A projective resolution and automorphisms for $\mathbf{Z}D_8$

Bachelor's Thesis

Maximilian Hofmann
Universität Stuttgart

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Contents

Conventions

Let X, Y and Z be sets.

- Composition of maps is written on the right, i.e., $X \xrightarrow{f} Y \xrightarrow{g} Z = X \xrightarrow{fg} Z$ and for $x \in X$ we write xf for the image of x under f.
- We denote the identity map on X by $1 := 1_X := \mathrm{id}_X$.
- We write $X \subseteq Y$ if X is a subset of Y.
- Let $X \xrightarrow{f} Y$ be a map. Let $A \subseteq X$ and $B \subseteq Y$. We write $A \xrightarrow{f|_A^B} B$ for the map given by $af|_A^B := af$ for $a \in A$. If $A = X$, we write $f|^B := f|_X^B$ and if $B = Y$ we write $f|_A := f|_A^Y$.
- If X is a finite set, write $|X|$ for the number of elements in X.
- For $a \in \mathbf{Z}$ we write $\mathbf{Z}_{\geq a} := \{z \in \mathbf{Z} : z \geq a\}$.
- For $a, b \in \mathbf{Z}$ let $[a, b] := \{z \in \mathbf{Z} : a \leq z \leq b\}$ denote the integer interval.
- If $x, y \in X$ then let $\partial_{x,y} := 1$ for $x = y$ and $\partial_{x,y} := 0$ for $x \neq y$.
- For $m \in \mathbf{Z}_{\geq 0}$, we write S_m for the symmetric group on m symbols.
- For $x, y \in \mathbf{Q}$ and $a \in \mathbf{Z}$ we write $x \equiv_a y$ for $x - y \in a\mathbf{Z}$.
- For a matrix A we write A^{T} for its transpose.
- All rings have a multiplicative identity.
- For a ring R we write $R^\times := R \setminus \{0\}$.
- For a ring R and $a \in R$ we write $R \xrightarrow{\dot{a}} R$ for the R-linear map given by $r \longmapsto ra$.
- We denote (left, right, two-sided) ideals in a ring R by fraktur letters $\mathfrak{a}, \mathfrak{b}, \ldots$. We call two-sided ideals simply ideals.
- Let R be a ring. We denote its centre by $\mathrm{Z}(R)$ and its group of units by $\mathrm{U}(R)$.
- Let R be a ring. Then by an R-module we understand a left R-module.
- Let R and S be rings. For an R-S-bimodule M we sometimes write ${}_R M_S := M$.
- Let R be a ring, $n \in \mathbf{Z}_{\geq 0}$ and let $(M_i)_{i \in [1,n]}$ be R-modules. We denote the R-linear inclusion into the k-th summand by $M_k \xrightarrow{\iota_k} \bigoplus_{i \in [1,n]} M_i$ and the R-linear projection onto the k-th summand by $\bigoplus_{i \in [1,n]} M_i \xrightarrow{\pi_k} M_k$.
- Let R be a ring. A complex of R-modules is a sequence of R-modules and R-linear maps of the form

  $$\cdots\to P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{d_{-1}} P_{-1} \xrightarrow{d_{-2}} P_{-2} \to\cdots,$$

  such that $d_k d_{k-1} = 0$ for $k \in \mathbf{Z}$. The maps d_k are called differentials.
  If there is some $n \in \mathbf{Z}$ such that $P_k = 0$ for $k < n$ we write

  $$\cdots\to P_{n+3} \xrightarrow{d_{n+2}} P_{n+2} \xrightarrow{d_{n+1}} P_{n+1} \xrightarrow{d_n} P_n \longrightarrow 0.$$

  If there is some $n \in \mathbf{Z}$ such that $P_k = 0$ for $k > n$ we write

  $$0 \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \xrightarrow{d_{n-2}} P_{n-2} \xrightarrow{d_{n-3}} P_{n-3} \to\cdots.$$

- Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories and $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{E}$ be functors. For a morphism $A \xrightarrow{f} B$ in $\mathcal{C}$ we write $FA \xrightarrow{Ff} FB$ for its image in $\mathcal{D}$. We compose functors in a traditional way, i.e. we write $\mathcal{C} \xrightarrow{G \circ F} \mathcal{E}$ for the composition of F with G.

Chapter 0

Introduction

0.1 Projective resolutions

Let R be a commutative ring and let A be an R-algebra. Let M be an A-module. A *projective resolution* P of M over A is a complex

$$P = \Big(\ldots \longrightarrow P_3 \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow 0 \Big)$$

of projective A-module P_k and A-linear maps d_k for $k \in \mathbf{Z}_{\geq 0}$ such that there is an A-linear map $P_0 \xrightarrow{\varepsilon} M$ and we have an acyclic complex

$$P' = \Big(\ldots \longrightarrow P_3 \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \Big) .$$

The latter complex P' is called the *augmented projective resolution* corresponding to P and ε is called the *augmentation map*.

Suppose given a finite group G. Then a special case of the construction above is given when $A := RG$ is the group ring of G over R and $M := R$ is the trivial RG-module.

For the case that G has a cyclic Sylow p-subgroup and a discrete valuation ring R with residue field of characteristic p it is known that there is a periodic projective resolution of the trivial RG-module R, cf. [**4**, Theorem (62.56)].

In [**1**, Theorems 5.14.2 and 5.14.5] Benson proposes a generalisation to the case of non-cyclic Sylow p-subgroups. For an arbitrary finite group G his construction gives a projective resolution of the trivial RG-module R that is given by the n-fold tensor product of eventually periodic complexes.

Let $\mathfrak{r} \subseteq A$ be the Jacobson radical of A. A projective resolution P of an A-module M is called minimal if $\operatorname{im}(d_k) \subseteq \mathfrak{r}P_k$ for all $k \in \mathbf{Z}_{\geq 0}$, cf. [**2**, Proposition 9, §3.6]. Benson's construction does not necessarily give a minimal projective resolution, cf. [**1**, p. 206].

0.2 Wedderburn images

Let G be a finite group. Let K be a field such that the characteristic of K does not divide the order of the group. Let m be the exponent of the group G, i.e. the least common multiple of all orders of element of G. Suppose that K contains all m-th roots of unity. In this case, the group algebra KG is split semisimple, i.e. there is an isomorphism of K-algebras

$$KG \xrightarrow[\sim]{\tilde{\omega}} \prod_{i=1}^{t} K_i^{n_i \times n_i},$$

where $t \in \mathbf{Z}_{\geq 1}$ and $n_i \in \mathbf{Z}_{\geq 1}$ for $i \in [1, t]$, cf. [**3**, Theorems (3.34), (17.1)]. We call $\tilde{\omega}$ a *Wedderburn isomorphism*.

Suppose $R \subseteq K$ is a subring. Restricting $\tilde{\omega}$, we obtain an embedding

$$RG \overset{\omega}{\hookrightarrow} \prod_{i=1}^{t} K_i^{n_i \times n_i}.$$

We call ω the *Wedderburn embedding* and its image $\mathrm{im}(\omega)$ the *Wedderburn image*.

Now let $\mathrm{D}_8 = \langle a, b \mid a^4, b^2, (ba)^2 \rangle$ be the dihedral group of order 8. A Wedderburn isomorphism for the group algebra $\mathbf{Q}\mathrm{D}_8$ is given by

$$\begin{array}{ccc} \mathbf{Q}\mathrm{D}_8 & \xrightarrow[\sim]{\tilde{\omega}} & \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}^{2\times 2} \\ & & \\ a & \longmapsto & \left(1, 1, -1, -1, \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)\right) \\ b & \longmapsto & \left(1, -1, 1, -1, \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right)\right). \end{array}$$

Write $\Gamma := \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2}$. Restricting $\tilde{\omega}$ to the integral group ring $\mathbf{Z}\mathrm{D}_8 \subseteq \mathbf{Q}\mathrm{D}_8$, we obtain a Wedderburn embedding $\mathbf{Z}\mathrm{D}_8 \xrightarrow{\omega} \Gamma$. In Lemma 15 we calculate the Wedderburn image to be

$$\Lambda := \mathrm{im}(\omega) = \left\{ \left(p, q, r, s, \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)\right) \in \Gamma : \begin{array}{c} p \equiv_2 q \equiv_2 r \equiv_2 s \equiv_2 t \equiv_2 w,\ q - r \equiv_4 v,\ q - s \equiv_4 2u, \\ p + q \equiv_4 r + s,\ p + q + r + s \equiv_8 2(t + w) \end{array} \right\}.$$

0.3 Projective resolution of $\mathbf{Z}$ over $\mathbf{Z}\mathrm{D}_8$ and localisation

We give a projective resolution of the trivial $\mathbf{Z}\mathrm{D}_8$-module $\mathbf{Z}$ over $\mathbf{Z}\mathrm{D}_8$, which can be written as the total complex of the following double complex.

$$\Xi = \left(\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots \to & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_-} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_-} & \mathbf{Z}\mathrm{D}_8 \\
 & & \downarrow C_+ & & \downarrow B_+ & & \downarrow B_+ & & \downarrow B_+ & & \downarrow B_+ \\
\cdots \to & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_-} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_-} & \mathbf{Z}\mathrm{D}_8 \\
 & & \downarrow C_- & & \downarrow C_- & & \downarrow B_- & & \downarrow B_- & & \downarrow B_- \\
\cdots \to & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_-} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_-} & \mathbf{Z}\mathrm{D}_8 \\
 & & \downarrow C_+ & & \downarrow C_+ & & \downarrow C_+ & & \downarrow B_+ & & \downarrow B_+ \\
\cdots \to & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_-} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{D_-} & \mathbf{Z}\mathrm{D}_8 \\
 & & \downarrow C_- & & \downarrow C_- & & \downarrow C_- & & \downarrow C_- & & \downarrow B_- \\
\cdots \to & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_-} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_+} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{A_-} & \mathbf{Z}\mathrm{D}_8
\end{array} \right)$$

The maps in this double complex are given by multiplication with the following elements of Λ. Here we identified the group ring $\mathbf{Z}\mathrm{D}_8$ with its Wedderburn image Λ along the Wedderburn embedding.

$$
\begin{aligned}
A_- &:= \left(0,2,0,2,\left(\begin{smallmatrix}0&0\\2&2\end{smallmatrix}\right)\right) &&= 1-b & B_- &:= \left(0,2,2,0,\left(\begin{smallmatrix}2&1\\0&0\end{smallmatrix}\right)\right) &&= 1-ba^3\\
A_+ &:= \left(2,0,2,0,\left(\begin{smallmatrix}2&0\\-2&0\end{smallmatrix}\right)\right) &&= 1+b & B_+ &:= \left(2,0,0,2,\left(\begin{smallmatrix}0&-1\\0&2\end{smallmatrix}\right)\right) &&= 1+ba^3\\
C_- &:= \left(0,4,-4,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) &&= a+a^3-b-ba^2 & D_- &:= \left(0,4,0,-4,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) &&= a+a^3-ba-ba^3\\
C_+ &:= \left(4,0,0,-4,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) &&= a+a^3+b+ba^2 & D_+ &:= \left(4,0,-4,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) &&= a+a^3+ba+ba^3
\end{aligned}
$$

Note that all rows and columns of the double complex Ξ become eventually periodic.

Theorem (cf. Theorems 20 and 30). The total complex of Ξ

$$P = \Big(\ldots \longrightarrow (\mathbf{Z}\mathrm{D}_8)^{\oplus 4} \xrightarrow{d_2} (\mathbf{Z}\mathrm{D}_8)^{\oplus 3} \xrightarrow{d_1} (\mathbf{Z}\mathrm{D}_8)^{\oplus 2} \xrightarrow{d_0} \mathbf{Z}\mathrm{D}_8 \longrightarrow 0\Big)$$

is a projective resolution of the trivial $\mathbf{Z}\mathrm{D}_8$-module $\mathbf{Z}$. By extending scalars from $\mathbf{Z}$ to $\mathbf{Z}_{(2)}$, we arrive at a minimal projective resolution $P_{(2)}$ of the trivial $\mathbf{Z}_{(2)}\mathrm{D}_8$-module $\mathbf{Z}_{(2)}$ over $\mathbf{Z}_{(2)}\mathrm{D}_8$.

To verify that P is indeed acyclic and hence a projective resolution, we construct a $\mathbf{Z}$-linear contracting homotopy for the corresponding augmented projective resolution P'.

In [**6**] and [**9**], the *Wall-Hamada resolution* of the trivial $\mathbf{Z}\mathrm{D}_8$-module $\mathbf{Z}$ over $\mathbf{Z}\mathrm{D}_8$ is constructed. The Wall-Hamada resolution can be written as the total complex of a double complex Ξ^{WH} and is minimal after localisation at (2). We use the sign convention of [**7**].

$$
\Xi^{\mathrm{WH}} = \left(\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\dashrightarrow & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{1+a+a^2+a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{1+a+a^2+a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8\\
 & & \downarrow{\scriptstyle b+1} & & \downarrow{\scriptstyle ba+1} & & \downarrow{\scriptstyle b-1} & & \downarrow{\scriptstyle ba-1} & & \downarrow{\scriptstyle b+1}\\
\dashrightarrow & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{-1-a-a^2-a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{-1-a-a^2-a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8\\
 & & \downarrow{\scriptstyle b-1} & & \downarrow{\scriptstyle ba-1} & & \downarrow{\scriptstyle b+1} & & \downarrow{\scriptstyle ba+1} & & \downarrow{\scriptstyle b-1}\\
\dashrightarrow & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{1+a+a^2+a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{1+a+a^2+a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8\\
 & & \downarrow{\scriptstyle b+1} & & \downarrow{\scriptstyle ba+1} & & \downarrow{\scriptstyle b-1} & & \downarrow{\scriptstyle ba-1} & & \downarrow{\scriptstyle b+1}\\
\dashrightarrow & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{-1-a-a^2-a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{-1-a-a^2-a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8\\
 & & \downarrow{\scriptstyle b-1} & & \downarrow{\scriptstyle ba-1} & & \downarrow{\scriptstyle b+1} & & \downarrow{\scriptstyle ba+1} & & \downarrow{\scriptstyle b-1}\\
\dashrightarrow & & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{1+a+a^2+a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{1+a+a^2+a^3} & \mathbf{Z}\mathrm{D}_8 & \xrightarrow{a-1} & \mathbf{Z}\mathrm{D}_8
\end{array}\right)
$$

Now both Ξ and its mirror image are not isomorphic to Ξ^{WH} as double complexes, cf. Remark 22.

0.4 Automorphisms

In order to describe symmetries of the double complex corresponding to our projective resolution, we investigate $\mathbf{Z}$-algebra automorphisms of the Wedderburn image Λ of $\mathbf{Z}\mathrm{D}_8$.

We start by using the isomorphism $\mathbf{Q} \otimes \Lambda \simeq \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}^{2\times 2}$ to see that any $\mathbf{Z}$-algebra automorphism of Λ is given by permutation on the first four entries and conjugation by an element of $\mathrm{GL}_2(\mathbf{Q})$ on the 2×2-matrix block, cf. Lemma 34.

After giving a description of the group of central automorphisms of Λ, i.e. the group of automorphism that fix the centre of Λ pointwise, we are able to give an isomorphism

$$\mathrm{Outcent}_{\mathbf{Z}\text{-alg}}(\Lambda) \xrightarrow{\sim} \mathrm{C}_2,$$

cf. Lemma 40. This result is also established in [**5**, Theorem 5.6] using different methods.

With knowledge of $\mathrm{Outcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$ at hand and using $\mathbf{Z}\mathrm{D}_8 \simeq \Lambda$, we obtain the

Theorem (cf. Theorem 43 and Corollary 44). We have

$$\mathrm{Out}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8) \xrightarrow{\sim} \mathrm{D}_8 \times \mathrm{C}_2$$

for the full outer automorphism group. Moreover, the residue class map

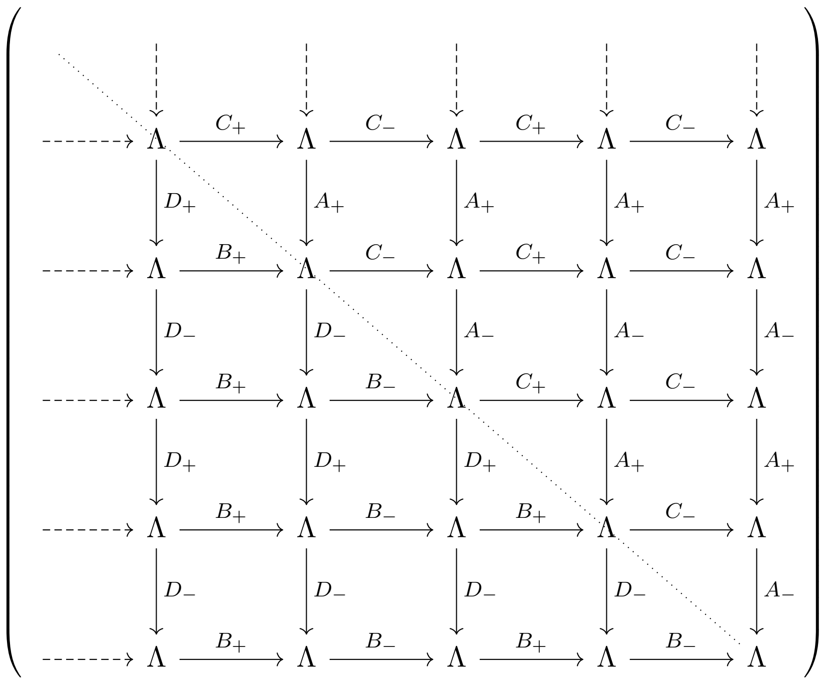
$$\mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8) \xrightarrow{\rho} \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8)$$

is a retraction.

We describe how certain double complexes of Λ-modules can be twisted using automorphisms of Λ. We show that some automorphisms of Λ describe symmetries of Ξ using the twisted double complex Ξ^{ψ}.

The following two diagrams show examples of such symmetries, the first one shows a reflection along the diagonal, the second one a shift along the diagonal.

$\Xi^{\psi_1} =$



$$
\Xi^{\psi_2} = \left(\begin{array}{ccccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dashrightarrow & & \Lambda & \xrightarrow{D_-} & \Lambda & \xrightarrow{D_+} & \Lambda & \xrightarrow{D_-} & \Lambda & \xrightarrow{D_+} & \Lambda \\
 & & \downarrow{\scriptstyle C_-} & & \downarrow{\scriptstyle B_-} & & \downarrow{\scriptstyle B_-} & & \downarrow{\scriptstyle B_-} & & \downarrow{\scriptstyle B_-} \\
\dashrightarrow & & \Lambda & \xrightarrow{A_-} & \Lambda & \xrightarrow{D_+} & \Lambda & \xrightarrow{D_-} & \Lambda & \xrightarrow{D_+} & \Lambda \\
 & & \downarrow{\scriptstyle C_+} & & \downarrow{\scriptstyle C_+} & & \downarrow{\scriptstyle B_+} & & \downarrow{\scriptstyle B_+} & & \downarrow{\scriptstyle B_+} \\
\dashrightarrow & & \Lambda & \xrightarrow{A_-} & \Lambda & \xrightarrow{A_+} & \Lambda & \xrightarrow{D_-} & \Lambda & \xrightarrow{D_+} & \Lambda \\
 & & \downarrow{\scriptstyle C_-} & & \downarrow{\scriptstyle C_-} & & \downarrow{\scriptstyle C_-} & & \downarrow{\scriptstyle B_-} & & \downarrow{\scriptstyle B_-} \\
\dashrightarrow & & \Lambda & \xrightarrow{A_-} & \Lambda & \xrightarrow{A_+} & \Lambda & \xrightarrow{A_-} & \Lambda & \xrightarrow{D_+} & \Lambda \\
 & & \downarrow{\scriptstyle C_+} & & \downarrow{\scriptstyle C_+} & & \downarrow{\scriptstyle C_+} & & \downarrow{\scriptstyle C_+} & & \downarrow{\scriptstyle B_+} \\
\dashrightarrow & & \Lambda & \xrightarrow{A_-} & \Lambda & \xrightarrow{A_+} & \Lambda & \xrightarrow{A_-} & \Lambda & \xrightarrow{A_+} & \Lambda
\end{array}\right)
$$

Since the Wedderburn image $\Lambda \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2}$ involves one 2×2-block, we have an antiautomorphism given by tranposition of this matrix and conjugation by a rational matrix to respect the congruences in Λ. On the other hand, in $\mathbf{Z}\mathrm{D}_8$ we have an antiautomorphism given by inversion on the group elements, which form a basis of the group ring.

In Remark 41, we relate these two antiautomorphisms using the Wedderburn embedding ω, an inner automorphism f_a and a non-inner central automorphism φ.

$$
\begin{array}{ccccc}
g & \quad & \mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\omega} \Lambda & \quad & \left(p, q, r, s, \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)\right) \\
\big\downarrow & & \wr\big\downarrow{\scriptstyle i} \qquad\qquad \wr\big\downarrow{\scriptstyle t} & & \big\downarrow \\
g^{-1} & & \mathbf{Z}\mathrm{D}_8 \qquad\qquad \Lambda & & \left(p, q, r, s, \left(\begin{smallmatrix} w & u \\ v & t \end{smallmatrix}\right)\right) \\
\big\downarrow & & \wr\big\downarrow{\scriptstyle f_a} \qquad\qquad \wr\big\downarrow{\scriptstyle \varphi} & & \big\downarrow \\
a^{-1}g^{-1}a & & \mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\omega} \Lambda & & \left(p, q, r, s, \left(\begin{smallmatrix} w & -u \\ -v & t \end{smallmatrix}\right)\right)
\end{array}
$$

0.5 Cohomology

In a final chapter, we calculate the groups $\mathrm{H}^k(\mathrm{D}_8)$ for $k \in \mathbf{Z}_{\geq 1}$ of the integral group cohomology for D_8 using our projective resolution from Chapter 2. In Lemma 54, we obtain the following result of Hamada, who used the Wall-Hamada resolution in [**6**, Corollary 1].

$$
\mathrm{H}^k(\mathrm{D}_8) \simeq \begin{cases} (\mathbf{Z}/2)^{\oplus k/2} \oplus \mathbf{Z}/4 & \text{if } k \equiv_4 0 \\ (\mathbf{Z}/2)^{\oplus (k-1)/2} & \text{if } k \equiv_4 1 \\ (\mathbf{Z}/2)^{\oplus (k+2)/2} & \text{if } k \equiv_4 2 \\ (\mathbf{Z}/2)^{\oplus (k-1)/2} & \text{if } k \equiv_4 3 \end{cases}
$$

Our computation of the cohomology groups can be seen as a test whether the constructed projective resolution P can be used for such calculations.

Chapter 1

Preliminaries

1.1 The dihedral group of order 8

Define the dihedral group D_8 by

$$\mathrm{D}_8 := \langle a, b \mid a^4, b^2, (ba)^2 \rangle.$$

Remark 1 We have an isomorphism of groups

$$\begin{array}{rcl} \mathrm{D}_8 & \xrightarrow[\sim]{\theta} & \langle (1,4,2,3), (3,4) \rangle \leq \mathrm{S}_4 \\ a & \longmapsto & (1,4,2,3) \\ b & \longmapsto & (3,4). \end{array}$$

Moreover, $|\mathrm{D}_8| = 8$.

Proof. To show that θ is a well-defined group morphism, we calculate.

$$\begin{aligned} (1,4,2,3)^4 &= \big((1,2)(3,4)\big)^2 = 1 \\ (3,4)^2 &= 1 \\ \big((3,4)(1,4,2,3)\big)^2 &= \big((1,4)(2,3)\big)^2 = 1 \end{aligned}$$

By construction, θ is surjective. We calculate.

$$\langle (1,4,2,3), (3,4) \rangle \supseteq \{1, (1,4,2,3), (1,2)(3,4), (1,3,2,4), (1,2), (3,4), (1,4)(2,3), (1,3)(2,4)\}$$

Hence $|\mathrm{D}_8| \geq 8$. From $(ba)^2 = 1$ it follows that $ba = a^{-1}b^{-1}$, and with $a^4 = b^2 = 1$ this shows that $ba = a^3b$. Hence an arbitrary element of D_8 can be written as a^ib^j with $i \in [0,3]$ and $j \in [0,1]$. Thus $|\mathrm{D}_8| \leq 8$. Therefore $|\mathrm{D}_8| = 8$ and θ is bijective. □

1.2 Complexes and scalar extensions

1.2.1 Complexes

Let R be a commutative ring and let A be an R-algebra.

Suppose given a *complex of A-modules* C, i.e. a sequence of A-modules C_k and A-linear maps $C_{k+1} \xrightarrow{d_k} C_k$, called *differentials*, such that $d_{k+1}d_k = 0$ for $k \in \mathbf{Z}$.

$$\ldots \longrightarrow C_{k+2} \xrightarrow{d_{k+1}} C_{k+1} \xrightarrow{d_k} C_k \xrightarrow{d_{k-1}} C_{k-1} \longrightarrow \ldots$$

If there is some $\ell \in \mathbf{Z}$ with $C_k = 0$ for $k < \ell$ we call the complex C bounded and we write

$$\ldots \longrightarrow C_{\ell+2} \xrightarrow{d_{\ell+1}} C_{\ell+1} \xrightarrow{d_\ell} C_\ell \longrightarrow 0.$$

Any A-module M becomes an R-module $M|_R$ by restricting the scalar multiplication, i.e. by $r \cdot m := (r1_A)m$ for $r \in R$ and $m \in M$. If $M \xrightarrow{f} N$ is an A-linear map, then $M|_R \xrightarrow{f} N|_R$ is R-linear and this assignment defines an additive functor $A\text{-Mod} \longrightarrow R\text{-Mod}$.

We write $C|_R$ for the restricted complex with differentials given by $C_{k+1}|_R \xrightarrow{d_k} C_k|_R$.

The complex of A-modules C is called *acyclic* if $\mathrm{im}(d_{k+1}) = \ker(d_k)$ for all $k \in \mathbf{Z}$. The complex C is acyclic if and only if the restricted complex $C|_R$ is acyclic as a complex of R-modules.

Furthermore, C is acyclic if and only if the sequence

$$\mathrm{im}(d_{k+1}) \xrightarrow{\iota_{k+1}} C_{k+1} \xrightarrow{\delta_k} \mathrm{im}(d_k) \qquad (*)$$

is short exact for all $k \in \mathbf{Z}$, were ι denotes the submodule embedding and $\delta_k := d_k|^{\mathrm{im}(d_k)}$. The complex of A-modules C is called *split acyclic* if the sequence $(*)$ is split short exact for all $k \in \mathbf{Z}$.

Suppose given an A-module M. An *augmented projective resolution of M* is an acyclic complex of the form

$$P' = \left(\ldots \longrightarrow P_3 \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{d_{-1}} M \longrightarrow 0\right),$$

where P_k is a projective A-module for $k \geq 0$. The map $P_0 \xrightarrow{d_{-1}} M$ is called the *augmentation map* and is often denoted by $\varepsilon := d_{-1}$. In this case, the complex of A-modules

$$P = \left(\ldots \longrightarrow P_3 \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow 0\right)$$

is called a *projective resolution of M*.

Suppose we are given a complex of R-modules D. A *contracting homotopy* $(h_k)_{k\in\mathbf{Z}}$ for D consists of R-linear maps $D_k \xrightarrow{h_k} D_{k+1}$ such that $h_{k+1}d_{k+1} + d_k h_k = 1$ for all $k \in \mathbf{Z}$.

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & D_{k+2} & \xrightarrow{d_{k+1}} & D_{k+1} & \xrightarrow{d_k} & D_k & \xrightarrow{d_{k-1}} & D_{k-1} & \longrightarrow \ldots \\
 & \Big| & \overset{h_{k+1}}{\swarrow} & \Big| & \overset{h_k}{\swarrow} & \Big| & \overset{h_{k-1}}{\swarrow} & \downarrow & \\
\ldots \longrightarrow & D_{k+2} & \xrightarrow{d_{k+1}} & D_{k+1} & \xrightarrow{d_k} & D_k & \xrightarrow{d_{k-1}} & D_{k-1} & \longrightarrow \ldots
\end{array}$$

Lemma 2 *A complex of R-modules D is split acyclic if and only if there is a contracting homotopy $(h_k)_{k\in\mathbf{Z}}$ for D.*

Proof. Ad $(\Rightarrow)$. Since D is split acyclic, for all $k \in \mathbf{Z}$ there is an isomorphism η_k such that the following diagram of R-modules commutes, where both rows are short exact.

$$\begin{array}{ccccc}
\mathrm{im}(d_k) & \xrightarrow{\iota_k} & D_k & \xrightarrow{\delta_{k-1}} & \mathrm{im}(d_{k-1}) \\
\| & & \eta_k \downarrow \wr & & \| \\
\mathrm{im}(d_k) & \xrightarrow{(1\ 0)} & \mathrm{im}(d_k) \oplus \mathrm{im}(d_{k-1}) & \xrightarrow{\binom{0}{1}} & \mathrm{im}(d_{k-1})
\end{array}$$

For $k \in \mathbf{Z}$ a define R-linear h_k maps by

$$h_k = \eta_k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \eta_{k+1}^{-1} \colon\ D_k \longrightarrow D_{k+1}.$$

We *claim* that $(h_k)_{k\in\mathbf{Z}}$ is a contracting homotopy for D. For $k \in \mathbf{Z}$ we compute using the commutativity of the diagram above and $d_k = \delta_k \iota_k$

$$\begin{aligned}
h_{k+1}d_{k+1} + d_k h_k &= \eta_{k+1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\eta_{k+2}^{-1}d_{k+1} + d_k\eta_k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\eta_{k+1}^{-1} \\
&= \eta_{k+1}\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\eta_{k+2}^{-1}\delta_{k+1}\iota_{k+1}\eta_{k+1} + \eta_{k+1}^{-1}\delta_k\iota_k\eta_k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\eta_{k+1}^{-1} \\
&= \eta_{k+1}\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\eta_{k+1}^{-1} \\
&= \eta_{k+1}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right)\eta_{k+1}^{-1} \\
&= 1.
\end{aligned}$$

Ad ($\Leftarrow$). Since D is a complex we have $d_{k+1}d_k = 0$ and hence $\mathrm{im}(d_{k+1}) \subseteq \ker(d_k)$. For the converse inclusion let $m \in \ker(d_k)$. We calculate

$$m = m1 = m(h_{k+1}d_{k+1} + d_k h_k) = mh_{k+1}d_{k+1} + md_k h_k = (mh_{k+1})d_{k+1}.$$

So $m \in \mathrm{im}(d_{k+1})$.

It remains to show that the short exact sequence

$$\mathrm{im}(d_{k+1}) \xrightarrow{\iota_{k+1}} M_{k+1} \xrightarrow{d_k} \mathrm{im}(d_k)$$

is split for all $k \in \mathbf{Z}$. To show this, we construct an R-linear map $\mathrm{im}(d_k) \xrightarrow{r_k} M_k$ with $r_k d_k = 1$.

Let $r_k = h_k|_{\mathrm{im}(d_k)}$. Then since $md_{k-1} = 0$ for $m \in \mathrm{im}(d_k) = \ker(d_{k-1})$

$$mr_k d_k = mh_k d_k = mh_k d_k + md_{k-1}h_{k-1} = m(h_k d_k + d_{k-1}h_{k-1}) = m.$$

So D is split acyclic. □

Remark 3 Suppose C is a complex of A-modules and we are given a R-linear contracting homotopy $(C_{k+1}|_R \xrightarrow{h_k} C_k|_R)_{k\in\mathbf{Z}}$ on the restricted complex $C|_R$. Lemma 2 then implies that the restricted complex $C|_R$ is split acyclic. Hence the complex of A-modules C is acyclic (but in general not split acyclic anymore).

1.2.2 Scalar extensions

Let R and S be commutative rings and let $R \xrightarrow{\varphi} S$ be a ring morphism. We extend φ to a ring morphism $RG \xrightarrow{\Phi} SG$ by $(rg)\Phi := (r\varphi)g$ for $r \in R$ and $g \in G$.

Let G be a finite group. All modules are assumed to be finitely generated.

Then S becomes a right R-module S_φ by $s * r := s(\varphi r)$ for $s \in S$ and $r \in R$.

Likewise, SG becomes a right RG-module SG_Φ by $y * x := y(\Phi x)$ for $y \in SG$ and $x \in RG$.

Remark 4 Consider RG as an R-module by restriction. We have an isomorphism of $\mathbf{Z}$-modules

$$\begin{array}{rcl}
SG_\Phi & \xrightarrow{\sim} & S_\varphi \underset{R}{\otimes} RG \\
sg & \longmapsto & s \otimes g \\
s(r\varphi)g = s(rg\Phi) & \longleftarrow & s \otimes rg.
\end{array}$$

Let M be an RG-module. Then we obtain an isomorphism of $\mathbf{Z}$-modules by

$$\begin{array}{rcl}
S_\varphi \underset{R}{\otimes} M & \xrightarrow[\sim]{\alpha} & SG_\Phi \underset{RG}{\otimes} M \\
s \otimes m & \longmapsto & s \otimes m \\
s \otimes gm & \longleftarrow & sg \otimes m.
\end{array}$$

Since SG is an SG-module by left-multiplication, we can regard $SG_\Phi \underset{RG}{\otimes} M$ as an SG-module and by transport of structure along the isomorphisms above we make the $\mathbf{Z}$-module $S_\varphi \underset{R}{\otimes} M$ into an SG-module. In concrete terms, for $s, t \in S$, $g \in G$ and $m \in M$ the scalar multiplication is given by

$$(sg) \cdot (t \otimes m) := ((sg)((t \otimes m)\alpha))\alpha^{-1} = ((sg)(t \otimes m))\alpha^{-1} = (stg \otimes m)\alpha^{-1} = st \otimes gm.$$

Let C be a complex of R-modules and $R\text{-mod} \xrightarrow{F} S\text{-mod}$ an additive functor. We write FC for the complex of S-modules one obtains by applying F to all modules and maps in C, i.e. the complex with modules $(FC_k)_{k \in \mathbf{Z}}$ and differentials Fd_k for $k \in \mathbf{Z}$.

Remark 5 Suppose $R\text{-mod} \xrightarrow{F} S\text{-mod}$ is an additive functor and C is a split acyclic complex of R-modules. Then FC is a split acyclic complex of S-modules.

Proof. By Lemma 2 there is a contracting homotopy $(h_k)_{k \in \mathbf{Z}}$ for C. But then $(Fh_k)_{k \in \mathbf{Z}}$ is a contracting homotopy for FC, since for $k \in \mathbf{Z}$ using additivity of F

$$1 = F1 = F(h_{k+1}d_{k+1} + d_k h_k) = Fh_{k+1}Fd_{k+1} + Fd_k Fh_k. \qquad \square$$

Lemma 6 *Let P be a projective resolution of an RG-module M. Suppose that the restriction $M|_R$ is projective as an R-module. Then $S_\varphi \underset{R}{\otimes} P$ is a projective resolution of the SG-module $S_\varphi \underset{R}{\otimes} M$.*

Proof. We *claim* that the restriction of the corresponding augmented projective resolution $P'|_R$ is split acylic.

We *claim* further that for $k \geq 0$ the module $\operatorname{im}(d_k)$ is projective as an R-module. We use induction on k. For base of the induction consider the short exact sequence of R-modules and R-linear maps

$$\operatorname{im}(d_0) \xrightarrow{\iota_0} P_0 \xrightarrow{d_{-1}} M.$$

Since M is projective as an R-module, this sequence splits. Hence $P_0 \simeq \operatorname{im}(d_0) \oplus M$ as R-modules and as a direct summand of a projective module $\operatorname{im}(d_0)$ is projective as an R-module.

Now let $k \geq 1$ and suppose that $\operatorname{im}(d_{k-1})$ is projective as an R-module. Consider the short exact sequence of R-modules

$$\operatorname{im}(d_k) \xrightarrow{\iota_k} P_k \xrightarrow{d_{k+1}} \operatorname{im}(d_{k-1}).$$

By assumption, $\operatorname{im}(d_{k-1})$ is projective an R-module, so the short exact sequence splits. Then we have $P_k \simeq \operatorname{im}(d_k) \oplus \operatorname{im}(d_{k-1})$ as R-modules and as a direct summand of a projective module $\operatorname{im}(d_k)$ is projective as an R-module.

Hence for all $k \geq -1$ the short exact sequence of R-modules

$$\operatorname{im}(d_{k+1}) \xrightarrow{\iota_{k+1}} P_{k+1} \xrightarrow{d_k} \operatorname{im}(d_k)$$

splits since $\operatorname{im}(d_k)$ is projective as an R-modules, so the restriction $P'|_R$ is split acyclic.

Consider the complex of S-modules

$$S_\varphi \underset{R}{\otimes} P' = \left(\ldots \longrightarrow S_\varphi \underset{R}{\otimes} P_2 \xrightarrow{1 \otimes d_1} S_\varphi \underset{R}{\otimes} P_1 \xrightarrow{1 \otimes d_0} S_\varphi \underset{R}{\otimes} P_0 \xrightarrow{1 \otimes d_{-1}} S_\varphi \underset{R}{\otimes} M \longrightarrow 0 \right).$$

By Remark 5 the complex $S_\varphi \underset{R}{\otimes} P'$ is a split acyclic over S, hence acyclic over SG.

It remains to show that $S_\varphi \underset{R}{\otimes} P_k$ is projective for $k \geq 0$. By construction (cf. Remark 4) there is an isomorphism of SG-modules $S_\varphi \underset{R}{\otimes} P_k \simeq SG_\Phi \underset{RG}{\otimes} P_k$. Since P_k is a finitely generated projective

RG-module projective, there is an $m_k \geq 1$ and an RG-module N_k such that $(RG)^{\oplus m_k} \simeq P_k \oplus N_k$. By additivity of the tensor product, we have

$$(SG)^{\oplus m_k} \simeq SG_\Phi \underset{RG}{\otimes} (RG)^{\oplus m_k} \simeq SG_\Phi \underset{RG}{\otimes} (P_k \oplus N_k) \simeq (SG_\Phi \underset{RG}{\otimes} P_k) \oplus (SG_\Phi \underset{RG}{\otimes} N_k).$$

Hence $S_\varphi \underset{R}{\otimes} P_k$ is isomorphic to a direct summand of a free SG-module, hence a projective SG-module. □

1.3 Automorphism groups

1.3.1 Generalities

Let R be a commutative ring and let A be an R-algebra.

Let $\mathrm{Aut}_{R\text{-alg}}(A) := \{A \xrightarrow{f} A : f\ R\text{-algebra isomorphism}\}$ be the *group of R-algebra automorphisms* of A. This is a group under composition of morphisms.

For any $u \in \mathrm{U}(A)$ an R-algebra automorphism f of A is given by $af_u := u^{-1}au$ for $a \in A$. Such an automorphism is called an *inner automorphism* of A. We write $\mathrm{Inn}_{R\text{-alg}}(A)$ for the group of inner R-algebra automorphism of A, cf. the following Remark 7.

Remark 7 The inner automorphisms $\mathrm{Inn}_{R\text{-alg}}(A)$ form a normal subgroup of $\mathrm{Aut}_{R\text{-alg}}(A)$.

Proof. Let $f_u, f_v \in \mathrm{Inn}_{R\text{-alg}}(A)$. Then for $a \in A$

$$af_u(f_v)^{-1} = (u^{-1}au)f_v^{-1} = vu^{-1}auv^{-1} = af_{uv^{-1}}.$$

So $\mathrm{Inn}_{R\text{-alg}}(A)$ is a subgroup of $\mathrm{Aut}_{R\text{-alg}}(A)$. Let $\alpha \in \mathrm{Aut}_{R\text{-alg}}(A)$. Then for $f_u \in \mathrm{Inn}_{R\text{-alg}}(A)$ and $a \in A$

$$a(\alpha^{-1}f_u\alpha) = (a\alpha^{-1})f_u\alpha = (u^{-1}(a\alpha^{-1})u)\alpha = (u\alpha)^{-1}a(u\alpha) = af_{u\alpha}.$$

So $\mathrm{Inn}_{R\text{-alg}}(A)$ is a normal subgroup of $\mathrm{Aut}_{R\text{-alg}}(A)$. □

Now we can define the *outer automorphism group* $\mathrm{Out}_{R\text{-alg}}(A) := \mathrm{Aut}_{R\text{-alg}}(A)/\,\mathrm{Inn}_{R\text{-alg}}(A)$.

We write $[f] \in \mathrm{Out}_{R\text{-alg}}(A)$ for the equivalence class of an automorphism $f \in \mathrm{Aut}_{R\text{-alg}}(A)$.

Remark 8 Let $A \xrightarrow{\varphi} B$ be an isomorphism of R-algebras. Then there is an isomorphism of groups

$$\begin{array}{rcl} \mathrm{Out}_{R\text{-alg}}(A) & \xrightarrow[\sim]{\psi} & \mathrm{Out}_{R\text{-alg}}(B) \\ [f] & \longmapsto & [\varphi^{-1}f\varphi] \end{array}$$

Proof. Consider the following bijection between the automorphism groups.

$$\begin{array}{rcl} \mathrm{Aut}_{R\text{-alg}}(A) & \xrightarrow[\sim]{\tilde{\psi}} & \mathrm{Aut}_{R\text{-alg}}(B) \\ f & \longmapsto & \varphi^{-1}f\varphi \\ \varphi g\varphi^{-1} & \longleftarrow & g \end{array}$$

This is an isomorphism of groups, because for $f, f' \in \mathrm{Aut}_{R\text{-alg}}(A)$ we have

$$(ff')\tilde{\psi} = \varphi^{-1}ff'\varphi = \varphi^{-1}f\varphi\varphi^{-1}f'\varphi = (f\tilde{\psi})(f'\tilde{\psi}).$$

For an inner automorphism $f_u \in \mathrm{Inn}_{R\text{-alg}}$ we have for $b \in B$

$$b(f_u\tilde{\psi}) = a(\varphi^{-1}f_u\varphi) = (u^{-1}(b\varphi^{-1})u)\varphi = (u\varphi)^{-1}b(u\varphi) = bf_{u\varphi}.$$

Since φ is an isomorphism the isomorphism $\tilde{\psi}$ restricts to an isomorphism

$$\begin{array}{rcl} \mathrm{Inn}_{R\text{-alg}}(A) & \xrightarrow[\sim]{\tilde{\psi}|_{\mathrm{Inn}(A)}^{\mathrm{Inn}(B)}} & \mathrm{Inn}_{R\text{-alg}}(B) \\ f_u & \longmapsto & f_{u\varphi} \\ g_{v\varphi^{-1}} & \longleftarrow & g_v. \end{array}$$

So $\tilde{\psi}$ induces the isomorphism $\mathrm{Out}_{R\text{-alg}}(A) \xrightarrow[\sim]{\psi} \mathrm{Out}_{R\text{-alg}}(B)$ between the outer automorphism groups. □

In $\mathrm{Aut}_{R\text{-alg}}(A)$ we have the subgroup of *central automorphisms*

$$\mathrm{Autcent}_{R\text{-alg}}(A) := \{f \in \mathrm{Aut}_{R\text{-alg}}(A) : zf = z \text{ for all } z \in \mathrm{Z}(A)\}$$

keeping the centre of A pointwise fixed. For an inner automorphism $f_u \in \mathrm{Inn}_{R\text{-alg}}(A)$ we have $zf_u = u^{-1}zu = u^{-1}uz = z$ for all $z \in \mathrm{Z}(U)$, hence the group of inner automorphisms $\mathrm{Inn}_{R\text{-alg}}(A) \leq \mathrm{Autcent}_{R\text{-alg}}(A)$ is a normal subgroup.

The quotient $\mathrm{Outcent}_{R\text{-alg}}(A) := \mathrm{Autcent}_{R\text{-alg}}(A)/\,\mathrm{Inn}_{R\text{-alg}}(A) \leq \mathrm{Out}_{R\text{-alg}}(A)$ is called the group of *outer central automorphisms.*

1.3.2 Direct products of matrix rings over fields

Let K be a field.

Let A be a finite-dimensional K-algebra.

Recall that an element $c \in A$ is called a *central idempotent* if $c \in \mathrm{Z}(A)$ and $c^2 = c$. We say that a set of central idempotents $\{c_i \in A : i \in [1, t]\}$ for some $t \in \mathbf{Z}_{\geq 1}$ is an *orthogonal decomposition of 1 into primitive central idempotents* if (1-3) hold.

(1) We have $1 = \sum_{i\in[1,t]} c_i$.

(2) For $i, j \in [1, t]$ we have $c_ic_j = 0$ for $i \neq j$. I.e. the central idempotents are orthogonal.

(3) For $i \in [1, t]$ and $c_i = d_i + d'_i$ with central idempotents $d_i, d'_i \in A$ satisfying $d_id'_i = 0$ we have $d_i = 0$ or $d'_i = 0$. I.e. the central idempotents are primitive in $\mathrm{Z}(A)$.

Lemma 9 *Let $\psi \in \mathrm{Aut}_{K\text{-alg}}(A)$ and suppose that $\{c_i \in A : i \in [1, t]\}$ is an orthogonal decomposition of 1 into primitive central idempotents.*

Then there is a permutation $\sigma \in \mathrm{S}_t$ with $c_i\psi = c_{i\sigma}$.

Proof. First note that for a central idempotent $c \in A$ one has $c \in \mathrm{Z}(A)$ and

$$(c\psi)(c\psi) = (cc)\psi = c\psi,$$

so $c\psi$ is again a central idempotent. Moreover, if $d', d'' \in A$ are orthogonal central idempotents, we have

$$(cd)(cd) = c^2d^2 = cd \quad \text{ and } \quad (cd)(cd') = c^2dd' = 0,$$

so cd and cd' are also orthogonal central idempotents.

For $i \in [1, t]$ let $d_i := c_i\psi$. We *claim* that $\{d_i : i \in [1, t]\}$ is also an orthogonal decomposition of 1 into primitive central idempotents. In fact, for $i, j \in [1, t]$ with $i \neq j$ we have

$$d_id_j = (c_i\psi)(c_j\psi) = (c_ic_j)\psi = 0,$$

and d_i is a central idempotents for $i \in [1, t]$ by the note above.

Let also $d', d'' \in \mathrm{Z}(A)$ be orthogonal central idempotents with $d_i = d' + d''$. Then $c_i = d_i\psi^{-1} = d'\psi^{-1} + d''\psi^{-1}$ with $(d'\psi^{-1})(d''\psi - 1) = (d'd'')\psi^{-1} = 0$. So $d'\psi^{-1}$ and $d''\psi^{-1}$ are central orthogonal idempotents, hence $c_i = d'\psi^{-1}$ or $c_i = d''\psi^{-1}$, because c_i is primitive. But then $d_i = d'$ or $d_i = d''$, hence d_i is also primitive.

Finally, we have

$$1_A = 1_A\psi = \left(\sum_{i\in[1,t]} c_i\right)\psi = \sum_{i\in[1,t]} d_i.$$

Hence the claim follows.

For $i \in [1, t]$ consider

$$d_i = d_i \cdot 1_A = d_i\left(\sum_{j\in[1,t]} c_j\right) = \sum_{i\in[1,t]} d_i c_j.$$

By the note at the beginning, the sum on the right-hand side consists of orthogonal central idempotents. Since d_i is a primitive central idempotent, we have $d_i = d_i c_j$ for some $j \in [1, t]$. On the other hand, we have

$$c_j = 1_A \cdot c_j = \left(\sum_{k\in[1,t]} d_k\right) c_j = \sum_{k\in[1,t]} d_k c_j.$$

Since $d_i = d_i c_j \neq 0$ takes part in the sum on the right-hand side and c_j is primitive, we have $c_j = d_i c_j$. So $d_i = c_i\psi = c_j$.

Hence the automorphism ψ restricts to a bijection on the set $\{c_i : i \in [1, t]\}$. Therefore there exists a permutation $\sigma \in \mathrm{S}_t$ with $c_i\psi = c_{i\sigma}$ for $i \in [1, t]$. □

Let $t \in \mathbf{Z}_{\geq 1}$ and $n_i \in \mathbf{Z}_{\geq 1}$ for $i \in [1, t]$ and let

$$\Omega := \prod_{i\in[1,t]} K^{n_i \times n_i}.$$

We will write $\omega = (\omega_i)_i$ with $\omega_i \in K^{n_i\times n_i}$ for a general element of Ω.

For $n \in \mathbf{Z}_{\geq 1}$ and $k, \ell \in [1, n]$ let $e_{k,\ell} := (\partial_{k,i}\partial_{\ell,j})_{i,j} \in K^{n\times n}$ be the matrix with 1 at position (k, ℓ) and 0 everywhere else. We will write $e_k := e_{k,k}$ for $k \in [1, n]$.

Note that $I_n = \sum_{k\in[1,n]} e_k$ is the identity matrix in $K^{n\times n}$.

Lemma 10 *Let $n \in \mathbf{Z}_{\geq 1}$.*

(1) *The K-algebra $K^{n\times n}$ is a simple algebra. I.e. its only ideals are* 0 *and $K^{n\times n}$.*

(2) *We have $\mathrm{Z}(K^{n\times n}) = KI_n$.*

(3) *We have $\mathrm{Z}(\Omega) = \prod_{i\in[1,t]} KI_{n_i}$.*

Proof. Ad (1). Suppose $0 \neq I \subseteq K^{n\times n}$ is a two-sided ideal. Then there is $X \in I$ with $X \neq 0$. Write $X = (x_{i,j})_{i,j}$. Hence there are $k, \ell \in [1, n]$ such that $x_{k,\ell} \neq 0$. Then we have

$$\frac{1}{x_{k,\ell}} e_{k,k} X e_{\ell,\ell} = \frac{1}{x_{k,\ell}} e_{k,k}\Big(\sum_{i,j} x_{i,j} e_{i,j}\Big) e_{\ell,\ell} = e_{k,\ell} \in I.$$

Now let $i, j \in [1, n]$. Then $e_{i,j} = e_{i,k}e_{k,\ell}e_{\ell,j} \in I$. So I contains a K-basis of $K^{n\times n}$, hence $I = K^{n\times n}$. So $K^{n\times n}$ is simple as a K-algebra.

Ad (2). We observe that $KI_n \subseteq \mathrm{Z}(K^{n\times n})$. For the other inclusion, let $X \in \mathrm{Z}(K^{n\times n})$ with $X = (x_{i,j})_{i,j}$. For $k, \ell \in [1, n]$ with $k \neq \ell$ we have

$$0 = 0 \cdot X = e_{k,\ell}e_{k,\ell}X = e_{k,\ell}Xe_{k,\ell} = e_{k,\ell}\Big(\sum_{i,j} x_{i,j}e_{i,j}\Big)e_{k,\ell} = e_{k,\ell}x_{\ell,k}e_{\ell,k}e_{k,\ell} = x_{\ell,k}e_{k,\ell}.$$

Therefore $x_{\ell,k} = 0$. Hence $X = \sum_i x_{i,i}e_{i,i}$. Again let $k, \ell \in [1, n]$ with $k \neq \ell$. Then

$$\begin{aligned} x_{k,k}e_{k,k} = e_{k,k}\Big(\sum_i x_{i,i}e_{i,i}\Big) &= e_{k,\ell}e_{\ell,k}X \\ &= e_{k,\ell}Xe_{\ell,k} = e_{k,\ell}\Big(\sum_i x_{i,i}e_{i,i}\Big)e_{\ell,k} = e_{k,\ell}x_{\ell,\ell}e_{\ell,\ell}e_{\ell,k} = x_{\ell,\ell}e_{k,k}. \end{aligned}$$

Therefore $x_{\ell,\ell} = x_{k,k}$. Hence $X = x_{1,1}I_n$ and the assertion follows.

Ad (3). Since multiplication in Ω is declared componentwise, an element $\omega = (\omega_i)_i$ is central in Ω if and only if ω_i is central in $K^{n_i\times n_i}$ for all $i \in [1, t]$.

Hence (2) implies that $\mathrm{Z}(\Omega) = \prod_{i\in[1,t]} \mathrm{Z}(K^{n_i\times n_i}) = \prod_{i\in[1,t]} KI_{n_i}$. □

Lemma 11 *The set $\{(\partial_{k,i}I_{n_i})_i : k \in [1, t]\}$ is an orthogonal decomposition of 1_Ω into primitive central idempotents.*

Proof. We observe that

$$1_\Omega = (I_{n_i})_i = \sum_{k\in[1,t]} (\partial_{k,i}I_{n_i})_i.$$

For $k, \ell \in [1, t]$ we have

$$(\partial_{k,i}I_{n_i})_i(\partial_{\ell,i}I_{n_i})_i = (\partial_{k,i}\partial_{\ell,i}I_{n_i})_i = \begin{cases} (\partial_{k,i}I_{n_i})_i & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases}$$

For primitivity, suppose that for $k \in [1, t]$ we have $(\partial_{k,i}I_{n_i})_i = c + d$ for central orthogonal idempotents c, d. Then there are $c_i, d_i \in K$ for $i \in [1, t]$ such that $c = (c_iI_{n_i})_i$ and $d = (d_iI_{n_i})$. Since $c + d = (\partial_{k,i}I_{n_i})_i$ we have $c_i + d_i = 0$ for $i \neq k$ and $c_i + d_i = 1$ for $i = k$. Since $cd = 0$ we have $c_id_i = 0$ for all $i \in [1, t]$. This implies $c_i = d_i = 0$ for $i \neq k$.

For $i = k$ note that $c^2 = c$ and $d^2 = d$ imply that $c_k^2 = c_k$ and $d_k^2 = d_k$ and thus $c_k, d_k \in \{0, 1\}$. Since $c_k + d_k = 1$ we either have $c_k = 1$ and $d_k = 0$ or $c_k = 0$ and $d_k = 1$, so either $c = 0$ or $d = 0$. □

All $K^{n\times n}$-modules are assumed to be finitely-generated.

Lemma 12 *Let $n \in \mathbf{Z}_{\geq 1}$.*

(1) *Suppose given $K^{n\times n}$-modules M and N. Then there is an isomorphism of K-vector spaces*

$$\begin{array}{ccc} \mathrm{Hom}_{K^{n\times n}}(M, N) & \xrightarrow[\sim]{\beta} & \mathrm{Hom}_K(e_1M, e_1N) \\ f & \longmapsto & f|_{e_1M}^{e_1N} \end{array}$$

(2) *Suppose given a $K^{n\times n}$-module M. If $M \neq 0$, then $e_1M \neq 0$.*

(3) *Suppose given simple $K^{n\times n}$-modules M and N. Then $M \simeq N$ as $K^{n\times n}$-modules.*

(4) *Any simple $K^{n\times n}$-module is isomorphic to K^n, where $K^n := K^{n\times 1}$ is the $K^{n\times n}$-module with operation given by matrix multiplication.*

Proof. *Ad* (1). For well-definedness of β note that for $m \in M$ and $f \in \mathrm{Hom}_{K^{n\times n}}(M,N)$ we have $(e_1 m)f = e_1(mf) \in e_1 N$, since f is $K^{n\times n}$-linear. Moreover, we observe that β is K-linear.

To show that β is an isomorphism of K-vector spaces, consider the map

$$\begin{array}{ccc} \mathrm{Hom}_{K^{n\times n}}(M,N) & \xleftarrow{\ \gamma\ } & \mathrm{Hom}_K(e_1M, e_1N) \\ \Big(m \longmapsto \sum\limits_{i\in[1,n]} e_{i,1}\big((e_{1,i}m)g\big)\Big) & \longleftarrow\!\!\shortmid & g \end{array}$$

For well-definedness of γ, note that for $i \in [1,n]$ we have $(e_{1,i}m)g = (e_{1,1}e_{1,i}m)g$ with $e_{1,1}e_{1,i}m = e_1(e_{1,i}m) \in e_1M$. Moreover, $g\gamma$ is $K^{n\times n}$-linear, since for a basis element $e_{k,\ell} \in K^{n\times n}$, where $k, \ell \in [1,n]$, and $m \in M$ we have

$$(e_{k,\ell}m)(g\gamma) = \sum_{i\in[1,n]} e_{i,1}\big((e_{1,i}e_{k,\ell}m)g\big) = e_{k,1}\big((e_{1,\ell}m)g\big)$$
$$e_{k,\ell}\big(m(g\gamma)\big) = e_{k,\ell}\Big(\sum_{i\in[1,n]} e_{i,1}\big((e_{1,i}m)g\big)\Big) = e_{k,1}\big((e_{1,\ell}m)g\big).$$

We want to show that $\beta\gamma = 1$. Suppose given $m \in M$ and $f \in \mathrm{Hom}_{K^{n\times n}}(M,N)$. Then, using that f is $K^{n\times n}$-linear, we obtain

$$\begin{aligned} m(f\beta\gamma) &= \sum_{i\in[1,n]} e_{i,1}\big((e_{1,i}m)(f\beta)\big) = \sum_{i\in[1,n]} e_{i,1}\Big((e_{1,i}m)f|_{e_1M}^{e_1N}\Big) = \sum_{i\in[1,n]} e_{i,1}\big((e_{1,i}m)f\big) \\ &= \sum_{i\in[1,n]} e_{i,1}e_{1,i}(mf) = \Big(\sum_{i\in[1,n]} e_{i,i}\Big)(mf) = I_n(mf) = mf. \end{aligned}$$

Hence $f\beta\gamma = f$, so $\beta\gamma = 1$.

We want to show that $\gamma\beta = 1$. Suppose given $g \in \mathrm{Hom}_K(e_1M, e_1N)$ and $e_1m \in e_1M$. Then

$$\begin{aligned} (e_1m)(g\gamma\beta) &= (e_1m)(g\gamma)|_{e_1M}^{e_1N} = (e_1m)(g\gamma) \\ &= \sum_{i\in[1,n]} e_{i,1}\big((e_{1,i}e_{1,1}m)g\big) = e_{1,1}\big((e_{1,1}m)g\big) = (e_1m)g, \end{aligned}$$

where $(e_1m)g \in e_1N$ and thus $e_1\big((e_1m)g\big) = (e_1m)g$. Hence $g\gamma\beta = g$, so $\gamma\beta = 1$.

Ad (2). If $M \neq 0$, then $\mathrm{Hom}_{K^{n\times n}}(M,M) \neq 0$, as the identity 1_M is $K^{n\times n}$-linear. Using the isomorphism in (1) this shows that $\mathrm{Hom}_K(e_1M, e_1M) \neq 0$, hence $e_1M \neq 0$.

Ad (3). Since M and N are simple, they are non-zero and by (2) we also have $e_1M \neq 0$ and $e_1N \neq 0$. Then $\dim_K(\mathrm{Hom}_K(e_1M, e_1N)) = \dim_K(e_1M)\dim_K(e_1N) \geq 1$, so $\mathrm{Hom}_K(e_1M, e_1N) \neq 0$. Using the isomorphism in (1) we conclude that $\mathrm{Hom}_{K^{n\times n}}(M,N) \neq 0$, hence there is a $K^{n\times n}$-linear map $M \xrightarrow{f} N$ with $f \neq 0$.

Since M is simple, we must have $\ker(f) = 0$ and since N is simple we must have $\mathrm{im}(f) = N$. Hence $M \xrightarrow{f} N$ is an isomorphism of $K^{n\times n}$-modules.

Ad (4). We show that K^n is a simple $K^{n\times n}$-module. Let M be a nonzero $K^{n\times n}$-module. Consider the K-algebra morphism

$$\begin{array}{ccc} K^{n\times n} & \xrightarrow{\ \alpha\ } & \mathrm{End}_K(M) \\ X & \longmapsto & (m \longmapsto Xm). \end{array}$$

Since $I_n\alpha = 1_M$ and $M \neq 0$, we have $\alpha \neq 0$. But then $\ker(\alpha)$ is a two-sided ideal properly contained in $K^{n\times n}$. Now $K^{n\times n}$ is a simple K-algebra by Lemma 10.(1), hence $\ker(\alpha) = 0$ and α must be injective. This implies $n^2 = \dim_K(K^{n\times n}) \leq \dim_K(\mathrm{End}_K(M)) = (\dim_K(M))^2$.

Hence the $K^{n\times n}$-module K^n is of minimal K-dimension for all non-zero $K^{n\times n}$-modules, therefore must be simple.

By (3), any two simple $K^{n\times n}$-modules are isomorphic, so any simple $K^{n\times n}$-module is isomorphic to K^n. □

Lemma 13 *Let $n \in \mathbf{Z}_{\geq 1}$. Suppose given a K-algebra automorphism $\psi \in \mathrm{Aut}_{K\text{-alg}}(K^{n\times n})$. Then there is $Q \in \mathrm{GL}_n(K)$ such that $X\psi = Q^{-1}XQ$ for all $X \in K^{n\times n}$.*

Proof. Let $K^n := K^{n\times 1}$ be the $K^{n\times n}$-module with operation given by matrix multiplication. By Lemma 12.(4) this is, up to isomorphism, the only simple $K^{n\times n}$-module.

Now consider the $K^{n\times n}$-module $K^{n,\psi^{-1}}$ with operation twisted by ψ, i.e. for $X \in K^{n\times n}$ and $v \in K^n$ we have $X \underset{K^{n,\psi^{-1}}}{\cdot} v := (X\psi)v$.

Since ψ is an automorphism, a subset $U \subseteq K^n$ is a $K^{n\times n}$ submodule of K^n if and only if it is a submodule of $K^{n,\psi^{-1}}$. Hence $K^{n,\psi^{-1}}$ is also a simple $K^{n\times n}$-module.

By Lemma 12.(4), there is an isomorphism of $K^{n\times n}$-modules $K^{n,\psi^{-1}} \xrightarrow[\sim]{f} K^n$. Let $Q \in \mathrm{GL}_n(K)$ be the representing matrix of f using the standard K-linear basis of K^n, so $vf = Qv$ for $v \in K^n$.

Since f is $K^{n\times n}$-linear, we have $XQv = X(vf) = (X \underset{K^{n,\psi^{-1}}}{\cdot} v)f = ((X\psi)v)f = Q(X\psi)v$ for all $v \in K^n$.

$$\begin{array}{ccc} K^{n,\psi^{-1}} & \xrightarrow{Q\cdot(-)} & K^n \\ {\scriptstyle X \underset{K^{n,\psi^{-1}}}{\cdot} (-)=(X\psi)\cdot(-)}\Big\downarrow & & \Big\downarrow{\scriptstyle X\cdot(-)} \\ K^{n,\psi^{-1}} & \xrightarrow{Q\cdot(-)} & K^n \end{array}$$

Hence $X\psi = Q^{-1}XQ$. □

Lemma 14 *Let $\psi \in \mathrm{Aut}_{K\text{-alg}}(\Omega)$. Then there is a unique permutation $\sigma \in \mathrm{S}_t$ with $n_{i\sigma} = n_i$ and invertible matrices $Q_i \in \mathrm{GL}_{n_i}(K)$ for $i \in [1, t]$, such that*

$$(\omega_i)_i\psi = (Q_i^{-1}\omega_{i\sigma}Q_i)_i.$$

Proof. By Lemma 11 we have an orthogonal decomposition of 1_Ω into primitive central idempotents given by

$$1_\Omega = \sum_{k\in[1,t]} (\partial_{k,i}I_{n_i})_i.$$

Let $\psi \in \mathrm{Aut}_{K\text{-alg}}(\Omega)$. By Lemma 9 there is a permutation $\sigma \in \mathrm{S}_t$ such that for all $k \in [1, t]$

$$(\partial_{k,i}I_{n_i})_i\psi = (\partial_{k\sigma^{-1},i}I_{n_i})_i.$$

Now ψ restricts to an isomorphism of K-algebras

$$\begin{array}{rcl} (\partial_{k,i}I_{n_i})_i\Omega & \xrightarrow[\sim]{\psi_k} & (\partial_{k\sigma^{-1},i}I_{n_i})_i\Omega \\ (\partial_{k,i}I_{n_i})_i(\omega_i)_i & \longmapsto & ((\partial_{k,i}I_{n_i})_i(\omega_i)_i)\psi = (\partial_{k\sigma^{-1},i}I_{n_i})_i((\omega_i)_i\psi). \end{array}$$

For each $k \in [1, t]$ there is an isomorphism of K-algebras

$$\begin{array}{rcl} (\partial_{k,i}I_{n_i})_i\Omega & \xrightarrow[\sim]{\theta_k} & K^{n_k\times n_k} \\ (\partial_{k,i}I_{n_i})_i(\omega_i)_i & \longmapsto & \omega_k. \end{array}$$

So

$$n_k^2 = \dim_K(\partial_{k,i}I_{n_i})_i\Omega = \dim_K(\partial_{k\sigma^{-1},i}I_{n_i})_i\Omega = n_{k\sigma^{-1}}^2.$$

Hence $n_{k\sigma} = n_k$ for $k \in [1, t]$. Moreover, the map $\theta_{k\sigma}^{-1}\psi_{k\sigma}\theta_k \colon K^{n_k\times n_k} = K^{n_{k\sigma}\times n_{k\sigma}} \longrightarrow K^{n_k\times n_k}$ is a K-algebra automorphism of a full matrix algebra. By Lemma 13 there is a $Q_k \in \mathrm{GL}_{n_k}(K)$ with

$$\omega_{k\sigma}\theta_{k\sigma}^{-1}\psi_{k\sigma}\theta_k = Q_k^{-1}\omega_{k\sigma}Q_k$$

for $\omega_k \in K^{n_k \times n_k}$. Now for $(\omega_i)_i \in \Omega$ we have

$$
\begin{aligned}
(\omega_i)_i \psi &= \left(\sum_{k \in [1,t]} (\partial_{k,i} \omega_i)_i \right) \psi \\
&= \sum_{k \in [1,t]} ((\partial_{k,i} I_{n_i})_i (\omega_i)_i) \psi \\
&= \sum_{k \in [1,t]} (\omega_k \theta_k^{-1}) \psi \\
&= \sum_{k \in [1,t]} \omega_k \theta_k^{-1} \psi_k \\
&= \sum_{k \in [1,t]} \omega_{k\sigma} \theta_{k\sigma}^{-1} \psi_{k\sigma} \\
&= \sum_{k \in [1,t]} \omega_{k\sigma} \theta_{k\sigma}^{-1} \psi_{k\sigma} \theta_k \theta_k^{-1} \\
&= \sum_{k \in [1,t]} (Q_k^{-1} \omega_{k\sigma} Q_k) \theta_k^{-1} \\
&= \sum_{k \in [1,t]} (\partial_{k,i} I_{n_i})_i (Q_i^{-1} \omega_{i\sigma} Q_i)_i \\
&= (Q_i^{-1} \omega_{i\sigma} Q_i)_i.
\end{aligned}
$$

Finally, uniqueness of σ follows since the permutation is already uniquely determined by the action of ψ on the central idempotents. □

Chapter 2

Projective resolutions

In this chapter, we construct a minimal projective resolution of the trivial $\mathbf{Z}_{(2)}\mathrm{D}_8$-module $\mathbf{Z}_{(2)}$.

In §2.1 we give a Wedderburn image Λ of the integral group ring $\mathbf{Z}\mathrm{D}_8$ with its Wedderburn embedding $\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\omega} \Lambda$.

In §2.2 we work out a projective resolution of the trivial $\mathbf{Z}\mathrm{D}_8$-module $\mathbf{Z}$. We present the resolution on both sides of the Wedderburn embedding.

Finally, §2.3 we extend scalars from $\mathbf{Z}$ to $\mathbf{Z}_{(2)}$ and show that the resulting projective resolution of the trivial $\mathbf{Z}_{(2)}\mathrm{D}_8$-module $\mathbf{Z}_{(2)}$ is minimal.

We use the following presentation of the dihedral group of order 8, cf. Remark 1.

$$\mathrm{D}_8 = \left\langle\, a, b : a^4, b^2, (ba)^2 \right\rangle$$

2.1 Wedderburn

Lemma 15 *Let $\Gamma := \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2}$ and define the following $\mathbf{Z}$-submodule of Γ.*

$$\Lambda := \left\{ (p,q,r,s,\left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)) \in \Gamma : \begin{array}{c} p \equiv_2 q \equiv_2 r \equiv_2 s \equiv_2 t \equiv_2 w,\ v \equiv_2 0,\ q - r \equiv_4 v, \\ q - s \equiv_4 2u,\ p + q \equiv_4 r + s,\ p + q + r + s \equiv_8 2(t + w) \end{array} \right\}$$

We have an injective morphism of $\mathbf{Z}$-algebras

$$\begin{array}{ccccccccccccc} \mathbf{Z}\mathrm{D}_8 & \xrightarrow{\tilde\omega} & & \mathbf{Z} & \times & \mathbf{Z} & \times & \mathbf{Z} & \times & \mathbf{Z} & \times & \mathbf{Z}^{2\times 2} & = \Gamma \\ a & \longmapsto & (& 1 & , & 1 & , & -1 & , & -1 & , & \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right) &) \\ b & \longmapsto & (& 1 & , & -1 & , & 1 & , & -1 & , & \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right) &) \ . \end{array}$$

Its image is given by $(\mathbf{Z}\mathrm{D}_8)\tilde\omega = \Lambda$. In particular, by $\omega := \omega|^{\Lambda}$ we obtain an isomorphism of $\mathbf{Z}$-algebras $\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\omega} \Lambda$.

Write $\otimes := \underset{\mathbf{Z}}{\otimes}$. Note that

$$\mathbf{Q}\mathrm{D}_8 \simeq \mathbf{Q}\otimes\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\mathbf{Q}\otimes\tilde\omega} \mathbf{Q}\otimes\Gamma \simeq \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}^{2\times 2}$$

is a Wedderburn isomorphism in the sense of Wedderburn's Theorem on finite-dimensional semisimple algebras over a field.

Proof. We verify that the images of a and b under $\tilde{\omega}$ fulfill all relations of D_8.

$$(1,1,-1,-1,\left(\begin{smallmatrix}1&1\\-2&-1\end{smallmatrix}\right))^4=(1,1,1,1,\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right))^2=(1,1,1,1,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))$$

$$(1,-1,1,-1,\left(\begin{smallmatrix}1&0\\-2&-1\end{smallmatrix}\right))^2=(1,1,1,1,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))$$

$$\Big((1,-1,1,-1,\left(\begin{smallmatrix}1&0\\-2&-1\end{smallmatrix}\right))(1,1,-1,-1,\left(\begin{smallmatrix}1&1\\-2&-1\end{smallmatrix}\right))\Big)^2=(1,-1,-1,1,\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right))^2=(1,1,1,1,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))$$

We conclude that $\tilde{\omega}$ is a well-defined $\mathbf{Z}$-algebra morphism.

Now we want to show that $(\mathbf{Z}D_8)\tilde{\omega}\subseteq\Lambda$. We calculate

$$1\tilde{\omega}=(1,1,1,1,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)) \qquad a\tilde{\omega}=(1,1,-1,-1,\left(\begin{smallmatrix}1&1\\-2&-1\end{smallmatrix}\right))$$

$$a^2\tilde{\omega}=(1,1,1,1,\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)) \qquad a^3\tilde{\omega}=(1,1,-1,-1,\left(\begin{smallmatrix}-1&-1\\2&1\end{smallmatrix}\right))$$

$$b\tilde{\omega}=(1,-1,1,-1,\left(\begin{smallmatrix}1&0\\-2&-1\end{smallmatrix}\right)) \qquad ba\tilde{\omega}=(1,-1,-1,1,\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right))$$

$$ba^2\tilde{\omega}=(1,-1,1,-1,\left(\begin{smallmatrix}-1&0\\2&1\end{smallmatrix}\right)) \qquad ba^3\tilde{\omega}=(1,-1,-1,1,\left(\begin{smallmatrix}-1&-1\\0&1\end{smallmatrix}\right))$$

and note that all images satisfy the congruences in the definition of Λ.

To show the equality $(\mathbf{Z}D_8)\tilde{\omega}=\Lambda$ consider $\mathbf{Z}D_8$ with the basis $(1,a,a^2,a^3,b,ba,ba^2,ba^3)$ and Γ with the standard basis, consisting of matrix tuples having entry 1 at one position 0 elsewhere. Using these bases, the map $\tilde{\omega}$ has the following matrix.

$$A_{\tilde{\omega}}=\begin{pmatrix}1&1&1&1&1&0&0&1\\1&1&-1&-1&1&1&-2&-1\\1&1&1&1&-1&0&0&-1\\1&1&-1&-1&-1&-1&2&1\\1&-1&1&-1&1&0&-2&-1\\1&-1&-1&1&1&1&0&-1\\1&-1&1&-1&-1&0&2&1\\1&-1&-1&1&-1&-1&0&1\end{pmatrix}$$

The matrix has determinant $\det(A_{\tilde{\omega}})=-1024$, hence $\tilde{\omega}$ is injective and one has

$$[\Gamma:(\mathbf{Z}D_8)\tilde{\omega}]=|\Gamma/(\mathbf{Z}D_8)\tilde{\omega}|=|\det(A_{\tilde{\omega}})|=1024.$$

We already know that $(\mathbf{Z}D_8)\tilde{\omega}\subseteq\Lambda$. To show equality consider the following $\mathbf{Z}$-linear basis of Λ.

$$a_1=(1,1,1,1,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)) \quad a_2=(0,2,0,-2,\left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)) \quad a_3=(0,0,2,-2,\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)) \quad a_4=(0,0,0,4,\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right))$$

$$a_5=(0,0,0,0,\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)) \quad a_6=(0,0,0,0,\left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right)) \quad a_7=(0,0,0,0,\left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)) \quad a_8=(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right))$$

Now the embedding $\Lambda\overset{\iota}{\hookrightarrow}\Gamma$ has the following matrix with respect to the chosen bases.

$$A_\iota=\begin{pmatrix}1&1&1&1&1&0&0&1\\0&2&0&-2&0&0&2&0\\0&0&2&-2&0&1&2&0\\0&0&0&4&2&0&0&0\\0&0&0&0&2&0&0&2\\0&0&0&0&0&2&0&0\\0&0&0&0&0&0&4&0\\0&0&0&0&0&0&0&4\end{pmatrix}$$

We calculate the determinant $\det(A_\iota)=1024$, which gives

$$[\Gamma:\Lambda]=|\Gamma/\Lambda|=|\det(A_\iota)|=1024.$$

Puttings these two results together, we obtain $[\Lambda:(\mathbf{Z}D_8)\tilde{\omega}]=[\Gamma:\Lambda]/[\Gamma:(\mathbf{Z}D_8)\tilde{\omega}]=1$. Since $(\mathbf{Z}D_8)\tilde{\omega}\subseteq\Lambda$ this implies $(\mathbf{Z}D_8)\tilde{\omega}=\Lambda$. Hence the restriction onto the image $\omega:=\tilde{\omega}|^{\Lambda}$ is well-defined and an isomorphism of $\mathbf{Z}$-algebras. □

Remark 16 Using [**8**, Proposition 1.1.5], we may calculate the index of Λ in Γ to be

$$[\Gamma : \Lambda] = \sqrt{\frac{8^8}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 2^4}} = 2^{10} = 1024,$$

which is in line with our results above.

Notation 17 Throughout this chapter, we identify $\mathbf{Z}\mathrm{D}_8$ and Λ via ω, cf. Lemma 15.

Remark 18 We have the following more condensed description of the Wedderburn image Λ in Γ, cf. Lemma 15.

$$\Lambda = \left\{ (p, q, r, s, \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)) \in \Gamma : \begin{array}{c} q \equiv_2 r, \ s \equiv_2 t \equiv_2 w, \ q - r \equiv_4 v, \\ q - s \equiv_4 2u, \ p + q + r + s \equiv_8 2(t + w) \end{array} \right\}$$

Proof. Since we only dropped some congruences in the definition of Λ in Lemma 15, the inclusion $\subseteq$ is clear. For the converse inclusion, we show that the dropped congruences are already implied by those we have left here.

Ad $r \equiv_2 s$. Using $q \equiv_2 r$ and $q - s \equiv_4 2u$ we have $0 \equiv_2 2u \equiv_2 q - s \equiv_2 r - s$.

Ad $p \equiv_2 q$. From $q \equiv_2 r \equiv_2 s$ and $p + q + r + s \equiv_8 2(t + w)$ we conclude

$$0 \equiv_2 2(t + w) \equiv_2 p + q + r + s \equiv_2 p + 3q \equiv_2 p - q.$$

Ad $v \equiv_2 0$. This one follows from $q - r \equiv_4 v$ using that $q - r \equiv_2 0$.

Ad $p + q \equiv_4 r + s$. Using $t + w \equiv_2 0$ and $p + q + r + s \equiv_8 2(t + w)$, as well as $r \equiv_2 s$, we obtain

$$0 \equiv_4 2(t + w) \equiv_4 p + q + r + s \equiv_4 p + q - r - s + 2(r + s) \equiv_4 p + q - r - s. \qquad \square$$

2.2 Projective resolution over $\mathbf{Z}\mathrm{D}_8$

2.2.1 Definitions

Recall $\Gamma = \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2}$ and the image Λ of $\mathbf{Z}\mathrm{D}_8$ under the Wedderburn embedding $\tilde{\omega}$, cf. Lemma 15.

$$\Lambda = \left\{ (p, q, r, s, \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)) \in \Gamma : \begin{array}{c} p \equiv_2 q \equiv_2 r \equiv_2 s \equiv_2 t \equiv_2 w, \ v \equiv_2 0, \ q - r \equiv_4 v, \\ q - s \equiv_4 2u, \ p + q \equiv_4 r + s, \ p + q + r + s \equiv_8 2(t + w) \end{array} \right\}$$

For $m \in \Lambda$ we denote by $\Lambda \xrightarrow{\dot{m}} \Lambda$ the Λ-linear map given by right-multiplication with m.

$$\begin{array}{ccc} \Lambda & \xrightarrow{\dot{m}} & \Lambda \\ \lambda & \longmapsto & \lambda m \end{array}$$

So we have $\lambda\dot{m} = \lambda m$ for $m, \lambda \in \Lambda$.

Definition 19 Define the following elements of Λ.

$$\begin{array}{llll} A_- := (0,2,0,2,\left(\begin{smallmatrix} 0 & 0 \\ 2 & 2 \end{smallmatrix}\right)) = 1 - b & & B_- := (0,2,2,0,\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right)) = 1 - ba^3 \\ A_+ := (2,0,2,0,\left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right)) = 1 + b & & B_+ := (2,0,0,2,\left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right)) = 1 + ba^3 \\ C_- := (0,4,-4,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = a + a^3 - b - ba^2 & & D_- := (0,4,0,-4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = a + a^3 - ba - ba^3 \\ C_+ := (4,0,0,-4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = a + a^3 + b + ba^2 & & D_+ := (4,0,-4,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = a + a^3 + ba + ba^3 \end{array}$$

Alternatively, for $i \in \mathbf{Z}_{\geq 0}$ we write for these elements

$$X_i := \begin{cases} X_+ & \text{if } i \equiv_2 0 \\ X_- & \text{if } i \equiv_2 1. \end{cases}$$

Now let $k \in \mathbf{Z}_{\geq 0}$ and let $\Lambda^{\oplus(2k+2)} \xrightarrow{d_{2k}} \Lambda^{\oplus(2k+1)}$ and $\Lambda^{\oplus(2k+3)} \xrightarrow{d_{2k+1}} \Lambda^{\oplus(2k+2)}$ be given by the formulas

$$d_{2k} := \left(\sum_{i=1}^{k+1} \pi_i \dot{A}_i \iota_i\right) + \left(\sum_{i=k+2}^{2k+1} \pi_i \dot{D}_i \iota_i\right) + \left(\sum_{i=1}^{k} \pi_{i+1} (-1)^i \dot{C}_i \iota_i\right) + \left(\sum_{i=k+1}^{2k+1} \pi_{i+1} (-1)^i \dot{B}_i \iota_i\right)$$

$$d_{2k+1} := \left(\sum_{j=1}^{k+1} \pi_j \dot{A}_{j+1} \iota_j\right) + \left(\sum_{j=k+2}^{2k+2} \pi_j \dot{D}_{j+1} \iota_j\right) - \left(\sum_{j=1}^{k+1} \pi_{j+1} (-1)^j \dot{C}_j \iota_j\right) - \left(\sum_{j=k+2}^{2k+2} \pi_{j+1} (-1)^j \dot{B}_j \iota_j\right).$$

As sums of Λ-linear maps the maps d_{2k} and d_{2k+1} are again Λ-linear for $k \in \mathbf{Z}_{\geq 0}$.

The first few differentials from Definition 19 can be visualised as matrices as follows.

$$d_0 = \begin{pmatrix} \dot{A}_- \\ -\dot{B}_- \end{pmatrix}$$

$$d_1 = \begin{pmatrix} \dot{A}_+ & 0 \\ \dot{C}_- & \dot{D}_- \\ 0 & -\dot{B}_+ \end{pmatrix}$$

$$d_2 = \begin{pmatrix} \dot{A}_- & 0 & 0 \\ -\dot{C}_- & \dot{A}_+ & 0 \\ 0 & \dot{B}_+ & \dot{D}_- \\ 0 & 0 & -\dot{B}_- \end{pmatrix}$$

$$d_3 = \begin{pmatrix} \dot{A}_+ & 0 & 0 & 0 \\ \dot{C}_- & \dot{A}_- & 0 & 0 \\ 0 & -\dot{C}_+ & \dot{D}_+ & 0 \\ 0 & 0 & \dot{B}_- & \dot{D}_- \\ 0 & 0 & 0 & -\dot{B}_+ \end{pmatrix}$$

$$d_4 = \begin{pmatrix} \dot{A}_- & 0 & 0 & 0 & 0 \\ -\dot{C}_- & \dot{A}_+ & 0 & 0 & 0 \\ 0 & \dot{C}_+ & \dot{A}_- & 0 & 0 \\ 0 & 0 & -\dot{B}_- & \dot{D}_+ & 0 \\ 0 & 0 & 0 & \dot{B}_+ & \dot{D}_- \\ 0 & 0 & 0 & 0 & -\dot{B}_- \end{pmatrix}$$

$$d_5 = \begin{pmatrix} \dot{A}_+ & 0 & 0 & 0 & 0 & 0 \\ \dot{C}_- & \dot{A}_- & 0 & 0 & 0 & 0 \\ 0 & -\dot{C}_+ & \dot{A}_+ & 0 & 0 & 0 \\ 0 & 0 & \dot{C}_- & \dot{D}_- & 0 & 0 \\ 0 & 0 & 0 & -\dot{B}_+ & \dot{D}_+ & 0 \\ 0 & 0 & 0 & 0 & \dot{B}_- & \dot{D}_- \\ 0 & 0 & 0 & 0 & 0 & -\dot{B}_+ \end{pmatrix}$$

$$d_6 = \begin{pmatrix} \dot{A}_- & 0 & 0 & 0 & 0 & 0 & 0 \\ -\dot{C}_- & \dot{A}_+ & 0 & 0 & 0 & 0 & 0 \\ 0 & \dot{C}_+ & \dot{A}_- & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{C}_- & \dot{A}_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{B}_+ & \dot{D}_- & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dot{B}_- & \dot{D}_+ & 0 \\ 0 & 0 & 0 & 0 & 0 & \dot{B}_+ & \dot{D}_- \\ 0 & 0 & 0 & 0 & 0 & 0 & -\dot{B}_- \end{pmatrix}$$

Here we marked the reappearance of d_0 in d_4, of d_1 in d_5 etc.

We denote a general element of $\mathbf{Z}\mathrm{D}_8$ by $\sum_{g \in \mathrm{D}_8} r_g g$ with $r_g \in \mathbf{Z}$ without further comment. Let $\mathbf{Z}$ be the trivial Λ-module. Recall the Λ-linear augmentation map

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} \\ (p, q, r, s, \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)) & \longmapsto & p \\ \sum_{g \in \mathrm{D}_8} r_g g & \longmapsto & \sum_{g \in \mathrm{D}_8} r_g. \end{array}$$

Theorem 20 *Consider the following sequences of* Λ*-modules and* Λ*-linear maps.*

$$P := \Big(\cdots\cdots\to \Lambda^{\oplus 5} \xrightarrow{d_3} \Lambda^{\oplus 4} \xrightarrow{d_2} \Lambda^{\oplus 3} \xrightarrow{d_1} \Lambda^{\oplus 2} \xrightarrow{d_0} \Lambda \longrightarrow 0 \Big)$$

$$P' := \Big(\cdots\cdots\to \Lambda^{\oplus 5} \xrightarrow{d_3} \Lambda^{\oplus 4} \xrightarrow{d_2} \Lambda^{\oplus 3} \xrightarrow{d_1} \Lambda^{\oplus 2} \xrightarrow{d_0} \Lambda \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0 \Big)$$

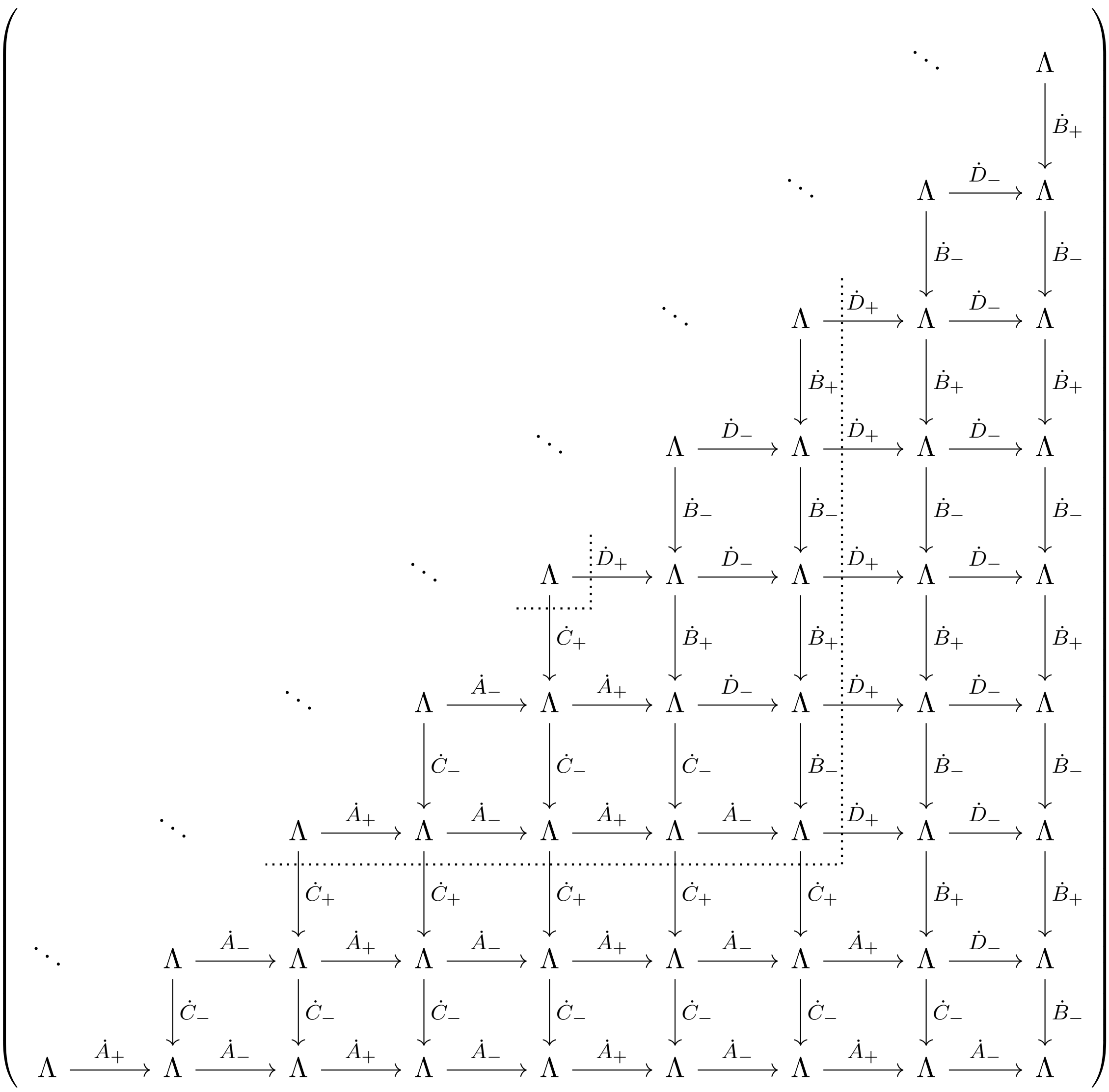
Then P *is a projective resolution of the trivial* Λ*-module* $\mathbf{Z}$ *and* P' *is the corresponding augmented projective resolution.*

Proof. All modules of P are free, hence projective.

We have to show that P' is an acyclic complex of Λ-modules. In Lemma 24 we show that P' (and thus also P) is a complex. Acyclicity of P' is verified in §2.2.3, Lemmas 27, 28 and 29. □

Remark 21 Consider the following diagram of Λ-modules and Λ-linear maps.

$\Xi =$



By Remark 23 in §2.2.2 Ξ is a double complex, i.e. a complex of complexes of Λ-modules. Hence all rows and columns in Ξ are complexes and all squares are commutative. One obtains the complex P from Theorem 20 as the total complex of the double complex Ξ. Here, our sign convention for the total complex is that we change signs in every second column starting from the right-most.

Note that the dotted lines in the diagram above describe a self-similarity of the double complex after shifting along the diagonal. This behaviour corresponds to the reappearence of earlier differentials after four steps as submatrices seen above after Definition 19. Additionally, note that all rows and columns in Ξ become eventually periodic.

In [**1**, Theorems 5.14.2 and 5.14.5] Benson shows that for any Noetherian ring R and finite group G, such that for any prime p dividing $|G|$ either p is invertible in R or $R/(p)$ is Artinian, there is a projective resolution of the trivial RG-module that can be expressed as the total complex of an n-fold complex, in which all rows, columns, etc. are periodic.

In general, a minimal projective resolution can not be written as the total complex of such an n-fold complex. This was observed by Benson, who gives a counterexample in [**1**, p. 206]. However, in [**1**, Remark at p. 205], Benson notes that the minimal projective resolution can always be written as the total of an n-fold complex in which all rows and columns become eventually periodic.

In Theorem 30 in §2.3 we will see that the complex P localised at (2) becomes a minimal projective resolution of the trivial $\mathbf{Z}_{(2)}\mathrm{D}_8$-module. One obtains the localised complex also as the total complex of the double complex Ξ localised at (2). So we construct a minimal projective resolution of the trivial $\mathbf{Z}_{(2)}\mathrm{D}_8$-module that can be expressed as the total complex of a double complex with eventually periodic rows and columns.

Remark 22 In [**6**] and [**9**], the *Wall-Hamada resolution* of the trivial Λ-module $\mathbf{Z}$ is constructed. The Wall-Hamada resolution can be written as the total complex of the following double complex.

$$\Xi^{\mathrm{WH}} = \left(\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\dashrightarrow & & \Lambda & \xrightarrow{(1+a+a^2+a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda & \xrightarrow{(1+a+a^2+a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda \\
 & & \downarrow{\scriptstyle (b+1)\cdot} & & \downarrow{\scriptstyle (ba+1)\cdot} & & \downarrow{\scriptstyle (b-1)\cdot} & & \downarrow{\scriptstyle (ba-1)\cdot} & & \downarrow{\scriptstyle (b+1)\cdot} \\
\dashrightarrow & & \Lambda & \xrightarrow{(-1-a-a^2-a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda & \xrightarrow{(-1-a-a^2-a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda \\
 & & \downarrow{\scriptstyle (b-1)\cdot} & & \downarrow{\scriptstyle (ba-1)\cdot} & & \downarrow{\scriptstyle (b+1)\cdot} & & \downarrow{\scriptstyle (ba+1)\cdot} & & \downarrow{\scriptstyle (b-1)\cdot} \\
\dashrightarrow & & \Lambda & \xrightarrow{(1+a+a^2+a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda & \xrightarrow{(1+a+a^2+a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda \\
 & & \downarrow{\scriptstyle (b+1)\cdot} & & \downarrow{\scriptstyle (ba+1)\cdot} & & \downarrow{\scriptstyle (b-1)\cdot} & & \downarrow{\scriptstyle (ba-1)\cdot} & & \downarrow{\scriptstyle (b+1)\cdot} \\
\dashrightarrow & & \Lambda & \xrightarrow{(-1-a-a^2-a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda & \xrightarrow{(-1-a-a^2-a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda \\
 & & \downarrow{\scriptstyle (b-1)\cdot} & & \downarrow{\scriptstyle (ba-1)\cdot} & & \downarrow{\scriptstyle (b+1)\cdot} & & \downarrow{\scriptstyle (ba+1)\cdot} & & \downarrow{\scriptstyle (b-1)\cdot} \\
\dashrightarrow & & \Lambda & \xrightarrow{(1+a+a^2+a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda & \xrightarrow{(1+a+a^2+a^3)\cdot} & \Lambda & \xrightarrow{(a-1)\cdot} & \Lambda
\end{array}\right)$$

Then the double complex Ξ^{WH} is neither isomorphic to Ξ, nor to the double complex Ξ^{T} obtained after reflecting Ξ along its top left to down right diagonal.

Suppose Ξ^{WH} is isomorphic to Ξ. Looking at the bottommost row of morphisms in the second column in both complexes, this implies that there are Λ-linear isomorphisms $\Lambda \xrightarrow{f,g} \Lambda$ such that the following diagram commutes.

$$\begin{array}{ccc} \Lambda & \xrightarrow{\dot{A}_+} & \Lambda \\ {\scriptstyle f}\downarrow\wr & & \wr\downarrow{\scriptstyle g} \\ \Lambda & \xrightarrow{(1+a+a^2+a^3)\cdot} & \Lambda \end{array}$$

Since $\mathrm{Hom}_\Lambda(\Lambda, \Lambda) \simeq \Lambda$, the isomorphisms f and g are just multiplication with units $\xi_f, \xi_g \in \mathrm{U}(\Lambda)$ from the right. Using the identification along the Wedderburn embedding ω from Lemma 15 we see that $1 + a + a^2 + a^3 = (4, 4, 0, 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))$. Since $A_+ = (2, 0, 2, 0, \left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right))$ has non-zero 2×2-matrix block, $A_+ \neq \xi_f(1 + a + a^2 + a^3)\xi_g^{-1}$.

Hence Ξ^{WH} and Ξ can not be isomorphic as double complexes.

Next suppose that Ξ^{WH} is isomorphic to Ξ^{T}. Again looking at the bottommost row of morphisms in the second column in both complexes, this implies that are Λ-linear isomorphisms $\Lambda \xrightarrow{f,g} \Lambda$ such that the following diagram commutes.

$$\begin{array}{ccc} \Lambda & \xrightarrow{\dot{B}_+} & \Lambda \\ {\scriptstyle f}\downarrow\wr & & \wr\downarrow{\scriptstyle g} \\ \Lambda & \xrightarrow{(1+a+a^2+a^3)\cdot} & \Lambda \end{array}$$

As above, the isomorphisms f and g are just multiplication with units $\xi_f, \xi_g \in \mathrm{U}(\Lambda)$ from the right. Since $B_+ = (2, 0, 0, 2, \left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right))$ has non-zero 2×2-matrix block, $B_+ \neq \xi_f(1+a+a^2+a^3)\xi_g^{-1}$.

Hence Ξ^{WH} and Ξ^{T} can not be isomorphic as double complexes.

2.2.2 Differential condition

Remark 23 Recall the notation used in Definition 19. Suppose given $i \in \mathbf{Z}_{\geq 0}$. Then (1-11) hold.

(1) $\dot{A}_i\dot{A}_{i+1} = 0$

(2) $\dot{B}_i\dot{B}_{i+1} = 0$

(3) $\dot{C}_i\dot{C}_{i+1} = 0$

(4) $\dot{D}_i\dot{D}_{i+1} = 0$

(5) $\dot{A}_i\dot{D}_{i+1} = 0$

(6) $\dot{B}_i\dot{C}_{i+1} = 0$

(7) $\dot{C}_i\dot{A}_i - \dot{D}_i\dot{B}_i = 0$

(8) $\dot{A}_i\dot{C}_i - \dot{C}_i\dot{A}_i = 0$

(9) $\dot{C}_i\dot{A}_{i+1} - \dot{A}_{i+1}\dot{C}_i = 0$

(10) $\dot{D}_i\dot{B}_i - \dot{B}_i\dot{D}_i = 0$

(11) $\dot{B}_i\dot{D}_{i+1} - \dot{D}_{i+1}\dot{B}_i = 0$

Proof. The map

$$\Lambda \xrightarrow{\dot{(-)}} \operatorname{Hom}_\Lambda(\Lambda, \Lambda), \quad m \longmapsto \dot{m}$$

is an isomorphism of rings, which allows us to prove (1-11) in Λ, effectively ignoring the dot. From their Wedderburn images in Definition 19 we see that $C_+, C_-, D_+, D_- \in \mathrm{Z}(\Lambda)$.

Ad (1). We have to consider the cases $i \equiv_2 0$ and $i \equiv_2 1$.

$$\begin{aligned} A_+A_- &= (2,0,2,0,\left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right))(0,2,0,2,\left(\begin{smallmatrix} 0 & 0 \\ 2 & 2 \end{smallmatrix}\right)) = 0 \\ A_-A_+ &= (0,2,0,2,\left(\begin{smallmatrix} 0 & 0 \\ 2 & 2 \end{smallmatrix}\right))(2,0,2,0,\left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right)) = 0 \end{aligned}$$

Ad (2). We have to consider the cases $i \equiv_2 0$ and $i \equiv_2 1$.

$$\begin{aligned} B_+B_- &= (2,0,0,2,\left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right))(0,2,2,0,\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right)) = 0 \\ B_-B_+ &= (0,2,2,0,\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right))(2,0,0,2,\left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right)) = 0 \end{aligned}$$

Ad (3). Since $C_+, C_- \in \mathrm{Z}(\Lambda)$, we only have to consider the case $i \equiv_2 0$.

$$C_+C_- = (4,0,0,-4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))(0,4,-4,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = 0$$

Ad (4). Since $D_+, D_- \in \mathrm{Z}(\Lambda)$, we only have to consider the case $i \equiv_2 0$.

$$D_+D_- = (4,0,-4,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))(0,4,0,-4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = 0$$

Ad (5). We have to consider the cases $i \equiv_2 0$ and $i \equiv_2 1$.

$$\begin{aligned} A_+D_- &= (2,0,2,0,\left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right))(0,4,0,-4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = 0 \\ A_-D_+ &= (0,2,0,2,\left(\begin{smallmatrix} 0 & 0 \\ 2 & 2 \end{smallmatrix}\right))(4,0,-4,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = 0 \end{aligned}$$

Ad (6). We have to consider the cases $i \equiv_2 0$ and $i \equiv_2 1$.

$$\begin{aligned} B_+C_- &= (2,0,0,2,\left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right))(0,4,-4,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = 0 \\ B_-C_+ &= (0,2,2,0,\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right))(4,0,0,-4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = 0 \end{aligned}$$

Ad (7). We have to consider the cases $i \equiv_2 0$ and $i \equiv_2 1$.

$$\begin{aligned} C_+A_+ - D_+B_+ &= (4,0,0,-4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))(2,0,2,0,\left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right)) - (4,0,-4,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))(2,0,0,2,\left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right)) \\ &= (8,0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) - (8,0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = 0 \\ C_-A_- - D_-B_- &= (0,4,-4,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))(0,2,0,2,\left(\begin{smallmatrix} 0 & 0 \\ 2 & 2 \end{smallmatrix}\right)) - (0,4,0,-4,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))(0,2,2,0,\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right)) \\ &= (0,8,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) - (0,8,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = 0 \end{aligned}$$

Ad (8-11). These follow since $C_+, C_-, D_+, D_- \in \mathrm{Z}(\Lambda)$. □

Lemma 24 *Recall the differentials from Definition 19 and the augmentation map* ε*.*
Suppose given $k \in \mathbf{Z}_{\geq 0}$*. Then* (1-3) *hold.*
In particular, the sequences P *and* P' *from Theorem 20 are complexes of* Λ*-modules.*

(1) $d_0\varepsilon = 0$

(2) $d_{2k+1}d_{2k} = 0$

(3) $d_{2k+2}d_{2k+1} = 0$

Proof. *Ad* (1). For $((p,q,r,s,\left(\begin{smallmatrix} t & u \\ v & w\end{smallmatrix}\right)),(p',q',r',s',\left(\begin{smallmatrix} t' & u' \\ v' & w'\end{smallmatrix}\right))) \in \Lambda^{\oplus 2}$ we obtain

$$\begin{aligned}\Big((p,q,r,s,\left(\begin{smallmatrix} t & u \\ v & w\end{smallmatrix}\right)),(p',q',r',s',\left(\begin{smallmatrix} t' & u' \\ v' & w'\end{smallmatrix}\right))\Big)\, d_0\varepsilon &= \Big((p,q,r,s,\left(\begin{smallmatrix} t & u \\ v & w\end{smallmatrix}\right))A_- - (p',q',r',s',\left(\begin{smallmatrix} t' & u' \\ v' & w'\end{smallmatrix}\right))B_-\Big)\,\varepsilon \\ &= \big(0, 2q-2q', -2r', 2s, \left(\begin{smallmatrix} 2u-2t' & 2u-t' \\ 2w-2v' & 2w-v'\end{smallmatrix}\right)\big)\varepsilon \\ &= 0,\end{aligned}$$

hence $d_0\varepsilon = 0$.

Ad (2). Recall the differentials from Definition 19.

$$d_{2k} := \left(\sum_{i=1}^{k+1} \pi_i \dot{A}_i \iota_i\right) + \left(\sum_{i=k+2}^{2k+1} \pi_i \dot{D}_i \iota_i\right) + \left(\sum_{i=1}^{k} \pi_{i+1}(-1)^i \dot{C}_i \iota_i\right) + \left(\sum_{i=k+1}^{2k+1} \pi_{i+1}(-1)^i \dot{B}_i \iota_i\right)$$

$$d_{2k+1} := \left(\sum_{j=1}^{k+1} \pi_j \dot{A}_{j+1} \iota_j\right) + \left(\sum_{j=k+2}^{2k+2} \pi_j \dot{D}_{j+1} \iota_j\right) - \left(\sum_{j=1}^{k+1} \pi_{j+1}(-1)^j \dot{C}_j \iota_j\right) - \left(\sum_{j=k+2}^{2k+2} \pi_{j+1}(-1)^j \dot{B}_j \iota_j\right)$$

We calculate using (1-11) from Remark 23.

$$\begin{aligned} d_{2k+1}d_{2k} &= \left(\sum_{i=1}^{k+1} \pi_i \dot{A}_{i+1}\dot{A}_i \iota_i\right) + \left(\sum_{i=2}^{k+1} \pi_i \dot{A}_{i+1}(-1)^{i-1}\dot{C}_{i-1}\iota_{i-1}\right) \\ &\quad + \left(\sum_{i=k+2}^{2k+1} \pi_i \dot{D}_{i+1}\dot{D}_i \iota_i\right) + \left(\sum_{i=k+2}^{2k+2} \pi_i \dot{D}_{i+1}(-1)^{i-1}\dot{B}_{i-1}\iota_{i-1}\right) \\ &\quad - \left(\sum_{i=1}^{k+1} \pi_{i+1}(-1)^i \dot{C}_i\dot{A}_i \iota_i\right) - \left(\sum_{i=2}^{k+1} \pi_{i+1}(-1)^i \dot{C}_i(-1)^{i-1}\dot{C}_{i-1}\iota_{i-1}\right) \\ &\quad - \left(\sum_{i=k+2}^{2k+1} \pi_{i+1}(-1)^i \dot{B}_i\dot{D}_i \iota_i\right) - \left(\sum_{i=k+2}^{2k+2} \pi_{i+1}(-1)^i \dot{B}_i(-1)^{i-1}\dot{B}_{i-1}\iota_{i-1}\right) \\ &= \left(\sum_{i=1}^{k+1} \pi_i \underbrace{\dot{A}_{i+1}\dot{A}_i}_{\overset{(1)}{=}0} \iota_i\right) + \left(\sum_{i=k+2}^{2k+1} \pi_i \underbrace{\dot{D}_{i+1}\dot{D}_i}_{\overset{(4)}{=}0} \iota_i\right) \\ &\quad + \left(\sum_{i=2}^{k+1} \pi_{i+1} \underbrace{\dot{C}_i\dot{C}_{i-1}}_{\overset{(3)}{=}0} \iota_{i-1}\right) + \left(\sum_{i=k+2}^{2k+2} \pi_{i+1} \underbrace{\dot{B}_i\dot{B}_{i-1}}_{\overset{(2)}{=}0} \iota_{i-1}\right) \\ &\quad + \left(\sum_{i=1}^{k} \pi_{i+1}(-1)^i \underbrace{\left(\dot{A}_i\dot{C}_i - \dot{C}_i\dot{A}_i\right)}_{\overset{(8)}{=}0} \iota_i\right) \\ &\quad + \left(\sum_{i=k+2}^{2k+1} \pi_{i+1}(-1)^i \underbrace{\left(\dot{D}_i\dot{B}_i - \dot{B}_i\dot{D}_i\right)}_{\overset{(10)}{=}0} \iota_i\right) \\ &\quad + \pi_{k+2}(-1)^k \underbrace{\left(\dot{C}_{k+1}\dot{A}_{k+1} - \dot{D}_{k+1}\dot{B}_{k+1}\right)}_{\overset{(7)}{=}0} \iota_{k+1} \\ &= 0 \end{aligned}$$

Ad (3). Recall the differentials from Definition 19.

$$d_{2k+1} = \left(\sum_{i=1}^{k+1} \pi_i \dot{A}_{i+1} \iota_i\right) + \left(\sum_{i=k+2}^{2k+2} \pi_i \dot{D}_{i+1} \iota_i\right) - \left(\sum_{i=1}^{k+1} \pi_{i+1} (-1)^i \dot{C}_i \iota_i\right) - \left(\sum_{i=k+2}^{2k+2} \pi_{i+1} (-1)^i \dot{B}_i \iota_i\right)$$

$$d_{2k+2} = \left(\sum_{j=1}^{k+2} \pi_j \dot{A}_j \iota_j\right) + \left(\sum_{j=k+3}^{2k+3} \pi_j \dot{D}_j \iota_j\right) + \left(\sum_{j=1}^{k+1} \pi_{j+1} (-1)^j \dot{C}_j \iota_j\right) + \left(\sum_{j=k+2}^{2k+3} \pi_{j+1} (-1)^j \dot{B}_j \iota_j\right)$$

We calculate using (1-11) from Remark 23.

$$\begin{aligned}
d_{2k+2} d_{2k+1} &= \left(\sum_{i=1}^{k+1} \pi_i \dot{A}_i \dot{A}_{i+1} \iota_i\right) + \pi_{k+2} \dot{A}_{k+2} \dot{D}_{k+3} \iota_{k+2} - \left(\sum_{i=2}^{k+2} \pi_i \dot{A}_i (-1)^{i-1} \dot{C}_{i-1} \iota_{i-1}\right) \\
&+ \left(\sum_{i=k+3}^{2k+2} \pi_i \dot{D}_i \dot{D}_{i+1} \iota_i\right) - \left(\sum_{i=k+3}^{2k+3} \pi_i \dot{D}_i (-1)^{i-1} \dot{B}_{i-1} \iota_{i-1}\right) \\
&+ \left(\sum_{i=1}^{k+1} \pi_{i+1} (-1)^i \dot{C}_i \dot{A}_{i+1} \iota_i\right) - \left(\sum_{i=2}^{k+1} \pi_{i+1} (-1)^i \dot{C}_i (-1)^{i-1} \dot{C}_{i-1} \iota_{i-1}\right) \\
&+ \left(\sum_{i=k+2}^{2k+2} \pi_{i+1} (-1)^i \dot{B}_i \dot{D}_{i+1} \iota_i\right) - \pi_{k+3} (-1)^{k+2} \dot{B}_{k+2} (-1)^{k+1} \dot{C}_{k+1} \iota_{k+1} \\
&- \left(\sum_{i=k+3}^{2k+3} \pi_{i+1} (-1)^i \dot{B}_i (-1)^{i-1} \dot{B}_{i-1} \iota_{i-1}\right) \\
&= \left(\sum_{i=1}^{k+1} \pi_i \underbrace{\dot{A}_i \dot{A}_{i+1}}_{\overset{(1)}{=} 0} \iota_i\right) + \pi_{k+2} \underbrace{\dot{A}_k \dot{D}_{k+1}}_{\overset{(5)}{=} 0} \iota_{k+2} - \left(\sum_{i=k+3}^{2k+2} \pi_i \underbrace{\dot{D}_i \dot{D}_{i+1}}_{\overset{(4)}{=} 0} \iota_i\right) \\
&+ \left(\sum_{i=2}^{k+1} \pi_{i+1} \underbrace{\dot{C}_i \dot{C}_{i-1}}_{\overset{(3)}{=} 0} \iota_{i-1}\right) + \pi_{k+3} \underbrace{\dot{B}_k \dot{C}_{k+1}}_{\overset{(6)}{=} 0} \iota_{k+1} + \left(\sum_{i=k+3}^{2k+3} \pi_{i+1} \underbrace{\dot{B}_i \dot{B}_{i-1}}_{\overset{(2)}{=} 0} \iota_{i-1}\right) \\
&+ \left(\sum_{i=1}^{k+1} \pi_{i+1} (-1)^i \underbrace{\left(\dot{C}_i \dot{A}_{i+1} - \dot{A}_{i+1} \dot{C}_i\right)}_{\overset{(9)}{=} 0} \iota_i\right) \\
&+ \left(\sum_{i=k+2}^{2k+2} \pi_{i+1} (-1)^i \underbrace{\left(\dot{B}_i \dot{D}_{i+1} - \dot{D}_{i+1} \dot{B}_i\right)}_{\overset{(11)}{=} 0} \iota_i\right) \\
&= 0
\end{aligned}$$

□

2.2.3 Acyclicity

Recall that we identify $\mathbf{Z}\mathrm{D}_8$ with its Wedderburn image Λ via ω, cf. Notation 17. So a general element of Λ can be written as $\sum_{g\in \mathrm{D}_8} r_g g$ with $r_g \in \mathbf{Z}$ for $g \in \mathrm{D}_8$.

Definition 25 Define the following $\mathbf{Z}$-linear maps from Λ to Λ.

$$
\begin{array}{lllll}
\Lambda & \xrightarrow{\alpha_-} & \Lambda, & \sum\limits_{g\in \mathrm{D}_8} r_g g & \longmapsto & (-r_{a^3}-r_b)1 + r_{a^2}a - r_{ba}a^3 + r_{ba}b - r_{a^2}ba^2 + (-r_a - r_{ba^2})ba^3 \\
\Lambda & \xrightarrow{\alpha_+} & \Lambda, & \sum\limits_{g\in \mathrm{D}_8} r_g g & \longmapsto & (-r_{a^3}+r_b)1 + r_{a^2}a + r_{ba}a^3 + r_{ba}b + r_{a^2}ba^2 + (r_a - r_{ba^2})ba^3 \\
\Lambda & \xrightarrow{\beta_-} & \Lambda, & \sum\limits_{g\in \mathrm{D}_8} r_g g & \longmapsto & (r_a+r_{ba^3})1 + r_{ba^2}a - r_{a^2}a^3 + (r_{a^3}+r_{ba})b + r_{a^2}ba - r_{ba^2}ba^3 \\
\Lambda & \xrightarrow{\beta_+} & \Lambda, & \sum\limits_{g\in \mathrm{D}_8} r_g g & \longmapsto & (-r_a+r_{ba^3})1 + r_{ba^2}a + r_{a^2}a^3 + (r_{a^3}-r_{ba})b + r_{a^2}ba + r_{ba^2}ba^3 \\
\Lambda & \xrightarrow{\gamma_-} & \Lambda, & \sum\limits_{g\in \mathrm{D}_8} r_g g & \longmapsto & (-r_{a^3}-r_b)1 + (-r_{a^2}-r_{ba})a^3 + (r_{a^2}+r_{ba})b + (-r_a - r_{ba^2})ba^3 \\
\Lambda & \xrightarrow{\gamma_+} & \Lambda, & \sum\limits_{g\in \mathrm{D}_8} r_g g & \longmapsto & (-r_{a^3}+r_b)1 + (-r_{a^2}+r_{ba})a^3 + (-r_{a^2}+r_{ba})b + (r_a - r_{ba^2})ba^3 \\
\Lambda & \xrightarrow{\delta_-} & \Lambda, & \sum\limits_{g\in \mathrm{D}_8} r_g g & \longmapsto & (-r_{a^2}+r_{ba^2})1 \\
\Lambda & \xrightarrow{\delta_+} & \Lambda, & \sum\limits_{g\in \mathrm{D}_8} r_g g & \longmapsto & (r_{a^2}+r_{ba^2})1
\end{array}
$$

Alternatively, for $i \in \mathbf{Z}_{\geq 0}$ we write for these maps

$$\xi_i := \begin{cases} \xi_+ & \text{if } i \equiv_2 0 \\ \xi_- & \text{if } i \equiv_2 1. \end{cases}$$

Define the $\mathbf{Z}$-linear map

$$\begin{array}{ccc} \mathbf{Z} & \xrightarrow{h_{-1}} & \Lambda \\ r & \longmapsto & r. \end{array}$$

For $k \in \mathbf{Z}_{\geq 0}$ define

$$h_{2k} = \left(\sum_{i=1}^{k} \pi_i \alpha_i \iota_i\right) + (\pi_{k+1}\gamma_{k+1}\iota_{k+1}) + \left(\sum_{i=k+1}^{2k+1} \pi_i \beta_i \iota_{i+1}\right) \;:\; \Lambda^{\oplus(2k+1)} \longrightarrow \Lambda^{\oplus(2k+2)}$$

$$h_{2k+1} = -\left(\sum_{j=1}^{k+1} \pi_j \alpha_j \iota_j\right) - (\pi_{k+1}\delta_{k+1}\iota_{k+2}) - \left(\sum_{j=k+2}^{2k+2} \pi_j \beta_{j+1} \iota_{j+1}\right) \;:\; \Lambda^{\oplus(2k+2)} \longrightarrow \Lambda^{\oplus(2k+3)}.$$

As sums of $\mathbf{Z}$-linear maps h_{2k} and h_{2k+1} are again $\mathbf{Z}$-linear for $k \in \mathbf{Z}_{\geq 0}$.

The first few maps can be visualised as matrices as follows. Note that we find h_0 as a submatrix in h_4, h_1 in h_5 and so on. We have already seen that the differentials show the same behaviour.

$$h_0 = \begin{pmatrix} \gamma_- & \beta_- \end{pmatrix}$$

$$h_1 = -\begin{pmatrix} \alpha_- & \delta_- & 0 \\ 0 & 0 & \beta_- \end{pmatrix}$$

$$h_2 = \begin{pmatrix} \alpha_- & 0 & 0 & 0 \\ 0 & \gamma_+ & \beta_+ & 0 \\ 0 & 0 & 0 & \beta_- \end{pmatrix}$$

$$h_3 = -\begin{pmatrix} \alpha_- & 0 & 0 & 0 & 0 \\ 0 & \alpha_+ & \delta_+ & 0 & 0 \\ 0 & 0 & 0 & \beta_+ & 0 \\ 0 & 0 & 0 & 0 & \beta_- \end{pmatrix}$$

$$h_4 = \begin{pmatrix} \alpha_- & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_+ & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_- & \beta_- & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_+ & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_- \end{pmatrix}$$

$$h_5 = -\begin{pmatrix} \alpha_- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_- & \delta_- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_- & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_- \end{pmatrix}$$

$$h_6 = \begin{pmatrix} \alpha_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_+ & \beta_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_- & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_- \end{pmatrix}$$

Now recall the asserted augmented projective resolution P' from Theorem 20.

$$P' = \Big(\cdots\cdots \to \Lambda^{\oplus 5} \xrightarrow{d_3} \Lambda^{\oplus 4} \xrightarrow{d_2} \Lambda^{\oplus 3} \xrightarrow{d_1} \Lambda^{\oplus 2} \xrightarrow{d_0} \Lambda \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0 \Big)$$

We will use Lemma 2 to show that P' is acyclic, i.e. we show that $(h_k)_{k \in \mathbf{Z}_{\geq -1}}$ defines a contracting homotopy for P'.

For this, we have to show that $h_0 d_0 + \varepsilon h_{-1} = 1$, $h_{2k+2} d_{2k+2} + d_{2k+1} h_{2k+1} = 1$ and $h_{2k+1} d_{2k+1} + d_{2k} h_{2k} = 1$ for $k \in \mathbf{Z}_{\geq 0}$.

In that case, note that by Lemma 2 the complex P' is also split acyclic.

Lemma 26 *Suppose we are given $i \in \mathbf{Z}_{\geq 0}$. Then* (1-12) *hold.*

(1) $\alpha_i \dot{A}_i - \dot{A}_{i+1} \alpha_i = 1$

(2) $-\alpha_i \dot{A}_{i+1} + \dot{A}_i \alpha_i = 1$

(3) $\beta_i (-1)^i \dot{B}_i - (-1)^i \dot{B}_{i+1} \beta_i = 1$

(4) $\beta_{i+1} (-1)^i \dot{B}_i - (-1)^i \dot{B}_{i+1} \beta_{i+1} = 1$

(5) $\beta_i \dot{D}_{i+1} - \dot{D}_{i+1} \beta_{i+1} = 0$

(6) $\alpha_i \dot{C}_{i+1} + \dot{C}_{i+1} \alpha_{i+1} = 0$

(7) $-\alpha_i \dot{A}_{i+1} + \delta_i (-1)^i \dot{C}_i + \dot{A}_i \gamma_i = 1$

(8) $\dot{B}_i \gamma_i = 0$

(9) $-\delta_i \dot{D}_i + \dot{A}_i \beta_i = 0$

(10) $\beta_i (-1)^i \dot{B}_i + \gamma_i \dot{A}_i - (-1)^i \dot{C}_{i+1} \delta_{i+1} = 1$

(11) $\dot{A}_i\delta_{i+1} = 0$

(12) $\gamma_i\dot{C}_{i+1} + \dot{C}_{i+1}\alpha_{i+1} = 0$

Proof. Write $(r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3}) := \sum_{g\in \mathrm{D}_8} r_g g$ for a general element of Λ. For each equation (1-12) we have to consider the cases $i \equiv_2 0$ and $i \equiv_2 1$.

(1) For $i \equiv_2 0$ we calculate

$$\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(\alpha_+\dot{A}_+ - \dot{A}_-\alpha_+)\\
&= (-r_{a^3} + r_b,\, r_{a^2},\, 0,\, r_{ba},\, r_{ba},\, 0,\, r_{a^2},\, r_a - r_{ba^2})\dot{A}_+\\
&\quad - (r_1 - r_b,\, r_a - r_{ba^3},\, r_{a^2} - r_{ba^2},\, r_{a^3} - r_{ba},\, -r_1 + r_b,\, -r_{a^3} + r_{ba},\, -r_{a^2} + r_{ba^2},\, -r_a + r_{ba^3})\alpha_+\\
&= (-r_{a^3} + r_b + r_{ba},\, r_a + r_{a^2} - r_{ba^2},\, r_{a^2},\, r_{ba},\, -r_{a^3} + r_b + r_{ba},\, r_{ba},\, r_{a^2},\, r_a + r_{a^2} - r_{ba^2})\\
&\quad - (-r_1 - r_{a^3} + r_b + r_{ba},\, r_{a^2} - r_{ba^2},\, 0,\, -r_{a^3} + r_{ba},\, -r_{a^3} + r_{ba},\, 0,\, r_{a^2} - r_{ba^2},\, r_a + r_{a^2} - r_{ba^2} - r_{ba^3})\\
&= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).
\end{aligned}$$

Hence $\alpha_+\dot{A}_+ - \dot{A}_-\alpha_+ = 1$.
For $i \equiv_2 1$ we calculate

$$\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(\alpha_-\dot{A}_- - \dot{A}_+\alpha_-)\\
&= (-r_{a^3} - r_b,\, r_{a^2},\, 0,\, -r_{ba},\, r_{ba},\, 0,\, -r_{a^2},\, -r_a - r_{ba^2})\dot{A}_-\\
&\quad - (r_1 + r_b,\, r_a + r_{ba^3},\, r_{a^2} + r_{ba^2},\, r_{a^3} + r_{ba},\, r_1 + r_b,\, r_{a^3} + r_{ba},\, r_{a^2} + r_{ba^2},\, r_a + r_{ba^3})\alpha_-\\
&= (-r_{a^3} - r_b - r_{ba},\, r_a + r_{a^2} + r_{ba^2},\, r_{a^2},\, -r_{ba},\, r_{a^3} + r_b + r_{ba},\, r_{ba},\, -r_{a^2},\, -r_a - r_{a^2} - r_{ba^2})\\
&\quad - (-r_1 - r_{a^3} - r_b - r_{ba},\, r_{a^2} + r_{ba^2},\, 0,\, -r_{a^3} - r_{ba},\, r_{a^3} + r_{ba},\, 0,\, -r_{a^2} - r_{ba^2},\, -r_a - r_{a^2} - r_{ba^2} - r_{ba^3})\\
&= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).
\end{aligned}$$

Hence $\alpha_-\dot{A}_- - \dot{A}_+\alpha_- = 1$.

(2) For $i \equiv_2 0$ we calculate

$$\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(-\alpha_+\dot{A}_- + \dot{A}_+\alpha_+)\\
&= -(-r_{a^3} + r_b,\, r_{a^2},\, 0,\, r_{ba},\, r_{ba},\, 0,\, r_{a^2},\, r_a - r_{ba^2})\dot{A}_-\\
&\quad + (r_1 + r_b,\, r_a + r_{ba^3},\, r_{a^2} + r_{ba^2},\, r_{a^3} + r_{ba},\, r_1 + r_b,\, r_{a^3} + r_{ba},\, r_{a^2} + r_{ba^2},\, r_a + r_{ba^3})\alpha_+\\
&= -(-r_{a^3} + r_b - r_{ba},\, -r_a + r_{a^2} + r_{ba^2},\, -r_{a^2},\, r_{ba},\, r_{a^3} - r_b + r_{ba},\, -r_{ba},\, r_{a^2},\, r_a - r_{a^2} - r_{ba^2})\\
&\quad + (r_1 - r_{a^3} + r_b - r_{ba},\, r_{a^2} + r_{ba^2},\, 0,\, r_{a^3} + r_{ba},\, r_{a^3} + r_{ba},\, 0,\, r_{a^2} + r_{ba^2},\, r_a - r_{a^2} - r_{ba^2} + r_{ba^3})\\
&= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).
\end{aligned}$$

Hence $-\alpha_+\dot{A}_- + \dot{A}_+\alpha_+ = 1$.
For $i \equiv_2 1$ we calculate

$$\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(-\alpha_-\dot{A}_+ + \dot{A}_-\alpha_-)\\
&= -(-r_{a^3} - r_b,\, r_{a^2},\, 0,\, -r_{ba},\, r_{ba},\, 0,\, -r_{a^2},\, -r_a - r_{ba^2})\dot{A}_+\\
&\quad + (r_1 - r_b,\, r_a - r_{ba^3},\, r_{a^2} - r_{ba^2},\, r_{a^3} - r_{ba},\, -r_1 + r_b,\, -r_{a^3} + r_{ba},\, -r_{a^2} + r_{ba^2},\, -r_a + r_{ba^3})\alpha_-\\
&= -(-r_{a^3} - r_b + r_{ba},\, -r_a + r_{a^2} - r_{ba^2},\, -r_{a^2},\, -r_{ba},\, -r_{a^3} - r_b + r_{ba},\, -r_{ba},\, -r_{a^2},\, -r_a + r_{a^2} - r_{ba^2})\\
&\quad + (r_1 - r_{a^3} - r_b + r_{ba},\, r_{a^2} - r_{ba^2},\, 0,\, r_{a^3} - r_{ba},\, -r_{a^3} + r_{ba},\, 0,\, -r_{a^2} + r_{ba^2},\, -r_a + r_{a^2} - r_{ba^2} + r_{ba^3})\\
&= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).
\end{aligned}$$

Hence $-\alpha_-\dot{A}_+ + \dot{A}_-\alpha_- = 1$.

(3) For $i \equiv_2 0$ we calculate

$$\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(\beta_+\dot{B}_+ - \dot{B}_-\beta_+)\\
&= (-r_a + r_{ba^3},\, r_{ba^2},\, 0,\, r_{a^2},\, r_{a^3} - r_{ba},\, r_{a^2},\, 0,\, r_{ba^2})\dot{B}_+\\
&\quad - (r_1 - r_{ba^3},\, r_a - r_{ba^2},\, r_{a^2} - r_{ba},\, r_{a^3} - r_b,\, -r_{a^3} + r_b,\, -r_{a^2} + r_{ba},\, -r_a + r_{ba^2},\, -r_1 + r_{ba^3})\beta_+
\end{aligned}$$

$$= (-r_a + r_{ba^2} + r_{ba^3},\, r_{ba^2},\, r_{a^2},\, r_{a^2} + r_{a^3} - r_{ba},\, r_{a^2} + r_{a^3} - r_{ba},\, r_{a^2},\, r_{ba^2},\, -r_a + r_{ba^2} + r_{ba^3})$$
$$- (-r_1 - r_a + r_{ba^2} + r_{ba^3},\, -r_a + r_{ba^2},\, 0,\, r_{a^2} - r_{ba},\, r_{a^2} + r_{a^3} - r_b - r_{ba},\, r_{a^2} - r_{ba},\, 0,\, -r_a + r_{ba^2})$$
$$= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).$$

Hence $\beta_+\dot{B}_+ - \dot{B}_-\beta_+ = 1$.
For $i \equiv_2 1$ we calculate

$$(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(-\beta_-\dot{B}_- + \dot{B}_+\beta_-)$$
$$= -(r_a + r_{ba^3},\, r_{ba^2},\, 0,\, -r_{a^2},\, r_{a^3} + r_{ba},\, r_{a^2},\, 0,\, -r_{ba^2})\dot{B}_-$$
$$+ (r_1 + r_{ba^3},\, r_a + r_{ba^2},\, r_{a^2} + r_{ba},\, r_{a^3} + r_b,\, r_{a^3} + r_b,\, r_{a^2} + r_{ba},\, r_a + r_{ba^2},\, r_1 + r_{ba^3})\beta_-$$
$$= -(r_a + r_{ba^2} + r_{ba^3},\, r_{ba^2},\, -r_{a^2},\, -r_{a^2} - r_{a^3} - r_{ba},\, r_{a^2} + r_{a^3} + r_{ba},\, r_{a^2},\, -r_{ba^2},\, -r_a - r_{ba^2} - r_{ba^3})$$
$$+ (r_1 + r_a + r_{ba^2} + r_{ba^3},\, r_a + r_{ba^2},\, 0,\, -r_{a^2} - r_{ba},\, r_{a^2} + r_{a^3} + r_b + r_{ba},\, r_{a^2} + r_{ba},\, 0,\, -r_a - r_{ba^2})$$
$$= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).$$

Hence $-\beta_-\dot{B}_- + \dot{B}_+\beta_- = 1$.

(4) For $i \equiv_2 0$ we calculate

$$(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(\beta_-\dot{B}_+ - \dot{B}_-\beta_-)$$
$$= (r_a + r_{ba^3},\, r_{ba^2},\, 0,\, -r_{a^2},\, r_{a^3} + r_{ba},\, r_{a^2},\, 0,\, -r_{ba^2})\dot{B}_+$$
$$- (r_1 - r_{ba^3},\, r_a - r_{ba^2},\, r_{a^2} - r_{ba},\, r_{a^3} - r_b,\, -r_{a^3} + r_b,\, -r_{a^2} + r_{ba},\, -r_a + r_{ba^2},\, -r_1 + r_{ba^3})\beta_-$$
$$= (r_a - r_{ba^2} + r_{ba^3},\, r_{ba^2},\, r_{a^2},\, -r_{a^2} + r_{a^3} + r_{ba},\, -r_{a^2} + r_{a^3} + r_{ba},\, r_{a^2},\, r_{ba^2},\, r_a - r_{ba^2} + r_{ba^3})$$
$$- (-r_1 + r_a - r_{ba^2} + r_{ba^3},\, -r_a + r_{ba^2},\, 0,\, -r_{a^2} + r_{ba},\, -r_{a^2} + r_{a^3} - r_b + r_{ba},\, r_{a^2} - r_{ba},\, 0,\, r_a - r_{ba^2})$$
$$= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).$$

Hence $\beta_-\dot{B}_+ - \dot{B}_-\beta_- = 1$.
For $i \equiv_2 1$ we calculate

$$(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(-\beta_+\dot{B}_- + \dot{B}_+\beta_+)$$
$$= -(-r_a + r_{ba^3},\, r_{ba^2},\, 0,\, r_{a^2},\, r_{a^3} - r_{ba},\, r_{a^2},\, 0,\, r_{ba^2})\dot{B}_-$$
$$+ (r_1 + r_{ba^3},\, r_a + r_{ba^2},\, r_{a^2} + r_{ba},\, r_{a^3} + r_b,\, r_{a^3} + r_b,\, r_{a^2} + r_{ba},\, r_a + r_{ba^2},\, r_1 + r_{ba^3})\beta_+$$
$$= -(-r_a - r_{ba^2} + r_{ba^3},\, r_{ba^2},\, -r_{a^2},\, r_{a^2} - r_{a^3} + r_{ba},\, -r_{a^2} + r_{a^3} - r_{ba},\, r_{a^2},\, -r_{ba^2},\, r_a + r_{ba^2} - r_{ba^3})$$
$$+ (r_1 - r_a - r_{ba^2} + r_{ba^3},\, r_a + r_{ba^2},\, 0,\, r_{a^2} + r_{ba},\, -r_{a^2} + r_{a^3} + r_b - r_{ba},\, r_{a^2} + r_{ba},\, 0,\, r_a + r_{ba^2})$$
$$= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).$$

Hence $-\beta_+\dot{B}_- + \dot{B}_+\beta_+ = 1$.

(5) For $i \equiv_2 0$ we calculate

$$(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(\beta_+\dot{D}_- - \dot{D}_-\beta_-)$$
$$= (-r_a + r_{ba^3},\, r_{ba^2},\, 0,\, r_{a^2},\, r_{a^3} - r_{ba},\, r_{a^2},\, 0,\, r_{ba^2})\dot{D}_-$$
$$- (r_a + r_{a^3} - r_{ba} - r_{ba^3},\, r_1 + r_{a^2} - r_b - r_{ba^2},\, r_a + r_{a^3} - r_{ba} - r_{ba^3},\, r_1 + r_{a^2} - r_b - r_{ba^2},$$
$$- r_a - r_{a^3} + r_{ba} + r_{ba^3},\, -r_1 - r_{a^2} + r_b + r_{ba^2},\, -r_a - r_{a^3} + r_{ba} + r_{ba^3},\, -r_1 - r_{a^2} + r_b + r_{ba^2})\beta_-$$
$$= (0,\, -r_a - r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, -r_a - r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, r_a + r_{a^3} - r_{ba} - r_{ba^3},\, 0,\, r_a + r_{a^3} - r_{ba} - r_{ba^3})$$
$$- (0,\, -r_a - r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, -r_a - r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, r_a + r_{a^3} - r_{ba} - r_{ba^3},\, 0,\, r_a + r_{a^3} - r_{ba} - r_{ba^3})$$
$$= 0.$$

Hence $\beta_+\dot{D}_- - \dot{D}_-\beta_- = 0$.
For $i \equiv_2 1$ we calculate

$$(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(\beta_-\dot{D}_+ - \dot{D}_+\beta_+)$$
$$= (r_a + r_{ba^3},\, r_{ba^2},\, 0,\, -r_{a^2},\, r_{a^3} + r_{ba},\, r_{a^2},\, 0,\, -r_{ba^2})\dot{D}_+$$
$$- (r_a + r_{a^3} + r_{ba} + r_{ba^3},\, r_1 + r_{a^2} + r_b + r_{ba^2},\, r_a + r_{a^3} + r_{ba} + r_{ba^3},\, r_1 + r_{a^2} + r_b + r_{ba^2},$$

$$
\begin{aligned}
&r_a + r_{a^3} + r_{ba} + r_{ba^3},\, r_1 + r_{a^2} + r_b + r_{ba^2},\, r_a + r_{a^3} + r_{ba} + r_{ba^3},\, r_1 + r_{a^2} + r_b + r_{ba^2})\beta_+ \\
&= (0,\, r_a + r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, r_a + r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, r_a + r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, r_a + r_{a^3} + r_{ba} + r_{ba^3}) \\
&\quad - (0,\, r_a + r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, r_a + r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, r_a + r_{a^3} + r_{ba} + r_{ba^3},\, 0,\, r_a + r_{a^3} + r_{ba} + r_{ba^3}) \\
&= 0.
\end{aligned}
$$

Hence $\beta_- \dot{D}_+ - \dot{D}_+ \beta_+ = 0$.

(6) For $i \equiv_2 0$ we calculate

$$
\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(\alpha_+ \dot{C}_- + \dot{C}_- \alpha_-) \\
&= (-r_{a^3} + r_b,\, r_{a^2},\, 0,\, r_{ba},\, r_{ba},\, 0,\, r_{a^2},\, r_a - r_{ba^2})\dot{C}_- \\
&\quad + (r_a + r_{a^3} - r_b - r_{ba^2},\, r_1 + r_{a^2} - r_{ba} - r_{ba^3},\, r_a + r_{a^3} - r_b - r_{ba^2},\, r_1 + r_{a^2} - r_{ba} - r_{ba^3}, \\
&\qquad - r_1 - r_{a^2} + r_{ba} + r_{ba^3},\, -r_a - r_{a^3} + r_b + r_{ba^2},\, -r_1 - r_{a^2} + r_{ba} + r_{ba^3},\, -r_a - r_{a^3} + r_b + r_{ba^2})\alpha_- \\
&= (0,\, -r_a - r_{a^3} + r_b + r_{ba^2},\, 0,\, -r_a - r_{a^3} + r_b + r_{ba^2},\, r_a + r_{a^3} - r_b - r_{ba^2},\, 0,\, r_a + r_{a^3} - r_b - r_{ba^2},\, 0) \\
&\quad + (0,\, r_a + r_{a^3} - r_b - r_{ba^2},\, 0,\, r_a + r_{a^3} - r_b - r_{ba^2},\, -r_a - r_{a^3} + r_b + r_{ba^2},\, 0,\, -r_a - r_{a^3} + r_b + r_{ba^2},\, 0) \\
&= 0.
\end{aligned}
$$

Hence $\alpha_+ \dot{C}_- + \dot{C}_- \alpha_- = 0$.
For $i \equiv_2 1$ we calculate

$$
\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(\alpha_- \dot{C}_+ + \dot{C}_+ \alpha_+) \\
&= (-r_{a^3} - r_b,\, r_{a^2},\, 0,\, -r_{ba},\, r_{ba},\, 0,\, -r_{a^2},\, -r_a - r_{ba^2})\dot{C}_+ \\
&\quad + (r_a + r_{a^3} + r_b + r_{ba^2},\, r_1 + r_{a^2} + r_{ba} + r_{ba^3},\, r_a + r_{a^3} + r_b + r_{ba^2},\, r_1 + r_{a^2} + r_{ba} + r_{ba^3}, \\
&\qquad r_1 + r_{a^2} + r_{ba} + r_{ba^3},\, r_a + r_{a^3} + r_b + r_{ba^2},\, r_1 + r_{a^2} + r_{ba} + r_{ba^3},\, r_a + r_{a^3} + r_b + r_{ba^2})\alpha_+ \\
&= (0,\, -r_a - r_{a^3} - r_b - r_{ba^2},\, 0,\, -r_a - r_{a^3} - r_b - r_{ba^2},\, -r_a - r_{a^3} - r_b - r_{ba^2},\, 0,\, -r_a - r_{a^3} - r_b - r_{ba^2},\, 0) \\
&\quad + (0,\, r_a + r_{a^3} + r_b + r_{ba^2},\, 0,\, r_a + r_{a^3} + r_b + r_{ba^2},\, r_a + r_{a^3} + r_b + r_{ba^2},\, 0,\, r_a + r_{a^3} + r_b + r_{ba^2},\, 0) \\
&= 0.
\end{aligned}
$$

Hence $\alpha_- \dot{C}_+ + \dot{C}_+ \alpha_+ = 0$.

(7) For $i \equiv_2 0$ we calculate

$$
\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(-\alpha_+ \dot{A}_- + \delta_+ \dot{C}_+ + \dot{A}_+ \gamma_+) \\
&= -(-r_{a^3} + r_b,\, r_{a^2},\, 0,\, r_{ba},\, r_{ba},\, 0,\, r_{a^2},\, r_a - r_{ba^2})\dot{A}_- \\
&\quad + (r_{a^2} + r_{ba^2},\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0)\dot{C}_+ \\
&\quad + (r_1 + r_b,\, r_a + r_{ba^3},\, r_{a^2} + r_{ba^2},\, r_{a^3} + r_{ba},\, r_1 + r_b,\, r_{a^3} + r_{ba},\, r_{a^2} + r_{ba^2},\, r_a + r_{ba^3})\gamma_+ \\
&= -(-r_{a^3} + r_b - r_{ba},\, -r_a + r_{a^2} + r_{ba^2},\, -r_{a^2},\, r_{ba},\, r_{a^3} - r_b + r_{ba},\, -r_{ba},\, r_{a^2},\, r_a - r_{a^2} - r_{ba^2}) \\
&\quad + (0,\, r_{a^2} + r_{ba^2},\, 0,\, r_{a^2} + r_{ba^2},\, r_{a^2} + r_{ba^2},\, 0,\, r_{a^2} + r_{ba^2},\, 0) \\
&\quad + (r_1 - r_{a^3} + r_b - r_{ba},\, 0,\, 0,\, -r_{a^2} + r_{a^3} + r_{ba} - r_{ba^2},\, -r_{a^2} + r_{a^3} + r_{ba} - r_{ba^2},\, 0,\, 0,\, r_a - r_{a^2} - r_{ba^2} + r_{ba^3}) \\
&= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).
\end{aligned}
$$

Hence $-\alpha_+ \dot{A}_- + \delta_+ \dot{C}_+ + \dot{A}_+ \gamma_+ = 1$.
For $i \equiv_2 1$ we calculate

$$
\begin{aligned}
&(r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3})(-\alpha_- \dot{A}_+ - \delta_- \dot{C}_- + \dot{A}_- \gamma_-) \\
&= -(-r_{a^3} - r_b,\, r_{a^2},\, 0,\, -r_{ba},\, r_{ba},\, 0,\, -r_{a^2},\, -r_a - r_{ba^2})\dot{A}_+ \\
&\quad - (-r_{a^2} + r_{ba^2},\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0)\dot{C}_- \\
&\quad + (r_1 - r_b,\, r_a - r_{ba^3},\, r_{a^2} - r_{ba^2},\, r_{a^3} - r_{ba},\, -r_1 + r_b,\, -r_{a^3} + r_{ba},\, -r_{a^2} + r_{ba^2},\, -r_a + r_{ba^3})\gamma_- \\
&= -(-r_{a^3} - r_b + r_{ba},\, -r_a + r_{a^2} - r_{ba^2},\, -r_{a^2},\, -r_{ba},\, -r_{a^3} - r_b + r_{ba},\, -r_{ba},\, -r_{a^2},\, -r_a + r_{a^2} - r_{ba^2}) \\
&\quad - (0,\, -r_{a^2} + r_{ba^2},\, 0,\, -r_{a^2} + r_{ba^2},\, +r_{a^2} - r_{ba^2},\, 0,\, +r_{a^2} - r_{ba^2},\, 0) \\
&\quad + (r_1 - r_{a^3} - r_b + r_{ba},\, 0,\, 0,\, -r_{a^2} + r_{a^3} - r_{ba} + r_{ba^2},\, r_{a^2} - r_{a^3} + r_{ba} - r_{ba^2},\, 0,\, 0,\, -r_a + r_{a^2} - r_{ba^2} + r_{ba^3}) \\
&= (r_1,\, r_a,\, r_{a^2},\, r_{a^3},\, r_b,\, r_{ba},\, r_{ba^2},\, r_{ba^3}).
\end{aligned}
$$

Hence $-\alpha_- \dot{A}_+ - \delta_- \dot{C}_- + \dot{A}_- \gamma_- = 1$.

(8) For $i \equiv_2 0$ we calculate

$$\begin{aligned}
&(r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})(\dot{B}_+\gamma_+)\\
&= (r_1 + r_{ba^3}, r_a + r_{ba^2}, r_{a^2} + r_{ba}, r_{a^3} + r_b, r_{a^3} + r_b, r_{a^2} + r_{ba}, r_a + r_{ba^2}, r_1 + r_{ba^3})\gamma_+\\
&= 0.
\end{aligned}$$

Hence $\dot{B}_+\gamma_+ = 0$.
For $i \equiv_2 1$ we calculate

$$\begin{aligned}
&(r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})(\dot{B}_-\gamma_-)\\
&= (r_1 - r_{ba^3}, r_a - r_{ba^2}, r_{a^2} - r_{ba}, r_{a^3} - r_b, -r_{a^3} + r_b, -r_{a^2} + r_{ba}, -r_a + r_{ba^2}, -r_1 + r_{ba^3})\gamma_-\\
&= 0.
\end{aligned}$$

Hence $\dot{B}_-\gamma_- = 0$.

(9) For $i \equiv_2 0$ we calculate

$$\begin{aligned}
&(r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})(-\delta_+\dot{D}_+ + \dot{A}_+\beta_+)\\
&= -(r_{a^2} + r_{ba^2}, 0, 0, 0, 0, 0, 0, 0)\dot{D}_+\\
&\quad + (r_1 + r_b, r_a + r_{ba^3}, r_{a^2} + r_{ba^2}, r_{a^3} + r_{ba}, r_1 + r_b, r_{a^3} + r_{ba}, r_{a^2} + r_{ba^2}, r_a + r_{ba^3})\beta_+\\
&= -(0, r_{a^2} + r_{ba^2}, 0, r_{a^2} + r_{ba^2}, 0, r_{a^2} + r_{ba^2}, 0, r_{a^2} + r_{ba^2})\\
&\quad + (0, r_{a^2} + r_{ba^2}, 0, r_{a^2} + r_{ba^2}, 0, r_{a^2} + r_{ba^2}, 0, r_{a^2} + r_{ba^2})\\
&= 0.
\end{aligned}$$

Hence $-\delta_+\dot{D}_+ + \dot{A}_+\beta_+ = 0$.
For $i \equiv_2 1$ we calculate

$$\begin{aligned}
&(r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})(-\delta_-\dot{D}_- + \dot{A}_-\beta_-)\\
&= -(-r_{a^2} + r_{ba^2}, 0, 0, 0, 0, 0, 0, 0)\dot{D}_-\\
&\quad + (r_1 - r_b, r_a - r_{ba^3}, r_{a^2} - r_{ba^2}, r_{a^3} - r_{ba}, -r_1 + r_b, -r_{a^3} + r_{ba}, -r_{a^2} + r_{ba^2}, -r_a + r_{ba^3})\beta_-\\
&= -(0, -r_{a^2} + r_{ba^2}, 0, -r_{a^2} + r_{ba^2}, 0, r_{a^2} - r_{ba^2}, 0, r_{a^2} - r_{ba^2})\\
&\quad + (0, -r_{a^2} + r_{ba^2}, 0, -r_{a^2} + r_{ba^2}, 0, r_{a^2} - r_{ba^2}, 0, r_{a^2} - r_{ba^2})\\
&= 0.
\end{aligned}$$

Hence $-\delta_-\dot{D}_- + \dot{A}_-\beta_- = 0$.

(10) For $i \equiv_2 0$ we calculate

$$\begin{aligned}
&(r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})(\beta_+\dot{B}_+ + \gamma_+\dot{A}_+ - \dot{C}_-\delta_-)\\
&= (-r_a + r_{ba^3}, r_{ba^2}, 0, r_{a^2}, r_{a^3} - r_{ba}, r_{a^2}, 0, r_{ba^2})\dot{B}_+\\
&\quad + (-r_{a^3} + r_b, 0, 0, -r_{a^2} + r_{ba}, -r_{a^2} + r_{ba}, 0, 0, r_a - r_{ba^2})\dot{A}_+\\
&\quad - (r_a + r_{a^3} - r_b - r_{ba^2}, r_1 + r_{a^2} - r_{ba} - r_{ba^3}, r_a + r_{a^3} - r_b - r_{ba^2}, r_1 + r_{a^2} - r_{ba} - r_{ba^3},\\
&\qquad - r_1 - r_{a^2} + r_{ba} + r_{ba^3}, -r_a - r_{a^3} + r_b + r_{ba^2}, -r_1 - r_{a^2} + r_{ba} + r_{ba^3}, -r_a - r_{a^3} + r_b + r_{ba^2})\delta_-\\
&= (-r_a + r_{ba^2} + r_{ba^3}, r_{ba^2}, r_{a^2}, r_{a^2} + r_{a^3} - r_{ba}, r_{a^2} + r_{a^3} - r_{ba}, r_{a^2}, r_{ba^2}, -r_a + r_{ba^2} + r_{ba^3})\\
&\quad + (-r_{a^2} - r_{a^3} + r_b + r_{ba}, r_a - r_{ba^2}, 0, -r_{a^2} + r_{ba}, -r_{a^2} - r_{a^3} + r_b + r_{ba}, -r_{a^2} + r_{ba}, 0, r_a - r_{ba^2})\\
&\quad - (-r_1 - r_a - r_{a^2} - r_{a^3} + r_b + r_{ba} + r_{ba^2} + r_{ba^3}, 0, 0, 0, 0, 0, 0, 0)\\
&= (r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3}).
\end{aligned}$$

Hence $\beta_+\dot{B}_+ + \gamma_+\dot{A}_+ - \dot{C}_-\delta_- = 1$.
For $i \equiv_2 1$ we calculate

$$\begin{aligned}
&(r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})(-\beta_-\dot{B}_- + \gamma_-\dot{A}_- + \dot{C}_+\delta_+)\\
&= -(r_a + r_{ba^3}, r_{ba^2}, 0, -r_{a^2}, r_{a^3} + r_{ba}, r_{a^2}, 0, -r_{ba^2})\dot{B}_-
\end{aligned}$$

$$
\begin{aligned}
&+(-r_{a^3}-r_b,\,0,\,0,\,-r_{a^2}-r_{ba},\,r_{a^2}+r_{ba},\,0,\,0,\,-r_a-r_{ba^2})\dot{A}_-\\
&+(r_a+r_{a^3}+r_b+r_{ba^2},\,r_1+r_{a^2}+r_{ba}+r_{ba^3},\,r_a+r_{a^3}+r_b+r_{ba^2},\,r_1+r_{a^2}+r_{ba}+r_{ba^3},\\
&\quad r_1+r_{a^2}+r_{ba}+r_{ba^3},\,r_a+r_{a^3}+r_b+r_{ba^2},\,r_1+r_{a^2}+r_{ba}+r_{ba^3},\,r_a+r_{a^3}+r_b+r_{ba^2})\delta_+\\
&=-(r_a+r_{ba^2}+r_{ba^3},\,r_{ba^2},\,-r_{a^2},\,-r_{a^2}-r_{a^3}-r_{ba},\,r_{a^2}+r_{a^3}+r_{ba},\,r_{a^2},\,-r_{ba^2},\,-r_a-r_{ba^2}-r_{ba^3})\\
&+(-r_{a^2}-r_{a^3}-r_b-r_{ba},\,r_a+r_{ba^2},\,0,\,-r_{a^2}-r_{ba},\,r_{a^2}+r_{a^3}+r_b+r_{ba},\,r_{a^2}+r_{ba},\,0,\,-r_a-r_{ba^2})\\
&+(r_1+r_a+r_{a^2}+r_{a^3}+r_b+r_{ba}+r_{ba^2}+r_{ba^3},\,0,\,0,\,0,\,0,\,0,\,0,\,0)\\
&=(r_1,\,r_a,\,r_{a^2},\,r_{a^3},\,r_b,\,r_{ba},\,r_{ba^2},\,r_{ba^3}).
\end{aligned}
$$

Hence $-\beta_-\dot{B}_- + \gamma_-\dot{A}_- + \dot{C}_+\delta_+ = 1$.

(11) For $i \equiv_2 0$ we calculate

$$
\begin{aligned}
&(r_1,\,r_a,\,r_{a^2},\,r_{a^3},\,r_b,\,r_{ba},\,r_{ba^2},\,r_{ba^3})(\dot{A}_+\delta_-)\\
&=(r_1+r_b,\,r_a+r_{ba^3},\,r_{a^2}+r_{ba^2},\,r_{a^3}+r_{ba},\,r_1+r_b,\,r_{a^3}+r_{ba},\,r_{a^2}+r_{ba^2},\,r_a+r_{ba^3})\delta_-\\
&=0.
\end{aligned}
$$

Hence $\dot{A}_+\delta_- = 0$.
For $i \equiv_2 1$ we calculate

$$
\begin{aligned}
&(r_1,\,r_a,\,r_{a^2},\,r_{a^3},\,r_b,\,r_{ba},\,r_{ba^2},\,r_{ba^3})(\dot{A}_-\delta_+)\\
&=(r_1-r_b,\,r_a-r_{ba^3},\,r_{a^2}-r_{ba^2},\,r_{a^3}-r_{ba},\,-r_1+r_b,\,-r_{a^3}+r_{ba},\,-r_{a^2}+r_{ba^2},\,-r_a+r_{ba^3})\delta_+\\
&=0.
\end{aligned}
$$

Hence $\dot{A}_-\delta_+ = 0$.

(12) For $i \equiv_2 0$ we calculate

$$
\begin{aligned}
&(r_1,\,r_a,\,r_{a^2},\,r_{a^3},\,r_b,\,r_{ba},\,r_{ba^2},\,r_{ba^3})(\gamma_+\dot{C}_- + \dot{C}_-\alpha_-)\\
&=(-r_{a^3}+r_b,\,0,\,0,\,-r_{a^2}+r_{ba},\,-r_{a^2}+r_{ba},\,0,\,0,\,r_a-r_{ba^2})\dot{C}_-\\
&+(r_a+r_{a^3}-r_b-r_{ba^2},\,r_1+r_{a^2}-r_{ba}-r_{ba^3},\,r_a+r_{a^3}-r_b-r_{ba^2},\,r_1+r_{a^2}-r_{ba}-r_{ba^3},\\
&\quad -r_1-r_{a^2}+r_{ba}+r_{ba^3},\,-r_a-r_{a^3}+r_b+r_{ba^2},\,-r_1-r_{a^2}+r_{ba}+r_{ba^3},\,-r_a-r_{a^3}+r_b+r_{ba^2})\alpha_-\\
&=(0,\,-r_a-r_{a^3}+r_b+r_{ba^2},\,0,\,-r_a-r_{a^3}+r_b+r_{ba^2},\,r_a+r_{a^3}-r_b-r_{ba^2},\,0,\,r_a+r_{a^3}-r_b-r_{ba^2},\,0)\\
&+(0,\,r_a+r_{a^3}-r_b-r_{ba^2},\,0,\,r_a+r_{a^3}-r_b-r_{ba^2},\,-r_a-r_{a^3}+r_b+r_{ba^2},\,0,\,-r_a-r_{a^3}+r_b+r_{ba^2},\,0)\\
&=0.
\end{aligned}
$$

Hence $\gamma_+\dot{C}_- + \dot{C}_-\alpha_- = 0$.
For $i \equiv_2 1$ we calculate

$$
\begin{aligned}
&(r_1,\,r_a,\,r_{a^2},\,r_{a^3},\,r_b,\,r_{ba},\,r_{ba^2},\,r_{ba^3})(\gamma_-\dot{C}_+ + \dot{C}_+\alpha_+)\\
&=(-r_{a^3}-r_b,\,0,\,0,\,-r_{a^2}-r_{ba},\,r_{a^2}+r_{ba},\,0,\,0,\,-r_a-r_{ba^2})\dot{C}_+\\
&+(r_a+r_{a^3}+r_b+r_{ba^2},\,r_1+r_{a^2}+r_{ba}+r_{ba^3},\,r_a+r_{a^3}+r_b+r_{ba^2},\,r_1+r_{a^2}+r_{ba}+r_{ba^3},\\
&\quad r_1+r_{a^2}+r_{ba}+r_{ba^3},\,r_a+r_{a^3}+r_b+r_{ba^2},\,r_1+r_{a^2}+r_{ba}+r_{ba^3},\,r_a+r_{a^3}+r_b+r_{ba^2})\alpha_+\\
&=(0,\,-r_a-r_{a^3}-r_b-r_{ba^2},\,0,\,-r_a-r_{a^3}-r_b-r_{ba^2},\,-r_a-r_{a^3}-r_b-r_{ba^2},\,0,\,-r_a-r_{a^3}-r_b-r_{ba^2},\,0)\\
&+(0,\,r_a+r_{a^3}+r_b+r_{ba^2},\,0,\,r_a+r_{a^3}+r_b+r_{ba^2},\,r_a+r_{a^3}+r_b+r_{ba^2},\,0,\,r_a+r_{a^3}+r_b+r_{ba^2},\,0)\\
&=0.
\end{aligned}
$$

Hence $\gamma_-\dot{C}_+ + \dot{C}_+\alpha_+ = 0$. □

Lemma 27 *We have $h_0d_0 + \varepsilon h_{-1} = 1$.*

Proof. We write $(r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3}) := \sum_{g\in \mathrm{D}_8} r_g g \in \Lambda$ for a general element of Λ and obtain

$$
(r_1,\,r_a,\,r_{a^2},\,r_{a^3},\,r_b,\,r_{ba},\,r_{ba^2},\,r_{ba^3})(h_0d_0+\varepsilon h_{-1})
$$

$$
\begin{aligned}
&= ((r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})\gamma_-, (r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})\beta_-)d_0 \\
&\quad + (r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3})\varepsilon h_{-1} \\
&= ((-r_{a^3} - r_b, 0, 0, -r_{a^2} - r_{ba}, r_{a^2} + r_{ba}, 0, 0, -r_a - r_{ba^2}), \\
&\qquad (r_a + r_{ba^3}, r_{ba^2}, 0, -r_{a^2}, r_{a^3} + r_{ba}, r_{a^2}, 0, -r_{ba^2}))d_0 \\
&\quad + (r_1 + r_a + r_{a^2} + r_{a^3} + r_b + r_{ba} + r_{ba^2} + r_{ba^3})h_{-1} \\
&= (-r_{a^3} - r_b, 0, 0, -r_{a^2} - r_{ba}, r_{a^2} + r_{ba}, 0, 0, -r_a - r_{ba^2})\dot{A}_- \\
&\quad - (r_a + r_{ba^3}, r_{ba^2}, 0, -r_{a^2}, r_{a^3} + r_{ba}, r_{a^2}, 0, -r_{ba^2})\dot{B}_- \\
&\quad + (r_1 + r_a + r_{a^2} + r_{a^3} + r_b + r_{ba} + r_{ba^2} + r_{ba^3})h_{-1} \\
&= (-r_{a^2} - r_{a^3} - r_b - r_{ba}, r_a + r_{ba^2}, 0, -r_{a^2} - r_{ba}, r_{a^2} + r_{a^3} + r_b + r_{ba}, r_{a^2} + r_{ba}, 0, -r_a - r_{ba^2}) \\
&\quad - (r_a + r_{ba^2} + r_{ba^3}, r_{ba^2}, -r_{a^2}, -r_{a^2} - r_{a^3} - r_{ba}, r_{a^2} + r_{a^3} + r_{ba}, r_{a^2}, -r_{ba^2}, -r_a - r_{ba^2} - r_{ba^3}) \\
&\quad + (r_1 + r_a + r_{a^2} + r_{a^3} + r_b + r_{ba} + r_{ba^2} + r_{ba^3}, 0, 0, 0, 0, 0, 0, 0) \\
&= (r_1, r_a, r_{a^2}, r_{a^3}, r_b, r_{ba}, r_{ba^2}, r_{ba^3}).
\end{aligned}
$$

Hence $h_0 d_0 + \varepsilon h_{-1} = 1$. □

Lemma 28 *For $k \in \mathbf{Z}_{\geq 0}$ we have $h_{2k+2}d_{2k+2} + d_{2k+1}h_{2k+1} = 1$.*

Proof. Recall the differentials from Definition 19 and the homotopies from Definition 25.

$$
d_{2k+1} = \left(\sum_{i=1}^{k+1} \pi_i \dot{A}_{i+1}\iota_i\right) + \left(\sum_{i=k+2}^{2k+2} \pi_i \dot{D}_{i+1}\iota_i\right) - \left(\sum_{i=1}^{k+1} \pi_{i+1}(-1)^i \dot{C}_i\iota_i\right) - \left(\sum_{i=k+2}^{2k+2} \pi_{i+1}(-1)^i \dot{B}_i\iota_i\right)
$$

$$
d_{2k+2} = \left(\sum_{j=1}^{k+2} \pi_j \dot{A}_j\iota_j\right) + \left(\sum_{j=k+3}^{2k+3} \pi_j \dot{D}_j\iota_j\right) + \left(\sum_{j=1}^{k+1} \pi_{j+1}(-1)^j \dot{C}_j\iota_j\right) + \left(\sum_{j=k+2}^{2k+3} \pi_{j+1}(-1)^j \dot{B}_j\iota_j\right)
$$

$$
h_{2k+1} = -\left(\sum_{j=1}^{k+1} \pi_j \alpha_j \iota_j\right) - (\pi_{k+1}\delta_{k+1}\iota_{k+2}) - \left(\sum_{j=k+2}^{2k+2} \pi_j \beta_{j+1}\iota_{j+1}\right)
$$

$$
h_{2k+2} = \left(\sum_{i=1}^{k+1} \pi_i \alpha_i \iota_i\right) + (\pi_{k+2}\gamma_{k+2}\iota_{k+2}) + \left(\sum_{i=k+2}^{2k+3} \pi_i \beta_i \iota_{i+1}\right)
$$

We calculate using (1-12) from Lemma 26.

$$
\begin{aligned}
&h_{2k+2}d_{2k+2} + d_{2k+1}h_{2k+1} \\
&= \left(\sum_{i=1}^{k+1} \pi_i \alpha_i \dot{A}_i \iota_i\right) + \left(\sum_{i=2}^{k+1} \pi_i \alpha_i (-1)^{i-1} \dot{C}_{i-1}\iota_{i-1}\right) + \left(\pi_{k+2}\gamma_{k+2}\dot{A}_{k+2}\iota_{k+2}\right) \\
&\quad + \left(\pi_{k+2}\gamma_{k+2}(-1)^{k+1}\dot{C}_{k+1}\iota_{k+1}\right) + \left(\sum_{i=k+2}^{2k+2} \pi_i \beta_i \dot{D}_{i+1}\iota_{i+1}\right) + \left(\sum_{i=k+2}^{2k+3} \pi_i \beta_i (-1)^i \dot{B}_i \iota_i\right) \\
&\quad - \left(\sum_{i=1}^{k+1} \pi_i \dot{A}_{i+1}\alpha_i \iota_i\right) - \left(\pi_{k+1}\dot{A}_{k+2}\delta_{k+1}\iota_{k+2}\right) - \left(\sum_{i=k+2}^{2k+2} \pi_i \dot{D}_{i+1}\beta_{i+1}\iota_{i+1}\right) \\
&\quad + \left(\sum_{i=1}^{k+1} \pi_{i+1}(-1)^i \dot{C}_i \alpha_i \iota_i\right) + \left(\pi_{k+2}(-1)^{k+1}\dot{C}_{k+1}\delta_{k+1}\iota_{k+2}\right) + \left(\sum_{i=k+2}^{2k+2} \pi_{i+1}(-1)^i \dot{B}_i \beta_{i+1}\iota_{i+1}\right) \\
&= \left(\sum_{i=1}^{k+1} \pi_i \underbrace{\left(\alpha_i \dot{A}_i - \dot{A}_{i+1}\alpha_i\right)}_{\overset{(1)}{=}1} \iota_i\right) + \left(\pi_{k+2} \underbrace{\left(\beta_k(-1)^k \dot{B}_k + \gamma_k \dot{A}_k - (-1)^k \dot{C}_{k+1}\delta_{k+1}\right)}_{\overset{(10)}{=}1} \iota_{k+2}\right)
\end{aligned}
$$

$$+\left(\sum_{i=k+3}^{2k+3}\pi_i\underbrace{\left(\beta_i(-1)^i\dot{B}_i-(-1)^i\dot{B}_{i+1}\beta_i\right)}_{\overset{(3)}{=}1}\iota_i\right)$$

$$+\left(\sum_{i=2}^{k+1}\pi_i(-1)^{i+1}\underbrace{\left(\alpha_i\dot{C}_{i+1}+\dot{C}_{i+1}\alpha_{i+1}\right)}_{\overset{(5)}{=}0}\iota_{i-1}\right)+(-1)^{k+1}\left(\pi_{k+2}\underbrace{\left(\gamma_k\dot{C}_{k+1}+\dot{C}_{k+1}\alpha_{k+1}\right)}_{\overset{(12)}{=}0}\iota_{k+1}\right)$$

$$+\left(\sum_{i=k+2}^{2k+2}\pi_i\underbrace{\left(\beta_i\dot{D}_{i+1}-\dot{D}_{i+1}\beta_{i+1}\right)}_{\overset{(6)}{=}0}\iota_{i+1}\right)-\left(\pi_{k+1}\underbrace{\dot{A}_k\delta_{k+1}}_{\overset{(11)}{=}0}\iota_{k+2}\right)$$

$$=\left(\sum_{i=1}^{2k+3}\pi_i\iota_i\right)$$

$$=1$$

□

Lemma 29 *For $k\in\mathbf{Z}_{\geq 0}$ we have $h_{2k+1}d_{2k+1}+d_{2k}h_{2k}=1$.*

Proof. Recall the differentials from Definition 19 and the homotopies from Definition 25.

$$d_{2k}:=\left(\sum_{i=1}^{k+1}\pi_i\dot{A}_i\iota_i\right)+\left(\sum_{i=k+2}^{2k+1}\pi_i\dot{D}_i\iota_i\right)+\left(\sum_{i=1}^{k}\pi_{i+1}(-1)^i\dot{C}_i\iota_i\right)+\left(\sum_{i=k+1}^{2k+1}\pi_{i+1}(-1)^i\dot{B}_i\iota_i\right)$$

$$d_{2k+1}:=\left(\sum_{j=1}^{k+1}\pi_j\dot{A}_{j+1}\iota_j\right)+\left(\sum_{j=k+2}^{2k+2}\pi_j\dot{D}_{j+1}\iota_j\right)-\left(\sum_{j=1}^{k+1}\pi_{j+1}(-1)^j\dot{C}_j\iota_j\right)-\left(\sum_{j=k+2}^{2k+2}\pi_{j+1}(-1)^j\dot{B}_j\iota_j\right)$$

$$h_{2k}=\left(\sum_{j=1}^{k}\pi_j\alpha_j\iota_j\right)+(\pi_{k+1}\gamma_{k+1}\iota_{k+1})+\left(\sum_{j=k+1}^{2k+1}\pi_j\beta_j\iota_{j+1}\right)$$

$$h_{2k+1}=-\left(\sum_{i=1}^{k+1}\pi_i\alpha_i\iota_i\right)-(\pi_{k+1}\delta_{k+1}\iota_{k+2})-\left(\sum_{i=k+2}^{2k+2}\pi_i\beta_{i+1}\iota_{i+1}\right)$$

We calculate using (1-12) from Lemma 26.

$$h_{2k+1}d_{2k+1}+d_{2k}h_{2k}$$

$$=-\left(\sum_{i=1}^{k+1}\pi_i\alpha_i\dot{A}_{i+1}\iota_i\right)+\left(\sum_{i=2}^{k+1}\pi_i\alpha_i(-1)^{i-1}\dot{C}_{i-1}\iota_{i-1}\right)-\left(\pi_{k+1}\delta_{k+1}\dot{D}_{k+3}\iota_{k+2}\right)$$

$$+\left(\pi_{k+1}\delta_{k+1}(-1)^{k+1}\dot{C}_{k+1}\iota_{k+1}\right)-\left(\sum_{i=k+2}^{2k+1}\pi_i\beta_{i+1}\dot{D}_{i+2}\iota_{i+1}\right)+\left(\sum_{i=k+2}^{2k+2}\pi_i\beta_{i+1}(-1)^i\dot{B}_i\iota_i\right)$$

$$+\left(\sum_{i=1}^{k}\pi_i\dot{A}_i\alpha_i\iota_i\right)+\left(\pi_{k+1}\dot{A}_{k+1}\gamma_{k+1}\iota_{k+1}\right)+\left(\pi_{k+1}\dot{A}_{k+1}\beta_{k+1}\iota_{k+2}\right)+\left(\sum_{i=k+2}^{2k+1}\pi_i\dot{D}_i\beta_i\iota_{i+1}\right)$$

$$+\left(\sum_{i=1}^{k}\pi_{i+1}(-1)^i\dot{C}_i\alpha_i\iota_i\right)+\left(\pi_{k+2}(-1)^{k+1}\dot{B}_{k+1}\gamma_{k+1}\iota_{k+1}\right)+\left(\sum_{i=k+1}^{2k+1}\pi_{i+1}(-1)^i\dot{B}_i\beta_i\iota_{i+1}\right)$$

$$=\left(\sum_{i=1}^{k}\pi_i\underbrace{\left(-\alpha_i\dot{A}_{i+1}+\dot{A}_i\alpha_i\right)}_{\overset{(2)}{=}1}\iota_i\right)+\pi_{k+1}\underbrace{\left(-\alpha_{k+1}\dot{A}_{k+2}+\delta_{k+1}(-1)^{k+1}\dot{C}_{k+1}+\dot{A}_{k+1}\gamma_{k+1}\right)}_{\overset{(7)}{=}1}\iota_{k+1}$$

$$
\begin{aligned}
&+ \left(\sum_{i=k+2}^{2k+2} \pi_i \underbrace{\left(\beta_{i+1}(-1)^i \dot{B}_i - (-1)^i \dot{B}_{i+1}\beta_{i+1}\right)}_{\overset{(4)}{=}1} \iota_i \right) \\
&+ \left(\sum_{i=2}^{k+1} \pi_i (-1)^{i+1} \underbrace{\left(\alpha_i \dot{C}_{i+1} + \dot{C}_{i+1}\alpha_{i+1}\right)}_{\overset{(6)}{=}0} \iota_{i-1} \right) + \pi_{k+1} \underbrace{\left(-\delta_{k+1}\dot{D}_{k+1} + \dot{A}_{k+1}\beta_{k+1}\right)}_{\overset{(9)}{=}0} \iota_{k+2} \\
&+ (-1)^{k+1} \left(\pi_{k+2} \underbrace{\left(\dot{B}_{k+1}\gamma_{k+1}\right)}_{\overset{(8)}{=}0} \iota_{k+1} \right) - \left(\sum_{i=k+2}^{2k+1} \pi_i \underbrace{\left(\beta_{i-1}\dot{D}_i - \dot{D}_i\beta_i\right)}_{\overset{(5)}{=}0} \iota_{i+1} \right) \\
&= \left(\sum_{i=1}^{2k+2} \pi_i \iota_i \right) \\
&= 1
\end{aligned}
$$

□

2.3 Projective resolution over $\mathbf{Z}_{(2)}\mathrm{D}_8$

2.3.1 Scalar extension

We extend scalars from $\mathbf{Z}$ to the discrete valuation ring $\mathbf{Z}_{(2)}$ and prove minimality of the resulting projective resolution of the trivial $\mathbf{Z}_{(2)}\mathrm{D}_8$-module.

We will make use of the results in §1.2.2. We will extend scalars from $\mathbf{Z}$ to $\mathbf{Z}_{(2)}$ along the injective ring morphism $\mathbf{Z} \xrightarrow{i} \mathbf{Z}_{(2)},\ z \longmapsto z/1$.

Hence we have to apply the functor $\mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} -$ to our projective resolution P.

For a $\mathbf{Z}\mathrm{D}_8$-module M recall the following isomorphism of $\mathbf{Z}_{(2)}\mathrm{D}_8$-modules from Remark 4.

$$
\begin{array}{rcl}
\mathbf{Z}_{(2)}\mathrm{D}_8 & \xrightarrow[\sim]{\alpha} & \mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}\mathrm{D}_8 \\
xg & \longmapsto & x \otimes g \\
xzg & \longleftarrow & x \otimes zg
\end{array}
$$

Now let $m \in \mathbf{Z}\mathrm{D}_8$. We identify $\mathbf{Z}\mathrm{D}_8 \subseteq \mathbf{Z}_{(2)}\mathrm{D}_8$ by identifying $\mathbf{Z} \subseteq \mathbf{Z}_{(2)}$ along injective ring morphism i.

We have the following commutative diagram of $\mathbf{Z}_{(2)}\mathrm{D}_8$-modules.

$$
\begin{array}{ccc}
\mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}\mathrm{D}_8 & \xrightarrow{1\otimes \dot{m}} & \mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}\mathrm{D}_8 \\
\alpha \uparrow \wr & & \alpha \uparrow \wr \\
\mathbf{Z}_{(2)}\mathrm{D}_8 & \xrightarrow{\dot{m}} & \mathbf{Z}_{(2)}\mathrm{D}_8
\end{array}
$$

Indeed, for $xg \in \mathbf{Z}_{(2)}\mathrm{D}_8$ we have

$$(xg)\alpha(1 \otimes \dot{m})\alpha^{-1} = (x \otimes g)(1 \otimes \dot{m})\alpha^{-1}(x \otimes gm)\alpha^{-1} = xgm = xg\dot{m}.$$

Now α gives rise to an isomorphism $(\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus n} \xrightarrow[\sim]{\alpha_n} \mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} (\mathbf{Z}\mathrm{D}_8)^{\oplus n}$ for $n \in \mathbf{Z}_{\geq 0}$, where we used additivity of the tensor product.

Define $(\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(2k+2)} \xrightarrow{d_{(2),2k}} (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(2k+1)}$ and $(\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(2k+3)} \xrightarrow{d_{(2),2k+1}} (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus 2k+2)}$ for $k \in \mathbf{Z}_{\geq 0}$ by the formulas.

$$d_{(2),2k} := \left(\sum_{i=1}^{k+1} \pi_i \dot{A}_i \iota_i \right) + \left(\sum_{i=k+2}^{2k+1} \pi_i \dot{D}_i \iota_i \right) + \left(\sum_{i=1}^{k} \pi_{i+1} (-1)^i \dot{C}_i \iota_i \right) + \left(\sum_{i=k+1}^{2k+1} \pi_{i+1} (-1)^i \dot{B}_i \iota_i \right)$$

$$d_{(2),2k+1} := \left(\sum_{j=1}^{k+1} \pi_j \dot{A}_{j+1}\iota_j\right) + \left(\sum_{j=k+2}^{2k+2} \pi_j \dot{D}_{j+1}\iota_j\right) - \left(\sum_{j=1}^{k+1} \pi_{j+1}(-1)^j \dot{C}_j\iota_j\right) - \left(\sum_{j=k+2}^{2k+2} \pi_{j+1}(-1)^j \dot{B}_j\iota_j\right)$$

Then the following diagram commutes for all $n \in \mathbf{Z}_{\geq 0}$.

$$\begin{array}{ccc} \mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} (\mathbf{Z}\mathrm{D}_8)^{\oplus(n+2)} & \xrightarrow{1\otimes d_n} & \mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} (\mathbf{Z}\mathrm{D}_8)^{\oplus(n+1)} \\ {\scriptstyle\alpha_{n+2}}\uparrow\wr & & {\scriptstyle\alpha_{n+1}}\uparrow\wr \\ (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(n+2)} & \xrightarrow{d_{(2),n}} & (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(n+1)} \end{array}$$

Let $\mathbf{Z}$ be the trivial $\mathbf{Z}\mathrm{D}_8$-module, let $\mathbf{Z}_{(2)}$ be the trivial $\mathbf{Z}_{(2)}\mathrm{D}_8$-module. We have an isomorphism of $\mathbf{Z}_{(2)}\mathrm{D}_8$-modules

$$\begin{array}{ccc} \mathbf{Z}_{(2)} & \xrightarrow[\sim]{\beta} & \mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} \mathbf{Z} \\ x & \longmapsto & x \otimes 1 \\ xz & \longleftarrow\!\shortmid & x \otimes z \end{array}$$

We verify $\mathbf{Z}_{(2)}\mathrm{D}_8$-linearity of β. But for $g \in G$ and $x \in \mathbf{Z}_{(2)}$ we have using Remark 4

$$(gx)\beta = x\beta = x \otimes 1 = x \otimes (g \cdot 1) = g(x \otimes 1) = g(x\beta).$$

Finally, let $\mathbf{Z}_{(2)}\mathrm{D}_8 \xrightarrow{\varepsilon_{(2)}} \mathbf{Z}_{(2)}$ be the $\mathbf{Z}_{(2)}\mathrm{D}_8$-linear augmentation map. Then the following diagram of $\mathbf{Z}_{(2)}\mathrm{D}_8$-modules commutes.

$$\begin{array}{ccc} \mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}\mathrm{D}_8 & \xrightarrow{1\otimes\varepsilon} & \mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} \mathbf{Z} \\ {\scriptstyle\alpha}\uparrow\wr & & {\scriptstyle\beta}\uparrow\wr \\ \mathbf{Z}_{(2)}\mathrm{D}_8 & \xrightarrow{\varepsilon_{(2)}} & \mathbf{Z}_{(2)} \end{array}$$

Indeed, for $xg \in \mathbf{Z}_{(2)}\mathrm{D}_8$ we have

$$(xg)\alpha(1 \otimes \varepsilon)\beta^{-1} = (x \otimes g)(1 \otimes \varepsilon)\beta^{-1} = (x \otimes 1)\beta^{-1} = x = (xg)\varepsilon_{(2)}.$$

Theorem 30 *Consider the following sequences of* $(\mathbf{Z}_{(2)}\mathrm{D}_8)$*-modules and* $(\mathbf{Z}_{(2)}\mathrm{D}_8)$*-linear maps.*

$$P_{(2)} := \left(\cdots\to (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus 4} \xrightarrow{d_{(2),2}} (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus 3} \xrightarrow{d_{(2),1}} (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus 2} \xrightarrow{d_{(2),0}} \mathbf{Z}_{(2)}\mathrm{D}_8 \longrightarrow 0 \right)$$

$$P'_{(2)} := \left(\cdots\to (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus 4} \xrightarrow{d_{(2),2}} (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus 3} \xrightarrow{d_{(2),1}} (\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus 2} \xrightarrow{d_{(2),0}} \mathbf{Z}_{(2)}\mathrm{D}_8 \xrightarrow{\varepsilon_{(2)}} \mathbf{Z}_{(2)} \longrightarrow 0 \right)$$

Then $P_{(2)}$ *is a minimal projective resolution of the trivial* $(\mathbf{Z}_{(2)}\mathrm{D}_8)$*-module* $\mathbf{Z}_{(2)}$ *and* $P'_{(2)}$ *is the corresponding augmented resolution.*

Proof. By construction, the complex $P'_{(2)}$ is isomorphic to the complex $\mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} P'$, where P' is the augmented projective resolution of the trivial $\mathbf{Z}\mathrm{D}_8$-modules $\mathbf{Z}$ from Theorem 20.

But Lemma 6 implies $\mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} P'$ is a projective resolution of the $\mathbf{Z}_{(2)}\mathrm{D}_8$-module $\mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}$.

Hence $P'_{(2)}$ is an augmented projective resolution of the trivial $\mathbf{Z}_{(2)}\mathrm{D}_8$-module $\mathbf{Z}_{(2)}$ and $P_{(2)}$ is the corresponding projective resolution.

Minimality of $P_{(2)}$ now follows from Lemma 32 below. □

2.3.2 Minimality

Let $\mathfrak{r} \subseteq \mathbf{Z}_{(2)}\mathrm{D}_8$ be the Jacobson radical of $\mathbf{Z}_{(2)}\mathrm{D}_8$. We will use [**2**, Proposition 9, §3.6] to show that $P_{(2)}$ from Theorem 30 is indeed minimal. Since all modules $P_{(2)}$ are finitely generated, it suffices to show that $\mathrm{im}(d_{(2),k}) \subseteq \mathfrak{r}\Big((\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(k+1)}\Big)$ for all $k \in \mathbf{Z}_{\geq 0}$.

Remark 31 We have the $\mathbf{Z}_{(2)}\mathrm{D}_8$-linear augmentation map $\mathbf{Z}_{(2)}\mathrm{D}_8 \xrightarrow{\varepsilon_{(2)}} \mathbf{Z}_{(2)}$. Let $\mathfrak{a} := \ker(\varepsilon_{(2)})$ be its kernel, called the *augmentation ideal* of $\mathbf{Z}_{(2)}\mathrm{D}_8$. We have

$$\mathfrak{a} = \left\{ \sum_{g\in \mathrm{D}_8} r_g g \in \mathbf{Z}_{(2)}\mathrm{D}_8 : \sum_{g\in \mathrm{D}_8} r_g = 0 \right\}$$

Now by [**3**, Corollary (5.25)] the radical of $\mathbf{Z}_{(2)}\mathrm{D}_8$ is given by $\mathfrak{r} = 2\mathbf{Z}_{(2)}\mathrm{D}_8 + \mathfrak{a}$, so

$$\mathfrak{r} = \left\{ \sum_{g\in \mathrm{D}_8} r_g g \in \mathbf{Z}_{(2)}\mathrm{D}_8 : \sum_{g\in \mathrm{D}_8} r_g \equiv_2 0 \right\}.$$

Lemma 32 *For $k \in \mathbf{Z}_{\geq 0}$ we have $\mathrm{im}(d_{(2),k}) \subseteq \mathfrak{r}\Big((\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(k+1)}\Big)$. In particular, the projective resolution $P_{(2)}$ from Theorem 30 is minimal.*

Proof. Recall that by identifying $\mathbf{Z}$ in $\mathbf{Z}_{(2)}$ we identify $\mathbf{Z}\mathrm{D}_8$ in $\mathbf{Z}_{(2)}\mathrm{D}_8$. Recall the following elements of $\mathbf{Z}\mathrm{D}_8 \subseteq \mathbf{Z}_{(2)}\mathrm{D}_8$ from Definition 19.

$$\begin{array}{llll} A_- := 1-b & B_- := 1-ba^3 & C_- := a+a^3-b-ba^2 & D_- := a+a^3-ba-ba^3 \\ A_+ := 1+b & B_+ := 1+ba^3 & C_+ := a+a^3+b+ba^2 & D_+ := a+a^3+ba+ba^3 \end{array}$$

We observe that all these generators are contained in $\mathfrak{r}$, cf. Remark 31.

Let $\mathfrak{b}$ be the left-ideal in $\mathbf{Z}_{(2)}\mathrm{D}_8$ generated by these eight elements. Then by the definition of the differentials $d_{(2),k}$ for $k \in \mathbf{Z}_{\geq 0}$ above, $\mathrm{im}(d_{(2),k}) \subseteq \mathfrak{b}\Big((\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(k+1)}\Big)$. Now all the generators of $\mathfrak{b}$ are contained in $\mathfrak{r}$, so $\mathfrak{b} \subseteq \mathfrak{r}$.

Therefore $\mathrm{im}(d_{(2),k}) = \mathfrak{b}\Big((\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(k+1)}\Big) \subseteq \mathfrak{r}\Big((\mathbf{Z}_{(2)}\mathrm{D}_8)^{\oplus(k+1)}\Big)$ for $k \in \mathbf{Z}_{\geq 0}$. Thus minimality of $P_{(2)}$ follows from [**2**, Proposition 9, §3.6]. □

Chapter 3

Automorphisms and Symmetries

We determine the outer automorphism group $\mathrm{Out}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8)$ and use some outer automorphisms to describe certain symmetries of our projective resolution from Chapter 2. Recall the isomorphism $\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\omega} \Lambda$ with the Wedderburn image Λ from Lemma 15.

$$\Lambda = \Big\{ (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2} :$$
$$x_1 \equiv_2 x_2 \equiv_2 x_3 \equiv_2 x_4 \equiv_2 x_{5;1,1} \equiv_2 x_{5;2,2},\ x_2 - x_4 \equiv_4 2x_{5;1,2},$$
$$x_2 - x_3 \equiv_4 x_{5;2,1},\ x_1 + x_2 \equiv_4 x_3 + x_4,\ x_1 + x_2 + x_3 + x_4 \equiv_8 2(x_{5;1,1} + x_{5;2,2}) \Big\}$$

The isomorphism ω gives rise to an isomorphism $\mathrm{Out}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8) \xrightarrow{\sim} \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\Lambda)$, cf. Remark 8. In this chapter, we will work in the Wedderburn image Λ exclusively.

3.1 Central automorphisms

We have the group $\mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda) := \{\psi \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda) : z\psi = z \text{ for all } z \in \mathrm{Z}(\Lambda)\}$ of automorphisms of Λ that fix the centre of Λ pointwise and the corresponding group of outer central automorphisms $\mathrm{Outcent}_{\mathbf{Z}\text{-alg}}(\Lambda) := \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)/\,\mathrm{Inn}_{\mathbf{Z}\text{-alg}}(\Lambda)$, see §1.3 for details.

In this section, we shall calculate $\mathrm{Outcent}_{\mathbf{Z}\text{-alg}}(\Lambda) \simeq \mathrm{C}_2$, cf. Lemma 40 below.

Remark 33 Suppose given $\sigma \in \mathrm{S}_4$ and $Q \in \mathrm{GL}_2(\mathbf{Q})$ and consider the map

$$\begin{array}{ccc} \Lambda & \xrightarrow{\tilde{\psi}} & \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}^{2\times 2} \\ (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) & \longmapsto & (x_{1\sigma}, x_{2\sigma}, x_{3\sigma}, x_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)Q). \end{array}$$

If $\Lambda\tilde{\psi} \subseteq \Lambda$, then $\psi \in \mathrm{End}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

Proof. We verify that ψ is an $\mathbf{Z}$-algebra morphism.

Let $\xi := (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)), \xi' := (x'_1, x'_2, x'_3, x'_4, \left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$. Then

$$\begin{aligned} (\xi\xi')\psi &= (x_1x'_1, x_2x'_2, x_3x'_3, x_4x'_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2} \end{smallmatrix}\right))\psi \\ &= (x_{1\sigma}x'_{1\sigma}, x_{2\sigma}x'_{2\sigma}, x_{3\sigma}x'_{3\sigma}, x_{4\sigma}x'_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2} \end{smallmatrix}\right)Q) \\ &= (x_{1\sigma}, x_{2\sigma}, x_{3\sigma}, x_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)Q)\,(x'_{1\sigma}, x'_{2\sigma}, x'_{3\sigma}, x'_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2} \end{smallmatrix}\right)Q) \\ &= (\xi\psi)(\xi'\psi). \end{aligned}$$

Moreover, for $z \in \mathbf{Z}$ we have

$$\begin{aligned}(z\xi+\xi')\psi &= \big(zx_1+x'_1, zx_2+x'_2, zx_3+x'_3, zx_4+x'_4, \left(\begin{smallmatrix} zx_{5;1,1} & zx_{5;1,2} \\ zx_{5;2,1} & zx_{5;2,2}\end{smallmatrix}\right)+\left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2}\end{smallmatrix}\right)\big)\psi \\ &= \big(zx_{1\sigma}+x'_{1\sigma}, zx_{2\sigma}+x'_{2\sigma}, zx_{3\sigma}+x'_{3\sigma}, zx_{4\sigma}+x'_{4\sigma}, Q^{-1}\big(\left(\begin{smallmatrix} zx_{5;1,1} & zx_{5;1,2} \\ zx_{5;2,1} & zx_{5;2,2}\end{smallmatrix}\right)+\left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2}\end{smallmatrix}\right)\big)Q\big) \\ &= z\big(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}, x_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)Q\big)+\big(x'_{1\sigma}, x'_{2\sigma}, x'_{3\sigma}, x'_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2}\end{smallmatrix}\right)Q\big) \\ &= z\xi\psi+\xi'\psi'.\end{aligned}$$

We conclude that $\psi \in \mathrm{End}_{\mathbf{Z}\text{-alg}}(\Lambda)$. $\square$

Lemma 34 (1) *Let $\psi \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$. Then there is a unique permutation $\sigma \in \mathrm{S}_4$ and an invertible matrix $Q \in \mathrm{GL}_2(\mathbf{Q})$ such that for all $(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)) \in \Lambda$*

$$(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right))\psi = (x_{1\sigma}, x_{2\sigma}, x_{3\sigma}, x_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)Q).$$

(2) *Suppose given $\sigma \in \mathrm{S}_4$ and $Q \in \mathrm{GL}_2(\mathbf{Q})$ with $Q^\ell = uI_2$ for some $\ell \in \mathbf{Z}_{\geq 0}$ and $u \in \mathbf{Q}$ and consider the map*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\tilde\psi} & \mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}^{2\times 2} \\ (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)) & \longmapsto & (x_{1\sigma}, x_{2\sigma}, x_{3\sigma}, x_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)Q). \end{array}$$

If $\Lambda\tilde\psi \subseteq \Lambda$, then $\psi := \tilde\psi|^{\Lambda} \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

Proof. Ad (1). Let $\otimes := \underset{\mathbf{Z}}{\otimes}$ and write $\Omega := \mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}^{2\times 2}$. Then $\mathbf{Q}\otimes\Lambda$ becomes a $\mathbf{Q}$-algebra by

$$(x\otimes\lambda)(x'\otimes\lambda') = xx'\otimes\lambda\lambda' \quad\text{and}\quad a(x\otimes\lambda) = ax\otimes\lambda \quad\text{for } a, x, x' \in \mathbf{Q},\ \lambda, \lambda' \in \Lambda.$$

We have the isomorphism of $\mathbf{Q}$-algebras

$$\begin{array}{ccc} \mathbf{Q}\otimes\Lambda & \xrightarrow[\sim]{\rho} & \Omega \\ x\otimes\lambda & \longmapsto & x\lambda. \end{array}$$

Let $\psi \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$. Then ψ induces $1\otimes\psi \in \mathrm{Aut}_{\mathbf{Q}\text{-alg}}(\mathbf{Q}\otimes\Lambda)$, which in turn gives the $\mathbf{Q}$-algebra automorphism $\rho^{-1}(1\otimes\psi)\rho \in \mathrm{Aut}_{\mathbf{Q}\text{-alg}}(\Omega)$.

By Lemma 14 there is $\sigma \in \mathrm{S}_4$ and $Q \in \mathrm{GL}_2(\mathbf{Q})$ such that for all $(y_1, y_2, y_3, y_4, \left(\begin{smallmatrix} y_{5;1,1} & y_{5;1,2} \\ y_{5;2,1} & y_{5;2,2}\end{smallmatrix}\right)) \in \Omega$

$$(y_1, y_2, y_3, y_4, \left(\begin{smallmatrix} y_{5;1,1} & y_{5;1,2} \\ y_{5;2,1} & y_{5;2,2}\end{smallmatrix}\right))\rho^{-1}(1\otimes\psi)\rho = (y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} y_{5;1,1} & y_{5;1,2} \\ y_{5;2,1} & y_{5;2,2}\end{smallmatrix}\right)Q).$$

But for $\lambda \in \Lambda \subseteq \Omega$ we have $\lambda\rho^{-1}(1\otimes\psi)\rho = (1\otimes\lambda)(1\otimes\psi)\rho = (1\otimes\lambda\psi) = \lambda\psi$, hence existence of σ and Q follows.

For uniqueness of σ, suppose there is $\sigma' \in \mathrm{S}_4$ and $Q' \in \mathrm{GL}_2(\mathbf{Q})$ and let $\psi' \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$ be given by

$$(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right))\psi' := (x_{1\sigma'}, x_{2\sigma'}, x_{3\sigma'}, x_{4\sigma'}, Q'^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)Q')$$

for $(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)) \in \Lambda$, such that $\psi' = \psi$. Then $\rho^{-1}(1\otimes\psi')\rho \in \mathrm{Aut}_{\mathbf{Q}\text{-alg}}(\mathbf{Q}\otimes\Lambda)$ is given by

$$(y_1, y_2, y_3, y_4, \left(\begin{smallmatrix} y_{5;1,1} & y_{5;1,2} \\ y_{5;2,1} & y_{5;2,2}\end{smallmatrix}\right))\rho^{-1}(1\otimes\psi)\rho = (y_{1\sigma'}, y_{2\sigma'}, y_{3\sigma'}, y_{4\sigma'}, Q'^{-1}\left(\begin{smallmatrix} y_{5;1,1} & y_{5;1,2} \\ y_{5;2,1} & y_{5;2,2}\end{smallmatrix}\right)Q')$$

for $(y_1, y_2, y_3, y_4, \left(\begin{smallmatrix} y_{5;1,1} & y_{5;1,2} \\ y_{5;2,1} & y_{5;2,2} \end{smallmatrix}\right)) \in \Omega$. But then by Lemma 14 the permutations $\sigma = \sigma'$ must be equal.

Ad (2). By assumption, we have for $\xi := (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$

$$\begin{aligned} \xi\psi^{|\langle\sigma\rangle|\ell} &= \big(x_{1\sigma^{|\langle\sigma\rangle|\ell}}, x_{2\sigma^{|\langle\sigma\rangle|\ell}}, x_{3\sigma^{|\langle\sigma\rangle|\ell}}, x_{4\sigma^{|\langle\sigma\rangle|\ell}}, Q^{-\ell|\langle\sigma\rangle|}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)Q^{\ell|\langle\sigma\rangle|}\big) \\ &= \big(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\big). \end{aligned}$$

Hence $\psi^{|\langle\sigma\rangle|\ell} = 1_\Lambda$, thus ψ is bijective. The claim follows with Remark 33. □

Lemma 35 *We have*

$$\mathrm{Z}(\Lambda) = \left\{ (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda : \begin{array}{c} x_{5;1,1} = x_{5;2,2}, \\ x_{5;1,2} = x_{5;2,1} = 0 \end{array} \right\}.$$

Proof. We observe that each element of the right-hand side is central. For the converse inclusion suppose given $(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \mathrm{Z}(\Lambda)$. Since $(0,0,0,0,\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right))$, $(0,0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right)) \in \Lambda$ we have

$$\begin{aligned} (0,0,0,0,\left(\begin{smallmatrix} 0 & 2x_{5;1,1} \\ 0 & 2x_{5;2,1} \end{smallmatrix}\right)) &= (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right))(0,0,0,0,\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)) \\ &= (0,0,0,0,\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right))(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \\ &= (0,0,0,0,\left(\begin{smallmatrix} 2x_{5;2,1} & 2x_{5;2,2} \\ 0 & 0 \end{smallmatrix}\right)) \\ (0,0,0,0,\left(\begin{smallmatrix} 4x_{5;1,2} & 0 \\ 4x_{5;2,2} & 0 \end{smallmatrix}\right)) &= (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right))(0,0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right)) \\ &= (0,0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right))(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} 4x_{5;1,1} & 4x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \\ &= (0,0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 4x_{5;1,1} & 4x_{5;1,2} \end{smallmatrix}\right)). \end{aligned}$$

We conclude that $x_{5;2,1} = x_{5;1,2} = 0$ and $x_{5;1,1} = x_{5;2,2}$. Hence every central element is contained in the right-hand side. □

Lemma 36 *Let* $\psi \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$. *Then there is an invertible matrix* $Q \in \mathrm{GL}_2(\mathbf{Q})$ *such that for all* $(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$

$$(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) = (x_1, x_2, x_3, x_4, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)Q).$$

Proof. By Lemma 34 (1) there is a permutation $\sigma \in \mathrm{S}_4$ and an invertible matrix $Q \in \mathrm{GL}_2(\mathbf{Q})$ such that for all $(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$

$$(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right))\psi = (x_{1\sigma}, x_{2\sigma}, x_{3\sigma}, x_{4\sigma}, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)Q).$$

Consider the following elements of Λ.

$$(8,0,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)),\quad (0,8,0,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)),\quad (0,0,8,0,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)),\quad (0,0,0,8,\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))$$

By Lemma 35, all these elements are central. Since central elements are fixed under ψ, it follows that $\sigma = \mathrm{id}$. □

Remark 37 Let $\psi \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$ such that for all $(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$

$$(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right))\psi = (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2} \end{smallmatrix}\right))$$

for some $x'_{5;1,1}$, $x'_{5;1,2}$, $x'_{5;2,1}$, $x'_{5;2,2} \in \mathbf{Z}$. Then $\psi \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

Proof. By Lemma 34 (1) there is $Q \in \mathrm{GL}_2(\mathbf{Q})$ such that

$$\begin{pmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2} \end{pmatrix} = Q^{-1} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} Q.$$

But since multiples of the identity matrix are fixed under conjugation, the automorphism ψ fixes the centre of Λ by Lemma 35, hence ψ is central. $\square$

Recall the $\mathbf{Z}$-linear basis $(a_1, \ldots, a_8)$ of Λ given by the following.

$$\begin{array}{llll} a_1 = \left(1,1,1,1,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right) & a_2 = \left(0,2,0,-2,\left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)\right) & a_3 = \left(0,0,2,-2,\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)\right) & a_4 = \left(0,0,0,4,\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)\right) \\ a_5 = \left(0,0,0,0,\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)\right) & a_6 = \left(0,0,0,0,\left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right)\right) & a_7 = \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)\right) & a_8 = \left(0,0,0,0,\left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right)\right) \end{array}$$

Let

$$\begin{array}{ccc} \Lambda & \xrightarrow{\pi_5} & \mathbf{Z}^{2\times 2} \\ \left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) & \longmapsto & \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \end{array}$$

be the projection onto the 2×2-matrix block. Then a $\mathbf{Z}$-linear basis of $\Lambda\pi_5$ is given by

$$\left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Moreover, we have

$$\Lambda\pi_5 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} a \equiv_2 d, \\ c \equiv_2 0 \end{array} \right\}.$$

Remark 38 We have

$$G := \mathrm{U}(\Lambda\pi_5) = \mathrm{GL}_2(\mathbf{Z}) \cap \Lambda\pi_5 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}) : c \equiv_2 0 \right\}.$$

Proof. For the second asserted equality, we only need to show the inclusion $\supseteq$. For this, note that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Z})$ with $c \equiv_2 0$ implies that $1 \equiv_2 ad - bc \equiv_2 ad$, so $a \equiv_2 d \equiv_2 1$.

To prove the first asserted equality, note that the inclusion $\subseteq$ follows since $\mathrm{GL}_2(\mathbf{Z}) = \mathrm{U}(\mathbf{Z}^{2\times 2})$ and $\Lambda\pi_5 \subseteq \mathbf{Z}^{2\times 2}$. For the converse inclusion let $Q := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Z}) \in \mathrm{GL}_2(\mathbf{Z}) \cap \Lambda\pi_5$. We need to show that Q is invertible in $\Lambda\pi_5$. We have

$$Q^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then $\frac{-c}{ad-bc} \equiv_2 c$, since $ad - bc \in \{1, -1\}$ and the diagonal entries $\frac{d}{ad-bc} \equiv_2 \frac{a}{ad-bc} \equiv_2 1$ are again odd numbers. So $Q^{-1} \in \Lambda\pi_5$. $\square$

Lemma 39 *Write $Z := \mathrm{U}(\mathbf{Z}) I_2 \leq \mathrm{GL}_2(\mathbf{Z})$. We have an isomorphism of groups*

$$\begin{array}{ccc} G/Z & \xrightarrow[\sim]{\alpha} & \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda) \\ QZ & \longmapsto & \left(\left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) \longrightarrow \left(x_1, x_2, x_3, x_4, Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) Q\right) \right). \end{array}$$

Proof. We *claim* the existence of a surjective group morphism

$$\begin{array}{ccc} G & \xrightarrow{\tilde\alpha} & \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda) \\ & & \\ Q & \longmapsto & \Big((x_1,x_2,x_3,x_4,\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)) \longrightarrow (x_1,x_2,x_3,x_4,Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)Q) \Big). \end{array}$$

Let $Q = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in G$ and let $\varepsilon = ad - bc = \det(Q) \in \mathrm{U}(\mathbf{Z}) = \{1,-1\}$. We have to show that $Q\tilde\alpha \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}$. For this, we have to show that for all $\xi := (x_1,x_2,x_3,x_4,\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)) \in \Lambda$ we also have $\xi(Q\alpha) \in \Lambda$. We calculate.

$$\begin{aligned} \xi(Q\alpha) &= (x_1,x_2,x_3,x_4,Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)Q) \\ &= (x_1,x_2,x_3,x_4,\varepsilon\left(\begin{smallmatrix} d & -b \\ -c & a\end{smallmatrix}\right)\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)) \\ &= (x_1,x_2,x_3,x_4,\varepsilon\left(\begin{smallmatrix} d & -b \\ -c & a\end{smallmatrix}\right)\left(\begin{smallmatrix} ax_{5;1,1}+cx_{5;1,2} & bx_{5;1,1}+dx_{5;1,2} \\ ax_{5;2,1}+cx_{5;2,2} & bx_{5;2,1}+dx_{5;2,2}\end{smallmatrix}\right)) \\ &= (x_1,x_2,x_3,x_4,\varepsilon\left(\begin{smallmatrix} adx_{5;1,1}+cdx_{5;1,2}-abx_{5;2,1}-bcx_{5;2,2} & bdx_{5;1,1}+d^2x_{5;1,2}-b^2x_{5;2,1}-bdx_{5;2,2} \\ -acx_{5;1,1}-c^2x_{5;1,2}+a^2x_{5;2,1}+acx_{5;2,2} & -bcx_{5;1,1}-cdx_{5;1,2}+abx_{5;2,1}+adx_{5;2,2}\end{smallmatrix}\right)) \end{aligned}$$

We have to verify the congruences in the definition of Λ. We use that $\xi \in \Lambda$ and $Q \in G$.

$$\begin{aligned} \varepsilon(adx_{5;1,1}+cdx_{5;1,2}-abx_{5;2,1}-bcx_{5;2,2}) &\equiv_2 \varepsilon adx_{5;1,1} \equiv_2 x_{5;1,1} \equiv_2 x_4 \\ \varepsilon(-bcx_{5;1,1}-cdx_{5;1,2}+abx_{5;2,1}+adx_{5;2,2}) &\equiv_2 \varepsilon adx_{5;2,2} \equiv_2 x_{5;2,2} \equiv_2 x_4 \\ 2\varepsilon(bdx_{5;1,1}+d^2x_{5;1,2}-b^2x_{5;2,1}-bdx_{5;2,2}) &\equiv_4 2\varepsilon bd(x_{5;1,1}-x_{5;2,2}) + 2\varepsilon d^2x_{5;1,2} \equiv_4 2x_{5;1,2} \equiv_4 x_2 - x_4 \\ \varepsilon(-acx_{5;1,1}-c^2x_{5;1,2}+a^2x_{5;2,1}+acx_{5;2,2}) &\equiv_4 \varepsilon ac(x_{5;2,2}-x_{5;1,1}) + a^2x_{5;2,1} \equiv_4 x_{5;2,1} \equiv_4 x_2 - x_3 \end{aligned}$$

The congruence modulo 8 is fulfilled since conjugation preserves matrix traces. By Remark 33 we have $Q\tilde\alpha \in \mathrm{End}_{\mathbf{Z}\text{-alg}}(\Lambda)$. Since its inverse is given by $Q^{-1}\tilde\alpha$ with $Q^{-1} \in G$, we have $Q\tilde\alpha \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$ and by Remark 37 also $Q\tilde\alpha \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$. Hence $\tilde\alpha$ is a well-defined group morphism.

For surjectivity of $\tilde\alpha$, let $\psi \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$. By Lemma 36 there is an invertible matrix $Q = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Q})$ such that

$$(x_1,x_2,x_3,x_4,\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right))\psi = (x_1,x_2,x_3,x_4,Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)Q).$$

Note that we obtain the same automorphism if we rescale Q, i.e. multiply the matrix by an element of $\mathbf{Q}^\times$. By rescaling, we can assume without loss of generality that $a,b,c,d \in \mathbf{Z}$ with $\gcd(a,b,c,d) = 1$.

For $\xi = (x_1,x_2,x_3,x_4,\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)) \in \Lambda$ write

$$(x_1,x_2,x_3,x_4,\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right))\psi =: (x_1,x_2,x_3,x_4,\left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2}\end{smallmatrix}\right)) =: \xi',$$

where $x'_{5;1,1},\ x'_{5;1,2},\ x'_{5;2,1},\ x'_{5;2,2} \in \mathbf{Z}$. Since ξ and ξ' are contained in Λ, we obtain the following congruences.

$$\begin{array}{ll} x'_{5;1,1} \equiv_2 x_4 \equiv_2 x_{5;1,1} & x'_{5;1,2} \equiv_2 \tfrac{1}{2}(x_2 - x_4) \equiv_2 x_{5;1,2} \\ x'_{5;2,1} \equiv_4 x_2 - x_3 \equiv_4 x_{5;2,1} & x'_{5;2,2} \equiv_2 x_4 \equiv_2 x_{5;2,2} \end{array} \qquad (*)$$

Calculating as above, we obtain with $\Delta = ad - bc = \det(Q) \in \mathbf{Z}$

$$Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2}\end{smallmatrix}\right)Q = \frac{1}{\Delta}\left(\begin{smallmatrix} adx_{5;1,1}+cdx_{5;1,2}-abx_{5;2,1}-bcx_{5;2,2} & bdx_{5;1,1}+d^2x_{5;1,2}-b^2x_{5;2,1}-bdx_{5;2,2} \\ -acx_{5;1,1}-c^2x_{5;1,2}+a^2x_{5;2,1}+acx_{5;2,2} & -bcx_{5;1,1}-cdx_{5;1,2}+abx_{5;2,1}+adx_{5;2,2}\end{smallmatrix}\right).$$

Write

$$\begin{array}{ccc} \Lambda & \xrightarrow{\pi_5} & \mathbf{Z}^{2\times 2} \\ (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) & \longmapsto & \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right). \end{array}$$

Then a $\mathbf{Z}$-linear basis of $\Lambda\pi_5$ is given by

$$\left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Consider the congruences $(*)$ for the first three of these matrices, as the identity matrix is fixed under conjugation. We obtain the following congruences for a, b, c, d, Δ.

For $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$: $\quad \frac{2ad}{\Delta} \equiv_2 0, \quad \frac{2bd}{\Delta} \equiv_2 0, \quad \frac{2ac}{\Delta} \equiv_4 0, \quad \frac{2bc}{\Delta} \equiv_2 0$

For $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$: $\quad \frac{cd}{\Delta} \equiv_2 0, \quad \frac{d^2}{\Delta} \equiv_2 1, \quad \frac{c^2}{\Delta} \equiv_4 0$

For $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$: $\quad \frac{2ab}{\Delta} \equiv_2 0, \quad \frac{2b^2}{\Delta} \equiv_2 0, \quad \frac{2a^2}{\Delta} \equiv_4 2$

In particular, it follows that Δ divides all elements of $\{a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, cd\}$. Since $\gcd(a, b, c, d) = 1$, there are $k, l, m, n \in \mathbf{Z}$ with $1 = ka + lb + mc + nd$. Hence

$1 = (ka+lb+mc+nd)^2 = k^2a^2 + l^2b^2 + m^2c^2 + n^2d^2 + 2klab + 2kmac + 2knad + 2lmbc + 2lnbd + 2mncd.$

Then Δ divides all summands on the right-hand side, hence $\Delta \in \{1, -1\}$. So $Q \in \mathrm{GL}_2(\mathbf{Z})$. Finally, from $\frac{c^2}{\Delta} \equiv_4 0$ we conclude that $c \equiv_2 0$. So $Q \in G$ with $\psi = Q\tilde{\alpha}$. Hence $\tilde{\alpha}$ is surjective.

To show that α is an isomorphism, we show that $\ker(\tilde{\alpha}) = Z$. Suppose $Q = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \ker(\tilde{\alpha})$. Then for all $\xi = (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$ we have $\xi(Q\tilde{\alpha}) = \xi$, hence

$$Q^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)Q = \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right), \quad \text{i.e.} \quad \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)Q = Q\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right).$$

Consider the elements $(0, 0, 0, 0, \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right))$, $(0, 0, 0, 0, \left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right)) \in \Lambda$. Then

$$\begin{aligned} \left(\begin{smallmatrix} 2c & 2d \\ 0 & 0 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 2a \\ 0 & 2c \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 4a & 4b \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4b & 0 \\ 4d & 0 \end{smallmatrix}\right). \end{aligned}$$

So $a = d$ and $b = c = 0$. Since $\det(Q) = ad - bc = a^2 \in \mathrm{U}(\mathbf{Z})$, we have $a \in \mathrm{U}(\mathbf{Z})$. Hence $Q \in Z$, therefore $\ker(Q) \subseteq Z$. The converse inclusion follows since conjugation with central elements gives the identity. □

Let $\mathrm{C}_2 = \langle c : c^2 \rangle$ be the cyclic group of order 2.

Lemma 40 *We have an isomorphism of groups*

$$\begin{array}{ccc} \mathrm{C}_2 & \xrightarrow{\sim} & \mathrm{Outcent}_{\mathbf{Z}\text{-alg}}(\Lambda) \\ c & \longmapsto & [\varphi], \end{array}$$

where $\varphi \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$ *is given by*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varphi} & \Lambda \\ (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) & \longmapsto & (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & -x_{5;1,2} \\ -x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)). \end{array}$$

Proof. We show that φ is a central automorphism of Λ. Let $Q := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then

$$\begin{aligned}
(x_1, x_2, x_3, x_4, Q^{-1}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} Q) &= (x_1, x_2, x_3, x_4, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \\
&= (x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ -x_{5;2,1} & -x_{5;2,2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \\
&= (x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & -x_{5;1,2} \\ -x_{5;2,1} & x_{5;2,2} \end{pmatrix}).
\end{aligned}$$

So by Lemma 39 the map φ is a central automorphism and the conjugating matrix Q is unique up to multiplication with a unit in $\mathbf{Z}$. Moreover, φ is of order 2 in $\mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

We show that φ is not an inner automorphism. *Assume* the contrary.

Then there are $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{1, -1\}$ such that $\xi := (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)) \in \Lambda$. Then $\varepsilon_2 - \varepsilon_3 \equiv_4 0$ and $\varepsilon_2 - \varepsilon_4 \equiv_4 0$, so $\varepsilon_2 = \varepsilon_3 = \varepsilon_4$. But then $\varepsilon_1 + \varepsilon_2 \equiv_4 \varepsilon_3 + \varepsilon_4$ implies that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4$. Hence we obtain

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \equiv_8 4 \not\equiv_8 0 \equiv_8 2(1-1),$$

in *contradiction* to $\xi \in \Lambda$.

We show that $\mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda) = 1\,\mathrm{Inn}_{\mathbf{Z}\text{-alg}}(\Lambda) \sqcup \varphi\,\mathrm{Inn}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

Let $\psi \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$. By Lemma 39 there is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Z})$ with $a \equiv_2 d \equiv_2 1$ and $c \equiv_2 0$ such that for all $(x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}) \in \Lambda$

$$(x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix})\psi = (x_1, x_2, x_3, x_4, \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}).$$

We distinguish eight cases (1-8). For the cases (5-8) consider $\varphi\psi \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$. Then $\varphi\psi$ is a central automorphism with conjugation of the 2×2-matrix block given by

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Case 1*:* $a + d \equiv_4 2,\ b \equiv_2 0,\ c \equiv_4 0.$

Then ψ is given by conjugation with $(1, 1, 1, 1, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \in \mathrm{U}(\Lambda)$.

Case 2*:* $a + d \equiv_4 0,\ b \equiv_2 0,\ c \equiv_4 2.$

Then ψ is given by conjugation with $(1, -1, 1, -1, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \in \mathrm{U}(\Lambda)$.

Case 3*:* $a + d \equiv_4 0,\ b \equiv_2 1,\ c \equiv_4 0.$

Then ψ is given by conjugation with $(1, -1, -1, 1, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \in \mathrm{U}(\Lambda)$.

Case 4*:* $a + d \equiv_4 0,\ b \equiv_2 1,\ c \equiv_4 2.$

Then ψ is given by conjugation with $(1, 1, -1, -1, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \in \mathrm{U}(\Lambda)$.

Case 5*:* $a + d \equiv_4 0,\ b \equiv_2 0,\ c \equiv_4 0.$

Then $a' + d' \equiv_4 a - d \equiv_4 2$, $b' \equiv_2 0$ and $c' \equiv_4 0$, so by Case 1 the automorphism $\varphi\psi$ is inner.

Case 6*:* $a + d \equiv_4 2,\ b \equiv_2 0,\ c \equiv_4 2.$

But then $a' + d' \equiv_4 a - d \equiv_4 0$, $b' \equiv_2 0$ and $c' \equiv_4 2$, so by Case 2 the automorphism $\varphi\psi$ is inner.

Case 7*:* $a + d \equiv_4 2,\ b \equiv_2 1,\ c \equiv_4 0.$

But then $a' + d' \equiv_4 a - d \equiv_4 0$, $b' \equiv_2 1$ and $c' \equiv_4 0$, so by Case 3 the automorphism $\varphi\psi$ is inner.

Case 8*:* $a + d \equiv_4 2,\ b \equiv_2 1,\ c \equiv_4 2.$

Then $a' + d' \equiv_4 a - d \equiv_4 0$, $b' \equiv_2 1$ and $c' \equiv_4 2$, so by Case 4 the automorphism $\varphi\psi$ is inner. □

Consider the following antiautomorphism of Λ.

$$\begin{array}{ccc} \Lambda & \xrightarrow[\sim]{t} & \Lambda \\ \left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) & \longmapsto & \left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;2,2} & x_{5;1,2} \\ x_{5;2,1} & x_{5;1,1} \end{smallmatrix}\right)\right). \end{array}$$

In fact, since

$$\begin{aligned} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)^{\mathrm{T}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_{5;1,1} & x_{5;2,1} \\ x_{5;1,2} & x_{5;2,2} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_{5;2,1} & x_{5;1,1} \\ x_{5;2,2} & x_{5;1,2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_{5;2,2} & x_{5;1,2} \\ x_{5;2,1} & x_{5;1,1} \end{smallmatrix}\right) \end{aligned}$$

and all congruences in the definition of Λ remain valid if we permute $x_{5;1,1}$ and $x_{5;2,2}$, this is indeed an antiautomorphism of Λ.

In $\mathbf{Z}\mathrm{D}_8$ with $\mathbf{Z}$-linear basis given by the group elements $\mathrm{D}_8 = \langle a, b \mid a^4, b^2, (ba)^2 \rangle$ we have the following antiautomorphism given by inversion on the group elements $g \in \mathrm{D}_8$

$$\begin{array}{ccc} \mathbf{Z}\mathrm{D}_8 & \xrightarrow[\sim]{i} & \mathbf{Z}\mathrm{D}_8 \\ g & \longmapsto & g^{-1}. \end{array}$$

The following remark relates these two antiautomorphisms.

Remark 41 Let $\Lambda \xrightarrow[\sim]{t} \Lambda$ and $\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{i} \mathbf{Z}\mathrm{D}_8$ be as above.

Let $\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\omega} \Lambda$ be the Wedderburn embedding from Lemma 15.

Let $\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{f_a} \mathbf{Z}\mathrm{D}_8,\ x \longmapsto a^{-1}xa$ denote the inner $\mathbf{Z}$-algebra automorphism given by conjugation with the group element $a \in \mathbf{Z}\mathrm{D}_8$.

Let $\Lambda \xrightarrow[\sim]{\varphi} \Lambda$ be the central automorphism from Lemma 40.

Then the following diagram of $\mathbf{Z}$-algebra automorphisms and antiautomorphisms commutes.

$$\begin{array}{ccc} \mathbf{Z}\mathrm{D}_8 & \xrightarrow[\sim]{\quad\omega\quad} & \Lambda \\ \wr\big\downarrow i & & \wr\big\downarrow t \\ \mathbf{Z}\mathrm{D}_8 & & \Lambda \\ \wr\big\downarrow f_a & & \wr\big\downarrow \varphi \\ \mathbf{Z}\mathrm{D}_8 & \xrightarrow[\sim]{\quad\omega\quad} & \Lambda \end{array}$$

Proof. We only have to show this on the $\mathbf{Z}$-algebra generators of $\mathbf{Z}\mathrm{D}_8$ given by the group elements $a, b \in \mathbf{Z}\mathrm{D}_8$. For a we have

$$\begin{aligned} a\omega t\varphi &= (1, 1, -1, -1, \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right))t\varphi = (1, 1, -1, -1, \left(\begin{smallmatrix} -1 & 1 \\ -2 & 1 \end{smallmatrix}\right))\varphi = (1, 1, -1, -1, \left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right)) \\ aif_a\omega &= a^3 f_a\omega = a^3\omega = (1, 1, -1, -1, \left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right)) \end{aligned}$$

Hence $a\omega t\varphi = aif_a\omega$. For b we have

$$\begin{aligned} b\omega t\varphi &= (1, -1, 1, -1, \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right))t\varphi = (1, -1, 1, -1, \left(\begin{smallmatrix} -1 & 0 \\ -2 & 1 \end{smallmatrix}\right))\varphi = (1, -1, 1, -1, \left(\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix}\right)) \\ bif_a\omega &= bf_a\omega = (ba^2)\omega = (1, -1, 1, -1, \left(\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix}\right)). \end{aligned}$$

hence $b\omega t\varphi = bif_a\omega$. We conclude that $\omega t\varphi = if_a\omega$, i.e. the diagram commutes. □

3.2 The outer automorphism group

We shall construct the following group morphisms. For the construction of τ see Lemma 42 below. For β being an isomorphism see Theorem 43 below. Here ρ denotes the quotient map.

$$\begin{array}{ccc} \mathrm{D}_8 \times \mathrm{C}_2 & \xrightarrow[\sim]{\beta} & \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\Lambda) \\ & {}_{\tau}\searrow \quad \nearrow_{\rho} & \\ & \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda) & \end{array}$$

Recall that we use the presentations $\mathrm{D}_8 = \langle a, b : a^4, b^2, (ba)^2 \rangle$ and $\mathrm{C}_2 = \langle c : c^2 \rangle$.

Lemma 42 *We have the following group morphism.*

$$\begin{array}{lcl} \mathrm{D}_8 \times \mathrm{C}_2 & \xrightarrow{\tau} & \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda) \\ (a,1) & \longmapsto & \left((x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}) \mapsto (x_4, x_3, x_1, x_2, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}) \right) \\ (b,1) & \longmapsto & \left((x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}) \mapsto (x_1, x_2, x_4, x_3, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}) \right) \\ (1,c) & \longmapsto & \left((x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}) \mapsto (x_1, x_2, x_3, x_4, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \right) \end{array}$$

Proof. We have to show that the asserted images of $(a,1)$ and $(b,1)$ under τ are automorphisms of Λ. For $(1,c)\tau$ this follows from Lemma 40, as $(1,c)\tau = \varphi$.

Suppose given $\xi = (x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}) \in \Lambda$. To see that $\xi((a,1)\tau) \in \Lambda$ we calculate.

$$\xi((a,1)\tau) = (x_4, x_3, x_1, x_2, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}) = (x_4, x_3, x_1, x_2, \begin{pmatrix} x_{5;2,2} & \frac{1}{2} x_{5;2,1} \\ 2x_{5;1,2} & x_{5;1,1} \end{pmatrix})$$

We observe that $\xi((a,1)\tau) \in \Lambda$, by a verification of the congruences in the definition of Λ. Moreover, we calculate

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_2. \tag{$*$}$$

Hence Lemma 34.(2) implies that $(a,1)\tau \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

For $\xi((b,1)\tau) \in \Lambda$ we calculate.

$$\xi((b,1)\tau) = (x_1, x_2, x_4, x_3, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}) = (x_1, x_2, x_4, x_3, \begin{pmatrix} x_{5;2,2} & \frac{1}{2} x_{5;2,1} \\ 2x_{5;1,2} & x_{5;1,1} \end{pmatrix})$$

Again we observe that $\xi((b,1)\tau) \in \Lambda$, by verifying all congruences in the definition of Λ. Using $(*)$ we conclude using Lemma 34.(2) that $(b,1)\tau \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

Furthermore, we have to verify the relations of the presentation of D_8 and C_2, as well as the relations

$$((a,1)\tau)((1,c)\tau) = ((1,c)\tau)((a,1)\tau) \quad \text{and} \quad ((b,1)\tau)((1,c)\tau) = ((1,c)\tau)((b,1)\tau)$$

arising from the direct product $\mathrm{D}_8 \times \mathrm{C}_2$. For $\xi = (x_1, x_2, x_3, x_4, \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}) \in \Lambda$ we calculate.

$$\begin{aligned} \xi((a,1))\tau)^4 &= (x_4, x_3, x_1, x_2, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix})((a,1))\tau)^3 \\ &= (x_2, x_1, x_4, x_3, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix})((a,1))\tau)^2 \\ &= (x_3, x_4, x_2, x_1, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix})((a,1))\tau) \end{aligned}$$

$$
\begin{aligned}
&= \left(x_1, x_2, x_3, x_4, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) \\
&= \xi \\
\xi((b,1))\tau)^2 &= \left(x_1, x_2, x_4, x_3, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\right)((b,1)\tau) \\
&= \left(x_1, x_2, x_3, x_4, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) \\
&= \xi \\
\xi\big(((b,1)\tau)((a,1)\tau)\big)^2 &= \left(x_1, x_2, x_4, x_3, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\right)((a,1)\tau)((b,1)\tau)((a,1)\tau) \\
&= \left(x_3, x_4, x_1, x_2, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)((b,1)\tau)((a,1)\tau) \\
&= \left(x_3, x_4, x_2, x_1, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\right)((a,1)\tau) \\
&= \left(x_1, x_2, x_3, x_4, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) \\
&= \xi \\
\xi\big(((a,1)\tau)((1,c)\tau)\big) &= \left(x_4, x_3, x_1, x_2, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\
&= \left(x_4, x_3, x_1, x_2, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\right) \\
&= \xi\big(((1,c)\tau)((a,1)\tau)\big) \\
\xi\big(((b,1)\tau)((1,c)\tau)\big) &= \left(x_1, x_2, x_4, x_3, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\
&= \left(x_1, x_2, x_4, x_3, \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\right) \\
&= \xi\big(((1,c)\tau)((b,1)\tau)\big)
\end{aligned}
$$

Hence the group morphism τ exists as asserted. □

Recall the representation of D_8 as a permutation group given by the following isomorphism, cf. Remark 1.

$$
\begin{array}{ccc}
\mathrm{D}_8 & \xrightarrow[\sim]{\theta} & \langle (1,4,2,3), (3,4) \rangle \leq \mathrm{S}_4 \\
a & \longmapsto & (1,4,2,3) \\
b & \longmapsto & (3,4)
\end{array}
$$

For the reminder of this section, we will identify D_8 as a subgroup in S_4 via θ.

Lemma 43 *Recall the group morphism τ from Lemma 42. Then we have an isomorphism of groups*

$$\beta := \tau\rho\colon \mathrm{D}_8 \times \mathrm{C}_2 \xrightarrow{\sim} \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\Lambda).$$

In particular, τ is injective.

Proof. We have to show injectivity and surjectivity of β.

Injectivity. Let $(\sigma, c^i) \in \ker(\beta)$ for $\sigma \in \mathrm{D}_8 \leq \mathrm{S}_4$ and $i \in [0,1]$. Then $(\sigma, c^i)\beta = [1]$, so we have $(\sigma, c^i)\tau \in \mathrm{Inn}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

Since inner automorphisms are central, Lemma 36 together with the uniqueness part of Lemma 34 implies that $\sigma = 1$. By Lemma 40 the automorphism $(1,c)\tau$ is a central automorphism that is not inner, so $i = 0$.

Hence $\ker(\beta) = 1$ and β is injective.

Surjectivity. Let $[\psi] \in \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\Lambda)$ with $\psi \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$. By Lemma 34.(1) there is a unique

$\sigma_\psi \in \mathrm{S}_4$ and $Q_\psi \in \mathrm{GL}_2(\mathbf{Q})$ such that for all $(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$ we have

$$(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right))\psi = (x_{1\sigma_\psi}, x_{2\sigma_\psi}, x_{3\sigma_\psi}, x_{4\sigma_\psi}, Q_\psi^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)Q_\psi).$$

Observe that for $\psi, \psi' \in \mathrm{Aut}_{Z\text{-alg}}(\Lambda)$ we have $\sigma_\psi \sigma_{\psi'} = \sigma_{\psi\psi'}$. Hence we obtain a group morphism

$$\begin{array}{ccc} \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda) & \xrightarrow{\ \delta\ } & \mathrm{S}_4 \\ \psi & \longmapsto & \sigma_\psi \end{array}$$

We *claim* that $\mathrm{im}(\delta) = \mathrm{D}_8 = \langle (1,4,2,3), (3,4) \rangle \leq \mathrm{S}_4$. Lemma 42 implies that $\mathrm{D}_8 \leq \mathrm{im}(\delta)$. Since $\mathrm{D}_8 \leq \mathrm{S}_4$ is a maximal subgroup, it suffices to show that $(2,3) \notin \mathrm{im}(\delta)$, as $(2,3) \notin \mathrm{D}_8$. *Assume the contrary.* Then there is $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Q})$ with

$$(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right))\psi = (x_1, x_3, x_2, x_4, \frac{1}{ps-qr}\left(\begin{smallmatrix} s & -q \\ -r & p \end{smallmatrix}\right)\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right))$$

for all $(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$. By rescaling the conjugating matrix, we can assume without loss of generality that $p, q, r, s \in \mathbf{Z}$ with $\gcd(p, q, r, s) = 1$.

For $\xi := (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$ let $\xi' := \xi\psi$ with

$$\begin{aligned} \xi' &=: (x_1, x_3, x_2, x_4, \left(\begin{smallmatrix} x'_{5;1,1} & x'_{5;1,2} \\ x'_{5;2,1} & x'_{5;2,2} \end{smallmatrix}\right)) \\ &= (x_1, x_3, x_2, x_4, \frac{1}{ps-qr}\left(\begin{smallmatrix} s & -q \\ -r & p \end{smallmatrix}\right)\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)) \\ &= (x_1, x_3, x_2, x_4, \frac{1}{ps-qr}\left(\begin{smallmatrix} psx_{5;1,1}+rsx_{5;1,2}-pqx_{5;2,1}-qrx_{5;2,2} & qsx_{5;1,1}+s^2x_{5;1,2}-q^2x_{5;2,1}-qsx_{5;2,2} \\ -prx_{5;1,1}-r^2x_{5;1,2}+p^2x_{5;2,1}+prx_{5;2,2} & -qrx_{5;1,1}-rsx_{5;1,2}+pqx_{5;2,1}+psx_{5;2,2} \end{smallmatrix}\right)) \end{aligned}$$

where $x'_{5;1,1}$, $x'_{5;1,2}$, $x'_{5;2,1}$, $x'_{5;2,2} \in \mathbf{Z}$. Since $\xi, \xi' \in \Lambda$, we obtain using the congruences in the definition of Λ

$$\begin{array}{ll} x'_{5;1,1} \equiv_2 x_4 \equiv_2 x_{5;1,1} & 2x'_{5;1,2} \equiv_4 x_4 - x_3 \\ x'_{5;2,1} \equiv_4 x_2 - x_3 \equiv_4 x_{5;2,1} & x'_{5;2,2} \equiv_2 x_4 \equiv_2 x_{5;2,2}. \end{array} \qquad (*)$$

Now consider the elements $(0, 2, 0, -2, \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right))$, $(0, -2, 2, 0, \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)) \in \Lambda$. We calculate.

$$\begin{aligned} (0, 2, 0, -2, \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right))\psi &= (0, 0, 2, -2, \frac{1}{ps-qr}\left(\begin{smallmatrix} -2pq & -2q^2 \\ 2p^2 & 2pq \end{smallmatrix}\right)) \\ (0, -2, 2, 0, \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right))\psi &= (0, 2, -2, 0, \frac{1}{ps-qr}\left(\begin{smallmatrix} rs & s^2 \\ -r^2 & -rs \end{smallmatrix}\right)) \end{aligned}$$

Using $(*)$, the calculations above give the following congruences for p, q, r, s.

$$\begin{array}{ll} -\dfrac{4q^2}{ps-qr} \equiv_4 2 & \dfrac{2p^2}{ps-qr} \equiv_4 2 \\ \dfrac{2s^2}{ps-qr} \equiv_4 2 & -\dfrac{r^2}{ps-qr} \equiv_4 0 \end{array}$$

From $-\frac{4q^2}{ps-qr} \equiv_4 2$ it follows that $\frac{2q^2}{ps-qr} \equiv_2 1$, so $2 \mid ps - qr$.

From $\frac{2p^2}{ps-qr} \equiv_4 2$ and $\frac{2s^2}{ps-qr} \equiv_4 2$ it follows that $\frac{p^2}{ps-qr} \equiv_2 1$ and $\frac{s^2}{ps-qr} \equiv_2 1$. Since $2 \mid ps - qr$, we conclude that $2 \mid p$ and $2 \mid s$.

Moreover, $-\frac{r^2}{ps-qr} \equiv_4 0$ implies that $4 \mid r$. Hence $4 \mid ps - qr$. But then $\frac{2q^2}{ps-qr} \equiv_2 1$ implies that $2 \mid q$, in *contradiction* to $\gcd(p, q, r, s) = 1$.

Hence we have shown that $\mathrm{im}(\delta) = \mathrm{D}_8$. Then we have $((\sigma^{-1}, 1)\tau) \cdot \psi \in \mathrm{Autcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$ by Remark 37. So $[((\sigma^{-1}, 1)\tau) \cdot \psi] \in \mathrm{Outcent}_{\mathbf{Z}\text{-alg}}(\Lambda)$. By Lemma 40 there is $x \in \mathrm{C}_2$ such that we have $[((\sigma^{-1}, 1)\tau) \cdot \psi] = [(1, x)\tau]$. Then $[\psi] = [(\sigma, 1)\tau][(1, x)\tau] = (\sigma, x)\beta$, therefore β is surjective. $\square$

Corollary 44 *The quotient map* $\mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda) \xrightarrow{\rho} \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\Lambda)$ *is a retraction with coretraction given by* $\mathrm{Out}_{\mathbf{Z}\text{-alg}}(\Lambda) \xrightarrow{\beta^{-1}\tau} \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$.

3.3 Symmetries

3.3.1 Twisted double complexes

Let R be a commutative ring and let A be an R-algebra. Suppose we are given a double complex of A-modules of the form

$$\Delta := \left(\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & A & \xrightarrow{\dot d^{\mathrm{h}}_{3,2}} & A & \xrightarrow{\dot d^{\mathrm{h}}_{3,1}} & A & \xrightarrow{\dot d^{\mathrm{h}}_{3,0}} & A \\
 & & \downarrow \dot d^{\mathrm{v}}_{2,3} & & \downarrow \dot d^{\mathrm{v}}_{2,2} & & \downarrow \dot d^{\mathrm{v}}_{2,1} & & \downarrow \dot d^{\mathrm{v}}_{2,0} \\
\cdots & \to & A & \xrightarrow{\dot d^{\mathrm{h}}_{2,2}} & A & \xrightarrow{\dot d^{\mathrm{h}}_{2,1}} & A & \xrightarrow{\dot d^{\mathrm{h}}_{2,0}} & A \\
 & & \downarrow \dot d^{\mathrm{v}}_{1,3} & & \downarrow \dot d^{\mathrm{v}}_{1,2} & & \downarrow \dot d^{\mathrm{v}}_{1,1} & & \downarrow \dot d^{\mathrm{v}}_{1,0} \\
\cdots & \to & A & \xrightarrow{\dot d^{\mathrm{h}}_{1,2}} & A & \xrightarrow{\dot d^{\mathrm{h}}_{1,1}} & A & \xrightarrow{\dot d^{\mathrm{h}}_{1,0}} & A \\
 & & \downarrow \dot d^{\mathrm{v}}_{0,3} & & \downarrow \dot d^{\mathrm{v}}_{0,2} & & \downarrow \dot d^{\mathrm{v}}_{0,1} & & \downarrow \dot d^{\mathrm{v}}_{0,0} \\
\cdots & \to & A & \xrightarrow{\dot d^{\mathrm{h}}_{0,2}} & A & \xrightarrow{\dot d^{\mathrm{h}}_{0,1}} & A & \xrightarrow{\dot d^{\mathrm{h}}_{0,0}} & A.
\end{array} \right)$$

Then for $i, j \in \mathbf{Z}_{\geq 0}$ the A-linear maps $\dot d^{\mathrm{v}}_{i,j}$ and $\dot d^{\mathrm{h}}_{i,j}$ are just multiplication with the element $d^{\mathrm{v}}_{i,j} \in A$ or $d^{\mathrm{v}}_{i,j} \in A$ from the right.

For $m \in A$ consider the map $\psi^{-1}\dot m\psi$. For $a \in A$ we have

$$a(\psi^{-1}\dot m\psi) = (a\psi^{-1})\dot m\psi = ((a\psi^{-1})m)\psi = a(m\psi).$$

So $\psi^{-1}\dot m\psi$ is given by multiplication with $m\psi$ from the right, i.e. $\psi^{-1}\dot m\psi = (m\psi)^{\cdot}$. Hence $\psi^{-1}\dot m\psi$ is A-linear.

Recall that Δ being a double complex means that for all $i, j \in \mathbf{Z}_{\geq 0}$ we have $\dot d^{\mathrm{h}}_{i,j+1}\dot d^{\mathrm{h}}_{i,j} = 0$, $\dot d^{\mathrm{v}}_{i+1,j}\dot d^{\mathrm{v}}_{i,j} = 0$ and $\dot d^{\mathrm{h}}_{i+1,j}\dot d^{\mathrm{v}}_{i,j} = \dot d^{\mathrm{v}}_{i,j+1}\dot d^{\mathrm{h}}_{i,j}$.

Now let $\psi \in \mathrm{Aut}_{R\text{-alg}}(A)$. Then we can define a double complex Δ^{ψ} by

$$\Delta^{\psi} := \left(\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{3,2}\psi} & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{3,1}\psi} & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{3,0}\psi} & A \\
 & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{2,3}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{2,2}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{2,1}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{2,0}\psi \\
\cdots & \to & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{2,2}\psi} & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{2,1}\psi} & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{2,0}\psi} & A \\
 & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{1,3}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{1,2}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{1,1}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{1,0}\psi \\
\cdots & \to & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{1,2}\psi} & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{1,1}\psi} & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{1,0}\psi} & A \\
 & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{0,3}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{0,2}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{0,1}\psi & & \downarrow \psi^{-1}\dot d^{\mathrm{v}}_{0,0}\psi \\
\cdots & \to & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{0,2}\psi} & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{0,1}\psi} & A & \xrightarrow{\psi^{-1}\dot d^{\mathrm{h}}_{0,0}\psi} & A.
\end{array} \right)$$

In this section, we will investigate such "twisted" double complex of the double complex Ξ from Remark 21, whose total complex is the projective resolution P of the trivial Λ-module $\mathbf{Z}$ from Theorem 20.

Remark 45 For any double complex Δ as above, we can form the total complex

$$\mathrm{T}(\Delta) = \Big(\dots \longrightarrow A^{\oplus 4} \xrightarrow{d_2} A^{\oplus 3} \xrightarrow{d_1} A^{\oplus 2} \xrightarrow{d_0} A \longrightarrow 0\Big).$$

Here the differentials d_k for $k \in \mathbf{Z}_{\geq 0}$ are given by

$$d_k = \sum_{i=0}^{k} \pi_{i+2}(-1)^{k-i+1} d^{\mathrm{v}}_{i,k-i}\iota_{i+1} + \sum_{i=0}^{k} \pi_{i+1} d^{\mathrm{h}}_{i,k-i}\iota_{i+1} \;:\; A^{\oplus(k+2)} \longrightarrow A^{\oplus(k+1)}.$$

Suppose $\psi \in \mathrm{Aut}_{R\text{-alg}}(A)$. For $k \in \mathbf{Z}_{\geq 0}$ write d^{ψ}_k for the differentials of the total complex $\mathrm{T}(\Delta^{\psi})$. We have an isomorphism of complexes of R-modules.

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & A^{\oplus 4} & \xrightarrow{d_2} & A^{\oplus 3} & \xrightarrow{d_1} & A^{\oplus 2} & \xrightarrow{d_0} & A \\
 & & \wr\downarrow \begin{pmatrix} \psi & 0 & 0 & 0 \\ 0 & \psi & 0 & 0 \\ 0 & 0 & \psi & 0 \\ 0 & 0 & 0 & \psi \end{pmatrix} & & \wr\downarrow \begin{pmatrix} \psi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi \end{pmatrix} & & \wr\downarrow \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix} & & \wr\downarrow \psi \\
\dots & \longrightarrow & A^{\oplus 4} & \xrightarrow{d^{\psi}_2} & A^{\oplus 3} & \xrightarrow{d^{\psi}_1} & A^{\oplus 2} & \xrightarrow{d^{\psi}_0} & A
\end{array}$$

Now suppose that the total complex $\mathrm{T}(\Delta)$ is a projective resolution of some A-module M, i.e. we have an augmentation map $A \xrightarrow{\varepsilon} M$ such that the following complex is acyclic.

$$\dots \longrightarrow A^{\oplus 4} \xrightarrow{d_2} A^{\oplus 3} \xrightarrow{d_1} A^{\oplus 2} \xrightarrow{d_0} A \xrightarrow{\varepsilon} M \longrightarrow 0$$

Let M^{ψ} be the A-module with operation twisted by ψ^{-1}, i.e. for $m \in M$ and $a \in A$ we have

$$a \underset{M^{\psi}}{\cdot} m := (a\psi^{-1}) \underset{M}{\cdot} m.$$

Then the R-linear map $A \xrightarrow{\psi^{-1}\varepsilon} M^{\psi}$ is also A-linear, since for $a, b \in A$ we have

$$(ab)\psi^{-1}\varepsilon = \big((a\psi^{-1})(b\psi^{-1})\big)\varepsilon = (a\psi^{-1}) \underset{M}{\cdot} ((b\psi^{-1})\varepsilon) = a \underset{M^{\psi}}{\cdot} (b\psi^{-1}\varepsilon).$$

We obtain the following isomorphism of complexes of R-modules.

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & A^{\oplus 4} & \xrightarrow{d_2} & A^{\oplus 3} & \xrightarrow{d_1} & A^{\oplus 2} & \xrightarrow{d_0} & A & \xrightarrow{\varepsilon} & M \\
 & & \wr\downarrow \begin{pmatrix} \psi & 0 & 0 & 0 \\ 0 & \psi & 0 & 0 \\ 0 & 0 & \psi & 0 \\ 0 & 0 & 0 & \psi \end{pmatrix} & & \wr\downarrow \begin{pmatrix} \psi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi \end{pmatrix} & & \wr\downarrow \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix} & & \wr\downarrow \psi & & \wr\downarrow 1 \\
\dots & \longrightarrow & A^{\oplus 4} & \xrightarrow{d^{\psi}_2} & A^{\oplus 3} & \xrightarrow{d^{\psi}_1} & A^{\oplus 2} & \xrightarrow{d^{\psi}_0} & A & \xrightarrow{\psi^{-1}\varepsilon} & M^{\psi}
\end{array}$$

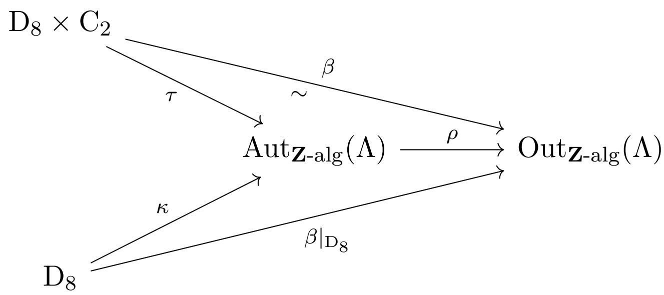
Moreover, this implies that the complex $\mathrm{T}(\Delta^{\psi})$ is a projective resolution of the A-module M^{ψ}.

3.3.2 Description of symmetries

Recall the group morphisms from Lemma 42 and 43.

$$\begin{array}{ccc}
\mathrm{D}_8 \times \mathrm{C}_2 & \xrightarrow[\sim]{\beta} & \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\Lambda) \\
 & {\scriptstyle\tau}\searrow \quad \nearrow{\scriptstyle\rho} & \\
 & \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda) &
\end{array}$$

In a first step, we shall construct a group morphism $\mathrm{D}_8 \xrightarrow{\kappa} \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$ such that $\kappa\rho = \beta|_{\mathrm{D}_8}$. The automorphisms in the image of κ have the property of acting nicely using our twisted double complex construction described above, cf. Remarks 48, 49, 50, 51. We shall remark that κ can not be extended to a group morphism $\mathrm{D}_8 \times \mathrm{C}_2 \xrightarrow{\kappa'} \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$ with $\kappa'\rho = \beta$, cf. Remark 47.



Lemma 46 *We have a group morphism.*

$$\begin{array}{ccc} \mathrm{D}_8 & \xrightarrow{\kappa} & \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda) \\ a & \longmapsto & \left((x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \mapsto (x_4, x_3, x_1, x_2, \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right)) \right) \\ b & \longmapsto & \left((x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \mapsto (x_1, x_2, x_4, x_3, \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)) \right) \end{array}$$

Moreover, we have $\kappa\rho = \beta|_{\mathrm{D}_8}$.

Proof. Note that the image of b under the asserted map κ is the same as $(b, 1)\tau$, hence $b\kappa \in \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$. cf. Lemma 42. For a we calculate for $\xi := (x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)) \in \Lambda$.

$$\begin{aligned} \xi(a\kappa) &= (x_4, x_3, x_1, x_2, \tfrac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ -2 & -2 \end{smallmatrix}\right) \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right)) \\ &= (x_4, x_3, x_1, x_2, \tfrac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ -2 & -2 \end{smallmatrix}\right) \left(\begin{smallmatrix} -2x_{5;1,1}+2x_{5;1,2} & -x_{5;1,1} \\ -2x_{5;2,1}+2x_{5;2,2} & -x_{5;2,1} \end{smallmatrix}\right)) \\ &= (x_4, x_3, x_1, x_2, \left(\begin{smallmatrix} -x_{5;2,1}+x_{5;2,2} & -\frac{1}{2}x_{5;2,1} \\ 2x_{5;1,1}-2x_{5;1,2}+2x_{5;2,1}-2x_{5;2,2} & x_{5;1,1}+x_{5;2,1} \end{smallmatrix}\right)) \end{aligned}$$

We verify the congruences in the definition of Λ to show that $\xi(a\kappa) \in \Lambda$. The congruence modulo 8 is satisfied since conjugation preserves matrix traces. Also note that congruences only involving the four 1×1-matrix entries are also satisfied by Lemma 42.

$$\begin{aligned} -x_{5;2,1} + x_{5;2,2} &\equiv_2 x_{5;2,2} \equiv_2 x_4 \\ x_{5;1,1} + x_{5;2,1} &\equiv_2 x_{5;1,1} \equiv_2 x_4 \\ 2 \cdot \left(-\tfrac{1}{2}x_{5;2,1}\right) &= -x_{5;2,1} \equiv_4 x_3 - x_2 \\ 2x_{5;1,1} - 2x_{5;1,2} + 2x_{5;2,1} - 2x_{5;2,2} &\equiv_4 2(x_{5;1,1} - x_{5;2,2} - 2x_{5;1,2} + 2x_{5;2,1} \\ &\equiv_4 2x_{5;1,2} \equiv_4 x_2 - x_4 \equiv_4 x_3 - x_1 \end{aligned}$$

We have

$$\begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}^4 = \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix}^2 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}.$$

By Lemma 34.(2) the map $a\kappa$ is indeed an automorphism of Λ.

To show that κ is a well-defined group morphism, recall the presentation $\mathrm{D}_8 = \langle a, b \mid a^4, b^2, (ba)^2 \rangle$. We calculate.

$$\xi(a\kappa)^4 = (x_4, x_3, x_1, x_2, \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right))(a\kappa)^3$$

$$
\begin{aligned}
&= \Big(x_2, x_1, x_4, x_3, \left(\begin{smallmatrix} 2 & 2 \\ -4 & -2 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 2 & 2 \\ -4 & -2 \end{smallmatrix}\right)\Big)(a\kappa)^2 \\
&= \Big(x_3, x_4, x_2, x_1, \left(\begin{smallmatrix} 0 & -2 \\ 4 & 4 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -2 \\ 4 & 4 \end{smallmatrix}\right)\Big)(a\kappa) \\
&= \Big(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} -4 & 0 \\ 0 & -4 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} -4 & 0 \\ 0 & -4 \end{smallmatrix}\right)\Big) \\
&= \xi \\
\xi(b\kappa)^2 &= \Big(x_1, x_2, x_4, x_3, \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)\Big)(b\kappa) \\
&= \Big(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\Big) \\
&= \xi \\
\xi\big((b\kappa)(a\kappa)\big)^2 &= \Big(x_1, x_2, x_4, x_3, \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)\Big)(a\kappa)(b\kappa)(a\kappa) \\
&= \Big(x_3, x_4, x_1, x_2, \left(\begin{smallmatrix} 2 & 0 \\ -4 & -2 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 2 & 0 \\ -4 & -2 \end{smallmatrix}\right)\Big)(b\kappa)(a\kappa) \\
&= \Big(x_3, x_4, x_2, x_1, \left(\begin{smallmatrix} 0 & 2 \\ -4 & -4 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 2 \\ -4 & -4 \end{smallmatrix}\right)\Big)(a\kappa) \\
&= \Big(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right) \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right)\Big) \\
&= \xi
\end{aligned}
$$

Hence κ is a well-defined group morphism.

To show that $\kappa\rho = \beta|_{\mathrm{D}_8}$ it suffices to show this for the generators a and b. Since $b\kappa = b\tau$ and $\tau\rho = \beta$ this follows for b by Lemma 42 and 43.

For a consider the unit element

$$\lambda := \Big(1, -1, -1, 1, \left(\begin{smallmatrix} -1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\Big) \in \mathrm{U}(\Lambda)$$

of order 2. Since

$$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

we have for $\xi \in \Lambda$

$$\xi((a,1)\tau) = (\lambda^{-1}\xi\lambda)(a\kappa) = \xi(f_\lambda \cdot (a\kappa)),$$

where f_λ denotes the inner automorphism given by conjugation with λ from the right. We conclude that $a\beta|_{\mathrm{D}_8} = (a,1)\tau\rho = a\kappa\rho$ and thus $\kappa\rho = \beta|_{\mathrm{D}_8}$. □

Remark 47 There is no group morphism $\mathrm{D}_8 \times \mathrm{C}_2 \xrightarrow{\kappa'} \mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\Lambda)$ with $\kappa'\rho = \beta$ and $\kappa'|_{\mathrm{D}_8} = \kappa$.

Proof. Assume the contrary. Let $A \in \mathrm{GL}_2(\mathbf{Q})$ be the conjugating matrix corresponding to the automorphism $(1,c)\kappa'$, cf. Lemma 34.(1). By assumption $(1,c)\kappa'\rho = (1,c)\beta = (1,c)\tau\rho$, cf. Lemma 42, so we have for $\Big(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\Big) \in \Lambda$

$$\Big(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\Big)(c\kappa') = \Big(x_1, x_2, x_3, x_4, A^{-1}\left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)A\Big).$$

Since $(1,c)\kappa'(a,1)\kappa' = (a,1)\kappa'(1,c)\kappa'$ and $(b,1)\kappa'(1,c)\kappa' = (1,c)\kappa'(b,1)\kappa'$ there are $u, v \in \mathbf{Q}$ such that

$$
\begin{aligned}
A\left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right) &= u\left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right)A \\
A\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right) &= v\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)A.
\end{aligned}
$$

Consider the linear maps

$$\begin{array}{cccccccc} \mathbf{Q}^{2\times 2} & \xrightarrow{g} & \mathbf{Q}^{2\times 2} & & \mathbf{Q}^{2\times 2} & \xrightarrow{h} & \mathbf{Q}^{2\times 2} \\ M & \longmapsto & \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} M \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right) & & M & \longmapsto & \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} M \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right). \end{array}$$

Then $Ag = uA$ and $Ah = vA$, i.e. A is an eigenvector of g resp. h for the eigenvalue u resp. v. Suppose $A = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ with p, q, r, $s \in \mathbf{Q}$. We calculate.

$$\begin{aligned} \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \left(\begin{smallmatrix} -2 & -1 \\ 2 & 0 \end{smallmatrix}\right) &= \tfrac{1}{2} \left(\begin{smallmatrix} 0 & 1 \\ -2 & -2 \end{smallmatrix}\right) \left(\begin{smallmatrix} -2p+2q & -p \\ -2r+2s & -r \end{smallmatrix}\right) \\ &= \tfrac{1}{2} \left(\begin{smallmatrix} -2r+2s & -r \\ 4p-4q+4r-4s & 2p+2r \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right) &= \tfrac{1}{2} \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 2q & p \\ 2s & r \end{smallmatrix}\right) \\ &= \tfrac{1}{2} \left(\begin{smallmatrix} 2s & r \\ 4q & 2p \end{smallmatrix}\right) \end{aligned}$$

Using the following $\mathbf{Q}$-linear basis of $\mathbf{Q}^{2\times 2}$

$$\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right)$$

the maps g and h have the following representing matrices.

$$M_g = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} \qquad M_h = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomials are given by

$$\chi_g(X) = (X-1)^2(X^2+1) \qquad \chi_h(X) = (X-1)^2(X+1)^2.$$

The eigenspaces for the rational eigenvalues are given by

$$\begin{aligned} \mathrm{Eig}_g(1) &= \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -2 & -2 \end{smallmatrix}\right) \right\rangle \\ \mathrm{Eig}_h(1) &= \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right) \right\rangle \\ \mathrm{Eig}_h(-1) &= \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -2 & 0 \end{smallmatrix}\right) \right\rangle. \end{aligned}$$

We have to consider two cases.

Case 1: $Ag = A$ and $Ah = A$. We have

$$A \in \mathrm{Eig}_g(1) \cap \mathrm{Eig}_h(1) = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -2 & -2 \end{smallmatrix}\right) \right\rangle \cap \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right) \right\rangle = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle.$$

But then the automorphism $(1, c)\kappa'$ is given by the identity automorphism, in *contradiction* to $(1, c)\kappa'\rho = (1, c)\beta \neq 1$, cf. Theorem 43.

Case 2: $Ag = A$ and $Ah = -A$. We have

$$A \in \mathrm{Eig}_g(1) \cap \mathrm{Eig}_h(-1) = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -2 & -2 \end{smallmatrix}\right) \right\rangle \cap \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -2 & 0 \end{smallmatrix}\right) \right\rangle = \left\langle \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right) \right\rangle.$$

Consider the unit element

$$\lambda = (1, 1, -1, -1, \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)) \in \mathrm{U}(\Lambda).$$

of order 4. Then the automorphism $(1, c)\kappa'$ is given by conjugation with λ, thus $(1, c)\kappa'$ is inner. Again we have a *contradiction* to $(1, c)\kappa'\rho = (1, c)\beta \neq 1$, cf. Theorem 43. □

We recall the projective resolution P of the trivial Λ-module from Chapter 2, cf. Definition 19, Theorem 20 and Remark 21. We have the following elements of Λ.

$$\begin{aligned}
A_- &= \left(0,2,0,2,\left(\begin{smallmatrix}0&0\\2&2\end{smallmatrix}\right)\right) & B_- &= \left(0,2,2,0,\left(\begin{smallmatrix}2&1\\0&0\end{smallmatrix}\right)\right)\\
A_+ &= \left(2,0,2,0,\left(\begin{smallmatrix}-2&0\\-2&0\end{smallmatrix}\right)\right) & B_+ &= \left(2,0,0,2,\left(\begin{smallmatrix}0&-1\\0&2\end{smallmatrix}\right)\right)\\
C_- &= \left(0,4,-4,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) & D_- &= \left(0,4,0,-4,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)\\
C_+ &= \left(4,0,0,-4,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) & D_+ &= \left(4,0,-4,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)
\end{aligned}$$

Then the projective resolution P of the trivial Λ-module $\mathbf{Z}$ is the total complex of the following double complex.

$$\Xi = \left(\begin{array}{ccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda \\
 & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda \\
 & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{B}_-} & & \downarrow{\scriptstyle \dot{B}_-} & & \downarrow{\scriptstyle \dot{B}_-} \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda \\
 & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda \\
 & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{B}_-} \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda
\end{array}\right)$$

Remark 48 Consider the automorphism $\psi_1 := b\kappa$, given by

$$\begin{array}{ccc}
\Lambda & \xrightarrow[\sim]{\psi_1} & \Lambda \\
\left(x_1,x_2,x_3,x_4,\left(\begin{smallmatrix}x_{5;1,1}&x_{5;1,2}\\x_{5;2,1}&x_{5;2,2}\end{smallmatrix}\right)\right) & \longmapsto & \left(x_1,x_2,x_4,x_3,\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}x_{5;1,1}&x_{5;1,2}\\x_{5;2,1}&x_{5;2,2}\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)\right).
\end{array}$$

We have

$$\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}x_{5;1,1}&x_{5;1,2}\\x_{5;2,1}&x_{5;2,2}\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right) = \tfrac{1}{2}\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)\left(\begin{smallmatrix}2x_{5;1,2}&x_{5;1,1}\\2x_{5;2,2}&x_{5;2,1}\end{smallmatrix}\right) = \left(\begin{smallmatrix}x_{5;2,2}&\frac{1}{2}x_{5;2,1}\\2x_{5;1,2}&x_{5;1,1}\end{smallmatrix}\right)$$

We calculate using that $\psi_1^2 = 1$ and hence $\psi_1 = \psi_1^{-1}$.

$$\begin{aligned}
A_-\psi_1 &= \left(0,2,2,0,\left(\begin{smallmatrix}2&1\\0&0\end{smallmatrix}\right)\right) = B_- & B_-\psi_1 &= A_-\\
A_+\psi_1 &= \left(2,0,0,2,\left(\begin{smallmatrix}0&-1\\0&2\end{smallmatrix}\right)\right) = B_+ & B_+\psi_1 &= A_+\\
C_-\psi_1 &= \left(0,4,0,-4,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) = D_- & D_-\psi_1 &= C_-\\
C_+\psi_1 &= \left(4,0,-4,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) = D_+ & D_+\psi_1 &= D_-
\end{aligned}$$

Observe that the double complex Ξ^{ψ_1} arises from Ξ by reflection along the top left to down right diagonal.

Since $\psi_1^{-1}\varepsilon = \varepsilon$ the total complex of the twisted double complex $\mathrm{T}(\Xi^{\psi_1})$ is again a projective resolution of the trivial module $\mathbf{Z}$, cf. Remark 45.

$$\Xi^{\psi_1} = \left(\begin{array}{ccccccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{C}_+} & \Lambda & \xrightarrow{\dot{C}_-} & \Lambda & \xrightarrow{\dot{C}_+} & \Lambda & \xrightarrow{\dot{C}_-} & \Lambda \\
 & \downarrow{\scriptstyle \dot{D}_+} & & \downarrow{\scriptstyle \dot{A}_+} & & \downarrow{\scriptstyle \dot{A}_+} & & \downarrow{\scriptstyle \dot{A}_+} & & \downarrow{\scriptstyle \dot{A}_+} \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{B}_+} & \Lambda & \xrightarrow{\dot{C}_-} & \Lambda & \xrightarrow{\dot{C}_+} & \Lambda & \xrightarrow{\dot{C}_-} & \Lambda \\
 & \downarrow{\scriptstyle \dot{D}_-} & & \downarrow{\scriptstyle \dot{D}_-} & & \downarrow{\scriptstyle \dot{A}_-} & & \downarrow{\scriptstyle \dot{A}_-} & & \downarrow{\scriptstyle \dot{A}_-} \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{B}_+} & \Lambda & \xrightarrow{\dot{B}_-} & \Lambda & \xrightarrow{\dot{C}_+} & \Lambda & \xrightarrow{\dot{C}_-} & \Lambda \\
 & \downarrow{\scriptstyle \dot{D}_+} & & \downarrow{\scriptstyle \dot{D}_+} & & \downarrow{\scriptstyle \dot{D}_+} & & \downarrow{\scriptstyle \dot{A}_+} & & \downarrow{\scriptstyle \dot{A}_+} \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{B}_+} & \Lambda & \xrightarrow{\dot{B}_-} & \Lambda & \xrightarrow{\dot{B}_+} & \Lambda & \xrightarrow{\dot{C}_-} & \Lambda \\
 & \downarrow{\scriptstyle \dot{D}_-} & & \downarrow{\scriptstyle \dot{D}_-} & & \downarrow{\scriptstyle \dot{D}_-} & & \downarrow{\scriptstyle \dot{D}_-} & & \downarrow{\scriptstyle \dot{A}_-} \\
\dashrightarrow & \Lambda & \xrightarrow{\dot{B}_+} & \Lambda & \xrightarrow{\dot{B}_-} & \Lambda & \xrightarrow{\dot{B}_+} & \Lambda & \xrightarrow{\dot{B}_-} & \Lambda
\end{array}\right)$$

Remark 49 Consider the automorphism $\psi_2 := a^2\kappa$ given by

$$\begin{array}{ccc}
\Lambda & \xrightarrow[\sim]{\psi_2} & \Lambda \\
\left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) & \longmapsto & \left(x_2, x_1, x_4, x_3, \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)\right)
\end{array}$$

We have

$$\begin{aligned}
\left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x_{5;1,1}-2x_{5;1,2} & x_{5;1,1}-x_{5;1,2} \\ x_{5;2,1}-2x_{5;2,2} & x_{5;2,1}-x_{5;2,2} \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} -x_{5;1,1}+2x_{5;1,2}-x_{5;2,1}+2x_{5;2,2} & -x_{5;1,1}+x_{5;1,2}-x_{5;2,1}+x_{5;2,2} \\ 2x_{5;1,1}-4x_{5;1,2}+x_{5;2,1}-2x_{5;2,2} & 2x_{5;1,1}-2x_{5;1,2}+x_{5;2,1}-x_{5;2,2} \end{smallmatrix}\right).
\end{aligned}$$

Note that $\psi_2^2 = 1$. We calculate.

$$\begin{array}{ll}
A_-\psi_2 = \left(2, 0, 2, 0, \left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right)\right) = A_+ & A_+\psi_2 = A_- \\
B_-\psi_2 = \left(2, 0, 0, 2, \left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right)\right) = B_+ & B_+\psi_2 = B_- \\
C_-\psi_2 = \left(4, 0, 0, -4, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = C_+ & C_+\psi_2 = C_- \\
D_-\psi_2 = \left(4, 0, -4, 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = D_+ & D_+\psi_2 = D_-
\end{array}$$

Observe that the double complex Ξ^{ψ_2} arises from Ξ be shifting the double complex to the right and downwards and cutting off one row and one column.

By Remark 45, the total complex $\mathrm{T}(\Xi^{\psi_2})$ is a projective resolution of the Λ-module $\mathbf{Z}^{\psi_2}$. Recall $\mathrm{D}_8 = \langle a, b \mid a^4, b^2, (ba)^2 \rangle$ and the Wedderburn embedding $\mathbf{Z}\mathrm{D}_8 \xrightarrow{\omega} \Lambda$. Then

$$\begin{aligned}
(a\omega) \underset{\mathbf{Z}^{\psi_2}}{\cdot} 1 &= (a\omega)\psi_2^{-1} \underset{\mathbf{Z}}{\cdot} 1 = (a\omega)\psi_2^{-1}\varepsilon = \left(1, 1, -1, -1, \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)\right)\psi_2^{-1}\varepsilon = \left(1, 1, -1, -1, \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)\right)\varepsilon = 1 \\
(b\omega) \underset{\mathbf{Z}^{\psi_2}}{\cdot} 1 &= (b\omega)\psi_2^{-1} \underset{\mathbf{Z}}{\cdot} 1 = (b\omega)\psi_2^{-1}\varepsilon = \left(1, -1, 1, -1, \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right)\right)\psi_2^{-1}\varepsilon = \left(-1, 1, -1, 1, \left(\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix}\right)\right)\varepsilon = -1
\end{aligned}$$

Hence the Λ-module $\mathbf{Z}^{\psi_2}$ corresponds to the representation

$$\begin{array}{ccc}
\mathrm{D}_8 & \longrightarrow & \mathrm{GL}_2(\mathbf{Z}) \\
a & \longmapsto & 1 \\
b & \longmapsto & -1.
\end{array}$$

$$
\Xi^{\psi_2} = \left(\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda \\
 & & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{B}_-} & & \downarrow{\scriptstyle \dot{B}_-} & & \downarrow{\scriptstyle \dot{B}_-} & & \downarrow{\scriptstyle \dot{B}_-} \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda \\
 & & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{D}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda \\
 & & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{C}_-} & & \downarrow{\scriptstyle \dot{B}_-} & & \downarrow{\scriptstyle \dot{B}_-} \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{D}_+} & \Lambda \\
 & & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{C}_+} & & \downarrow{\scriptstyle \dot{B}_+} \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda
\end{array} \right)
$$

Remark 50 Consider the automorphism $\psi_3 := (ba)\kappa$ given by

$$
\begin{array}{ccc}
\Lambda & \xrightarrow[\sim]{\psi_3} & \Lambda \\
\left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) & \longmapsto & \left(x_3, x_4, x_1, x_2, \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right)\right)
\end{array}
$$

We have

$$
\begin{aligned}
\left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x_{5;1,1}-2x_{5;1,2} & -x_{5;1,2} \\ x_{5;2,1}-2x_{5;2,2} & -x_{5;2,2} \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} x_{5;1,1}-2x_{5;1,2} & -x_{5;1,2} \\ -2x_{5;1,1}+4x_{5;1,2}-x_{5;2,1}+2x_{5;2,2} & 2x_{5;1,2}+x_{5;2,2} \end{smallmatrix}\right).
\end{aligned}
$$

Note that $\psi_3^2 = 1$. We calculate.

$$
\begin{array}{ll}
A_-\psi_3 = (0, 2, 0, 2, \left(\begin{smallmatrix} 0 & 0 \\ 2 & 2 \end{smallmatrix}\right)) = A_- & A_+\psi_3 = (2, 0, 2, 0, \left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right)) = A_+ \\
B_-\psi_3 = (2, 0, 0, 2, \left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right)) = B_+ & B_+\psi_3 = B_- \\
C_-\psi_3 = (-4, 0, 0, 4, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = -C_+ & C_+\psi_3 = -C_- \\
D_-\psi_3 = (0, -4, 0, 4, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = -D_- & D_+\psi_3 = (-4, 0, 4, 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = -D_+
\end{array}
$$

By Remark 45 the total complex $\mathrm{T}(\Xi^{\psi_3})$ is a projective resolution of the Λ-module $\mathbf{Z}^{\psi_3}$. As in the previous remark, we calculate.

$$
\begin{aligned}
(a\omega) \underset{\mathbf{Z}^{\psi_3}}{\cdot} 1 &= (a\omega)\psi_3^{-1} \underset{\mathbf{Z}}{\cdot} 1 = (a\omega)\psi_3^{-1}\varepsilon = \left(1, 1, -1, -1, \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)\right)\psi_3^{-1}\varepsilon = \left(-1, -1, 1, 1, \left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right)\right)\varepsilon = -1 \\
(b\omega) \underset{\mathbf{Z}^{\psi_3}}{\cdot} 1 &= (b\omega)\psi_3^{-1} \underset{\mathbf{Z}}{\cdot} 1 = (b\omega)\psi_3^{-1}\varepsilon = \left(1, -1, 1, -1, \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right)\right)\psi_3^{-1}\varepsilon = \left(1, -1, 1, -1, \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right)\right)\varepsilon = 1
\end{aligned}
$$

Hence the module $\mathbf{Z}^{\psi_3}$ corresponds to the representation

$$
\begin{array}{ccc}
\mathrm{D}_8 & \longrightarrow & \mathrm{GL}_2(\mathbf{Z}) \\
a & \longmapsto & -1 \\
b & \longmapsto & 1.
\end{array}
$$

$$
\Xi^{\psi_3} = \left(\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots & \dashrightarrow & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda \\
 & & \downarrow {\scriptstyle -\dot{C}_-} & & \downarrow {\scriptstyle \dot{B}_-} & & \downarrow {\scriptstyle \dot{B}_-} & & \downarrow {\scriptstyle \dot{B}_-} & & \downarrow {\scriptstyle \dot{B}_-} \\
\cdots & \dashrightarrow & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda \\
 & & \downarrow {\scriptstyle -\dot{C}_+} & & \downarrow {\scriptstyle -\dot{C}_+} & & \downarrow {\scriptstyle \dot{B}_+} & & \downarrow {\scriptstyle \dot{B}_+} & & \downarrow {\scriptstyle \dot{B}_+} \\
\cdots & \dashrightarrow & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda \\
 & & \downarrow {\scriptstyle -\dot{C}_-} & & \downarrow {\scriptstyle -\dot{C}_-} & & \downarrow {\scriptstyle -\dot{C}_-} & & \downarrow {\scriptstyle \dot{B}_-} & & \downarrow {\scriptstyle \dot{B}_-} \\
\cdots & \dashrightarrow & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda \\
 & & \downarrow {\scriptstyle -\dot{C}_+} & & \downarrow {\scriptstyle -\dot{C}_+} & & \downarrow {\scriptstyle -\dot{C}_+} & & \downarrow {\scriptstyle -\dot{C}_+} & & \downarrow {\scriptstyle \dot{B}_+} \\
\cdots & \dashrightarrow & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda
\end{array}\right)
$$

Remark 51 Consider the automorphism $\psi_4 := (ba^3)\kappa$ given by

$$
\begin{array}{ccc}
\Lambda & \xrightarrow[\sim]{\psi_4} & \Lambda \\
\left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) & \longmapsto & \left(x_4, x_3, x_2, x_1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)\right)
\end{array}
$$

We have

$$
\begin{aligned}
\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,1} - x_{5;1,2} \\ x_{5;2,1} & x_{5;2,1} - x_{5;2,2} \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} x_{5;1,1} + x_{5;2,1} & x_{5;1,1} - x_{5;1,2} + x_{5;2,1} - x_{5;2,2} \\ -x_{5;2,1} & -x_{5;2,1} + x_{5;2,2} \end{smallmatrix}\right)
\end{aligned}
$$

Note that $\psi_4^2 = 1$. We calculate.

$$
\begin{array}{ll}
A_-\psi_4 = (2, 0, 2, 0, \left(\begin{smallmatrix} 2 & 0 \\ -2 & 0 \end{smallmatrix}\right)) = A_+ & A_+\psi_4 = A_- \\
B_-\psi_4 = (0, 2, 2, 0, \left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right)) = B_- & B_+\psi_4 = (2, 0, 0, 2, \left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right)) = B_+ \\
C_-\psi_4 = (0, -4, 4, 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = -C_- & C_+\psi_4 = (-4, 0, 0, 4, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = -C_+ \\
D_-\psi_4 = (-4, 0, 4, 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = -D_+ & D_+\psi_4 = (0, -4, 0, 4, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = -D_+
\end{array}
$$

By Remark 45 the total complex $\mathrm{T}(\Xi^{\psi_4})$ is a projective resolution of the Λ-module $\mathbf{Z}^{\psi_4}$. As in the previous remarks, we calculate.

$$
\begin{aligned}
(a\omega) \underset{\mathbf{Z}^{\psi_4}}{\cdot} 1 = (a\omega)\psi_4^{-1} \underset{\mathbf{Z}}{\cdot} 1 = (a\omega)\psi_4^{-1}\varepsilon = \left(1, 1, -1, -1, \left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right)\right)\psi_4^{-1}\varepsilon = \left(-1, -1, 1, 1, \left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right)\right)\varepsilon = -1 \\
(b\omega) \underset{\mathbf{Z}^{\psi_4}}{\cdot} 1 = (b\omega)\psi_4^{-1} \underset{\mathbf{Z}}{\cdot} 1 = (b\omega)\psi_4^{-1}\varepsilon = \left(1, -1, 1, -1, \left(\begin{smallmatrix} 1 & 0 \\ -2 & -1 \end{smallmatrix}\right)\right)\psi_4^{-1}\varepsilon = \left(-1, 1, -1, 1, \left(\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix}\right)\right)\varepsilon = -1
\end{aligned}
$$

Hence the module $\mathbf{Z}^{\psi_4}$ corresponds to the representation

$$
\begin{array}{ccc}
\mathrm{D}_8 & \longrightarrow & \mathrm{GL}_2(\mathbf{Z}) \\
a & \longmapsto & -1 \\
b & \longmapsto & -1.
\end{array}
$$

$$
\Xi^{\psi_4} = \left(\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dashrightarrow & & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda \\
 & & \downarrow{\scriptstyle -\dot{C}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda \\
 & & \downarrow{\scriptstyle -\dot{C}_-} & & \downarrow{\scriptstyle -\dot{C}_-} & & \downarrow{\scriptstyle \dot{B}_-} & & \downarrow{\scriptstyle \dot{B}_-} & & \downarrow{\scriptstyle \dot{B}_-} \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{-\dot{D}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda \\
 & & \downarrow{\scriptstyle -\dot{C}_+} & & \downarrow{\scriptstyle -\dot{C}_+} & & \downarrow{\scriptstyle -\dot{C}_+} & & \downarrow{\scriptstyle \dot{B}_+} & & \downarrow{\scriptstyle \dot{B}_+} \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{-\dot{D}_+} & \Lambda \\
 & & \downarrow{\scriptstyle -\dot{C}_-} & & \downarrow{\scriptstyle -\dot{C}_-} & & \downarrow{\scriptstyle -\dot{C}_-} & & \downarrow{\scriptstyle -\dot{C}_-} & & \downarrow{\scriptstyle \dot{B}_-} \\
\dashrightarrow & & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda & \xrightarrow{\dot{A}_-} & \Lambda & \xrightarrow{\dot{A}_+} & \Lambda
\end{array}\right)
$$

Chapter 4

Cohomology

We calculate the integral cohomology groups $\mathrm{H}^n(\mathrm{D}_8)$ for $n \in \mathbf{Z}_{\geq 1}$ using our projective resolution from Chapter 2. These have already been calculated by Hamada using the Wall-Hamada resolution in [**6**, Corollary 1].

4.1 Preparations

Recall the isomorphism $\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\omega} \Lambda$ with the Wedderburn image Λ from Lemma 15.

$$\Lambda = \Bigg\{ \left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2} : \\ x_1 \equiv_2 x_2 \equiv_2 x_3 \equiv_2 x_4 \equiv_2 x_{5;1,1} \equiv_2 x_{5;2,2},\ x_2 - x_4 \equiv_4 2x_{5;1,2}, \\ x_2 - x_3 \equiv_4 x_{5;2,1},\ x_1 + x_2 \equiv_4 x_3 + x_4,\ x_1 + x_2 + x_3 + x_4 \equiv_8 2(x_{5;1,1} + x_{5;2,2}) \Bigg\}$$

Let $\mathbf{Z}$ be the trivial Λ-module. Then for $\lambda := \left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) \in \Lambda$ and $z \in \mathbf{Z}$ we have $\lambda z = x_1 z$.

To compute cohomology, we have to apply the contravariant additive functor $\mathrm{Hom}_\Lambda(-, \mathbf{Z})$ to our projective resolution of $\mathbf{Z}$ over Λ, cf. Theorem 20.

Note that we have an isomorphism of Λ-modules

$$\begin{array}{rcl} \mathrm{Hom}_\Lambda(\Lambda, \mathbf{Z}) & \xrightarrow[\sim]{\varphi} & \mathbf{Z} \\ f & \longmapsto & 1f \\ (\xi \longmapsto \xi z) & \longleftarrow\!\shortmid & z. \end{array}$$

We will identify along φ.

Remark 52 Suppose given $\lambda = \left(x_1, x_2, x_3, x_4, \left(\begin{smallmatrix} x_{5;1,1} & x_{5;1,2} \\ x_{5;2,1} & x_{5;2,2} \end{smallmatrix}\right)\right) \in \Lambda$. The Λ-linear map $\Lambda \xrightarrow{\dot{\lambda}} \Lambda$ induces the $\mathbf{Z}$-linear map

$$\begin{array}{rcl} \mathbf{Z} & \xleftarrow{\mathrm{Hom}_\Lambda(\dot{\lambda}, \mathbf{Z})} & \mathbf{Z} \\ x_1 z & \longleftarrow\!\shortmid & z. \end{array}$$

Proof. Let $z \in \mathbf{Z}$. Writing out the identification along φ, we have

$$\begin{aligned} z\varphi \,\mathrm{Hom}_\Lambda(\dot{\lambda}, \mathbf{Z})\varphi^{-1} &= (\xi \longmapsto \xi z)\,\mathrm{Hom}_\Lambda(\dot{\lambda}, \mathbf{Z})\varphi^{-1} \\ &= (\xi \longmapsto (\xi\dot{\lambda})z)\varphi^{-1} = (\xi \longmapsto \xi(\lambda z))\varphi^{-1} \\ &= \lambda z = x_1 z. \end{aligned}$$

□

We recall the projective resolution P of $\mathbf{Z}$ over Λ, cf. §2.2, Definition 19 and Theorem 20.
We have the following elements of Λ.

$$\begin{aligned} A_- &= \left(0,2,0,2,\left(\begin{smallmatrix}0&0\\2&2\end{smallmatrix}\right)\right) & B_- &= \left(0,2,2,0,\left(\begin{smallmatrix}2&1\\0&0\end{smallmatrix}\right)\right)\\ A_+ &= \left(2,0,2,0,\left(\begin{smallmatrix}2&0\\-2&0\end{smallmatrix}\right)\right) & B_+ &= \left(2,0,0,2,\left(\begin{smallmatrix}0&-1\\0&2\end{smallmatrix}\right)\right)\\ C_- &= \left(0,4,-4,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) & D_- &= \left(0,4,0,-4,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)\\ C_+ &= \left(4,0,0,-4,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) & D_+ &= \left(4,0,-4,0,\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) \end{aligned}$$

Then the projective resolution is given by the complex

$$P = \left(\dots \longrightarrow \Lambda^{\oplus 4} \xrightarrow{d_2} \Lambda^{\oplus 3} \xrightarrow{d_1} \Lambda^{\oplus 2} \xrightarrow{d_0} \Lambda \longrightarrow 0\right),$$

where for $\ell \in \mathbf{Z}_{\geq 0}$ the differentials are given by

$$d_{2\ell} := \left(\sum_{i=1}^{\ell+1} \pi_i \dot{A}_i \iota_i\right) + \left(\sum_{i=\ell+2}^{2\ell+1} \pi_i \dot{D}_i \iota_i\right) + \left(\sum_{i=1}^{\ell} \pi_{i+1}(-1)^i \dot{C}_i \iota_i\right) + \left(\sum_{i=\ell+1}^{2\ell+1} \pi_{i+1}(-1)^i \dot{B}_i \iota_i\right)$$

$$d_{2\ell+1} := \left(\sum_{j=1}^{\ell+1} \pi_j \dot{A}_{j+1} \iota_j\right) + \left(\sum_{j=\ell+2}^{2\ell+2} \pi_j \dot{D}_{j+1} \iota_j\right) - \left(\sum_{j=1}^{\ell+1} \pi_{j+1}(-1)^j \dot{C}_j \iota_j\right) - \left(\sum_{j=\ell+2}^{2\ell+2} \pi_{j+1}(-1)^j \dot{B}_j \iota_j\right).$$

Recall the convention $A_i := A_+$ for $i \equiv_2 0$ and $A_i := A_-$ for $i \equiv_2 1$, for the other elements analogously, cf. Definition 19.
Applying the contravariant functor $\mathrm{Hom}_\Lambda(-, \mathbf{Z})$, we obtain the complex

$$\mathrm{Hom}_\Lambda(P', \mathbf{Z}) = \left(\dots \longleftarrow \mathbf{Z}^{\oplus 4} \xleftarrow{\delta_2} \mathbf{Z}^{\oplus 3} \xleftarrow{\delta_1} \mathbf{Z}^{\oplus 2} \xleftarrow{\delta_0} \mathbf{Z} \longleftarrow 0\right).$$

Define the following elements of $\mathbf{Z}$.

$$p_+ := 2 \qquad p_- := 0 \qquad q_+ := 4 \qquad q_- := 0$$

Using additivity of $\mathrm{Hom}_\Lambda(-, \mathbf{Z})$ and Remark 52 for $\ell \in \mathbf{Z}_{\geq 0}$ the differentials are given by

$$\delta_{2\ell} := \left(\sum_{i=1}^{\ell+1} \pi_i \dot{p}_i \iota_i\right) + \left(\sum_{i=\ell+2}^{2\ell+1} \pi_i \dot{q}_i \iota_i\right) + \left(\sum_{i=1}^{\ell} \pi_i (-1)^i \dot{q}_i \iota_{i+1}\right) + \left(\sum_{i=\ell+1}^{2\ell+1} \pi_i (-1)^i \dot{p}_i \iota_{i+1}\right)$$

$$\delta_{2\ell+1} := \left(\sum_{j=1}^{\ell+1} \pi_j \dot{p}_{j+1} \iota_j\right) + \left(\sum_{j=\ell+2}^{2\ell+2} \pi_j \dot{q}_{j+1} \iota_j\right) - \left(\sum_{j=1}^{\ell+1} \pi_j (-1)^j \dot{q}_j \iota_{j+1}\right) - \left(\sum_{j=\ell+2}^{2\ell+2} \pi_j (-1)^j \dot{p}_j \iota_{j+1}\right).$$

We can visualise the first few differentials by matrices as follows.

$$\delta_0 = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad \delta_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \delta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\delta_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \quad \delta_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \delta_5 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

4.2 Calculation of the cohomology groups

Lemma 53 *The cohomology groups* $\mathrm{H}^k(\mathrm{D}_8)$ *for* $k \in [0,4]$ *are given by the following.*

(0) $\mathrm{H}^0(\mathrm{D}_8) \simeq \mathbf{Z}$

(1) $\mathrm{H}^1(\mathrm{D}_8) \simeq 0$

(2) $\mathrm{H}^2(\mathrm{D}_8) \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/2$

(3) $\mathrm{H}^3(\mathrm{D}_8) \simeq \mathbf{Z}/2$

(4) $\mathrm{H}^4(\mathrm{D}_8) \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/4$

Proof. Ad (0). We have $\mathrm{H}^0(\mathrm{D}_8) = \mathrm{H}^0(\mathrm{D}_8, \mathbf{Z}) \simeq \mathbf{Z}^{\mathrm{D}_8} = \mathbf{Z}$, since $\mathbf{Z}$ is the trivial $\mathbf{Z}\mathrm{D}_8$-module.

Ad (1). Observe that $\ker(\delta_1) = 0$. So $\mathrm{H}^1(\mathrm{D}_8) \simeq \ker(\delta_1)/\operatorname{im}(\delta_0) = 0$.

Ad (2). We have $\operatorname{im}(\delta_1) = 2\mathbf{Z} \oplus 0 \oplus 2\mathbf{Z} \subseteq \mathbf{Z}^{\oplus 3}$ and $\ker(\delta_2) = \mathbf{Z} \oplus 0 \oplus \mathbf{Z} \subseteq \mathbf{Z}^{\oplus 3}$. Hence we have $\mathrm{H}^2(\mathrm{D}_8) = \ker(\delta_2)/\operatorname{im}(\delta_1) \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/2$.

Ad (3). Consider the following.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathbf{Z}^{\oplus 3} & \xrightarrow[=\delta_2]{\begin{pmatrix} 0&0&0&0 \\ 0&2&2&0 \\ 0&0&0&0 \end{pmatrix}} & \mathbf{Z}^{\oplus 4} & \xrightarrow[=\delta_3]{\begin{pmatrix} 2&0&0&0&0 \\ 0&0&-4&0&0 \\ 0&0&4&0&0 \\ 0&0&0&0&-2 \end{pmatrix}} & \mathbf{Z}^{\oplus 5} & \longrightarrow & \cdots \\
 & & \Big\downarrow 1 & & \wr\Big\downarrow \begin{pmatrix} 1&0&0&0 \\ 0&1&-1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} & & \Big\downarrow 1 & & \\
 & & \mathbf{Z}^{\oplus 3} & \xrightarrow[=:\tilde{\delta}_2]{\begin{pmatrix} 0&0&0&0 \\ 0&2&0&0 \\ 0&0&0&0 \end{pmatrix}} & \mathbf{Z}^{\oplus 4} & \xrightarrow[=:\tilde{\delta}_3]{\begin{pmatrix} 2&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&4&0&0 \\ 0&0&0&0&-2 \end{pmatrix}} & \mathbf{Z}^{\oplus 5} & &
\end{array}
$$

We have $\operatorname{im}(\tilde{\delta}_2) = 0 \oplus 2\mathbf{Z} \oplus 0 \oplus 0 \subseteq \mathbf{Z}^{\oplus 4}$ and $\ker(\tilde{\delta}_3) = 0 \oplus \mathbf{Z} \oplus 0 \oplus 0 \subseteq \mathbf{Z}^{\oplus 4}$. Hence we have $\mathrm{H}^3(\mathrm{D}_8) = \ker(\delta_3)/\operatorname{im}(\delta_2) \simeq \ker(\tilde{\delta}_3)/\operatorname{im}(\tilde{\delta}_2) \simeq \mathbf{Z}/2$.

Ad (4). Consider the following.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathbf{Z}^{\oplus 4} & \xrightarrow[=\delta_3]{\begin{pmatrix} 2&0&0&0&0 \\ 0&0&-4&0&0 \\ 0&0&4&0&0 \\ 0&0&0&0&-2 \end{pmatrix}} & \mathbf{Z}^{\oplus 5} & \xrightarrow[=\delta_4]{\begin{pmatrix} 0&0&0&0&0&0 \\ 0&2&4&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&4&2&0 \\ 0&0&0&0&0&0 \end{pmatrix}} & \mathbf{Z}^{\oplus 6} & \longrightarrow & \cdots \\
 & & \wr\Big\downarrow \begin{pmatrix} 1&0&0&0 \\ 0&1&-1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} & & \Big\downarrow 1 & & \Big\downarrow 1 & & \\
 & & \mathbf{Z}^{\oplus 4} & \xrightarrow[=:\tilde{\delta}_3]{\begin{pmatrix} 2&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&4&0&0 \\ 0&0&0&0&-2 \end{pmatrix}} & \mathbf{Z}^{\oplus 5} & \xrightarrow[=:\tilde{\delta}_4]{\begin{pmatrix} 0&0&0&0&0&0 \\ 0&2&4&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&4&2&0 \\ 0&0&0&0&0&0 \end{pmatrix}} & \mathbf{Z}^{\oplus 6} & &
\end{array}
$$

We have $\operatorname{im}(\tilde{\delta}_3) = 2\mathbf{Z} \oplus 0 \oplus 4\mathbf{Z} \oplus 0 \oplus 2\mathbf{Z} \subseteq \mathbf{Z}^{\oplus 5}$ and $\ker(\tilde{\delta}_4) = \mathbf{Z} \oplus 0 \oplus \mathbf{Z} \oplus 0 \oplus \mathbf{Z} \subseteq \mathbf{Z}^{\oplus 5}$. Hence we have $\mathrm{H}^4(\mathrm{D}_8) = \ker(\delta_4)/\operatorname{im}(\delta_3) \simeq \ker(\tilde{\delta}_4)/\operatorname{im}(\tilde{\delta}_3) \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/4$. □

Suppose $m, n \in \mathbf{Z}_{\geq 1}$. For $p \in [1, m]$ and $k \in [1, 1+n-p]$ we write

$$\iota_{[k,k+p-1]} := \sum_{i \in [1,p]} \pi_i \iota_{i+k-1} \; : \; \mathbf{Z}^{\oplus p} \longrightarrow \mathbf{Z}^{\oplus n}$$

and for $q \in [1, n]$ and $\ell \in [1, 1+m-q]$ we write

$$\pi_{[\ell,\ell+q-1]} := \sum_{i \in [1,q]} \pi_{i+\ell-1} \iota_i \; : \; \mathbf{Z}^{\oplus m} \longrightarrow \mathbf{Z}^{\oplus q}.$$

We will make use of these maps in the following way. Suppose we are given a $\mathbf{Z}$-linear map $\mathbf{Z}^{\oplus p} \xrightarrow{f} \mathbf{Z}^{\oplus q}$ given by $f = \sum_{i \in [1,p],\, j \in [1,q]} \pi_i f_{i,j} \iota_j$ and k, ℓ as before.

Then we can build a $\mathbf{Z}$-linear map $\mathbf{Z}^{\oplus m} \xrightarrow{g} \mathbf{Z}^{\oplus n}$ with f as a block at position (k, ℓ) by

$$\begin{aligned} g = \pi_{[k,k+p-1]} f \iota_{[\ell,\ell+q-1]} &= \Big(\sum_{i \in [1,p]} \pi_{i+k-1} \iota_i \Big) \Big(\sum_{i \in [1,p],\, j \in [1,q]} \pi_i f_{i,j} \iota_j \Big) \Big(\sum_{i \in [1,q]} \pi_i \iota_{i+\ell-i} \Big) \\ &= \sum_{i \in [1,p],\, j \in [1,q]} \pi_{i+k-1} f_{i,j} \iota_{j+\ell-1} \\ &= \sum_{i \in [k,k+p-1],\, j \in [\ell,\ell+q-1]} \pi_i f_{i-k+1,j-\ell+1} \iota_j. \end{aligned}$$

Lemma 54 (cf. [**6**, Corollary 1]) *The cohomology groups for $k \in \mathbf{Z}_{\geq 1}$ are given by the following.*

$$\mathrm{H}^k(\mathrm{D}_8) \simeq \begin{cases} (\mathbf{Z}/2)^{\oplus k/2} \oplus \mathbf{Z}/4 & \text{if } k \equiv_4 0 \\ (\mathbf{Z}/2)^{\oplus (k-1)/2} & \text{if } k \equiv_4 1 \\ (\mathbf{Z}/2)^{\oplus (k+2)/2} & \text{if } k \equiv_4 2 \\ (\mathbf{Z}/2)^{\oplus (k-1)/2} & \text{if } k \equiv_4 3 \end{cases}$$

Proof. For $k \in \mathbf{Z}_{\geq 0}$ define $\mathbf{Z}$-linear maps $\mathbf{Z}^{\oplus (k+2)} \xrightarrow{\eta_k} \mathbf{Z}^{\oplus (k+2)}$ by

$$\eta_k := \begin{cases} 1 & \text{if } k \in [0,3] \\ 1 - 2\pi_2 \iota_3 - 2\pi_{k+1} \iota_k & \text{if } k \geq 4 \text{ and } k \equiv_2 0 \\ 1 & \text{if } k \geq 5 \text{ and } k \equiv_2 1 \end{cases}$$

Note that η_k is an isomorphism for all $k \in \mathbf{Z}_{\geq 0}$. We only have to show this for $k \geq 4$ and $k \equiv_2 0$. We calculate.

$$\begin{aligned} \eta_k(1 + 2\pi_2\iota_3 + 2\pi_{k+1}\iota_k) &= 1 - 2\pi_2\iota_3 + 2\pi_2\iota_3 - 2\pi_{k+1}\iota_k + 2\pi_{k+1}\iota_k = 1 \\ (1 + 2\pi_2\iota_3 + 2\pi_{k+1}\iota_k)\eta_k &= 1 + 2\pi_2\iota_3 - 2\pi_2\iota_3 + 2\pi_{k+1}\iota_k - 2\pi_{k+1}\iota_k = 1 \end{aligned}$$

It follows that η_k is an isomorphism with inverse given by

$$\eta_k^{-1} = 1 + 2\pi_2\iota_3 + 2\pi_{k+1}\iota_k.$$

Now define a complex of $\mathbf{Z}$-modules $\tilde{P}$ with $\tilde{P}_k = 0$ for $k \in \mathbf{Z}_{<0}$ and $\tilde{P}_k = \mathbf{Z}^{\oplus (k+1)}$ for $k \geq \mathbf{Z}_{\geq 0}$ with differentials $\mathbf{Z}^{\oplus (k+1)} \xrightarrow{\tilde{\delta}_k} \mathbf{Z}^{\oplus (k+2)}$ given by

$$\tilde{\delta}_k := \begin{cases} \delta_k & \text{if } k \in [0,3] \\ \delta_k \eta_k & \text{if } k \geq 4 \text{ and } k \equiv_2 0 \\ \eta_{k-1}^{-1} \delta_k & \text{if } k \geq 5 \text{ and } k \equiv_2 1 \end{cases}$$

Then $\eta = (\eta_k)_{k\in\mathbf{Z}_{\geq 0}}$ defines an isomorphism of complexes between $\operatorname{Hom}_\Lambda(P,\mathbf{Z})$ and $\tilde{P}$.

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\delta_0} & \mathbf{Z}^{\oplus 2} & \xrightarrow{\delta_1} & \mathbf{Z}^{\oplus 3} & \xrightarrow{\delta_2} & \mathbf{Z}^{\oplus 4} & \xrightarrow{\delta_3} & \mathbf{Z}^{\oplus 5} & \xrightarrow{\delta_4} & \mathbf{Z}^{\oplus 6} & \longrightarrow & \dots \\
\wr\downarrow & & \wr\downarrow 1 & & \wr\downarrow 1 & & \wr\downarrow 1 & & \wr\downarrow 1 & & \wr\downarrow 1 & & \wr\downarrow \eta_4 & & \\
0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\tilde{\delta}_0=\delta_0} & \mathbf{Z}^{\oplus 2} & \xrightarrow{\tilde{\delta}_1=\delta_1} & \mathbf{Z}^{\oplus 3} & \xrightarrow{\tilde{\delta}_2=\delta_2} & \mathbf{Z}^{\oplus 4} & \xrightarrow{\tilde{\delta}_3=\delta_3} & \mathbf{Z}^{\oplus 5} & \xrightarrow{\tilde{\delta}_4=\delta_4\eta_4} & \mathbf{Z}^{\oplus 6} & \longrightarrow & \dots
\end{array}$$

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathbf{Z}^{\oplus(2\ell+2)} & \xrightarrow{\delta_{2\ell+1}} & \mathbf{Z}^{\oplus(2\ell+3)} & \xrightarrow{\delta_{2\ell+2}} & \mathbf{Z}^{\oplus(2\ell+4)} & \xrightarrow{\delta_{2\ell+3}} & \mathbf{Z}^{\oplus(2\ell+5)} & \longrightarrow & \dots \\
 & & \wr\downarrow \eta_{2\ell} & & \wr\downarrow 1 & & \wr\downarrow \eta_{2\ell+2} & & \wr\downarrow 1 & & \\
\dots & \longrightarrow & \mathbf{Z}^{\oplus(2\ell+2)} & \xrightarrow[=\eta_{2\ell}^{-1}\delta_{2\ell+1}]{\tilde{\delta}_{2\ell+1}} & \mathbf{Z}^{\oplus(2\ell+3)} & \xrightarrow[=\delta_{2\ell+2}\eta_{2\ell+2}]{\tilde{\delta}_{2\ell+2}} & \mathbf{Z}^{\oplus(2\ell+4)} & \xrightarrow[=\eta_{2\ell+2}^{-1}\delta_{2\ell+3}]{\tilde{\delta}_{2\ell+3}} & \mathbf{Z}^{\oplus(2\ell+5)} & \longrightarrow & \dots
\end{array}$$

Here we have $\ell \geq 2$.

Define the following $\mathbf{Z}$-linear maps.

$$\beta_+ := 2\pi_1\iota_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \ : \ \mathbf{Z}^{\oplus 2} \longrightarrow \mathbf{Z}^{\oplus 2} \qquad \beta_- := 2\pi_2\iota_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \ : \ \mathbf{Z}^{\oplus 2} \longrightarrow \mathbf{Z}^{\oplus 2}$$

Note that we have $\beta_+\beta_- = \beta_-\beta_+ = 0$ with $\ker(\beta_+)/\operatorname{im}(\beta_-) \simeq \mathbf{Z}/2$ and $\ker(\beta_-)/\operatorname{im}(\beta_+) \simeq \mathbf{Z}/2$.

Now suppose that $\ell \in \mathbf{Z}_{\geq 2}$. We calculate.

$$\begin{aligned}
\delta_{2\ell}\eta_{2\ell} &= \left(\left(\sum_{i=1}^{\ell+1} \pi_i\dot{p}_i\iota_i\right) + \left(\sum_{i=\ell+2}^{2\ell+1} \pi_i\dot{q}_i\iota_i\right) + \left(\sum_{i=1}^{\ell} \pi_i(-1)^i\dot{q}_i\iota_{i+1}\right) + \left(\sum_{i=\ell+1}^{2\ell+1} \pi_i(-1)^i\dot{p}_i\iota_{i+1}\right)\right)\eta_{2\ell} \\
&= \delta_{2\ell} - 2\pi_2\dot{p}_+\iota_3 + 2\pi_1\dot{q}_-\iota_3 - 2\pi_{2\ell+1}\dot{q}_-\iota_{2\ell} - 2\pi_{2\ell}\dot{p}_+\iota_{2\ell} \\
&= \delta_{2\ell} - 4\pi_2\iota_3 - 4\pi_{2\ell}\iota_{2\ell} \\
&= 2\pi_2\iota_2 + \left(\sum_{i=3}^{\ell+1} \pi_i\dot{p}_i\iota_i\right) + \left(\sum_{i=\ell+2}^{2\ell-1} \pi_i\dot{q}_i\iota_i\right) + 4\pi_{2\ell}\iota_{2\ell} \\
&\quad + 4\pi_2\iota_3 + \left(\sum_{i=3}^{\ell} \pi_i(-1)^i\dot{q}_i\iota_{i+1}\right) + \left(\sum_{i=\ell+1}^{2\ell-1} \pi_i(-1)^i\dot{p}_i\iota_{i+1}\right) + 2\pi_{2\ell}\iota_{2\ell+1} \\
&\quad - 4\pi_2\iota_3 - 4\pi_{2\ell}\iota_{2\ell} \\
&= \pi_{[1,2]}2\pi_2\iota_2\iota_{[1,2]} \\
&\quad + \pi_{[3,2\ell-1]}\left(\left(\sum_{i=1}^{\ell-1} \pi_i\dot{p}_i\iota_i\right) + \left(\sum_{i=\ell}^{2\ell-3} \pi_i\dot{q}_i\iota_i\right)\right. \\
&\quad \left. + \left(\sum_{i=1}^{\ell-2} \pi_i(-1)^i\dot{q}_i\iota_{i+1}\right) + \left(\sum_{i=\ell-1}^{2\ell-3} \pi_i(-1)^i\dot{p}_i\iota_{i+1}\right)\right)\iota_{[3,2\ell]} \\
&\quad + \pi_{[2\ell,2\ell+1]}2\pi_1\iota_1\iota_{[2\ell+1,2\ell+2]} \\
&= \pi_{[1,2]}\beta_-\iota_{[1,2]} + \pi_{[3,2\ell-1]}\delta_{2(\ell-2)}\iota_{[3,2\ell]} + \pi_{[2\ell,2\ell+1]}\beta_+\iota_{[2\ell+1,2\ell+2]} \\
\eta_{2\ell}^{-1}\delta_{2\ell+1} &= \eta_{2\ell}^{-1}\left(\left(\sum_{j=1}^{\ell+1} \pi_j\dot{p}_{j+1}\iota_j\right) + \left(\sum_{j=\ell+2}^{2\ell+2} \pi_j\dot{q}_{j+1}\iota_j\right)\right. \\
&\quad \left. - \left(\sum_{j=1}^{\ell+1} \pi_j(-1)^j\dot{q}_j\iota_{j+1}\right) - \left(\sum_{j=\ell+2}^{2\ell+2} \pi_j(-1)^j\dot{p}_j\iota_{j+1}\right)\right) \\
&= \delta_{2\ell+1} + 2\pi_2\dot{p}_+\iota_3 + 2\pi_2\dot{q}_-\iota_4 + 2\pi_{2\ell+1}\dot{q}_-\iota_{2\ell} - 2\pi_{2\ell+1}\dot{p}_+\iota_{2\ell+1} \\
&= \delta_{2\ell+1} + 4\pi_2\iota_3 - 4\pi_{2\ell+1}\iota_{2\ell+1} \\
&= 2\pi_1\iota_1 + \left(\sum_{j=3}^{\ell+1} \pi_j\dot{p}_{j+1}\iota_j\right) + \left(\sum_{j=\ell+2}^{2\ell} \pi_j\dot{q}_{j+1}\iota_j\right) + 4\pi_{2\ell+1}\iota_{2\ell+1}
\end{aligned}$$

$$
\begin{aligned}
&- 4\pi_2\iota_3 - \left(\sum_{j=3}^{\ell+1} \pi_j(-1)^j \dot{q}_j\iota_{j+1}\right) - \left(\sum_{j=\ell+2}^{2\ell} \pi_j(-1)^j \dot{p}_j\iota_{j+1}\right) - 2\pi_{2\ell+2}\iota_{2\ell+3} \\
&+ 4\pi_2\iota_3 - 4\pi_{2\ell+1}\iota_{2\ell+1} \\
&= \pi_{[1,2]} 2\pi_1\iota_1\iota_{[1,2]} \\
&+ \pi_{[3,2\ell]}\left(\left(\sum_{j=1}^{\ell-1} \pi_j\dot{p}_{j+1}\iota_j\right) + \left(\sum_{j=\ell}^{2\ell-2} \pi_j\dot{q}_{j+1}\iota_j\right)\right. \\
&\left. - \left(\sum_{j=1}^{\ell-1} \pi_j(-1)^j \dot{q}_j\iota_{j+1}\right) - \left(\sum_{j=\ell}^{2\ell-2} \pi_j(-1)^j \dot{p}_j\iota_{j+1}\right)\right)\iota_{[3,2\ell+1]} \\
&- \pi_{[2\ell+1,2\ell+2]} 2\pi_2\iota_2\iota_{[2\ell+2,2\ell+3]} \\
&= \pi_{[1,2]}\beta_+\iota_{[1,2]} + \pi_{[3,2\ell]}\delta_{2(\ell-2)+1}\iota_{[3,2\ell+1]} - \pi_{[2\ell+1,2\ell+2]}\beta_-\iota_{[2\ell+2,2\ell+3]}
\end{aligned}
$$

We *claim* that $\mathrm{H}^k(\mathrm{D}_8) = (\mathbf{Z}/2)^{\oplus 2} \oplus \mathrm{H}^{k-4}(\mathrm{D}_8)$ for $k \in \mathbf{Z}_{\geq 5}$. $(*)$

Case 1: $k = 2\ell$ for some $\ell \in \mathbf{Z}_{\geq 3}$. Using the calculations above we have the following commutative diagram of $\mathbf{Z}$-modules.

$$
\begin{array}{ccccc}
\mathbf{Z}^{\oplus 2\ell} & \xrightarrow{\delta_{2\ell-1}} & \mathbf{Z}^{\oplus(2\ell+1)} & \xrightarrow{\delta_{2\ell}} & \mathbf{Z}^{\oplus(2\ell+2)} \\
\downarrow{\wr}\,\eta_{2\ell-2} & & \downarrow 1 & & \downarrow{\wr}\,\eta_{2\ell} \\
\mathbf{Z}^{\oplus 2} \oplus \mathbf{Z}^{\oplus(2\ell-4)} \oplus \mathbf{Z}^{\oplus 2} & \xrightarrow[=\tilde{\delta}_{2\ell-1}=\eta_{2\ell-2}^{-1}\delta_{2\ell-1}]{\begin{pmatrix} \beta_+ & 0 & 0 \\ 0 & \delta_{2\ell-5} & 0 \\ 0 & 0 & -\beta_- \end{pmatrix}} & \mathbf{Z}^{\oplus 2} \oplus \mathbf{Z}^{\oplus(2\ell-3)} \oplus \mathbf{Z}^{\oplus 2} & \xrightarrow[=\tilde{\delta}_{2\ell}=\delta_{2\ell}\eta_{2\ell}]{\begin{pmatrix} \beta_- & 0 & 0 \\ 0 & \delta_{2\ell-4} & 0 \\ 0 & 0 & \beta_+ \end{pmatrix}} & \mathbf{Z}^{\oplus 2} \oplus \mathbf{Z}^{\oplus(2\ell-2)} \oplus \mathbf{Z}^{\oplus 2}
\end{array}
$$

It follows that

$$
\begin{aligned}
\mathrm{H}^k(\mathrm{D}_8) = \mathrm{H}^{2\ell}(\mathrm{D}_8) &= \ker(\delta_{2\ell})/\operatorname{im}(\delta_{2\ell-1}) \\
&\simeq \ker(\tilde{\delta}_{2\ell})/\operatorname{im}(\tilde{\delta}_{2\ell-1}) \\
&\simeq \ker(\beta_-)/\operatorname{im}(\beta_+) \oplus \ker(\delta_{2\ell-4})/\operatorname{im}(\delta_{2\ell-5}) \oplus \ker(\beta_+)/\operatorname{im}(\beta_-) \\
&\simeq \mathbf{Z}/2 \oplus \mathrm{H}^{2\ell-4}(\mathrm{D}_8) \oplus \mathbf{Z}/2 \\
&\simeq (\mathbf{Z}/2)^{\oplus 2} \oplus \mathrm{H}^{k-4}(\mathrm{D}_8).
\end{aligned}
$$

Case 2: $k = 2\ell + 1$ for some $\ell \in \mathbf{Z}_{\geq 2}$. Using the calculations above we have the following commutative diagram of $\mathbf{Z}$-modules.

$$
\begin{array}{ccccc}
\mathbf{Z}^{\oplus(2\ell+1)} & \xrightarrow{\delta_{2\ell}} & \mathbf{Z}^{\oplus(2\ell+2)} & \xrightarrow{\delta_{2\ell+1}} & \mathbf{Z}^{\oplus(2\ell+3)} \\
\downarrow 1 & & \downarrow{\wr}\,\eta_{2\ell} & & \downarrow 1 \\
\mathbf{Z}^{\oplus 2} \oplus \mathbf{Z}^{\oplus(2\ell-3)} \oplus \mathbf{Z}^{\oplus 2} & \xrightarrow[=\tilde{\delta}_{2\ell}=\delta_{2\ell}\eta_{2\ell}]{\begin{pmatrix} \beta_- & 0 & 0 \\ 0 & \delta_{2\ell-4} & 0 \\ 0 & 0 & \beta_+ \end{pmatrix}} & \mathbf{Z}^{\oplus 2} \oplus \mathbf{Z}^{\oplus(2\ell-2)} \oplus \mathbf{Z}^{\oplus 2} & \xrightarrow[=\tilde{\delta}_{2\ell+1}=\eta_{2\ell}^{-1}\delta_{2\ell+1}]{\begin{pmatrix} \beta_+ & 0 & 0 \\ 0 & \delta_{2\ell-3} & 0 \\ 0 & 0 & -\beta_- \end{pmatrix}} & \mathbf{Z}^{\oplus 2} \oplus \mathbf{Z}^{\oplus(2\ell-1)} \oplus \mathbf{Z}^{\oplus 2}
\end{array}
$$

It follows that

$$
\begin{aligned}
\mathrm{H}^k(\mathrm{D}_8) = \mathrm{H}^{2\ell+1}(\mathrm{D}_8) &= \ker(\delta_{2\ell+1})/\operatorname{im}(\delta_{2\ell}) \\
&\simeq \ker(\tilde{\delta}_{2\ell+1})/\operatorname{im}(\tilde{\delta}_{2\ell}) \\
&\simeq \ker(\beta_+)/\operatorname{im}(\beta_-) \oplus \ker(\delta_{2\ell-3})/\operatorname{im}(\delta_{2\ell-4}) \oplus \ker(\beta_-)/\operatorname{im}(\beta_+) \\
&\simeq \mathbf{Z}/2 \oplus \mathrm{H}^{2\ell-3}(\mathrm{D}_8) \oplus \mathbf{Z}/2 \\
&\simeq (\mathbf{Z}/2)^{\oplus 2} \oplus \mathrm{H}^{k-4}(\mathrm{D}_8).
\end{aligned}
$$

This shows the claim $(*)$.

Now let $k \in \mathbf{Z}_{\geq 1}$. Consider the following four cases.

Case $k \equiv_4 0$. Let $m \in \mathbf{Z}_{\geq 1}$ with $k = 4m$.

Applying $(*)$ and Lemma 53.(4) we obtain

$$\begin{aligned}
\mathrm{H}^k(\mathrm{D}_8) &= \mathrm{H}^{4m}(\mathrm{D}_8) \\
&\simeq (\mathbf{Z}/2)^{\oplus 2(m-1)} \oplus \mathrm{H}^4(\mathrm{D}_8) \\
&\simeq (\mathbf{Z}/2)^{\oplus 2(m-1)} \oplus (\mathbf{Z}/2)^{\oplus 2} \oplus (\mathbf{Z}/4) \\
&= (\mathbf{Z}/2)^{\oplus 2m} \oplus \mathbf{Z}/4 \\
&= (\mathbf{Z}/2)^{\oplus k/2} \oplus \mathbf{Z}/4.
\end{aligned}$$

Case $k \equiv_4 1$. Let $m \in \mathbf{Z}_{\geq 0}$ with $k = 4m+1$.

Applying $(*)$ and Lemma 53.(1) we obtain

$$\begin{aligned}
\mathrm{H}^k(\mathrm{D}_8) &= \mathrm{H}^{4m+1}(\mathrm{D}_8) \\
&\simeq (\mathbf{Z}/2)^{\oplus 2m} \oplus \mathrm{H}^1(\mathrm{D}_8) \\
&\simeq (\mathbf{Z}/2)^{\oplus 2m} \oplus 0 \\
&= (\mathbf{Z}/2)^{\oplus 2m} \\
&= (\mathbf{Z}/2)^{\oplus (k-1)/2}.
\end{aligned}$$

Case $k \equiv_4 2$. Let $m \in \mathbf{Z}_{\geq 0}$ with $k = 4m+2$.

Applying $(*)$ and Lemma 53.(2) we obtain

$$\begin{aligned}
\mathrm{H}^k(\mathrm{D}_8) &= \mathrm{H}^{4m+2}(\mathrm{D}_8) \\
&\simeq (\mathbf{Z}/2)^{\oplus 2m} \oplus \mathrm{H}^2(\mathrm{D}_8) \\
&\simeq (\mathbf{Z}/2)^{\oplus 2m} \oplus (\mathbf{Z}/2)^{\oplus 2} \\
&= (\mathbf{Z}/2)^{\oplus (2m+2)} \\
&= (\mathbf{Z}/2)^{\oplus (k+2)/2}.
\end{aligned}$$

Case $k \equiv_4 3$. Let $m \in \mathbf{Z}_{\geq 0}$ with $k = 4m+3$.

Applying $(*)$ and Lemma 53.(3) we obtain

$$\begin{aligned}
\mathrm{H}^k(\mathrm{D}_8) &= \mathrm{H}^{4m+3}(\mathrm{D}_8) \\
&\simeq (\mathbf{Z}/2)^{\oplus 2m} \oplus \mathrm{H}^3(\mathrm{D}_8) \\
&\simeq (\mathbf{Z}/2)^{\oplus 2m} \oplus \mathbf{Z}/2 \\
&= (\mathbf{Z}/2)^{\oplus (2m+1)} \\
&= (\mathbf{Z}/2)^{\oplus (k-1)/2}.
\end{aligned}$$

□

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Zusammenfassung

Sei D_8 die Diedergruppe von Ordnung 8 und $\mathbf{Z}\mathrm{D}_8$ der ganzzahlige Gruppenring über D_8.

Wir geben einen Wedderburnisomorphismus für $\mathbf{Z}\mathrm{D}_8$, d.h. einen Ringisomorphismus $\mathbf{Z}\mathrm{D}_8 \xrightarrow[\sim]{\omega} \Lambda$, wobei das Wedderburnbild $\Lambda \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^{2\times 2} =: \Gamma$ von endlichem Index in Γ als abelsche Gruppen ist.

Wir konstruieren eine projektive Auflösung P des trivialen $\mathbf{Z}\mathrm{D}_8$-Moduls $\mathbf{Z}$ der Form

$$\ldots \longrightarrow (\mathbf{Z}\mathrm{D}_8)^{\oplus 4} \xrightarrow{d_2} (\mathbf{Z}\mathrm{D}_8)^{\oplus 3} \xrightarrow{d_1} (\mathbf{Z}\mathrm{D}_8)^{\oplus 2} \xrightarrow{d_0} \mathbf{Z}\mathrm{D}_8 \longrightarrow 0.$$

Die projektive Auflösung P lässt sich als Totalkomplex eines Doppelkomplexes Ξ von freien $\mathbf{Z}\mathrm{D}_8$-Moduln darstellen, dessen Zeilen und Spalten letztendlich periodisches Verhalten zeigen. Um zu zeigen, dass P tatsächlich eine projektive Auflösung ist, konstruieren wir eine $\mathbf{Z}$-lineare kontrahierende Homotopie auf der zugehörigen augmentierten Auflösung P'.

Durch eine Erweiterung der Skalare von $\mathbf{Z}$ zu $\mathbf{Z}_{(2)}$, d.h. Anwenden des Funktors $\mathbf{Z}_{(2)} \underset{\mathbf{Z}}{\otimes} -$ auf P, erhalten wir eine projektive Auflösung $P_{(2)}$ des trivialen $\mathbf{Z}_{(2)}\mathrm{D}_8$-Moduls $\mathbf{Z}_{(2)}$. Wir zeigen, dass $P_{(2)}$ eine minimale projektive Auflösung ist.

Der Doppelkomplex Ξ von freien $\mathbf{Z}\mathrm{D}_8$-Moduln zeigt gewisse Symmetrieeigenschaften, welche sich durch Konjugation der Abbildungen in Ξ mit Automorphismen von $\mathbf{Z}\mathrm{D}_8$ als $\mathbf{Z}$-Algebra beschreiben lassen.

Ausgehend vom Isomorphismus von $\mathbf{Q}$-Algebren $\mathbf{Q} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}\mathrm{D}_8 \simeq \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}^{2\times 2}$ nutzen wir die Kenntnis von Automorphismen von K-Algebren von direkten Produkten von Matrixringen über Körpern K für eine Beschreibung von Automorphismen von $\mathbf{Z}\mathrm{D}_8$ als $\mathbf{Z}$-Algebra. Wir arbeiten durchgehend auf der Bildseite des Wedderburnisomorphismus ω bei der Untersuchung von Automorphismen von $\mathbf{Z}\mathrm{D}_8$.

Wir beginnen mit der Untersuchung von zentralen Automorphismen von $\mathbf{Z}\mathrm{D}_8$, d.h. Automorphismen, welche das Zentrum von $\mathbf{Z}\mathrm{D}_8$ punktweise fixieren. Wir erhalten einen Isomorphismus

$$\mathrm{C}_2 \xrightarrow{\sim} \mathrm{Outcent}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8),$$

wobei C_2 die zyklische Gruppe von Ordnung 2 und $\mathrm{Outcent}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8)$ die Gruppe der äußeren zentralen Automorphismen von $\mathbf{Z}\mathrm{D}_8$ als $\mathbf{Z}$-Algebra ist, d.h. der Quotient aus zentralen Automorphismen mit inneren Automorphismen. Ausgehend von diesem Resultat geben wir einen Isomorphismus der ganzen äußeren Automorphismengruppe an.

$$\mathrm{D}_8 \times \mathrm{C}_2 \xrightarrow{\sim} \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8).$$

Weiterhin zeigen wir, dass die Quotientenabbildung $\mathrm{Aut}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8) \longrightarrow \mathrm{Out}_{\mathbf{Z}\text{-alg}}(\mathbf{Z}\mathrm{D}_8)$ eine Retraktion ist.

Wir vergleichen den Antiautomorphismus von $\mathbf{Z}\mathrm{D}_8$, der gegeben ist durch Inversion auf den Gruppenelementen, mit einem Antiautomorphismus des Wedderburnbildes Λ, welcher durch eine Transposition im 2×2-Matrixblock zustande kommt. Wir erhalten, dass diese beiden Antiautomorphismen sich um einen zentralen Automorphismus von $\mathbf{Z}\mathrm{D}_8$ unterscheiden, welcher kein innerer Automorphismus ist.

In einem letzten Abschnitt berechnen wir aus unserer projektiven Auflösung P des trivialen $\mathbf{Z}\mathrm{D}_8$-Moduls $\mathbf{Z}$ die Gruppen $\mathrm{H}^k(\mathrm{D}_8)$, $k \geq 1$, der ganzzahligen Gruppenkohomologie der Diedergruppe D_8. Die erfolgreiche Berechnung aller Kohomologiegruppen zeigt, dass unsere projektive Auflösung P solche praktischen Berechnungen zulässt.

Hiermit versichere ich,

(1) dass ich meine Arbeit selbstständig verfasst habe,

(2) dass ich keine anderen als die angegeben Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,

(3) dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und

(4) dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, November 2016

Maximilian Hofmann