Quasi-isomorphisms and quasi-cyclic graphs

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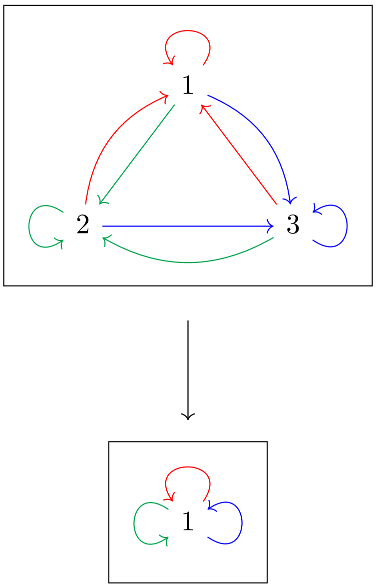

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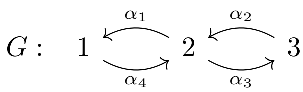
Contents

1 Introduction

1.1 Graphs and graph morphisms

The *graphs* considered in this text are directed and consist of vertices and edges. Suppose given a graph G, we denote its vertex set by V_G and its edge set by E_G. The direction of each edge $e \in \mathrm{E}_G$ is given by maps $\mathrm{s}_G, \mathrm{t}_G : \mathrm{E}_G \to \mathrm{V}_G$, and is always from the source vertex $e\,\mathrm{s}_G$ to the target vertex $e\,\mathrm{t}_G$. The following example illustrates what a graph G may look like:



The vertex set of G is $\mathrm{V}_G = \{1, 2, 3\}$, and the edge set is $\mathrm{E}_G = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. To explain the direction of an edge of G, we use arrows, which are pointed from source vertex to target vertex. Take α_2 as an example. The source vertex of α_2 is $\alpha_2\,\mathrm{s}_G = 3$, and the target vertex of α_2 is $\alpha_2\,\mathrm{t}_G = 2$. The direction of α_2 is from 3 to 2.

A *graph morphism* $f : G \to H$ maps vertices in G to vertices in H, and edges in G to edges in H, while respecting sources and targets. We use $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ and $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ to denote the maps between vertices and between edges, respectively. Then for every $e \in \mathrm{E}_G$ with source vertex $e\,\mathrm{s}_G$ and target vertex $e\,\mathrm{t}_G$ in G, its image $e\,\mathrm{E}_f$ in H must satisfy $e\,\mathrm{E}_f\,\mathrm{s}_H = e\,\mathrm{s}_G\,\mathrm{V}_f$ and $e\,\mathrm{E}_f\,\mathrm{t}_H = e\,\mathrm{t}_G\,\mathrm{V}_f$. The following diagram illustrates what a graph morphism $f : G \to H$ may look like:

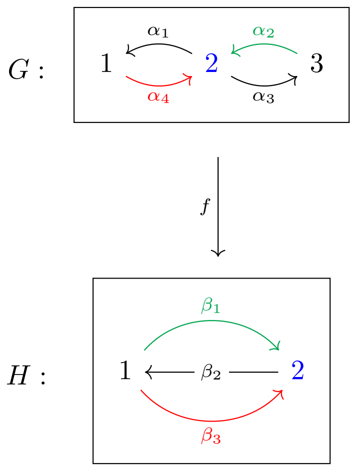


Figure 1.1

The vertex map V_f and the edge map E_f of this graph morphism f are given by

$$\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H : \left\{ \begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 1 \end{array} \right\} \quad \text{and} \quad \mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H : \left\{ \begin{array}{ccc} \alpha_1 & \mapsto & \beta_2 \\ \alpha_2 & \mapsto & \beta_1 \\ \alpha_3 & \mapsto & \beta_2 \\ \alpha_4 & \mapsto & \beta_3 \end{array} \right\}.$$

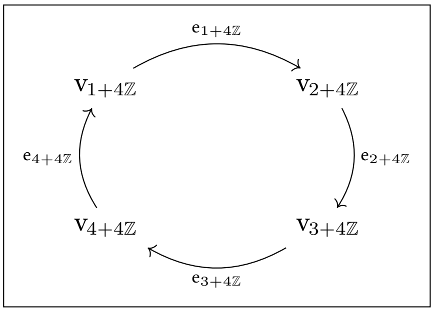
A graph morphism $f : G \to H$ is a *monomorphism* (respectively, *epimorphism*, *isomorphism*) if and only if both its vertex map V_f and edge map E_f are injective (respectively, surjective, bijective). If there is an isomorphism between G and H, we say the graphs G and H are isomorphic.

Since the composites of graph morphisms are graph morphisms, graphs together with graph morphisms form a category, denoted by **Gph**.

1.2 Quasi-isomorphisms based on cyclic graphs

In the category **Gph**, the *cyclic graph* C_n, where $n \in \mathbb{Z}_{\geqslant 1}$, has n edges and n vertices, and is of a cyclic shape. We label the n vertices and edges of C_n with the aid of the elements of $\mathbb{Z}/n\mathbb{Z}$, and for each $\mathrm{e}_{i+n\mathbb{Z}} \in \mathrm{E}_{\mathrm{C}_n}$, we set $\mathrm{e}_{i+n\mathbb{Z}}\,\mathrm{s}_{\mathrm{C}_n} = \mathrm{v}_{i+n\mathbb{Z}}$ and $\mathrm{e}_{i+n\mathbb{Z}}\,\mathrm{t}_{\mathrm{C}_n} = \mathrm{v}_{i+1+n\mathbb{Z}}$. Take C_4 as an example:

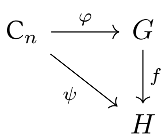
C_4 :



The cyclic subgraphs of a graph G are the subgraphs of G that are isomorphic to some cyclic graph C_n, where $n \in \mathbb{Z}_{\geqslant 1}$.

Every graph morphism $f : G \to H$ together with a cyclic graph C_n induces a map

$$(\mathrm{C}_n, f) : (\mathrm{C}_n, G)_{\mathrm{Gph}} \to (\mathrm{C}_n, H)_{\mathrm{Gph}}, \varphi \mapsto \varphi \cdot f.$$



If the map (C_n, f) is bijective for every $n \in \mathbb{Z}_{\geqslant 1}$, we say that $f : G \to H$ is a *quasi-isomorphism*. This definition is due to T. Bisson and A. Tsemo, where the original term used is "acyclics" ; cf. [**1,** Definition 4.2, Definition 4.3]. The graph morphism $f : G \to H$ in Figure 1.1 is a example of quasi-isomorphism; cf. Example 4.9. Using quasi-isomorphisms as weak equivalences, Bisson and Tsemo have defined a model category structure on **Gph**; cf. [**1,** Corollary 4.8], cf. also [**4,** Proposition 204].

However, determining whether a graph morphism f is a quasi-isomorphism based solely on the definition can be challenging, since the determination process is generally infinite. Therefore, we shall provide several lifting results for quasi-isomorphisms; see section 1.4 below. Using one of them, we give a criterion that allows to decide in finitely many steps whether a graph morphism between finite graphs is a quasi-isomorphism.

1.3 Quasi-cyclic graphs

To investigate the properties of quasi-isomorphisms, quasi-cyclic graphs can be employed. A graph C is said to be *quasi-cyclic* if it can be covered by finitely many cyclic subgraphs $(C_i)_{i=1}^{m}$, $m \in \mathbb{Z}_{\geqslant 1}$, where every C_i in this sequence has at least a common vertex with the union graph of $C_1, \ldots, C_{i-1}$. The quasi-cyclic subgraphs of a graph G are the subgraphs of G that are isomorphic to some quasi-cyclic graph.

As examples, both G and H in Figure 1.1 are quasi-cyclic.

Quasi-cyclic graphs are connected in a strong sense. The discussion of the connectivity of general undirected graphs in **[2,** Chapter 3**]** gave rise to the idea of discussing the connectivity of quasi-cyclic graphs. The connectivity in this book corresponds to the weak connectivity in this text, that is, if a graph G is weakly connected, then for every $v, w \in \mathrm{V}_G$, we have a vertex sequence $(v_i)_{i=0}^{n}$, with $v_0 = v$ and $v_n = w$, satisfying that for any $0 \leqslant i \leqslant n-1$, there exists $e_i \in \mathrm{E}_G$ such that v_i, v_{i+1} are both end vertices of e_i. This definition is equivalent to stating that G cannot be written as the union of two disjoint nonempty subgraphs. If one demands that all e_i point into the target direction, one obtains a stronger version of the notion of connectivity.

To this end, we first define the notion of a *path* in our graphs. A path of length $n \in \mathbb{Z}_{\geqslant 0}$ in a graph G is a sequence $(e_i)_{i=0}^{n-1} \subseteq \mathrm{E}_G$ of n edges starting from a certain vertex $v_0 \in \mathrm{V}_G$ such that $e_0\,\mathrm{s}_G = v_0$ and $e_i\,\mathrm{s}_G = e_{i-1}\,\mathrm{t}_G$ for $1 \leqslant i \leqslant n-1$. When $n = 0$, we consider the path to be the single vertex v_0. For $n > 0$, the direction of the path $(e_i)_{i=0}^{n-1}$ is from $v_0 := e_0\,\mathrm{s}_G$ to $v_n := e_{n-1}\,\mathrm{t}_G$. We say that a graph G is *strongly connected* if for every $v, w \in \mathrm{V}_G$, there exists a path from v to w and a path from w to v. All strongly connected graphs are weakly connected, and indeed, all quasi-cyclic graphs are also strongly connected.

On the other hand, each quasi-cyclic graph is derived from a cyclic graph through a graph morphism. More precisely, every graph morphism from a cyclic graph C_n has a quasi-cyclic image graph; cf. Lemma 3.8. Conversely, for every given quasi-cyclic graph C, we can construct an epimorphism $f : \mathrm{C}_n \to C$ from a certain cyclic graph C_n; cf. Lemma 3.9. In this way, starting from any vertex in the quasi-cyclic graph C, we can find such a path that passes through all the vertices and edges in C and finally returns to the starting point. Therefore, we define the circumferential length $\ell(C)$ of a quasi-cyclic graph as the length of the shortest path that satisfies the above conditions, that is,

$$\ell(C) := \min\{n \in \mathbb{Z}_{\geqslant 1} \;:\; \text{there exists a graph epimorphism from } \mathrm{C}_n \text{ to } C\}\;.$$

It is worth mentioning here that we cannot predict the circumferential length of a quasi-cyclic graph by the number of its edges. In fact, for every $k \in \mathbb{Z}_{\geqslant 1}$, there exists a quasi-cyclic graph C such that $\frac{\ell(C)}{|\,\mathrm{E}_C\,|} > k$; so this ratio is unbounded. In Section 3.3, we provide a reasonably fast method for calculating the circumferential length of a quasi-cyclic graph through the covering method; cf. Algorithm 5.14.

1.4 Lifting results for quasi-isomorphisms

In Chapter 4, we present several lifting results for a quasi-isomorphism $f : G \to H$, which can also serve as inspiration for constructing examples of quasi-isomorphisms. In fact, one can view those results as necessary conditions for a graph morphism to be a quasi-isomorphism. Moreover, a sufficient condition for a morphism f to be quasi-isomorphic has been provided in **[4]**.

Suppose that we are given a quasi-isomorphism

$$f : G \xrightarrow{\approx} H$$

between graphs G, H. Here, the graphs G, H can also be infinite. We denote the image graph of a subgraph $G' \leqslant G$ under f by $(G')f$; cf. Remark 2.10.

Then we conclude that for every cyclic subgraph $C^{(H)} \leqslant H$, there exists a unique cyclic subgraph $C^{(G)} \leqslant G$ such that $\left(C^{(G)}\right) f = C^{(H)}$; and the restriction $f\big|_{C^{(G)}}^{C^{(H)}}$ of f is an isomorphism.

If, in addition, G is a finite graph, then the following relationship holds among those pairs of such $C^{(G)}$ and $C^{(H)}$: Suppose that $C_1^{(H)}$, $C_2^{(H)}$ are cyclic subgraphs of H sharing a common vertex. Let $C_1^{(G)}$, $C_2^{(G)}$ denote the cyclic subgraphs of G corresponding to $C_1^{(H)}$, $C_2^{(H)}$, respectively. The cyclic $C_1^{(G)}$, $C_2^{(G)}$ subgraphs may have no common vertex in G, but they must be contained in a common quasi-cyclic subgraph of G; cf. Lemma 4.19, cf. Example 4.20.

Next, we consider the lifting results for quasi-cyclic subgraphs along quasi-isomorphisms. For every quasi-cyclic subgraph $Q^{(H)} \leqslant H$, there exists a quasi-cyclic subgraph $Q^{(G)} \leqslant G$ of the same circumferential length as $Q^{(H)}$ such that the restriction $f\big|_{Q^{(G)}}^{Q^{(H)}} : Q^{(G)} \to Q^{(H)}$ is an epimorphism. However, such a $Q^{(G)}$ may not be the unique, and the restriction $f\big|_{Q^{(G)}}^{Q^{(H)}}$ is not necessarily a quasi-isomorphism; cf. Example 4.15, cf. Remark 4.16. But if we restrict f to the subgraph

$$G^{\left(Q^{(H)}\right)} := \bigcup \left\{C \leqslant G \ : \ C \text{ is cyclic and } (C)f \leqslant Q^{(H)}\right\}, \tag{1.1}$$

we obtain a quasi-isomorphism $f\big|_{G^{\left(Q^{(H)}\right)}}^{Q^{(H)}} : G^{\left(Q^{(H)}\right)} \to Q^{(H)}$; cf. Lemma 4.17.

In Section 4.4, we restrict our discussion of quasi-isomorphisms to finite graphs. A subgraph G' of G is called *maximal quasi-cyclic* if G' is quasi-cyclic and the only quasi-cyclic subgraph that includes G' is G' itself. Write the set of all maximal quasi-cyclic subgraphs of G as $\mathrm{S}_{\mathrm{qc}}^{\max}(G)$, if G, H are finite graphs, then the map

$$\begin{array}{rccc} \hat{f}: & \mathrm{S}_{\mathrm{qc}}^{\max}(G) & \to & \mathrm{S}_{\mathrm{qc}}^{\max}(H) \\ & \hat{G} & \mapsto & (\hat{G})f \end{array}$$

induced by f is bijective and every restriction $f\big|_{\hat{G}}^{(\hat{G})f} : \hat{G} \to (\hat{G})f$ is a quasi-isomorphism. Conversely, every graph morphism between finite graphs that satisfies these properties is a quasi-isomorphism; cf. Proposition 4.22.

Furthermore, if G, H are finite graphs, then for every quasi-cyclic subgraph $Q^{(H')} \leqslant H$, the subgraph $G^{\left(Q^{(H)}\right)}$ defined in Equation (1.1) is also a quasi-cyclic subgraph of G. This results in section 4.4 also enable us to restrict our investigation of quasi-isomorphisms on finite graphs to quasi-cyclic graphs.

1.5 A characterization of quasi-isomorphisms between finite graphs

In section 4.5, we focus on the graph morphisms from a quasi-cyclic graph to a finite graph.

Let $f : G \to H$ be a graph morphism, where G is quasi-cyclic and H is finite. The pullback graph $P_f^{(G)}$ of

$$\begin{array}{ccc} & & G \\ & & \downarrow f \\ G & \xrightarrow{f} & H \end{array} \quad ,$$

can be constructed as follows. Let

$$P_f^{(G)} = \left(\mathrm{V}_{P_f^{(G)}}, \mathrm{E}_{P_f^{(G)}}; \mathrm{s}_{P_f^{(G)}}, \mathrm{t}_{P_f^{(G)}} \right) , \tag{1.2}$$

where

$$\mathrm{V}_{P_f^{(G)}} := \left\{ (v^+, v^-) \in \mathrm{V}_G \times \mathrm{V}_G \; : \; v^+ \, \mathrm{V}_f = v^- \, \mathrm{V}_f \right\} ;$$
$$\mathrm{E}_{P_f^{(G)}} := \left\{ (e^+, e^-) \in \mathrm{E}_G \times \mathrm{E}_G \; : \; e^+ \, \mathrm{E}_f = e^- \, \mathrm{E}_f \right\} ;$$

and where for every $(e^+, e^-) \in \mathrm{E}_{P_f^{(G)}}$, we set

$$\mathrm{s}_{P_f^{(G)}} : (e^+, e^-) \longmapsto (e^+ \, \mathrm{s}_G, e^- \, \mathrm{s}_G) ;$$
$$\mathrm{t}_{P_f^{(G)}} : (e^+, e^-) \longmapsto (e^+ \, \mathrm{t}_G, e^- \, \mathrm{t}_G) .$$

With this construction, we conclude that the map (C_n, f) is injective for all $n \in \mathbb{Z}_{\geqslant 1}$ if and only if the following conditions hold (see Proposition 4.25):

(i) For $e_1, e_2 \in \mathrm{E}_G$ with $e_1 \neq e_2$, $e_1 \, \mathrm{s}_G = e_2 \, \mathrm{s}_G$ and $e_1 \, \mathrm{t}_G = e_2 \, \mathrm{t}_G$, we have $e_1 \, \mathrm{E}_f \neq e_2 \, \mathrm{E}_f$.

(ii) For every $C \in \mathrm{S_c}(P_f^{(G)})$ and every $(v^+, v^-) \in \mathrm{V}_{P_f^{(G)}}$ with $v^+ \neq v^-$, we have $(v^+, v^-) \notin \mathrm{V}_C$.

To ensure that (C_n, f) is also surjective, we employ the notion of adjacency matrices of finite graphs. We label the vertices of G and of H as $v_1^{(G)}, \ldots, v_{|\mathrm{V}_G|}^{(G)}$ and $v_1^{(H)}, \ldots, v_{|\mathrm{V}_H|}^{(H)}$, respectively. The *adjacency matrix* $A_G \in \mathbb{Q}^{|\mathrm{V}_G| \times |\mathrm{V}_G|}$ of G is a square matrix with

$$(A_G)_{ij} = \left| \left\{ e \in \mathrm{E}_G \; : \; e \, \mathrm{s}_G = v_i^{(G)} \text{ and } e \, \mathrm{t}_G = v_j^{(G)} \right\} \right|$$

for $i, j \in [1, |\mathrm{V}_G|]$. We denote by A_H the adjacency matrix of H in the same sense. We also define the matrix $A_f \in \mathbb{Q}^{|\mathrm{V}_G| \times |\mathrm{V}_H|}$ by

$$(A_f)_{ij} := \begin{cases} 1 & \text{if } v_i^{(G)} \, \mathrm{V}_f = v_j^{(H)} \\ 0 & \text{if } v_i^{(G)} \, \mathrm{V}_f \neq v_j^{(H)} \end{cases}$$

for $i \in [1, |\mathrm{V}_G|]$ and $j \in [1, |\mathrm{V}_H|]$. Here is our main result:

Theorem 4.32

Let $f : G \to H$ be a graph morphism between finite graphs, where G is quasi-cyclic. The following conditions (1) and (2) are equivalent:

(1) The graph morphism $f : G \to H$ is a quasi-isomorphism.

(2) (i) For $e_1, e_2 \in \mathrm{E}_G$ with $e_1 \neq e_2$, $e_1 \, \mathrm{s}_G = e_2 \, \mathrm{s}_G$ and $e_1 \, \mathrm{t}_G = e_2 \, \mathrm{t}_G$, we have $e_1 \, \mathrm{E}_f \neq e_2 \, \mathrm{E}_f$.

(ii) For every $C \in \mathrm{S_c}(P_f^{(G)})$ and every $(v^+, v^-) \in \mathrm{V}_{P_f^{(G)}}$ with $v^+ \neq v^-$, we have $(v^+, v^-) \notin \mathrm{V}_C$.

(iii) We have $A_f^{\mathrm{T}} (A_G^k)^{\mathrm{diag}} A_f = (A_H^k)^{\mathrm{diag}}$ for every $1 \leqslant k \leqslant |\mathrm{V}_G| + |\mathrm{V}_H|$.

Combining with Proposition 4.22, we may decide whether a graph morphism between finite graphs is a quasi-isomorphism in finitely many steps:

Theorem 4.33

Let $f : G \to H$ be a graph morphism between finite graphs. The following conditions (1) and (2) are equivalent:

(1) The graph morphism $f : G \to H$ is a quasi-isomorphism.

(2) (a) The map

$$\begin{array}{rccc} \hat{f}: & \mathrm{S}^{\max}_{\mathrm{qc}}(G) & \to & \mathrm{S}^{\max}_{\mathrm{qc}}(H) \\ & \hat{G} & \mapsto & (\hat{G})f \end{array}$$

is bijective.

(b) For every restriction $f^{\hat{G}} := f|_{\hat{G}}^{(\hat{G})f} : \hat{G} \to (\hat{G})f$ with $\hat{G} \in \mathrm{S}^{\max}_{\mathrm{qc}}(G)$, the following conditions hold:

(i) For $e_1, e_2 \in \mathrm{E}_{\hat{G}}$ with $e_1 \neq e_2$, $e_1\,\mathrm{s}_G = e_2\,\mathrm{s}_G$ and $e_1\,\mathrm{t}_G = e_2\,\mathrm{t}_G$, we have $e_1\,\mathrm{E}_f \neq e_2\,\mathrm{E}_f$.

(ii) For every cyclic subgraph $C \leqslant P^{(\hat{G})}_{f^{\hat{G}}}$ and every $(v^+, v^-) \in \mathrm{V}_{P^{(\hat{G})}_{f^{\hat{G}}}}$ with $v^+ \neq v^-$, we have $(v^+, v^-) \notin \mathrm{V}_C$.

(iii) We have $A^{\mathrm{T}}_{f^{\hat{G}}}(A^k_{\hat{G}})^{\mathrm{diag}} A_{f^{\hat{G}}} = (A^k_{(\hat{G})f})^{\mathrm{diag}}$ for every $1 \leqslant k \leqslant |\,\mathrm{V}_{\hat{G}}\,| + |\,\mathrm{V}_{(\hat{G})f}\,|$.

1.6 Conventions

Throughout this paper, we adopt the following conventions for commonly used mathematical symbols.

- Let A be a set. We denote the cardinality of A as $|A|$. If A is a finite set, then $|A|$ is the count of elements in A.

- Let A be a set. For $n \in \mathbb{Z}_{\geqslant 1}$, we denote the cartesian product $A^{\times n} := \underbrace{A \times \cdots \times A}_{\text{n times}}$.

- For integers $n, m \in \mathbb{Z}$, we denote

$$[n, m] := \{k \in \mathbb{Z} \mid n \leqslant k \leqslant m\}\,,$$

which is understood as integer interval.

- For a map $f : X \to Y$ and $x \in X$, we write the image of x under f as xf, or as $(x)f$ if necessary. Similarly, the image of $X' \subseteq X$ under f is written as $X'f$, or as $(X')f$ if necessary.

- Composites of maps are written from left to right. So for maps $f : X \to Y$ and $g : Y \to Z$, the composite of f and g is written as $f \cdot g : X \to Z$. For $x \in X$, we have

$$x(f \cdot g) := ((x)f)g\,.$$

- Let X, Y be sets, and $f : X \to Y$ be a map. Suppose that $X' \subseteq X$ and $Y' \subseteq Y$ such that $X'f \subseteq Y'$. The restriction $f|_{X'}^{Y'} : X' \to Y'$ of f is the map with $xf|_{X'}^{Y'} = xf$ for every $x \in X'$. If $X' = X$, we may simply write $f|^{Y'} := f|_X^{Y'}$; If $Y' = Y$, we may simply write $f|_{X'} := f|_{X'}^{Y}$.

- A graph G consists of a vertex set V_G, an edge set E_G, a source map $\mathrm{s}_G : \mathrm{E}_G \to \mathrm{V}_G$ and a target map $\mathrm{t}_G : \mathrm{E}_G \to \mathrm{V}_G$; cf. Definition 2.1.

- Given a graph G, we write $G' \leqslant G$ to express that G' is a subgraph of G; cf. Definition 2.6.

- Given two graphs G and H, a graph morphism $f : G \to H$ consists of a vertex map $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ and an edge map $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ such that $\mathrm{E}_f \cdot \mathrm{s}_H = \mathrm{s}_G \cdot \mathrm{V}_f$ and $\mathrm{E}_f \cdot \mathrm{t}_H = \mathrm{t}_G \cdot \mathrm{V}_f$; cf. Definition 2.9. Let $G' \leqslant G$. The image graph of G' under f is written as $(G')f$; cf. Remark 2.10.

- Composites of graph morphisms are also written from left to right; cf. Remark 2.11.

- Let G, H be graphs, and $f : G \to H$ be a graph morphism. Suppose that $G' \leqslant G$ and $H' \leqslant H$ such that $(G')f \leqslant H'$. The restriction $f|_{G'}^{H'} : G' \to H'$ is the graph morphism between G' and H' consisting the vertex map $\mathrm{V}_f|_{\mathrm{V}_{G'}}^{\mathrm{V}_{H'}}$ and the edge map $\mathrm{E}_f|_{\mathrm{E}_{G'}}^{\mathrm{E}_{H'}}$. If $G' = G$, we may simply write $f|^{H'} := f|_G^{H'} : G \to H'$; If $H' = H$, we may simply write $f|_{G'} := f|_{G'}^{H} : G' \to H$.

- Let A be a matrix. The symbol A^{T} states for the transformation matrix of A.

- Let $A \in \mathbb{Q}^{n\times n}$ be a $n \times n$-matrix with $n \in \mathbb{Z}_{\geqslant 1}$. We denote by $(A)^{\mathrm{diag}} \in \mathbb{Q}^{n\times n}$ the matrix with

$$(A)^{\mathrm{diag}}_{ij} = \begin{cases} 0 & \text{if } i = j \\ A_{ij} & \text{if } i \neq j \end{cases} .$$

2 Preliminaries

2.1 Graphs

In this section, we present the mathematical framework for describing graphs and investigating their internal structure.

Definition 2.1 (Graphs)
A *graph* $G = (\mathrm{V}_G, \mathrm{E}_G; \mathrm{s}_G, \mathrm{t}_G)$ is a 4-tuple, where

- V_G is the *vertex set* of G,
- E_G is the *edge set* of G,
- $\mathrm{s}_G : \mathrm{E}_G \to \mathrm{V}_G$ and $\mathrm{t}_G : \mathrm{E}_G \to \mathrm{V}_G$ are the *source map* and *target map* of G, and specify the source vertex and target vertex of each edge $e \in \mathrm{E}_G$, respectively.

Each edge $e \in \mathrm{E}_G$ is considered to be directed from $e\,\mathrm{s}_G$ to $e\,\mathrm{t}_G$. Depending on the context and the information to be provided, we may also write $G = (\mathrm{V}_G, \mathrm{E}_G)$, or simply G.

In the literature, the type of graph considered in this text is often referred to as a directed graph. However, the term "graph" throughout this text will exclusively refer to an object satisfying Definition 2.1.

Remark 2.2
Generally, graphs can be represented visually. For a graph $G = (\mathrm{V}_G, \mathrm{E}_G; \mathrm{s}_G, \mathrm{t}_G)$, the vertices are depicted as the names of the elements of V_G. An edge $e \in \mathrm{E}_G$ is represented by an arrow from $e\,\mathrm{s}_G$ to $e\,\mathrm{t}_G$, labeled by e, that is,

$$e\,\mathrm{s}_G \xrightarrow{e} e\,\mathrm{t}_G\,.$$

Moreover, we may also define a graph directly by providing a picture, from which all the necessary information for constructing this graph can be extracted.

Example 2.3
To illustrate how the tuple definition of a graph corresponds to its diagrammatic representation, we consider the following example. Let $G = (\mathrm{V}_G, \mathrm{E}_G;\ \mathrm{s}_G, \mathrm{t}_G)$ be a graph, where

- $\mathrm{V}_G = \{v_1, v_2, v_3, v_4, v_5\}$
- $\mathrm{E}_G = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$

- $\mathrm{s}_G = \left\{ \begin{array}{l} e_1 \mapsto v_1 \\ e_2 \mapsto v_2 \\ e_3 \mapsto v_3 \\ e_4 \mapsto v_4 \\ e_5 \mapsto v_3 \\ e_6 \mapsto v_1 \\ e_7 \mapsto v_4 \\ e_8 \mapsto v_4 \\ e_9 \mapsto v_1 \end{array} \right\}$ and $\mathrm{t}_G = \left\{ \begin{array}{l} e_1 \mapsto v_2 \\ e_2 \mapsto v_3 \\ e_3 \mapsto v_4 \\ e_4 \mapsto v_2 \\ e_5 \mapsto v_1 \\ e_6 \mapsto v_4 \\ e_7 \mapsto v_4 \\ e_8 \mapsto v_1 \\ e_9 \mapsto v_2 \end{array} \right\}$.

The graph G possesses also a diagrammatic representation given in Figure 2.1.

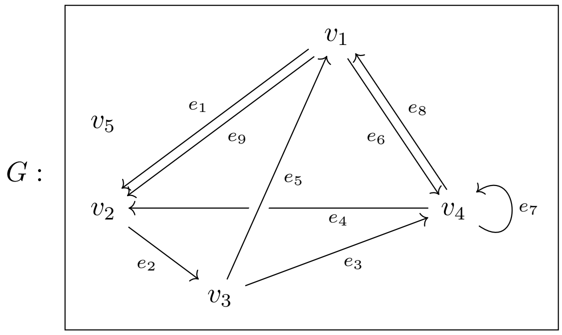


Figure 2.1

Definition 2.4

A graph G is said to be

(1) *finite* if both V_G and E_G are finite.

(2) *simple* if for every given $e_1, e_2 \in \mathrm{E}_G$ the following is valid:

$$(e_1\,\mathrm{s}_G = e_2\,\mathrm{s}_G) \wedge (e_1\,\mathrm{t}_G = e_2\,\mathrm{t}_G) \quad \Rightarrow \quad e_1 = e_2.$$

Definition 2.5

For a graph $G = (\mathrm{V}_G, \mathrm{E}_G; \mathrm{s}_G, \mathrm{t}_G)$ and $v \in \mathrm{V}_G$, we define the following sets:

(1) The set of all edges in G with source vertex v is denoted as

$$\mathrm{E}_G(v, *) := \{e \in \mathrm{E}_G \mid e\,\mathrm{s}_G = v\}.$$

(2) The set of all vertices in G reachable from v via a single edge is

$$\mathrm{V}_G(v, *) := \{w \in \mathrm{V}_G \ : \ \exists e \in \mathrm{E}_G \text{ such that } e\,\mathrm{s}_G = v,\ e\,\mathrm{t}_G = w\}.$$

(3) The set of all edges in G with target vertex v is

$$\mathrm{E}_G(*, v) := \{e \in \mathrm{E}_G \mid e\,\mathrm{t}_G = v\}.$$

(4) The set of all source vertices of edges in $\mathrm{E}_G(*, v)$ is

$$\mathrm{V}_G(*, v) := \{w \in \mathrm{V}_G \ : \ \exists e \in \mathrm{E}_G \text{ such that } e\,\mathrm{t}_G = v,\ e\,\mathrm{s}_G = w\}.$$

With these notations, we say that a vertex $w \in \mathrm{V}_G$ is *adjacent* to v if

$$w \in \mathrm{V}_G(v,*) \cup \mathrm{V}_G(*,v).$$

The vertex v is called *isolated* if

$$\mathrm{E}_G(v,*) \cup \mathrm{E}_G(*,v) = \emptyset.$$

Definition 2.6 (Subgraph)
Let $G = (\mathrm{V}_G, \mathrm{E}_G; \mathrm{s}_G, \mathrm{t}_G)$ be a graph. A graph $G' = (\mathrm{V}_{G'}, \mathrm{E}_{G'}; \mathrm{s}_{G'}, \mathrm{t}_{G'})$ is called *subgraph* of G, denoted by $G' \leqslant G$, if the following conditions hold:

- $\mathrm{V}_{G'} \subseteq \mathrm{V}_G$ and $\mathrm{E}_{G'} \subseteq \mathrm{E}_G$,
- $\mathrm{E}_{G'}\, \mathrm{s}_G \subseteq \mathrm{V}_{G'}$ and $\mathrm{E}_{G'}\, \mathrm{t}_G \subseteq \mathrm{V}_{G'}$,
- $\mathrm{s}_{G'} = \mathrm{s}_G |_{\mathrm{E}_{G'}}^{\mathrm{V}_{G'}}$ and $\mathrm{t}_{G'} = \mathrm{t}_G |_{\mathrm{E}_{G'}}^{\mathrm{V}_{G'}}$.

We denote by $\mathrm{S}(G)$ the set of all subgraphs of G. Moreover, if $G' \leqslant G$ and $G' \neq G$, we write $G' < G$.

Definition 2.7 (Union of subgraphs)
Let $G = (\mathrm{V}_G, \mathrm{E}_G; \mathrm{s}_G, \mathrm{t}_G)$ be a graph. For a given index set Λ and a tuple $(G_\lambda)_{\lambda\in\Lambda}$ of subgraphs of G we define their *union*

$$\hat{G} := \bigcup_{\lambda\in\Lambda} G_\lambda = (\mathrm{V}_{\hat{G}}, \mathrm{E}_{\hat{G}}; \mathrm{s}_{\hat{G}}, \mathrm{t}_{\hat{G}})$$

by setting

- $\mathrm{V}_{\hat{G}} = \bigcup_{\lambda\in\Lambda} \mathrm{V}_{G_\lambda}$,
- $\mathrm{E}_{\hat{G}} = \bigcup_{\lambda\in\Lambda} \mathrm{E}_{G_\lambda}$,
- $\mathrm{s}_{\hat{G}} = \mathrm{s}_G \Big|_{\mathrm{E}_{\hat{G}}}^{\mathrm{V}_{\hat{G}}}$ and $\mathrm{t}_{\hat{G}} = \mathrm{t}_G \Big|_{\mathrm{E}_{\hat{G}}}^{\mathrm{V}_{\hat{G}}}$.

The union $\hat{G} = \bigcup_{\lambda\in\Lambda} G_\lambda$ of subgraphs of G is itself a subgraph of G. Since for every $e \in \mathrm{E}_G$ there is a $\lambda \in \Lambda$ such that $e \in \mathrm{E}_{G_\lambda}$, we have $e\, \mathrm{s}_G = e\, \mathrm{s}_G \Big|_{\mathrm{E}_{G_\lambda}}^{\mathrm{V}_{G_\lambda}} = e\, \mathrm{s}_{G_\lambda} \in \mathrm{V}_{G_\lambda} \subseteq \mathrm{V}_{\hat{G}}$. Similar argument for $e\, \mathrm{t}_G \in \mathrm{V}_{\hat{G}}$.

In the case $|\Lambda| = 2$, for instance $\Lambda = \{\lambda_1, \lambda_2\}$, we may also write $\bigcup_{\lambda\in\Lambda} G_\lambda$ as $G_{\lambda_1} \cup G_{\lambda_2}$. Moreover, for a given $M \subseteq \mathrm{S}(G)$, we also use $\bigcup M$ to denote the union of all subgraphs in M.

Definition 2.8 (Partition)
Let G be a graph, and let $(G_\lambda)_{\lambda\in\Lambda}$ be a tuple of nonempty subgraphs of G, where Λ is an index set. We say that $(G_\lambda)_{\lambda\in\Lambda}$ is a *partition* of G if the following conditions are satisfied:

- $\bigcup_{\lambda\in\Lambda} G_\lambda = G$,
- $\mathrm{V}_{G_{\lambda_1}} \cap \mathrm{V}_{G_{\lambda_2}} = \emptyset$ for $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$.

Note that the second condition ensure that

$$\mathrm{E}_{G_{\lambda_1}} \cap \mathrm{E}_{G_{\lambda_2}} = \emptyset \quad \text{for } \lambda_1, \lambda_2 \in \Lambda \text{ with } \lambda_1 \neq \lambda_2.$$

In this case, we write $G = [G_\lambda]_{\lambda\in\Lambda}$; or $G = [G_{\lambda_1}, \ldots, G_{\lambda_n}]$ if $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, $n \in \mathbb{Z}_{\geqslant 1}$, is finite.

2.2 Graph Morphisms

In this section, we review the definition of graph morphisms and discuss some of their properties. In particular, we give some equivalent conditions for graph morphisms to be monomorphisms, epimorphisms, or isomorphisms.

Definition 2.9 (Graph morphisms)
Let $G = (\mathrm{V}_G, \mathrm{E}_G; \mathrm{s}_G, \mathrm{t}_G)$ and $H = (\mathrm{V}_H, \mathrm{E}_H; \mathrm{s}_H, \mathrm{t}_H)$ be graphs. A *graph morphism* $f : G \to H$ consists of a vertex map $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ and an edge map $\mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H$ such that the following conditions hold:

- $\mathrm{E}_f \cdot \mathrm{s}_H = \mathrm{s}_G \cdot \mathrm{V}_f$
- $\mathrm{E}_f \cdot \mathrm{t}_H = \mathrm{t}_G \cdot \mathrm{E}_f$

These conditions ensure that the direction of each edge is respected by f, that is, if e is an edge of G with source vertex $e\,\mathrm{s}_G$ and target vertex $e\,\mathrm{t}_G$, then $e\,\mathrm{E}_f$ is an edge of H with source vertex $e\,\mathrm{s}_G\,\mathrm{V}_f$ and target vertex $e\,\mathrm{t}_G\,\mathrm{V}_f$.

We denote by

$$(G, H)_{\mathsf{Gph}} := \{f : G \to H \mid f \text{ is a graph morphism}\}$$

the set of all graph morphisms from G to H.

Remark 2.10 (Image graphs)
Let $f : G \to H$ be a graph morphism between graphs G, H. We define the *image graph* of G under f as

$$(G)f := \left(\mathrm{V}_{(G)f}, \mathrm{E}_{(G)f}; \mathrm{s}_{(G)f}, \mathrm{t}_{(G)f}\right),$$

where

$$\mathrm{V}_{(G)f} := \left\{v^{(H)} \in \mathrm{V}_H \;:\; \exists v^{(G)} \in \mathrm{V}_G :\; v^{(G)}\,\mathrm{V}_f = v^{(H)}\right\},$$
$$\mathrm{E}_{(G)f} := \left\{e^{(H)} \in \mathrm{E}_H \;:\; \exists e^{(G)} \in \mathrm{E}_G :\; e^{(G)}\,\mathrm{E}_f = e^{(H)}\right\},$$

and

$$\mathrm{s}_{(G)f} := \mathrm{s}_H\big|_{\mathrm{E}_{(G)f}}^{\mathrm{V}_{(G)f}},$$
$$\mathrm{t}_{(G)f} := \mathrm{t}_H\big|_{\mathrm{E}_{(G)f}}^{\mathrm{V}_{(G)f}}.$$

The image graph $(G)f$ is a subgraph of H.

Proof. Let $e^{(H)} \in \mathrm{E}_{(G)f}$, and $e^{(G)} \in \mathrm{E}_G$ be an edge in G mapped to H. Then we have

$$e^{(H)}\,\mathrm{s}_{(G)f} = e^{(G)}\,\mathrm{E}_f\,\mathrm{s}_{(G)f} = e^{(G)}\,\mathrm{E}_f\,\mathrm{s}_H = e^{(G)}\,\mathrm{s}_G\,\mathrm{V}_f \in \mathrm{V}_{(G)f}$$
$$e^{(H)}\,\mathrm{t}_{(G)f} = e^{(G)}\,\mathrm{E}_f\,\mathrm{t}_{(G)f} = e^{(G)}\,\mathrm{E}_f\,\mathrm{t}_H = e^{(G)}\,\mathrm{t}_G\,\mathrm{V}_f \in \mathrm{V}_{(G)f}.$$

Hence $(G)f$ is a subgraph of H. □

Remark 2.11
For any given graph morphisms $f : G \to H$ and $g : H \to K$, we write $f \cdot g$ for the composite of f and g; cf. **[4,** Definition 60**]**. Then $f \cdot g$ is a graph morphism with $\mathrm{V}_{f \cdot g} = \mathrm{V}_f \cdot \mathrm{V}_g$ and $\mathrm{E}_{f \cdot g} = \mathrm{E}_f \cdot \mathrm{E}_g$; cf. **[4,** Definition 60**]**.

Remark 2.12

For every given graph G, we define

(1) the *identity* $\mathrm{id}_G : G \to G$ as the graph morphism consisting of the vertex map $\mathrm{V}_{\mathrm{id}_G} := \mathrm{id}_{\mathrm{V}_G}$ and edge map $\mathrm{E}_{\mathrm{id}_G} := \mathrm{id}_{\mathrm{E}_G}$.

(2) the *embedding* of a subgraph $G' \leqslant G$ to G as the graph morphism $\iota^G_{G'} : G' \to G$ consisting of the vertex map $\mathrm{V}_{\iota^G_{G'}}$ and edge map $\mathrm{E}_{\iota^G_{G'}}$, which are both inclusion maps in the usual sense. Sometimes, we use $G' \xhookrightarrow{\iota^G_{G'}} G$ to denote the embedding of G' to G.

Remark 2.13

With graphs as objects, and with graph morphisms as morphisms, we obtain a category **Gph**; see **[4, Definition 64]**.

Definition 2.14 (Monomorphisms, epimorphisms, isomorphisms)

Let G, H be graphs. A graph morphism $f \in (G, H)_{\mathrm{Gph}}$ is called

(1) a *monomorphism* if for every given graph X and every graph morphisms $g_1, g_2 : X \to G$, we have

$$g_1 \cdot f = g_2 \cdot f \quad \Rightarrow \quad g_1 = g_2.$$

(2) an *epimorphism* if for every given graph Y and every graph morphisms $h_1, h_2 : H \to Y$, we have

$$f \cdot h_1 = f \cdot h_2 \quad \Rightarrow \quad h_1 = h_2.$$

(3) an *isomorphism*, denoted as $G \xrightarrow[\sim]{f} H$, if there exists a graph morphism $f^{-1} : H \to G$ such that $f \cdot f^{-1} = \mathrm{id}_G$ and $f^{-1} \cdot f = \mathrm{id}_H$. We say that G and H are *isomorphic*, denoted as $G \simeq H$, if there exists a graph isomorphism between G and H. Furthermore, if $G = H$, we call the isomorphism f an *automorphism*.

Remark 2.15

Let G, H be graphs. A graph morphism $f : G \to H$ is an isomorphism if and only if the vertex map V_f and the edge map E_f are bijective.

Proof. Suppose that f ia isomorphism; so there exists a graph morphism $f^{-1} : H \to G$ such that $f \cdot f^{-1} = \mathrm{id}_G$ and $f^{-1} \cdot f = \mathrm{id}_H$. Then we have

$$\mathrm{V}_f \cdot \mathrm{V}_{f^{-1}} = \mathrm{V}_{f \cdot f^{-1}} = \mathrm{V}_{\mathrm{id}_G} \quad \text{and} \quad \mathrm{V}_{f^{-1}} \cdot \mathrm{V}_f = \mathrm{V}_{f^{-1} \cdot f} = \mathrm{V}_{\mathrm{id}_H}.$$

Hence V_f is bijective. The bijectivity of E_f can be proved similarly.

To show the converse implication, suppose that V_f and E_f are bijective. We define f^{-1} by

$$\mathrm{V}_{f^{-1}} := (\mathrm{V}_f)^{-1},\ \mathrm{E}_{f^{-1}} := (\mathrm{E}_f)^{-1}.$$

We claim that f^{-1} is a graph morphism. The property $\mathrm{s}_G \cdot \mathrm{V}_f = \mathrm{E}_f \cdot \mathrm{s}_H$ implies that

$$\mathrm{E}_{f^{-1}} \cdot \mathrm{s}_G = \mathrm{E}_{f^{-1}} \cdot \mathrm{s}_G \cdot \mathrm{id}_{\mathrm{V}_G} = \mathrm{E}_{f^{-1}} \cdot \mathrm{s}_G \cdot \mathrm{V}_f \cdot \mathrm{V}_{f^{-1}} = \mathrm{E}_{f^{-1}} \cdot \mathrm{E}_f \cdot \mathrm{s}_H \cdot \mathrm{V}_{f^{-1}} = \mathrm{id}_{\mathrm{E}_G} \cdot \mathrm{s}_H \cdot \mathrm{V}_{f^{-1}} = \mathrm{s}_H \cdot \mathrm{V}_{f^{-1}}.$$

Analogously, we obtain

$$\mathrm{E}_{f^{-1}} \cdot \mathrm{t}_G = \mathrm{t}_H \cdot \mathrm{V}_{f^{-1}}$$

on the target side. Hence f^{-1} is a graph morphism. Since $f^{-1} \cdot f = \mathrm{id}_H$ and $f \cdot f^{-1} = \mathrm{id}_G$, we conclude that f is an isomorphism. □

Lemma 2.16

Let G, H be graphs, and $f \in (G, H)_{\mathrm{Gph}}$ be a graph morphism. Then f is an epimorphism if and only if V_f and E_f are surjective.

Proof. The statement that surjectivity of $\mathrm{V}_f, \mathrm{E}_f$ implies that f is an epimorphism is given in [4, Remark 72]. Here, we prove only the reverse direction.

An example illustrating the proof principle is given in Example 2.17.

Suppose that f is an epimorphism. We show that $\mathrm{V}_f, \mathrm{E}_f$ are surjective maps.

First, we *assume* that E_f is not surjective; so we may choose an edge $e_H \in \mathrm{E}_H$ that is not in the image $(G)f$. By taking a copy of H and inserting two additional edges from $e_H\,\mathrm{s}_H$ to $e_H\,\mathrm{t}_H$, We construct a new graph $\hat{H}$ with

$$\mathrm{V}_{\hat{H}} := \mathrm{V}_H,\ \mathrm{E}_{\hat{H}} := \mathrm{E}_H \,\dot{\cup} \left\{ e_H^{(1)}, e_H^{(2)} \right\},$$

and

$$e\,\mathrm{s}_{\hat{H}} := \begin{cases} e\,\mathrm{s}_H & \text{for } e \in \mathrm{E}_H \\ e_H\,\mathrm{s}_H & \text{for } e \in \left\{ e_H^{(1)}, e_H^{(2)} \right\}, \end{cases}$$

$$e\,\mathrm{t}_{\hat{H}} := \begin{cases} e\,\mathrm{t}_H & \text{for } e \in \mathrm{E}_H \\ e_H\,\mathrm{t}_H & \text{for } e \in \left\{ e_H^{(1)}, e_H^{(2)} \right\}. \end{cases}$$

We consider the graph morphisms h_1, h_2 defined as

$$\begin{array}{llcl} h_1 : & H & \to & \hat{H} \\ \mathrm{V}_{h_1} : & v & \mapsto & v \\ \mathrm{E}_{h_1} : & e & \mapsto & e\,\mathrm{E}_{h_1} \end{array} \quad \text{and} \quad \begin{array}{llcl} h_2 : & H & \to & \hat{H} \\ \mathrm{V}_{h_2} : & v & \mapsto & v \\ \mathrm{E}_{h_2} : & e & \mapsto & e\,\mathrm{E}_{h_2} \end{array},$$

where

$$e\,\mathrm{E}_{h_1} := \begin{cases} e & \text{for } e \in \mathrm{E}_H \setminus \{e_H\} \\ e_H^{(1)} & \text{for } e = e_H \end{cases},$$

and

$$e\,\mathrm{E}_{h_2} = \begin{cases} e & \text{for } e \in \mathrm{E}_H \setminus \{e_H\} \\ e_H^{(2)} & \text{for } e = e_H \end{cases}.$$

Since $\mathrm{V}_{h_1}\big|_{\mathrm{V}(G)f} = \mathrm{V}_{h_2}\big|_{\mathrm{V}(G)f}$ and $\mathrm{E}_{h_1}\big|_{\mathrm{E}(G)f} = \mathrm{E}_{h_2}\big|_{\mathrm{E}(G)f}$, we have $f \cdot h_1 = f \cdot h_2$. However $h_1 \neq h_2$, which *contradicts* f being an epimorphism. Therefore, E_f is surjective.

Now *assume* that V_f is not surjective. Then we may choose $v_H \in \mathrm{V}_H$ that is not a vertex of $(G)f$. Since E_f is surjective, v_H is an isolated vertex in H. We duplicate v_H in order to obtaining another vertex $v'_H \notin \mathrm{V}_H$ and construct a new graph $\hat{H}'$ by defining

$$\mathrm{V}_{\hat{H}'} := V_H \dot{\cup} \{v'_H\},\ \mathrm{E}_{\hat{H}'} := \mathrm{E}_H\,;$$

and for every $e \in \mathrm{E}_{\hat{H}'}$, we set

$$e\,\mathrm{s}_{\hat{H}'} := e\,\mathrm{s}_H,\ e\,\mathrm{t}_{\hat{H}'} := e\,\mathrm{t}_H\ .$$

Then H is a subgraph of $\hat{H}'$, and v_H, v'_H are isolated vertices in $\hat{H}'$. We consider the graph morphisms $h'_1, h'_2 : H \to \hat{H}'$, where h'_1 satisfies

$$\mathrm{V}_{h'_1}\Big|_{\mathrm{V}_H}^{\mathrm{V}_H} = \mathrm{id}_{\mathrm{V}_H},\ \mathrm{E}_{h'_1} = \mathrm{id}_{\mathrm{E}_H}$$

and h_2' is defined as

$$\mathrm{V}_{h_2'}\big|_{\mathrm{V}_H \setminus \{v_H\}} := \mathrm{V}_{h_1'}\big|_{\mathrm{V}_H \setminus \{v_H\}}\,,\; v_H\,\mathrm{V}_{h_2'} := v_H',\; \mathrm{E}_{h_2'} := \mathrm{id}_{\mathrm{E}_H}\,.$$

Since v_H is not a vertex of $(G)f$, we have $f \cdot h_1' = f \cdot h_2'$. However, $h_1' \neq h_2'$, which *contradicts* f being a graph epimorphism. Thus V_f is also surjective, which completes the proof. □

Example 2.17

This example illustrates the proof of Lemma 2.16.

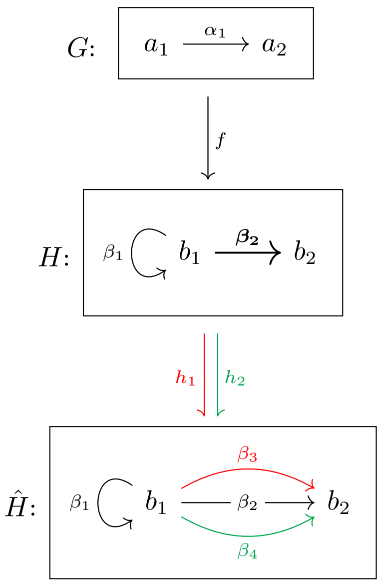


Figure 2.2

The graph morphism $f : G \to H$ is defined as

$$f : \left\{ \begin{array}{l} a_1 \overset{\mathrm{V}_f}{\longmapsto} b_1 \\ a_2 \overset{\mathrm{V}_f}{\longmapsto} b_1 \\ \alpha_1 \overset{\mathrm{E}_f}{\longmapsto} \beta_1 \end{array} \right\}.$$

We choose the edge β_2 in H as e_H. By adding the copies of β_2, namely β_3, β_4, we obtain $\hat{H}$ such that $H \leqslant \hat{H}$. We define $h_1, h_2 : H \to \hat{H}$ as follows:

$$h_1 : \left\{ \begin{array}{l} b_1 \overset{\mathrm{V}_{h_1}}{\longmapsto} b_1 \\ b_2 \overset{\mathrm{V}_{h_1}}{\longmapsto} b_2 \\ \beta_1 \overset{\mathrm{E}_{h_1}}{\longmapsto} \beta_1 \\ \beta_2 \overset{\mathrm{E}_{h_1}}{\longmapsto} \beta_3 \end{array} \right\} \quad \text{and} \quad h_2 : \left\{ \begin{array}{l} b_1 \overset{\mathrm{V}_{h_2}}{\longmapsto} b_1 \\ b_2 \overset{\mathrm{V}_{h_2}}{\longmapsto} b_2 \\ \beta_1 \overset{\mathrm{E}_{h_2}}{\longmapsto} \beta_1 \\ \beta_2 \overset{\mathrm{E}_{h_2}}{\longmapsto} \beta_4. \end{array} \right\}.$$

Note that we have $f \cdot h_1 = f \cdot h_2$, but $h_1 \neq h_2$; so f is not an epimorphism.

Lemma 2.18

Let G, H be graphs. A graph morphism $f \in (G, H)_{\mathrm{Gph}}$ is a monomorphism if and only if V_f and E_f are injective.

Proof. Suppose that V_f and E_f are injective. For an given graph X and $g_1, g_2 \in (X, G)_{\mathrm{Gph}}$ satisfying $g_1 f = g_2 f$, we show that $g_1 = g_2$. We claim that $\mathrm{V}_{g_1} = \mathrm{V}_{g_2}$ and $\mathrm{E}_{g_1} = \mathrm{E}_{g_2}$. We have

$$\mathrm{V}_{g_1} \cdot \mathrm{V}_f = \mathrm{V}_{g_1 \cdot f} = \mathrm{V}_{g_2 \cdot f} = \mathrm{V}_{g_2} \cdot \mathrm{V}_f,$$

and

$$\mathrm{E}_{g_1} \cdot \mathrm{E}_f = \mathrm{E}_{g_1 \cdot f} = \mathrm{E}_{g_2 \cdot f} = \mathrm{E}_{g_2} \cdot \mathrm{E}_f\,.$$

Since V_f and E_f are injective, it follows that $\mathrm{V}_{g_1} = \mathrm{V}_{g_2}$ and $\mathrm{E}_{g_1} = \mathrm{E}_{g_2}$. Hence $g_1 = g_2$, and f is a monomorphism.

To show the converse direction, we *assume* that f is a monomorphism, but V_f or E_f is not injective.

In the case where E_f is not injective, we choose $e_1, e_2 \in \mathrm{E}_G$, with $e_1 \neq e_2$, such that $e_1\,\mathrm{E}_f = e_2\,\mathrm{E}_f$. We consider the graph

$$X : \boxed{\;1 \xrightarrow[e]{} 2\;}$$

and two different graph morphisms $g_1, g_2 : X \to G$ defined as

$$g_1 : \left\{ \begin{array}{cccc} \mathrm{V}_{g_1} : & \mathrm{V}_X & \to & \mathrm{V}_G \\ & 1 & \mapsto & e_1\,\mathrm{s}_G \\ & 2 & \mapsto & e_1\,\mathrm{t}_G \\ \mathrm{E}_{g_1} : & \mathrm{E}_X & \to & \mathrm{E}_G \\ & e & \mapsto & e_1 \end{array} \right\} \quad \text{and} \quad g_2 : \left\{ \begin{array}{cccc} \mathrm{V}_{g_2} : & \mathrm{V}_X & \to & \mathrm{V}_G \\ & 1 & \mapsto & e_2\,\mathrm{s}_G \\ & 2 & \mapsto & e_2\,\mathrm{t}_G \\ \mathrm{E}_{g_2} : & \mathrm{E}_X & \to & \mathrm{E}_G \\ & e & \mapsto & e_2 \end{array} \right\}.$$

We show that $g_1 f = g_2 f$:

$$\begin{aligned} &e\,\mathrm{E}_{g_1 f} = e\,\mathrm{E}_{g_1}\,\mathrm{E}_f = e_1\,\mathrm{E}_f = e_2\,\mathrm{E}_f = e\,\mathrm{E}_{g_2}\,\mathrm{E}_f = e\,\mathrm{E}_{g_2 f} \\ &1\,\mathrm{V}_{g_1 f} = 1\,\mathrm{V}_{g_1}\,\mathrm{V}_f = e_1\,\mathrm{s}_G\,\mathrm{V}_f = e_1\,\mathrm{E}_f\,\mathrm{s}_H = e_2\,\mathrm{E}_f\,\mathrm{s}_H = e_2\,\mathrm{s}_G\,\mathrm{V}_f = 1\,\mathrm{V}_{g_2}\,\mathrm{V}_f = 1\,\mathrm{V}_{g_2 f} \\ &2\,\mathrm{V}_{g_2 f} = 2\,\mathrm{V}_{g_2}\,\mathrm{V}_f = e_1\,\mathrm{t}_G\,\mathrm{V}_f = e_1\,\mathrm{E}_f\,\mathrm{t}_H = e_2\,\mathrm{E}_f\,\mathrm{t}_H = e_1\,\mathrm{t}_G\,\mathrm{V}_f = 2\,\mathrm{V}_{g_2}\,\mathrm{V}_f = 2\,\mathrm{V}_{g_2 f} \end{aligned}$$

Since f is a monomorphism, we obtain $g_1 = g_2$. But $g_1 \neq g_2$. Hence this case does occur.

Now, we consider the case where V_f is not injective. We choose $v_1, v_2 \in \mathrm{E}_g$, with $v_1 \neq v_2$, such that $v_1\,\mathrm{V}_f = v_2\,\mathrm{V}_f$. Let X be the graph with exactly one vertex and no edges. We denote this vertex as v. We consider two different graph morphisms $g_1, g_2 : X \to G$ defined as

$$g_1 : \left\{ \begin{array}{cccc} \mathrm{V}_{g_1} : & \mathrm{V}_X & \to & \mathrm{V}_G \\ & v & \mapsto & v_1 \end{array} \right\} \quad \text{and} \quad g_2 : \left\{ \begin{array}{cccc} \mathrm{V}_{g_2} : & \mathrm{V}_X & \to & \mathrm{V}_G \\ & v & \mapsto & v_2 \end{array} \right\}$$

We show that $g_1 f = g_2 f$:

$$v\,\mathrm{V}_{g_1 f} = v\,\mathrm{V}_{g_1}\,\mathrm{V}_f = v_1\,\mathrm{V}_f = v_2\,\mathrm{V}_f = v\,\mathrm{V}_{g_2}\,\mathrm{V}_f = v\,\mathrm{V}_{g_2 f}\,.$$

Since f is a monomorphism, we obtain $g_1 = g_2$. However, $g_1 \neq g_2$; so this case cannot occur. As neither case is possible, we reach a *contradiction* to our assumption that V_f or E_f is not injective. Hence, if f is a monomorphism, then both E_f and V_f must be injective. □

Lemma 2.19

Let G, H be graphs, with G being simple. Let $f : G \to H$ be a graph morphism. If the vertex map V_f of f is injective, then the edge map E_f is also injective. More precisely, the restriction $f|_G^{(G)f}$ is an isomorphism.

Proof. Let $e_1, e_2 \in \mathrm{E}_G$ be such that $e_1\,\mathrm{E}_f = e_2\,\mathrm{E}_f \in \mathrm{E}_H$. Then we have

$$e_1\,\mathrm{s}_G\,\mathrm{V}_f = e_1\,\mathrm{E}_f\,\mathrm{s}_H = e_2\,\mathrm{E}_f\,\mathrm{s}_H = e_2\,\mathrm{s}_G\,\mathrm{V}_f$$

and

$$e_1\,\mathrm{t}_G\,\mathrm{V}_f = e_1\,\mathrm{E}_f\,\mathrm{t}_H = e_2\,\mathrm{E}_f\,\mathrm{t}_H = e_2\,\mathrm{t}_G\,\mathrm{V}_f \ .$$

Since V_f is injective, we have $e_1\,\mathrm{s}_G = e_2\,\mathrm{s}_G$ and $e_1\,\mathrm{t}_G = e_2\,\mathrm{t}_G$. As G is a simple graph, we have $e_1 = e_2$. Hence E_f is injective.

By Remark 2.10 we know that the vertex map $\mathrm{V}_f\,|_{\mathrm{V}_G}^{\mathrm{V}_{(G)f}}$ and the edge map $\mathrm{E}_f\,|_{\mathrm{E}_G}^{\mathrm{E}_{(G)f}}$ of $f|_G^{(G)f}$ are surjective. Since they are also injective as restrictions of V_f and E_f, we conclude that $f|_G^{(G)f}$ is a graph isomorphism; cf. Remark 2.15. □

3 Quasi-cyclic Subgraphs

3.1 From cyclic graphs to quasi-cyclic graphs

Definition 3.1 (Cyclic graphs)
Let $n \in \mathbb{Z}_{\geqslant 1}$. The *cyclic graph* $\mathrm{C}_n = (\mathrm{V}_{\mathrm{C}_n}, \mathrm{E}_{\mathrm{C}_n}; \mathrm{s}_{\mathrm{C}_n}, \mathrm{t}_{\mathrm{C}_n})$ is defined by

$$\mathrm{V}_{\mathrm{C}_n} := \{\mathrm{v}_{i+n\mathbb{Z}} \mid i \in \mathbb{Z}\}\,,$$
$$\mathrm{E}_{\mathrm{C}_n} := \{\mathrm{e}_{i+n\mathbb{Z}} \mid i \in \mathbb{Z}\}\,,$$

where $\mathrm{e}_{i+n\mathbb{Z}}\, \mathrm{s}_{\mathrm{C}_n} = \mathrm{v}_{i+n\mathbb{Z}}$ and $\mathrm{e}_{i+n\mathbb{Z}}\, \mathrm{t}_{\mathrm{C}_n} = \mathrm{v}_{i+1+n\mathbb{Z}}$ for all $i \in \mathbb{Z}$.

Throughout this text, if known from the context that we are discussing the vertices and edges in the cyclic graph C_n, we may abbreviate

$$\mathrm{v}_i := \mathrm{v}_{i+n\mathbb{Z}}\,,$$
$$\mathrm{e}_i := \mathrm{e}_{i+n\mathbb{Z}}\,.$$

For example, the cyclic graph C_4 is as shown below:

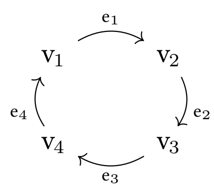


Figure 3.1: the cyclic graph C_4

Lemma 3.2
Let $\mathrm{C}_n, \mathrm{C}_m$ be cyclic graphs, where $n, m \in \mathbb{Z}_{\geqslant 1}$. Then we have $(\mathrm{C}_n, \mathrm{C}_m)_{\mathrm{Gph}} \neq \emptyset$ if and only if m is a divisor of n.

Proof. Suppose that $(\mathrm{C}_n, \mathrm{C}_m)_{\mathrm{Gph}} \neq \emptyset$, and we choose a graph morphism $u \in (\mathrm{C}_n, \mathrm{C}_m)_{\mathrm{Gph}}$. Suppose that the vertex $\mathrm{v}_{0+n\mathbb{Z}} \in \mathrm{V}_{\mathrm{C}_n}$ is mapped to $\mathrm{v}_{j+n\mathbb{Z}} \in \mathrm{V}_{\mathrm{C}_m}$. We first show that for every $i \in \mathbb{Z}_{\geqslant 0}$, the vertex $\mathrm{v}_{i+n\mathbb{Z}} \in \mathrm{V}_{\mathrm{C}_n}$ is mapped to $\mathrm{v}_{i+j+m\mathbb{Z}} \in \mathrm{V}_{\mathrm{C}_m}$, that is,

$$\mathrm{v}_{i+n\mathbb{Z}}\, \mathrm{V}_u = \mathrm{v}_{i+j+m\mathbb{Z}} \quad \text{for every } i \in \mathbb{Z}_{\geqslant 0}\,.$$

Due to the structure of cyclic graphs and the property of graph morphisms, we have

$$\mathrm{e}_{0+m\mathbb{Z}}\, \mathrm{E}_u\, \mathrm{s}_{\mathrm{C}_m} = \mathrm{e}_{0+m\mathbb{Z}}\, \mathrm{s}_{\mathrm{C}_n}\, \mathrm{V}_u = \mathrm{v}_{0+n\mathbb{Z}}\, \mathrm{V}_f = \mathrm{v}_{j+m\mathbb{Z}}\,.$$

Since $\mathrm{e}_{j+m\mathbb{Z}} \in \mathrm{E}_{\mathrm{C}_m}$ is the unique edge in $\mathrm{E}_{\mathrm{C}_m}$ that have the source vertex $\mathrm{v}_{j+m\mathbb{Z}}$, we have $\mathrm{e}_{0+n\mathbb{Z}}\, \mathrm{E}_u = \mathrm{e}_{j+m\mathbb{Z}}$. We proceed one step further by considering the target of $\mathrm{e}_{0+n\mathbb{Z}} \in \mathrm{E}_{\mathrm{C}_n}$ and $\mathrm{e}_{j+m\mathbb{Z}} \in \mathrm{E}_{\mathrm{C}_m}$:

$$\mathrm{v}_{1+n\mathbb{Z}}\, \mathrm{V}_u = \mathrm{e}_{0+n\mathbb{Z}}\, \mathrm{t}_{\mathrm{C}_n}\, \mathrm{V}_u = \mathrm{e}_{0+n\mathbb{Z}}\, \mathrm{E}_f\, \mathrm{t}_{\mathrm{C}_m} = \mathrm{e}_{j+m\mathbb{Z}}\, \mathrm{t}_{\mathrm{C}_m} = \mathrm{e}_{1+j+m\mathbb{Z}}\,.$$

Therefore, we obtain the following implication:

$$\mathrm{v}_{0+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{v}_{0+j+m\mathbb{Z}} \quad \Rightarrow \quad \mathrm{v}_{1+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{v}_{1+j+m\mathbb{Z}} \ . \tag{3.1}$$

By iterating the implication (3.1), we may conclude that

$$\mathrm{v}_{i+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{v}_{i+j+m\mathbb{Z}} \quad \text{for every } i \in \mathbb{Z}_{\geqslant 0} \ .$$

In particular, for $i = n$, we have

$$\mathrm{v}_{n+j+m\mathbb{Z}} = \mathrm{v}_{n+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{v}_{0+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{v}_{j+m\mathbb{Z}} \ .$$

Note that we have $j + m\mathbb{Z} = j + n + m\mathbb{Z}$ in $\mathbb{Z}/m\mathbb{Z}$ if and only if m is a divisor of n, which proves the implication.

To show the converse implication, we suppose that $n = am$ with $a \in \mathbb{Z}_{\geqslant 1}$. We give a graph morphism $u : \mathrm{C}_n \to \mathrm{C}_m$, where the vertex map V_u and the edge map E_u is defined as follows:

$$\begin{aligned} \mathrm{V}_u :\ \ & \mathrm{V}_{\mathrm{C}_n} \to \mathrm{V}_{\mathrm{C}_m} \\ & \mathrm{v}_{i+n\mathbb{Z}} \mapsto \mathrm{v}_{i+m\mathbb{Z}} \ ; \end{aligned}$$

$$\begin{aligned} \mathrm{E}_u :\ \ & \mathrm{E}_{\mathrm{C}_n} \to \mathrm{E}_{\mathrm{C}_m} \\ & \mathrm{e}_{i+n\mathbb{Z}} \mapsto \mathrm{e}_{i+m\mathbb{Z}} \ . \end{aligned}$$

- We verify that V_u and E_u are well-defined. For $x, y \in \mathbb{Z}$ with $\mathrm{v}_{x+n\mathbb{Z}} = \mathrm{v}_{y+n\mathbb{Z}}$, there is a $k \in \mathbb{Z}$ such that $x = y + nk$. Then we have $x = y + mak$, which implies

$$\mathrm{v}_{x+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{v}_{x+m\mathbb{Z}} = \mathrm{v}_{y+mak+m\mathbb{Z}} = \mathrm{v}_{y+m\mathbb{Z}} = \mathrm{v}_{y+n\mathbb{Z}}\,\mathrm{V}_u \ .$$

  Hence V_u is well-defined. The verification for E_u is analogous.

- We verify that u is a graph morphism. Let $\mathrm{e}'_i \in \mathrm{E}_{\mathrm{C}_n}$. Then we have:

$$\begin{aligned} \mathrm{e}_{i+n\mathbb{Z}}\,\mathrm{E}_u\,\mathrm{s}_{\mathrm{C}_m} &= \mathrm{e}_{i+m\mathbb{Z}}\,\mathrm{s}_{\mathrm{C}_m} = \mathrm{v}_{i+m\mathbb{Z}} = \mathrm{v}_{i+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{e}_{i+n\mathbb{Z}}\,\mathrm{s}_{\mathrm{C}_n}\,\mathrm{V}_u \ ; \\ \mathrm{e}_{i+n\mathbb{Z}}\,\mathrm{E}_u\,\mathrm{t}_{\mathrm{C}_m} &= \mathrm{e}_{i+m\mathbb{Z}}\,\mathrm{t}_{\mathrm{C}_m} = \mathrm{v}_{i+1+m\mathbb{Z}} = \mathrm{v}_{i+1+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{e}_{i+n\mathbb{Z}}\,\mathrm{t}_{\mathrm{C}_n}\,\mathrm{V}_u \ . \end{aligned}$$

Hence we find a graph morphism $u \in (\mathrm{C}_n, \mathrm{C}_m)_{\mathrm{Gph}}$, so $(\mathrm{C}_n, \mathrm{C}_m)_{\mathrm{Gph}} \neq \emptyset$. □

Remark 3.3

Let $n, m \in \mathbb{Z}_{\geqslant 1}$ and C_n, C_m be two cyclic graphs. Suppose that m is a divisor of n. Then each $u \in (\mathrm{C}_n, \mathrm{C}_m)_{\mathrm{Gph}}$ is an epimorphism. Furthermore, if $n = m$, then each $u \in (\mathrm{C}_n, \mathrm{C}_m)_{\mathrm{Gph}}$ is an automorphism.

Proof. Let $u \in (\mathrm{C}_n, \mathrm{C}_m)_{\mathrm{Gph}}$. *Assume* that u is not an epimorphism. It follows from Lemma 2.16 that V_u or E_u is not surjective.

We first discuss the case that V_u is not surjective. Suppose that $\mathrm{v}_{i+m\mathbb{Z}} \in \mathrm{V}_{\mathrm{C}_m}$ is not in the image $(\mathrm{C}_n)u$. Then $\mathrm{e}_{i+m\mathbb{Z}}$ is not in the image $(\mathrm{C}_n)u$, for otherwise, we may choose $\mathrm{e}_{j+n\mathbb{Z}} \in \mathrm{E}_{\mathrm{C}_n}$ with $\mathrm{e}_{j+n\mathbb{Z}}\,\mathrm{E}_u = \mathrm{e}_{i+m\mathbb{Z}}$ such that we have

$$\mathrm{v}_{j+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{e}_{j+n\mathbb{Z}}\,\mathrm{s}_{\mathrm{C}_n}\,\mathrm{V}_u = \mathrm{e}_{j+n\mathbb{Z}}\,\mathrm{E}_u\,\mathrm{s}_{\mathrm{C}_m} = \mathrm{v}_{i+m\mathbb{Z}} \ ,$$

which is inconsistent with $\mathrm{v}_{i+m\mathbb{Z}} \notin \mathrm{V}_{(\mathrm{C}_n)u}$. Therefore, E_u is not surjective. Furthermore, if $\mathrm{e}_{i+m\mathbb{Z}} \notin \mathrm{E}_{(\mathrm{C}_n)u}$, then $\mathrm{v}_{i+1+m\mathbb{Z}} \notin \mathrm{V}_{(\mathrm{C}_n)u}$. Otherwise, there exists a vertex $\mathrm{v}_{k+n\mathbb{Z}} \in \mathrm{V}_{\mathrm{C}_n}$ such that $\mathrm{v}_{k+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{v}_{i+1+m\mathbb{Z}}$. Then we have

$$\mathrm{e}_{k-1+n\mathbb{Z}}\,\mathrm{E}_u\,\mathrm{t}_{\mathrm{C}_m} = \mathrm{e}_{k-1+n\mathbb{Z}}\,\mathrm{t}_{\mathrm{C}_n}\,\mathrm{V}_u = \mathrm{v}_{k+n\mathbb{Z}}\,\mathrm{V}_u = \mathrm{v}_{i+1+m\mathbb{Z}} \ .$$

Since $\mathrm{e}_{i+m\mathbb{Z}}$ is the only edge in $\mathrm{E}_{\mathrm{C}_m}$ having the target vertex $\mathrm{v}_{i+1+m\mathbb{Z}}$, we have $\mathrm{e}_{k-1+n\mathbb{Z}}\,\mathrm{E}_u = \mathrm{e}_{i+m\mathbb{Z}}$, which is inconsistent with $\mathrm{e}_{i+m\mathbb{Z}} \notin \mathrm{E}_{(\mathrm{C}_n)u}$. From the above, we obtain the following implications:

$$\mathrm{e}_{i+m\mathbb{Z}} \notin \mathrm{E}_{(\mathrm{C}_n)u} \Rightarrow \mathrm{v}_{i+1+m\mathbb{Z}} \notin \mathrm{V}_{(\mathrm{C}_n)u} \Rightarrow \mathrm{e}_{i+1+m\mathbb{Z}} \notin \mathrm{E}_{(\mathrm{C}_n)u} \ .$$

By iterating this argument, we conclude that

$$\mathrm{V}_{\mathrm{C}_m} \cap \mathrm{V}_{(\mathrm{C}_n)u} = \emptyset \ .$$

This *contradicts* the existence of u. Hence both V_u and E_u are bijective, and hence u is an epimorphism.

If $n = m$, then E_u and V_u are maps between sets with the same cardinality. Since $\mathrm{V}_{\mathrm{C}_n}$ and $\mathrm{E}_{\mathrm{C}_n}$ are finite, the surjective maps E_u and V_u are bijective. Hence u is an automorphism. □

Definition 3.4 (Cyclic subgraphs)

Let G be a graph. We say a subgraph $G' \leqslant G$ *cyclic* if there exists a cyclic graph C_k, $k \in \mathbb{Z}_{\geqslant 1}$ such that $\mathrm{C}_k \simeq G'$. We denote the set of all cyclic subgraphs of G by $\mathrm{S_c}(G)$.

Definition 3.5 (Quasi-cyclic subgraphs)

Let G be a graph.

(1) A subgraph $G' \leqslant G$ is called *quasi-cyclic* in G if G' is the union of finitely many cyclic subgraphs $G_1, G_2, \ldots, G_n \in \mathrm{S_c}(G)$ for some $n \in \mathbb{Z}_{\geqslant 1}$ such that the condition

$$\mathrm{V}_{G_i} \cap \left(\bigcup_{j=1}^{i-1} \mathrm{V}_{G_j} \right) \neq \emptyset \tag{3.2}$$

holds for all $i \in [2, n]$. In this case, we write $G' = G_1 \oplus G_2 \oplus \cdots \oplus G_n$, or we may abbreviate $G' = \bigoplus_{i=1}^{n} G_i$. We denote the set of all quasi-cyclic subgraphs of G by $\mathrm{S_{qc}}(G)$.

Note that the subgraphs $G_1, \ldots, G_n \leqslant G$ in the definition above are not necessarily pairwise distinct.

(2) If G is quasi-cyclic as a subgraph of itself, we say G is a *quasi-cyclic graph*.

Remark 3.6

In Definition 3.5 the order of the given sequence of cyclic subgraphs informs us about how the quasi-cyclic subgraph is constructed. Therefore, this order cannot be arbitrarily changed. In this text, the given order of the cyclic subgraphs always corresponds to the property defined via Equation (3.2). For example, the following graph

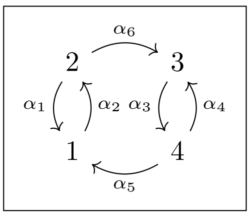
G :


Figure 3.2

is a quasi-cyclic graph, and the subgraphs

$$\begin{aligned} &G_1 \text{ with } \mathrm{V}_{G_1} = \{1,2\}\,,\ \mathrm{E}_{G_1} = \{\alpha_1,\alpha_2\}\,,\\ &G_2 \text{ with } \mathrm{V}_{G_2} = \{1,2,3,4\}\,,\ \mathrm{E}_{G_2} = \{\alpha_2,\alpha_6,\alpha_3,\alpha_5\}\,,\\ &G_3 \text{ with } \mathrm{V}_{G_3} = \{3,4\}\,,\ \mathrm{E}_{G_3} = \{\alpha_3,\alpha_4\} \end{aligned}$$

are cyclic subgraphs of G. Then we can write $G = G_1 \oplus G_2 \oplus G_3$ or $G = G_2 \oplus G_1 \oplus G_3$. However, the expression $G = G_1 \oplus G_3 \oplus G_2$ is incorrect, since G_1 and G_3 do not share a common vertex.

Definition 3.7 (Maximal quasi-cyclic)
Let G be a graph. A subgraph $G' \leqslant G$ is said to be *maximal quasi-cyclic* if $G' \in \mathrm{S}_{\mathrm{qc}}(G)$ and there does not exist $G'' \in \mathrm{S}_{\mathrm{qc}}(G)$ such that $G' < G''$. We denote the set of all maximal quasi-cyclic subgraphs of G by $\mathrm{S}_{\mathrm{qc}}^{\max}(G)$.

Lemma 3.8 (Image of cyclic graphs)
Let G be a graph. For every $k \in \mathbb{Z}_{\geqslant 1}$ and every $u \in (\mathrm{C}_k, G)_{\mathrm{Gph}}$, the image graph $(\mathrm{C}_k)\,u$ is a quasi-cyclic subgraph of G, i.e., $(\mathrm{C}_k)u \in \mathrm{S}_{\mathrm{qc}}(G)$. More precisely, there exists cyclic subgraphs $G_1, \dots, G_n \leqslant G$, for some $n \in \mathbb{Z}_{\geqslant 1}$, such that $(\mathrm{C}_k)u = \bigoplus_{i=1}^n G_i$ and such that $k = \sum_{i=1}^n |\mathrm{V}_{G_i}|$.

Proof. *Assume* that the set

$$K := \left\{ k \in \mathbb{Z}_{\geqslant 1} \;\middle|\; \begin{array}{l} \text{there exists } u \in (\mathrm{C}_k, G)_{\mathrm{Gph}} \text{ such that for every } n \in \mathbb{Z}_{\geqslant 1} \text{ and} \\ \text{every } (G_i)_{i=1}^n \in \mathrm{S}_{\mathrm{c}}(G)^{\times n}, \text{ we have } (\mathrm{C}_k)u \neq \bigoplus_{i=1}^n G_i \text{ or } k \neq \sum_{i=1}^n |\,\mathrm{V}_{G_i}\,|. \end{array} \right\}$$

is nonempty; so we may choose the smallest $k_0 \in K$ and consider a $u_0 \in (\mathrm{C}_{k_0}, G)_{\mathrm{Gph}}$ satisfying the condition from the definition of K. We proceed by distinguishing two cases depending on whether V_{u_0} is injective or not.

Case: V_{u_0} is injective. Since C_{k_0} is a simple graph, it follows from Lemma 2.19 that $(\mathrm{C}_{k_0})\,u_0$ is isomorphic to C_{k_0}. Hence $(\mathrm{C}_{k_0})\,u_0$ is a cyclic subgraph of G. Writing $G_1 = (\mathrm{C}_{k_0})\,u_0$, we have $(\mathrm{C}_{k_0})\,u_0 = \bigoplus_{i=1}^1 G_i$ and $k_0 = \sum_{i=1}^1 |\mathrm{V}_{G_i}|$, which implies that $k_0 \notin K$; so this case does not occur.

Case: V_{u_0} is not injective. We shall construct $\hat{C} \in \mathrm{S}_{\mathrm{c}}(G)$ and $\hat{G} \in \mathrm{S}_{\mathrm{qc}}(G)$ such that $(\mathrm{C}_{k_0})u_0 = \hat{C} \cup \hat{G}$ and $\mathrm{V}_{\hat{C}} \cap \mathrm{V}_{\hat{G}} \neq \emptyset$.

Ad construction of $\hat{C}$: Since V_{u_0} is not injective, we have $k_0 > 1$; and in the sequence $(\mathrm{v}_i\,\mathrm{V}_{u_0})_{i=1}^{k_0}$ there exist $n, m \in [1, k_0]$, with $n \neq m$, such that

$$\mathrm{v}_n\,\mathrm{V}_{u_0} = \mathrm{v}_m\,\mathrm{V}_{u_0}\,,$$

where $(\mathrm{v}_i\,\mathrm{V}_{u_0})_{i=1}^{k_0}$ is the image sequence of vertices $\mathrm{v}_i \in \mathrm{V}_{\mathrm{C}_{k_0}}$ under V_{u_0}. Hence we may choose the minimal $\hat{m} \in [1, k_0]$ such that there exists a $\hat{n} \in [1, \hat{m}-1]$ satisfying $\mathrm{v}_{\hat{n}}\,\mathrm{V}_{u_0} = \mathrm{v}_{\hat{m}}\,\mathrm{V}_{u_0}$. Note that, since $\hat{m}$ is the minimal element satisfying this condition, the element $\hat{n}$ in the range $[1, \hat{m}-1]$ that satisfies $\mathrm{v}_{\hat{n}}\,\mathrm{V}_{u_0} = \mathrm{v}_{\hat{m}}\,\mathrm{V}_{u_0}$ is unique. It follows that V_{u_0} is injective on $\{\mathrm{v}_{\hat{n}}, \dots, \mathrm{v}_{\hat{m}-1}\}$. Now, we construct a graph morphism $u_1 : \mathrm{C}_{\hat{m}-\hat{n}} \to G$, in order to get

$$\hat{C} = (\mathrm{C}_{\hat{m}-\hat{n}})u_1\,.$$

The vertex map and edge map of u_1 are defined as follows:

$$\begin{array}{rccl} \mathrm{V}_{u_1}: & \mathrm{V}_{\mathrm{C}_{\hat{m}-\hat{n}}} & \to & \mathrm{V}_G \\ & \mathrm{v}_i^{(1)} & \mapsto & \mathrm{v}_{i+\hat{n}-1}\,\mathrm{V}_{u_0}, \quad \text{for } i \in [1, \hat{m}-\hat{n}], \end{array}$$

and

$$\begin{array}{rccl} \mathrm{E}_{u_1}: & \mathrm{E}_{\mathrm{C}_{\hat{m}-\hat{n}}} & \to & \mathrm{E}_G \\ & \mathrm{e}_i^{(1)} & \mapsto & \mathrm{e}_{i+\hat{n}-1}\,\mathrm{E}_{u_0}, \quad \text{for } i \in [1, \hat{m}-\hat{n}], \end{array}$$

where $\mathrm{v}_i^{(1)}$ and $\mathrm{e}_i^{(1)}$ denote the vertex $\mathrm{v}_{i+(\hat{m}-\hat{n})\mathbb{Z}}$ and the edge $\mathrm{e}_{i+(\hat{m}-\hat{n})\mathbb{Z}}$ of $\mathrm{C}_{\hat{m}-\hat{n}}$, respectively. We verify that $u_1 = (\mathrm{V}_{u_1}, \mathrm{E}_{u_1})$ is a well-defined graph morphism. For $i \in [1, \hat{m}-\hat{n}]$, we have:

$$\mathrm{e}_i^{(1)}\,\mathrm{E}_{u_1}\,\mathrm{s}_G = \mathrm{e}_{i+\hat{n}-1}\,\mathrm{E}_{u_0}\,\mathrm{s}_G = \mathrm{e}_{i+\hat{n}-1}\,\mathrm{s}_{\mathrm{C}_{k_0}}\,\mathrm{V}_{u_0} = \mathrm{v}_{i+\hat{n}-1}\,\mathrm{V}_{u_0} = \mathrm{v}_i^{(1)}\,\mathrm{V}_{u_1} = \mathrm{e}_i^{(1)}\,\mathrm{s}_{\mathrm{C}_{\hat{m}-\hat{n}}}\,\mathrm{V}_{u_1},$$

$$\mathrm{e}_i^{(1)}\,\mathrm{E}_{u_1}\,\mathrm{t}_G = \mathrm{e}_{i+\hat{n}-1}\,\mathrm{E}_{u_0}\,\mathrm{t}_G = \mathrm{e}_{i+\hat{n}-1}\,\mathrm{t}_{\mathrm{C}_{k_0}}\,\mathrm{V}_{u_0} = \mathrm{v}_{i+\hat{n}}\,\mathrm{V}_{u_0} \overset{(*)}{=} \mathrm{v}_{i+1}^{(1)}\,\mathrm{V}_{u_1} = \mathrm{e}_i^{(1)}\,\mathrm{t}_{\mathrm{C}_{\hat{m}-\hat{n}}}\,\mathrm{V}_{u_1},$$

where the equality $(*)$ holds in particular for $i = \hat{m}-\hat{n}$ because

$$\mathrm{v}_{i+\hat{n}}\,\mathrm{V}_{u_0} = \mathrm{v}_{\hat{m}}\,\mathrm{V}_{u_0} = \mathrm{v}_{\hat{n}}\,\mathrm{V}_{u_0} = \mathrm{v}_1^{(1)}\,\mathrm{V}_{u_1} = \mathrm{v}_{\hat{m}-\hat{n}+1}^{(1)}\,\mathrm{V}_{u_1} = \mathrm{v}_{i+1}^{(1)}\,\mathrm{V}_{u_1}\,.$$

The injectivity of V_{u_0} on $\{\mathrm{v}_{\hat{n}}, \ldots, \mathrm{v}_{\hat{m}-1}\}$ yields the injectivity of V_{u_1}. Write $\hat{C} := (C_{\hat{m}-\hat{n}})\,u_1$. As $\mathrm{C}_{\hat{m}-\hat{n}}$ is simple, it follows from Lemma 2.19 that $u_1|^{\hat{C}} : \mathrm{C}_{\hat{m}-\hat{n}} \xrightarrow{\sim} \hat{C}$ is an isomorphism. Therefore, $\hat{C}$ is a cyclic subgraph of G.

Ad construction of $\hat{G}$: We construct a graph morphism $u_2 : \mathrm{C}_{k_0-\hat{m}+\hat{n}} \to G$, in order to get

$$\hat{G} = (\mathrm{C}_{k_0-\hat{m}+\hat{n}})u_2.$$

The vertex map and edge map of u_2 are defined as follows:

$$\begin{array}{rccl} \mathrm{V}_{u_2}: & \mathrm{V}_{\mathrm{C}_{k_0-\hat{m}+\hat{n}}} & \to & \mathrm{V}_G \\ & \mathrm{v}_i^{(2)} & \mapsto & \mathrm{v}_{i+\hat{m}-1}\,\mathrm{V}_{u_0}, \quad \text{for } i \in [1, k_0-\hat{m}+\hat{n}], \end{array}$$

and

$$\begin{array}{rccl} \mathrm{E}_{u_2}: & \mathrm{E}_{\mathrm{C}_{k_0-\hat{m}+\hat{n}}} & \to & \mathrm{E}_G \\ & \mathrm{e}_i^{(2)} & \mapsto & \mathrm{e}_{i+\hat{m}-1}\,\mathrm{E}_{u_0}, \quad \text{for } i \in [1, k_0-\hat{m}+\hat{n}], \end{array}$$

where $\mathrm{v}_i^{(2)}$ and $\mathrm{e}_i^{(2)}$ denote the vertex $\mathrm{v}_{i+(k_0-\hat{m}+\hat{n})\mathbb{Z}}$ and the edge $\mathrm{e}_{i+(k_0-\hat{m}+\hat{n})\mathbb{Z}}$ of $\mathrm{C}_{k_0-\hat{m}+\hat{n}}$, respectively. We verify that $u_2 = (\mathrm{V}_{u_2}, \mathrm{E}_{u_2})$ is a well-defined graph morphism. For $i \in [1, k_0-\hat{m}+\hat{n}]$, we have:

$$\mathrm{e}_i^{(2)}\,\mathrm{E}_{u_2}\,\mathrm{s}_G = \mathrm{e}_{i+\hat{m}-1}\,\mathrm{E}_{u_0}\,\mathrm{s}_G = \mathrm{e}_{i+\hat{m}-1}\,\mathrm{s}_{\mathrm{C}_{k_0}}\,\mathrm{V}_{u_0} = \mathrm{v}_{i+\hat{m}-1}\,\mathrm{V}_{u_0} = \mathrm{v}_i^{(2)}\,\mathrm{V}_{u_2} = \mathrm{e}_i^{(2)}\,\mathrm{s}_{\mathrm{C}_{k_0-\hat{m}+\hat{n}}}\,\mathrm{V}_{u_2},$$

$$\mathrm{e}_i^{(2)}\,\mathrm{E}_{u_2}\,\mathrm{t}_G = \mathrm{e}_{i+\hat{m}-1}\,\mathrm{E}_{u_0}\,\mathrm{t}_G = \mathrm{e}_{i+\hat{m}-1}\,\mathrm{t}_{\mathrm{C}_{k_0}}\,\mathrm{V}_{u_0} = \mathrm{v}_{i+\hat{m}}\,\mathrm{V}_{u_0} \overset{(\triangle)}{=} \mathrm{v}_{i+1}^{(2)}\,\mathrm{V}_{u_2} = \mathrm{e}_i^{(2)}\,\mathrm{t}_{\mathrm{C}_{k_0-\hat{m}+\hat{n}}}\,\mathrm{V}_{u_2},$$

where the equality $(\triangle)$ holds in particular for $i = k_0-\hat{m}+\hat{n}$ because

$$\mathrm{v}_{i+\hat{m}}\,\mathrm{V}_{u_0} = \mathrm{v}_{k_0+\hat{n}}\,\mathrm{V}_{u_0} = \mathrm{v}_{\hat{n}}\,\mathrm{V}_{u_0} = \mathrm{v}_{\hat{m}}\,\mathrm{V}_{u_0} = \mathrm{v}_1^{(2)}\,\mathrm{V}_{u_2} = \mathrm{v}_{k_0-\hat{m}+\hat{n}+1}^{(2)}\,\mathrm{V}_{u_2} = \mathrm{v}_{i+1}^{(2)}\,\mathrm{V}_{u_2}\;.$$

Note that, since $k_0-\hat{m}+\hat{n} < k_0$, $\hat{G}$ is a quasi-cyclic subgraph of G.

Ad $(\mathrm{C}_{k_0})\,u_0 = \hat{C} \cup \hat{G}$: Based on the above construction, we have $(\mathrm{C}_{k_0})u_0 \geqslant \hat{C} \cup \hat{G}$. It remains to prove the reverse inclusion. According to the definition of u_1, u_2, for each $\mathrm{v}_i \in \mathrm{V}_{\mathrm{C}_{k_0}}$, we have

$$\mathrm{v}_i\,\mathrm{V}_{u_0} = \begin{cases} \mathrm{v}_{i-\hat{m}+1+k_0}^{(2)}\,\mathrm{V}_{u_2} \in \mathrm{V}_{\hat{G}} & \text{if } i \in [1, \hat{n}-1] \\ \mathrm{v}_{i-\hat{n}+1}^{(1)}\,\mathrm{V}_{u_1} \in \mathrm{V}_{\hat{C}} & \text{if } i \in [\hat{n}, \hat{m}-1] \\ \mathrm{v}_{i-\hat{m}+1}^{(2)}\,\mathrm{V}_{u_2} \in \mathrm{V}_{\hat{G}} & \text{if } i \in [\hat{m}, k_0]. \end{cases}$$

Hence we have $\mathrm{V}_{(\mathrm{C}_{k_0})u_0} \subseteq \mathrm{V}_{\hat{C}} \cup \mathrm{V}_{\hat{G}}$. Similarly, for each $\mathrm{e}_i \in \mathrm{E}_{\mathrm{C}_{k_0}}$, we have

$$\mathrm{e}_i\,\mathrm{E}_{u_0} = \begin{cases} \mathrm{e}^{(2)}_{i-\hat{m}+1+k_0}\,\mathrm{E}_{u_2} \in \mathrm{E}_{\hat{G}} & \text{if } i \in [1, \hat{n}-1] \\ \mathrm{e}^{(1)}_{i-\hat{n}+1}\,\mathrm{E}_{u_1} \in \mathrm{E}_{\hat{C}} & \text{if } i \in [\hat{n}, \hat{m}-1] \\ \mathrm{e}^{(2)}_{i-\hat{m}+1}\,\mathrm{E}_{u_2} \in \mathrm{E}_{\hat{G}} & \text{if } i \in [\hat{m}, k_0]. \end{cases}$$

Hence we have $\mathrm{E}_{(\mathrm{C}_{k_0})u_0} \subseteq \mathrm{E}_{\hat{C}} \cup \mathrm{E}_{\hat{G}}$. Hence, $(\mathrm{C}_{k_0})u_0 = \hat{C} \cup \hat{G}$.

Ad $\mathrm{V}_{\hat{C}} \cap \mathrm{V}_{\hat{G}} \neq \emptyset$: We have

$$\mathrm{v}^{(1)}_1\,\mathrm{V}_{u_1} = \mathrm{v}_{\hat{n}}\,\mathrm{V}_{u_0} = \mathrm{v}_{\hat{m}}\,\mathrm{V}_{u_0} = \mathrm{v}^{(2)}_1\,\mathrm{V}_{u_2} \in \mathrm{V}_{\hat{C}} \cap \mathrm{V}_{\hat{G}}\,.$$

Since $\hat{G} = (\mathrm{C}_{k_0-\hat{m}+\hat{n}})u_2$ and $k_0 - \hat{m} + \hat{n} < k_0$, we can write $\hat{G} = \bigoplus_{i=1}^{l} G_i$ for some $l \in \mathbb{Z}_{\geqslant 1}$, where $G_1, \ldots, G_l \in \mathrm{S_c}(G)$ and $k_0 - \hat{m} + \hat{n} = \sum_{i=1}^{l} |\,\mathrm{V}_{G_i}\,|$. Since $\mathrm{V}_{\hat{C}} \cap \mathrm{V}_{\hat{G}} \neq \emptyset$, we have $(\mathrm{C}_{k_0})u_0 = \left(\bigoplus_{i=1}^{l} G_i\right) \oplus \hat{C}$ and $\sum_{i=1}^{l} |\,\mathrm{V}_{G_i}\,| + |\mathrm{V}_{\hat{C}}| = k_0 - \hat{m} + \hat{n} + (\hat{m} - \hat{n}) = k_0$, which yields $k_0 \notin K$; so this case does not occur.

Since both case do not occur, we have a *contradiction*. This finishes the proof. □

Lemma 3.9

Suppose given a graph G and a quasi-cyclic subgraph $G' \leqslant G$ with $G' = \bigoplus_{i=1}^{n} G_i$, where G_i are cyclic subgraphs of G for $i \in [1, n]$; cf. Definition 3.5. Then there exists an epimorphism $f : \mathrm{C}_m \to G'$, where $m = \sum_{i=1}^{n} |\mathrm{E}_{G_i}|$.

Proof. To prove this statement, we show by induction that for all $l \in [1, n]$, we can find an epimorphism from C_k to $\bigoplus_{i=1}^{1} G_i$ with $k = \sum_{i=1}^{l} |\mathrm{E}_{G_1}|$.

For $l = 1$, the subgraph $\bigoplus_{i=1}^{1} G_i = G_1$ is a cyclic subgraph of G, which is isomorphic to $\mathrm{C}_{|\,\mathrm{E}_{G_1}\,|} = \mathrm{C}_k$. Any isomorphism from the set $(\mathrm{C}_k, G_1)_{\mathrm{Gph}}$ can be chosen, for it is an epimorphism.

Suppose that for a given $l \in [1, n-1]$, we can find an epimophism u from a cyclic graph C_k to $\bigoplus_{i=1}^{l} G_i$, where $k = \sum_{i=1}^{l} |\mathrm{E}_{G_i}|$. We show that our statement still holds for $l+1$.

We denote $C^{(1)} := \mathrm{C}_k$, and $C^{(2)} := \mathrm{C}_{|\mathrm{E}_{G_{l+1}}|}$. Then $C^{(2)}$ is isomorphic to G_{l+1}. In addition, we denote

$$\begin{aligned} \mathrm{v}^{(1)}_i &:= \mathrm{v}_{i+k\mathbb{Z}} \ \text{ in } C^{(1)}, \text{ for } i \in [1, k] \\ \mathrm{e}^{(1)}_i &:= \mathrm{e}_{i+k\mathbb{Z}} \ \text{ in } C^{(1)}, \text{ for } i \in [1, k] \\ \mathrm{v}^{(2)}_j &:= \mathrm{v}_{j+|\mathrm{E}_{G_{l+1}}|\mathbb{Z}} \ \text{ in } C^{(2)}, \text{ for } j \in \left[1, |\mathrm{E}_{G_{l+1}}|\right] \\ \mathrm{e}^{(2)}_j &:= \mathrm{e}_{j+|\mathrm{E}_{G_{l+1}}|\mathbb{Z}} \ \text{ in } C^{(2)}, \text{ for } j \in \left[1, |\mathrm{E}_{G_{l+1}}|\right]. \end{aligned}$$

By Definition 3.5, we may choose a vertex

$$v \in \left(\bigcup_{i=1}^{l} \mathrm{V}_{G_i}\right) \cap \mathrm{V}_{G_{l+1}}\,.$$

Then, we choose an automorphism $a \in \left(C^{(1)}, C^{(1)}\right)_{\mathrm{Gph}}$ such that

$$\left(\mathrm{v}^{(1)}_1\right) \underbrace{(a \cdot u)}_{=: u_1} = v,$$

and an isomorphism $u_2 \in \left(C^{(2)}, G_{l+1}\right)_{\mathrm{Gph}}$ satisfying $\mathrm{v}^{(2)}_1\, u_2 = v$. We now consider the cyclic graph $\mathrm{C}_{k'}$ with

$$k' := k + |\,\mathrm{E}_{G_{l+1}}\,| = \sum_{i=1}^{l+1} |\mathrm{E}_{G_i}|$$

and define the graph morphism $u' : \mathrm{C}_{k'} \to \bigoplus_{i=1}^{l+1} G_i$ as follows:

$$\mathrm{V}_{u'} : \begin{cases} \mathrm{v}_{i+k'\mathbb{Z}} & \mapsto & \mathrm{v}_i^{(1)} \, \mathrm{V}_{u_1} & \text{if } i \in [1,k] \\ \mathrm{v}_{i+k'\mathbb{Z}} & \mapsto & \mathrm{v}_{i-k}^{(2)} \, \mathrm{V}_{u_2} & \text{if } i \in [k+1,k'] \end{cases}$$

$$\mathrm{E}_{u'} : \begin{cases} \mathrm{e}_{i+k'\mathbb{Z}} \, \mathrm{E}_{u'} & \mapsto & \mathrm{e}_i^{(1)} \, \mathrm{E}_{u_1} & \text{if } i \in [1,k] \\ \mathrm{e}_{i+k'\mathbb{Z}} \, \mathrm{E}_{u'} & \mapsto & \mathrm{e}_{i-k}^{(2)} \, \mathrm{E}_{u_2} & \text{if } i \in [k+1,k'] \end{cases}$$

We provide an illustration to the construction of u' by Example 3.10.

We show that u' is an epimorphism in $\left(\mathrm{C}_{k'}, \bigoplus_{i=1}^{l+1} G_i\right)_{\mathrm{Gph}}$. Write $G^* := \bigoplus_{i=1}^{l+1} G_i$. According to the definition of u', we have

$$\mathrm{v}_{k+1+k'\mathbb{Z}} \, \mathrm{V}_{u'} = \mathrm{v}_1^{(2)} \, \mathrm{V}_{u_2} = v = \mathrm{v}_1^{(1)} \, \mathrm{V}_{u_1} = \mathrm{v}_1 \, \mathrm{V}_{u'} \,. \tag{3.3}$$

Using equation (3.3), we can now prove that u' is a graph morphism. For $i \in [1,k]$, the map u' satisfies the following properties

$$\begin{aligned} \mathrm{e}_i \, \mathrm{E}_{u'} \, \mathrm{s}_{G^*} &= \mathrm{e}_i^{(1)} \, \mathrm{E}_{u_1} \, \mathrm{s}_{G^*} = \mathrm{e}_i^{(1)} \, \mathrm{s}_{C^{(1)}} \, \mathrm{V}_{u_1} = \mathrm{v}_i^{(1)} \, \mathrm{V}_{u_1} = \mathrm{v}_i \, \mathrm{V}_{u'} = \mathrm{e}_i \, \mathrm{s}_{\mathrm{C}_{k'}} \, \mathrm{V}_{u'} \; ; \\ \mathrm{e}_i \, \mathrm{E}_{u'} \, \mathrm{t}_{G^*} &= \mathrm{e}_i^{(1)} \, \mathrm{E}_{u_1} \, \mathrm{t}_{G^*} = \mathrm{e}_i^{(1)} \, \mathrm{t}_{C^{(1)}} \, \mathrm{V}_{u_1} = \mathrm{v}_{i+1}^{(1)} \, \mathrm{V}_{u_1} = \mathrm{v}_{i+1} \, \mathrm{V}_{u'} = \mathrm{e}_i \, \mathrm{t}_{\mathrm{C}_{k'}} \, \mathrm{V}_{u'} \; . \end{aligned} \tag{3.4}$$

In particular, for $i = k$, the penultimate equation in (3.4) is valid, since via (3.3) we have

$$\mathrm{v}_{k+1}^{(1)} \, \mathrm{V}_{u_1} = \mathrm{v}_1^{(1)} \, \mathrm{V}_{u_1} = \mathrm{v}_{k+1} \, \mathrm{V}_{u'} \,.$$

Moreover, for $i \in [k+1, k']$, we have

$$\begin{aligned} \mathrm{e}_i \, \mathrm{E}_{u'} \, \mathrm{s}_{G^*} &= \mathrm{e}_{i-k}^{(2)} \, \mathrm{E}_{u_2} \, \mathrm{s}_{G^*} = \mathrm{e}_{i-k}^{(2)} \, \mathrm{s}_{C^{(2)}} \, \mathrm{V}_{u_2} = \mathrm{v}_{i-k}^{(2)} \, \mathrm{V}_{u_2} = \mathrm{v}_i \, \mathrm{V}_{u'} = \mathrm{e}_i \, \mathrm{s}_{\mathrm{C}_{k'}} \, \mathrm{V}_{u'} \; ; \\ \mathrm{e}_i \, \mathrm{E}_{u'} \, \mathrm{t}_{G^*} &= \mathrm{e}_{i-k}^{(2)} \, \mathrm{E}_{u_2} \, \mathrm{t}_{G^*} = \mathrm{e}_{i-k}^{(2)} \, \mathrm{t}_{C^{(2)}} \, \mathrm{V}_{u_2} = \mathrm{v}_{i-k+1}^{(2)} \, \mathrm{V}_{u_2} = \mathrm{v}_{i+1} \, \mathrm{V}_{u'} = \mathrm{e}_i \, \mathrm{t}_{\mathrm{C}_{k'}} \, \mathrm{V}_{u'} \; . \end{aligned} \tag{3.5}$$

In particular, for $i = k'$, the penultimate equation in (3.5) is valid, since via (3.3) we have

$$\mathrm{v}_{k'-k+1}^{(2)} \, \mathrm{V}_{u_2} = \mathrm{v}_1^{(2)} \, \mathrm{V}_{u_2} = \mathrm{v}_{1+k'\mathbb{Z}} \, \mathrm{V}_{u'} = \mathrm{v}_{k+1+k'\mathbb{Z}} \, \mathrm{V}_{u'} \; .$$

Since the image of u_1 is $\bigoplus_{i=1}^{l} G_i$ and the image of u_2 is G_{l+1}, the image of u' is

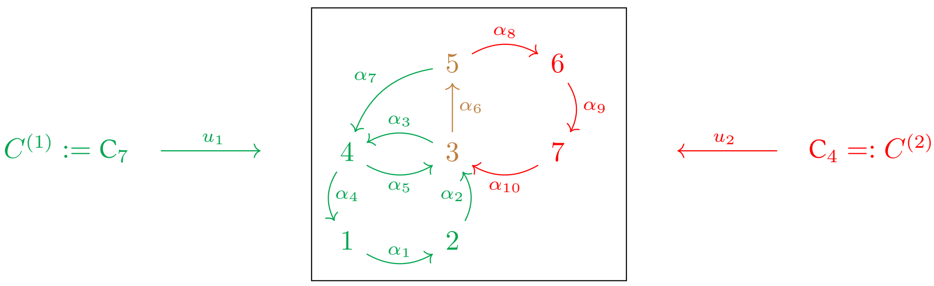
$$\left(\bigoplus_{i=1}^{l} G_i \right) \cup G_{l+1} = G^* \,.$$

Hence u' is an epimorphism.

With the principle of induction we obtain especially for $l = n$ an epimorphism im $(\mathrm{C}_m, G')_{\mathrm{Gph}}$, where $m = \sum_{i=1}^{n} |\mathrm{E}_{G_i}|$. □

Example 3.10

We give an example for the induction step in the proof of Lemma 3.9.



G'

We choose the common vertex $v = 3$ of image graphs $(\mathrm{C}_7)u_1$ and $(\mathrm{C}_4)u_2$. The definitions of u_1, u_2 and u' are as follow. Since for a graph morphism f from a cyclic graph to an arbitrary graph, the edge map E_f determines the vertex map V_f via source and target maps, we provide here only the edge maps of u_1, u_2 and u':

$$u_1 := \begin{cases} e_1^{(1)} \mapsto \alpha_6 \\ e_2^{(1)} \mapsto \alpha_7 \\ e_3^{(1)} \mapsto \alpha_5 \\ e_4^{(1)} \mapsto \alpha_3 \\ e_5^{(1)} \mapsto \alpha_4 \\ e_6^{(1)} \mapsto \alpha_1 \\ e_7^{(1)} \mapsto \alpha_2 \end{cases} \qquad u' := \begin{cases} e_1 \mapsto \alpha_6 \\ e_2 \mapsto \alpha_7 \\ e_3 \mapsto \alpha_5 \\ e_4 \mapsto \alpha_3 \\ e_5 \mapsto \alpha_4 \\ e_6 \mapsto \alpha_1 \\ e_7 \mapsto \alpha_2 \\ e_8 \mapsto \alpha_6 \\ e_9 \mapsto \alpha_8 \\ e_{10} \mapsto \alpha_9 \\ e_{11} \mapsto \alpha_{10} \end{cases}$$

$$u_2 := \begin{cases} e_1^{(2)} \mapsto \alpha_6 \\ e_2^{(2)} \mapsto \alpha_8 \\ e_3^{(2)} \mapsto \alpha_9 \\ e_4^{(2)} \mapsto \alpha_{10} \end{cases}$$

Remark 3.11

Let $f : G \to H$ be a graph morphism and $G' \leqslant G$ be a quasi-cyclic subgraph. Then $H' := (G')f$ is quasi-cyclic subgraph of H.

Proof. By Lemma 3.9, we may construct an epimorphism $u : \mathrm{C}_n \to G'$ from a certain cyclic graph C_n. Then $u \cdot f_{G'}^{H'} : \mathrm{C}_n \to H'$ is an epimorphism as a composite of two epimorphisms; and we have

$$(G')f = (G')f_{G'}^{H'} = ((\mathrm{C}_n)u)f_{G'}^{H'} = (\mathrm{C}_n)\left(u \cdot f_{G'}^{H'}\right) .$$

By Lemma 3.8, $(G')f$ is a quasi-cyclic subgraph of H. ☐

Based on Lemmas 3.8 and 3.9, we may measure the size of quasi-cyclic subgraphs:

Definition 3.12 (circumferential length)

The *circumferential length* of a quasi-cyclic subgraph $G' \leqslant G$ is defined as follows:

$$\ell(G') := \min\left\{k \in \mathbb{Z}_{\geqslant 1} \ : \ \exists f \in (\mathrm{C}_k, G')_{\mathrm{Gph}} \text{ such that } f \text{ is an epimorphism}\right\}$$

Remark 3.13

Let G' be a quasi-cyclic subgraph of G. If there is an epimorphism $f \in (\mathrm{C}_k, G')_{\mathrm{Gph}}$, then it follows that $k \geqslant \ell(G')$.

Remark 3.14

Let G, H be graphs and $f : G \to H$ be a graph morphism. Suppose that $G' \leqslant G$, $H' \leqslant H$ are quasi-cyclic subgraphs such that $(G')f = H'$. Then we have $\ell(G') \geqslant \ell(H')$.

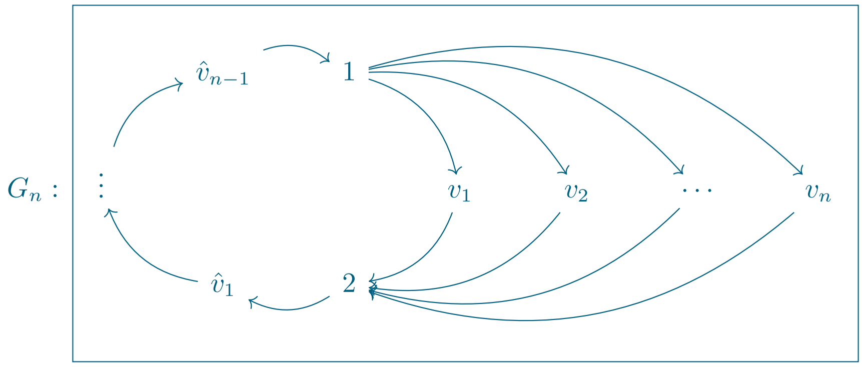
Proof. According to Definition 3.12, we may choose an epimorphism $u \in \left(\mathrm{C}_{\ell(G')}, G'\right)_{\mathrm{Gph}}$. Then we have $\left(\mathrm{C}_{\ell(G')}\right)\left(u \cdot f|_{G'}^{H'}\right) = (G')f = H'$, so that $u \cdot f|_{G'}^{H'} \in \left(\mathrm{C}_{\ell(G')}, H'\right)_{\mathrm{Gph}}$ is an epimorphism. The Remark 3.13 implies $\ell(G') \geqslant \ell(H')$. □

Remark 3.15

For a given quasi-cyclic graph G, the edge count $|\mathrm{E}_G|$ does not provide a bound for its circumferential length $\ell(G)$. In fact, for every $k \in \mathbb{Z}_{\geqslant 1}$, there exists a simple quasi-cyclic graph G such that

$$\frac{\ell(G)}{|\mathrm{E}_G|} > k.$$

Proof. For each $n \in \mathbb{Z}_{\geqslant 2}$, we set G_n as the following simple quasi-cyclic graph:



Then we have $|\mathrm{E}_{G_n}| = 3n$ and $\ell(G_n) = n(n+2)$. For each $k \in \mathbb{Z}_{\geqslant 1}$, we choose $n = 3k$. Then we have

$$\frac{\ell(G_n)}{|\mathrm{E}_{G_n}|} = \frac{n(n+2)}{3n} = \frac{3k+2}{3} > k,$$

which completes the proof. □

3.2 Connectivity of quasi-cyclic graphs

In the following, we characterize quasi-cyclic graphs by describing their connectivity properties. This characterization will later be used in Algorithm 5.13 to determine whether a given graph is quasi-cyclic.

Definition 3.16 (Weakly connected graphs)

We say a graph G is *weakly connected* if we have

$$\left\{ f \in (G, H)_{\mathrm{Gph}} \ : \ \mathrm{V}_f \ \text{ is surjective} \right\} = \emptyset,$$

where H is the graph presented in Figure 3.3.

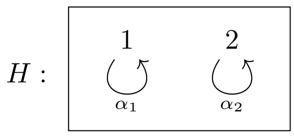


Figure 3.3

Remark 3.17

A nonempty weakly connected graph G admits a unique partition into subgraphs, namely $G = [G]$.

Proof. *Assume* that $[G_\lambda]_{\lambda\in\Lambda}$ is another partition of G, where Λ is a index set with $|\Lambda| > 1$. Suppose that $\lambda_1 \in \Lambda$ and write $G' := \bigcup_{\lambda\in\Lambda\setminus\{\lambda_1\}} G_\lambda$. Then $[G_{\lambda_1}, G']$ is a partition of G, and we may define a graph morphism $u \in (G,H)_{\text{Gph}}$, where H is the graph presented in Figure 3.3, by setting:

$$\begin{aligned} \mathrm{V}_u : \mathrm{V}_G &\to \mathrm{V}_H \\ v &\mapsto 1, \quad \text{for } v \in \mathrm{V}_{G_{\lambda_1}}, \\ v &\mapsto 2, \quad \text{for } v \in \mathrm{V}_{G'}, \end{aligned}$$

$$\begin{aligned} \mathrm{E}_u : \mathrm{E}_G &\to \mathrm{E}_H \\ e &\mapsto \alpha_1, \quad \text{for } e \in \mathrm{E}_{G_{\lambda_1}}, \\ e &\mapsto \alpha_2, \quad \text{for } e \in \mathrm{E}_{G'}. \end{aligned}$$

Since for $e \in \mathrm{E}_{G_{\lambda_1}}$, we have

$$\underbrace{e\,\mathrm{s}_G}_{\in \mathrm{V}_{G_{\lambda_1}}}\mathrm{V}_u = 1 = e\,\mathrm{E}_u\,\mathrm{s}_H \quad \text{and} \quad \underbrace{e\,\mathrm{t}_G}_{\in \mathrm{V}_{G_{\lambda_1}}}\mathrm{V}_u = 1 = e\,\mathrm{E}_u\,\mathrm{t}_H,$$

and since for $e \in \mathrm{E}_{G'}$, we have

$$\underbrace{e\,\mathrm{s}_G}_{\in \mathrm{V}_{G'}}\mathrm{V}_u = 2 = e\,\mathrm{E}_u\,\mathrm{s}_H \quad \text{and} \quad \underbrace{e\,\mathrm{t}_G}_{\in \mathrm{V}_{G'}}\mathrm{V}_u = 2 = e\,\mathrm{E}_u\,\mathrm{t}_H,$$

the graph morphism u is well-defined. However, the surjectivity of V_u *contradicts* the weak connectivity of G. Hence, G does not admit a partition indexed by Λ with $|\Lambda| > 1$, completing the proof. □

Since the graphs under consideration are directed, imposing restrictions on the direction of paths allows us to define stronger notions of connectivity. To this end, we first introduce the definition of paths in a graph.

Definition 3.18 (Directed path graphs)

For $n \in \mathbb{Z}_{\geqslant 0}$, we define the *directed path graph* $\mathrm{D}_n = (\mathrm{V}_{\mathrm{D}_n}, \mathrm{E}_{\mathrm{D}_n}; \mathrm{s}_{\mathrm{D}_n}, \mathrm{t}_{\mathrm{D}_n})$ of length n as follows:

$$\begin{aligned} \mathrm{V}_{\mathrm{D}_n} &= \{\hat{\mathrm{v}}_i \ : \ i \in [0,n]\}\,; \\ \mathrm{E}_{\mathrm{D}_n} &= \{\hat{\mathrm{e}}_i \ : \ i \in [0,n-1]\}\,; \end{aligned}$$

and for each $i \in [0, n-1]$ we set

$$\hat{\mathrm{e}}_i\,\mathrm{s}_{\mathrm{D}_n} = \hat{\mathrm{v}}_i, \quad \hat{\mathrm{e}}_i\,\mathrm{t}_{\mathrm{D}_n} = \hat{\mathrm{v}}_{i+1}.$$

An example for directed path graph D_n:

$$\mathrm{D}_4 : \quad \hat{\mathrm{v}}_0 \xrightarrow{\hat{\mathrm{e}}_0} \hat{\mathrm{v}}_1 \xrightarrow{\hat{\mathrm{e}}_1} \hat{\mathrm{v}}_2 \xrightarrow{\hat{\mathrm{e}}_2} \hat{\mathrm{v}}_3 \xrightarrow{\hat{\mathrm{e}}_3} \hat{\mathrm{v}}_4$$

For $i_0, j_0 \in [0,n]$ with $i_0 \leqslant j_0$, we use $\mathrm{D}_n^{[i_0,j_0]}$ to denote the subgraph of D_n defined by

$$\mathrm{V}_{\mathrm{D}_n^{[i_0,j_0]}} = \{\hat{\mathrm{v}}_i \ : \ i \in [i_0, j_0]\} \quad \text{and} \quad \mathrm{E}_{\mathrm{D}_n^{[i_0,j_0]}} = \{\hat{\mathrm{e}}_i \ : \ i \in [i_0, j_0 - 1]\}\,.$$

An example for $\mathrm{D}_4^{[1,3]} \leqslant \mathrm{D}_4$:

$$\mathrm{D}_4^{[1,3]} : \quad \boxed{\hat{\mathrm{v}}_1 \xrightarrow{\hat{\mathrm{e}}_1} \hat{\mathrm{v}}_2 \xrightarrow{\hat{\mathrm{e}}_2} \hat{\mathrm{v}}_3}$$

For further properties of directed path graphs, we refer the reader to [**4,** Definition 56].

Definition 3.19 **(Paths)**
Let G be a graph. A *path* in the graph G is defined as a graph morphism $p : \mathrm{D}_n \to G$ for some $n \in \mathbb{Z}_{\geqslant 0}$, where D_n denotes the directed path graph of length n. The *start vertex* and *end vertex* of path p are given by $p_\mathrm{s} := \hat{\mathrm{v}}_0\, \mathrm{V}_p$ and $p_\mathrm{t} := \hat{\mathrm{v}}_n\, \mathrm{V}_p$, respectively. Furthermore, we denote

$$\overline{\mathrm{V}_G(v,*)} := \bigcup_{n=0}^{\infty} \left\{ v' \in \mathrm{V}_G \;:\; \exists p \in (\mathrm{D}_n, G)_{\mathrm{Gph}} \text{ such that } p_\mathrm{s} = v,\ p_\mathrm{t} = v' \right\}$$

as the set of all vertices that appear as the end vertex of a path in G starting with v.

Definition 3.20 **(Star-connected graphs and strongly connected graphs)**
A graph G is said to be *star-connected* if there is a vertex $v \in \mathrm{V}_G$ such that

$$\mathrm{V}_G = \overline{\mathrm{V}_G(v,*)} .$$

A nonempty graph G is called *strongly connected* if for every vertex $v \in \mathrm{V}_G$, it holds that

$$\mathrm{V}_G = \overline{\mathrm{V}_G(v,*)} .$$

Remark 3.21
From Definitions 3.16 and 3.20, we can conclude that

(1) every nonempty strongly connected graph is star-connected,

(2) every star-connected graph is weakly connected.

However, the converses do not hold in general:

(3) There exists a graph that is star-connected, but not weakly connected.

(4) There exists a nonempty graph that is star-connected, but not strongly connected.

For instance, in Figure 3.4, G_1 is weakly connected but not star-connected; G_2 is star-connected but not strongly connected.

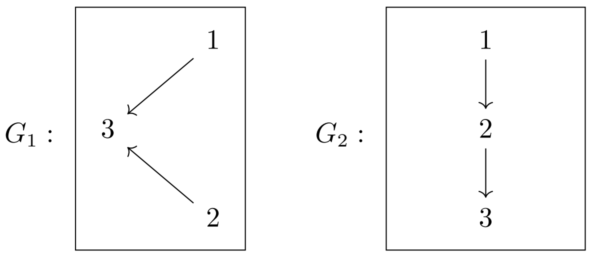


Figure 3.4: Graphs illustrating different types of connectivity.

Proof. Here, we give a proof to (2). Let G be a star-connected graph, so we may choose $v \in \mathrm{V}_G$ such that $\mathrm{V}_G \subseteq \overline{\mathrm{V}_G(v,*)}$. We prove that there does not exist any $f \in (G,H)_{\mathrm{Gph}}$ with H defined in Figure 3.3 such that V_f is surjective:

Let $f : G \to H$ be a graph morphism, and $v' \in \mathrm{V}_G$. As G is star-connected, we have $v' \in \overline{\mathrm{V}_G(v,*)}$ and we may consider a path p in G with length $n \in \mathbb{Z}_{\geqslant 0}$, which has start vertex $p_\mathrm{s} = v$ and end vertex $p_\mathrm{t} = v'$. Then, the composite $p \cdot f : \mathrm{D}_n \to H$ is a graph morphism, and for each $i \in [0, n-1]$ we have

$$\hat{\mathrm{v}}_i\, \mathrm{V}_{pf} = \hat{\mathrm{e}}_i\, \mathrm{s}_{\mathrm{D}_n}\, \mathrm{V}_{pf} = \hat{\mathrm{e}}_i\, \mathrm{E}_{pf}\, \mathrm{s}_H = \hat{\mathrm{e}}_i\, \mathrm{E}_{pf}\, \mathrm{t}_H = \hat{\mathrm{e}}_i\, \mathrm{t}_{\mathrm{D}_n}\, \mathrm{V}_{pf} = \hat{\mathrm{v}}_{i+1}\, \mathrm{V}_{pf} \ .$$

This implies $\hat{\mathrm{v}}_0\, \mathrm{V}_{pf} = \hat{\mathrm{v}}_n\, \mathrm{V}_{pf}$. Hence

$$v\, \mathrm{V}_f = \hat{\mathrm{v}}_0\, \mathrm{V}_p \cdot \mathrm{V}_f = \hat{\mathrm{v}}_n\, \mathrm{V}_p \cdot \mathrm{V}_f = v'\, \mathrm{V}_f \,.$$

As $v' \in \mathrm{V}_G$ was chosen arbitrarily, it follows that all the vertices of G are mapped to the same vertex in H. Hence, V_f is not surjective, which completes our proof. □

Before we come to the characterization of quasi-cyclic graphs, we first prove the following lemma, which serves as a preparation:

Lemma 3.22

Let G be a graph and $G^{(1)}, G^{(2)} \leqslant G$ be quasi-cyclic subgraphs such that $\mathrm{V}_{G^{(1)}} \cap \mathrm{V}_{G^{(2)}} \neq \emptyset$. Then the union $G^{(1)} \cup G^{(2)}$ is also a quasi-cyclic subgraph of G.

Proof. According to Definition 3.5, we may suppose that $G^{(1)} = \bigoplus_{i=1}^{n} G_i^{(1)}$ and $G^{(2)} = \bigoplus_{j=1}^{m} G_j^{(2)}$, where $n, m \in \mathbb{Z}_{\geqslant 1}$, $G_1^{(1)}, \dots, G_n^{(1)}$ and $G_1^{(2)}, \dots, G_m^{(2)}$ are cyclic subgraphs of $G^{(1)}$ and $G^{(2)}$, respectively. Choose $v \in \mathrm{V}_{G^{(1)}} \cap \mathrm{V}_{G^{(2)}}$. Then there exists $j \in [1, m]$ such that $v \in \mathrm{V}_{G_j^{(2)}}$, and we let

$$j_0 := \min \left\{ j \in [1,m] \ : \ v \in \mathrm{V}_{G_j^{(2)}} \right\},$$

so that

$$G^{(1)} \cup G_{j_0}^{(2)} \ = \ \left(\bigoplus_{i=1}^{n} G_i^{(1)} \right) \oplus G_{j_0}^{(2)}$$

is a quasi-cyclic subgraph.

In the case $j_0 = 1$: We have $G_{j_0}^{(2)} = G_1^{(2)}$, which implies

$$G^{(1)} \cup G^{(2)} = G^{(1)} \cup G_{j_0}^{(2)} \cup G^{(2)} = G^{(1)} \cup G_1^{(2)} \cup G^{(2)} = \left(\bigoplus_{i=1}^{n} G_i^{(1)} \right) \oplus G_1^{(2)} \oplus \left(\bigoplus_{j=1}^{m} G_j^{(2)} \right).$$

Therefore, $G^{(1)} \cup G^{(2)}$ is a quasi-cyclic subgraph.

In the case $j_0 > 1$: Since $\bigoplus_{j=1}^{j_0} G_j^{(2)}$ is a quasi-cyclic subgraph of $G^{(2)}$, by Definition 3.5, we can choose $j_1 \in [1, j_0 - 1]$ such that $\mathrm{V}_{G_{j_1}^{(2)}} \cap \mathrm{V}_{G_{j_0}^{(2)}} \neq \emptyset$. If $j_1 \neq 1$, we can similarly find $j_2 \in [1, j_1 - 1]$ such that $\mathrm{V}_{G_{j_2}^{(2)}} \cap \mathrm{V}_{G_{j_1}^{(2)}} \neq \emptyset$. Proceeding inductively, if $j_i \neq 1$, we choose $j_{i+1} \in [1, j_i - 1]$ such that $\mathrm{V}_{G_{j_{i+1}}^{(2)}} \cap \mathrm{V}_{G_{j_i}^{(2)}} \neq \emptyset$. We continue this process until, for some $k \geqslant 1$, we reach $j_k = 1$. Now, we have obtained a monotonically decreasing sequence $1 = j_k < j_{k-1} < \cdots < j_1 < j_0$ that satisfies

$$\mathrm{V}_{G_{j_l}^{(2)}} \cap \mathrm{V}_{G_{j_{l+1}}^{(2)}} \neq \emptyset \quad \text{for all } l \in [0, k-1].$$

Finally, we have

$$\begin{aligned} G^{(1)} \cup G^{(2)} &= G^{(1)} \cup \left(\bigoplus_{l=0}^{k} G^{(2)}_{j_l} \right) \cup G^{(2)} \\ &= \left(\bigoplus_{i=1}^{n} G^{(1)}_{i} \right) \cup \left(\bigoplus_{l=0}^{k} G^{(2)}_{j_l} \right) \cup \left(\bigoplus_{j=1}^{m} G^{(2)}_{j} \right) \\ &= \left(\bigoplus_{i=1}^{n} G^{(1)}_{i} \right) \oplus \left(\bigoplus_{l=0}^{k} G^{(2)}_{j_l} \right) \oplus \left(\bigoplus_{j=1}^{m} G^{(2)}_{j} \right), \end{aligned}$$

which indicates that $G^{(1)} \cup G^{(2)}$ is a quasi-cyclic graph. □

By iteratively applying Lemma 3.22, we obtain the following corollary:

Corollary 3.23 (to Lemma 3.22)
Let G be a graph, and let $G_1, G_2, \dots, G_n$ be quasi-cyclic subgraphs of G satisfying the following conditions:

- $\mathrm{V}_{G_1} \cap \mathrm{V}_{G_2} \neq \emptyset$,
- $(\mathrm{V}_{G_1} \cup \mathrm{V}_{G_2}) \cap \mathrm{V}_{G_3} \neq \emptyset$,
- $\dots$,
- $\left(\mathrm{V}_{G_1} \cup \mathrm{V}_{G_2} \cup \dots \cup \mathrm{V}_{G_{n-1}}\right) \cap \mathrm{V}_{G_n} \neq \emptyset$.

Then $\bigcup_{i=1}^{n} G_i$ is a quasi-cyclic subgraph of G.

Lemma 3.24 (Characterization of quasi-cyclic graphs)
For a given graph G, the following are equivalent:

(1) G is a quasi-cyclic graph.

(2) G is a finite, strongly connected graph with $\mathrm{E}_G \neq \emptyset$.

Proof. (1) ⇒ (2): Let G be a quasi-cyclic graph and $v, w \in \mathrm{V}_G$. Then G is finite and has $\mathrm{E}_G \neq \emptyset$; cf. Definition 3.5. We show that G is strongly connected. By Lemma 3.9, we may consider a graph epimorphism $f : \mathrm{C}_n \to G$ for a certain $n \in \mathbb{Z}_{\geqslant 1}$. Suppose that $\mathrm{v}_{i+n\mathbb{Z}}, \mathrm{v}_{j+n\mathbb{Z}} \in \mathrm{V}_{\mathrm{C}_n}$ for some $i, j \in \mathbb{Z}$ such that $\mathrm{v}_{i+n\mathbb{Z}}\, \mathrm{V}_f = v$ and $\mathrm{v}_{j+n\mathbb{Z}}\, \mathrm{V}_f = w$. We choose $i_0 \in i + n\mathbb{Z}$ and $j_0 \in i + n\mathbb{Z}$ that satisfy $0 \leqslant j_0 - i_0 < n$. Now we consider the graph morphism $u : \mathrm{D}_{j_0 - i_0} \to \mathrm{C}_n$, where $\mathrm{D}_{j_0-i_0}$ is a directed path graph, defined by

$$\begin{array}{rccc} \mathrm{V}_u : & \mathrm{V}_{\mathrm{D}_{j_0-i_0}} & \to & \mathrm{V}_{\mathrm{C}_n} \\ & \hat{\mathrm{v}}_k & \mapsto & \mathrm{v}_{k+i_0} \end{array}$$

and

$$\begin{array}{rccc} \mathrm{E}_u : & \mathrm{E}_{\mathrm{D}_{j_0-i_0}} & \to & \mathrm{E}_{\mathrm{C}_n} \\ & \hat{\mathrm{e}}_k & \mapsto & \mathrm{e}_{k+i_0} \end{array}.$$

We claim that u is a graph morphism: For every $k \in [0, j_0 - i_0 - 1]$, we have

$$\begin{aligned} \hat{\mathrm{e}}_k\, \mathrm{E}_u\, \mathrm{s}_{\mathrm{C}_n} &= \mathrm{e}_{k+i_0}\, \mathrm{s}_{\mathrm{C}_n} = \mathrm{v}_{k+i_0} = \hat{\mathrm{v}}_k\, \mathrm{V}_u = \hat{\mathrm{e}}_k\, \mathrm{s}_{\mathrm{D}_{j_0-i_0}}\, \mathrm{V}_u\ , \\ \hat{\mathrm{e}}_k\, \mathrm{E}_u\, \mathrm{t}_{\mathrm{C}_n} &= \mathrm{e}_{k+i_0}\, \mathrm{t}_{\mathrm{C}_n} = \mathrm{v}_{k+i_0+1} = \hat{\mathrm{v}}_{k+1}\, \mathrm{V}_u = \hat{\mathrm{e}}_k\, \mathrm{t}_{\mathrm{D}_{j_0-i_0}}\, \mathrm{V}_u\ . \end{aligned}$$

It follows that the composite $u \cdot f : \mathrm{D}_{j_0-i_0} \to G$ is a path with start vertex v and end vertex w. Hence, we may conclude that for arbitrary $v, w \in \mathrm{V}_G$, we can find a path p such that its start vertex $p_\mathrm{s} = v$ and its end vertex $p_{\mathrm{t}} = w$. Hence, for every $v \in \mathrm{V}_G$, we have $\mathrm{V}_G \subseteq \overline{\mathrm{V}_G(v,*)}$. By Definition 3.20, G is strongly connected.

(2) $\Rightarrow$ (1): Let G be a finite, strongly connected graph with $\mathrm{E}_G \neq \emptyset$. First, we prove that for each $e \in \mathrm{E}_G$ there exists a quasi-cyclic subgraph $G_e \leqslant G$ such that $e \in \mathrm{E}_{G_e}$. As G is strongly connected, there is a path $p : \mathrm{D}_n \to G$, with a certain directed path graph D_n of length $n \in \mathbb{Z}_{\geqslant 0}$, such that $p_{\mathrm{s}} = \hat{\mathrm{v}}_0 \,\mathrm{V}_p = e \,\mathrm{t}_G$ and $p_{\mathrm{t}} = \hat{\mathrm{v}}_n \,\mathrm{V}_p = e \,\mathrm{s}_G$. We claim that the subgraph $G_e = (\mathrm{V}_{G_e}, \mathrm{E}_{G_e})$ defined by $\mathrm{V}_{G_e} := \mathrm{V}_{(\mathrm{D}_n)p}$, $\mathrm{E}_{G_e} := \mathrm{E}_{(\mathrm{D}_n)p} \cup \{e\}$ is a quasi-cyclic subgraph. For this purpose, we construct a graph morphism $\hat{u} : \mathrm{C}_{n+1} \to G_e$ with

$$\begin{array}{rccll} \mathrm{V}_{\hat{u}} : & \mathrm{V}_{\mathrm{C}_{n+1}} & \to & \mathrm{V}_{G_e} & \\ & \mathrm{v}_i & \mapsto & \hat{\mathrm{v}}_{i-1}\,\mathrm{V}_p & \text{for } i \in [1, n+1] \end{array}$$

and

$$\begin{array}{rccll} \mathrm{E}_{\hat{u}} : & \mathrm{E}_{\mathrm{C}_{n+1}} & \to & \mathrm{E}_{G_e} & \\ & \mathrm{e}_i & \mapsto & \hat{\mathrm{e}}_{i-1}\,\mathrm{E}_p, & \text{for } i \in [1, n] \\ & \mathrm{e}_{n+1} & \mapsto & e\,. & \end{array}$$

We verify that $\hat{u}$ is a graph morphism: For $i \in [1, n]$ we have

$$\begin{aligned} \mathrm{e}_i \,\mathrm{E}_{\hat{u}} \,\mathrm{s}_{G_e} &= \hat{\mathrm{e}}_{i-1} \,\mathrm{E}_p \,\mathrm{s}_{G_e} = \hat{e}_{i-1} \,\mathrm{s}_{\mathrm{D}_n} \,\mathrm{V}_p = \hat{\mathrm{v}}_{i-1} \,\mathrm{V}_p = \mathrm{v}_i \,\mathrm{V}_{\hat{u}} = \mathrm{e}_i \,\mathrm{s}_{\mathrm{C}_{n+1}} \,\mathrm{V}_{\hat{u}}\ , \\ \mathrm{e}_i \,\mathrm{E}_{\hat{u}} \,\mathrm{t}_{G_e} &= \hat{\mathrm{e}}_{i-1} \,\mathrm{E}_p \,\mathrm{t}_{G_e} = \hat{e}_{i-1} \,\mathrm{t}_{\mathrm{D}_n} \,\mathrm{V}_p = \hat{\mathrm{v}}_i \,\mathrm{V}_p = \mathrm{v}_{i+1} \,\mathrm{V}_{\hat{u}} = \mathrm{e}_i \,\mathrm{t}_{\mathrm{C}_{n+1}} \,\mathrm{V}_{\hat{u}}\ ; \end{aligned}$$

and for e_{n+1} we have

$$\begin{aligned} \mathrm{e}_{n+1} \,\mathrm{E}_{\hat{u}} \,\mathrm{s}_{G_e} &= e \,\mathrm{s}_{G_e} = \hat{\mathrm{v}}_n \,\mathrm{V}_p = \mathrm{v}_{n+1} \,\mathrm{V}_{\hat{u}} = \mathrm{e}_{n+1} \,\mathrm{s}_{\mathrm{C}_{n+1}} \,\mathrm{V}_{\hat{u}}\ , \\ \mathrm{e}_{n+1} \,\mathrm{E}_{\hat{u}} \,\mathrm{t}_{G_e} &= e \,\mathrm{t}_{G_e} = \hat{\mathrm{v}}_0 \,\mathrm{V}_p = \mathrm{v}_1 \,\mathrm{V}_{\hat{u}} = \mathrm{v}_{n+2} \,\mathrm{V}_{\hat{u}} = \mathrm{e}_{n+1} \,\mathrm{t}_{\mathrm{C}_{n+1}} \,\mathrm{V}_{\hat{u}}\ . \end{aligned}$$

Hence, $\hat{u}$ is a graph morphism. Since $G_e = (\mathrm{C}_{n+1})\,\hat{u}$, it follows from Lemma 3.8 that G_e is a quasi-cyclic subgraph with $e \in \mathrm{E}_{G_e}$.

As G is a strongly connected graph with $|\,\mathrm{E}_G\,| \geqslant 1$, no vertex in G is isolated. Therefore,

$$G = \bigcup_{e \in \mathrm{E}_G} G_e = \bigcup_{x \in \mathrm{V}_G} \bigcup_{e \in G(x,*)} G_e.$$

Corollary 3.23 implies that for every $x \in \mathrm{V}_G$, the subgraph

$$G^{(x)} := \bigcup_{e \in G(x,*)} G_e$$

is quasi-cyclic. It remains to show that the union $\bigcup_{x \in \mathrm{V}_G} G^{(x)}$ is quasi-cyclic.

We begin by proving that for each $V \subsetneq \mathrm{V}_G$, $V \neq \emptyset$, there exists a $y \in \mathrm{V}_G \setminus V$ such that $G^{(y)}$ and $\bigcup_{x \in V} G^{(x)}$ share at least one vertex. Since $\mathrm{V}_G \setminus V \neq \emptyset$, we can choose a vertex $z \in \mathrm{V}_G \setminus V$. For a given $x \in V$, we consider a path $p' : \mathrm{D}_m \to G$ of length $m \in \mathbb{Z}_{\geqslant 1}$, which has start vertex $p'_{\mathrm{s}} = x$ and end vertex $p'_{\mathrm{t}} = z$. Next, we let

$$l_0 := \min \{ l \in [1, m] \ : \ \hat{\mathrm{v}}_l \,\mathrm{V}_{p'} \in \mathrm{V}_G \setminus V \}$$

and set $x_0 = \hat{\mathrm{v}}_{l_0 - 1} \,\mathrm{V}_{p'}$, $y = \hat{\mathrm{v}}_{l_0} \,\mathrm{V}_{p'}$. Then we have

$$\begin{aligned} \hat{\mathrm{e}}_{l_0-1} \,\mathrm{E}_{p'} \,\mathrm{t}_G &= \hat{\mathrm{e}}_{l_0-1} \,\mathrm{t}_{\mathrm{D}_m} \,\mathrm{V}_{p'} = \hat{\mathrm{v}}_{l_0} \,\mathrm{V}_p = y \in \mathrm{V}_G \setminus V, \\ \hat{\mathrm{e}}_{l_0-1} \,\mathrm{E}_{p'} \,\mathrm{s}_G &= \hat{\mathrm{e}}_{l_0-1} \,\mathrm{s}_{\mathrm{D}_m} \,\mathrm{V}_{p'} = \hat{\mathrm{v}}_{l_0-1} \,\mathrm{V}_p = x_0 \in V\,, \end{aligned}$$

which indicates that $\hat{e}_{l_0-1}\,\mathrm{E}_{p'} \in G(x_0,*)$ and y is a vertex of $G^{(x_0)}$. Hence, y is a vertex of $\bigcup_{x\in V} G^{(x)}$. Since y is also a vertex of $G^{(y)}$, we conclude that y is a vertex of $G^{(y)}$ and of $\bigcup_{x\in V} G^{(x)}$.

Now we prove that $G = \bigcup_{x\in \mathrm{V}_G} G^{(x)}$ is a quasi-cyclic graph. Choose an arbitrary vertex $v_1 \in \mathrm{V}_G$. If $\{v_1\} \subsetneq \mathrm{V}_G$, we can select $v_2 \in \mathrm{V}_G \setminus \{v_1\}$ such that

$$\mathrm{V}_{G^{(v_2)}} \cap \bigcup_{v\in\{v_1\}} \mathrm{V}_{G^{(v)}} \neq \emptyset.$$

If $\{v_1, v_2\} \subsetneq \mathrm{V}_G$, we may choose $v_3 \in \mathrm{V}_G \setminus \{v_1, v_2\}$ such that

$$\mathrm{V}_{G^{(v_3)}} \cap \bigcup_{v\in\{v_1,v_2\}} \mathrm{V}_{G^{(v)}} \neq \emptyset.$$

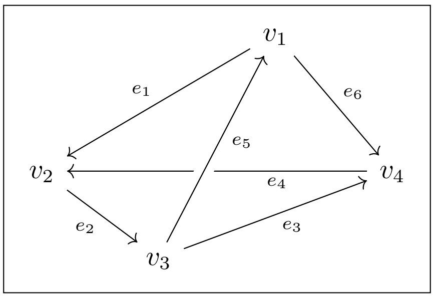
Continuing in this way, we construct a sequence $(v_r)_{r=1}^{|\mathrm{V}_G|}$ such that $\{v_1, \ldots, v_{|\mathrm{V}_G|}\} = \mathrm{V}_G$, and each corresponding subgraph $G^{(v_r)}$ satisfies

$$\mathrm{V}_{G^{(v_r)}} \cap \bigcup_{v\in\{v_1,\ldots,v_{r-1}\}} \mathrm{V}_{G^{(v)}} \neq \emptyset, \quad \text{for all } r \in [2, |\mathrm{V}_G|].$$

By Corollary 3.23, the graph $G = \bigcup_{r=1}^{|\mathrm{V}_G|} G^{(v_r)}$ is quasi-cyclic. □

Example 3.25

According to Lemma 3.24, the graph with pyramid structure illustrated in the following figure is quasi-cyclic.



3.3 Covering quasi-cyclic graphs

In this section, we present an alternative method for computing the circumferential length of a quasi-cyclic graph G. Compared to the direct approach based on the definition, this method allows for a relatively more efficient computation; cf. Algorithm 5.14.

Let G be a quasi-cyclic graph. From the definition, we know that G is the union of finitely many subgraphs in $\mathrm{S_c}(G)$. Therefore, G is a finite graph with $G \subseteq \bigcup \mathrm{S_c}(G)$, which also implies that the set $\mathrm{S_c}(G)$ is nonempty and finite, i.e., $0 < |\mathrm{S_c}(G)| < \infty$. Based on these premises, we can derive the following lemma:

Lemma 3.26

For a quasi-cyclic graph G, we have

$$\ell(G) = \min \left\{ \sum_{C\in M} |\mathrm{E}_C| \;\middle|\; M \subseteq \mathrm{S_c}(G) \text{ such that } \bigcup M \supseteq G \right\}$$

Proof. *Ad* "$\leqslant$". Let $M \subseteq \mathrm{S_c}(G)$ satisfying $\bigcup M \supseteq G$. First, we prove that for every nonempty subset $M' \subsetneq M$ we can choose $\hat{C} \in M \setminus M'$ such that

$$\mathrm{V}_{\hat{C}} \cap \bigcup_{C' \in M'} \mathrm{V}_{C'} \neq \emptyset.$$

To prove this claim, we distinguish two cases.

Case $\bigcup M' = G$. Since $\mathrm{V}_{\hat{C}} \subseteq \mathrm{V}_G = \bigcup_{C' \in M'} \mathrm{V}_{C'}$ and $\mathrm{V}_{\hat{C}} \neq \emptyset$, we have

$$\mathrm{V}_{\hat{C}} \cap \bigcup_{C' \in M'} \mathrm{V}_{C'} = \mathrm{V}_{\hat{C}} \cap \mathrm{V}_G \neq \emptyset.$$

Case $\bigcup M' \neq G$. By Lemma 3.24, the quasi-cyclic graph G is strongly connected with $|\,\mathrm{E}_G\,| \geqslant 1$, which implies that for every $v \in \mathrm{V}_G$,

$$\mathrm{E}_G(v, *) \neq \emptyset \quad \text{and} \quad \mathrm{E}_G(*, v) \neq \emptyset,$$

and we may conclude that $\mathrm{E}_{\bigcup M'} \subsetneq \mathrm{E}_G$: *Assume* that $\mathrm{E}_{\bigcup M'} = \mathrm{E}_G$. Then we have

$$\mathrm{V}_G = \mathrm{E}_G\, \mathrm{t}_G = \mathrm{E}_{\bigcup M'}\, \mathrm{t}_G \subseteq \mathrm{V}_{\bigcup M'},$$

which implies that $\bigcup M' = G$, *contradicting* the precondition $\bigcup M' \neq G$.

We claim that there exists $e \in \mathrm{E}_G \setminus \mathrm{E}_{\bigcup M'}$ such that

$$\{e\, \mathrm{s}_G, e\, \mathrm{t}_G\} \cap \bigcup_{C' \in M'} \mathrm{V}_{C'} \neq \emptyset. \tag{3.6}$$

Assume not. We consider the subgraph $\hat{G} \leqslant G$ given by

$$\mathrm{E}_{\hat{G}} = \mathrm{E}_G \setminus \mathrm{E}_{\bigcup M'} \quad \text{and} \quad \mathrm{V}_{\hat{G}} = \mathrm{E}_{\hat{G}}\, \mathrm{s}_G \cup \mathrm{E}_{\hat{G}}\, \mathrm{t}_G\,.$$

As every $e \in \mathrm{E}_{\hat{G}}$ satisfies $\{e\, \mathrm{s}_G, e\, \mathrm{t}_G\} \cap \bigcup_{C' \in M'} \mathrm{V}_{C'} = \emptyset$, we have $\mathrm{V}_{\hat{G}} \cap \mathrm{V}_{\bigcup M'} = \emptyset$. Moreover, $\mathrm{E}_{\hat{G}} \cup \mathrm{E}_{\bigcup M'} = G$ implies that $\hat{G} \cup \bigcup M' = G$. Therefore we can partition G into $\left[\hat{G}, \bigcup M'\right]$, which *contradicts* to G being strongly connected; cf. Remark 3.17. So we may choose $e \in \mathrm{E}_G \setminus \mathrm{E}_{\bigcup M'}$ satisfying (3.6). Since $G = \bigcup M$, there exists $\hat{C} \in M$ such that $e \in \mathrm{E}_{\hat{C}}$ and $\hat{C} \in M \setminus M'$. Hence, we find $\hat{C} \in M \setminus M'$ satisfying

$$\mathrm{V}_{\hat{C}} \cap \bigcup_{C' \in M'} \mathrm{V}_{C'} \neq \emptyset.$$

Now, we choose $C_1 \in M$. If $|M| \geqslant 2$, select $C_2 \in M \setminus \{C_1\}$ such that

$$\mathrm{V}_{C_2} \cap \mathrm{V}_{C_1} \neq \emptyset.$$

If $\{C_1, C_2\} \subsetneq M$, then select $C_3 \in M \setminus \{C_1, C_2\}$ such that

$$\mathrm{V}_{C_3} \cap \bigcup_{i \in [1,2]} \mathrm{V}_{C_i} \neq \emptyset.$$

Continue this process until $\{C_1, \dots, C_n\} = M$ for some $n \in \mathbb{Z}_{\geqslant 1}$. Then we may write $G = \bigoplus_{i=1}^{n} C_i$. By Lemma 3.9, there exists a graph epimorphism u from cyclic graph C_L to G, where

$$L = \sum_{i=1}^{n} |\mathrm{E}_{C_i}|\,.$$

By Remark 3.13, we have $\ell(G) \leqslant L$. As M is chosen arbitrarily, we complete the proof of "$\leqslant$" direction.

Ad "$\geqslant$". We consider a graph epimorphism $u \in \left(\mathrm{C}_{\ell(G)}, G\right)_{\mathrm{Gph}}$, from the cyclic graph $\mathrm{C}_{\ell(G)}$. By Lemma 3.8, there exists a sequence $(C_i)_{i=1}^n$ of cyclic subgraphs of G such that $G = \bigoplus_{i=1}^n C_i$ and $\ell(G) = \sum_{i=1}^n |\mathrm{E}_{C_i}|$. We denote $\hat{M} = \{C_i \ : \ i \in [1,n]\}$. Note that C_i, $i \in [1,n]$, are not necessarily to be pairwise distinct; hence we have

$$\ell(G) = \sum_{i=1}^{n} |\mathrm{E}_{C_i}| \geqslant \sum_{C \in \hat{M}} |\mathrm{E}_C| \geqslant \min \left\{ \sum_{C \in M} |\mathrm{E}_C| \,\middle|\, M \subseteq \mathrm{S_c}(G) \text{ such that } \bigcup M \supseteq G \right\},$$

which completes our proof. □

4 Quasi-isomorphisms

4.1 Notion of a quasi-isomorphism

Definition 4.1

Let G, H be graphs and $f : G \to H$ be a graph morphism. For an arbitrary graph X we define the map (X, f) as follows:

$$(X,f): \begin{array}{ccc} (X,G)_{\mathrm{Gph}} & \longrightarrow & (X,H)_{\mathrm{Gph}} \\ u & \mapsto & u \cdot f \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{u} & G \\ & \searrow^{u\cdot f} & \downarrow f \\ & & H \end{array}$$

Figure 4.1: Illustration for the map (X, f).

Definition 4.2 (quasi-isomorphism)

Let G, H be graphs. A graph morphism $f \in (G,H)_{\mathrm{Gph}}$ is called a *quasi-isomorphism*, and denoted as $G \xrightarrow[\approx]{f} H$, if for every $k \in \mathbb{Z}_{\geqslant 1}$, the map (C_k, f) is bijective; that is, for each given graph morphism $v \in (\mathrm{C}_k, H)_{\mathrm{Gph}}$, there exists a unique graph morphism $u \in (\mathrm{C}_k, G)_{\mathrm{Gph}}$ such that $u \cdot f = v$.

$$\begin{array}{ccc} \mathrm{C}_k & \xrightarrow{\exists ! u} & G \\ & \searrow^{v} & \wr\downarrow f \\ & & H \end{array}$$

Figure 4.2: Graph morphism f is quasi-isomorphism, if the map (C_k, f) is bijective for all $k \in \mathbb{Z}_{\geqslant 1}$.

Example 4.3 (Quasi-isomorphism)

Let $f : G \to H$ be a graph morphism, where G is an arbitrary graph and H is a graph with $\mathrm{S_c}(H) = \emptyset$. Then we have $\mathrm{S_{qc}}(H) = \emptyset$, and hence $(\mathrm{C}_k, H)_{\mathrm{Gph}} = \emptyset$ for every $k \in \mathbb{Z}_{\geqslant 1}$ by Lemma 3.8. On the other hand, we have $\mathrm{S_{qc}}(G) = \emptyset$, for otherwise, by Lemma 3.9, we would have a graph morphism $u : \mathrm{C}_k \to G$ from some cyclic graph C_k; and there would be a graph morphism $u \cdot f$ in $(\mathrm{C}_k, H)_{\mathrm{Gph}}$. Therefore, we also have $(\mathrm{C}_k, G)_{\mathrm{Gph}} = \emptyset$ for every $k \in \mathbb{Z}_{\geqslant 1}$. Altogether, f is a quasi-isomorphism.

4.2 A review of a sufficient condition to be a quasi-isomorphism

J. Hess provided a sufficient condition for a graph morphism to be a quasi-isomorphism; [**4,** Chapter 7]. We briefly review this method here.

Definition 4.4 (Etale fibration [4, Definition 127.(2)])
A graph morphism $f : G \to H$ is said to be an *etale fibration* if for every $v \in \mathrm{V}_G$, the map

$$\begin{array}{rccc} \mathrm{E}_{f,v} : & \mathrm{E}_G(v,*) & \to & \mathrm{E}_H(v\,\mathrm{V}_f,*) \\ & e & \mapsto & e\,\mathrm{E}_f \end{array}$$

is bijective.

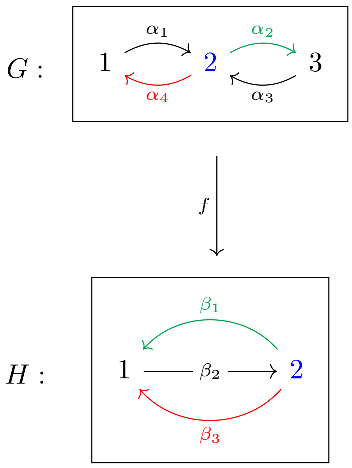
Definition 4.5 (Unitargeting [4, Definition 206])
For a given graph morphism $f : G \to H$, an edge $e_H \in \mathrm{E}_H$ is called *unitargeting* with respect to f if

$$|\{e_G\,\mathrm{t}_G \ : \ e_G \in \mathrm{E}_G,\ e_G\,\mathrm{E}_f = e_H\}| = 1.$$

Proposition 4.6 ([4, Proposition 210])
Let G, H be graphs and $f : G \to H$ be an etale fibration. If for every $n \in \mathbb{Z}_{\geqslant 1}$ and every $u \in (\mathrm{C}_n, H)_{\mathrm{Gph}}$, there exists $i \in \mathbb{Z}$ such that $\mathrm{e}_i\,\mathrm{E}_u \in \mathrm{E}_H$ is unitargeting with respect to f, where $\mathrm{e}_i = \mathrm{e}_{i+n\mathbb{Z}} \in \mathrm{E}_{\mathrm{C}_n}$, then the graph morphism $f : G \to H$ is a quasi-isomorphism.

Example 4.7
We provide an example demonstrating how Proposition 4.6 can be used to decide that a graph morphism is a quasi-isomorphisms.

Let $f : G \to H$ be the graph morphism between the graphs in following diagram.



The vertex map V_f and edge map E_f of f are given by

$$\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H : \left\{ \begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 1 \end{array} \right\} \quad \text{and} \quad \mathrm{E}_f : \mathrm{E}_G \to \mathrm{E}_H : \left\{ \begin{array}{ccc} \alpha_1 & \mapsto & \beta_2 \\ \alpha_2 & \mapsto & \beta_1 \\ \alpha_3 & \mapsto & \beta_2 \\ \alpha_4 & \mapsto & \beta_3 \end{array} \right\}.$$

We show that $f : G \to H$ is a quasi-isomorphism. First, we observe that f is an etale fibration. It remains to prove that for every graph morphism $u : \mathrm{C}_n \to H$ from a cyclic graph C_n, there exists a unitargeting edge in the image graph $(\mathrm{C}_n)u$ with respect to f. Let $u : \mathrm{C}_n \to H$ be a graph morphism from a certain cyclic graph C_n. Then the image graph $(\mathrm{C}_n)u$ is a quasi-cyclic subgraph of H; cf. Lemma 3.8.

Since all cyclic subgraphs of H contain β_2, the edge β_2 is also contained in $(\mathrm{C}_n)u$; so there exists $i \in \mathbb{Z}$ such that the edge $\mathrm{e}_{i+n\mathbb{Z}} \in \mathrm{E}_{\mathrm{C}_n}$ is mapped to β_2, which is unitargeting in $(\mathrm{C}_n)u \leqslant H$ with respect to f. Hence f is a quasi-isomorphism by Proposition 4.6.

Remark 4.8

For a given graph $G = (\mathrm{V}_G, \mathrm{E}_G; \mathrm{s}_G, \mathrm{t}_G)$, its opposite graph $G^{\mathrm{op}} = (\mathrm{V}_{G^{\mathrm{op}}}, \mathrm{E}_{G^{\mathrm{op}}}; \mathrm{s}_{G^{\mathrm{op}}}, \mathrm{t}_{G^{\mathrm{op}}})$ is defined as the graph having the identical vertex set and edge set as G, and the direction of every edge $e \in \mathrm{E}_{G^{\mathrm{op}}}$ is reversed, that is, $e\,\mathrm{s}_{G^{\mathrm{op}}} = e\,\mathrm{t}_G$ and $e\,\mathrm{t}_{G^{\mathrm{op}}} = e\,\mathrm{s}_G$ for $e \in \mathrm{E}_{G^{\mathrm{op}}}$. So we have

$$G^{\mathrm{op}} = (\mathrm{V}_{G^{\mathrm{op}}}, \mathrm{E}_{G^{\mathrm{op}}}; \mathrm{s}_{G^{\mathrm{op}}}, \mathrm{t}_{G^{\mathrm{op}}}) := (\mathrm{V}_G, \mathrm{E}_G; \mathrm{t}_G, \mathrm{s}_G).$$

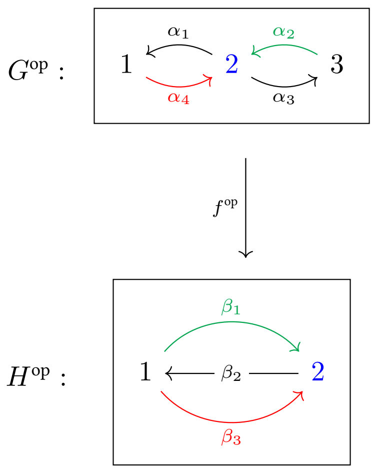
For a given graph morphism $f : G \to H$, we may also define its opposite graph morphism

$$f^{\mathrm{op}} = (\mathrm{V}_{f^{\mathrm{op}}}, \mathrm{E}_{f^{\mathrm{op}}}) : G^{\mathrm{op}} \to H^{\mathrm{op}}$$

as $\mathrm{V}_{f^{\mathrm{op}}} := \mathrm{V}_f$ and $\mathrm{E}_{f^{\mathrm{op}}} := \mathrm{E}_f$. It has been reasoned in **[4,** Definition 211.(2)**]** that f^{op} is a well-defined graph morphism. Moreover, if $f : G \to H$ is a quasi-isomorphism, its opposite graph morphism f^{op} is also a quasi-isomorphism; cf. **[4,** Remark 214**]**. In Example 4.9 we demonstrate how to determine a quasi-isomorphism by this criterion.

Example 4.9

Let $f : G \to H$ be the graph morphism defined in Example 4.7. The graph morphism shown in the following diagram is the opposite graph morphism of f. Since f is a quasi-isomorphism, the following graph morphism is a quasi-isomorphism as well. However, it should be note that f^{op} is not an etal fibration.



4.3 Lifting results for quasi-isomorphisms of graphs

Lemma 4.10

Let G, H be graphs and $f : G \to H$ be a quasi-isomorphism. Then for each $H' \in \mathrm{S_c}(H)$, there exists $G' \in \mathrm{S_c}(G)$ such that $(G')f \leqslant H'$ and $f|_{G'}^{H'}$ is an isomorphism.

Proof. We consider a cyclic subgraph $H' \leqslant H$. Let $\iota_{H'}^{H}$ be the embedding. According to the definition of a cyclic subgraph, there exists a unique $k \in \mathbb{Z}_{\geqslant 1}$ such that $\mathrm{C}_k \simeq H'$, namely $k = |\mathrm{E}_{H'}|$. Hence we may choose a graph isomorphism $v \in (\mathrm{C}_k, H')_{\mathrm{Gph}}$. Since f is a quasi-isomorphism, there exists a unique $u \in (\mathrm{C}_k, G)_{\mathrm{Gph}}$ such that $u \cdot f = v \cdot \iota^H_{H'}$. Since $v \cdot \iota^H_{H'}$ is composed of an isomorphism and a monomorphism, the composites $\mathrm{V}_v \cdot \mathrm{V}_{\iota^H_{H'}}$ and $\mathrm{E}_v \cdot \mathrm{E}_{\iota^H_{H'}}$ are injective. It follows that u also consists of injective maps V_u and E_u, and hence u is a monomorphism; cf. Lemma 2.18. Write

$$G' = (\mathrm{C}_k)u,$$

then $u|^{G'}_{\mathrm{C}_k}$ is an isomorphism, which implies that G' is a cyclic subgraph; and we have

$$(G')f = (\mathrm{C}_k)(u \cdot f) = (\mathrm{C}_k)\left(v \cdot \iota^H_{H'}\right) \leqslant H' .$$

From the above, we obtain the following diagram:

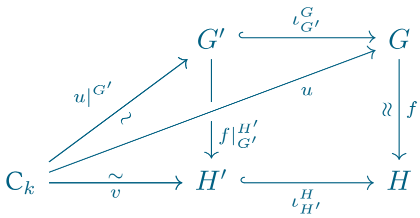


Figure 4.3

Now we verify that $f|^{H'}_{G'}$ is an isomorphism. We have

$$\begin{aligned} & u \cdot f = u|^{G'} \cdot \iota^G_{G'} \cdot f = v \cdot \iota^H_{H'} \\ \Rightarrow\quad & u|^{G'} \cdot f|^{H'}_{G'} \cdot \iota^H_{H'} = v \cdot \iota^H_{H'} \\ \Rightarrow\quad & u|^{G'} \cdot f|^{H'}_{G'} = v \\ \Rightarrow\quad & f|^{H'}_{G'} = \left(u|^{G'}\right)^{-1} \cdot v . \end{aligned}$$

Therefore, $f|^{H'}_{G'}$ is the composite of two isomorphisms, which implies that $f|^{H'}_{G'}$ is itself an isomorphism. □

Lemma 4.11

Let G, H be graphs and $f : G \to H$ be a quasi-isomorphism. Suppose that $G' \leqslant G$ and $H' \leqslant H$ are cyclic subgraphs of G and H, respectively, such that $(G')f \leqslant H'$. Then the restriction $f|^{H'}_{G'}$ is a graph isomorphism.

Proof. Let $\mathrm{C}_m, \mathrm{C}_n$ with $n, m \in \mathbb{Z}_{\geqslant 1}$ be cyclic graphs such that $\mathrm{C}_m \simeq G'$ and $\mathrm{C}_n \simeq H'$. So we may choose graph isomorphisms $g \in (\mathrm{C}_m, G')$ and $h \in (\mathrm{C}_n, H')_{\mathrm{Gph}}$. Then the composite $g \cdot f|^{H'}_{G'} \cdot h^{-1}$ is a graph morphism in $(\mathrm{C}_m, \mathrm{C}_n)_{\mathrm{Gph}}$, which implies by Lemma 3.2 that n is a divisor of m. That means there exists a $k \in \mathbb{Z}_{\geqslant 1}$ such that we can write $m = k \cdot n$. Furthermore, it follows from Remark 3.3 that $g \cdot f|^{H'}_{G'} \cdot h^{-1}$ is an epimorphism. Since g and h^{-1} are isomorphisms, $f|^{H'}_{G'}$ is an epimorphism. From the above, we obtain the following commutative diagram. Recall that $\iota^G_{G'}$ and $\iota^H_{H'}$ are embeddings from G' to G and from H' to H, respectively.

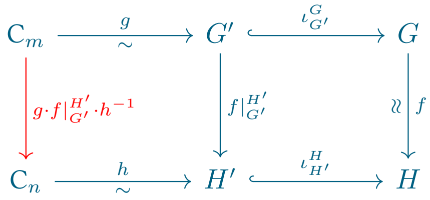


Figure 4.4: $G' \leqslant G$, $H' \leqslant H$ are cyclic subgraphs.

Note that by Lemma 4.10, we may choose a cyclic subgraph $\tilde{G}' \leqslant G$ such that $f|_{\tilde{G}'}^{H'}$ is a graph isomorphism and we have

$$f|_{\tilde{G}'}^{H'} \cdot \iota_{H'}^{H} = \iota_{\tilde{G}'}^{G} \cdot f \;, \tag{4.1}$$

and hence

$$\iota_{H'}^{H} = \left(f|_{\tilde{G}'}^{H'}\right)^{-1} \cdot \iota_{\tilde{G}'}^{G} \cdot f \;. \tag{4.2}$$

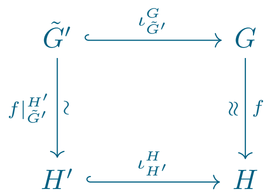


Figure 4.5: Illustration of Equations (4.1) and (4.2)

Now we discuss the relationship between G' and $\tilde{G}'$.

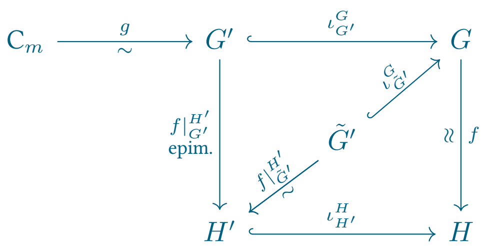


Figure 4.6: Relationship between G' and $\tilde{G}'$

According to Figure 4.6 and Equation (4.2), we have

$$g \cdot \iota_{G'}^{G} \cdot f = g \cdot f|_{G'}^{H'} \cdot \iota_{H'}^{H} = g \cdot f|_{G'}^{H'} \cdot \left(f|_{\tilde{G}'}^{H'}\right)^{-1} \cdot \iota_{\tilde{G}'}^{G} \cdot f \;.$$

Since f is a quasi-isomorphism, we have

$$g \cdot \iota_{G'}^{G} = g \cdot f|_{G'}^{H'} \cdot \left(f|_{\tilde{G}'}^{H'}\right)^{-1} \cdot \iota_{\tilde{G}'}^{G} \;.$$

When two graph morphisms are equal, their images are the same. Moreover, note that $f|_{G'}^{H'}$ is an epimorphism and the images of all the maps in the above equation are known. Thus, we obtain

$$G' = (\mathrm{C}_m)\left(g \cdot \iota_{G'}^{G}\right) = (\mathrm{C}_m)\left(g \cdot f|_{G'}^{H'} \cdot \left(f|_{\tilde{G}'}^{H'}\right)^{-1} \cdot \iota_{\tilde{G}'}^{G}\right) = \tilde{G}' \;,$$

as subgraphs of G. The equality $G' = \tilde{G}'$ implies that

$$\mathrm{C}_m \simeq G' = \tilde{G}' \simeq H' \simeq \mathrm{C}_n \;.$$

Therefore, $m = n$ and $g \cdot f|_{G'}^{H'} \cdot h^{-1}$ is a graph isomorphism; cf. Remark 3.3. Then $f|_{G'}^{H'}$ is equal to the composite $g^{-1} \cdot \left(g \cdot f|_{G'}^{H'} \cdot h^{-1}\right) \cdot h$, which is composed of graph isomorphisms. Thus $f|_{G'}^{H'}$ is a graph isomorphism. □

Proposition 4.12

Let G, H be graphs and $f : G \to H$ be a quasi-isomorphism. For each $H' \in \mathrm{S_c}(H)$, there is a unique $G' \in \mathrm{S_c}(G)$ such that $(G')f \leqslant H'$. In this case, $f|_{G'}^{H'}$ is a graph isomorphism.

Proof. Lemma 4.10 states that there exists $G' \in \mathrm{S_c}(G)$ such that $(G')f \leqslant H'$ and $f|_{G'}^{H'}$ is an isomorphism. We prove the uniqueness of such a cyclic subgraph G'.

Suppose given $G'' \in \mathrm{S_c}(G)$ such that $(G'')f \leqslant H'$. By Lemma 4.11, $f|_{G''}^{H'}$ is a graph isomorphism. We consider the cyclic graph C_n, for a certain $n \in \mathbb{Z}_{\geqslant 1}$, that is isomorphic to H'. Choose a graph isomorph $v \in (\mathrm{C}_n, H')_{\mathrm{Gph}}$. Then

$$u := v \cdot \left(f|_{G'}^{H'}\right)^{-1} \cdot \iota_{G'}^{G} : \mathrm{C}_n \to G$$

is a graph morphism such that $u \cdot f = v \cdot \iota_{H'}^{H}$. On the other hand,

$$u' := v \cdot \left(f|_{G''}^{H'}\right)^{-1} \cdot \iota_{G''}^{G} : \mathrm{C}_n \to G$$

is a graph morphism which also satisfies $u' \cdot f = v \cdot \iota_{H'}^{H}$. Since f is a quasiisomorphism, we have $u = u'$. Hence $G' = (\mathrm{C}_n)u = (\mathrm{C}_n)u' = G''$. This completes our proof of uniqueness.

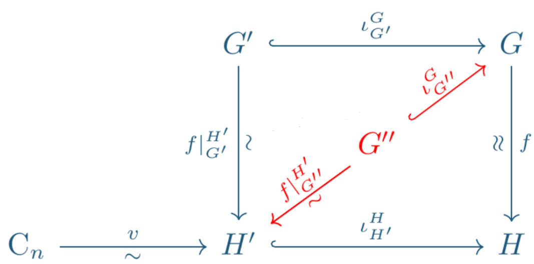


Figure 4.7: An illustration to the proof of Lemma 4.12

□

Remark 4.13

Proposition 4.12 implies that if $f : G \to H$ is a quasi-isomorphism, then for every $H' \in \mathrm{S_c}(H)$, there exists a unique $G' \in \mathrm{S_{qc}}(G)$ such that $(G')f \leqslant H'$. Furthermore, for every $G'' \in \mathrm{S_{qc}}(G) \setminus \mathrm{S_c}(G)$, the image $(G'')f$ is in $\mathrm{S_{qc}}(H) \setminus \mathrm{S_c}(H)$.

Proof. Let $H' \in \mathrm{S_c}(H)$. By Proposition 4.12, there is a unique $G' \in \mathrm{S_c}(G)$ such that $(G')f \leqslant H'$. *Assume* that there exists $G'' \in \mathrm{S_{qc}}(G) \setminus \mathrm{S_c}(G)$ such that $(G'')f \leqslant H'$. Then we have $|\mathrm{S_c}(G'')| \geqslant 2$; and for every $C \in \mathrm{S_c}(G'') \subseteq \mathrm{S_c}(G)$, we have $(C)f \leqslant H'$. This is a *contradiction*. Hence G' is the unique quasi-cyclic subgraph of G such that $(G')f \leqslant H'$. As H' has been chosen arbitrarily, we may also conclude that for every $G'' \in \mathrm{S_{qc}}(G) \setminus \mathrm{S_c}(G)$, the image graph $(G'')f$ is not cyclic in H, however, quasi-cyclic by Lemma 3.8. □

Lemma 4.14

Let G and H be graphs, and let $f : G \to H$ be a quasi-isomorphism. For each quasi-cyclic subgraph

$H' \leqslant H$, there exists a quasi-cyclic subgraph $G' \leqslant G$ such that $\ell(G') = \ell(H')$ and such that the restriction $f|_{G'}^{H'}$ is an epimorphism.

Proof. By Lemma 3.9 and Definition 3.12, we can choose an epimorphism $v \in \left(\mathrm{C}_{\ell(H')}, H'\right)_{\mathrm{Gph}}$. Since f is a quasi-isomorphism, there exists a graph morphism $u \in \left(\mathrm{C}_{\ell(H')}, G\right)_{\mathrm{Gph}}$ such that $u \cdot f = v \cdot \iota_{H'}^{H}$. Let $G' = \left(\mathrm{C}_{\ell(H')}\right) u$. By Lemma 3.8 and Remark 3.13, G' is a quasi-cyclic subgraph of G with $\ell(G') \leqslant \ell(H')$. Moreover, we have

$$(G')f = \left(\mathrm{C}_{\ell(H')}\right)(u \cdot f) = \left(\mathrm{C}_{\ell(H')}\right)\left(v \cdot \iota_{H'}^{H}\right) = H',$$

which implies that $f|_{G'}^{H'}$ exists and is an epimorphism.

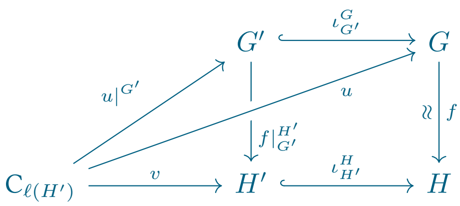
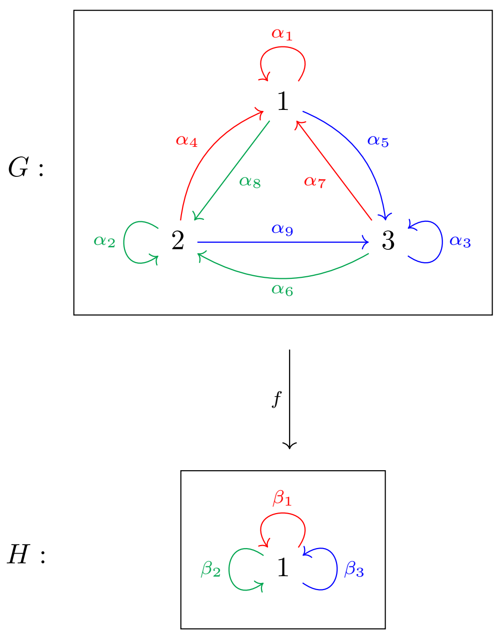


Figure 4.8: Commutative diagram for Lemma 4.14

Remark 3.14 asserts that $\ell(G') \geqslant \ell(H')$. Altogether, we have $\ell(G') = \ell(H')$. Therefore, G' is a quasi-cyclic subgraph of G with $\ell(G') = \ell(H')$, and $f|_{G'}^{H'}$ is an epimorphism. □

Remark 4.15

For a given quasi-isomorphism $f : G \to H$, Proposition 4.12 and Remark 4.13 have shown that for every $C_H \in \mathrm{S_c}(H)$, there exists a unique $C_G \in \mathrm{S_{qc}}(G)$ such that $(C_G)f = C_H$; and $f|_{C_G}^{C_H}$ is an isomorphism. In comparison, for $Q_H \in \mathrm{S_{qc}}(H)$, Lemma 4.14 states the existence of $Q_G \in \mathrm{S_{qc}}(G)$ that satisfies $(Q_G)f = Q_H$ and $\ell(Q_G) = \ell(Q_H)$, whereas the uniqueness of such a Q_G can not be guaranteed. In fact, the following graph morphism gives a counterexample:



The graph morphism f is defined by

$$\mathrm{V}_f := \left\{\begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 1 \end{array}\right\} \quad \text{and} \quad \mathrm{E}_f := \left\{\begin{array}{ccc} \alpha_1 & \mapsto & \beta_1 \\ \alpha_2 & \mapsto & \beta_2 \\ \alpha_3 & \mapsto & \beta_3 \\ \alpha_4 & \mapsto & \beta_1 \\ \alpha_5 & \mapsto & \beta_3 \\ \alpha_6 & \mapsto & \beta_2 \\ \alpha_7 & \mapsto & \beta_1 \\ \alpha_8 & \mapsto & \beta_2 \\ \alpha_9 & \mapsto & \beta_3 \end{array}\right\}.$$

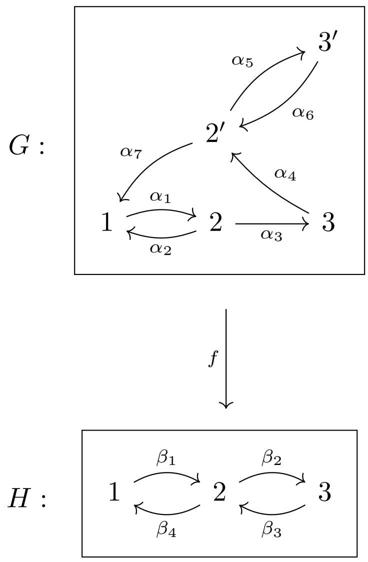
Since for every $v \in \mathrm{V}_G$, the map $\mathrm{E}_{f,v} : \mathrm{E}_G(v,*) \to \mathrm{E}_H(v\,\mathrm{V}_f,*)$ is bijective, and since for every $e_H \in \mathrm{E}_H$, we have $|\{e_G\,\mathrm{t}_G \,:\, e_G \in \mathrm{E}_G,\ e_G\,\mathrm{E}_f = e_H\}| = 1$, that is, all $e_H \in \mathrm{E}_H$ are unitargeting, f is a quasi-isomorphism by Proposition 4.6. Note that the graph H is quasi-cyclic. Meanwhile $G' = (\{1,2,3\}, \{\alpha_4,\alpha_5,\alpha_6\})$ and $G'' = (\{1,2,3\}, \{\alpha_7,\alpha_8,\alpha_9\})$ are cyclic subgraphs in G that satisfy

$$(G')f = (G'')f = H \quad \text{and} \quad \ell(G') = \ell(G'') = \ell(H) = 3.$$

However, $G' \neq G''$. Hence we do not obtain uniqueness.

Remark 4.16

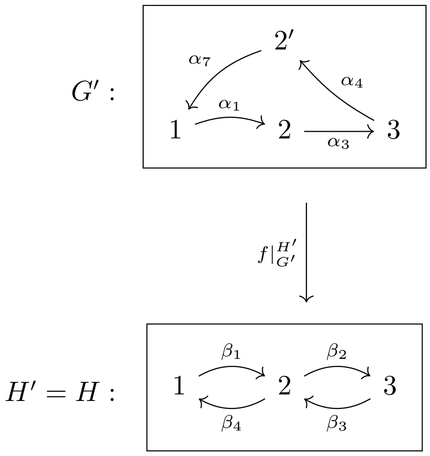
In Lemma 4.14, the restriction $f|_{G'}^{H'}$ is in general not a quasi-isomorphism. The graph morphism presented in the following figure gives a counterexample.



The graph morphism $f : G \to H$ is defined by

$$\mathrm{V}_f := \left\{\begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 2' & \mapsto & 2 \\ 3 & \mapsto & 3 \\ 3' & \mapsto & 3 \end{array}\right\} \quad \text{and} \quad \mathrm{E}_f := \left\{\begin{array}{ccc} \alpha_1 & \mapsto & \beta_1 \\ \alpha_2 & \mapsto & \beta_4 \\ \alpha_3 & \mapsto & \beta_2 \\ \alpha_4 & \mapsto & \beta_3 \\ \alpha_5 & \mapsto & \beta_2 \\ \alpha_6 & \mapsto & \beta_3 \\ \alpha_7 & \mapsto & \beta_4 \end{array}\right\}.$$

By **[4, Example 215]**, f is a quasi-isomorphism. We consider a restriction $f|_{G'}^{H'}$ as follows.



Note that G', H' are quasi-cyclic subgraphs of G and H, respectively, both having the same circumferential length $\ell(G') = \ell(H') = 4$. At the same time, we have $(G')f = H'$; so $f|_{G'}^{H'}$ is an epimorphism. However, Proposition 4.12 implies that $f|_{G'}^{H'}$ is not a quasi-isomorphism. In fact, for the cyclic subgraph $C_{H'} \leqslant H'$, with $\mathrm{V}_{C_{H'}} = \{1, 2\}$ and $\mathrm{E}_{C_{H'}} = \{\beta_1, \beta_4\}$, there exists no cyclic subgraph in G' that is isomorphic to it.

Lemma 4.17

Let G, H be graphs and $f : G \to H$ be a graph morphism. For $H' \in \mathrm{S}_{\mathrm{qc}}(H)$, we define the subgraph $G^{(H')} \leqslant G$ as

$$G^{(H')} = \bigcup \{C \in \mathrm{S}_{\mathrm{c}}(G) \; : \; (C)f \leqslant H'\},$$

and denote

$$f^{(H')} := f\big|_{G^{(H')}}^{H'} : G^{(H')} \to H'.$$

Then f is a quasi-isomorphism if and only if for every $H' \in \mathrm{S}_{\mathrm{qc}}(H)$, the graph morphism $f^{(H')}$ is a quasi-isomorphism.

Proof. Let f be a quasi-isomorphism. Suppose a subgraph $H' \in \mathrm{S}_{\mathrm{qc}}(H)$ is given. We show that $f^{(H')}$ is a quasi-isomorphism, that is, that the map $\left(\mathrm{C}_n, f^{(H')}\right)$ is bijective for every $n \in \mathbb{Z}_{\geqslant 1}$.

We begin by proving that for every $n \in \mathbb{Z}_{\geqslant 1}$, the map $\left(\mathrm{C}_n, f^{(H')}\right)$ is injective. For a given $n \in \mathbb{Z}_{\geqslant 1}$, let $u^{(1)}_{G^{(H')}}, u^{(2)}_{G^{(H')}} \in \left(\mathrm{C}_n, G^{(H')}\right)_{\mathrm{Gph}}$ be such that $u^{(1)}_{G^{(H')}} \cdot f^{(H')} = u^{(2)}_{G^{(H')}} \cdot f^{(H')}$. Then we have

$$u^{(1)}_{G^{(H')}} \cdot \iota^{G}_{G^{(H')}} \cdot f = u^{(1)}_{G^{(H')}} \cdot f^{(H')} \cdot \iota^{H}_{H'} = u^{(2)}_{G^{(H')}} \cdot f^{(H')} \cdot \iota^{H}_{H'} = u^{(2)}_{G^{(H')}} \cdot \iota^{G}_{G^{(H')}} \cdot f,$$

where $\iota^{G}_{G^{(H')}}, \iota^{H}_{H'}$ are embeddings; cf. Definition 2.12. Since f is a quasi-isomorphism, we have

$$u^{(1)}_{G^{(H')}} \cdot \iota^{G}_{G^{(H')}} = u^{(2)}_{G^{(H')}} \cdot \iota^{G}_{G^{(H')}},$$

and since $\iota^{G}_{G^{(H')}}$ is a monomorphism, we have $u^{(1)}_{G^{(H')}} = u^{(2)}_{G^{(H')}}$. Hence $\left(\mathrm{C}_n, f^{(H')}\right)$ is injective for every $n \in \mathbb{Z}_{\geqslant 1}$.

Next, we prove that for every $n \in \mathbb{Z}_{\geqslant 1}$, the map $\left(\mathrm{C}_n, f^{(H')}\right)$ is surjective. For a given $n \in \mathbb{Z}_{\geqslant 1}$, let $u_{H'} \in (\mathrm{C}_n, H')_{\mathrm{Gph}}$. Then $u_H := u_{H'} \cdot \iota^{H}_{H'}$ is a graph morphism in $(\mathrm{C}_n, H)_{\mathrm{Gph}}$. Since f is a quasi-isomorphism, there is a unique $u_G \in (\mathrm{C}_n, G)_{\mathrm{Gph}}$ such that $u_G \cdot f = u_H$. By Lemma 3.8, $(\mathrm{C}_n)u_G$ is a quasi-cyclic subgraph of G; so there exist $C_1, \ldots, C_m \in \mathrm{S}_{\mathrm{c}}(G)$ for some $m \in \mathbb{Z}_{\geqslant 1}$ such that $(\mathrm{C}_n)u_G = \bigcup_{i=1}^{m} C_i$. Then for each $i \in [1, m]$, we have

$$(C_i)f \leqslant ((\mathrm{C}_n)u_G)f = (\mathrm{C}_n)u_H \leqslant H'.$$

It follows from the definition of $G^{(H')}$ that $C_i \leqslant G^{(H')}$. Hence

$$(\mathrm{C}_n)u_G = \bigcup_{i=1}^{m} C_i \leqslant G^{(H')}.$$

Therefore, we may let $u_{G^{(H')}} := u_G|^{G^{(H')}}$. Then $u_G = u_G|^{G^{(H')}} \cdot \iota^G_{G^{(H')}}$. Hence we have

$$u_{G^{(H')}} \cdot f^{(H')} \cdot \iota^H_{H'} = u_{G^{(H')}} \cdot \iota^G_{G^{(H')}} \cdot f = u_G \cdot f = u_H = u_{H'} \cdot \iota^H_{H'},$$

which implies that $u_{G^{(H')}} \cdot f^{(H')} = u_{H'}$. Hence $\left(\mathrm{C}_n, f^{(H')}\right)$ is surjective for every $n \in \mathbb{Z}_{\geqslant 1}$.

Combining the results above, we conclude that $f^{(H')}$ is a quasi-isomorphism.

To show the converse implication, we first prove that for every $n \in \mathbb{Z}_{\geqslant 1}$, the map (C_n, f) is surjective. Let $u_H \in (\mathrm{C}_n, H)_{\mathsf{Gph}}$. Write $H' = (\mathrm{C}_n)u_H$. By Lemma 3.8, H' is a quasi-cyclic subgraph of H. Since $\iota^H_{H'}$ is a monomorphism, the restriction $u_H|^{H'}$ is the unique graph morphism in $(\mathrm{C}_n, H')_{\mathsf{Gph}}$ satisfying $u_H|^{H'} \cdot \iota^H_{H'} = u_H$. Note that $f^{(H')}$ is a quasi-isomorphism; so there exists a unique $u_{G^{(H')}} \in (\mathrm{C}_n, G^{(H')})_{\mathsf{Gph}}$ such that $u_{G^{(H')}} \cdot f^{(H')} = u_H|^{H'}$. Set $u_G := u_{G^{(H')}} \cdot \iota^G_{G^{(H')}}$. Then we have

$$u_G \cdot f = u_{G^{(H')}} \cdot \iota^G_{G^{(H')}} \cdot f = u_{G^{(H')}} \cdot f^{(H')} \cdot \iota^H_{H'} = u_H|^{H'} \cdot \iota^H_{H'} = u_H\ .$$

Hence (C_n, f) is surjective for every $n \in \mathbb{Z}_{\geqslant 1}$.

Now, it remains to show that for every $n \in \mathbb{Z}_{\geqslant 1}$, the map (C_n, f) is injective. For a given $n \in \mathbb{Z}_{\geqslant 1}$, let $u^{(1)}_G, u^{(2)}_G \in (\mathrm{C}_n, G)_{\mathsf{Gph}}$ be such that $u^{(1)}_G \cdot f = u_H = u^{(2)}_G \cdot f$ for some $u_H \in (\mathrm{C}_n, H)_{\mathsf{Gph}}$. We denote $H' = (\mathrm{C}_n)u_H$. We claim that $(\mathrm{C}_n)u^{(1)}_G$, $(\mathrm{C}_n)u^{(2)}_G \leqslant G^{(H')}$. Since $(\mathrm{C}_n)u^{(1)}_G$ is a quasi-cyclic subgraph of G, we may choose $C^{(1)}_1, \cdots, C^{(1)}_{m^{(1)}} \in \mathrm{S_c}(G)$, for some $m^{(1)} \in \mathbb{Z}_{\geqslant 1}$ such that $(\mathrm{C}_n)u^{(1)}_G = \bigcup_{i=1}^{m^{(1)}} C^{(1)}_i$. Since $C^{(1)}_i f \leqslant (\mathrm{C}_n)u^{(1)}_G f = (\mathrm{C}_n)u_H = H'$, we have $(\mathrm{C}_n)u^{(1)}_G = \bigcup_{i=1}^{m^{(1)}} C^{(1)}_i \leqslant G^{(H')}$. Similarly, $(\mathrm{C}_n)u^{(2)}_G \leqslant G^{(H')}$ follows as well. For each $u^{(i)}_G$, $i \in [1, 2]$, there is a unique graph morphism $u^{(i)}_{G^{(H')}} = u^{(i)}_G|^{G^{(H')}}$ in $(\mathrm{C}_n, G^{(H')})_{\mathsf{Gph}}$ such that $u^{(i)}_{G^{(H')}} \cdot \iota^G_{G^{(H')}} = u^{(i)}_G$, and hence $u^{(i)}_{G^{(H')}} \cdot \iota^G_{G^{(H')}} \cdot f = u^{(i)}_G \cdot f = u_H$. Using the commutativity, we also have

$$u^{(1)}_{G^{(H')}} \cdot f^{(H')} \cdot \iota^H_{H'} = u^{(1)}_{G^{(H')}} \cdot \iota^G_{G^{(H')}} \cdot f = u^{(1)}_G \cdot f = u_H = u^{(2)}_G \cdot f = u^{(2)}_{G^{(H')}} \cdot \iota^G_{G^{(H')}} \cdot f = u^{(2)}_{G^{(H')}} \cdot f^{(H')} \cdot \iota^H_{H'}\ .$$

Since $\iota^H_{H'}$ is a monomorphism, we have $u^{(1)}_{G^{(H')}} \cdot f^{(H')} = u^{(2)}_{G^{(H')}} \cdot f^{(H')}$. Since $f^{(H')}$ is a quasi-isomorphism, we have $u^{(1)}_{G^{(H')}} = u^{(2)}_{G^{(H')}}$. Hence $u^{(1)}_G = u^{(1)}_{G^{(H')}} \cdot \iota^G_{G^{(H')}} = u^{(2)}_{G^{(H')}} = \iota^G_{G^{(H')}} = u^{(2)}_G$. This completes the proof of the injectivity of (C_n, f) for every $n \in \mathbb{Z}_{\geqslant 1}$.

So finally, f is a quasi-isomorphism.

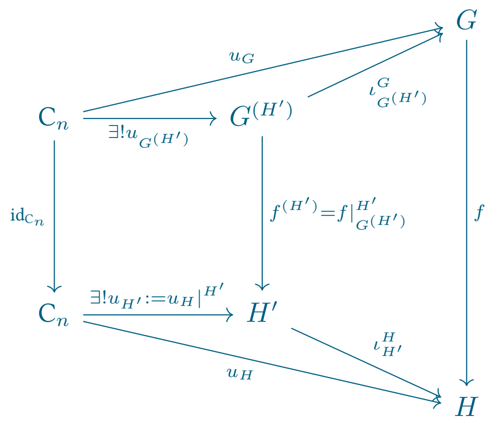


Figure 4.9: Commutative diagram for Lemma 4.17

□

Remark 4.18

In Lemma 4.17, if G is a finite graph and $f : G \to H$ is a quasi-isomorphism, then for every $H' \in \mathrm{S}_{\mathrm{qc}}(H)$, the subgraph $G^{(H')} \leqslant G$ is quasi-cyclic, as will be shown in Remark 4.23 below.

4.4 Lifting results for quasi-isomorphisms of finite graphs

Lemma 4.19

Let $f : G \to H$ be a quasi-isomorphism of graphs, where the graph G is finite. Then for every $C', C'' \in \mathrm{S}_{\mathrm{c}}(G)$ satisfying

$$\mathrm{V}_{(C')f} \cap \mathrm{V}_{(C'')f} \neq \emptyset,$$

there exist paths $p' : \mathrm{D}_x \to G$ and $p'' : \mathrm{D}_y \to G$, for some $x, y \in \mathbb{Z}_{\geqslant 0}$, such that

$$p'_{\mathrm{s}} \in \mathrm{V}_{C'}\,,\ p'_{\mathrm{t}} \in \mathrm{V}_{C''} \quad \text{and} \quad p''_{\mathrm{s}} \in \mathrm{V}_{C''}\,,\ p''_{\mathrm{t}} \in \mathrm{V}_{C'}$$

Proof. Let $N := |\,\mathrm{V}_G\,|$. Since $C', C'' \in \mathrm{S}_{\mathrm{c}}(G)$, we have $N \geqslant 1$. We denote $H' := (C')f$, $H'' := (C'')f$, and choose a vertex $v \in \mathrm{V}_{H'} \cap \mathrm{V}_{H''}$.

Let $M' := N \cdot |\,\mathrm{E}_{C'}\,|$, $M'' := N \cdot |\,\mathrm{E}_{C''}\,|$, and $M := M' + M''$. We first construct a specific graph morphism $u_H \in (\mathrm{C}_M, H)_{\mathrm{Gph}}$. Recall that as f is a quasi-isomorphism, there is a unique $u_G \in (\mathrm{C}_M, G)_{\mathrm{Gph}}$ such that $u_G \cdot f = u_H$. Later, we will show that both C' and C'' share at least one vertex with the image graph $(\mathrm{C}_M)u_G \leqslant G$.

To construct u_H, let $\varphi' \in (\mathrm{C}_{M'}, C')_{\mathrm{Gph}}$ and $\varphi'' \in (\mathrm{C}_{M''}, C'')_{\mathrm{Gph}}$ be such that

$$\mathrm{v}_{0+M'\mathbb{Z}}(\mathrm{V}_{\varphi'} \cdot \mathrm{V}_{\iota^G_{C'}} \cdot \mathrm{V}_f) = v = \mathrm{v}_{0+M''\mathbb{Z}}(\mathrm{V}_{\varphi''} \cdot \mathrm{V}_{\iota^G_{C''}} \cdot \mathrm{V}_f).$$

The graph morphism $u_H : \mathrm{C}_M \to H$ is defined as follows:

$$\begin{array}{rlcll} \mathrm{V}_{u_H}: & \mathrm{V}_{\mathrm{C}_M} & \to & \mathrm{V}_H & \\ & \mathrm{v}_{i+M\mathbb{Z}} & \mapsto & \mathrm{v}_{i+M'\mathbb{Z}}\,\mathrm{V}_{\varphi'}\,\mathrm{V}_f & \text{if } i \in [0, M'] \\ & \mathrm{v}_{i+M\mathbb{Z}} & \mapsto & \mathrm{v}_{i-M'+M''\mathbb{Z}}\,\mathrm{V}_{\varphi''}\,\mathrm{V}_f & \text{if } i \in [M', M] \end{array}$$

$$\begin{array}{rlcll} \mathrm{E}_{u_H}: & \mathrm{E}_{\mathrm{C}_M} & \to & \mathrm{V}_H & \\ & \mathrm{e}_{i+M\mathbb{Z}} & \mapsto & \mathrm{e}_{i+M'\mathbb{Z}}\,\mathrm{E}_{\varphi'}\,\mathrm{E}_f & \text{if } i \in [0, M'-1] \\ & \mathrm{e}_{i+M\mathbb{Z}} & \mapsto & \mathrm{e}_{i-M'+M''\mathbb{Z}}\,\mathrm{E}_{\varphi''}\,\mathrm{E}_f & \text{if } i \in [M', M-1] \end{array}$$

This is possible, since

$$\mathrm{v}_{0+M'\mathbb{Z}}\,\mathrm{V}_{\varphi'}\,\mathrm{V}_f = v = \mathrm{v}_{M-M'+M''\mathbb{Z}}\,\mathrm{V}_{\varphi''}\,\mathrm{V}_f$$

and

$$\mathrm{v}_{M'+M'\mathbb{Z}}\,\mathrm{V}_{\varphi'}\,\mathrm{V}_f = v = \mathrm{v}_{M'-M'+M''\mathbb{Z}}\,\mathrm{V}_{\varphi''}\,\mathrm{V}_f\,.$$

Next, let $u_G \in (\mathrm{C}_M, G)_{\mathrm{Gph}}$ be the unique graph morphism with $u_G \cdot f = u_H$. We prove that $\mathrm{V}_{C'} \cap \mathrm{V}_{(\mathrm{C}_M)u_G} \neq \emptyset$ and $\mathrm{V}_{C''} \cap \mathrm{V}_{(\mathrm{C}_M)u_G} \neq \emptyset$. We first discuss the C' side. We consider a path $p^{(1)} : \mathrm{D}_{M'} \to G$ in G with

$$\begin{array}{rlcl} \mathrm{V}_{p^{(1)}}: & \mathrm{V}_{\mathrm{D}_{M'}} & \to & \mathrm{V}_G \\ & \hat{\mathrm{v}}_i & \mapsto & \mathrm{v}_{i+M\mathbb{Z}}\,\mathrm{V}_{u_G} \end{array} \quad \text{and} \quad \begin{array}{rlcl} \mathrm{E}_{p^{(1)}}: & \mathrm{E}_{\mathrm{D}_{M'}} & \to & \mathrm{E}_G \\ & \hat{\mathrm{e}}_i & \mapsto & \mathrm{e}_{i+M\mathbb{Z}}\,\mathrm{E}_{u_G} \end{array}\,.$$

Then $p^{(1)}$ is also a path in the subgraph $(\mathrm{C}_M)u_G \leqslant G$.

Assume that $\mathrm{V}_{(\mathrm{C}_M)u_G} \cap \mathrm{V}_{C'} = \emptyset$. Then $\mathrm{V}_{(\mathrm{D}_{M'})p^{(1)}} \cap \mathrm{V}_{C'} = \emptyset$. We claim that this implies the set

$$I := \left\{ \hat{\mathrm{v}}_i \, \mathrm{V}_{p^{(1)}} \in \mathrm{V}_G \;\middle|\; i \in [0, M'] \text{ and } i \in |\,\mathrm{E}_{C'}\,| \mathbb{Z} \right\}$$

has cardinality $|I| = N + 1$. Recall that $M' = N \cdot |\,\mathrm{E}_{C'}\,|$. Assume that $\hat{\mathrm{v}}_{i_0} \, \mathrm{V}_{p^{(1)}} = \hat{\mathrm{v}}_{j_0} \, \mathrm{V}_{p^{(1)}}$, where $0 \leqslant i_0 < j_0 \leqslant N \cdot |\,\mathrm{E}_{C'}\,| = M'$ are divisible by $|\,\mathrm{E}_{C'}\,|$. Write

$$Q' := \left(\mathrm{D}_{M'}^{[i_0, j_0]}\right) p^{(1)}.$$

Then $Q' \leqslant G$ is a quasi-cyclic subgraph. Let $k := \dfrac{j_0 - i_0}{|\,\mathrm{E}_{C'}\,|} \in \mathbb{Z}_{\geqslant 1}$; so $k \cdot |\,\mathrm{E}_{C'}\,| = j_0 - i_0$. We construct the graph epimorphism

$$\alpha : \mathrm{C}_{k \cdot |\,\mathrm{E}_{C'}\,|} \to Q'$$

by defining

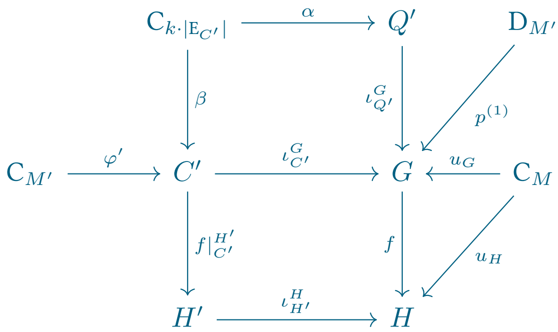
$$\begin{array}{rccc} \mathrm{V}_\alpha : & \mathrm{V}_{\mathrm{C}_{k \cdot |\mathrm{E}_{C'}|}} & \to & \mathrm{V}_{Q'} \\ & \mathrm{v}_{l + k|\mathrm{E}_{C'}|\mathbb{Z}} & \mapsto & \hat{\mathrm{v}}_{l+i_0} \, \mathrm{V}_{p^{(1)}} \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathrm{E}_\alpha : & \mathrm{E}_{\mathrm{C}_{k \cdot |\mathrm{E}_{C'}|}} & \to & \mathrm{E}_{Q'} \\ & \mathrm{e}_{l + k|\mathrm{E}_{C'}|\mathbb{Z}} & \mapsto & \hat{\mathrm{e}}_{l+i_0} \, \mathrm{E}_{p^{(1)}} \end{array} ,$$

for $l \in \mathbb{Z}$. We construct the graph epimorphism

$$\beta : \mathrm{C}_{k \cdot |\,\mathrm{E}_{C'}\,|} \to C'$$

by defining

$$\begin{array}{rccc} \mathrm{V}_\beta : & \mathrm{V}_{\mathrm{C}_{k \cdot |\mathrm{E}_{C'}|}} & \to & \mathrm{V}_{C'} \\ & \mathrm{v}_{l + k|\mathrm{E}_{C'}|\mathbb{Z}} & \mapsto & \mathrm{v}_{l+i_0+M'\mathbb{Z}} \, \mathrm{V}_{\varphi'} \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathrm{E}_\beta : & \mathrm{E}_{\mathrm{C}_{k \cdot |\mathrm{E}_{C'}|}} & \to & \mathrm{E}_{C'} \\ & \mathrm{e}_{l + k|\mathrm{E}_{C'}|\mathbb{Z}} & \mapsto & \mathrm{e}_{l+i_0+M'\mathbb{Z}} \, \mathrm{E}_{\varphi'} \end{array} .$$



We verify that $\alpha \cdot \iota^G_{Q'} \cdot f = \beta \cdot \iota^G_{C'} \cdot f \in \left(\mathrm{C}_{k \cdot |\,\mathrm{E}_{C'}\,|}, H\right)_{\mathrm{Gph}}$. For every $l \in [0, k \cdot |\mathrm{E}_{C'}| - 1]$, since $l + i_0 \in [i_0, j_0 - 1] \subseteq [0, M' - 1]$, we have

$$\begin{aligned} \mathrm{e}_{l + k|\mathrm{E}_{C'}|\mathbb{Z}} \, \mathrm{E}_\alpha \, \mathrm{E}_{\iota^G_{Q'}} \, \mathrm{E}_f &= \hat{\mathrm{e}}_{l+i_0} \, \mathrm{E}_{p^{(1)}} \, \mathrm{E}_{\iota^G_{Q'}} \, \mathrm{E}_f \\ &= \hat{\mathrm{e}}_{l+i_0} \, \mathrm{E}_{p^{(1)}} \, \mathrm{E}_f \\ &= \mathrm{e}_{l+i_0+M\mathbb{Z}} \, \mathrm{E}_{u_G} \, \mathrm{E}_f \\ &= \mathrm{e}_{l+i_0+M\mathbb{Z}} \, \mathrm{E}_{u_H} \\ &= \mathrm{e}_{l+i_0+M'\mathbb{Z}} \, \mathrm{E}_{\varphi'} \, \mathrm{E}_f \\ &= \mathrm{e}_{l+k|\,\mathrm{E}_{C'}\,|\mathbb{Z}} \, \mathrm{E}_\beta \, \mathrm{E}_f = \mathrm{e}_{l+k|\,\mathrm{E}_{C'}\,|\mathbb{Z}} \, \mathrm{E}_\beta \, \mathrm{E}_{\iota^G_{C'}} \, \mathrm{E}_f \end{aligned}$$

and

$$\begin{aligned}
\mathrm{v}_{l+k|\mathrm{E}_{C'}|\mathbb{Z}} \mathrm{V}_\alpha \mathrm{V}_{\iota^G_{Q'}} \mathrm{V}_f &= \hat{\mathrm{v}}_{l+i_0} \mathrm{V}_{p^{(1)}} \mathrm{V}_{\iota^G_{Q'}} \mathrm{V}_f \\
&= \hat{\mathrm{v}}_{l+i_0} \mathrm{V}_{p^{(1)}} \mathrm{V}_f \\
&= \mathrm{v}_{l+i_0+M\mathbb{Z}} \mathrm{V}_{u_G} \mathrm{V}_f \\
&= \mathrm{v}_{l+i_0+M\mathbb{Z}} \mathrm{V}_{u_H} \\
&= \mathrm{v}_{l+i_0+M'\mathbb{Z}} \mathrm{V}_{\varphi'} \mathrm{V}_f \\
&= \mathrm{v}_{l+k|\,\mathrm{E}_{C'}|\mathbb{Z}} \mathrm{V}_\beta \mathrm{V}_f = \mathrm{v}_{l+k|\,\mathrm{E}_{C'}|\mathbb{Z}} \mathrm{V}_\beta \mathrm{V}_{\iota^G_{C'}} \mathrm{V}_f \ .
\end{aligned}$$

Hence $\alpha \cdot \iota^G_{Q'} \cdot f = \beta \cdot \iota^G_{C'} \cdot f$. Since $\left(\mathrm{C}_{k\cdot|\,\mathrm{E}_{C'}|}\right)\left(\alpha \cdot \iota^G_{Q'}\right) = Q' = \left(\mathrm{D}^{[i_0,j_0]}_{M'}\right) p^{(1)} \leqslant \left(\mathrm{D}_{M'}\right) p^{(1)}$ and $\left(\mathrm{C}_{k\cdot|\,\mathrm{E}_{C'}|}\right)\left(\beta \cdot \iota^G_{C'}\right) = C'$, and since $\mathrm{V}_{(\mathrm{D}_{M'})p^{(1)}} \cap \mathrm{V}_{C'} = \emptyset$, we have

$$\alpha \cdot \iota^G_{Q'} \neq \beta \cdot \iota^G_{C'} \,.$$

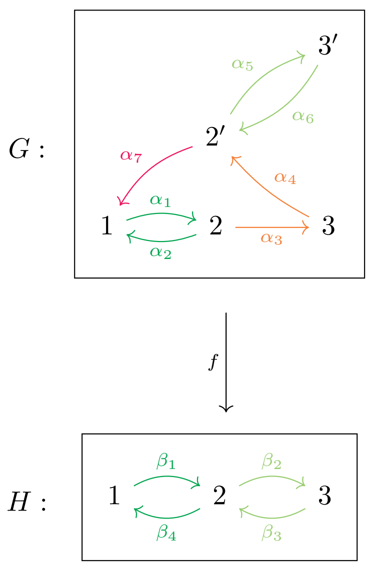
This contradicts f being a quasi-isomorphism. Hence $|I| = N+1$. Recall that $N = |\,\mathrm{V}_G\,|$, which is less than $|I| = N+1$; so such a subset $I \subseteq \mathrm{V}_G$ can not exist. This is a *contradiction*. Therefore, we have $\mathrm{V}_{(\mathrm{C}_M)u_G} \cap \mathrm{V}_{C'} \neq \emptyset$. On the C'' side, $\mathrm{V}_{(\mathrm{C}_M)u_G} \cap \mathrm{V}_{C''} \neq \emptyset$ follows analogously by considering the path $p^{(2)} : \mathrm{D}_{M''} \to G$ in G with

$$\begin{array}{rccc} \mathrm{V}_{p^{(2)}} : & \mathrm{V}_{\mathrm{D}_{M''}} & \to & \mathrm{V}_G \\ & \hat{\mathrm{v}}_i & \mapsto & \mathrm{v}_{i+M'+M\mathbb{Z}} \mathrm{V}_{u_G} \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathrm{E}_{p^{(2)}} : & \mathrm{E}_{\mathrm{D}_{M''}} & \to & \mathrm{E}_G \\ & \hat{\mathrm{e}}_i & \mapsto & \mathrm{e}_{i+M'+M\mathbb{Z}} \mathrm{E}_{u_G} \end{array} .$$

Now, let $v' \in \mathrm{V}_{(\mathrm{C}_M)u_G} \cap \mathrm{V}_{C'}$ and $v'' \in \mathrm{V}_{(\mathrm{C}_M)u_G} \cap \mathrm{V}_{C''}$. Since $(\mathrm{C}_M)\,u_G$ is a quasi-cyclic subgraph of G, it is also strongly connected by Lemma 3.24. Hence, there exist paths $p' : \mathrm{D}_x \to G$ and $p'' : \mathrm{D}_y \to G$ such that $p'_{\mathrm{s}} = v'$, $p'_{\mathrm{t}} = v''$ and $p''_{\mathrm{s}} = v''$, $p''_{\mathrm{t}} = v'$, completing the proof. □

Example 4.20 **(For Lemma 4.19)**

We provide an example to illustrate the result of Lemma 4.19. Consider the quasi-isomorphism $f : G \to H$ given in the Remark 4.16.



Let C' be the cyclic subgraph of G with $\mathrm{V}_{C'} = \{1, 2\}$. Let C'' be the cyclic subgraph of G with $\mathrm{V}_{C''} = \{3', 2'\}$. The image graphs $(C')f, (C'')f \leqslant H$ have a common vertex $2 \in \mathrm{V}_H$. Although C' and C''

are disjoint subgraphs of G, there exist paths

$$p'\colon\ \mathrm{D}_2 \to G \quad \text{from } C' \text{ to } C'', \qquad \text{and} \qquad p''\colon\ \mathrm{D}_1 \to G \quad \text{from } C'' \text{ to } C',$$

such that the image graphs are

$$(\mathrm{D}_2)p' = (\{2,3,2'\}, \{\alpha_3, \alpha_4\})\,, \quad \text{and} \quad (\mathrm{D}_1)p'' = (\{2',1\}, \{\alpha_7\})\,.$$

Corollary 4.21

Let G be a finite graph and $f : G \to H$ be a quasi-isomorphism. Let $G', G'' \leqslant G$ be quasi-cyclic subgraphs such that

$$\mathrm{V}_{(G')f} \cap \mathrm{V}_{(G'')f} \neq \emptyset.$$

Then there exists a quasi-cyclic subgraph $\hat{G} \leqslant G$ such that $G', G'' \leqslant \hat{G}$.

Proof. Let $v_H \in \mathrm{V}_{(G')f} \cap \mathrm{V}_{(G'')f}$. Since G' and G'' can be covered by cyclic subgraphs, we may choose $C' \leqslant \mathrm{S_c}(G')$ and $C'' \leqslant \mathrm{S_c}(G'')$ such that $v_H \in \mathrm{V}_{(C')f} \cap \mathrm{V}_{(C'')f}$. By Lemma 4.19, we may choose paths $p' : \mathrm{D}_x \to G$ and $p'' : \mathrm{D}_y \to G$, with $x, y \in \mathbb{Z}_{\geqslant 0}$, such that

$$p'_\mathrm{s} \in \mathrm{V}_{C'} \subseteq \mathrm{V}_{G'},\ p'_\mathrm{t} \in \mathrm{V}_{C''} \subseteq \mathrm{V}_{G''} \quad \text{and} \quad p''_\mathrm{s} \in \mathrm{V}_{C''} \subseteq \mathrm{V}_{G''},\ p''_\mathrm{t} \in \mathrm{V}_{C'} \subseteq \mathrm{V}_{G'}\,.$$

By Lemma 3.24, there also exist paths $p^{(G')} : \mathrm{D}_{n'} \to G'$ and $p^{(G'')} : \mathrm{D}_{n''} \to G''$, with $n', n'' \in \mathbb{Z}_{\geqslant 0}$ such that

$$p^{(G')}_\mathrm{s} = p''_\mathrm{t},\ p^{(G')}_\mathrm{t} = p'_\mathrm{s} \quad \text{and} \quad p^{(G'')}_\mathrm{s} = p'_\mathrm{t},\ p^{(G'')}_\mathrm{t} = p''_\mathrm{s}.$$

Note that $(\mathrm{D}_{n'})p^{(G')} \cup (\mathrm{D}_x)p' \cup (\mathrm{D}_y)p'' \cup (\mathrm{D}_{n''})p^{(G'')}$ is a quasi-cyclic subgraph of G. We define

$$\hat{G} := G' \cup ((\mathrm{D}_{n'})p^{(G')} \cup (\mathrm{D}_x)p' \cup (\mathrm{D}_y)p'' \cup (\mathrm{D}_{n''})p^{(G'')}) \cup G''.$$

It follows from Corollary 3.23 that $\hat{G}$ is quasi-cyclic subgraph of G that satisfies $G', G'' \leqslant \hat{G}$. □

Proposition 4.22

Let G, H be finite graphs and $f : G \to H$ be a graph morphism. The following are equivalent:

(1) f is a quasi-isomorphism.

(2) The map

$$\begin{array}{rccc} \hat{f}: & \mathrm{S}^{\max}_{\mathrm{qc}}(G) & \to & \mathrm{S}^{\max}_{\mathrm{qc}}(H) \\ & \hat{G} & \mapsto & (\hat{G})f \end{array}$$

is bijective, and for each $\hat{G} \in \mathrm{S}^{\max}_{\mathrm{qc}}(G)$, the restriction $f|^{(\hat{G})f}_{\hat{G}}$ is a quasi-isomorphism.

Proof. *Ad* $(1) \Rightarrow (2)$: Suppose that f is a quasi-isomorphism.

We begin by proving that $\hat{f}$ is well-defined, that is, for every $\hat{G} \in \mathrm{S}^{\max}_{\mathrm{qc}}(G)$, we have $(\hat{G})f \in \mathrm{S}^{\max}_{\mathrm{qc}}(H)$. By Remark 3.11, $(\hat{G})f$ is quasi-cyclic. It remains to show the maximality of $(\hat{G})f$ as a quasi-cyclic subgraph in H. *Assume* that $(\hat{G})f \notin \mathrm{S}^{\max}_{\mathrm{qc}}(H)$. Then there exists $\hat{H} \in \mathrm{S_{qc}}(H)$ such that $(\hat{G})f < \hat{H}$. This implies that $\mathrm{E}_{(\hat{G})f} \subsetneq \mathrm{E}_{\hat{H}}$, for otherwise, due to Lemma 3.9 we have

$$\mathrm{V}_{(\hat{G})f} = \mathrm{E}_{(\hat{G})f}\, \mathrm{s}_H = \mathrm{E}_{\hat{H}}\, \mathrm{s}_H = \mathrm{V}_{\hat{H}}\ ,$$

which indicates $(\hat{G})f = \hat{H}$. This is not the case. Then for a given $e \in \mathrm{E}_{\hat{H}} \setminus \mathrm{E}_{(\hat{G})f}$, there is a path $p^{(e)} : \mathrm{D}_m \to \hat{H}$, with $m \in \mathbb{Z}_{\geqslant 1}$, such that $p^{(e)}_\mathrm{s} \in \mathrm{V}_{(\hat{G})f}$ and $p^{(e)}_\mathrm{t} = e\,\mathrm{s}_H$. If $(\mathrm{D}_m)p^{(e)} \leqslant (\hat{G})f$, we retain the notation e

for this given edge; otherwise, we replace e with the edge $\hat{\mathrm{e}}_i\,\mathrm{E}_{p^{(e)}}$, where i is minimal in $[0, m-1]$ with $\hat{\mathrm{e}}_i\,\mathrm{E}_{p^{(e)}} \notin \mathrm{E}_{(\hat{G})f}$. Then $e \in \mathrm{E}_{\hat{H}} \setminus \mathrm{E}_{(\hat{G})f}$ is an edge with $e\,\mathrm{s}_H \in \mathrm{V}_{(\hat{G})f}$. Since $\hat{H} \in \mathrm{S}_{\mathrm{qc}}(H)$ can be covered by its cyclic subgraphs, we may choose $C_H^{(e)} \in \mathrm{S}_{\mathrm{c}}(\hat{H})$ such that $e \in \mathrm{E}_{C_H^{(e)}}$. By Proposition 4.12, there is a unique $C_G^{(e)} \in \mathrm{S}_{\mathrm{c}}(G)$ such that $(C_G^{(e)})f = C_H^{(e)}$. Since $e \in \mathrm{E}_{(C_G^{(e)})f} \setminus \mathrm{E}_{(\hat{G})f}$, we have $(C_G^{(e)})f \not\leqslant (\hat{G})f$, and thus $C_G^{(e)} \not\leqslant \hat{G}$. Then $\hat{G}, C_G^{(e)} \leqslant G$ are quasi-cyclic subgraphs with $e\,\mathrm{s}_H \in \mathrm{V}_{(\hat{G})f} \cap \mathrm{V}_{(C_G^{(v)})f}$. By Corollary 4.21, there exists $\hat{\hat{G}} \in \mathrm{S}_{\mathrm{qc}}(G)$ such that $\hat{G} \leqslant \hat{\hat{G}}$ and $C_G^{(e)} \leqslant \hat{\hat{G}}$. Since $C_G^{(e)} \not\leqslant \hat{G}$, we have $\hat{G} < \hat{\hat{G}}$. This *contradicts* $\hat{G}$ being maximal quasi-cyclic in G. Therefore, $\hat{G}\hat{f} \in \mathrm{S}_{\mathrm{qc}}^{\max}(H)$.

Next, we prove the bijectivity of $\hat{f}$.

Injectivity of f: Let $\hat{G}', \hat{G}'' \in \mathrm{S}_{\mathrm{qc}}^{\max}(G)$ such that $\hat{G}'\hat{f} = (\hat{G}')f = (\hat{G}'')f = \hat{G}''\hat{f}$. Then, by Corollary 4.21, there is a $\hat{\hat{G}} \in \mathrm{S}_{\mathrm{qc}}(G)$ such that $\hat{G}', \hat{G}'' \leqslant \hat{\hat{G}}$. The maximality of $\hat{G}', \hat{G}''$ implies that $\hat{G}' = \hat{\hat{G}} = \hat{G}''$. Hence $\hat{f}$ is injective.

Surjectivity of $\hat{f}$: Let $\hat{H} \in \mathrm{S}_{\mathrm{qc}}^{\max}(H)$. It follows from Lemma 4.14 that there exists $\hat{G} \in \mathrm{S}_{\mathrm{qc}}(G)$ such that $(\hat{G})f = \hat{H}$. Since G is finite, there exists $\hat{\hat{G}} \in \mathrm{S}_{\mathrm{qc}}^{\max}(G)$ such that $\hat{G} \leqslant \hat{\hat{G}}$. Then we have $\hat{\hat{G}}\hat{f} \in \mathrm{S}_{\mathrm{qc}}^{\max}(H)$ and $\hat{H} = (\hat{G})f \leqslant (\hat{\hat{G}})f = \hat{\hat{G}}\hat{f}$. Hence $\hat{H} = \hat{\hat{G}}\hat{f}$; and $\hat{f}$ is surjective.

Now, let $\hat{G} \in \mathrm{S}_{\mathrm{qc}}^{\max}(G)$. We show that $f|_{\hat{G}}^{(\hat{G})f}$ is a quasi-isomorphism. Denote $\hat{H} := (\hat{G})f$. We claim that

$$G^{(\hat{H})} := \bigcup \left\{ C \in \mathrm{S}_{\mathrm{c}}(G) \;:\; (C)f \leqslant \hat{H} \right\}$$

is a subgraph of $\hat{G}$. By Corollary 4.21, for every $C \in \mathrm{S}_{\mathrm{c}}(G)$ with $(C)f \leqslant \hat{H}$, there exists $\hat{\hat{G}} \in \mathrm{S}_{\mathrm{qc}}(G)$ such that $\hat{G}, C \leqslant \hat{\hat{G}}$. Due to the maximality of $\hat{G}$, we have $C \leqslant \hat{\hat{G}} = \hat{G}$. Hence $G^{(\hat{H})}$, being the union of such C, is a subgraph of $\hat{G}$. Since every $C_{\hat{G}} \in \mathrm{S}_{\mathrm{c}}(\hat{G})$ is a cyclic subgraph of $G^{(\hat{H})}$ as well, we have $\hat{G} = G^{(\hat{H})}$. By Lemma 4.17, the restriction $f|_{\hat{G}}^{\hat{H}} = f|_{G^{(\hat{H})}}^{\hat{H}}$ is a quasi-isomorphism.

Ad (2) $\Leftarrow$ (1): We show that for every $n \in \mathbb{Z}_{\geqslant 1}$, the map (C_n, f) is bijective.

Surjectivity of (C_n, f): Let $u_H : \mathrm{C}_n \to H$ be a graph morphism. Then $(\mathrm{C}_n)u_H$ is a quasi-cyclic subgraph in H. Since H is finite, there exists $\hat{H} \in \mathrm{S}_{\mathrm{qc}}^{\max}(H)$ such that $(\mathrm{C}_n)u_H \leqslant \hat{H}$. Let $\hat{G} \in \mathrm{S}_{\mathrm{qc}}^{\max}(G)$ with $\hat{G}\hat{f} = \hat{H}$, using the bijectivity of $\hat{f}$. Since $f|_{\hat{G}}^{\hat{H}}$ is a quasi-isomorphism, there is a $u_{\hat{G}} \in (\mathrm{C}_n, \hat{G})_{\mathrm{Gph}}$ such that $u_{\hat{G}} \cdot f|_{\hat{G}}^{\hat{H}} = u_H|^{\hat{H}}$. Then we have $u_{\hat{G}} \cdot \iota_{\hat{G}}^{G} \cdot f = u_{\hat{G}} \cdot f|_{\hat{G}}^{\hat{H}} \cdot \iota_{\hat{H}}^{H} = u_H|^{\hat{H}} \cdot \iota_{\hat{H}}^{H} = u_H$. Write $u_G := u_{\hat{G}} \cdot \iota_{\hat{G}}^{G}$. Then u_G is a graph morphism in $(\mathrm{C}_n, G)_{\mathrm{Gph}}$ satisfying $u_G f = u_H$. Hence (C_n, f) is surjective.

Injectivity of (C_n, f): Let $u_G^{(1)}, u_G^{(2)} \in (\mathrm{C}_n, G)_{\mathrm{Gph}}$ such that $u_G^{(1)} \cdot f = u_G^{(2)} \cdot f$. Then, by Corollary 4.21, and since G is finite, there exist $\hat{G} \in \mathrm{S}_{\mathrm{qc}}^{\max}(G)$ such that $(\mathrm{C}_n)u_G^{(1)}, (\mathrm{C}_n)u_G^{(2)} \leqslant \hat{G}$. Write $\hat{H} := (\hat{G})\hat{f}$. It follows that

$$u_G^{(1)}|^{\hat{G}} \cdot f|_{\hat{G}}^{\hat{H}} \cdot \iota_{\hat{H}}^{H} = u_G^{(1)} \cdot f = u_G^{(2)} \cdot f = u_G^{(2)}|^{\hat{G}} \cdot f|_{\hat{G}}^{\hat{H}} \cdot \iota_{\hat{H}}^{H}\,.$$

Since $\iota_{\hat{H}}^{H}$ is a monomorphism, we have $u_G^{(1)}|^{\hat{G}} \cdot f|_{\hat{G}}^{\hat{H}} = u_G^{(2)}|^{\hat{G}} \cdot f|_{\hat{G}}^{\hat{H}}$. Since $f|_{\hat{G}}^{\hat{H}}$ is a quasi-isomorphism, we have $u_G^{(1)}|^{\hat{G}} = u_G^{(2)}|^{\hat{G}}$. Hence $u_G^{(1)} = u_G^{(1)}|^{\hat{G}} \cdot \iota_{\hat{G}}^{G} = u_G^{(2)}|^{\hat{G}} \cdot \iota_{\hat{G}}^{G} = u_G^{(2)}$, and hence (C_n, f) is injective.

Therefore, f is a quasi-isomorphism. □

Remark 4.23

In the situation of Lemma 4.17, suppose that G is finite and that $f : G \to H$ is a quasi-isomorphism. Then for every $H' \in \mathrm{S}_{\mathrm{qc}}(H)$, the subgraph $G^{(H')} \leqslant G$ is quasi-cyclic.

Proof. By Lemma 4.17, $f|_{G^{(H')}}^{H'}$ is a quasi-isomorphism. We remark that the unique maximal quasi-cyclic subgraph of H' is H' itself; so by Proposition 4.22, $G^{(H')}$ also has only one maximal quasi-cyclic subgraph, and we denote this as $\hat{G} \leqslant G^{(H')}$. This implies every $C \in \mathrm{S}_{\mathrm{c}}(G^{(H')})$ is a subgraph of $\hat{G}$. Hence by definition of $G^{(H')}$, we have $C \leqslant \hat{G}$ for every $C \in \mathrm{S_c}(G)$ with $(C)f \leqslant H'$. Hence $G^{(H')} \leqslant \hat{G}$. It follows that $G^{(H')} = \hat{G}$, which indicates $G^{(H')}$ is a quasi-cyclic graph. □

4.5 A characterization of quasi-isomorphisms between finite graphs

In Proposition 4.22, we have proved that to verify a graph morphism $f : G \to H$ between finite graphs is a quasi-isomorphism, a key step is to verify for every maximal quasi-cyclic subgraph $\hat{G}$ in G, the restriction $f|_{\hat{G}}^{(\hat{G})f} : \hat{G} \to (\hat{G})f$ is a quasi-isomorphism. In this section, we provide a method for a verification whether a graph morphisms from a quasi-cyclic graph is a quasi-isomorphism.

To begin with, we need the notion of a pullback for the later construction.

Definition 4.24 (Pullback)
Let X_1, X_2, Y be graphs, and $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ be graph morphisms. In the category **Gph**, a graph P, together with graph morphisms $g_1 : P \to X_1$ and $g_2 : P \to X_2$, is called a *pullback* of f_1, f_2, or of the diagram

$$\begin{array}{ccc} & & X_1 \\ & & \downarrow{\scriptstyle f_1} \\ X_2 & \xrightarrow{f_2} & Y \end{array} \quad ,$$

if the following conditions hold:

(1) We have $g_1 f_1 = g_2 f_2$, i.e., the following square is commutative:

$$\begin{array}{ccc} P & \xrightarrow{g_1} & X_1 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\ X_2 & \xrightarrow{f_2} & Y \end{array}$$

(2) For every graph W and graph morphisms $w_1 : W \to X_1$, $w_2 : W \to X_2$ such that $w_1 f_1 = w_2 f_2$, there exists a unique graph morphism $\omega : W \to P$ such that $w_1 = \omega g_1$ and $w_2 = \omega g_2$.

$$\begin{array}{cccc} W & & \xrightarrow{\quad w_1 \quad} & \\ & \overset{\exists!\omega}{\searrow} & & \searrow \\ & P & \xrightarrow{g_1} & X_1 \\ {\scriptstyle w_2}\searrow & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\ & X_2 & \xrightarrow{f_2} & Y \end{array}$$

Given graphs G, H and a graph morphism $f : G \to H$, we construct a graph

$$P_f^{(G)} = \left(\mathrm{V}_{P_f^{(G)}}, \mathrm{E}_{P_f^{(G)}}; \mathrm{s}_{P_f^{(G)}}, \mathrm{t}_{P_f^{(G)}}\right) , \tag{4.3}$$

where

$$\mathrm{V}_{P_f^{(G)}} := \left\{(v^+, v^-) \in \mathrm{V}_G \times \mathrm{V}_G \ : \ v^+ \,\mathrm{V}_f = v^- \,\mathrm{V}_f\right\} ;$$
$$\mathrm{E}_{P_f^{(G)}} := \left\{(e^+, e^-) \in \mathrm{E}_G \times \mathrm{E}_G \ : \ e^+ \,\mathrm{E}_f = e^- \,\mathrm{E}_f\right\} ;$$

and where for every $(e^+, e^-) \in \mathrm{E}_{P_f^{(G)}}$, we set

$$\mathrm{s}_{P_f^{(G)}} : (e^+, e^-) \longmapsto (e^+ \,\mathrm{s}_G, e^- \,\mathrm{s}_G)\,;$$
$$\mathrm{t}_{P_f^{(G)}} : (e^+, e^-) \longmapsto (e^+ \,\mathrm{t}_G, e^- \,\mathrm{t}_G)\,.$$

We have $(e^+ \,\mathrm{s}_G, e^- \,\mathrm{s}_G), (e^+ \,\mathrm{t}_G, e^- \,\mathrm{t}_G) \in \mathrm{V}_{P_f^{(G)}}$, since $e^+ \,\mathrm{s}_G \,\mathrm{V}_f = e^+ \,\mathrm{E}_f \,\mathrm{s}_H = e^- \,\mathrm{E}_f \,\mathrm{s}_H = e^- \,\mathrm{s}_G \,\mathrm{V}_f$, and the same argument applies to $(e^+ \,\mathrm{t}_G, e^- \,\mathrm{t}_G)$.

We may also construct graph morphisms $g_+, g_- : P_f^{(G)} \to G$ such that the diagram

$$\begin{array}{ccc} P_f^{(G)} & \xrightarrow{g_+} & G \\ \Big\downarrow{\scriptstyle g_-} & & \Big\downarrow{\scriptstyle f} \\ G & \xrightarrow{f} & H \end{array} \tag{4.4}$$

is commutative. The graph morphism $g_+ = (\mathrm{V}_{g_+}, \mathrm{E}_{g_+})$ is given by

$$\mathrm{V}_{g_+} : \begin{array}{ccc} \mathrm{V}_{P_f^{(G)}} & \longrightarrow & \mathrm{V}_G \\ (v^+, v^-) & \longmapsto & v^+ \end{array} \quad \text{and} \quad \mathrm{E}_{g_+} : \begin{array}{ccc} \mathrm{E}_{P_f^{(G)}} & \longrightarrow & \mathrm{E}_G \\ (e^+, e^-) & \longmapsto & e^+ \end{array} .$$

The graph morphism $g_- = (\mathrm{V}_{g_-}, \mathrm{E}_{g_-})$ is given by

$$\mathrm{V}_{g_-} : \begin{array}{ccc} \mathrm{V}_{P_f^{(G)}} & \longrightarrow & \mathrm{V}_G \\ (v^+, v^-) & \longmapsto & v^- \end{array} \quad \text{and} \quad \mathrm{E}_{g_-} : \begin{array}{ccc} \mathrm{E}_{P_f^{(G)}} & \longrightarrow & \mathrm{E}_G \\ (e^+, e^-) & \longmapsto & e^- \end{array} .$$

The graph morphisms g_+, g_- are well-defined, since for every $(e^+, e^-) \in \mathrm{E}_{P_f^{(G)}}$, we have

$$(e^+, e^-)\,\mathrm{E}_{g_+} \,\mathrm{s}_G = e^+ \,\mathrm{s}_G = (e^+ \,\mathrm{s}_G, e^- \,\mathrm{s}_G)\,\mathrm{V}_{g_+} = (e^+, e^-)\,\mathrm{s}_{P_f^{(G)}} \,\mathrm{V}_{g_+}\,;$$
$$(e^+, e^-)\,\mathrm{E}_{g_+} \,\mathrm{t}_G = e^+ \,\mathrm{t}_G = (e^+ \,\mathrm{t}_G, e^- \,\mathrm{t}_G)\,\mathrm{V}_{g_+} = (e^+, e^-)\,\mathrm{t}_{P_f^{(G)}} \,\mathrm{V}_{g_+}\,;$$

and since $(e^+, e^-)\,\mathrm{E}_{g_-} \,\mathrm{s}_G = (e^+, e^-)\,\mathrm{s}_{P_f^{(G)}} \,\mathrm{V}_{g_-}$, as well as $(e^+, e^-)\,\mathrm{E}_{g_-} \,\mathrm{t}_G = (e^+, e^-)\,\mathrm{t}_{P_f^{(G)}} \,\mathrm{V}_{g_-}$ can be verified analogously.

Our construction employs the method given by J. Hess in **[4,** Construction 97**]**, and Hess has proved that the square (4.4) obtained in this way is a pullback.

Proposition 4.25

Let $f : G \to H$ be a graph morphism between finite graphs, where G is quasi-cyclic. The following conditions (1) and (2) are equivalent:

(1) The map $(\mathrm{C}_n, f) : (\mathrm{C}_n, G)_{\mathrm{Gph}} \to (\mathrm{C}_n, H)_{\mathrm{Gph}}$ is injective for every $n \in \mathbb{Z}_{\geqslant 1}$.

(2) (i) For $e_1, e_2 \in \mathrm{E}_G$ with$e_1 \neq e_2$, $e_1 \,\mathrm{s}_G = e_2 \,\mathrm{s}_G$ and $e_1 \,\mathrm{t}_G = e_2 \,\mathrm{t}_G$, we have $e_1 \,\mathrm{E}_f \neq e_2 \,\mathrm{E}_f$.

(ii) For every $C \in \mathrm{S_c}(P_f^{(G)})$ and every $(v^+, v^-) \in \mathrm{V}_{P_f^{(G)}}$ with $v^+ \neq v^-$, we have $(v^+, v^-) \notin \mathrm{V}_C$.

Proof. Suppose that (C_n, f) is injective for every $n \in \mathbb{Z}_{\geqslant 1}$. We *assume* that (2) does not hold.

If (2).(i) does not hold, then we may choose $e_1, e_2 \in \mathrm{E}_G$, with $e_1 \neq e_2$, such that $e_1 \,\mathrm{s}_G = e_2 \,\mathrm{s}_G$, $e_1 \,\mathrm{t}_G = e_2 \,\mathrm{t}_G$ and $e_1 \,\mathrm{E}_f = e_2 \,\mathrm{E}_f$. We denote the source vertex of e_1, e_2 as v and the target vertex as v'.

Since G is quasi-cyclic, thus strongly connected, there is a path $p : \mathrm{D}_m \to G$, for some $m \in \mathbb{Z}_{\geqslant 0}$, such that $p_\mathrm{s} = v'$ and $p_\mathrm{t} = v$; cf. 3.24. We consider graph morphisms $\varphi_1, \varphi_2 : \mathrm{C}_{m+1} \to G$ given by

$$\varphi_1 : \left\{ \begin{array}{lcclr} \mathrm{V}_{\varphi_1} : & \mathrm{V}_{\mathrm{C}_{m+1}} & \to & \mathrm{V}_G & \\ & \mathrm{v}_{i+(m+1)\mathbb{Z}} & \mapsto & \hat{\mathrm{v}}_i \, \mathrm{V}_p & \text{for } i \in [0, m] \\ \mathrm{E}_{\varphi_1} : & \mathrm{E}_{\mathrm{C}_{m+1}} & \to & \mathrm{E}_G & \\ & \mathrm{e}_{i+(m+1)\mathbb{Z}} & \mapsto & \hat{\mathrm{e}}_i \, \mathrm{E}_p & \text{for } i \in [0, m-1] \\ & \mathrm{e}_{i+(m+1)\mathbb{Z}} & \mapsto & e_1 & \text{for } i = m \end{array} \right\}$$

and

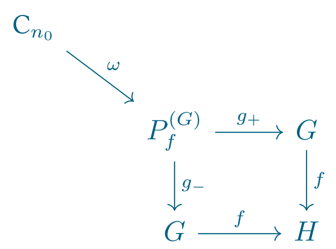
$$\varphi_2 : \left\{ \begin{array}{lcclr} \mathrm{V}_{\varphi_2} : & \mathrm{V}_{\mathrm{C}_{m+1}} & \to & \mathrm{V}_G & \\ & \mathrm{v}_{i+(m+1)\mathbb{Z}} & \mapsto & \hat{\mathrm{v}}_i \, \mathrm{V}_p & \text{for } i \in [0, m] \\ \mathrm{E}_{\varphi_2} : & \mathrm{E}_{\mathrm{C}_{m+1}} & \to & \mathrm{E}_G & \\ & \mathrm{e}_{i+(m+1)\mathbb{Z}} & \mapsto & \hat{\mathrm{e}}_i \, \mathrm{E}_p & \text{for } i \in [0, m-1] \\ & \mathrm{e}_{i+(m+1)\mathbb{Z}} & \mapsto & e_2 & \text{for } i = m \end{array} \right\} .$$

Since φ_1 and φ_2 differ only at $\mathrm{e}_{m+(m+1)\mathbb{Z}} \in \mathrm{E}_{\mathrm{C}_{m+1}}$, and since $\mathrm{e}_{m+(m+1)\mathbb{Z}} \, \mathrm{E}_{\varphi_1} \, \mathrm{E}_f = e_1 \, \mathrm{E}_f = e_2 \, \mathrm{E}_f = \mathrm{e}_{m+(m+1)\mathbb{Z}} \, \mathrm{E}_{\varphi_2}$, we have $\varphi_1 \cdot f = \varphi_2 \cdot f$. This is impossible, since (C_{m+1}, f) is injective. Therefore, (2).(i) must hold.

If (2).(ii) does not hold, we may choose $C \in \mathrm{S}_\mathrm{c}(P_f^{(G)})$ that possesses a vertex $(v^+, v^-) \in \mathrm{V}_C$ with $v^+ \neq v^-$ in V_G. Suppose that $C \simeq \mathrm{C}_{n_0}$, with $n_0 \in \mathbb{Z}_{\geqslant 1}$; so we may choose a graph morphism

$$\omega : \mathrm{C}_{n_0} \to P_f^{(G)}$$

such that $(\mathrm{C}_{n_0})\omega = C$, such that $\mathrm{V}_{0+n_0\mathbb{Z}} \, \mathrm{V}_\omega = (v^+, v^-)$ and such that $\omega|^C$ is an isomorphism.



Then we have $\omega \cdot g_+, \omega \cdot g_- \in (\mathrm{C}_{n_0}, G)_{\mathrm{Gph}}$. Since the square (4.4) is commutative, we have

$$(\omega \cdot g_+) \cdot f = (\omega \cdot g_-) \cdot f.$$

However, we also have $\mathrm{v}_{0+n_0\mathbb{Z}} \, \mathrm{V}_\omega \, \mathrm{V}_{g_+} = v^+ \neq v^- = \mathrm{v}_{0+n_0\mathbb{Z}} \, \mathrm{V}_\omega \, \mathrm{V}_{g_-}$, which yields $\omega \cdot g_+ \neq \omega \cdot g_-$. This is impossible, since (C_{m+1}, f) is injective. Therefore, (2).(ii) must hold.

Since both (2).(i) and (2).(ii) hold, we obtain a *contradiction*.

Now, we show the converse direction. Suppose that (2) is true. We derive that (C_n, f) is injective for every $n \in \mathbb{Z}_{\geqslant 1}$. *Assume* that (C_{n_*}, f) is not injective for some $n_* \in \mathbb{Z}_{\geqslant 1}$; so we may choose $w_+, w_- : \mathrm{C}_{n_*} \to G$ with $w_+ \neq w_-$ such that $w_+ f = w_- f$. We proceed by distinguishing two cases depending on whether $\mathrm{V}_{w_+}, \mathrm{V}_{w_-}$ are equal.

Case $\mathrm{V}_{w_+} = \mathrm{V}_{w_-}$: To ensure $w_+ \neq w_-$, we must have $\mathrm{E}_{w_+} \neq \mathrm{E}_{w_-}$; so we may choose $i \in [0, n_*]$ with $\mathrm{e}_{i+n_*\mathbb{Z}} \, \mathrm{E}_{w_+} \neq \mathrm{e}_{i+n_*\mathbb{Z}} \, \mathrm{E}_{w_-} \in \mathrm{E}_G$. The edges $\mathrm{e}_{i+n_*\mathbb{Z}} \, \mathrm{E}_{w_+}$ and $\mathrm{e}_{i+n_*\mathbb{Z}} \, \mathrm{E}_{w_-}$ have the same source and target vertices, since

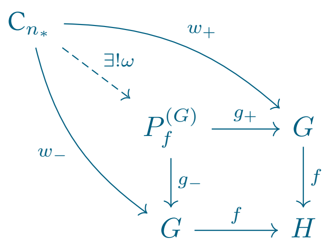
$$\mathrm{e}_{i+n_*\mathbb{Z}} \, \mathrm{E}_{w_+} \, \mathrm{s}_G = \mathrm{e}_{i+n_*\mathbb{Z}} \, \mathrm{s}_{\mathrm{C}_{n_*}} \, \mathrm{V}_{w_+} = \mathrm{e}_{i+n_*\mathbb{Z}} \, \mathrm{s}_{\mathrm{C}_{n_*}} \, \mathrm{V}_{w_-} = \mathrm{e}_{i+n_*\mathbb{Z}} \, \mathrm{E}_{w_-} \, \mathrm{s}_G$$

and

$$\mathrm{e}_{i+n_*\mathbb{Z}}\,\mathrm{E}_{w_+}\,\mathrm{t}_G = \mathrm{e}_{i+n_*\mathbb{Z}}\,\mathrm{t}_{\mathrm{C}_{n_*}}\,\mathrm{V}_{w_+} = \mathrm{e}_{i+n_*\mathbb{Z}}\,\mathrm{t}_{\mathrm{C}_{n_*}}\,\mathrm{V}_{w_-} = \mathrm{e}_{i+n_*\mathbb{Z}}\,\mathrm{E}_{w_-}\,\mathrm{t}_G \ .$$

However, we have $(\mathrm{e}_{i+n_*\mathbb{Z}}\,\mathrm{E}_{w_+})\,\mathrm{E}_f = \mathrm{e}_{i+n_*\mathbb{Z}}\,\mathrm{E}_{w_+f} = \mathrm{e}_{i+n_*\mathbb{Z}}\,\mathrm{E}_{w_-f} = (\mathrm{e}_{i+n_*\mathbb{Z}}\,\mathrm{E}_{w_-})\,\mathrm{E}_f$. The condition (2).(i) ensures that this case does not occur.

Case $\mathrm{V}_{w_+} \neq \mathrm{V}_{w_-}$: We choose $i \in [0, n_* - 1]$ such that $\mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{w_+} \neq \mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{w_-}$. Since square (4.4) is a pullback, by Definition 4.24, there exists a unique graph morphism $\omega : \mathrm{C}_{n_*} \to P_f^{(G)}$ such that $\omega \cdot g_+ = w_+$ and $\omega \cdot g_- = w_-$.



Note that $\mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_\omega = (\mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{\omega_+}, \mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{\omega_-})$, since $\mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_\omega\,\mathrm{V}_{g_+} = \mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{\omega_+}$ and $\mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_\omega\,\mathrm{V}_{g_-} = \mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{\omega_-}$. By Lemma 3.8, the image $(\mathrm{C}_{n_*})\omega$ is a quasi-cyclic subgraph of $P_f^{(G)}$. Hence there exist cyclic subgraphs $C_1, \ldots, C_N \leqslant P_f^{(G)}$ with $N \in \mathbb{Z}_{\geqslant 1}$ such that $(\mathrm{C}_{n_*})\omega = \bigcup_{k=1}^N C_k$; see Definition 3.5. Hence there is $k_0 \in [1, N]$ such that $(\mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{w_+}, \mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{w_-}) \in \mathrm{V}_{C_{k_0}}$, whereas $\mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{w_+} \neq \mathrm{v}_{i+n_*\mathbb{Z}}\,\mathrm{V}_{w_-}$ The condition (2).(ii) ensure that this case also does not occur.

Altogether, whether $\mathrm{V}_{w_+} = \mathrm{V}_{w_-}$ or $\mathrm{V}_{w_+} \neq \mathrm{V}_{w_-}$, neither can occur under condition (2). We obtain a *contradiction* to (C_{n_*}, f) not being injective. Therefore, we conclude that (C_n, f) is injective for all $n \in \mathbb{Z}_{\geqslant 1}$. □

To ensure (C_n, f) being bijective for all $n \in \mathbb{Z}$, we still need a condition for the surjectivity. In the rest of this section, we give a possible condition using adjacency matrices of graphs.

Given a finite graph G, we label the vertices of G as $v_1^{(G)}, \ldots, v_{|\,\mathrm{V}_G\,|}^{(G)}$.

Definition 4.26 (Adjacency matrices)
The *adjacency matrix* $A_G \in \mathbb{Q}^{|\,\mathrm{V}_G\,| \times |\,\mathrm{V}_G\,|}$ of G is a square matrix with

$$(A_G)_{ij} = \left|\left\{ e \in \mathrm{E}_G \ : \ e\,\mathrm{s}_G = v_i^{(G)} \text{ and } e\,\mathrm{t}_G = v_j^{(G)} \right\}\right|$$

for $i, j \in [1, |\,\mathrm{V}_G\,|]$.

Lemma 4.27
Let A_G be the adjacency matrix of G. For $k \in \mathbb{Z}_{\geqslant 1}$ and $i, j \in [1, |\,\mathrm{V}_G\,|]$, we have

$$\left(A_G^k\right)_{ij} = \left|\left\{ p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = v_i^{(G)} \text{ and } p_{\mathrm{t}} = v_j^{(G)} \right\}\right| \ , \tag{4.5}$$

where $p : \mathrm{D}_k \to G$ is a path in G of length k. In particular, if $i = j$, then we have

$$\left(A_G^k\right)_{ii} = \left|\left\{ p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = p_{\mathrm{t}} = v_i^{(G)} \right\}\right| \ . \tag{4.6}$$

Proof. Fix $i, j \in [1, |\mathrm{V}_G|]$. We prove the first statement of this lemma by induction on k.

For $k = 1$, we consider the map

$$a : \begin{array}{ccc} \left\{p \in (\mathrm{D}_1, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = v_i^{(G)},\ p_{\mathrm{t}} = v_j^{(G)}\right\} & \to & \left\{e \in \mathrm{E}_G \ : \ e\,\mathrm{s}_G = v_i^{(G)},\ e\,\mathrm{t}_G = v_j^{(G)}\right\} \\ p & \mapsto & \hat{\mathrm{e}}_0\,\mathrm{E}_p \end{array} .$$

The map a is well-defined and bijective. Hence (4.5) holds for $k = 1$.

Suppose that (4.5) holds for $k = k_0 \in \mathbb{Z}_{\geqslant 1}$. We prove that it remains valid for $k = k_0 + 1$. In fact, we obtain

$$\begin{aligned} (A_G^{k_0+1})_{ij} &= \sum_{l=1}^{|\mathrm{V}_G|} (A_G^{k_0})_{il}(A_G)_{lj} \\ &= \sum_{l=1}^{|\mathrm{V}_G|} \left|\left\{p \in (\mathrm{D}_{k_0}, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = v_i^{(G)},\ p_{\mathrm{t}} = v_l^{(G)}\right\}\right| \cdot \left|\left\{p \in (\mathrm{D}_1, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = v_l^{(G)},\ p_{\mathrm{t}} = v_j^{(G)}\right\}\right| \\ &= \left|\left\{p \in (\mathrm{D}_{k_0+1}, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = v_i^{(G)},\ p_{\mathrm{t}} = v_j^{(G)}\right\}\right| . \end{aligned}$$

Therefore, the Equation (4.5) is valid for every $k \in \mathbb{Z}_{\geqslant 1}$. □

Lemma 4.28

For a graph morphism $f : G \to H$ between finite graphs G and H, we define the matrix $A_f \in \mathbb{Q}^{|\mathrm{V}_G| \times |\mathrm{V}_H|}$ by

$$(A_f)_{ij} := \begin{cases} 1 & \text{if } v_i^{(G)}\,\mathrm{V}_f = v_j^{(H)} \\ 0 & \text{if } v_i^{(G)}\,\mathrm{V}_f \neq v_j^{(H)} \end{cases}$$

for $i \in [1, |\mathrm{V}_G|]$ and $j \in [1, |\mathrm{V}_H|]$. Then for $k \in \mathbb{Z}_{\geqslant 1}$ and $i, j \in [1, |\mathrm{V}_H|]$, we have

(1) $(A_f^{\mathrm{T}} A_G^k A_f)_{ij} = \left|\left\{p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}}\,\mathrm{V}_f = v_i^{(H)},\ p_{\mathrm{t}}\,\mathrm{V}_f = v_j^{(H)}\right\}\right|$;

(2) $(A_f^{\mathrm{T}} (A_G^k)^{\mathrm{diag}} A_f)_{ii} = \left|\left\{p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = p_{\mathrm{t}},\ p_{\mathrm{s}}\,\mathrm{V}_f = p_{\mathrm{t}}\,\mathrm{V}_f = v_i^{(H)}\right\}\right|$;

(3) $A_f^{\mathrm{T}} (A_G^k)^{\mathrm{diag}} A_f = (A_f^{\mathrm{T}} (A_G^k)^{\mathrm{diag}} A_f)^{\mathrm{diag}}$.

Proof. Let $k \in \mathbb{Z}_{\geqslant 1}$.

To prove (1), we fix $i, j \in [1, |\mathrm{V}_H|]$. We have

$$\begin{aligned}
(A_f^{\mathrm{T}} A_G^k A_f)_{ij} &= \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left(A_G^k A_f\right)_{xj} \\
&= \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left(\sum_{y=1}^{|\mathrm{V}_G|} (A_G^k)_{xy} (A_f)_{yj} \right) \\
&= \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left(\sum_{y \in [1,|\mathrm{V}_G|],\ v_y^{(G)} \mathrm{V}_f = v_j^{(H)}} (A_G^k)_{xy} \right) \\
&\overset{\text{Lemma 4.27}}{=} \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left(\sum_{y \in [1,|\mathrm{V}_G|],\ v_y^{(G)} \mathrm{V}_f = v_j^{(H)}} \left| \{ p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = v_x^{(G)},\ p_{\mathrm{t}} = v_y^{(G)} \} \right| \right) \\
&= \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left| \left\{ p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = v_x^{(G)},\ p_{\mathrm{t}} \mathrm{V}_f = v_j^{(H)} \right\} \right| \\
&= \sum_{x \in [1,|\mathrm{V}_G|],\ v_x^{(G)} \mathrm{V}_f = v_i^{(H)}} \left| \left\{ p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = v_x^{(G)},\ p_{\mathrm{t}} \mathrm{V}_f = v_j^{(H)} \right\} \right| \\
&= \left| \left\{ p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} \mathrm{V}_f = v_i^{(H)},\ p_{\mathrm{t}} \mathrm{V}_f = v_j^{(H)} \right\} \right| ,
\end{aligned}$$

completing the proof of (1).

To prove (2), we fix $i \in [1, |\mathrm{V}_H|]$. We have

$$\begin{aligned}
(A_f^{\mathrm{T}} (A_G^k)^{\mathrm{diag}} A_f)_{ii} &= \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left((A_G^k)^{\mathrm{diag}} A_f\right)_{xi} \\
&= \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left(\sum_{y=1}^{|\mathrm{V}_G|} (A_G^k)^{\mathrm{diag}}_{xy} (A_f)_{yi} \right) \\
&= \sum_{x \in [1,|\mathrm{V}_G|],\ v_x^{(G)} \mathrm{V}_f = v_i^{(H)}} (A_f^{\mathrm{T}})_{ix} (A_G^k)^{\mathrm{diag}}_{xx} \\
&= \sum_{x \in [1,|\mathrm{V}_G|],\ v_x^{(G)} \mathrm{V}_f = v_i^{(H)}} (A_G^k)_{xx} \\
&= \left| \left\{ p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = p_{\mathrm{t}},\ p_{\mathrm{s}} \mathrm{V}_f = p_{\mathrm{t}} \mathrm{V}_f = v_i^{(H)} \right\} \right| ,
\end{aligned}$$

completing the proof of (2).

To prove (3), we fix $i, j \in [1, |\mathrm{V}_H|]$ with $i \neq j$. We have

$$\begin{aligned}
(A_f^{\mathrm{T}} (A_G^k)^{\mathrm{diag}} A_f)_{ij} &= \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left((A_G^k)^{\mathrm{diag}} A_f\right)_{xj} \\
&= \sum_{x=1}^{|\mathrm{V}_G|} (A_f^{\mathrm{T}})_{ix} \left(\sum_{y=1}^{|\mathrm{V}_G|} (A_G^k)^{\mathrm{diag}}_{xy} (A_f)_{yj} \right) \\
&= \sum_{x \in [1,|\mathrm{V}_G|],\ v_x^{(G)} \mathrm{V}_f = v_j^{(H)}} (A_f^{\mathrm{T}})_{ix} (A_G^k)^{\mathrm{diag}}_{xx} \\
&= \sum_{x \in [1,|\mathrm{V}_G|],\ v_x^{(G)} \mathrm{V}_f = v_j^{(H)},\ v_x^{(G)} \mathrm{V}_f = v_i^{(H)}} (A_G^k)_{xx} \\
&= 0 .
\end{aligned}$$

Hence $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}}A_f$ is a diagonal matrix; so we obtain $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}}A_f = (A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}}A_f)^{\mathrm{diag}}$. This completes the proof of (3). □

Remark 4.29

Let G be a graph, which may also be infinite. For every $k \in \mathbb{Z}_{\geqslant 1}$ and $v \in \mathrm{V}_G$, the following maps are bijective:

$$\begin{array}{llcl} a_{k,v}: & \{p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \;:\; p_{\mathrm{s}} = p_{\mathrm{t}} = v\} & \to & \{u \in (\mathrm{C}_k, G)_{\mathrm{Gph}} \;:\; \mathrm{v}_{0+k\mathbb{Z}}\,\mathrm{V}_u = v\} \\ & p: \left\{\begin{array}{ll} \mathrm{V}_p : \hat{\mathrm{v}}_i \mapsto \hat{\mathrm{v}}_i\,\mathrm{V}_p & \text{for } i \in [0,k] \\ \mathrm{E}_p : \hat{\mathrm{e}}_i \mapsto \hat{\mathrm{e}}_i\,\mathrm{E}_p & \text{for } i \in [0,k-1] \end{array}\right\} & \mapsto & u: \left\{\begin{array}{ll} \mathrm{V}_u : \mathrm{v}_{i+k\mathbb{Z}} \mapsto \hat{\mathrm{v}}_i\,\mathrm{V}_p & \text{for } i \in [0,k-1] \\ \mathrm{E}_u : \mathrm{e}_{i+k\mathbb{Z}} \mapsto \hat{\mathrm{e}}_i\,\mathrm{E}_p & \text{for } i \in [0,k-1] \end{array}\right\}, \end{array}$$

$$\begin{array}{llcl} a_{k,v}^{-1}: & \{u \in (\mathrm{C}_k, G)_{\mathrm{Gph}} \;:\; \mathrm{v}_{0+k\mathbb{Z}}\,\mathrm{V}_u = v\} & \to & \{p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \;:\; p_{\mathrm{s}} = p_{\mathrm{t}} = v\} \\ & u: \left\{\begin{array}{ll} \mathrm{V}_u : \mathrm{v}_{i+k\mathbb{Z}} \mapsto \mathrm{v}_{i+k\mathbb{Z}}\,\mathrm{V}_u & \text{for } i \in [0,k-1] \\ \mathrm{E}_u : \mathrm{e}_{i+k\mathbb{Z}} \mapsto \mathrm{e}_{i+k\mathbb{Z}}\,\mathrm{E}_u & \text{for } i \in [0,k-1] \end{array}\right\} & \mapsto & p: \left\{\begin{array}{ll} \mathrm{V}_p : \hat{\mathrm{v}}_i \mapsto \mathrm{v}_{i+k\mathbb{Z}}\,\mathrm{V}_u & \text{for } i \in [0,k] \\ \mathrm{E}_p : \hat{\mathrm{e}}_i \mapsto \mathrm{e}_{i+k\mathbb{Z}}\,\mathrm{E}_u & \text{for } i \in [0,k-1] \end{array}\right\}. \end{array}$$

Proposition 4.30

Let $f : G \to H$ be a graph morphism between finite graphs, where G is quasi-cyclic. The following conditions (1) and (2) are equivalent:

(1) The graph morphism $f : G \to H$ is a quasi-isomorphism.

(2) (i) For $e_1, e_2 \in \mathrm{E}_G$ with $e_1 \neq e_2$, $e_1\,\mathrm{s}_G = e_2\,\mathrm{s}_G$ and $e_1\,\mathrm{t}_G = e_2\,\mathrm{t}_G$, we have $e_1\,\mathrm{E}_f \neq e_2\,\mathrm{E}_f$.

(ii) For every $C \in \mathrm{S}_{\mathrm{c}}(P_f^{(G)})$ and every $(v^+, v^-) \in \mathrm{V}_{P_f^{(G)}}$ with $v^+ \neq v^-$, we have $(v^+, v^-) \notin \mathrm{V}_C$; cf. (4.3).

(iii) We have $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}}A_f = (A_H^k)^{\mathrm{diag}}$ for every $k \in \mathbb{Z}_{\geqslant 1}$.

Proof. In Proposition 4.25, we have proved that the maps $(\mathrm{C}_k, f) : (\mathrm{C}_k, G)_{\mathrm{Gph}} \to (\mathrm{C}_k, H)_{\mathrm{Gph}}$ are injective for all $k \in \mathbb{Z}_{\geqslant 1}$ if and only if the conditions (2).(i) and (2).(ii) hold. It remains to show the implication from (1) to (2).(iii) and how condition (2).(iii) ensure the surjectivity of all the maps (C_k, f) with $k \in \mathbb{Z}_{\geqslant 1}$. To this end, we first provide an equivalent statement of (2).(iii).

We fix a $k \in \mathbb{Z}_{\geqslant 1}$. By Lemma 4.28.(3), the matrix $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}}A_f \in \mathbb{Q}^{|\mathrm{V}_H| \times |\mathrm{V}_H|}$ is a diagonal matrix. Hence (2).(iii) is valid if and only if $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}}A_f$ and (A_H^k) have the same diagonal, that is,

$$(A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}}A_f)_{ii} = (A_H^k)_{ii} \tag{4.7}$$

for every $i \in [1, |\mathrm{V}_H|]$.

We consider the left side of Equation (4.7). By Lemma 4.28.(2) and Remark 4.29, we have

$$\begin{aligned} (A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}}A_f)_{ii} &= \left|\left\{p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \;:\; p_{\mathrm{s}} = p_{\mathrm{t}},\ p_{\mathrm{s}}\,\mathrm{V}_f = p_{\mathrm{t}}\,\mathrm{V}_f = v_i^{(H)}\right\}\right| \\ &= \sum_{v_G \in \mathrm{V}_G,\ v_G\,\mathrm{V}_f = v_i^{(H)}} \left|\{p \in (\mathrm{D}_k, G)_{\mathrm{Gph}} \;:\; p_{\mathrm{s}} = p_{\mathrm{t}} = v_G\}\right| \\ &= \sum_{v_G \in \mathrm{V}_G,\ v_G\,\mathrm{V}_f = v_i^{(H)}} \left|\{u_G \in (\mathrm{C}_k, G)_{\mathrm{Gph}} \;:\; \mathrm{v}_{0+k\mathbb{Z}}\,\mathrm{V}_{u_G} = v_G\}\right| \\ &= \left| \bigcup_{\substack{v_G \in \mathrm{V}_G, \\ v_G\,\mathrm{V}_f = v_i^{(H)}}} \{u_G \in (\mathrm{C}_k, G)_{\mathrm{Gph}} \;:\; \mathrm{v}_{0+k\mathbb{Z}}\,\mathrm{V}_{u_G} = v_G\} \right| \end{aligned}$$

for every $i \in [1, |\mathrm{V}_H|]$. We consider the right side of Equation (4.7). By Equation (4.6) and Remark 4.29, we have

$$
\begin{aligned}
(A_H^k)_{ii} &= \left| \left\{ p \in (\mathrm{D}_k, H)_{\mathrm{Gph}} \ : \ p_{\mathrm{s}} = p_{\mathrm{t}} = v_i^{(H)} \right\} \right| \\
&= \left| \left\{ u_H \in (\mathrm{C}_k, H)_{\mathrm{Gph}} \ : \ \mathrm{v}_{0+k\mathbb{Z}} \, \mathrm{V}_{u_H} = v_i^{(H)} \right\} \right|
\end{aligned}
$$

for every $i \in [1, |\mathrm{V}_H|]$. Hence (2).(iii) is valid if and only if we have

$$
\left| \bigcup_{\substack{v_G \in \mathrm{V}_G, \\ v_G \, \mathrm{V}_f = v_i^{(H)}}} \left\{ u_G \in (\mathrm{C}_k, G)_{\mathrm{Gph}} \ : \ \mathrm{v}_{0+k\mathbb{Z}} \, \mathrm{V}_{u_G} = v_G \right\} \right| = \left| \left\{ u_H \in (\mathrm{C}_k, H)_{\mathrm{Gph}} \ : \ \mathrm{v}_{0+k\mathbb{Z}} \, \mathrm{V}_{u_H} = v_i^{(H)} \right\} \right| \tag{4.8}
$$

for every $i \in [1, |\mathrm{V}_H|]$.

Now, we show the implication from (1) to (2).(iii). Suppose that $f : G \to H$ is a quasi-isomorphism. We show that Equation (4.8) is valid for every $i \in [1, |\mathrm{V}_H|]$. Write

$$
M_1^{(i)} := \bigcup_{\substack{v_G \in \mathrm{V}_G, \\ v_G \, \mathrm{V}_f = v_i^{(H)}}} \left\{ u_G \in (\mathrm{C}_k, G)_{\mathrm{Gph}} \ : \ \mathrm{v}_{0+k\mathbb{Z}} \, \mathrm{V}_{u_G} = v_G \right\}
$$

and write

$$
M_2^{(i)} := \left\{ u_H \in (\mathrm{C}_k, H)_{\mathrm{Gph}} \ : \ \mathrm{v}_{0+k\mathbb{Z}} \, \mathrm{V}_{u_H} = v_i^{(H)} \right\} .
$$

Then the restriction $(C_k, f)|_{M_1^{(i)}}^{M_2^{(i)}}$ is bijective for every $i \in [1, |\mathrm{V}_H|]$. Hence Equation (4.8) is valid for every $i \in [1, |\mathrm{V}_H|]$. Due to the equivalence between (2).(iii) and (4.8), we conclude that (2).(iii) is valid.

Next, we show the implication from (2) to (1). Suppose that all conditions in (2) hold. By Proposition 4.25, the map (C_k, f) is injective. It remains to show that (C_k, f) is surjective. Since

$$
|(\mathrm{C}_k, G)_{\mathrm{Gph}}| = \sum_{i=1}^{|\mathrm{V}_H|} \left| \bigcup_{\substack{v_G \in \mathrm{V}_G, \\ v_G \, \mathrm{V}_f = v_i^{(H)}}} \left\{ u_G \in (\mathrm{C}_k, G)_{\mathrm{Gph}} \ : \ \mathrm{v}_{0+k\mathbb{Z}} \, \mathrm{V}_{u_G} = v_G \right\} \right|
$$

and since

$$
|(\mathrm{C}_k, H)_{\mathrm{Gph}}| = \sum_{i=1}^{|\mathrm{V}_H|} \left| \left\{ u_H \in (\mathrm{C}_k, H)_{\mathrm{Gph}} \ : \ \mathrm{v}_{0+k\mathbb{Z}} \, \mathrm{V}_{u_H} = v_i^{(H)} \right\} \right| ,
$$

we have

$$
|(\mathrm{C}_k, G)_{\mathrm{Gph}}| = |(\mathrm{C}_k, H)_{\mathrm{Gph}}|
$$

due to the equivalence between (2).(iii) and (4.8). Since $(\mathrm{C}_k, G)_{\mathrm{Gph}}$ and $(\mathrm{C}_k, H)_{\mathrm{Gph}}$ are finite sets, and since $(\mathrm{C}_k, f) : (\mathrm{C}_k, G)_{\mathrm{Gph}} \to (\mathrm{C}_k, H)_{\mathrm{Gph}}$ is injective, it follows that (C_k, f) is bijective. Since our $k \in \mathbb{Z}_{\geqslant 1}$ is chosen arbitrarily, (C_k, f) is bijective for all $k \in \mathbb{Z}_{\geqslant 1}$. Hence $f : G \to H$ is a quasi-isomorphism. □

To verify $f \in (G, H)_{\mathrm{Gph}}$ being a quasi-isomophism, the condition (2).(iii) given in Theorem 4.30 is still infinite. In the following lemma, We reduce this infinite condition to a finite one.

Lemma 4.31

Let $f : G \to H$ be a graph morphism between finite graphs. The following are equivalent:

(1) $A_f^{\mathrm{T}} (A_G^k)^{\mathrm{diag}} A_f = (A_H^k)^{\mathrm{diag}}$ for every $k \in \mathbb{Z}_{\geqslant 1}$.

(2) $A_f^{\mathrm{T}} (A_G^k)^{\mathrm{diag}} A_f = (A_H^k)^{\mathrm{diag}}$ for every $k \in [1, |\mathrm{V}_G| + |\mathrm{V}_H|]$.

Proof. We have to show the implication (2) ⇒ (1). Let $\mu_G(X), \mu_H(X) \in \mathbb{Q}[X]$ be minimal polynomials of adjacency matrices A_G and A_H, respectively. Write

$$\mu(X) = \sum_{i=0}^{k_0} a_i X^i := \mu_G(X) \cdot \mu_H(X) \,,$$

where $k_0 = \deg(\mu_G(X)) + \deg(\mu_H(X))$, where the coefficients $a_0, \ldots, a_{k_0-1}$ are in $\mathbb{Q}$ and where $a_{k_0} = 1$. Since for a matrix, the minimal polynomial is a divisor of the characteristic polynomial, we have

$$k_0 = \deg(\mu_G(X)) + \deg(\mu_H(X)) \leqslant |\,\mathrm{V}_G\,| + |\,\mathrm{V}_H\,| \,;$$

so according to the precondition given in (2), we have $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}} A_f = (A_H^k)^{\mathrm{diag}}$ for every $k \in [1, k_0]$. We proceed to prove by induction that for given $k_1 \in \mathbb{Z}_{>k_0}$, if $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}} A_f = (A_H^k)^{\mathrm{diag}}$ is known for $k \in [1, k_1 - 1]$, then $A_f^{\mathrm{T}}(A_G^{k_1})^{\mathrm{diag}} A_f = (A_H^{k_1})^{\mathrm{diag}}$ follows.

Let $k_1 \in \mathbb{Z}_{>k_0}$, and let $h := k_1 - k_0$. We have $\mu(A_G) = 0$ and $\mu(A_H) = 0$. These imply that $\mu(A_G) \cdot A_G^h = \sum_{i=0}^{k_0} a_i A_G^{i+h} = 0$ and $\mu(A_H) \cdot A_H^h = \sum_{i=0}^{k_0} a_i A_H^{i+h} = 0$. Hence we may represent $A_G^{k_1}$ and $A_H^{k_1}$ in the following form:

$$A_G^{k_1} = A_G^{k_0+h} = -\sum_{i=0}^{k_0-1} a_i A_G^{i+h} \quad \text{and} \quad A_H^{k_1} = A_H^{k_0+h} = -\sum_{i=0}^{k_0-1} a_i A_H^{i+h} \,. \tag{4.9}$$

Substituting (4.9) into the equation to be proved, we obtain

$$\begin{aligned}
A_f^{\mathrm{T}}(A_G^{k_1})^{\mathrm{diag}} A_f &= A_f^{\mathrm{T}} \left(-\sum_{i=0}^{k_0-1} a_i A_G^{i+h} \right)^{\mathrm{diag}} A_f \\
&= A_f^{\mathrm{T}} \left(-\sum_{i=0}^{k_0-1} a_i \left(A_G^{i+h}\right)^{\mathrm{diag}} \right) A_f \\
&= -\sum_{i=0}^{k_0-1} a_i \left(A_f^{\mathrm{T}} \left(A_G^{i+h}\right)^{\mathrm{diag}} A_f \right) \\
&= -\sum_{i=0}^{k_0-1} a_i \left(A_H^{i+h}\right)^{\mathrm{diag}} \\
&= \left(A_H^{k_1}\right)^{\mathrm{diag}} \,.
\end{aligned}$$

Therefore, if we have $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}} A_f = (A_H^k)^{\mathrm{diag}}$ for $k \in [1, |\,\mathrm{V}_G\,| + |\,\mathrm{V}_H\,|]$, it actually holds for $k \in \mathbb{Z}_{\geqslant 1}$. This complete the proof for the implication (2) ⇒ (1). □

Theorem 4.32

Let $f : G \to H$ be a graph morphism between finite graphs, where G is quasi-cyclic. The following conditions (1) and (2) are equivalent:

(1) The graph morphism $f : G \to H$ is a quasi-isomorphism.

(2) (i) For $e_1, e_2 \in \mathrm{E}_G$ with $e_1 \neq e_2$, $e_1\,\mathrm{s}_G = e_2\,\mathrm{s}_G$ and $e_1\,\mathrm{t}_G = e_2\,\mathrm{t}_G$, we have $e_1\,\mathrm{E}_f \neq e_2\,\mathrm{E}_f$.

(ii) For every $C \in \mathrm{S_c}(P_f^{(G)})$ and every $(v^+, v^-) \in \mathrm{V}_{P_f^{(G)}}$ with $v^+ \neq v^-$, we have $(v^+, v^-) \notin \mathrm{V}_C$; cf. (4.3).

(iii) We have $A_f^{\mathrm{T}}(A_G^k)^{\mathrm{diag}} A_f = (A_H^k)^{\mathrm{diag}}$ for every $k \in [1, |\,\mathrm{V}_G\,| + |\,\mathrm{V}_H\,|]$.

Proof. This follows from Proposition 4.30 and Lemma 4.31. □

Together with Proposition 4.22, we can determine whether a given graph morphism $f : G \to H$ is quasi-isomorphism for any finite graphs G, H. First, we need to verify $\hat{f} : \mathrm{S}_{\mathrm{qc}}^{\max}(G) \to \mathrm{S}_{\mathrm{qc}}^{\max}(H), \hat{G} \mapsto (\hat{G})f$ is bijective. Second, we verify that $f|_{\hat{G}}^{(\hat{G})f}$ is a quasi-isomorphism for every $\hat{G} \in \mathrm{S}_{\mathrm{qc}}^{\max}(G)$ by Corollary 4.32.

Theorem 4.33

Let $f : G \to H$ be a graph morphism between finite graphs. The following conditions (1) and (2) are equivalent:

(1) The graph morphism $f : G \to H$ is a quasi-isomorphism.

(2) (a) The map

$$\begin{array}{rccc} \hat{f} : & \mathrm{S}_{\mathrm{qc}}^{\max}(G) & \to & \mathrm{S}_{\mathrm{qc}}^{\max}(H) \\ & \hat{G} & \mapsto & (\hat{G})f \end{array}$$

is bijective.

(b) For every restriction $f^{\hat{G}} := f|_{\hat{G}}^{(\hat{G})f} : \hat{G} \to (\hat{G})f$ with $\hat{G} \in \mathrm{S}_{\mathrm{qc}}^{\max}(G)$, the following conditions hold:

(i) For $e_1, e_2 \in \mathrm{E}_{\hat{G}}$ with $e_1 \neq e_2$, $e_1\, \mathrm{s}_G = e_2\, \mathrm{s}_G$ and $e_1\, \mathrm{t}_G = e_2\, \mathrm{t}_G$, we have $e_1\, \mathrm{E}_f \neq e_2\, \mathrm{E}_f$.

(ii) For every cyclic subgraph $C \leqslant P_{f^{\hat{G}}}^{(\hat{G})}$ and every $(v^+, v^-) \in \mathrm{V}_{P_{f^{\hat{G}}}^{(\hat{G})}}$ with $v^+ \neq v^-$, we have $(v^+, v^-) \notin \mathrm{V}_C$.

(iii) We have $A_{f^{\hat{G}}}^{\mathrm{T}} (A_{\hat{G}}^k)^{\mathrm{diag}} A_{f^{\hat{G}}} = (A_{(\hat{G})f}^k)^{\mathrm{diag}}$ for every $1 \leqslant k \leqslant |\,\mathrm{V}_{\hat{G}}\,| + |\,\mathrm{V}_{(\hat{G})f}\,|$.

Proof. This follows from Proposition 4.22 and Theorem 4.32. □

5 Algorithms

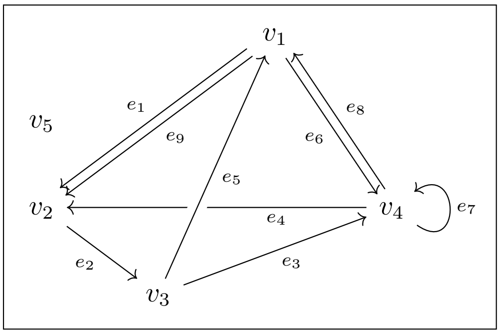
In this chapter, we introduce several useful algorithms for finite directed graphs (hereafter simply referred to as graphs), with a particular focus on quasi-cyclic graphs.

The algorithms presented in this text are implemented in the programming language Magma **[3]**. For further details, we refer to the official documentation of Magma:

https://magma.maths.usyd.edu.au/magma/documentation/.

Besides the built-in functions provided by Magma, the implementations in this text are also based on the framework described in **[4,** Chapter 10**]**. For example, given a graph G defined as follows:

$G:$



Then in Magma it is represented by:

```
G := <[1,2,3,4,5],[<1,1,2>,<2,2,3>,<3,3,4>,<4,4,2>,<3,5,1>,
<1,6,4>,<4,7,4>,<4,8,1>,<1,9,2>]>;
```

Here, the first list G[1] stores the vertices of G, and the second list G[2] contains the edges. There is no prescribed order for the elements in G[1] and G[2]. As for the labeling of edges in G[2], we adopt the following convention: for example, the edge <2,1,3> represents an edge labeled α_1, starting at vertex 2 and ending at vertex 3. For clarity and readability, the notation for vertices and edges in the text may differ slightly when necessary. However, all such modifications are made in a consistent manner that preserves the correspondence between the subscripts used in the textual descriptions and the indices employed in the underlying code representation of the graph.

It should be noted that the following Magma functions from J. Hess **[4,** Chapter 10**]** are directly used in this paper, whereas their explicit implementations are not provided here:

- CyclicGraph **[4,** p. 240**]**
- ListGraphMorphisms **[4,** pp. 243-246**]**
- Is_Injective **[4,** p. 246**]**
- Is_Surjective **[4,** p. 246**]**
- Is_Bijective **[4,** p. 246**]**

- `IsIsomorphic` **[4,** p. 247**]**

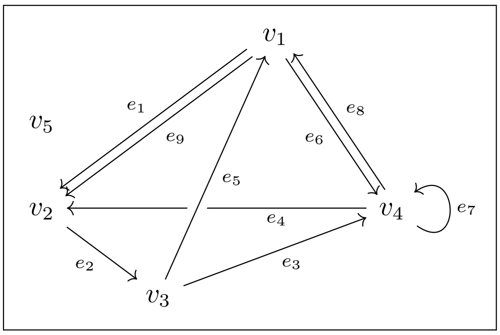
The algorithms below, together with the mentioned algorithms from **[4,** Chapter 10**]**, can be loaded via: `load "electronic_appendix_danning_liu.m"` .

5.1 For general graphs

Algorithm 5.1

Let G be a graph and $v \in \mathrm{V}_G$. For example:

G :



With Magma:

```
G := <[1,2,3,4,5],[<1,1,2>,<2,2,3>,<3,3,4>,<4,4,2>,<3,5,1>,
<1,6,4>,<4,7,4>,<4,8,1>,<1,9,2>]>;
```

i | The function `GetEdgesFromVertex` returns $\mathrm{E}_G(v, *)$, that is, the list of all edges in the graph G starting at the vertex v; see Definition 2.5.

```
// Input:
//   G - a graph represented as a pair <vertices, edges>
//   v - a vertex of G
//
// Output:
//   A list of all edges in G with source vertex v
GetEdgesFromVertex := function(v,G)
    return [e : e in G[2] | e[1] eq v];
end function;
```

For example, we choose $v = 1$:

```
> GetEdgesFromVertex(1,G);
[ <1, 1, 2>, <1, 6, 4>, <1, 9, 2> ]
```

ii | The function `GetEdgesTargetVertex` returns $\mathrm{E}_G(*, v)$, that is, the list of all edges in the graph G pointing to the vertex v; see Definition 2.5.

```
// Input:
//   G - a graph represented as a pair <vertices, edges>
//   v - a vertex of G
//
// Output:
```

```
//   A list of all edges in G with target vertex v
GetEdgesTargetVertex := function(v,G)
    return [e : e in G[2] | e[3] eq v];
end function;
```

For example, we choose $v = 2$:

```
> GetEdgesTargetVertex(2,G);
[ <1, 1, 2>, <4, 4, 2>, <1, 9, 2> ]
```

iii The function `GetAdjacentVertices` returns the list of vertices adjacent to a given vertex v in graph G, i.e., all the vertices in $\mathrm{V}_G(*.v) \cup \mathrm{V}_G(v,*)$; see Definition 2.5.

```
// Input:
//   G - a graph represented as a pair <vertices, edges>
//   v - a vertex of G
//
// Output:
//   A list of all vertices that are adjacent to v.
GetAdjacentVertices := function (v,G)
    adjacent := [];
    for e in G[2] do
        if e[3] eq v then
            adjacent := adjacent cat [e[1]];
        end if;
        if e[1] eq v then
            adjacent := adjacent cat [e[3]];
        end if;
    end for;
    return SetToSequence (Set( adjacent ));
end function;
```

For example, we choose $v = 3$:

```
> GetAdjacentVertices(3,G);
[ 1, 2, 4 ]
```

iv The function `GetAllReachableVertices` returns the set of all vertices in G that are reachable from v via a directed path; see Definition 3.18 and Definition 3.19.

```
// Input:
//   G - a graph
//   v - a vertex of G
//
// Output:
//   Set of all vertices in G that are reachable
//   from v via a directed path
GetAllReachableVertices := function(v, G)
    reachable := {v};
    frontier := {v};
    while not IsEmpty(frontier) do
        nextFrontier := {};
```

```
                for u in frontier do
                    for e in GetEdgesFromVertex(u, G) do
                        w := e[3];
                        if w notin reachable then
                            Include(~reachable, w);
                            Include(~nextFrontier, w);
                        end if;
                    end for;
                end for;
                frontier := nextFrontier;
            end while;
            return reachable;
        end function;
```

For example, we choose $v = 5$:

```
> GetAllReachableVertices(5,G);
{ 5 }
```

Or, we choose $v = 2$:

```
> GetAllReachableVertices(2,G);
{ 1, 2, 3, 4 }
```

Algorithm 5.2

i | The function IsWeaklyConnected checks if a given graph G is weakly connected (see Definition 3.16):

```
// Input:
//    G - a graph
//
// Output:
//    true if G is weakly connected;
//    false otherwise
IsWeaklyConnected := function(G)
    if #G[1] le 1 then
        return true;
    else
        reachable := {G[1][1]};
        frontier := reachable;
        while not IsEmpty(frontier) do
            nextFrontier := {};
            for u in frontier do
                nextFrontier := nextFrontier join (Set(
                    GetAdjacentVertices(u,G)) diff reachable);
            end for;
            reachable := reachable join nextFrontier;
            frontier := nextFrontier;
        end while;
    end if;
    if reachable eq Set(G[1]) then
```

```
            return true;
        else
            return false;
        end if;
    end function;
```

For example, if G is the graph given in Algorithm 5.1:

```
> IsWeaklyConnected(G);
false
```

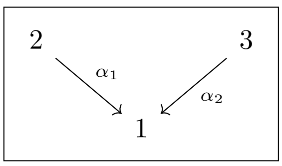
If G is the graph given in Remark 4.15:

```
> G := <[1,2,3],[<1,1,1>,<2,2,2>,<3,3,3>,<2,4,1>,<1,5,3>\
,<3,6,2>,<3,7,1>,<1,8,2>,<2,9,3>]>;
> IsWeaklyConnected(G);
true
```

ii The function `IsStarConnected` checks if a given graph G is star-connected (see Definition 3.20):

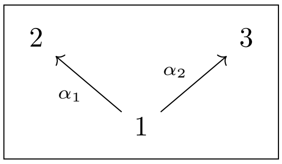
```
// Input:
//    G - a graph
//
// Output:
//    true if G is star-connected;
//    false otherwise
IsStarConnected := function(G)
    for vertex in G[1] do
        if Set(GetAllReachableVertices(vertex,G)) eq Set(G[1])
           then
           return true;
        end if;
    end for;
    return false;
end function;
```

For example, if G is the following graph:

$G:$ 


```
> G := <[1,2,3],[<2,1,1>,<3,2,1>]>;
> IsStarConnected(G);
false
```

If G is the following graph:

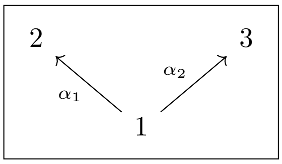
G : 

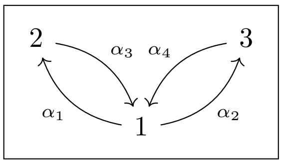

```
> G := <[1,2,3],[<1,1,2>,<1,2,3>]>;
> IsStarConnected(G);
true
```

iii The function `IsStronglyConnected` checks if a given graph G is strongly connected (see Definition 3.20):

```
// Input:
//   G - a graph
//
// Output:
//   true if G is strongly connected;
//   false otherwise
IsStronglyConnected := function(G)
    if #G[1] eq 0 then
        return false;
    end if;
    for v in G[1] do
        if Set(G[1]) ne GetAllReachableVertices(v,G) then
            return false;
        end if;
    end for;
    return true;
end function;
```

For example, let G_1, G_2 be the following graphs:

G_1 : 

G_2 : 

Then in Magma:

```
> G1 := <[1,2,3],[<1,1,2>,<1,2,3>]>;
> G2 := <[1,2,3],[<1,1,2>,<1,2,3>, <2,3,1>,<3,4,1>]>;
> IsStronglyConnected(G1);
false
> IsStronglyConnected(G2);
true
```

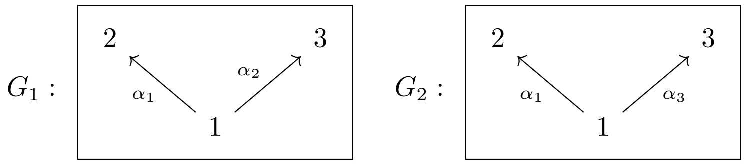
5.2 For collections of graphs

Algorithm 5.3

Given graphs G_1 and G_2, the function `AreGraphsEqual` determines if G_1 and G_2 are identical.

```
// Input:
//   G1,G2 - graphs
//
// Output:
//   true if G1 and G2 are equal;
//   false otherwise
AreGraphsEqual := function(G1,G2)
    return Set(G1[1]) eq Set(G2[1]) and Set(G1[2]) eq Set(G2[2]);
end function;
```

For example, let G_1 and G_2 be graphs given in the following figure



Then in Magma, we have

```
> G1 := <[1,2,3],[<1,1,2>,<1,2,3>]>;
> G2 := <[1,2,3],[<1,1,2>,<1,3,3>]>;
> AreGraphsEqual(G1,G2);
false
```

Algorithm 5.4

i | Given a graph G and a set of graphs, the function `IsGraphInSet` checks if G is contained in the given set.

```
// Input:
//   G    - a graph
//   set  - a set of graphs
//
// Output:
//   true if G is already in the set;
//   false otherwise
IsGraphInSet := function(G,set)
    for graph in set do
        if AreGraphsEqual(G,graph) eq true then
            return true;
        end if;
    end for;
    return false;
end function;
```

For example:

```
> GraphSet := {<[1,2,3],[<1,1,2>,<1,2,3>]>, <[1,2,3],[<1,1,2>,<1,2,3>, <2,3,1>\
,<3,4,1>]>, <[1,2,3],[<2,1,1>,<3,2,1>]>};
> G := <[1,2,3],[<1,1,2>,<1,3,3>]>;
> IsGraphInSet(G,GraphSet);
false
```

ii Given a graph G and a list of graphs, the function `IsGraphInList` checks if G is contained in the given list.

```
// Input:
//   G     - a graph
//   list  - a list of graphs
//
// Output:
//   true if G is already in the list;
//   false otherwise
IsGraphInList := function(G,list)
    for graph in list do
        if AreGraphsEqual(G,graph) eq true then
            return true;
        end if;
    end for;
    return false;
end function;
```

For example:

```
> GraphList := [<[1,2,3],[<1,1,2>,<1,2,3>]>, <[1,2,3],[<1,1,2>,<1,2,3>, <2,3,1>\
,<3,4,1>]>, <[1,2,3],[<2,1,1>,<3,2,1>]>];
> G := <[1,2,3],[<1,2,3>,<1,1,2>]>;
> IsGraphInList(G,GraphList);
true
```

Algorithm 5.5

The function `ListToSetForGraphs` eliminates duplicate graphs from a list by converting it into a set. Note that the argument should not be an empty list.

```
// Input:
//   list - a list of graphs
//
// Output:
//   A set containing all distinct graphs from the input list.
//
// Error:
//   empty_list_error - the input list must not be empty.
ListToSetForGraphs := function(list)
    graphset := {list[1]};
```

```
        for graph in list do
            if not IsGraphInSet(graph,graphset) then
                graphset := graphset join {graph};
            end if;
        end for;
        return graphset;
    end function;
```

For example, in Magma, we have

```
> GraphList := [<[1,2,3],[<1,1,2>,<1,2,3>]>, <[1,3,2],[<1,1,2>,<1,2,3>]>, <[1,\
2,3],[<1,1,2>,<1,2,3>]>];
> ListToSetForGraphs(GraphList);
{<[ 1, 2, 3 ], [ <1, 1, 2>, <1, 2, 3> ]>}
```

Algorithm 5.6

Suppose given a graph G and $G_1, G_2 \leqslant G$. The function `GetUnion` computes the union of G_1, G_2.

```
// Input:
//   G1, G2 - subgraphs of a graph G, each represented as a pair
//            <vertex list, edge list>
//
// Output:
//   A subgraph given by the union of vertices as well as edges
    from G1 and G2.
//   The result is returned as a pair <vertex list, edge list>.
GetUnion := function(G1,G2)
    verticesInUnion := Set(G1[1]) join Set(G2[1]);
    edgesInUnion := Set(G1[2]) join Set(G2[2]);
    union := <SetToSequence(verticesInUnion),SetToSequence(
        edgesInUnion)>;
    return union;
end function;
```

For example, let G be the graph in our frontispiece:

G :

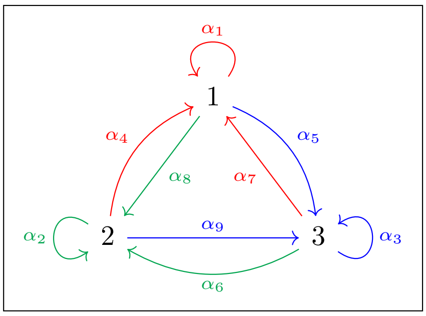


Figure 5.1

Let

$$G_{\text{red}} = (\mathrm{V}_{G_{\text{red}}}, \mathrm{E}_{G_{\text{red}}}) \leqslant G,$$

where $V_{G_{red}} = \{1, 2, 3\}$ and $E_{G_{red}} = \{\alpha_1, \alpha_4, \alpha_7\}$; and let

$$G_{green} = (V_{G_{green}}, E_{G_{green}}) \leqslant G,$$

where $V_{G_{green}} = \{1, 2, 3\}$ and $E_{G_{green}} = \{\alpha_2, \alpha_6, \alpha_8\}$. We compute the union $G_{red} \cup G_{green}$ in Magma:

```
> G := <[1,2,3],[<1,1,1>,<2,2,2>,<3,3,3>,<2,4,1>,<1,5,3>,<3,6,2>,<3,7,1>,<1,8,\
2>,<2,9,3>]>;
> G_red := <[1,2,3],[<1,1,1>,<2,4,1>,<3,7,1>]>;
> G_green := <[1,2,3],[<2,2,2>,<3,6,2>,<1,8,2>]>;
> GetUnion(G_red,G_green);
<[ 1, 2, 3 ], [ <1, 1, 1>, <2, 4, 1>, <3, 7, 1>, <3, 6, 2>, <1, 8, 2>, <2, 2, 2>]>
```

ii Given a graph G and a list of graphs. Suppose that all graphs in this list are subgraphs of G. The function GetUnionList computes the union of these subgraphs in the given list.

```
// Input:
//   G_list - a list of subgraphs of a graph G,
//            where each subgraph is represented as a pair
//            <vertex list, edge list>
//
// Output:
//   The union of all subgraphs in G_list.
//   The result is returned as a pair <vertex list, edge list>.
GetUnionList := function(G_list)
    verticesInUnion := &join[Set(x[1]) : x in G_list];
    edgesInUnion := &join[Set(x[2]) : x in G_list];
    union := <SetToSequence(verticesInUnion),SetToSequence(
        edgesInUnion)>;
    return union;
end function;
```

As an example, we still consider the graph G given in Figure 5.1 and the subgraphs G_{red} and G_{green} defined in **i** . Let $G_{blue} = (V_{G_{blue}}, E_{G_{blue}}) \leqslant G$, where $V_{G_{blue}} = \{1, 2, 3\}$ and $E_{G_{blue}} = \{\alpha_3, \alpha_5, \alpha_9\}$. We compute the union of the subgraphs in list [G_red, G_green, G_blue] with Magma:

```
> G := <[1,2,3],[<1,1,1>,<2,2,2>,<3,3,3>,<2,4,1>,<1,5,3>,<3,6,2>,<3,7,1>,<1,8,\
2>,<2,9,3>]>;
> G_red := <[1,2,3],[<1,1,1>,<2,4,1>,<3,7,1>]>;
> G_green := <[1,2,3],[<2,2,2>,<3,6,2>,<1,8,2>]>;
> G_blue := <[1,2,3],[<3,3,3>,<1,5,3>,<2,9,3>]>;
> union := GetUnionList([G_red, G_green, G_blue]);
> union;
<[ 1, 2, 3 ], [ <1, 5, 3>, <1, 1, 1>, <3, 3, 3>, <2, 9, 3>, <2, 4, 1>, <3, 7,
1>, <3, 6, 2>, <1, 8, 2>, <2, 2, 2> ]>
> AreGraphsEqual(union, G);
true
```

5.3 Algorithms for computing the image of a graph morphism

Algorithm 5.7

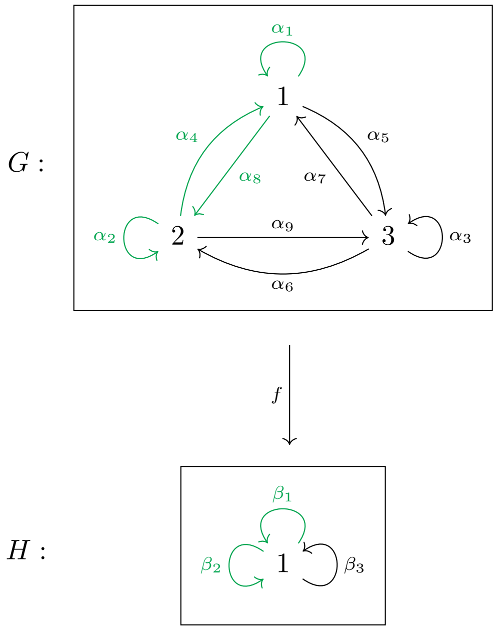
For a given graph morphism $f : G \to H$, the function `ComputeImage` computes the image of a subgraph of G under f; cf. Remark 2.10.

```
// Input:
//   subgraph - a subgraph of a graph G, represented as a pair
//              <vertex list, edge list>
//   f        - a morphism from graph G to an arbitrary graph H,
//              represented as a pair <vertex mapping, edge mapping>,
//              where:
//              - f[1] is a list of vertex mappings <v, vV_f>
//              - f[2] is a list of edge mappings <e, eE_f>
//
// Output:
//   A new subgraph of H representing the image of the input subgraph
//   under the morphism f. The image consists of all vertices and
//   edges that are images of the original vertices and edges.
ComputeImage := function(subgraph,f)
    verticesInImage := [];
    edgesInImage := [];
    for vertexMap in f[1] do
        if vertexMap[1] in subgraph[1] then
            verticesInImage := verticesInImage cat [vertexMap[2]];
        end if;
    end for;
    for edgeMap in f[2] do
        if edgeMap[1] in subgraph[2] then
            edgesInImage := edgesInImage cat [edgeMap[2]];
        end if;
    end for;
    verticesInImage := Set(verticesInImage);
    edgesInImage := Set(edgesInImage);
    verticesInImage := SetToSequence(verticesInImage);
    edgesInImage := SetToSequence(edgesInImage);
    return <verticesInImage,edgesInImage>;
end function;
```

For example, let $f : G \to H$ be the graph morphism given in Remark 4.15. We consider the subgraph $G_1 \leqslant G$ given by

$$\mathrm{V}_{G_1} := \{1, 2\} \quad \text{and} \quad \mathrm{E}_{G_1} := \{\alpha_1, \alpha_2, \alpha_4, \alpha_8\} \ .$$

The following diagram depicts G_1 and the image of G_1 in H:



In `Magma`, we have:

```
> G := <[1,2,3],[<1,1,1>,<2,2,2>,<3,3,3>,<2,4,1>,<1,5,3>,<3,6,2>,<3,7,1>,<1,8,\
2>,<2,9,3>]>;
> H := <[1], [<1,1,1>,<1,2,1>,<1,3,1>]>;
> f := <[<1,1>, <2,1>, <3,1>], [<<1,1,1>,<1,1,1>>, <<2,2,2>,<1,2,1>>, <<3,3,3>\
,<1,3,1>>, <<2,4,1>,<1,1,1>>, <<1,5,3>,<1,3,1>>, <<3,6,2>,<1,2,1>>,
<<3,7,1>,<1,1,1>>, <<1,8,2>,<1,2,1>>, <<2,9,3>,<1,3,1>>]>;
> G1 := <[1,2], [<1,1,1>,<2,4,1>,<2,2,2>,<2,8,1>]>;
> ComputeImage(G1,f);
<[ 1 ], [ <1, 1, 1>, <1, 2, 1> ]>
```

Algorithm 5.8

Given a graph morphism $f : G \to H$ and a vertex v of graph G, the function `GetImageOfVertex` returns the vertex in H that v is mapped to.

```
// Input:
//   f - a graph morphism from G to H.
//   v - a vertex in the domain of vertex map of f.
// Output:
//   The vertex in H that v is mapped to.
GetImageOfVertex := function(f,v)
    for vmap in f[1] do
        if vmap[1] eq v then
            return vmap[2];
        end if;
    end for;
end function;
```

For example, let f be the graph morphism given in the Remark 4.16. We compute the images of vertices 1, 2, $2'$ with Magma:

```
> G := <[ "1", "2", "3", "2'", "3'" ], [ <"1", "1", "2">, <"2", "2", "1">,
> <"2", "3", "3">, <"2'", "4", "3'">, <"3'", "5", "2'">, <"3", "6", "2'">,
> <"2'", "7", "1"> ]>;
> H := <[ "1", "2", "3" ], [ <"1", "1", "2">, <"2", "2", "3">, <"3", "3", "2">,
> <"2", "4", "1"> ]>;
> f := <[ <"1", "1">, <"2", "2">, <"3", "3">, <"2'", "2">, <"3'", "3"> ],
> [ <<"1", "1", "2">, <"1", "1", "2">>, <<"2", "2", "1">, <"2", "4", "1">>,
> <<"2", "3", "3">, <"2", "2", "3">>, <<"2'", "4", "3'">, <"2", "2", "3">>,
> <<"3'", "5", "2'">, <"3", "3", "2">>, <<"3", "6", "2'">, <"3", "3", "2">>,
> <<"2'", "7", "1">, <"2", "4", "1">> ]>;
> GetImageOfVertex(f,"1");
1
> GetImageOfVertex(f,"2");
2
> GetImageOfVertex(f,"2'");
2
```

Algorithm 5.9

Given a graph morphism $f : G \to H$ and an edge e of graph G, the function `GetImageOfEdge` returns the edge in H that e is mapped to.

```
// Input:
//   f - a graph morphism from G to H.
//   e - an edge in the domain of edge map of f.
// Output:
//   The edge in H that e is mapped to.
GetImageOfEdge := function(f,e)
    for emap in f[2] do
        if emap[1][2] eq e then
            return emap[2];
        end if;
    end for;
end function;
```

For example, let f be the graph morphism given in the Remark 4.16. We compute the images of edges 1, 2 with Magma:

```
> G := <[ "1", "2", "3", "2'", "3'" ], [ <"1", "1", "2">, <"2", "2", "1">,
> <"2", "3", "3">, <"2'", "4", "3'">, <"3'", "5", "2'">, <"3", "6", "2'">,
> <"2'", "7", "1"> ]>;
> H := <[ "1", "2", "3" ], [ <"1", "1", "2">, <"2", "2", "3">, <"3", "3", "2">,
> <"2", "4", "1"> ]>;
> f := <[ <"1", "1">, <"2", "2">, <"3", "3">, <"2'", "2">, <"3'", "3"> ],
```

```
> [ <<"1", "1", "2">, <"1", "1", "2">>, <<"2", "2", "1">, <"2", "4", "1">>,
> <<"2", "3", "3">, <"2", "2", "3">>, <<"2'", "4", "3'">, <"2", "2", "3">>,
> <<"3'", "5", "2'">, <"3", "3", "2">>, <<"3", "6", "2'">, <"3", "3", "2">>,
> <<"2'", "7", "1">, <"2", "4", "1">> ]>;
> GetImageOfEdge(f,"1");
<"1", "1", "2">
> GetImageOfEdge(f,"2");
<"2", "4", "1">
> GetImageOfEdge(f,"3");
<"2", "2", "3">
> GetImageOfEdge(f,"4");
<"2", "2", "3">
```

5.4 Algorithms for quasi-cyclic graphs

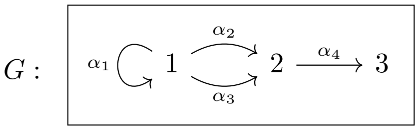
Algorithm 5.10

Given a graph G and a vertex v of G, the algorithm `IsVertexinCycle` determines if v is a vertex of cyclic subgraph of G.

```
// Input:
//   G - a graph represented as a pair <vertex list, edge list>.
//   v - a vertex in the vertex list of G.
// Output:
//   true if v is a vertex of a cyclic subgraph of G.
//   false if none of the cyclic subgraphs of G contains v.
IsVertexinCycle := function(G,v)
    reachable := Set([]);
    frontier := {v};
    while not IsEmpty(frontier) do
        nextFrontier := {};
        for u in frontier do
            for e in GetEdgesFromVertex(u,G) do
                w := e[3];
                if w notin reachable then
                    Include(~reachable,w);
                    Include(~nextFrontier,w);
                end if;
            end for;
        end for;
        frontier := nextFrontier;
    end while;
    if v in reachable then
        return true;
    else
        return false;
    end if;
```

```
end function;
```

For example, let G be the following graph:



In Magma, we have

```
> G := <[1,2,3],[<1,1,1>,<1,2,2>,<1,3,2>,<2,4,3>]>;
> IsVertexinCycle(G,1);
true
> IsVertexinCycle(G,2);
false
```

Algorithm 5.11

Given a graph G, the function GiveCyclicSubgraphs computes all cyclic subgraphs of G.

```
// Input:
//   G - a graph represented as a pair <vertex list, edge list>
//
// Output:
//   A list of all cyclic subgraphs of G.
GiveCyclicSubgraphs := function(G)
    cyclicSubgraphs := [];
    n := #G[1];
    for i in [1..n] do
        C := CyclicGraph(i);
        list_inj := [f : f in ListGraphMorphisms(C,G) | Is_Injective(f,C,
            G)];
        if #list_inj ge 1 then
            cyclicSubgraphs cat := SetToSequence(ListToSetForGraphs([
                ComputeImage(C,f) : f in list_inj]));
        end if;
    end for;
    return cyclicSubgraphs;
end function;
```

For example, let G be the graph given in Figure 5.1. We compute all the cyclic subgraphs of G with Magma:

```
> G := <[1,2,3],[<1,1,1>,<2,2,2>,<3,3,3>,<2,4,1>,<1,5,3>,<3,6,2>,<3,7,1>,<1,8,\
2>,<2,9,3>]>;
> GiveCyclicSubgraphs(G);
[
<[ 1 ], [ <1, 1, 1> ]>,
<[ 3 ], [ <3, 3, 3> ]>,
<[ 2 ], [ <2, 2, 2> ]>,
```

```
<[ 1, 3 ], [ <1, 5, 3>, <3, 7, 1> ]>,
<[ 2, 3 ], [ <2, 9, 3>, <3, 6, 2> ]>,
<[ 1, 2 ], [ <2, 4, 1>, <1, 8, 2> ]>,
<[ 1, 2, 3 ], [ <2, 9, 3>, <3, 7, 1>, <1, 8, 2> ]>,
<[ 1, 2, 3 ], [ <1, 5, 3>, <3, 6, 2>, <2, 4, 1> ]>
]
```

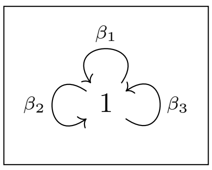
Algorithm 5.12

Given a graph G, the function `GiveQuasiCyclicSubgraphs` computes all quasi-cyclic subgraphs of G.

```
// Input:
//   G - a graph represented as a pair <vertex list, edge list>
//
// Output:
//   A list of all quasi-cyclic subgraphs of G.
GiveQuasiCyclicSubgraphs := function(G)
    quasiCyclicSubgraphs := GiveCyclicSubgraphs(G);
    for graph1 in quasiCyclicSubgraphs do
      for graph2 in quasiCyclicSubgraphs do
        if not IsEmpty(Set(graph1[1]) meet Set(graph2[1])) then
          verticesNewGraph := Set(graph1[1]) join Set(graph2[1]);
          edgesNewGraph := Set(graph1[2]) join Set(graph2[2]);
          newGraph := <SetToSequence(verticesNewGraph),SetToSequence(
              edgesNewGraph)>;
          if not IsGraphInList(newGraph,quasiCyclicSubgraphs) then
            quasiCyclicSubgraphs := quasiCyclicSubgraphs cat [newGraph];
          end if;
        end if;
      end for;
    end for;
  return quasiCyclicSubgraphs;
end function;
```

For example, let G be the following graph:

H :



We compute all the quasi-cyclic subgraphs of G with `Magma`:

```
> H := <[1], [<1,1,1>,<1,2,1>,<1,3,1>]>;
> GiveQuasiCyclicSubgraphs(H);
[
<[ 1 ], [ <1, 1, 1> ]>,
<[ 1 ], [ <1, 3, 1> ]>,
```

```
<[ 1 ], [ <1, 2, 1> ]>,
<[ 1 ], [ <1, 3, 1>, <1, 1, 1> ]>,
<[ 1 ], [ <1, 1, 1>, <1, 2, 1> ]>,
<[ 1 ], [ <1, 3, 1>, <1, 2, 1> ]>,
<[ 1 ], [ <1, 3, 1>, <1, 1, 1>, <1, 2, 1> ]>
]
```

Algorithm 5.13

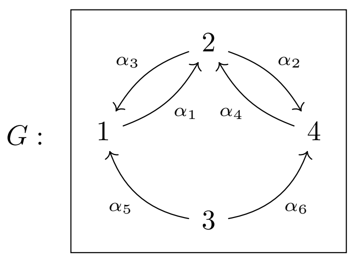
Given a graph G, the function `IsQuasiCyclicGraph` determines whether G is quasi-cyclic.

```
// Input:
//   G - a graph represented as a pair <vertex list, edge list>
//
// Output:
//   true if G is quasi-cyclic;
//   false otherwise.
IsQuasiCyclicGraph := function(G)
    if #G[2] eq 0 then
        return false;
    end if;
    if IsStronglyConnected(G) then
        return true;
    else
        return false;
    end if;
end function;
```

For example, if G is the graph given in Figure 5.1, then in `Magma` we have:

```
> G := <[1,2,3],[<1,1,1>,<2,2,2>,<3,3,3>,<2,4,1>,<1,5,3>,<3,6,2>,<3,7,1>,<1,8,\
2>,<2,9,3>]>;
> IsQuasiCyclicGraph(G);
true
```

If G is the following graph:



Then in `Magma` we have:

```
> G := <[1,2,3,4],[<1,1,2>,<2,2,4>,<2,3,1>,<4,4,2>,<3,5,1>,<3,6,4>]>;
> IsQuasiCyclicGraph(G);
false
```

Algorithm 5.14

Given a graph G, the function `GetCircumferentialLength` computes the circumferential length of a quasi-cyclic graph G. Note that the circumferential length is a property possessed only by quasi-cyclic subgraphs, so the argument of this function should be a quasi-cyclic graph.

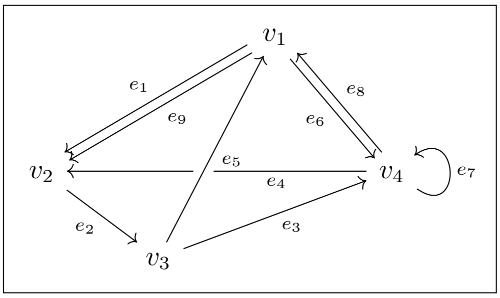
```
// Input:
//   G - a quasi-cyclic graph
//
// Output:
//   The circumferential length of G
GetCircumferentialLength := function(G)
    allCyclicSubgraphs := Set(GiveCyclicSubgraphs(G));
    subsetCyclicSubgraphs := Subsets(allCyclicSubgraphs);
    lengthStacks := [];
    for set in subsetCyclicSubgraphs do
        if #&join{Set(C[2]) : C in set} eq #G[2] then
            length := &+[#C[2] : C in set];
            Append(~lengthStacks, length);
        end if;
    end for;
    min := Min(lengthStacks);
    return min;
end function;
```

For example, if G is the graph given in Figure 5.1, we compute the circumferential length of G with `Magma`:

```
> G := <[1,2,3],[<1,1,1>,<2,2,2>,<3,3,3>,<2,4,1>,<1,5,3>,<3,6,2>,<3,7,1>,<1,8,\
2>,<2,9,3>]>;
> GetCircumferentialLength(G);
9
```

Suppose that G is the graph given as follow:

$G:$



We compute the circumferential length of G with `Magma`:

```
G := <[1,2,3,4],[<1,1,2>,<2,2,3>,<3,3,4>,<4,4,2>,<3,5,1>,
<1,6,4>,<4,7,4>,<4,8,1>,<1,9,2>]>;
> GetCircumferentialLength(G);
12
```

5.5 Algorithms for quasi-isomorphisms

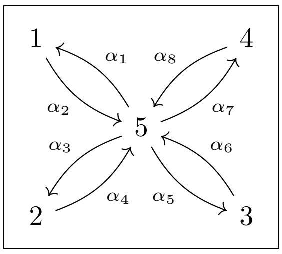
Algorithm 5.15

Given a graph G, the function `GetAdjacencyMatrix` returns the adjacency matrix of G; cf. Definition 4.26.

```
// Input:
//   G - a graph
//
// Output:
//   The adjacency matrix of G.
GetAdjacencyMatrix := function(G) // G: Graph
    n := #G[1];
    AG := ZeroMatrix(Rationals(), n, n);
    for e in G[2] do
        AG[Index(G[1],e[1]), Index(G[1],e[3])] := AG[Index(G[1],e[1]),
            Index(G[1],e[3])] + 1;
    end for;
    return AG;
end function;
```

For example, let G be the following graph:

G :



We compute the adjacency matrix of G via `Magma`:

```
> G := <[1,2,3,4,5],[<5,1,1>,<1,2,5>,<5,3,2>,<2,4,5>,<5,5,3>,<3,6,5>,<5,7,4>,<\
4,8,5>]>;
> GetAdjacencyMatrix(G);
[0 0 0 0 1]
[0 0 0 0 1]
[0 0 0 0 1]
[0 0 0 0 1]
[1 1 1 1 0]
```

Algorithm 5.16

Given a graph morphism $f : G \to H$ between finite graphs G and H, the function GetMorphismMatrix returns the matrix $A_f \in \mathbb{Q}^{|\,V_G\,| \times |\,V_H\,|}$ (see Lemma 4.28) with

$$(A_f)_{ij} := \begin{cases} 1 & \text{if } v_i^{(G)}\,V_f = v_j^{(H)} \\ 0 & \text{if } v_i^{(G)}\,V_f \neq v_j^{(H)} \end{cases}$$

for $i \in [1, |\,V_G\,|]$ and $j \in [1, |\,V_H\,|]$.

```
// Input:
//   G - a graph
//   H - a graph
//   f - a graph morphism from G to H.
// Output:
//
GetMorphismMatrix := function(G,H,f)
    Af := ZeroMatrix(Rationals(), #G[1], #H[1]);
    for vmap in f[1] do
        Af[Index(G[1], vmap[1]), Index(H[1],vmap[2])] := 1;
    end for;
    return Af;
end function;
```

For example, we consider the graph morphism given in the following diagram:

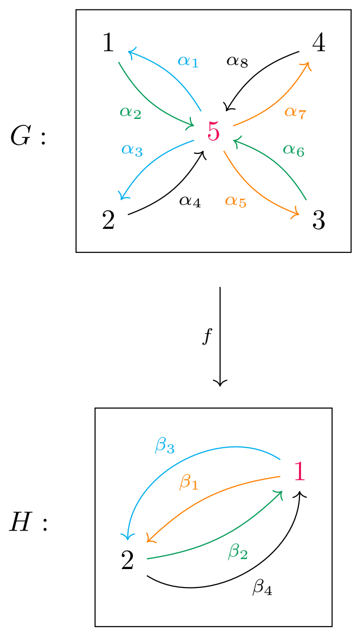


Figure 5.2

The graph morphism $f : G \to H$ is defined by

$$\mathrm{V}_f := \left\{ \begin{array}{ccc} 1 & \mapsto & 2 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 2 \\ 4 & \mapsto & 2 \\ 5 & \mapsto & 1 \end{array} \right\} \quad \text{and} \quad \mathrm{E}_f := \left\{ \begin{array}{ccc} \alpha_1 & \mapsto & \beta_3 \\ \alpha_2 & \mapsto & \beta_2 \\ \alpha_3 & \mapsto & \beta_3 \\ \alpha_4 & \mapsto & \beta_4 \\ \alpha_5 & \mapsto & \beta_1 \\ \alpha_6 & \mapsto & \beta_2 \\ \alpha_7 & \mapsto & \beta_1 \\ \alpha_8 & \mapsto & \beta_4 \end{array} \right\}.$$

We compute the matrix of A_f via Magma:

```
> G := <[1,2,3,4,5],[<5,1,1>,<1,2,5>,<5,3,2>,<2,4,5>,<5,5,3>,<3,6,5>,<5,7,4>,<\
4,8,5>]>;
> H := <[1,2],[<1,1,2>, <1,3,2>, <2,2,1>, <2,4,1>]>;
> f := <[<5,1>, <1,2>, <2,2>, <3,2>, <4,2>],[<<5,1,1>,<1,3,2>>, <<1,2,5>,<2,2,1>>,
<<5,3,2>,<1,3,2>>, <<2,4,5>,<2,4,1>>, <<5,5,3>,<1,1,2>>, <<3,6,5>,<2,2,1>>,
<<5,7,4>,<1,1,2>>, <<4,8,5>,<2,4,1>>]>;
> GetMorphismMatrix(G,H,f);
[0 1]
[0 1]
[0 1]
[0 1]
[1 0]
```

Algorithm 5.17

Given a graph morphism $f : G \to H$, the function IsQuasiisomorphism determines whether f is a quasi-isomorphism.

```
// Input:
//   G - a quasi-cyclic graph represented as a pair <vertex list, edge
    list>
//   H - a graph represented as a pair <vertex list, edge list>
//   f - a morphism from graph G to H, represented as a pair <vertex
    mapping, edge mapping>,
//   where:
//     - f[1] is a list of vertex mappings <v, vV_f>
//     - f[2] is a list of edge mappings <e, eE_f>
// Output:
//   true if f is a quasi-isomorphism;
//   false otherwise.
IsQuasiisomorphism := function(G,H,f)
    for e1 in G[2] do
        for e2 in G[2] do
            if e1 ne e2 then
                if e1[1] eq e2[1] and e1[3] eq e2[3] then
```

```
                    if GetImageOfEdge(f,e1[2]) eq GetImageOfEdge(f,e1[2])
                        then
                        return false;
                    end if;
                end if;
            end if;
        end for;
    end for;

    AG := GetAdjacencyMatrix(G);
    AH := GetAdjacencyMatrix(H);
    Af := GetMorphismMatrix(G,H,f);
    n := #G[1] + #H[1];
    for k in [1..n] do
        Diag_AG_k := DiagonalMatrix([(AG^k)[i][i] : i in [1..Nrows(AG^k
            )]]);
        Diag_AH_k := DiagonalMatrix([(AH^k)[i][i] : i in [1..Nrows(AH^k
            )]]);
        M := Transpose(Af) * Diag_AG_k  * Af - Diag_AH_k;
        if not IsZero(M) then
            return false;
        end if;
    end for;

    vertices := [];
    edges := [];
    for v_plus in G[1] do
        for v_minus in G[1] do
            if GetImageOfVertex(f,v_plus) eq GetImageOfVertex(f,v_minus)
                then
                Append(~vertices, <v_plus,v_minus>);
            end if;
        end for;
    end for;
    vertices := SetToSequence(Set(vertices));
    for e_plus in G[2] do
        for e_minus in G[2] do
            if GetImageOfEdge(f,e_plus[2]) eq GetImageOfEdge(f,e_minus
                [2]) then
                Append(~edges, <<e_plus[1],e_minus[1]>,<e_plus[2],e_minus
                    [2]>,<e_plus[3],e_minus[3]>>);
            end if;
        end for;
    end for;
    P_f_G := <vertices, edges>;
    edges := SetToSequence(Set(edges));
    for v in P_f_G[1] do
        if v[1] ne v[2] then
            if IsVertexinCycle(P_f_G,v) then
                return false;
```

```
                end if;
            end if;
        end for;

        return true;
    end function;
```

For example, we verify the graph morphism f given in the Figure 5.2 via `Magma`:

```
> G := <[1,2,3,4,5],[<5,1,1>,<1,2,5>,<5,3,2>,<2,4,5>,<5,5,3>,<3,6,5>,<5,7,4>,<\
4,8,5>]>;
> H := <[1,2],[<1,1,2>, <1,3,2>, <2,2,1>, <2,4,1>]>;
> f := <[<5,1>, <1,2>, <2,2>, <3,2>, <4,2>],[<<5,1,1>,<1,3,2>>, <<1,2,5>,<2,2,1>>,
<<5,3,2>,<1,3,2>>, <<2,4,5>,<2,4,1>>, <<5,5,3>,<1,1,2>>, <<3,6,5>,<2,2,1>>,
<<5,7,4>,<1,1,2>>, <<4,8,5>,<2,4,1>>]>;
> IsQuasiisomorphism(G,H,f);
true
```

List of symbols

Symbol	Meaning
V_G	the vertex set of graph G
E_G	the edge set of graph G
s_G	source map of graph G
t_G	target map of graph G
$\mathrm{E}_G(v,*)$	all edges in G with source vertex v
$\mathrm{E}_G(*,v)$	all edges in G with target vertex v
$\mathrm{V}_G(v,*)$	all target vertices of edges in $\mathrm{E}_G(v,*)$
$\mathrm{V}_G(*,v)$	all source vertices of edges in $\mathrm{E}_G(*,v)$
$G' \leqslant G$	G' is subgraph of G
$G' < G$	$G' \leqslant G$ and $G' \neq G$
$G = [G_\lambda]_{\lambda\in\Lambda}$	$(G_\lambda)_{\lambda\in\Lambda}$ is a partition of G
$(G,H)_{\mathrm{Gph}}$	the set of graph morphisms from G to H
V_f	vertex map of morphism f
E_f	edge map of morphism f
$(G)f$	image graph of graph G under f
C_n	cyclic graph with n edges, where $n \in \mathbb{Z}_{\geqslant 1}$
$\mathrm{v}_i (\in \mathrm{V}_{\mathrm{C}_n})$	vertex $\mathrm{v}_{i+n\mathbb{Z}}$ of C_n
$\mathrm{e}_i (\in \mathrm{V}_{\mathrm{C}_n})$	edge $\mathrm{e}_{i+n\mathbb{Z}}$ of C_n
$\mathrm{S}_{\mathrm{c}}(G)$	cyclic subgraphs of G
$\mathrm{S}_{\mathrm{qc}}(G)$	quasi-cyclic subgraphs of G
$\mathrm{S}_{\mathrm{qc}}^{\max}(G)$	maximal quasi-cyclic subgraphs of G
D_n	directed path graph with n edges, where $n \in \mathbb{Z}_{\geqslant 0}$
$\mathrm{D}_n^{[i_0,j_0]}$, $0 \leqslant i_0 \leqslant j_0 \leqslant n$	subgraph of D_n from $\hat{\mathrm{v}}_{i_0}$ to $\hat{\mathrm{v}}_{j_0}$
$\hat{\mathrm{v}}_i (\in \mathrm{V}_{\mathrm{D}_n})$	the i-th vertex of D_n
$\hat{\mathrm{e}}_i (\in \mathrm{E}_{\mathrm{D}_n})$	the i-th edge of D_n
p_{s}	start vertex of path p
p_{t}	end vertex of path p
$\overline{\mathrm{V}_G(v,*)}$	end vertices of all paths in G starting with v
(X,f) $(= (X, f : G \to H))$	$(X,f) : (X,G)_{\mathrm{Gph}} \to (X,H)_{\mathrm{Gph}}$, $u \mapsto uf$
$G \simeq H$	graphs G and H are isomorphic
$G \xrightarrow[\sim]{f} H$	$f : G \to H$ is an isomorphism
$G \xrightarrow[\approx]{f} H$	$f : G \to H$ is a quasi-isomorphism
$P_f^{(G)}$	given $f \in (G,H)_{\mathrm{Gph}}$, graph $P_f^{(G)}$ is the pullback of f and f; cf. (4.3)
A_G	adjacency matrix of graph G

References

[1] Bisson, T., Tsemo, A. *A homotopical algebra of graphs related to zeta series.* Homology, Homotopy and Applications 11(1), p. 171-184, 2009.

[2] Bondy, J. A., Murty, U. S. R. *Graph Theory.* Springer, New York, 2008.

[3] Bosma, W., Cannon, J. J., Fieker, C., Steel, A. (eds.) *Handbook of Magma functions.* Edition v2.21-3.

[4] Hess, J. *On the Bisson-Tsemo model category of graphs.* Master thesis, University of Stuttgart, 2022.

Zusammenfassung

Die hier behandelten Graphen G sind gerichtet und bestehen aus einer Knotenmenge V_G und einer Kantenmenge E_G. Wir verwenden Abbildungen $\mathrm{s}_G, \mathrm{t}_G : \mathrm{E}_G \to \mathrm{V}_G$, die als Quelle bzw. Ziel bezeichnet werden, um die Richtung einer Kante $e \in \mathrm{E}_G$, nämlich von $e\,\mathrm{s}_G$ nach $e\,\mathrm{t}_G$, festzustellen. Ein Graphmorphismus $f : G \to H$, von einem Graphen G nach einem Graphen H, besteht aus einer Knotenabbildung $\mathrm{V}_f : \mathrm{V}_G \to \mathrm{V}_H$ und einer Kantenabbildung $\mathrm{E}_G : \mathrm{E}_G \to \mathrm{E}_H$, die die Richtung jeder Kante während der Abbildung beibehalten.

Für $n \geqslant 1$ haben wir den zyklischen Graph C_n. Jeder Graphmorphismus $f : G \to H$ induziert durch Komposition eine Abbildung

$$(\mathrm{C}_n, f) : (\mathrm{C}_n, G)_{\mathrm{Gph}} \to (\mathrm{C}_n, H)_{\mathrm{Gph}}, \quad \varphi \mapsto \varphi \cdot f.$$

$$\begin{array}{ccc} \mathrm{C}_n & \xrightarrow{\varphi} & G \\ & \searrow^{\psi} & \downarrow f \\ & & H \end{array}$$

Ist der Morphismus (C_n, f) bijektiv für jedes $n \in \mathbb{Z}_{\geqslant 1}$, so bezeichnen wir $f : G \to H$ als *Quasiisomorphismus.*

Die Bestimmung, ob ein Graphmorphismus f ein Quasiisomorphismus ist, kann jedoch schwierig sein, da der Bestimmungsprozess anhand der Definition die Überprüfung der Bijektivität von (C_n, f) für $n \in \mathbb{Z}_{\geqslant 1}$ erfordert, und also kein endliches Verfahren ist.

Zur Untersuchung der Eigenschaften von Quasiisomorphismen können quasizyklische Graphen verwendet werden. Ein Graph C heißt *quasizyklisch*, wenn er Bild eines zyklischen Graphen unter einem Graphmorphismus ist. Ein quasizyklischer Teilgraph von einem Graph, welcher in keinen weiteren quasizyklischen Teilgraphen echt enthalten ist, heißt maximal.

Sei $f : G \to H$ ein Quasiisomorphismus.

Jeder zyklische Teilgraph von H kann eindeutig zu einem zyklischen Teilgraphen von G gehoben werden. Falls G endlich ist und zwei zyklische Teilgraphen von H mit wenigstens einem gemeinsamen Punkt vorliegen, dann müssen die zu ihnen korrespondierenden Teilgraphen von G nicht unbedingt einen gemeinsamen Vertex haben. Sie sind aber in einem gemeinsamen quasizyklischen Graphen von G enthalten.

Jeder quasizyklische Teilgraph von H kann zu einem quasizyklischen Teilgraph von G gehoben werden. Falls G und H endlich sind, dann kann jeder maximale quasizyklische Teilgraph von H auf eindeutige Weise zu einem maximalen quasizyklischen Teilgraph von G gehoben werden.

Es wird ein in endlich vielen Schritten überprüfbares Kriterium dafür angegeben, zu entscheiden, ob ein Morphismus zwischen endlichen Graphen ein Quasiisomorphismus ist.

Versicherung

Hiermit versichere ich,

1. dass ich meine Arbeit selbstständig verfasst habe,
2. dass ich keine anderen als die angegeben Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,
3. dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und
4. dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart 30.09.2025
Danning Liu