xi_g(f(hk,l)) \cdot \xi_g(f(h,kl))^{-1} \cdot (\xi_g \circ \xi_h)(f(k,l))^{-1} \\ & = & \xi_g(f(h,k)) \cdot \xi_g(f(hk,l)) \cdot \xi_g(f(h,kl))^{-1} \cdot (\mathrm{Int}(f(g,h)) \circ \xi_{gh})(f(k,l))^{-1} \\ & = & \xi_g(f(h,k)) \cdot \xi_g(f(hk,l)) \cdot \xi_g(f(h,kl))^{-1} \cdot f(g,h) \cdot \xi_{gh}(f(k,l))^{-1} \cdot f(g,h)^{-1} \end{array}$$

and

$$\begin{array}{rcl} c(g,h,k) & = & f(g,h)\cdot f(gh,k)\cdot f(g,hk)^{-1}\cdot \xi_g(f(h,k))^{-1} \\ c(g,hk,l) & = & f(g,hk)\cdot f(ghk,l)\cdot f(g,hkl)^{-1}\cdot \xi_g(f(hk,l))^{-1} \\ c(g,h,kl) & = & f(g,hkl)^{-1}\cdot \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot f(gh,kl) \\ c(gh,k,l) & = & f(gh,kl)^{-1}\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(gh,k)\cdot f(ghk,l)\ . \end{array}$$

Note that $\xi_g(c(h,k,l))\cdot c(g,h,k)\in \mathrm{Z}(A)$, whence

$$\xi_g(c(h,k,l))\cdot c(g,h,k) = y^{-1}\cdot \xi_g(c(h,k,l))\cdot c(g,h,k)\cdot y$$

for $y\in A$. Therefore

$$\begin{array}{rl} & \xi_g(c(h,k,l))\cdot c(g,h,k) \\ = & \xi_g(f(h,k))\cdot \xi_g(f(hk,l))\cdot \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(g,h)^{-1} \\ & \cdot f(g,h)\cdot f(gh,k)\cdot f(g,hk)^{-1}\cdot \xi_g(f(h,k))^{-1} \\ = & \xi_g(f(hk,l))\cdot \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(g,h)^{-1} \\ & \cdot f(g,h)\cdot f(gh,k)\cdot f(g,hk)^{-1} \\ = & \xi_g(f(hk,l))\cdot \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(gh,k)\cdot f(g,hk)^{-1}\ . \end{array}$$

Consequently, the left side is

$$\begin{array}{rl} & \xi_g(c(h,k,l))\cdot c(g,h,k)\cdot c(g,hk,l) \\ = & \xi_g(f(hk,l))\cdot \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(gh,k)\cdot f(g,hk)^{-1} \\ & \cdot f(g,hk)\cdot f(ghk,l)\cdot f(g,hkl)^{-1}\cdot \xi_g(f(hk,l))^{-1} \\ = & \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(gh,k)\cdot f(g,hk)^{-1} \\ & \cdot f(g,hk)\cdot f(ghk,l)\cdot f(g,hkl)^{-1} \\ = & \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(gh,k)\cdot f(ghk,l)\cdot f(g,hkl)^{-1}\ . \end{array}$$

Moreover, the right side is

$$\begin{array}{rl} & c(g,h,kl)\cdot c(gh,k,l) \\ = & f(g,hkl)^{-1}\cdot \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot f(gh,kl)\cdot f(gh,kl)^{-1}\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(gh,k)\cdot f(ghk,l) \\ = & \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot f(gh,kl)\cdot f(gh,kl)^{-1}\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(gh,k)\cdot f(ghk,l)\cdot f(g,hkl)^{-1} \\ = & \xi_g(f(h,kl))^{-1}\cdot f(g,h)\cdot \xi_{gh}(f(k,l))^{-1}\cdot f(gh,k)\cdot f(ghk,l)\cdot f(g,hkl)^{-1} \end{array}$$

So both sides agree.

Ad (3). We have to show independence of $c\cdot \mathrm{B}^3(G,\mathrm{Z}(A))$ of the choice of ξ and of f.

First, we keep ξ, but vary f. That is, we suppose given a map $f' : G\times G\to A$ such that $\xi_g\circ\xi_h = \mathrm{Int}(f'(g,h))\circ\xi_{gh}$ for $g,\, h\in G$. We let $c' : G\times G\times G\to \mathrm{Z}(A)$ be defined by

$$c'(g,h,k) \;:=\; f'(g,h)\cdot f'(gh,k)\cdot f'(g,hk)^{-1}\cdot \xi_g(f'(h,k))^{-1}\ .$$

We have to show that $c\cdot \mathrm{B}^3(G,\mathrm{Z}(A)) \overset{!}{=} c'\cdot \mathrm{B}^3(G,\mathrm{Z}(A))$.

Given $g,\, h\in G$, we have

$$\mathrm{Int}(f(g,h)) = \xi_g\circ\xi_h\circ\xi_{gh}^{-1} = \mathrm{Int}(f'(g,h))\ .$$

So we get

$$v(g,h) \ := \ f(g,h)\cdot f'(g,h)^{-1} \ \in \ \mathrm{Z}(A)$$

Further we have $v(g,h) = f(g,h)\cdot f'(g,h)^{-1} = f'(g,h)^{-1}\cdot f(g,h)$ by conjugation.

So for $g,\ h,\ k\ \in G$, we get

$$\begin{array}{cl} & c(g,h,k)^{-1}\cdot c'(g,h,k) \\ = & \xi_g(f(h,k))\cdot f(g,hk)\cdot f(gh,k)^{-1}\cdot f(g,h)^{-1}\cdot f'(g,h)\cdot f'(gh,k)\cdot f'(g,hk)^{-1}\cdot \xi_g(f'(h,k))^{-1} \\ = & \xi_g(f(h,k))\cdot f(g,hk)\cdot f(gh,k)^{-1}\cdot v(g,h)^{-1}\cdot f'(gh,k)\cdot f'(g,hk)^{-1}\cdot \xi_g(f'(h,k))^{-1} \\ = & \xi_g(f(h,k))\cdot f(g,hk)\cdot v(gh,k)^{-1}\cdot f'(g,hk)^{-1}\cdot \xi_g(f'(h,k))^{-1}\cdot v(g,h)^{-1} \\ = & \xi_g(f(h,k))\cdot v(g,hk)\cdot \xi_g(f'(h,k))^{-1}\cdot v(gh,k)^{-1}\cdot v(g,h)^{-1} \\ = & \xi_g(v(h,k))\cdot v(gh,k)^{-1}\cdot v(g,hk)\cdot v(g,h)^{-1}\ . \end{array}$$

Therefore $c^{-1}\cdot c' \in \mathrm{B}^3(G,\mathrm{Z}(A))$; cf. Remark 3.3.(3.b). I.e. $c\cdot \mathrm{B}^3(G,\mathrm{Z}(A)) = c'\cdot \mathrm{B}^3(G,\mathrm{Z}(A))$.

Second, we vary ξ. That is, we suppose given a map $\tilde{\xi} : G \to \mathrm{Aut}(A)$ such that $\rho\circ\tilde{\xi} = \varpi = \rho\circ\xi$ and such that $\tilde{\xi}_1 = \mathrm{id}_A$.

We shall let $\tilde{c} : G\times G\times G \to \mathrm{Z}(A)$ be defined by

$$\tilde{c}(g,h,k) \ := \ \tilde{f}(g,h)\cdot \tilde{f}(gh,k)\cdot \tilde{f}(g,hk)^{-1}\cdot \tilde{\xi}_g(\tilde{f}(h,k))^{-1}$$

for a particular choice of a map $\tilde{f} : G\times G \to A$ such that $\tilde{\xi}_g\circ\tilde{\xi}_h = \mathrm{Int}(\tilde{f}(g,h))\circ\tilde{\xi}_{gh}$ for $g,\ h\ \in G$ and such that $\tilde{f}(g,1) = 1$ and $\tilde{f}(1,h) = 1$ for $g,\ h\ \in G$.

We then shall show that $c\cdot \mathrm{B}^3(G,\mathrm{Z}(A)) \overset{!}{=} \tilde{c}\cdot \mathrm{B}^3(G,\mathrm{Z}(A))$ by showing that actually $c \overset{!}{=} \tilde{c}$.

By the first step, this will suffice to show that $c\cdot \mathrm{B}^3(G,\mathrm{Z}(A))$ is independent of the choice of ξ and f.

Since $\rho\circ\tilde{\xi} = \rho\circ\xi$, we have $\tilde{\xi}_g\circ\mathrm{Int}(A) = \xi_g\circ\mathrm{Int}(A)$ in $\mathrm{Out}(A)$, for $g\in G$. So there exists a map $\mu : G\to A$ such that $\mu(1) = 1$ and such that

$$\tilde{\xi}_g = \xi_g\circ\mathrm{Int}(\mu(g))\ .$$

Now we have, using Remark 3.4 repeatedly,

$$\begin{array}{rcl} \tilde{\xi}_g\circ\tilde{\xi}_h\circ\tilde{\xi}_{gh}^{-1} & = & \xi_g\circ\mathrm{Int}(\mu(g))\circ\xi_h\circ\mathrm{Int}(\mu(h))\circ\mathrm{Int}(\mu(gh))^{-1}\circ\xi_{gh}^{-1} \\ & = & \mathrm{Int}(\xi_g(\mu(g)))\circ\xi_g\circ\xi_h\circ\mathrm{Int}(\mu(h))\circ\mathrm{Int}(\mu(gh)^{-1})\circ\xi_{gh}^{-1} \\ & = & \mathrm{Int}(\xi_g(\mu(g)))\circ\mathrm{Int}(\xi_g(\xi_h(\mu(h))))\circ\mathrm{Int}(\xi_g(\xi_h(\mu(gh)^{-1})))\circ\xi_g\circ\xi_h\circ\xi_{gh}^{-1} \\ & = & \mathrm{Int}(\ \xi_g(\mu(g))\cdot\xi_g(\xi_h(\mu(h)))\cdot\xi_g(\xi_h(\mu(gh)))^{-1}\ )\circ\xi_g\circ\xi_h\circ\xi_{gh}^{-1} \\ & = & \mathrm{Int}(\ \xi_g(\mu(g))\cdot\xi_g(\xi_h(\mu(h)))\cdot\xi_g(\xi_h(\mu(gh)))^{-1}\cdot f(g,h)\ )\ . \end{array}$$

Let

$$\begin{array}{rcl} \tilde{f}\ :\ G\times G & \to & A \\ (g,h) & \mapsto & \tilde{f}(g,h)\ :=\ \xi_g(\mu(g))\cdot\xi_g(\xi_h(\mu(h)))\cdot\xi_g(\xi_h(\mu(gh)))^{-1}\cdot f(g,h)\ . \end{array}$$

Then $\tilde{\xi}_g\circ\tilde{\xi}_h = \mathrm{Int}(\tilde{f}(g,h))\circ\tilde{\xi}_{gh}$ for $g,\ h\ \in G$.

Moreover, $\tilde{f}(g,1) = 1$ and $\tilde{f}(1,h) = 1$ for $g,\ h\ \in G$.

Recall that $\xi_g \circ \xi_h = \mathrm{Int}(f(g,h)) \circ \xi_{gh}$ for $g,\ h\ \in\ G$.

For $g,\ h\ \in\ G$, we have

$$\begin{array}{rcl} \tilde{f}(g,h) & = & \xi_g(\mu(g)) \cdot \xi_g(\xi_h(\mu(h))) \cdot \xi_g(\xi_h(\mu(gh)))^{-1} \cdot f(g,h) \\ & = & \xi_g(\mu(g)) \cdot (\mathrm{Int}(f(g,h)) \circ \xi_{gh})(\mu(h)) \cdot (\mathrm{Int}(f(g,h)) \circ \xi_{gh})(\mu(gh))^{-1} \cdot f(g,h) \\ & = & \xi_g(\mu(g)) \cdot f(g,h) \cdot \xi_{gh}(\mu(h)) \cdot \xi_{gh}(\mu(gh))^{-1} \ . \end{array}$$

Suppose given $g,\ h,\ k\ \in\ G$. We calculate.

$$\begin{array}{rcl} \tilde{c}(g,h,k) & = & \tilde{f}(g,h) \cdot \tilde{f}(gh,k) \cdot \tilde{f}(g,hk)^{-1} \cdot \tilde{\xi}_g(\tilde{f}(h,k))^{-1} \\ & = & \xi_g(\mu(g)) \cdot f(g,h) \cdot \xi_{gh}(\mu(h)) \cdot \xi_{gh}(\mu(gh))^{-1} \\ & & \cdot\, \xi_{gh}(\mu(gh)) \cdot f(gh,k) \cdot \xi_{ghk}(\mu(k)) \cdot \xi_{ghk}(\mu(ghk))^{-1} \\ & & \cdot\, \xi_{ghk}(\mu(ghk)) \cdot \xi_{ghk}(\mu(hk))^{-1} \cdot f(g,hk)^{-1} \cdot \xi_g(\mu(g))^{-1} \\ & & \cdot\, \tilde{\xi}_g(\ \xi_{hk}(\mu(hk)) \cdot \xi_{hk}(\mu(k))^{-1} \cdot f(h,k)^{-1} \cdot \xi_h(\mu(h))^{-1}\ ) \end{array}$$

Note that for $a \in A$ we have

$$\tilde{\xi}_g(a) = (\xi_g \circ \mathrm{Int}(\mu(g)))(a) = \xi_g(\mu(g) \cdot a \cdot \mu(g)^{-1}) = \xi_g(\mu(g)) \cdot \xi_g(a) \cdot \xi_g(\mu(g))^{-1} \ .$$

So we can continue our calculation.

$$\begin{array}{rcl} \tilde{c}(g,h,k) & = & \xi_g(\mu(g)) \cdot f(g,h) \cdot \xi_{gh}(\mu(h)) \cdot \xi_{gh}(\mu(gh))^{-1} \\ & & \cdot\, \xi_{gh}(\mu(gh)) \cdot f(gh,k) \cdot \xi_{ghk}(\mu(k)) \cdot \xi_{ghk}(\mu(ghk))^{-1} \\ & & \cdot\, \xi_{ghk}(\mu(ghk)) \cdot \xi_{ghk}(\mu(hk))^{-1} \cdot f(g,hk)^{-1} \cdot \xi_g(\mu(g))^{-1} \\ & & \cdot\, \xi_g(\mu(g)) \cdot \xi_g(\ \xi_{hk}(\mu(hk)) \cdot \xi_{hk}(\mu(k))^{-1} \cdot f(h,k)^{-1} \cdot \xi_h(\mu(h))^{-1}\ ) \cdot \xi_g(\mu(g))^{-1} \\ & = & \xi_g(\mu(g)) \cdot f(g,h) \cdot \xi_{gh}(\mu(h)) \\ & & \cdot\, f(gh,k) \cdot \xi_{ghk}(\mu(k)) \\ & & \cdot\, \xi_{ghk}(\mu(hk))^{-1} \cdot f(g,hk)^{-1} \\ & & \cdot\, \xi_g(\ \xi_{hk}(\mu(hk)) \cdot \xi_{hk}(\mu(k))^{-1} \cdot f(h,k)^{-1} \cdot \xi_h(\mu(h))^{-1}\ ) \cdot \xi_g(\mu(g))^{-1} \\ & = & \xi_g(\mu(g)) \cdot f(g,h) \cdot \xi_{gh}(\mu(h)) \\ & & \cdot\, f(gh,k) \cdot \xi_{ghk}(\mu(k)) \\ & & \cdot\, \xi_{ghk}(\mu(hk))^{-1} \cdot f(g,hk)^{-1} \\ & & \cdot\, \xi_g(\xi_{hk}(\mu(hk) \cdot \mu(k)^{-1})) \cdot \xi_g(f(h,k))^{-1} \cdot \xi_g(\xi_h(\mu(h)))^{-1} \cdot \xi_g(\mu(g))^{-1} \\ & = & \xi_g(\mu(g)) \cdot f(g,h) \cdot \xi_{gh}(\mu(h)) \\ & & \cdot\, f(gh,k) \cdot \xi_{ghk}(\mu(k)) \\ & & \cdot\, \xi_{ghk}(\mu(hk))^{-1} \cdot f(g,hk)^{-1} \\ & & \cdot\, f(g,hk) \cdot \xi_{ghk}(\mu(hk) \cdot \mu(k)^{-1}) \cdot f(g,hk)^{-1} \cdot \xi_g(f(h,k))^{-1} \\ & & \cdot\, f(g,h) \cdot \xi_{gh}(\mu(h))^{-1} \cdot f(g,h)^{-1} \cdot \xi_g(\mu(g))^{-1} \end{array}$$

By conjugation, we can move the product $f(g,h)\cdot\xi_{gh}(\mu(h))^{-1}\cdot f(g,h)^{-1}\cdot\xi_g(\mu(g))^{-1}$ to the first place without changing $\tilde{c}(g,h,k)\in \mathrm{Z}(A)$. So we can continue our calculation as follows.

$$
\begin{array}{rcl}
\tilde{c}(g,h,k) & = & f(g,h)\cdot\xi_{gh}(\mu(h))^{-1}\cdot f(g,h)^{-1}\cdot\xi_g(\mu(g))^{-1} \\
& & \cdot\,\xi_g(\mu(g))\cdot f(g,h)\cdot\xi_{gh}(\mu(h)) \\
& & \cdot\, f(gh,k)\cdot\xi_{ghk}(\mu(k)) \\
& & \cdot\,\xi_{ghk}(\mu(hk))^{-1}\cdot f(g,hk)^{-1} \\
& & \cdot\, f(g,hk)\cdot\xi_{ghk}(\mu(hk)\cdot\mu(k)^{-1})\cdot f(g,hk)^{-1}\cdot\xi_g(f(h,k))^{-1} \\
& = & f(g,h) \\
& & \cdot\, f(gh,k)\cdot\xi_{ghk}(\mu(k)) \\
& & \cdot\,\xi_{ghk}(\mu(hk))^{-1} \\
& & \cdot\,\xi_{ghk}(\mu(hk)\cdot\mu(k)^{-1})\cdot f(g,hk)^{-1}\cdot\xi_g(f(h,k))^{-1} \\
& = & f(g,h) \\
& & \cdot\, f(gh,k)\cdot\xi_{ghk}(\mu(k)) \\
& & \cdot\,\xi_{ghk}(\mu(k)^{-1})\cdot f(g,hk)^{-1}\cdot\xi_g(f(h,k))^{-1} \\
& = & f(g,h)\cdot f(gh,k)\cdot f(g,hk)^{-1}\cdot\xi_g(f(h,k))^{-1} \\
& = & c(g,h,k)\ .
\end{array}
$$

Hence $\tilde{c}=c$, as was to be shown. □

Example 3.6. Consider the group morphism $!:G\to\mathrm{Out}(A)$, sending all elements to 1.

As map $\xi:G\to\mathrm{Aut}(A)$ satisfying $\rho\circ\xi=\,!$ and $\xi_1=\mathrm{id}_A$, we may choose $\xi_g:=\mathrm{id}_A$ for $g\in G$.

As map $f:G\times G\to A$ satisfying $\xi_g\circ\xi_h=\mathrm{Int}(f(g,h))\circ\xi_{gh}$ and $f(g,1)=1$ and $f(1,h)=1$ for $g,\ h\ \in\ G$, we may choose $f(g,h):=1$ for $g,\ h\ \in\ G$.

Therefore,

$$
c(g,h,k)\ =\ f(g,h)\cdot f(gh,k)\cdot f(g,hk)^{-1}\cdot\xi_g(f(h,k))^{-1}\ =\ 1
$$

for $g,\ h,\ k\ \in\ G$.

Hence $\zeta_!=c\cdot\mathrm{B}^3(G,\mathrm{Z}(A))=1$.

Theorem 3.7. There is a group extension of A by G inducing the group morphism

$$
\varpi\ :\ G\ \to\ \mathrm{Out}(A)
$$

if and only if

$$
\zeta_\varpi=1
$$

in $\mathrm{H}^3(G,\mathrm{Z}(A))$. Cf. Remark 3.1, Lemma 3.5.(3).

Proof. Recall that ζ_ϖ has been constructed as follows. We choose a map $\xi:G\to\mathrm{Aut}(A)$, $g\mapsto\xi_g$, such that $\rho\circ\xi=\varpi$. In particular, we choose $\xi_1:=\mathrm{id}_A$. We choose a map $f:G\times G\to A$ such that $\xi_g\circ\xi_h=\mathrm{Int}(f(g,h))\circ\xi_{gh}$ for $g,\ h\ \in\ G$. In particular, we choose $f(g,1):=1$ and $f(1,h):=1$ for $g,\ h\ \in\ G$. Let

$$
\begin{array}{rcl}
G\times G\times G & \stackrel{c}{\to} & A \\
(g,h,k) & \mapsto & c(g,h,k)\ :=\ f(g,h)\cdot f(gh,k)\cdot f(g,hk)^{-1}\cdot\xi_g(f(h,k))^{-1}\ .
\end{array}
$$

Then $\zeta_\varpi = c \cdot \mathrm{B}^3(G, \mathrm{Z}(A))$.

Note that $c(1,h,k) = f(1,h) \cdot f(1 \cdot h, k) \cdot f(1, hk)^{-1} \cdot \xi_1(f(h,k))^{-1} = 1$ for $h,\ k\ \in\ G$.

Note that $c(g,1,k) = f(g,1) \cdot f(g \cdot 1, k) \cdot f(g, 1 \cdot k)^{-1} \cdot \xi_g(f(1,k))^{-1} = 1$ for $g,\ k\ \in\ G$.

Note that $c(g,h,1) = f(g,h) \cdot f(gh,1) \cdot f(g, h \cdot 1)^{-1} \cdot \xi_g(f(h,1))^{-1} = 1$ for $g,\ h\ \in\ G$.

First, if $\zeta_\varpi = c \cdot \mathrm{B}^3(G, \mathrm{Z}(A)) = 1$ in $\mathrm{H}^3(G, \mathrm{Z}(A))$, then there is a map $\tilde{b} : G \times G \to \mathrm{Z}(A)$ such that

$$c(g,h,k) = \xi_g(\tilde{b}(h,k)) \cdot \tilde{b}(gh,k)^{-1} \cdot \tilde{b}(g,hk) \cdot \tilde{b}(g,h)^{-1} .$$

In particular,

$$1 \;=\; c(g,h,1) \;=\; \xi_g(\tilde{b}(h,1)) \cdot \tilde{b}(gh,1)^{-1} \cdot \tilde{b}(g,h\cdot 1) \cdot \tilde{b}(g,h)^{-1} \;=\; \xi_g(\tilde{b}(h,1)) \cdot \tilde{b}(gh,1)^{-1} .$$

for $g,\ h\ \in\ G$. Moreover,

$$1 \;=\; c(1,1,k) \;=\; \xi_1(\tilde{b}(1,k)) \cdot \tilde{b}(1\cdot 1,k)^{-1} \cdot \tilde{b}(1,1\cdot k) \cdot \tilde{b}(1,1)^{-1} \;=\; \tilde{b}(1,k) \cdot \tilde{b}(1,1)^{-1}$$

for $k \in G$.

Let

$$\begin{array}{rcl} b \;:\; G \times G & \to & \mathrm{Z}(A) \\ (g,h) & \mapsto & b(g,h) \;:=\; \tilde{b}(g,h) \cdot \tilde{b}(g,1)^{-1} . \end{array}$$

Then

$$\begin{array}{rl} & \xi_g(b(h,k)) \cdot b(gh,k)^{-1} \cdot b(g,hk) \cdot b(g,h)^{-1} \\ = & \xi_g(\tilde{b}(h,k)) \cdot \tilde{b}(gh,k)^{-1} \cdot \tilde{b}(g,hk) \cdot \tilde{b}(g,h)^{-1} \cdot \xi_g(\tilde{b}(h,1))^{-1} \cdot \tilde{b}(gh,1) \cdot \tilde{b}(g,1)^{-1} \cdot \tilde{b}(g,1) \\ = & c(g,h,k) \end{array}$$

for $g,\ h,\ k\ \in\ G$.

Moreover, $b(g,1) = 1$ for $g \in G$. Furthermore, $b(1,g) = \tilde{b}(1,g) \cdot \tilde{b}(1,1)^{-1} = 1$ for $g \in G$.

We define the map

$$\begin{array}{rcl} f' \;:\; G \times G & \to & A \\ (g,h) & \mapsto & f'(g,h) := f(g,h) \cdot b(g,h) . \end{array}$$

Because $b(g,h) \in \mathrm{Z}(A)$ and therefore $\mathrm{Int}(b(g,h)) = \mathrm{id}$, we have

$$\mathrm{Int}(f'(g,h)) = \mathrm{Int}(f(g,h)) \circ \mathrm{Int}(b(g,h)) = \mathrm{Int}(f(g,h))$$

for $g,\ h\ \in\ G$.

Moreover we have

$$\begin{array}{rcl} f'(g,h) \cdot f'(gh,k) & = & f(g,h) \cdot b(g,h) \cdot f(gh,k) \cdot b(gh,k) \\ & = & f(g,h) \cdot f(gh,k) \cdot b(g,h) \cdot b(gh,k) \\ & = & c(g,h,k) \cdot \xi_g(f(h,k)) \cdot f(g,hk) \cdot b(g,h) \cdot b(gh,k) \\ & = & \xi_g(b(h,k)) \cdot b(gh,k)^{-1} \cdot b(g,hk) \cdot b(g,h)^{-1} \cdot \xi_g(f(h,k)) \\ & & \cdot\, f(g,hk) \cdot b(g,h) \cdot b(gh,k) \\ & = & \xi_g(b(h,k)) \cdot \xi_g(f(h,k)) \cdot f(g,hk) \cdot b(g,hk) \\ & = & \xi_g(f'(h,k)) \cdot f'(g,hk) . \end{array}$$

We summarize.

(1) We have $\xi_g \circ \xi_h = \mathrm{Int}(f'(g,h)) \circ \xi_{gh}$ for $g,\ h\ \in\ G$ by Lemma 3.5 and the calculation above.

(2) We have $\xi_1 = \mathrm{id}_A$.

(3) We have $f'(g,h) \cdot f'(gh,k) = \xi_g(f'(h,k)) \cdot f'(g,hk)$ for $g,\ h,\ k\ \in\ G$ by the calculation above.

(4) We have $f'(1,g) = f(1,g) \cdot b(1,g) = 1$ and $f'(g,1) = f(g,1) \cdot b(g,1) = 1$ for $g \in G$ by the calculation above.

According to lemma 2.7 we have $(\xi, f') \in \mathrm{z}^2(G,A)$. Write $\beta([\xi, f']) =: [1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1] \in \mathrm{Ext}(G,A)$, where $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ is a short exact sequence of groups.

The group morphism induced by $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ is then

$$(\gamma \circ \alpha)([1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1]) \;=\; (\gamma \circ \alpha)(\beta([\xi, f'])) \;=\; \gamma([\xi, f']) \;=\; \rho \circ \xi \;=\; \varpi$$

as required; cf. Remark 3.1 and Theorem 2.17.

Second, suppose that we have a group extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ that induces ϖ. Choose a map $s : G \to E$ such that $\pi \circ s = \mathrm{id}_G$ and $s(1) = 1$. Then $\varpi = \rho \circ \omega^{(\iota,\pi),s}$.

We claim that $\zeta_\varpi \overset{!}{=} 1$; cf. Lemma 3.5.

Concerning properties of the pair $(\omega^{(\iota,\pi),s}, \mathrm{f}^{(\iota,\pi),s})$, we refer to Lemma 2.6, or, by choice, Lemma 2.7.

To calculate ζ_ϖ, we may use $\xi := \omega^{(\iota,\pi),s}$ since $\varpi = \rho \circ \omega^{(\iota,\pi),s}$. Moreover, we may use $f := \mathrm{f}^{(\iota,\pi),s}$ by loc. cit. (1, 4). For $g,\ h,\ k\ \in\ G$, we obtain

$$c(g,h,k) \;=\; f(g,h) \cdot f(gh,k) \cdot f(g,hk)^{-1} \cdot \omega_g^{(\iota,\pi),s}(f(h,k))^{-1} \overset{\text{loc. cit. (3)}}{=} 1\ .$$

Hence $\zeta_\varpi = c \cdot \mathrm{B}^3(G, \mathrm{Z}(A)) = 1$. □

Remark 3.8. Suppose that A is abelian.

Then $\rho : \mathrm{Aut}(A) \to \mathrm{Out}(A)$ is an isomorphism, which we identify with the identity.

Moreover, $\mathrm{Z}(A) = A$.

Consider the situation of Lemma 3.5.

The only choice for ξ is to let $\xi = \varpi$. In particular, $\xi : G \to \mathrm{Aut}(A)$ is a group morphism.

We have to choose a map $f : G \times G \to A$ such that $\xi_g \circ \xi_h = \mathrm{Int}(f(g,h)) \circ \xi_{gh}$ holds for $g,\ h\ \in\ G$. Since ξ is a group morphism, we may choose $f(g,h) = 1$ for $g,\ h\ \in\ G$.

Therefore, $c(g,h,k) = 1$ for $g,\ h,\ k\ \in\ G$. Thus

$$\zeta_\varpi \;=\; 1\ .$$

To summarize, if A is abelian, then ζ maps each element of $\mathrm{Mor}(G, \mathrm{Out}(A))$ to $1 \in \mathrm{H}^3(G,A)$.

In fact, an extension inducing ϖ does exist, namely the semidirect product of A with G with respect to ϖ. This confirms Theorem 3.7 in this case.

3.4 Classification of group extensions inducing ϖ

Suppose given a group morphism $\varpi : G \to \mathrm{Out}(A)$. We write $\varpi_g \in \mathrm{Out}(A)$ for the image of $g \in G$ under ϖ. So we have an induced group morphism $\varpi^{\mathrm{Z}} : G \to \mathrm{Aut}(\mathrm{Z}(A))$ by means of which $\mathrm{Z}(A)$ becomes a G-module; cf. Remark 3.2. Using ϖ^{Z}, we form $\mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$, cf. Definition 1.11 and Remark 3.3.(2.c).

Lemma 3.9 (and definition)**.** We define

$$\mathrm{h}^2_\varpi(G, A) \; := \; \gamma^{-1}(\{\varpi\}) \; = \; \{[\omega, f] \in \mathrm{h}^2(G, A) \; : \; \rho \circ \omega = \varpi\} \; ;$$

cf. Remark 3.1. Note that for $[\omega, f] \in \mathrm{h}^2_\varpi(G, A)$, we have $\omega_g \circ \mathrm{Int}(A) = \varpi_g$ for $g \in G$.

We define

$$\begin{array}{rl} & \mathrm{Ext}_\varpi(G, A) \\ := & (\gamma \circ \alpha)^{-1}(\{\varpi\}) \\ = & \big\{[1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1] \in \mathrm{Ext}(G, A) \; : \; 1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1 \text{ induces } \varpi\big\} \\ = & \big\{[1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1] \in \mathrm{Ext}(G, A) \; : \; G \xrightarrow{s} E \text{ is a map such that } \pi \circ s = \mathrm{id}_G \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{and } s(1) = 1, \text{ and } \rho \circ \omega^{(\iota,\pi),s} = \varpi\big\} \; ; \end{array}$$

cf. Theorem 2.17, Remark 3.1. Note that for $[1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1] \in \mathrm{Ext}_\varpi(G, A)$ and for s as above, we have $\omega^{(\iota,\pi),s}_g \circ \mathrm{Int}(A) = \varpi_g$ for $g \in G$.

Then $\alpha_\varpi := \alpha|^{\mathrm{h}^2_\varpi(G,A)}_{\mathrm{Ext}_\varpi(G,A)}$ and $\beta_\varpi := \beta|^{\mathrm{Ext}_\varpi(G,A)}_{\mathrm{h}^2_\varpi(G,A)}$ exist and are mutually inverse bijections

$$\mathrm{Ext}_\varpi(G, A) \underset{\beta_\varpi}{\overset{\alpha_\varpi}{\underset{\sim}{\rightleftarrows}}} \mathrm{h}^2_\varpi(G, A)$$

Proof. We have

$$\alpha\big(\mathrm{Ext}_\varpi(G, A)\big) \; = \; \alpha\big((\gamma \circ \alpha)^{-1}(\{\varpi\})\big) \; = \; \gamma^{-1}(\{\varpi\}) \; = \; \mathrm{h}^2_\varpi(G, A)$$

and

$$\beta\big(\mathrm{h}^2_\varpi(G, A)\big) \; = \; \alpha^{-1}\big(\gamma^{-1}(\{\varpi\})\big) \; = \; (\gamma \circ \alpha)^{-1}(\{\varpi\}) \; = \; \mathrm{Ext}_\varpi(G, A) \; .$$

□

Lemma 3.10. Suppose given $(\xi, f_0) \in \mathrm{z}^2(G, A)$ such that $[\xi, f_0] \in \mathrm{h}^2_\varpi(G, A)$.

The map

$$\begin{array}{rrcl} \eta^{\xi, f_0} \; : & \mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) & \to & \mathrm{h}^2_\varpi(G, A) \\ & z \cdot \mathrm{B}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) & \mapsto & [\xi, z \cdot f_0] \end{array}$$

is well-defined, where $(z \cdot f_0)(g, h) := z(g, h) \cdot f_0(g, h)$ for $g, h \in G$.

Proof. First of all we need to show that the pair $(\xi, z \cdot f_0)$ is an element of $\mathrm{z}^2(G, A)$. We show conditions (1) to (4) from Lemma 2.7, using that these conditions are satisfied for (ξ, f_0).

Ad (1). Since z takes values in $\mathrm{Z}(A)$, we get $\mathrm{Int}(z(g, h) \cdot f_0(g, h)) = \mathrm{Int}(f_0(g, h))$ and thus

$$\mathrm{Int}((z \cdot f_0)(g, h)) \circ \xi_{gh} \; = \; \mathrm{Int}(z(g, h) \cdot f_0(g, h)) \circ \xi_{gh} \; = \; \mathrm{Int}(f_0(g, h)) \circ \xi_{gh} \; = \; \xi_g \circ \xi_h$$

for $g,\, h\, \in\, G$.

Ad (2). We have $\xi_1 = \mathrm{id}$.

Ad (3). For $g,\, h,\, k\, \in\, G$, we have

$$\begin{array}{rcl}
(z\cdot f_0)(g,h)\cdot(z\cdot f_0)(gh,k) & = & z(g,h)\cdot f_0(g,h)\cdot z(gh,k)\cdot f_0(gh,k) \\
& = & z(g,h)\cdot z(gh,k)\cdot f_0(g,h)\cdot f_0(gh,k) \\
& = & z(g,h)\cdot z(gh,k)\cdot \xi_g(f_0(h,k))\cdot f_0(g,hk) \\
& = & z(g,h)\cdot z(gh,k)\cdot \xi_g(z(h,k))^{-1}\cdot z(g,hk)^{-1} \\
& & \cdot\, \xi_g(z(h,k)\cdot f_0(h,k))\cdot z(g,hk)\cdot f_0(g,hk) \\
& = & 1\cdot \xi_g((z\cdot f_0)(h,k))\cdot (z\cdot f_0)(g,hk)\ .
\end{array}$$

Ad (4). Due to the definition of $\mathrm{H}^2_{\mathrm{norm}}(G,\mathrm{Z}(A))$ we have

$$\begin{array}{rcl}
(z\cdot f_0)(1,g) & = & z(1,g)\cdot f_0(1,g) = 1\cdot 1 = 1 \\
(z\cdot f_0)(g,1) & = & z(g,1)\cdot f_0(g,1) = 1\cdot 1 = 1
\end{array}$$

Note that $\gamma([\xi, z\cdot f_0]) = \gamma([\xi, f_0]) = \varpi$, so that $[\xi, z\cdot f_0]\in \mathrm{h}^2_{\varpi}(G,A)$.

Suppose given $z,\, z'\, \in\, \mathrm{Z}^2(G,\mathrm{Z}(A))$ with $z\cdot \mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A)) = z'\cdot \mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A))$. We have to show that $[\xi, z\cdot f_0] \overset{!}{=} [\xi, z'\cdot f_0]$, i.e. that $(\xi, z\cdot f_0) \overset{!}{\sim} (\xi, z'\cdot f_0)$; cf. Lemma 2.7.

There exists a map $v : G\to \mathrm{Z}(A)$ such that

$$z(g,h)^{-1}\cdot z'(g,h)\ =\ (z^{-1}\cdot z')(g,h)\ =\ \xi_g(v(h))\cdot v(gh)^{-1}\cdot v(g)$$

for $g,\, h\, \in\, G$ and such that $v(1) = 1$; cf. Remark 3.3.(2.b).

We have $\xi_g = \mathrm{Int}(v(g))\circ \xi_g$ for $g\in G$.

Moreover, we have

$$\begin{array}{rcl}
z'(g,h)\cdot f_0(g,h) & = & z(g,h)\cdot \xi_g(v(h))\cdot v(gh)^{-1}\cdot v(g)\cdot f_0(g,h) \\
& = & v(g)\cdot \xi_g(v(h))\cdot \big(z(g,h)\cdot f_0(g,h)\big)\cdot v(gh)^{-1}
\end{array}$$

Hence $(\xi, c\cdot f_0)\sim(\xi, c'\cdot f_0)$. □

Lemma 3.11. Suppose given $(\xi, f_0)\in \mathrm{z}^2(G,A)$ such that $[\xi, f_0]\in \mathrm{h}^2_{\varpi}(G,A)$. Then, for each $[\omega, f]\in \mathrm{h}^2_{\varpi}(G,A)$, there exists a map $f' : G\times G\to A$, such that $[\xi, f'] = [\omega, f]$ in $\mathrm{h}^2_{\varpi}(G,A)$.

Proof. We have $\rho\circ\omega = \varpi = \rho\circ\xi$, i.e $\omega\circ\mathrm{Int}(A) = \xi\circ\mathrm{Int}(A)$.

So there is a map $t : G\to A$ with $t(1) = 1$ such that

$$\xi_g = \mathrm{Int}(t(g))\circ\omega_g$$

for $g\in G$.

We define

$$f'(g,h)\ :=\ t(g)\cdot\omega_g(t(h))\cdot f(g,h)\cdot t(gh)^{-1}$$

for $g,\, h\, \in\, G$.

We show that $(\xi, f') \overset{!}{\in} \mathrm{z}^2(G, A)$; cf. Lemma 2.7.

Ad (1). Using Remark 3.4, we obtain

$$\begin{aligned}
\xi_g \circ \xi_h &= \mathrm{Int}(t(g)) \circ \omega_g \circ \mathrm{Int}(t(h)) \circ \omega_h \\
&= \mathrm{Int}(t(g) \cdot \omega_g(t(h))) \circ \omega_g \circ \omega_h \\
&= \mathrm{Int}(t(g) \cdot \omega_g(t(h))) \circ \mathrm{Int}(f(g,h)) \circ \omega_{gh} \\
&= \mathrm{Int}(t(g) \cdot \omega_g(t(h))) \circ \mathrm{Int}(f(g,h)) \circ \mathrm{Int}(t(gh)^{-1}) \circ \xi_{gh} \\
&= \mathrm{Int}(t(g) \cdot \omega_g(t(h)) \cdot f(g,h) \cdot t(gh)^{-1}) \circ \xi_{gh} \\
&= \mathrm{Int}(f'(g,h)) \circ \xi_{gh}
\end{aligned}$$

for $g,\ h\ \in\ G$.

Ad (2). We have $\xi_1 = \mathrm{id}_A$ since $(\xi, f_0) \in \mathrm{z}^2(G, A)$.

Ad (3). We have

$$\begin{aligned}
& f'(g,h) \cdot f'(gh,k) \\
= \;& t(g) \cdot \omega_g(t(h)) \cdot f(g,h) \cdot t(gh)^{-1} \cdot t(gh) \cdot \omega_{gh}(t(k)) \cdot f(gh,k) \cdot t(ghk)^{-1} \\
= \;& t(g) \cdot \omega_g(t(h)) \cdot f(g,h) \cdot \omega_{gh}(t(k)) \cdot f(gh,k) \cdot t(ghk)^{-1} \\
= \;& t(g) \cdot \omega_g(t(h)) \cdot f(g,h) \cdot \omega_{gh}(t(k)) \cdot f(g,h)^{-1} \cdot f(g,h) \cdot f(gh,k) \cdot t(ghk)^{-1} \\
= \;& t(g) \cdot \omega_g(t(h)) \cdot (\mathrm{Int}(f(g,h)) \circ \omega_{gh})(t(k)) \cdot f(g,h) \cdot f(gh,k) \cdot t(ghk)^{-1} \\
= \;& t(g) \cdot \omega_g(t(h)) \cdot \omega_g(\omega_h(t(k))) \cdot \omega_g(f(h,k)) \cdot f(g,hk) \cdot t(ghk)^{-1} \\
= \;& t(g) \cdot \omega_g(t(h)) \cdot \omega_g(\omega_h(t(k))) \cdot \omega_g(f(h,k)) \cdot \omega_g(t(hk))^{-1} \cdot t(g)^{-1} \\
& \cdot t(g) \cdot \omega_g(t(hk)) \cdot f(g,hk) \cdot t(ghk)^{-1} \\
= \;& (\mathrm{Int}(t(g)) \circ \omega_g)(t(h) \cdot \omega_h(t(k)) \cdot f(h,k) \cdot t(hk)^{-1}) \cdot f'(g,hk) \\
= \;& \xi_g(f'(h,k)) \cdot f'(g,hk)
\end{aligned}$$

for $g,\ h,\ k\ \in\ G$.

Ad (4). We have

$$f'(1,h) \;=\; t(1) \cdot \omega_1(t(h)) \cdot f(1,h) \cdot t(1 \cdot h)^{-1} \;=\; 1$$

for $h \in G$. We have

$$f'(g,1) \;=\; t(g) \cdot \omega_g(t(1)) \cdot f(g,1) \cdot t(g \cdot 1)^{-1} \;=\; 1$$

for $g \in G$.

So $(\xi, f') \in \mathrm{z}^2(G, A)$.

By construction, we have $(\omega, f) \sim (\xi, f')$; cf. Lemma 2.7. So $[\omega, f] = [\xi, f']$ in $\mathrm{h}^2(G, A)$. Hence $[\omega, f] = [\xi, f']$ in $\mathrm{h}^2_\varpi(G, A)$. $\square$

Lemma 3.12. Suppose given $(\xi, f_0) \in \mathrm{z}^2(G, A)$ such that $[\xi, f_0] \in \mathrm{h}^2_\varpi(G, A)$.

By Lemma 3.11, each element of $\mathrm{h}^2_\varpi(G, A)$ can be written in the form $[\xi, f]$ for some map $f : G \times G \to A$ such that $(\xi, f) \in \mathrm{z}^2(G, A)$.

Define $f_0^{-1} : G \times G \to A$, $(g,h) \mapsto f_0^{-1}(g,h) := f_0(g,h)^{-1}$.

Then the map

$$\begin{aligned}
\vartheta^{\xi, f_0} \;:\; \mathrm{h}^2_\varpi(G, A) &\to \mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) \\
[\xi, f] &\mapsto (f \cdot f_0^{-1}) \cdot \mathrm{B}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) .
\end{aligned}$$

is well-defined.

Proof. First of all we need to show that $f \cdot f_0^{-1}$ is an element of $\mathrm{Z}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$.
We have

(a) $$f_0(g,h) \cdot f_0(gh,k) = \xi_g(f_0(h,k)) \cdot f_0(g,hk)$$

and

(b) $$f(g,h) \cdot f(gh,k) = \xi_g(f(h,k)) \cdot f(g,hk)$$

for $g,\ h,\ k\ \in\ G$.
Let $z := f \cdot f_0^{-1}$. So $z(g,h) := f(g,h) \cdot f_0(g,h)^{-1}$ for $g,\ h\ \in\ G$. Since

$$\mathrm{Int}(f(g,h)) = \xi_g \circ \xi_h \circ \xi_{gh}^{-1} = \mathrm{Int}(f_0(g,h))\ ,$$

we know that $z(g,h) \in \mathrm{Z}(A)$.
In particular, note that

$$z(g,h) = f(g,h) \cdot f_0(g,h)^{-1} = {}^{f_0(g,h)^{-1}}(f(g,h) \cdot f_0(g,h)^{-1}) = f_0(g,h)^{-1} \cdot f(g,h)$$

for $g,\ h\ \in\ G$.
We have to show that $z \overset{!}{\in} \mathrm{Z}^2(G, \mathrm{Z}(A))$. In fact, we have

$$\begin{array}{cl}
& \xi_g(z(h,k)) \cdot z(gh,k)^{-1} \cdot z(g,hk) \cdot z(g,h)^{-1} \\
= & z(g,hk) \cdot \xi_g(z(h,k)) \cdot z(gh,k)^{-1} \cdot z(g,h)^{-1} \\
= & f(g,hk) \cdot f_0(g,hk)^{-1} \cdot \xi_g(f_0(h,k))^{-1} \cdot \xi_g(f(h,k)) \cdot f_0(gh,k) \cdot f(gh,k)^{-1} \\
& \cdot f_0(g,h) \cdot f(g,h)^{-1} \\
= & f_0(g,hk)^{-1} \cdot \xi_g(f_0(h,k))^{-1} \cdot \xi_g(f(h,k)) \cdot f(g,hk) \cdot f_0(gh,k) \cdot f(gh,k)^{-1} \\
& \cdot f_0(g,h) \cdot f(g,h)^{-1} \\
\overset{(a),(b)}{=} & f_0(gh,k)^{-1} \cdot f_0(g,h)^{-1} \cdot f(g,h) \cdot f(gh,k) \cdot f_0(gh,k) \cdot f(gh,k)^{-1} \\
& \cdot f_0(g,h) \cdot f(g,h)^{-1} \\
= & f_0(gh,k)^{-1} \cdot f_0(g,h)^{-1} \cdot f(g,h) \cdot f(gh,k) \cdot f(gh,k)^{-1} \cdot f_0(gh,k) \\
& \cdot f_0(g,h) \cdot f(g,h)^{-1} \\
= & f_0(gh,k)^{-1} \cdot f_0(g,h)^{-1} \cdot f(g,h) \cdot f_0(gh,k) \\
& \cdot f_0(g,h) \cdot f(g,h)^{-1} \\
= & f_0(g,h)^{-1} \cdot f(g,h) \cdot f_0(g,h) \cdot f(g,h)^{-1} \\
= & f(g,h) \cdot f_0(g,h)^{-1} \cdot f_0(g,h) \cdot f(g,h)^{-1} \\
= & 1
\end{array}$$

for $g,\ h,\ k\ \in\ G$. Hence $z \in \mathrm{Z}^2(G, \mathrm{Z}(A))$; cf. Remark 3.3.(1.a).
Further we have

$$z(1,g) = f(1,g) \cdot f_0(1,g) = 1$$

and

$$z(g,1) = f(g,1) \cdot f_0(g,1) = 1$$

for $g \in G$.

Hence $z = f \cdot f_0^{-1}$ is in $\mathrm{Z}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$.

Now we need to show that the image of $[\xi, f]$ is independent of the chosen representative.

Suppose given a map $\tilde{f} : G \times G \to A$ such that $(\xi, f) \in \mathrm{z}^2(G, A)$ and such that

$$[\xi, f] \;=\; [\xi, \tilde{f}]$$

in $\mathrm{h}^2_\varpi(G, A)$.

Since $(\xi, f) \sim (\xi, \tilde{f})$, we have a map $t : G \to A$ with $t(1) = 1$ such that

$$\xi_g = \mathrm{Int}(t(g)) \circ \xi_g$$

for $g \in G$ and

$$\tilde{f}(g, h) := t(g) \cdot \xi_g(t(h)) \cdot f(g, h) \cdot t(gh)^{-1}$$

for $g,\ h\ \in\ G$.

Then we have $\mathrm{Int}(t(g)) = \mathrm{id}_A$ and thus $t(g) \in \mathrm{Z}(A)$ for $g \in G$.

So we have

$$\begin{aligned} \xi_g(t(h)) \cdot t(gh)^{-1} \cdot t(g) \;&=\; \tilde{f}(g, h) \cdot f(g, h)^{-1} \\ &=\; (\tilde{f}(g, h) \cdot f_0^{-1}(g, h)) \cdot (f(g, h) \cdot f_0^{-1}(g, h))^{-1} \end{aligned}$$

for $g,\ h\ \in\ G$.

So we have $(\tilde{f} \cdot f_0^{-1}) \cdot (f \cdot f_0^{-1})^{-1} \in \mathrm{B}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$ and therefore

$$(\tilde{f} \cdot f_0^{-1}) \cdot \mathrm{B}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) \;=\; (f \cdot f_0^{-1}) \cdot \mathrm{B}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$$

in $\mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$. □

Theorem 3.13. Recall that $\varpi : G \to \mathrm{Out}(A)$ is a group morphism. Recall that we have an induced group morphism $\varpi^{\mathrm{Z}} : G \to \mathrm{Aut}(\mathrm{Z}(A))$ used to form $\mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$; cf. Remark 3.2, Definition 1.11 and Remark 3.3.(2.c).

Note that $\mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) \xrightarrow[\sim]{\varphi} \mathrm{H}^2(G, \mathrm{Z}(A))$, $z \cdot \mathrm{B}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) \mapsto z \cdot \mathrm{B}^2(G, \mathrm{Z}(A))$; cf. Lemma 1.13.

Suppose that there exists a group extension of A by G inducing the given group morphism $\varpi : G \to \mathrm{Out}(A)$.

Therefore $\mathrm{h}^2_\varpi(G, A) \neq \emptyset$; cf. Lemma 3.9. So we may choose (ξ, f_0) in $\mathrm{z}^2(G, A)$ such that $[\xi, f_0] \in \mathrm{h}^2_\varpi(G, A)$.

Then we have the diagram

$$\mathrm{Ext}_\varpi(G, A) \; \underset{\beta_\varpi}{\overset{\alpha_\varpi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \; \mathrm{h}^2_\varpi(G, A) \; \underset{\eta^{\xi, f_0}}{\overset{\vartheta^{\xi, f_0}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \; \mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$$

where α_ϖ and β_ϖ are mutually inverse bijections and where ϑ^{ξ, f_0} and η^{ξ, f_0} are mutually inverse bijections. Cf. Lemmata 3.10 and 3.12.

Proof. By Lemma 3.9, the maps α_ϖ and β_ϖ are mutually inverse bijections.

Consider ϑ^{ξ,f_0} and η^{ξ,f_0}.

Suppose given an element in $\mathrm{h}^2_\varpi(G,A)$, which we may write in the form $[\xi,f]$ by Lemma 3.11. By Lemmata 3.12 and 3.10, we obtain

$$(\eta^{\xi,f_0}\circ\vartheta^{\xi,f_0})([\xi,f]) \;=\; \eta^{\xi,f_0}\big((f\cdot f_0^{-1})\cdot \mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A))\big) \;=\; [\xi,(f\cdot f_0^{-1})\cdot f_0] \;=\; [\xi,f]\,.$$

Suppose given $z\in \mathrm{Z}^2_{\mathrm{norm}}(G,\mathrm{Z}(A))$. By Lemmata 3.10 and 3.12, we obtain

$$\begin{aligned}(\vartheta^{\xi,f_0}\circ\eta^{\xi,f_0})(z\cdot \mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A))) \;&=\; \vartheta^{\xi,f_0}([\xi,z\cdot f_0]) \\ &=\; ((z\cdot f_0)\cdot f_0^{-1})\cdot \mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A)) \;=\; z\cdot \mathrm{B}^2_{\mathrm{norm}}(G,\mathrm{Z}(A))\,.\end{aligned}$$

So the maps ϑ^{ξ,f_0} and η^{ξ,f_0} are mutually inverse bijections. □

Corollary 3.14. Suppose that $\mathrm{Z}(A)=1$.

Recall that ϖ is a group morphism from G to $\mathrm{Out}(A)$.

Then the assertions (1) and (2) hold.

(1) There exists a group extension $1\to A\xrightarrow{\iota} E\xrightarrow{\pi} G\to 1$ inducing ϖ; cf. Theorem 3.7.

(2) Suppose given group extensions $1\to A\xrightarrow{\iota'} E'\xrightarrow{\pi'} G\to 1$ and $1\to A\xrightarrow{\iota''} E''\xrightarrow{\pi''} G\to 1$ that induce ϖ. Then these two group extensions are equivalent; cf. Theorem 3.13.

Remark 3.15.

(1) Corollary 3.14 has been shown by other means by Baer; cf. [**1,** p. 375].

(2) Mac Lane remarks in [**5,** Ch. IV, Th. 9.1] that Corollary 3.14.(1) can be shown by a direct construction as follows.

Retain the supposition that $\mathrm{Z}(A)=1$.

Consider the subgroup

$$E \;:=\; \{\,(\alpha,g)\in \mathrm{Aut}(A)\times G \;:\; \alpha\circ \mathrm{Int}(A)=\varpi_g\,\} \;\leq\; \mathrm{Aut}(A)\times G\,,$$

which is in fact a subgroup since $\rho:\mathrm{Aut}(A)\to\mathrm{Out}(A)$ and $\varpi:G\to\mathrm{Out}(A)$ are group morphisms.

We choose a map $\xi:G\to\mathrm{Aut}(A)$ such that $\xi_g\circ\mathrm{Int}(A)=\varpi_g$ for $g\in G$, i.e. such that $\rho\circ\xi=\varpi$.

Consider the following maps.

$$\begin{array}{ccccc} A & \xrightarrow{\iota} & E & \xrightarrow{\pi} & G \\ a & \mapsto & (\mathrm{Int}(a),1) & & \\ & & (\alpha,g) & \mapsto & g \end{array}$$

Note that $(\mathrm{Int}(a),1)\in E$, that $a\mapsto(\mathrm{Int}(a),1)$ is a group morphism and that the kernel of ι is $\{\,a\in A \;:\; \mathrm{Int}(a)=1\,\}=\mathrm{Z}(A)=1$. Hence ι is an injective group morphism.

Note that π is a group morphism. Note that $(\xi_g, g) \in E$ for $g \in G$. Hence π is a surjective group morphism.

We have $\pi \circ \iota = !$, the trivial morphism.

An element $(\alpha, g) \in E$ is in the kernel of π if and only if $g = 1$. But then $\alpha \circ \mathrm{Int}(A) = \varpi_1 = \mathrm{id}_A \circ \mathrm{Int}(A)$, hence $\alpha \in \mathrm{Int}(A)$. So we may write $\alpha = \mathrm{Int}(a)$ for some $a \in A$. Thus $(\alpha, g) = (\alpha, 1) = (\mathrm{Int}(a), 1) = \iota(a)$.

Altogether, $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ is a group extension.

It remains to show that this group extension induces ϖ.

Let $s : G \to E$, $g \to s(g) := (\xi_g, g)$.

For $g \in G$ and $a \in G$, we have

$$\begin{array}{rcl} s(g) \cdot \iota(a) \cdot s(g)^{-1} & = & (\xi_g, g) \cdot (\mathrm{Int}(a), 1) \cdot (\xi_g, g)^{-1} \\ & = & (\xi_g \circ \mathrm{Int}(a) \circ \xi_g^{-1}, g \cdot 1 \cdot g^{-1}) \\ & \overset{\text{R. 3.4}}{=} & (\mathrm{Int}(\xi_g(a)) \circ \xi_g \circ \xi_g^{-1}, 1) \\ & = & (\mathrm{Int}(\xi_g(a)), 1) \\ & = & \iota(\xi_g(a)) \ . \end{array}$$

Thus $\omega^{(\iota,\pi)} = \xi$; cf. 2.5.(2). Hence the group extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ induces $\rho \circ \omega^{(\iota,\pi)} = \rho \circ \xi = \varpi$; cf. Remark 3.1.

Remark 3.16. Suppose that A is abelian. Cf. Remark 3.8.

Then a group extension of A by G inducing ϖ exists; cf. Remark 3.8.

The bijections from Theorem 3.14 yield a bijection between $\mathrm{Ext}_\varpi(G, A)$ and $\mathrm{H}^2(G, A)$, recovering the assertion from the theory of extensions with abelian kernel; cf. e.g. [**2,** IV, Th. 3.12].

3.5 An example

The following example was constructed with the help of Magma [**4**].

Example 3.17. We define groups A, E and G as follows.

$$\begin{array}{rcl} A & := & \mathrm{D}_8 \;=\; \langle\, a, b \,:\, a^4,\, b^2,\, (ba)^2 \,\rangle \\ E & := & \langle\, u,\, x \,:\, u^8,\, x^2,\, xux^{-1}u^{-3} \,\rangle \\ G & := & \mathrm{C}_2 \;=\; \langle\, d \,:\, d^2 \rangle \end{array}$$

Note that $ba = a^3 b$, that $|A| = 8$, that $A = \{\, a^i b^j \,:\, i \in [0,3],\, j \in [0,1] \,\}$ and that for $i,\, i' \in [0,3]$ and $j,\, j' \in [0,1]$, we have $a^i b^j = a^{i'} b^{j'}$ if and only if $i = i'$ and $j = j'$.

Note that $\mathrm{Z}(A) = \langle a^2 \rangle$.

Note that $xu = u^3 x$ in E. Hence $E = \{\, u^i x^j \,:\, i \in [0,7],\, j \in [0,1] \,\}$, so $|E| \leq 16$.

Consider the map

$$\begin{array}{rcl} \{u, x\} & \xrightarrow{\tilde{h}} & \mathrm{S}_8 \\ u & \mapsto & (1,2,3,4,5,6,7,8) \\ x & \mapsto & (2,4)(3,7)(6,8) \ . \end{array}$$

We have $\tilde{h}(u^8) = \tilde{h}(x^2) = \tilde{h}(xux^{-1}u^{-3}) = 1$. Thus there is a group morphism

$$\begin{array}{rcl} E & \overset{h}{\to} & S_8 \\ u & \mapsto & (1,2,3,4,5,6,7,8) \\ x & \mapsto & (2,4)(3,7)(6,8)\ . \end{array}$$

Note that $|\langle\,(1,2,3,4,5,6,7,8),\ (2,4)(3,7)(6,8)\rangle| = 16$. Thus $|E| \geq 16$.

Altogether, we have $|E| = 16$. Moreover, for $i,\ i' \in [0,7]$ and $j,\ j' \in [0,1]$, we have $u^i x^j = u^{i'} x^{j'}$ if and only if $i = i'$ and $j = j'$.

Consider the map

$$\begin{array}{rcl} \{a,\,b\} & \overset{\tilde{\iota}}{\to} & E \\ a & \mapsto & u^2 \\ b & \mapsto & x \end{array}$$

We have $\tilde{\iota}(a^4) = \tilde{\iota}(b^2) = \tilde{\iota}((ba)^2) = 1$. Thus there is a group morphism

$$\begin{array}{rcl} A & \overset{\iota}{\to} & E \\ a & \mapsto & u^2 \\ b & \mapsto & x \end{array}$$

Analogously, we have the group morphism

$$\begin{array}{rcl} E & \overset{\pi}{\to} & G \\ u & \mapsto & d \\ x & \mapsto & 1\ . \end{array}$$

Then π is surjective and $\ker(\pi) = \{\, u^{2k}x^l \ :\ k \in [0,3],\ l \in [0,1]\,\} = \iota(A)$.

Since $\ker(\pi) = \iota(A)$ we have

$$\mathrm{C}_2 \;=\; \pi(E) \;\simeq\; E/\iota(A)$$

hence $|\iota(A)| = 16/2 = 8 = |A|$ and so ι is injective.

Altogether we have the group extension

$$1 \to A \overset{\iota}{\to} E \overset{\pi}{\to} G \to 1\ .$$

Consider the map

$$\begin{array}{rcl} G & \overset{s}{\to} & E \\ 1 & \mapsto & 1 \\ d & \mapsto & u \end{array}$$

We have $\pi \circ s = \mathrm{id}_G$.

The map $f := \mathrm{f}^{(\iota,\pi),s} : G \times G \to A$ is determined by its value on (d,d) ; cf. Definition 2.5.(3), Lemma 2.6.(4). We obtain

$$\begin{array}{rcl} \iota(f(d,d)) & = & s(d)\cdot s(d)\cdot s(d\cdot d)^{-1} \\ & = & u^2 \\ & = & \iota(a)\ . \end{array}$$

So

$$f(d,d) \;=\; a\;.$$

Write $\omega := \omega^{(\iota,\pi),s} : G \to \mathrm{Aut}(A)$; cf. Definition 2.5.(2). We have $\omega_1 = \mathrm{id}_A$; cf. Lemma 2.6.(2). The automorphism $\omega_d : A \to A$ is determined by its values on a and b. We obtain

$$\begin{aligned} \iota(\omega_d(a)) &= s(d)\cdot\iota(a)\cdot s(d)^{-1} \\ &= u\cdot u^2\cdot u^{-1} \\ &= u^2 \\ &= \iota(a) \end{aligned}$$

and

$$\begin{aligned} \iota(\omega_d(b)) &= s(d)\cdot\iota(b)\cdot s(d)^{-1} \\ &= u\cdot x\cdot u^{-1} \\ &= u\cdot u^{-3}\cdot x \\ &= u^6 x \\ &= \iota(a^3 b)\;. \end{aligned}$$

So

$$\omega_d(a) \;=\; a\;, \qquad \omega_d(b) \;=\; a^3 b\;.$$

Altogether, we have calculated that

$$\alpha([1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1]) \;=\; [\omega, f]\;.$$

Note that $(\omega_d)^2(b) = \omega_d(a^3 b) = a^3 a^3 b = a^2 b$, so that $(\omega_d)^2 \neq \mathrm{id}_A = \omega_1$.

Therefore, $\omega : G \to \mathrm{Aut}(A)$ is not a group morphism.

Consider $\rho : \mathrm{Aut}(A) \to \mathrm{Out}(A)$.

First, we consider $\mathrm{Aut}(A)$. Suppose given $\sigma \in \mathrm{Aut}(A)$. Then $\sigma(a)$ is an element of order 4, so $\sigma(a) = a^{2j+1}$ for some $j \in [0,1]$. Moreover, $\sigma(b)$ is an element of order 2 that conjugates $\sigma(a)$ to $\sigma(a)^{-1}$. So $\sigma(b) = a^i b$ for some $i \in [0,3]$. Conversely, the relations defining A show that for all $i \in \mathbf{Z}$ and all $j \in \mathbf{Z}$, an automorphism

$$\begin{array}{ccc} A & \xrightarrow{\sigma_{i,j}} & A \\ a & \mapsto & a^{2j+1} \\ b & \mapsto & a^i b \end{array}$$

exists. So we have

$$\mathrm{Aut}(A) \;=\; \{\,\sigma_{i,j} \,:\, i \in [0,3],\ j \in [0,1]\,\}\;.$$

Note that $\mathrm{Int}(a^k b^\ell) = \sigma_{2k,\ell}$ for $k \in [0,3]$ and $\ell \in [0,1]$. So

$$\mathrm{Int}(A) \;=\; \{\,\sigma_{2k,\ell} \,:\, k \in [0,1],\ \ell \in [0,1]\,\}\;.$$

is of order 4, whence

$$\mathrm{Out}(A) \;=\; \mathrm{Aut}(A)/\,\mathrm{Int}(A)$$

is of order 2, the element of order 2 being the coset $\sigma_{1,0} \circ \mathrm{Int}(A) = \{\,\sigma_{1,0}\,,\,\sigma_{3,0}\,,\,\sigma_{1,1}\,,\,\sigma_{3,1}\,\}$.

In particular, $\omega_d = \sigma_{0,3}$, whence $\rho(\omega_d)$ is of order 2.

We have two group morphisms from G to $\mathrm{Out}(A)$, namely the group morphism ! and the group morphism i that sends d to the element of order 2. So

$$\mathrm{Mor}(G, \mathrm{Out}(A)) \;=\; \{\,!\,,\, i\,\}$$

Therefore, our extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ induces the group morphism

$$\rho \circ \omega \;=\; i\,;$$

cf. Remark 3.1.

So, by Theorem 3.7, we know that $\zeta_i = 1$. We want to verify this equality in this example by a direct calculation.

We have to choose a map $\xi : G \to \mathrm{Aut}(A)$ such that $\xi_1 = \mathrm{id}_A$ and $\rho \circ \xi = i$. We choose $\xi := \omega$. Using the map f from above, we remark that $\omega_g \circ \omega_h = \mathrm{Int}(f(g,h)) \circ \omega_{gh}$ and $f(g,1) = f(1,h) = 1$ for $g,\, h \in G$; cf. Lemma 2.7.(1, 4). So we may use this map f. Thus, letting

$$c(g,h,k) \;:=\; f(g,h) \cdot f(gh,k) \cdot f(g,hk)^{-1} \cdot \xi_g(f(h,k))^{-1}$$

for $g,\, h,\, k \in G$, we obtain $\zeta_i := c \cdot \mathrm{B}^3(G, \mathrm{Z}(A))$.

By Lemma 2.7.(3), we have $c = !$.

Alternatively, we have

$$c(1,h,k) = c(g,1,k) = c(g,h,1) = 1$$

for $g,\, h,\, k \in G$; cf. Lemma 3.5, proof of Theorem 3.7. Moreover,

$$\begin{aligned} c(d,d,d) &= f(d,d) \cdot f(1,d) \cdot f(d,1)^{-1} \cdot \xi_d(f(d,d))^{-1} \\ &= a \cdot 1 \cdot 1 \cdot \omega_d(a)^{-1} \\ &= a \cdot \omega_d(a)^{-1} \\ &= 1 \end{aligned}$$

So also this direct calculation shows that $c = !$.

Hence

$$\zeta_i \;=\; c \cdot \mathrm{B}^3(G, \mathrm{Z}(A)) \;=\; 1\,.$$

Example 3.6 shows that $\zeta_! = 1$.

Since $G = \mathrm{C}_2$ and $\mathrm{Z}(A) = \langle a^2 \rangle \simeq \mathrm{C}_2$, we have $\mathrm{H}^3(G, \mathrm{Z}(A)) \simeq \mathrm{C}_2$, as it is well-known, e.g. via the package HAP of GAP [**3**].

Altogether, we have the map

$$\begin{array}{rcl} \overbrace{\mathrm{Mor}(G, \mathrm{Out}(A))}^{\{\,!\,,\,i\,\}} & \xrightarrow{\zeta} & \overbrace{\mathrm{H}^3(G, \mathrm{Z}(A))}^{\simeq\, \mathrm{C}_2} \\ ! & \mapsto & \zeta_! \;=\; 1 \\ i & \mapsto & \zeta_i \;=\; 1\,. \end{array}$$

So this map is trivial in our example. Note that $|\,\mathrm{Mor}(G, \mathrm{Out}(A))| \neq 1$, that $|\,\mathrm{H}^3(G, \mathrm{Z}(A))| \neq 1$ and that $1 < \mathrm{Z}(A) < A$. Cf. Remark 3.8.

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Zusammenfassung

Die folgenden Resultate gehen zurück auf Schreier, Morandi, Eilenberg und Mac Lane; siehe **[8]**, **[7]**, **[6]**.

Seien A und G Gruppen, nicht notwendig abelsch.

Wir führen die Menge $\mathrm{h}^2(G, A)$ der Äquivalenzklassen normalisierter verallgemeinerter 2-Cozyklen ein; siehe Lemma 2.7. Wir schreiben $\mathrm{Ext}(G, A)$ für die Menge der Äquivalenzklassen der Gruppenerweiterungen von A mit G; siehe Definition 2.3.

Theorem 2.17. Wir konstruieren wechselseitig inverse Bijektionen

$$\mathrm{Ext}(G, A) \underset{\beta}{\overset{\alpha}{\underset{\sim}{\rightleftarrows}}} \mathrm{h}^2(G, A) \;;$$

siehe Propositionen 2.9 und 2.14.

Sei ein Gruppenmorphismus $\varpi : G \to \mathrm{Out}(A)$ gegeben. Dann induziert ϖ einen Gruppenmorphismus $\varpi^{\mathrm{Z}} : G \to \mathrm{Aut}(\mathrm{Z}(A))$; siehe Bemerkung 3.2. Cohomologiegruppen von G mit Werten in $\mathrm{Z}(A)$ sind bezüglich ϖ^{Z} zu bilden.

Wir ordnen ϖ ein Element $\zeta_\varpi \in \mathrm{H}^3(G, \mathrm{Z}(A))$ zu; siehe Lemma 3.5.(3).

Theorem 3.7. Genau dann gibt es eine Gruppenerweiterung von A mit G, die den Gruppenmorphismus

$$\varpi \;:\; G \;\to\; \mathrm{Out}(A)$$

induziert, wenn sich in $\mathrm{H}^3(G, \mathrm{Z}(A))$

$$\zeta_\varpi = 1$$

ergibt.

Wir geben ein Beispiel, in welchem die Abbildung ζ trivial ist, obwohl $|\,\mathrm{Mor}(G, \mathrm{Out}(A))| \neq 1$ und $|\,\mathrm{H}^3(G, \mathrm{Z}(A))| \neq 1$ und $1 < \mathrm{Z}(A) < A$ ist; siehe Beispiel 3.17.

Sei $\mathrm{Ext}_\varpi(G, A)$ die Teilmenge von $\mathrm{Ext}(G, A)$, die aus den Äquivalenzklassen der Gruppenerweiterungen von A mit G besteht, die ϖ induzieren im Sinne von Bemerkung 3.1.

Sei $\mathrm{h}^2_\varpi(G, A)$ die Teilmenge von $\mathrm{h}^2(G, A)$, die aus den Äquivalenzklassen der normalisierten verallgemeinerten 2-Cozyklen (ω, f) besteht, für welche ω eine Hebung von ϖ ist; siehe Lemmata 2.7 und 3.9.

Dann schränken die Bijektionen von Theorem 2.17 ein auf die Teilmengen $\mathrm{Ext}_\varpi(G, A)$ und $\mathrm{h}^2_\varpi(G, A)$. Die Einschränkungen werden α_ϖ und β_ϖ geschrieben.

Wir verwenden eine normalisierte Variante $\mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A))$ von $\mathrm{H}^2(G, \mathrm{Z}(A))$, in welcher die Cohomologieklassen von normalisierten 2-Cozyklen repräsentiert werden; siehe Bemerkung 3.3, Teil 2. Dank Lemma 1.13 ist

$$\mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) \overset{\varphi}{\underset{\sim}{\to}} \mathrm{H}^2(G, \mathrm{Z}(A)) \;.$$

Theorem 3.13. Es existiere eine Gruppenerweiterung von A mit G, die den Gruppenmorphismus $\varpi : G \to \mathrm{Out}(A)$ induziert.

Somit ist $\mathrm{h}^2_\varpi(G, A) \neq \emptyset$; siehe Lemma 3.9. Wir können also ein Element (ξ, f_0) in $\mathrm{z}^2(G, A)$ wählen mit $[\xi, f_0] \in \mathrm{h}^2_\varpi(G, A)$.

Wir konstruieren wechselseitig inverse Bijektionen ϑ^{ξ, f_0} und η^{ξ, f_0} derart, dass wir insgesamt folgendes Diagramm erhalten.

$$\begin{tikzcd}
\mathrm{Ext}_\varpi(G, A) \arrow[r, shift left, "\alpha_\varpi"] \arrow[r, phantom, "\sim"] & \mathrm{h}^2_\varpi(G, A) \arrow[l, shift left, "\beta_\varpi"] \arrow[r, shift left, "\vartheta^{\xi, f_0}"] \arrow[r, phantom, "\sim"] & \mathrm{H}^2_{\mathrm{norm}}(G, \mathrm{Z}(A)) \arrow[l, shift left, "\eta^{\xi, f_0}"] \arrow[r, "\varphi", "\sim"'] & \mathrm{H}^2(G, \mathrm{Z}(A))
\end{tikzcd}$$

Ist insbesondere A abelsch, so erhalten wir als Spezialfall die Theorie der Gruppenerweiterungen mit abelschem Kern; siehe Bemerkungen 3.8 und 3.16.

Versicherung

Hiermit versichere ich,

1. dass ich meine Arbeit selbstständig verfasst habe,
2. dass ich keine anderen als die angegeben Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,
3. dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und
4. dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, Juni 2017

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