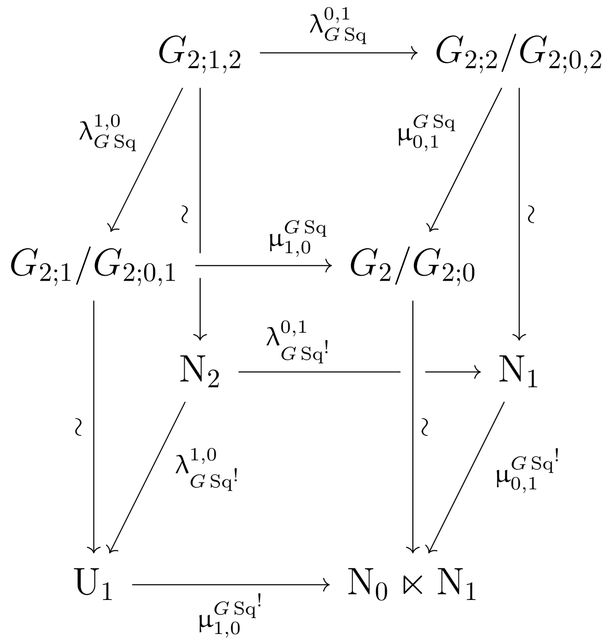
From simplicial groups to crossed squares

Bachelor's Thesis



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0 Introduction

0.1 Simplicial Groups

The simplex category $\triangle$ is the category of linearly ordered finite sets of the form $\{0, \ldots, n\}$, where $n \in \mathbb{Z}_{\geqslant 0}$, and monotone maps between them.

A simplicial group is a functor from $\triangle^{\mathrm{op}}$ to the category of groups.

This means, a simplicial group G consists of groups $\ldots, G_3, G_2, G_1, G_0$, of face maps $\mathrm{d}_i = \mathrm{d}_i^{G,n} : G_n \to G_{n-1}$ and of degeneracy maps $\mathrm{s}_j = \mathrm{s}_j^{G,n} : G_n \to G_{n+1}$ between them, which are group morphisms and which satisfy certain relations.

The category of simplicial groups is denoted by $\mathcal{S}imp\mathcal{G}rp$.

Connected topological spaces can be modelled by simplicial groups as follows.

We have an equivalence between homotopy category of topological spaces and the homotopy category of simplicial sets; cf. [**2,** I, Th. 11.4].

Moreover, we have an equivalence between the homotopy category of reduced simplicial sets and the homotopy category of simplicial groups; cf. [**2,** V, Cor. 6.4].

Altogether, a simplicial group corresponds to a connected topological space via these equivalences.

0.2 Truncating simplicial groups to $\lfloor 2, 0 \rfloor$-simplicial groups

A $\lfloor 2, 0 \rfloor$-simplicial group H consists of groups H_2, H_1, H_0 and face and degeneracy morphisms as follows

$$H_2 \; \begin{matrix} \xrightarrow{\mathrm{d}_2^{H,2}} \\ \xleftarrow{\mathrm{s}_1^{H,1}} \\ \xrightarrow{\mathrm{d}_1^{H,2}} \\ \xleftarrow{\mathrm{s}_0^{H,1}} \\ \xrightarrow{\mathrm{d}_0^{H,2}} \end{matrix} \; H_1 \; \begin{matrix} \xrightarrow{\mathrm{d}_1^{H,1}} \\ \xleftarrow{\mathrm{s}_0^{H,0}} \\ \xrightarrow{\mathrm{d}_0^{H,1}} \end{matrix} \; H_0,$$

satisfying certain relations and the Conduché condition

$$\left[\ker(H_2 \xrightarrow{\mathrm{d}_i^{H,2}} H_1), \bigcap_{j \in \{0,1,2\} \setminus \{i\}} \ker(H_2 \xrightarrow{\mathrm{d}_j^{H,2}} H_1) \right] = 1$$

for $i \in \{0, 1, 2\}$.

The category of $\lfloor 2, 0 \rfloor$-simplicial groups is denoted by $\lfloor 2, 0 \rfloor$-$\mathcal{S}imp\mathcal{G}rp$.

Given a simplicial group G, we consider the normal subgroup

$$G\mathrm{NB}_2 := \left(\bigcap_{i \in \{1,2,3\}} \ker(G_3 \xrightarrow{\mathrm{d}_i^{G,3}} G_2) \right) \mathrm{d}_0 \trianglelefteqslant G_2.$$

We construct the truncation

$$G\,\mathrm{Trunc} := \left(\begin{array}{ccccc} & \xrightarrow{\check{\mathrm{d}}_2^{G,2}} & & & \\ & \xleftarrow{\mathrm{s}_1^{G,1}\,\blacktriangle\,\rho^G} & & \xrightarrow{\mathrm{d}_1^{G,1}} & \\ G_2/G\mathrm{NB}_2 & \xrightarrow{\check{\mathrm{d}}_1^{G,2}} & G_1 & \xleftarrow{\mathrm{s}_0^{G,0}} & G_0 \\ & \xleftarrow{\mathrm{s}_0^{G,1}\,\blacktriangle\,\rho^G} & & \xrightarrow{\mathrm{d}_0^{G,1}} & \\ & \xrightarrow{\check{\mathrm{d}}_0^{G,2}} & & & \end{array} \right).$$

Here $\check{\mathrm{d}}_i^{G,1}$ is induced by $\mathrm{d}_i^{G,1}$ for $i \in \{0,1,2\}$.

Then $G\,\mathrm{Trunc}$ is a $\lfloor 2, 0\rfloor$-simplicial group.

We have obtained the truncation functor

$$\begin{array}{ccc} \mathcal{SimpGrp} & \xrightarrow{\mathrm{Trunc}} & \lfloor 2,0\rfloor\text{-}\mathcal{SimpGrp} \\ G & \longmapsto & G\,\mathrm{Trunc}\,. \end{array}$$

0.3 Crossed Squares

A crossed square is a commutative quadrangle of groups

$$\begin{array}{ccc} L & \longrightarrow & M' \\ \downarrow & & \downarrow \\ M & \longrightarrow & P, \end{array}$$

where in addition M and M' act on L, where P acts on M, M' and on L, together with a map

$$\begin{array}{ccc} M \times M' & \xrightarrow{\chi} & L \\ (m, m') & \longmapsto & \lceil m, m' \rceil, \end{array}$$

satisfying a list of compatibilities.

This notion is due to Loday [**3,** Def. 5.1].

The category of crossed squares is denoted by $\mathcal{CrSq}$.

For instance, given a group P and normal subgroups M, $M' \trianglelefteq P$, we obtain the crossed square

$$\begin{array}{ccc} M \cap M' & \hookrightarrow & M' \\ \downarrow & & \downarrow \\ M & \hookrightarrow & P, \end{array}$$

in which all actions are given by conjugation and for which

$$\lceil m, m' \rceil := [m, m'] \in M \cap M'$$

is the commutator bracket, for $m \in M$ and $m' \in M'$.

In a sense, crossed squares are the next bigger version of crossed modules.

In turn, crossed modules may be used to formulate the conditions appearing in the definition of a crossed square in a concise way.

0.4 A functor Sq from $\lfloor 2,0 \rfloor$-simplicial groups to crossed squares

Suppose given a $\lfloor 2,0 \rfloor$-simplicial group G.

For $X \subseteq \{0,1,2\}$, we write

$$G_{2;X} := \bigcap_{i \in X} \ker(G_2 \xrightarrow{\mathrm{d}_i^{G_2}} G_1) \trianglelefteqslant G_2.$$

Dropping the curly brackets in notation, we obtain e.g.

$$G_{2;0,2} = \ker(\mathrm{d}_0^{G,2}) \cap \ker(\mathrm{d}_2^{G,2}).$$

Following Porter [**5,** Prop. 7, Proof of Lem. A], we define the crossed square

$$G\,\mathrm{Sq} := \left(\begin{array}{ccc} G_{2;1,2} & \xrightarrow{\lambda^{0,1}_{G\,\mathrm{Sq}}} & G_{2;2}/G_{2;0,2} \\ \Big\downarrow{\scriptstyle \lambda^{1,0}_{G\,\mathrm{Sq}}} & & \Big\downarrow{\scriptstyle \mu^{G\,\mathrm{Sq}}_{0,1}} \\ G_{2;1}/G_{2;0,1} & \xrightarrow[\mu^{G\,\mathrm{Sq}}_{1,0}]{} & G_2/G_{2;0} \end{array} \right),$$

in which all maps are identical on representatives, in which all actions are induced by conjugation and for which $\chi^{G\,\mathrm{Sq}}$ is given by the commutator of the representatives.

This yields the functor

$$\lfloor 2,0 \rfloor\text{-}\mathcal{SimpGrp} \xrightarrow{\mathrm{Sq}} \mathcal{CrSq}.$$

0.5 An isomorphic copy $G\,\mathrm{Sq}^!$ of $G\,\mathrm{Sq}$

Following Porter [**5,** Prop. 8], we isomorphically replace $G\,\mathrm{Sq}$ by the crossed square

$$G\,\mathrm{Sq}^! := \left(\begin{array}{ccc} \mathrm{N}_2 & \xrightarrow{\lambda^{0,1}_{G\,\mathrm{Sq}^!}} & \mathrm{N}_1 \\ \Big\downarrow{\scriptstyle \lambda^{1,0}_{G\,\mathrm{Sq}^!}} & & \Big\downarrow{\scriptstyle \mu^{G\,\mathrm{Sq}^!}_{0,1}} \\ \mathrm{U}_1 & \xrightarrow[\mu^{G\,\mathrm{Sq}^!}_{1,0}]{} & \mathrm{N}_0 \ltimes \mathrm{N}_1 \end{array} \right).$$

Here $\mathrm{N}_2 := G_{2;1,2}$, $\mathrm{N}_1 := G_{1;1}$, $\mathrm{N}_0 := G_0$ and $\mathrm{U}_1 := \{(n_1^- \,\mathrm{d}_0, n_1) : n_1 \in \mathrm{N}_1\} \leqslant G_1$.

Cf. Conduché [**1,** Ex. 3.6].

Compared with $G\,\mathrm{Sq}$, the variant $G\,\mathrm{Sq}^!$ has the advantage of working with subgroups instead of factor groups.

As a disadvantage, $G\,\mathrm{Sq}^!$ consists of more involved morphisms and actions as $G\,\mathrm{Sq}$.

E.g. $(n_1^- \,\mathrm{d}_0, n_1) \in \mathrm{U}_1$ acts on $n_2 \in \mathrm{N}_2$ via

$$n_2^{(n_1^- \,\mathrm{d}_0, n_1)} = n_2^{(n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0}};$$

cf. Lemma 81.

0.6 Conclusion

Altogether, we have obtained the following functors.

$$\mathcal{SimpGrp} \xrightarrow{\text{Trunc}} \lfloor 2,0 \rfloor\text{-}\mathcal{SimpGrp} \xrightarrow{\text{Sq}} \mathcal{CrSq}$$

The functor Sq is not an equivalence.

In fact, using an abelian group $A \neq 1$, we may construct a crossed square

$$\begin{array}{ccc} A & \xrightarrow{\lambda'=\mathrm{id}_A} & A \\ {\scriptstyle \lambda=\mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mu'} \\ A & \xrightarrow[\mu]{} & 1\,. \end{array}$$

But a crossed square of the form $G\,\mathrm{Sq}$ has an injective lower horizontal morphism.

This to be seen in contrast to the situation for $\lfloor 1,0 \rfloor$-simplicial groups, where there exists an equivalence

$$\lfloor 1,0 \rfloor\text{-}\mathcal{SimpGrp} \xrightarrow{\sim} \mathcal{CrMod}\,.$$

1 Conventions

Suppose given a group G.

- We use $(\blacktriangle)$ as the symbol of composition.

  We compose on the right, i.e. the composite of $X \xrightarrow{u} Y \xrightarrow{v} Z$ is written $X \xrightarrow{u \blacktriangle v} Z$.

  Here, $u \blacktriangle v$ reads "u comp v".
- Categories are supposed to be small with respect to a suitable universe.
- For $n \in \mathbb{Z}_{\geqslant 0}$, we write $[n] := [0, n] = \{a \in \mathbb{Z} : 0 \leqslant a \leqslant n\}$.
- For $n \in \mathbb{Z}_{\geqslant 0}$, we write $\lfloor n, 0 \rfloor := \{a \in \mathbb{Z} : 0 \leqslant a \leqslant n\}$ ordered decreasingly.
- We write $g^- := g^{-1}$ for the inverse element of $g \in G$.
- Suppose given g, $h \in G$.

  We write $h^g := g^- \cdot h \cdot g$ and $[g, h] := g^- \cdot h^- \cdot g \cdot h$.
- Suppose given subgroups M, $N \leqslant G$.

  We write $[M, N] := \langle [m,\, n] : m \in M,\, n \in N \rangle \leqslant G$.

  Note that if M, $N \trianglelefteqslant G$, then $[M,\, N] \trianglelefteqslant G$.
- Given a group isomorphism $\varphi : G \xrightarrow{\sim} H$, we often write

$$\begin{array}{rcl} \varphi : G & \longrightarrow & H \\ g & \longmapsto & g\varphi \\ h\varphi^- & \longleftarrow\!\!\dashv & h. \end{array}$$

- Given $N \trianglelefteqslant G$, we write

$$\begin{array}{rcl} \rho = \rho_{G,N} : G & \longrightarrow & G/N \\ g & \longmapsto & gN \end{array}$$

  for the residue class morphism.

2 Preliminaries

2.1 Categories

Definition 1 A *category* $\mathcal{C}$ consists of a set of objects $\mathrm{Ob}(\mathcal{C})$, a set of morphisms $\mathrm{Mor}(\mathcal{C})$, maps

$$\begin{array}{llll} \mathrm{Ob}(\mathcal{C}) & \xrightarrow{\boldsymbol{i}=\boldsymbol{i}_{\mathcal{C}}} & \mathrm{Mor}(\mathcal{C}) & \text{(identity)} \\ \mathrm{Ob}(\mathcal{C}) & \xleftarrow{\boldsymbol{s}=\boldsymbol{s}_{\mathcal{C}}} & \mathrm{Mor}(\mathcal{C}) & \text{(source)} \\ \mathrm{Ob}(\mathcal{C}) & \xleftarrow{\boldsymbol{t}=\boldsymbol{t}_{\mathcal{C}}} & \mathrm{Mor}(\mathcal{C}) & \text{(target)} \end{array}$$

and a map, called composition,

$$\begin{array}{rcl} \mathrm{Mor}_2(\mathcal{C}) := \{(f,g) \in \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C}) : f\boldsymbol{t} = g\boldsymbol{s}\} & \xrightarrow{(\blacktriangle)=(\blacktriangle_{\mathcal{C}})} & \mathrm{Mor}(\mathcal{C}) \\ (f,g) & \longmapsto & f \blacktriangle g \end{array}$$

subject to the following properties.

(1) We have

$$\boldsymbol{i} \blacktriangle \boldsymbol{s} = \mathrm{id}_{\mathrm{Ob}(\mathcal{C})}$$

and

$$\boldsymbol{i} \blacktriangle \boldsymbol{t} = \mathrm{id}_{\mathrm{Ob}(\mathcal{C})} .$$

(2) Suppose given $(f,g) \in \mathrm{Mor}_2(\mathcal{C})$.

Then we have

$$(f \blacktriangle g)\boldsymbol{s} = f\boldsymbol{s}$$

and

$$(f \blacktriangle g)\boldsymbol{t} = g\boldsymbol{t}.$$

(3) Suppose given $f \in \mathrm{Mor}(\mathcal{C})$.

Then we have

$$f \blacktriangle f\boldsymbol{t}\boldsymbol{i} = f$$

and

$$f\boldsymbol{s}\boldsymbol{i} \blacktriangle f = f.$$

(4) Suppose given f, g, $h \in \mathrm{Mor}(\mathcal{C})$ such that (f,g), $(g,h) \in \mathrm{Mor}_2(\mathcal{C})$.

Then we have

$$(f \blacktriangle g) \blacktriangle h = f \blacktriangle (g \blacktriangle h).$$

Notation.

- Suppose given $f \in \mathrm{Mor}(\mathcal{C})$.

  For $X := f\boldsymbol{s}_{\mathcal{C}}$ and $Y := f\boldsymbol{t}_{\mathcal{C}}$ in $\mathrm{Ob}(\mathcal{C})$, we write $X \xrightarrow{f} Y$ or $f : X \to Y$.
- For $X \in \mathrm{Ob}(\mathcal{C})$, we write $\mathrm{id}_X := X\boldsymbol{i}$.
- For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we write ${}_{\mathcal{C}}(X,Y) := \{f \in \mathrm{Mor}(\mathcal{C}) : f\boldsymbol{s} = X, f\boldsymbol{t} = Y\}$.

Remark 2 The previous properties can be rewritten as follows.

(1) Given $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$\mathrm{id}_X\, s = X i s = X$$

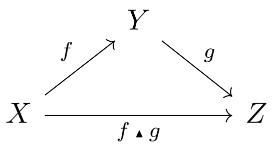
and

$$\mathrm{id}_X\, t = X i t = X.$$

Altogether,

$$X \xrightarrow{\mathrm{id}_X} X.$$

(2) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, we obtain a diagram as follows.



(3) Given $X \xrightarrow{f} Y$ in $\mathcal{C}$, we obtain

$$f t i = Y i = \mathrm{id}_Y \Rightarrow f \blacktriangle \mathrm{id}_Y = f$$

and

$$f s i = X i = \mathrm{id}_X \Rightarrow \mathrm{id}_X \blacktriangle f = f.$$

(4) Given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} E$ in $\mathcal{C}$, we have

$$f \blacktriangle g \blacktriangle h := (f \blacktriangle g) \blacktriangle h = f \blacktriangle (g \blacktriangle h).$$

Example 3 Let $\mathcal{C} = \mathcal{Grp}$ be the category of groups.

Then $\mathrm{Ob}(\mathcal{Grp})$ consists of all groups (in a given universe) and $\mathrm{Mor}(\mathcal{Grp})$ consists of all group morphisms between objects of $\mathcal{Grp}$.

Definition 4 Suppose given a category $\mathcal{C}$.

The *opposite category* $\mathcal{C}^{\mathrm{op}}$ consists of the following data.

(1) $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{C})$
$\mathrm{Mor}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Mor}(\mathcal{C})$

(2) $i_{\mathcal{C}^{\mathrm{op}}} := i_{\mathcal{C}}$
$s_{\mathcal{C}^{\mathrm{op}}} := t_{\mathcal{C}}$
$t_{\mathcal{C}^{\mathrm{op}}} := s_{\mathcal{C}}$

(3) $f \blacktriangle_{\mathcal{C}^{\mathrm{op}}} g := g \blacktriangle_{\mathcal{C}} f$

Notation.

- Given $f \in \mathrm{Mor}(\mathcal{C})$, we often write $f^{\mathrm{op}} := f$, if we consider it as an element of $\mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$.

  So $X \xrightarrow{f} Y$ in $\mathcal{C}$ is written $X \xleftarrow{f^{\mathrm{op}}} Y$, when considered in $\mathcal{C}^{\mathrm{op}}$.
- Let $(f, g) \in \mathrm{Mor}_2(\mathcal{C}^{\mathrm{op}})$. Then $f^{\mathrm{op}} \blacktriangle g^{\mathrm{op}} = (g \blacktriangle f)^{\mathrm{op}}$.

  In detail, $f^{\mathrm{op}} \blacktriangle_{\mathcal{C}^{\mathrm{op}}} g^{\mathrm{op}} = f \blacktriangle_{\mathcal{C}^{\mathrm{op}}} g = g \blacktriangle_{\mathcal{C}} f = (g \blacktriangle_{\mathcal{C}} f)^{\mathrm{op}}$.

Remark 5 Suppose given a category $\mathcal{C}$.

Then the opposite category $\mathcal{C}^{\mathrm{op}}$ is also a category.

Proof. We have to verify that the opposite category fulfills the properties of Definition 1.

Ad (1). We have

$$\begin{aligned} i_{\mathcal{C}^{\mathrm{op}}} \blacktriangle s_{\mathcal{C}^{\mathrm{op}}} &= i_{\mathcal{C}} \blacktriangle t_{\mathcal{C}} \\ &= \mathrm{id}_{\mathrm{Ob}(\mathcal{C})} \\ &= \mathrm{id}_{\mathrm{Ob}(\mathcal{C}^{\mathrm{op}})} \end{aligned}$$

and

$$\begin{aligned} i_{\mathcal{C}^{\mathrm{op}}} \blacktriangle t_{\mathcal{C}^{\mathrm{op}}} &= i_{\mathcal{C}} \blacktriangle s_{\mathcal{C}} \\ &= \mathrm{id}_{\mathrm{Ob}(\mathcal{C})} \\ &= \mathrm{id}_{\mathrm{Ob}(\mathcal{C}^{\mathrm{op}})} . \end{aligned}$$

Ad (2). We have

$$\begin{aligned} (f \blacktriangle_{\mathcal{C}^{\mathrm{op}}} g) s_{\mathcal{C}^{\mathrm{op}}} &= (g \blacktriangle_{\mathcal{C}} f) t_{\mathcal{C}} \\ &= f t_{\mathcal{C}} \\ &= f s_{\mathcal{C}^{\mathrm{op}}} \end{aligned}$$

and

$$\begin{aligned} (f \blacktriangle_{\mathcal{C}^{\mathrm{op}}} g) t_{\mathcal{C}^{\mathrm{op}}} &= (g \blacktriangle_{\mathcal{C}} f) s_{\mathcal{C}} \\ &= g s_{\mathcal{C}} \\ &= g t_{\mathcal{C}^{\mathrm{op}}} \end{aligned}$$

for $(f, g) \in \mathrm{Mor}_2(\mathcal{C}^{\mathrm{op}})$.

Ad (3). We have

$$\begin{aligned} f \blacktriangle_{\mathcal{C}^{\mathrm{op}}} f t_{\mathcal{C}^{\mathrm{op}}} i_{\mathcal{C}^{\mathrm{op}}} &= f \blacktriangle_{\mathcal{C}^{\mathrm{op}}} f s_{\mathcal{C}} i_{\mathcal{C}} \\ &= f s_{\mathcal{C}} i_{\mathcal{C}} \blacktriangle_{\mathcal{C}} f \\ &= f \end{aligned}$$

and

$$\begin{aligned} f s_{\mathcal{C}^{\mathrm{op}}} i_{\mathcal{C}^{\mathrm{op}}} \blacktriangle_{\mathcal{C}^{\mathrm{op}}} f &= f t_{\mathcal{C}} i_{\mathcal{C}} \blacktriangle_{\mathcal{C}^{\mathrm{op}}} f \\ &= f \blacktriangle_{\mathcal{C}} f t_{\mathcal{C}} i_{\mathcal{C}} \\ &= f \end{aligned}$$

for $f \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$.

Ad (4). We have

$$\begin{aligned} (f \blacktriangle_{\mathcal{C}^{\mathrm{op}}} g) \blacktriangle_{\mathcal{C}^{\mathrm{op}}} h &= (g \blacktriangle_{\mathcal{C}} f) \blacktriangle_{\mathcal{C}^{\mathrm{op}}} h \\ &= h \blacktriangle_{\mathcal{C}} (g \blacktriangle_{\mathcal{C}} f) \\ &= (h \blacktriangle_{\mathcal{C}} g) \blacktriangle_{\mathcal{C}} f \\ &= (g \blacktriangle_{\mathcal{C}^{\mathrm{op}}} h) \blacktriangle_{\mathcal{C}} f \\ &= f \blacktriangle_{\mathcal{C}^{\mathrm{op}}} (g \blacktriangle_{\mathcal{C}^{\mathrm{op}}} h) \end{aligned}$$

for f, g, $h \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$ such that (f, g), $(g, h) \in \mathrm{Mor}_2(\mathcal{C}^{\mathrm{op}})$. □

2.2 Functors

Definition 6 Suppose given categories $\mathcal{C}$ and $\mathcal{D}$.

A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of maps

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &\xrightarrow{\mathrm{Ob}(F)} \mathrm{Ob}(\mathcal{D}) \\ \mathrm{Mor}(\mathcal{C}) &\xrightarrow{\mathrm{Mor}(F)} \mathrm{Mor}(\mathcal{D}) \end{aligned}$$

such that the following properties hold.

(1) (a) $\mathrm{Ob}(F) \blacktriangle i_{\mathcal{D}} = i_{\mathcal{C}} \blacktriangle \mathrm{Mor}(F)$

(b) $\mathrm{Mor}(F) \blacktriangle s_{\mathcal{D}} = s_{\mathcal{C}} \blacktriangle \mathrm{Ob}(F)$

(c) $\mathrm{Mor}(F) \blacktriangle t_{\mathcal{D}} = t_{\mathcal{C}} \blacktriangle \mathrm{Ob}(F)$

(2) Suppose given $(f, g) \in \operatorname{Mor}_2(\mathcal{C})$. Then we have

$$f \operatorname{Mor}(F) \blacktriangle g \operatorname{Mor}(F) = (f \blacktriangle g) \operatorname{Mor}(F).$$

Notation. We often write $F := \operatorname{Ob}(F)$ and $F := \operatorname{Mor}(F)$, by abuse of notation.

Remark 7 The previous properties can be rewritten as follows.

(1) (a) Given $X \in \operatorname{Ob}(\mathcal{C})$, we have

$$X(\operatorname{Ob}(F) \blacktriangle \boldsymbol{i}_{\mathcal{D}}) = XF\boldsymbol{i}_{\mathcal{D}} = \mathrm{id}_{XF}$$

and

$$X(\boldsymbol{i}_{\mathcal{C}} \blacktriangle \operatorname{Mor}(F)) = X\boldsymbol{i}_{\mathcal{C}}F = \mathrm{id}_X F.$$

So

$$\mathrm{id}_{XF} = \mathrm{id}_X F : XF \to XF.$$

(b) Given $X \xrightarrow{f} Y$ in $\mathcal{C}$, we have

$$f(\operatorname{Mor}(F) \blacktriangle \boldsymbol{s}_{\mathcal{D}}) = (fF)\boldsymbol{s}_{\mathcal{D}}$$

and

$$f(\boldsymbol{s}_{\mathcal{C}} \blacktriangle \operatorname{Ob}(F)) = (f\boldsymbol{s}_{\mathcal{C}})F = XF.$$

Altogether,

$$(fF)\boldsymbol{s}_{\mathcal{D}} = XF.$$

(c) Given $X \xrightarrow{f} Y$ in $\mathcal{C}$, we have

$$f(\operatorname{Mor}(F) \blacktriangle \boldsymbol{t}_{\mathcal{D}}) = (fF)\boldsymbol{t}_{\mathcal{D}}$$

and

$$f(\boldsymbol{t}_{\mathcal{C}} \blacktriangle \operatorname{Ob}(F)) = (f\boldsymbol{t}_{\mathcal{C}})F = YF$$

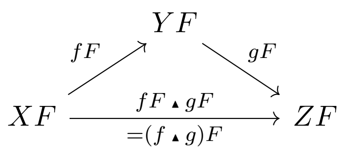
Altogether,

$$(fF)\boldsymbol{t}_{\mathcal{D}} = YF.$$

With (b) and (c), we come to the conclusion that

$$(X \xrightarrow{f} Y)F = (XF \xrightarrow{fF} YF).$$

(2) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, we obtain the following diagram in $\mathcal{D}$.

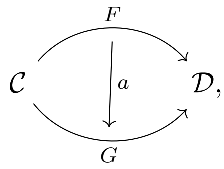


2.3 Transformations

Suppose given categories $\mathcal{C}$ and $\mathcal{D}$.

Definition 8 Suppose given functors F and G from $\mathcal{C}$ to $\mathcal{D}$.

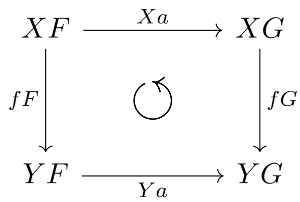
A tuple $a = (Xa)_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms in $\mathcal{C}$ is called a *transformation* from F to G, displayed as



if the following properties $(1, 2)$ hold.

(1) We have $XF \xrightarrow{Xa} XG$ for $X \in \mathrm{Ob}(\mathcal{C})$.

(2) The quadrangle



commutes for $X \xrightarrow{f} Y$ in $\mathcal{C}$.

Remark 9 The commutativity of the quadrangle from property (2) of Definition 8 means that

$$Xa \blacktriangle fG = fF \blacktriangle Ya,$$

for $X \xrightarrow{f} Y$ in $\mathcal{C}$.

Remark 10 Suppose given $F \xrightarrow{a} G \xrightarrow{b} H$.

Then $a \blacktriangle b := (Xa \blacktriangle Xb)_{X \in \mathrm{Ob}(\mathcal{C})}$ is a transformation from F to H.

So we have

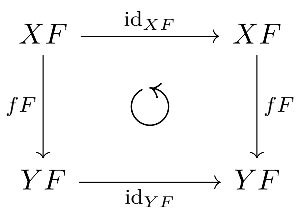
$$X(a \blacktriangle b) = Xa \blacktriangle Xb$$

for $X \in \mathrm{Ob}(\mathcal{C})$.

Remark 11 Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$.

Then we have the identity transformation $\mathrm{id}_F := (\mathrm{id}_{XF})_{X \in \mathrm{Ob}(\mathcal{C})}$.

So we have $X \,\mathrm{id}_F = \mathrm{id}_{XF}$ for $X \in \mathrm{Ob}(\mathcal{C})$.



Definition 12 We write $[\mathcal{C}, \mathcal{D}]$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$.

The set of objects $\mathrm{Ob}([\mathcal{C}, \mathcal{D}])$ consists of the functors from $\mathcal{C}$ to $\mathcal{D}$.

The set of morphisms $\mathrm{Mor}([\mathcal{C}, \mathcal{D}])$ consists of the transformations between such functors.

Composition and identity are those given in Remark 10 and Remark 11.

2.4 The simplex category

Definition 13 The *simplex category* $\triangle$ has $\mathrm{Ob}(\triangle) := \{[n] : n \in \mathbb{Z}_{\geqslant 0}\}$, where $[n] = [0, n]$ is considered as a linearly ordered set.

The set of morphisms $\mathrm{Mor}(\triangle)$ consists of all the monotonic maps between these objects.

Notation. For $n \in \mathbb{Z}_{\geqslant 0}$, we write $[n] := [0, n]$.

Definition 14 Suppose given $n \in \mathbb{Z}_{\geqslant 1}$.

For $i \in [0, n]$, we let

$$\begin{aligned} \delta_i^n : [n-1] &\longrightarrow [n] \\ j &\longmapsto \begin{cases} j+1 & \text{if} \quad j \in [i,\, n-1] \\ j & \text{if} \quad j \in [0,\, i-1]. \end{cases} \end{aligned}$$

So δ_i^n is an injective monotone map from $[n-1]$ to $[n]$ with $[n-1]\delta_i^n = [n] \backslash \{i\}$.

So $(\delta_i^n)^{\mathrm{op}}$ is a morphism in $\triangle^{\mathrm{op}}$ from $[n]$ to $[n-1]$.

For $i \in [0, n]$, we let

$$\begin{aligned} \sigma_i^n : [n+1] &\longrightarrow [n] \\ j &\longmapsto \begin{cases} j-1 & \text{if} \quad j \in [i+1,\, n] \\ j & \text{if} \quad j \in [0, i]. \end{cases} \end{aligned}$$

So σ_i^n is a surjective monotone map from $[n+1]$ to $[n]$ with $i\sigma_i^n = (i+1)\sigma_i^n = i$.

So $(\sigma_i^n)^{\mathrm{op}}$ is a morphism in $\triangle^{\mathrm{op}}$ from $[n]$ to $[n+1]$.

Remark 15 The morphisms in the simplex category $\triangle$ defined in Definition 14 satisfy the following relations.

Suppose given $n \in \mathbb{Z}_{\geqslant 0}$.

(1) $\delta_i^n \blacktriangle \delta_j^{n+1} = \delta_{j-1}^n \blacktriangle \delta_i^{n+1}$ for $i \in [0, n]$, $j \in [1,\, n+1]$ such that $i < j$

(2) $\delta_i^{n+1} \blacktriangle \sigma_j^n = \sigma_{j-1}^{n-1} \blacktriangle \delta_i^n$ for $i \in [0,\, n+1]$, $j \in [1, n]$ such that $i < j$

(3) $\delta_i^{n+1} \blacktriangle \sigma_j^n = \mathrm{id}_{[n]}$ for $i \in [0, n+1]$, $j \in [0, n]$ such that $i \in \{j,\, j+1\}$

(4) $\delta_i^{n+1} \blacktriangle \sigma_j^n = \sigma_j^{n-1} \blacktriangle \delta_{i-1}^n$ for $i \in [1,\, n+1]$, $j \in [0, n]$ such that $i > j+1$

(5) $\sigma_i^{n+1} \blacktriangle \sigma_j^n = \sigma_{j+1}^{n+1} \blacktriangle \sigma_i^n$ for $i \in [0,\, n+1]$, $j \in [0, n]$ such that $i \leqslant j$

Proof. This follows e.g. from [**4,** Def. 1.1]. □

Remark 16 Every morphism $\alpha : [m] \to [n]$ of the simplex category $\triangle$ can be written as a composite

$$\alpha = \sigma_{j_1}^{m-1} \blacktriangle \cdots \blacktriangle \sigma_{j_k}^{m-k} \blacktriangle \delta_{i_1}^{n-l+1} \blacktriangle \cdots \blacktriangle \delta_{i_l}^{n}$$

for some $k \in [0, m]$ and $l \in [0, n]$ such that $m - k = n - l$ and for suitable $j_a \in [0,\, m-a]$ for $a \in [1, k]$ and suitable $i_b \in [0,\, n-l+b]$ for $b \in [1, l]$.

2.5 Groups

2.5.1 An induced action

Lemma 17 Suppose given a group G.

Suppose given subgroups M, N, $P \leqslant G$ with $M \trianglelefteqslant G$, $N \trianglelefteqslant G$, $P \trianglelefteqslant N$ and $[M, N] \leqslant P$.

Then we have the group morphism

$$\begin{array}{rcl} \gamma : G/M & \rightarrow & \operatorname{Aut}(N/P) \\ gM & \mapsto & (nP \mapsto n^g P). \end{array}$$

We often write $(nP)^{gM} := (n)((gM)\gamma) = n^g P$ for $n \in N$ and $g \in G$.

Proof. We have the group morphism

$$\begin{array}{rcl} \gamma_0 : G & \longrightarrow & \operatorname{Aut}(N) \\ g & \longmapsto & (n \mapsto n^g). \end{array}$$

We *claim* that for $g \in G$ there is a unique group morphism $g\gamma_1 : N/P \mapsto N/P$ such that

$$\begin{array}{ccc} N/P & \xrightarrow{\;g\gamma_1\;} & N/P \\ \uparrow{\scriptstyle\rho} & \circlearrowleft & \uparrow{\scriptstyle\rho} \\ N & \xrightarrow[\;g\gamma_0\;]{} & N\,. \end{array}$$

In fact, we have

$$\begin{array}{rcl} (P)(g\gamma_0)\rho & = & (P^g)\rho \\ & \overset{P \trianglelefteqslant G}{=} & P\rho \\ & = & 1_{N/P}, \end{array}$$

so that there exists a unique group morphism $g\gamma_1$ making this quadrangle commutative.

This proves the *claim*.

For $n \in N$ and $g \in G$, we have

$$\begin{array}{rcl} (nP)(g\gamma_1) & = & ((n)\rho)(g\gamma_1) \\ & = & ((n)(g\gamma_0))\rho \\ & = & n^g P. \end{array}$$

We have $1\gamma_1 = \mathrm{id}_{N/P}$.

For g, $\tilde{g} \in G$, we have

$$(g \cdot \tilde{g})\gamma_1 = g\gamma_1 \blacktriangle \tilde{g}\gamma_1,$$

since for $n \in N$, we obtain

$$(nP)((g \cdot \tilde{g})\gamma_1) = n^{g \cdot \tilde{g}} P$$

and

$$\begin{array}{rcl} (nP)(g\gamma_1 \blacktriangle \tilde{g}\gamma_1) & = & (n^g P)(\tilde{g}\gamma_1) \\ & = & (n^g)^{\tilde{g}} P, \end{array}$$

which is the same.

In particular,

$$\begin{array}{rcl} g\gamma_1 \blacktriangle g^{-}\gamma_1 & = & (g \cdot g^{-})\gamma_1 \\ & = & 1_{\gamma_1} \\ & = & \mathrm{id}_{N/P} \end{array}$$

and

$$\begin{array}{rcl} g^{-}\gamma_1 \blacktriangle g\gamma_1 & = & (g^{-} \cdot g)\gamma_1 \\ & = & 1_{\gamma_1} \\ & = & \mathrm{id}_{N/P}, \end{array}$$

so that $g\gamma_1 \in \mathrm{Aut}(N/P)$.

Moreover,

$$\begin{array}{rcl} \gamma_1 : G & \longrightarrow & \mathrm{Aut}(N/P) \\ g & \longmapsto & (g\gamma_1 : nP \mapsto n^g P). \end{array}$$

is a group morphism.

To show that there exists a unique group morphism $\gamma : G/M \to \mathrm{Aut}(N/P)$ such that

$$\begin{array}{ccc} G/M & \xrightarrow{\gamma} & \mathrm{Aut}(N/P) \\ {\scriptstyle\rho}\uparrow & \nearrow_{\gamma_1} & \\ G & & \end{array}$$

commutes, we have to show that $M\gamma_1 = 1$.

Suppose given $m \in M$. We have to show that $m\gamma_1 = 1 = \mathrm{id}_{N/P}$.

In fact, we have

$$\begin{array}{rcl} (nP)(m\gamma_1) & = & (n^m)P \\ & = & nP \end{array}$$

for $n \in N$, because $n^{-}n^{m} = n^{-}m^{-}nm = [n, m] \in P$.

So γ uniquely exists as described.

Then $(gM)\gamma = g\rho\gamma = g\gamma_1$ maps nP to $(nP)(g\gamma_1) = n^g P$, as stated in the lemma. □

2.5.2 An induced isomorphism on the automorphisms groups

Remark 18 Suppose given a group isomorphism

$$\mathfrak{q} : G \xrightarrow{\sim} H.$$

Then we have the group isomorphism

$$\begin{array}{rcl} \hat{\mathfrak{q}} : \mathrm{Aut}(G) & \xrightarrow{\sim} & \mathrm{Aut}(H) \\ \alpha & \overset{\hat{\mathfrak{q}}}{\longmapsto} & \mathfrak{q}^{-} \blacktriangle \alpha \blacktriangle \mathfrak{q} \\ \mathfrak{q} \blacktriangle \beta \blacktriangle \mathfrak{q}^{-} & \overset{\hat{\mathfrak{q}}^{-}}{\longleftarrow\!\shortmid} & \beta. \end{array}$$

2.5.3 Semidirect products

Definition 19 Suppose given groups H and K.

Suppose given a group morphism

$$\begin{array}{rcl} \gamma : H & \longrightarrow & \mathrm{Aut}(K) \\ h & \longmapsto & (h\gamma : k \mapsto k(h\gamma) =: k^h). \end{array}$$

The *semidirect product* $H \ltimes_\gamma K = \{(h, k) : h \in H,\, k \in K\}$ of H and K with respect to γ carries the multiplication $(h, k) \cdot (\tilde{h}, \tilde{k}) := (h \cdot \tilde{h},\, k^{\tilde{h}} \cdot \tilde{k})$, where $(h, k), (\tilde{h}, \tilde{k}) \in H \ltimes_\gamma K$.

Notation. We often write $H \ltimes K := H \ltimes_\gamma K$.

Lemma 20 The semidirect product $H \ltimes K$ is a group.

Proof. Suppose given (a, b), (c, d), (e, f) and (h, k) in $H \ltimes K$. Then we have

$$\begin{array}{rcl} ((a,b)\cdot(c,d))\cdot(e,f) & = & (ac, b^c d)\cdot(e,f) \\ & = & (ace, (b^c d)^e f) \\ & = & (ace, b^{ce} d^e f) \\ & = & (a,b)\cdot(ce, d^e f) \\ & = & (a,b)\cdot((c,d)\cdot(e,f)) \end{array}$$

and

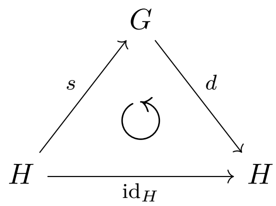
$$\begin{array}{rcl} (h,k)\cdot(1,1) & = & (h\cdot 1, k^1\cdot 1) \\ & = & (h,k). \end{array}$$

Furthermore, we have

$$\begin{array}{rcl} (h,k)\cdot(h^-, (k^-)^{h^-}) & = & (hh^-, k^{h^-}(k^-)^{h^-}) \\ & = & (hh^-, (kk^-)^{h^-}) \\ & = & (1,1). \end{array}$$

So the multiplication on $H \ltimes K$ is associative with the neutral element $1_{H\ltimes K} = (1,1)$ and the respective inverse element $(h,k)^- = (h^-, (k^-)^{h^-})$. $\square$

Lemma 21 Suppose given groups H and G and the following commutative triangle of group morphisms.



Let $K := \ker(d) \trianglelefteq G$.

We have the group morphism

$$\begin{array}{rcl} \gamma : H & \longrightarrow & \mathrm{Aut}(K) \\ h & \longmapsto & (h\gamma : k \mapsto k^h := k^{hs}). \end{array}$$

We consider the semidirect product $H \ltimes K := H \ltimes_\gamma K$.

So we have the group isomorphism

$$\begin{array}{rcl} \varphi : H \ltimes K & \longrightarrow & G \\ (h,k) & \longmapsto & hs\cdot k, \end{array}$$

with inverse

$$\begin{array}{rcl} \varphi^- : G & \longrightarrow & H \ltimes K \\ g & \longmapsto & (gd, g^- ds\cdot g). \end{array}$$

Proof. The map γ is a group morphism as the composite of the group morphisms

$$\begin{array}{ccccc} H & \xrightarrow{s} & G & \longrightarrow & \mathrm{Aut}(K) \\ & & g & \longmapsto & (k \mapsto k^g). \end{array}$$

Suppose given $h, \tilde{h} \in H$, $k, \tilde{k} \in K$ and $g, \tilde{g} \in G$.

Then we have

$$\begin{aligned}((h,k)\cdot(\tilde h,\tilde k))\varphi &= (h\cdot\tilde h, k^{\tilde h}\cdot\tilde k)\varphi\\ &= (h\cdot\tilde h)s\cdot k^{\tilde h}\cdot\tilde k\\ &= hs\cdot\tilde h s\cdot k^{\tilde h s}\cdot\tilde k\\ &= hs\cdot\tilde h s\cdot(\tilde h s)^{-}\cdot k\cdot\tilde h s\cdot\tilde k\\ &= (hs\cdot k)\cdot(\tilde h s\cdot\tilde k)\\ &= (h,k)\varphi\cdot(\tilde h,\tilde k)\varphi.\end{aligned}$$

Moreover, the claimed inverse φ^{-} has

$$g\varphi^{-} = (g\,\mathrm{d}, g^{-}\,\mathrm{d}\,\mathrm{s}\cdot g)$$

with

$$\begin{aligned}(g^{-}\,\mathrm{d}\,\mathrm{s}\cdot g)\,\mathrm{d} &= g^{-}\,\mathrm{d}\,\mathrm{s}\,\mathrm{d}\cdot g\,\mathrm{d}\\ &= 1,\end{aligned}$$

i.e. $g^{-}\,\mathrm{d}\,\mathrm{s}\cdot g\in K$.

It yields

$$\begin{aligned}(h,k)(\varphi\blacktriangle\varphi^{-}) &= (hs\cdot k)(\varphi^{-})\\ &= ((hs\cdot k)d, (hs\cdot k)^{-}ds\cdot(hs\cdot k))\\ &= (hsd\cdot kd, (hsds\cdot kds)^{-}\cdot hs\cdot k)\\ &= (h,(hs)^{-}\cdot hs\cdot k)\\ &= (h,k)\\ &= (h,k)\,\mathrm{id}_{H\ltimes K}\end{aligned}$$

and

$$\begin{aligned}(g)(\varphi^{-}\blacktriangle\varphi) &= (gd, g^{-}ds\cdot g)\varphi\\ &= gds\cdot g^{-}ds\cdot g\\ &= gds\cdot(gds)^{-}\cdot g\\ &= g = g\,\mathrm{id}_G\,.\end{aligned}$$

So

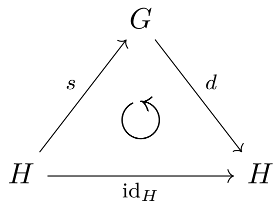
$$\varphi\blacktriangle\varphi^{-1} = \mathrm{id}_{H\ltimes K}$$

and

$$\varphi^{-1}\blacktriangle\varphi = \mathrm{id}_G\,.$$

□

Lemma 22 Suppose given groups H and G and the following commutative triangle of group morphisms.



Let $K := \ker(d) \trianglelefteq G$.

Then we have the group isomorphism

$$\begin{aligned}\varphi : G/K &\longrightarrow H\\ gK &\longmapsto gd,\end{aligned}$$

with inverse

$$\begin{aligned}\varphi^{-} : H &\longrightarrow G/K\\ h &\longmapsto hsK.\end{aligned}$$

Proof. Since $K = \ker(d)$, the injective group morphism φ exists as claimed in the assertion.

As the morphism d is surjective, also the morphism φ is surjective.

For $h \in H$, the claimed inverse φ^- yields

$$\begin{array}{rcl} (h)(\varphi^- \blacktriangle \varphi) & = & (h\varphi^-)\varphi \\ & = & (hsK)\varphi \\ & = & hsd \\ & = & h \\ & = & h\,\mathrm{id}_H\,. \end{array}$$

So $\varphi^- \blacktriangle \varphi = \mathrm{id}_H$ and thus φ^- is in fact the inverse to φ. □

2.6 Crossed modules

Definition 23 Suppose given groups M and B.

Suppose given a group morphism

$$\begin{array}{rcl} \gamma : B & \longrightarrow & \mathrm{Aut}(M) \\ b & \longmapsto & b\gamma. \end{array}$$

We write

$$\begin{array}{rcl} b\gamma : M & \longrightarrow & M \\ m & \longmapsto & m(b\gamma) =: m^b. \end{array}$$

We have $(m\tilde{m})^b = m^b \cdot \tilde{m}^b$ and $m^{b\cdot\tilde{b}} = (m^b)^{\tilde{b}}$ for $m,\, \tilde{m} \in M,\, b,\, \tilde{b} \in B$.

Suppose given a group morphism $f : M \to B$.

Suppose the following properties (CM 1, 2) to hold.

(CM 1) $(m^b)f = (mf)^b$ for $m \in M$ and $b \in B$.

(CM 2) $m^n = m^{nf}$ for $m,\, n \in M$ (Peiffer identity).

Then the quadruple

$$(M,\, B,\, \gamma,\, f)$$

is called a *crossed module*.

Notation. We often write just $(M,\, B)$ to denote this crossed module.

Example 24 Suppose given a group B and a normal subgroup $M \trianglelefteq B$ with the inclusion morphism

$$f : M \hookrightarrow B : m \mapsto m.$$

We have the morphism

$$\begin{array}{rcl} \gamma : B & \longrightarrow & \mathrm{Aut}(M) \\ b & \longmapsto & (m \mapsto m^b), \end{array}$$

given by conjugation in B.

Then

$$(M,\, B,\, \gamma,\, f)$$

is a crossed module.

Proof.

Ad (CM 1). Suppose given $m \in M$ and $b \in B$. Then we have

$$\begin{array}{rcl} (m^b)f & = & m^b \\ & = & (mf)^b. \end{array}$$

Ad (CM 2). Suppose given $m,\, n \in M$. Then we have

$$m^n = m^{nf}.$$

□

Example 25 Suppose given abelian groups M and B.

Suppose given a group morphism

$$f : M \to B.$$

Let

$$\begin{array}{rcl} \gamma : B & \longrightarrow & \mathrm{Aut}(M) \\ b & \longmapsto & \mathrm{id}_M\,. \end{array}$$

Then

$$(M,\, B,\, \gamma,\, f)$$

is a crossed module.

Proof.

Ad (CM 1). Suppose given $m \in M$ and $b \in B$.

Then we have

$$\begin{array}{rcl} (m^b)f & = & mf \\ & = & (mf)^b. \end{array}$$

Ad (CM 2). Suppose given $m,\, n \in M$.

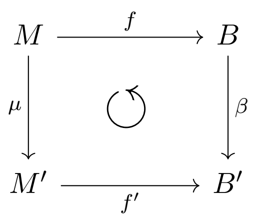
Then we have

$$\begin{array}{rcl} m^n & = & m \\ & = & m^{nf}. \end{array}$$

□

Definition 26 Suppose given crossed modules $(M,\, B,\, \gamma,\, f)$ and $(M',\, B',\, \gamma',\, f')$.

Suppose given group morphisms $\mu : M \to M'$ and $\beta : B \to B'$ such that $f \blacktriangle \beta = \mu \blacktriangle f'$ and such that $(m^b)\mu = (m\mu)^{b\beta}$ for $m \in M$ and $b \in B$.



Then we call

$$(\mu,\, \beta) : (M,\, B,\, \gamma,\, f) \to (M',\, B',\, \gamma',\, f')$$

a *morphism of crossed modules.*

Remark 27 Suppose given morphisms of crossed modules

$$(M,\, B,\, \gamma,\, f) \xrightarrow{(\mu,\beta)} (M',\, B',\, \gamma',\, f') \xrightarrow{(\mu',\beta')} (M'',\, B'',\, \gamma'',\, f'').$$

Then the composite

$$(\mu, \beta) \blacktriangle (\mu', \beta') := (\mu \blacktriangle \mu',\, \beta \blacktriangle \beta')$$

is a morphism of crossed modules from $(M,\, B,\, \gamma,\, f)$ to $(M'',\, B'',\, \gamma'',\, f'')$.

$$\begin{array}{ccc}
M & \xrightarrow{\quad f \quad} & B \\
{\scriptstyle \mu}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \beta} \\
M' & \xrightarrow[\quad f' \quad]{} & B' \\
{\scriptstyle \mu'}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \beta'} \\
M'' & \xrightarrow[\quad f'' \quad]{} & B''
\end{array}$$

Proof. We have

$$\begin{array}{rcl}
f \blacktriangle (\beta \blacktriangle \beta') & = & (f \blacktriangle \beta) \blacktriangle \beta' \\
& = & (\mu \blacktriangle f') \blacktriangle \beta' \\
& = & \mu \blacktriangle (f' \blacktriangle \beta') \\
& = & \mu \blacktriangle (\mu' \blacktriangle f'') \\
& = & (\mu \blacktriangle \mu') \blacktriangle f''.
\end{array}$$

Suppose given $m \in M$ and $b \in B$. We have

$$\begin{array}{rcl}
(m^b)(\mu \blacktriangle \mu') & = & (m^b\mu)\mu' \\
& = & ((m\mu)^{b\beta})\mu' \\
& = & ((m\mu)\mu')^{(b\beta)\beta'} \\
& = & (m(\mu \blacktriangle \mu'))^{b(\beta \blacktriangle \beta')}.
\end{array}$$

□

Remark 28 Suppose given a crossed module $(M,\, B,\, \gamma,\, f)$.

Then its identity, given by

$$\mathrm{id}_{(M,\, B,\, \gamma,\, f)} := (\mathrm{id}_M, \mathrm{id}_B) : (M,\, B,\, \gamma,\, f) \to (M,\, B,\, \gamma,\, f)$$

is a morphism of crossed modules.

Remark 29 We have the *category of crossed modules*, written $\mathcal{CrMod}$.

It has crossed modules as objects and crossed modules morphisms as morphisms; cf. Definition 23 and Definition 26.

The composite of morphisms is described in Remark 27.

The identity on an object is described in Remark 28.

Remark 30 Suppose given a morphism of crossed modules

$$(\mu,\, \beta) : (M,\, B,\, \gamma,\, f) \to (M',\, B',\, \gamma',\, f').$$

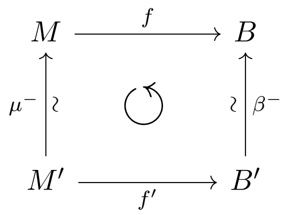
Then (μ, β) is an isomorphism in $\mathcal{CrMod}$ if and only if μ and β are group isomorphisms.

In this case, we have $(\mu, \beta)^- = (\mu^-, \beta^-)$ in $\mathcal{CrMod}$.

Proof. It suffices to show that if μ and β are group isomorphisms, then

$$(\mu^-,\, \beta^-) : (M',\, B',\, \gamma',\, f') \to (M,\, B,\, \gamma,\, f)$$

is a morphism of crossed modules.



We have

$$\begin{array}{rcl} \mu^- \blacktriangle f & = & \mu^- \blacktriangle f \blacktriangle \beta \blacktriangle \beta^- \\ & = & \mu^- \blacktriangle \mu \blacktriangle f' \blacktriangle \beta^- \\ & = & f' \blacktriangle \beta^-. \end{array}$$

Suppose given $m' \in M'$ and $b' \in B'$.

Then we have

$$\begin{array}{rcl} (m'^{b'})\mu^- & = & (m'\mu^-\mu)^{b'\beta^-\beta}\mu^- \\ & = & ((m'\mu^-)^{b'\beta^-})\mu\mu^- \\ & = & (m'\mu^-)^{b'\beta^-}. \end{array}$$

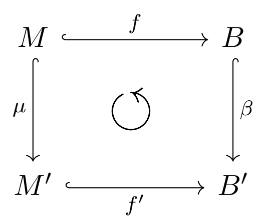
□

Example 31 Suppose given $M \trianglelefteq B \leqslant B'$ and $M \leqslant M' \trianglelefteq B'$

Then we have the crossed modules $(M,\, B,\, \gamma,\, f)$, $(M',\, B',\, \gamma',\, f')$ where $f,\, f'$ are the inclusion morphisms and $\gamma,\, \gamma'$ are given by conjugation, $(m)(b\gamma) = m^b$ and $(m')(b'\gamma') = m'^{b'}$, where $m \in M$, $b \in B$, $m' \in M'$ and $b' \in B'$.

Cf. Example 24.

We have inclusion morphisms $\mu : M \to M'$ and $\beta : B \to B'$ with $f \blacktriangle \beta = \mu \blacktriangle f'$.



Then

$$(\mu,\, \beta) : (M,\, B,\, \gamma,\, f) \to (M',\, B',\, \gamma',\, f')$$

is a morphism of crossed modules.

Proof. Suppose given $m \in M$ and $b \in B$.

Then we have

$$\begin{array}{rcl} (m^b)\mu & = & m^b \\ & = & (m\mu)^b \\ & = & (m\mu)^{b\beta}. \end{array}$$

□

Example 32 Suppose given crossed modules $(M,\, B,\, \gamma,\, f)$ and $(M',\, B',\, \gamma',\, f')$, with M, M', B, B' abelian groups, as in Example 25.

So we have the trivial group morphisms

$$\begin{array}{rcl} \gamma : B & \to & \mathrm{Aut}(M) \\ b & \mapsto & \mathrm{id}_M \end{array}$$

and

$$\begin{array}{rcl} \gamma' : B' & \to & \mathrm{Aut}(M') \\ b' & \mapsto & \mathrm{id}_{M'}\,. \end{array}$$

Suppose given group morphisms $\mu : M \to M'$ and $\beta : B \to B'$ such that $f \blacktriangle \beta = \mu \blacktriangle f'$ for $m \in M$.

Then

$$(\mu,\, \beta) : (M,\, B,\, \gamma,\, f) \to (M',\, B',\, \gamma',\, f')$$

is a morphism of crossed modules.

Proof. Suppose given $m \in M$ and $b \in B$.

Then we have

$$\begin{array}{rcl} (m^b)\mu & = & m\mu \\ & = & (m\mu)^{b\beta}. \end{array}$$

□

Remark 33 Suppose given a crossed module $(M,\, B,\, \gamma,\, f)$.

Suppose given group isomorphisms $M \xrightarrow[\sim]{\mu} M'$ and $B \xrightarrow[\sim]{\beta} B'$.

Let $f' := \mu^- \blacktriangle f \blacktriangle \beta : M' \to B'$.

$$\begin{array}{ccc} M & \xrightarrow{f} & B \\ \mu \downarrow \wr & & \wr \downarrow \beta \\ M' & \xrightarrow[f']{} & B' \end{array}$$

Let $\gamma' := \beta^- \blacktriangle \gamma \blacktriangle \hat{\mu} : B' \to \mathrm{Aut}(M')$; cf. Remark 18.

$$\begin{array}{ccc} B & \xrightarrow{\gamma} & \mathrm{Aut}(M) \\ \beta \downarrow \wr & & \wr \downarrow \hat{\mu} \\ B' & \xrightarrow[\gamma']{} & \mathrm{Aut}(M') \end{array}$$

(1) Then $(M',\, B',\, \gamma',\, f')$ is a crossed module.

(2) Moreover, $(\mu,\, \beta)$ is an isomorphism of crossed modules from $(M,\, B,\, \gamma,\, f)$ to $(M',\, B',\, \gamma',\, f')$.

Proof.

Ad (1). We have to show that $(M',\, B',\, \gamma',\, f')$ is a crossed module; cf. Definition 23.

Ad (CM 1). Suppose given $m' \in M'$ and $b' \in B'$. Then we have

$$\begin{array}{rcl} ({m'}^{b'})f' & = & ((m')(b'\gamma'))f' \\ & = & ((m')(b'(\beta^- \blacktriangle \gamma \blacktriangle \hat{\mu})))f' \\ & = & ((m')(b'\beta^-\gamma\hat{\mu}))f' \\ & \overset{\text{R.18}}{=} & ((m')(\mu^- \blacktriangle b'\beta^-\gamma \blacktriangle \mu))f' \\ & = & ((m'\mu^-)(b'\beta^-\gamma)\mu)f' \\ & = & ((m'\mu^-)(b'\beta^-\gamma)\mu)(\mu^- \blacktriangle f \blacktriangle \beta) \\ & = & ((m'\mu^-)(b'\beta^-\gamma))f\beta \\ & = & ((m'\mu^-)^{b'\beta^-})f\beta \\ & = & ((m'\mu^- f)^{b'\beta^-})\beta \\ & = & (m'\mu^- f\beta)^{b'\beta^-\beta} \\ & = & (m'(\mu^- \blacktriangle f \blacktriangle \beta))^{b'} \\ & = & (m'f')^{b'}. \end{array}$$

Ad (CM 2). Suppose given m', $n' \in M'$. Then we have

$$\begin{array}{rcl} {m'}^{n'f'} & = & (m')(n'f'\gamma') \\ & = & (m')(n'(\mu^- \blacktriangle f \blacktriangle \beta)\gamma') \\ & = & (m')(n'\mu^- f\beta\gamma') \\ & = & (m')(n'\mu^- f\beta(\beta^- \blacktriangle \gamma \blacktriangle \hat{\mu})) \\ & = & (m')(n'\mu^- f\gamma\hat{\mu}) \\ & \overset{\text{R.18}}{=} & (m')(\mu^- \blacktriangle (n'\mu^- f\gamma) \blacktriangle \mu) \\ & = & ((m'\mu^-)(n'\mu^- f\gamma))\mu \\ & = & ((m'\mu^-)^{n'\mu^- f})\mu \\ & = & ((m'\mu^-)^{n'\mu^-})\mu \\ & = & (m'\mu^-\mu)^{n'\mu^-\mu} \\ & = & {m'}^{n'}. \end{array}$$

Ad (2). We have to show that $(\mu,\, \beta)$ is a morphism of crossed modules; cf. Definition 26 and Remark 30.

We have

$$\begin{array}{rcl} \mu \blacktriangle f' & = & \mu \blacktriangle \mu^- \blacktriangle f \blacktriangle \beta \\ & = & f \blacktriangle \beta. \end{array}$$

Suppose given $m \in M$ and $b \in B$. Then we have

$$\begin{array}{rcl} (m\mu)^{b\beta} & = & (m\mu)(b\beta\gamma') \\ & = & (m\mu)(b\beta(\beta^- \blacktriangle \gamma \blacktriangle \hat{\mu})) \\ & = & (m\mu)(b\gamma\hat{\mu}) \\ & \overset{\text{R.18}}{=} & (m\mu)(\mu^- \blacktriangle b\gamma \blacktriangle \mu) \\ & = & ((m)(b\gamma))\mu \\ & = & (m^b)\mu. \end{array}$$

□

3 Simplicial groups and $\lfloor 2, 0 \rfloor$-simplicial groups

3.1 Simplicial groups

Definition 34 A *simplicial group* is a functor G from $\triangle^{\mathrm{op}}$ to $\mathcal{G}rp$.

A *morphism* of simplicial groups is a transformation between such functors.

We denote the category of simplicial groups by $\mathcal{S}imp\mathcal{G}rp := [\triangle^{\mathrm{op}}, \mathcal{G}rp]$.

Definition 35 Suppose given a morphism $G \xrightarrow{\varphi} H$ of simplicial groups.

For $k \geqslant 0$, we write $G_k := [k]G$ and $\varphi_k := [k]\varphi : G_k \to H_k$. So $\varphi = (\varphi_n)_{n \geqslant 0} : G \to H$.

The *face morphisms* are $\mathrm{d}_i^{G,n} := (\delta_i^n)^{\mathrm{op}} G$ from G_n to G_{n-1}, where $n \in \mathbb{Z}_{\geqslant 1}$ and $i \in [0, n]$.

The *degeneracy morphisms* are $\mathrm{s}_i^{G,n} := (\sigma_i^n)^{\mathrm{op}} G$ from G_n to G_{n+1}, where $n \in \mathbb{Z}_{\geqslant 0}$ and $i \in [0, n]$.

We often write $\mathrm{d}_i := \mathrm{d}_i^{G,n}$ and $\mathrm{s}_i := \mathrm{s}_i^{G,n}$.

Remark 36 Suppose given simplicial groups G, H and for $n \geqslant 0$ a group morphism $\varphi_n : G_n \to H_n$.

Then the following assertions are equivalent.

(1) The tuple $(\varphi_n)_{n \geqslant 0}$ is a morphism of simplicial groups.

(2) We have the commutative quadrangles

$$\begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & H_n \\ {\scriptstyle \mathrm{d}_i^{G,n}} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \mathrm{d}_i^{H,n}} \\ G_{n-1} & \xrightarrow[\varphi_{n-1}]{} & H_{n-1} \end{array}$$

for $n \geqslant 1$ and $i \in [0, n]$ and

$$\begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & H_n \\ {\scriptstyle \mathrm{s}_j^{G,n}} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \mathrm{s}_j^{H,n}} \\ G_{n+1} & \xrightarrow[\varphi_{n+1}]{} & H_{n+1} \end{array}$$

for $n \geqslant 0$ and $j \in [0, n]$.

Proof. *Ad* ((2) $\Rightarrow$ (1)). Suppose given a morphism $\alpha : [m] \to [n]$ in $\triangle$.

As we know from Remark 16, α is a composite of morphisms of the form σ_j and δ_i.

Therefore, $G_\alpha := \alpha^{\mathrm{op}} G$ is a composite of morphisms of the form $\sigma_j^{\mathrm{op}} G = \mathrm{s}_j$ and $\delta_i^{\mathrm{op}} G = \mathrm{d}_i$.

Likewise for H_α.

So the quadrangle

$$\begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & H_n \\ {\scriptstyle G_\alpha}\downarrow & \circlearrowleft & \downarrow{\scriptstyle H_\alpha} \\ G_m & \xrightarrow[\varphi_m]{} & H_m \end{array}$$

commutes as a composite of commutative quadrangles.

Therefore $\varphi = (\varphi_n)_{n \geqslant 0} : G \to H$ is a morphism of simplicial groups.

Ad ((1) $\Rightarrow$ (2)). Assertion (2) is a special case of (1) because of

$$\begin{array}{rcl} \mathrm{d}_i^{G,n} & = & \delta_i^n G \\ \mathrm{d}_i^{H,n} & = & \delta_i^n H \\ \mathrm{s}_j^{G,n} & = & \sigma_j^n G \\ \mathrm{s}_j^{H,n} & = & \sigma_j^n H. \end{array}$$

Cf. Definition 8. □

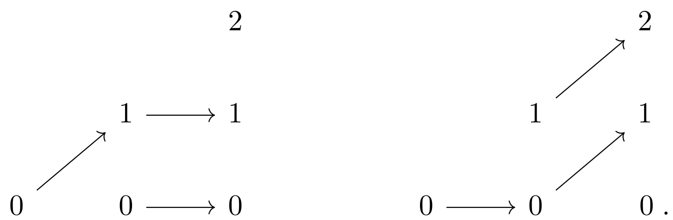
Example 37 We consider the composites

$$\delta_0^1 \blacktriangle \delta_2^2$$

and

$$\delta_1^1 \blacktriangle \delta_0^2$$

and we see that they both map 0 to 1:



So we have

$$\delta_0^1 \blacktriangle \delta_2^2 = \delta_1^1 \blacktriangle \delta_0^2.$$

This means

$$(\delta_2^2)^{\mathrm{op}} \blacktriangle (\delta_0^1)^{\mathrm{op}} = (\delta_0^2)^{\mathrm{op}} \blacktriangle (\delta_1^1)^{\mathrm{op}}.$$

So for a simplicial group $G : \triangle^{\mathrm{op}} \to \mathcal{G}rp$, we obtain

$$\begin{array}{rcl} \mathrm{d}_2 \blacktriangle \mathrm{d}_0 & = & (\delta_2^2)^{\mathrm{op}} G \blacktriangle (\delta_0^1)^{\mathrm{op}} G \\ & = & (\delta_0^2)^{\mathrm{op}} G \blacktriangle (\delta_1^1)^{\mathrm{op}} G \\ & = & \mathrm{d}_0 \blacktriangle \mathrm{d}_1 . \end{array}$$

Remark 38 The face and degeneracy morphisms satisfy the following relations.

Suppose given $n \in \mathbb{Z}_{\geqslant 0}$ and a simplicial group $G : \triangle^{\mathrm{op}} \to \mathcal{G}rp$.

(1) $\mathrm{d}_j^{G,n+1} \blacktriangle \mathrm{d}_i^{G,n} = \mathrm{d}_i^{G,n+1} \blacktriangle \mathrm{d}_{j-1}^{G,n}$ for $i \in [0,n]$, $j \in [1,\, n+1]$ such that $i < j$

(2) $\mathrm{s}_j^{G,n} \blacktriangle \mathrm{d}_i^{G,n+1} = \mathrm{d}_i^{G,n} \blacktriangle \mathrm{s}_{j-1}^{G,n-1}$ for $i \in [0,\, n+1]$, $j \in [1,n]$ such that $i < j$

(3) $\mathrm{s}_j^{G,n} \blacktriangle \mathrm{d}_i^{G,n+1} = \mathrm{id}_{G_n}$ for $i \in [0, n+1]$, $j \in [0,n]$ such that $i \in \{j,\, j+1\}$

(4) $\mathrm{s}_j^{G,n} \blacktriangle \mathrm{d}_i^{G,n+1} = \mathrm{d}_{i-1}^{G,n} \blacktriangle \mathrm{s}_j^{G,n-1}$ for $i \in [1,\, n+1]$, $j \in [0, n]$ such that $i > j+1$

(5) $\mathrm{s}_j^{G,n} \blacktriangle \mathrm{s}_i^{G,n+1} = \mathrm{s}_i^{G,n} \blacktriangle \mathrm{s}_{j+1}^{G,n+1}$ for $i \in [0,\, n+1]$, $j \in [0, n]$ such that $i \leqslant j$

Proof. This follows e.g. from May [**4,** Def. 2.1, Def. 1.1]. □

3.2 $\lfloor 2,0 \rfloor$-simplicial groups

Definition 39 Suppose given groups G_0, G_1 and G_2.

Furthermore, suppose given the following group morphisms.

$$\begin{array}{rcl} \mathrm{d}_0^{G,2}, \mathrm{d}_1^{G,2}, \mathrm{d}_2^{G,2} : G_2 & \to & G_1 \\ \mathrm{s}_0^{G,1}, \mathrm{s}_1^{G,1} : G_1 & \to & G_2 \\ \mathrm{d}_0^{G,1}, \mathrm{d}_1^{G,1} : G_1 & \to & G_0 \\ \mathrm{s}_0^{G,0} : G_0 & \to & G_1 \end{array}$$

We display these data as follows.

$$\begin{array}{ccccc} & \xrightarrow{\mathrm{d}_2^{G,2}} & & & \\ & \xleftarrow{\mathrm{s}_1^{G,1}} & & \xrightarrow{\mathrm{d}_1^{G,1}} & \\ G_2 & \xrightarrow{\mathrm{d}_1^{G,2}} & G_1 & \xleftarrow{\mathrm{s}_0^{G,0}} & G_0 \\ & \xleftarrow{\mathrm{s}_0^{G,1}} & & \xrightarrow{\mathrm{d}_0^{G,1}} & \\ & \xrightarrow{\mathrm{d}_0^{G,2}} & & & \end{array}$$

Suppose that the conditions $(1, 2)$ hold.

(1) We have

$$\begin{array}{rcl} \mathrm{s}_0^{G,0} \blacktriangle \mathrm{d}_0^{G,1} & = & \mathrm{id}_{G_0} \\ \mathrm{s}_0^{G,0} \blacktriangle \mathrm{d}_1^{G,1} & = & \mathrm{id}_{G_0} \\ \mathrm{s}_0^{G,1} \blacktriangle \mathrm{d}_0^{G,2} & = & \mathrm{id}_{G_1} \\ \mathrm{s}_0^{G,1} \blacktriangle \mathrm{d}_1^{G,2} & = & \mathrm{id}_{G_1} \\ \mathrm{s}_1^{G,1} \blacktriangle \mathrm{d}_1^{G,2} & = & \mathrm{id}_{G_1} \\ \mathrm{s}_1^{G,1} \blacktriangle \mathrm{d}_2^{G,2} & = & \mathrm{id}_{G_1} \\ \mathrm{s}_0^{G,1} \blacktriangle \mathrm{d}_2^{G,2} & = & \mathrm{d}_1^{G,1} \blacktriangle \mathrm{s}_0^{G,0} \\ \mathrm{s}_1^{G,1} \blacktriangle \mathrm{d}_0^{G,2} & = & \mathrm{d}_0^{G,1} \blacktriangle \mathrm{s}_0^{G,0} \\ \mathrm{d}_1^{G,2} \blacktriangle \mathrm{d}_0^{G,1} & = & \mathrm{d}_0^{G,2} \blacktriangle \mathrm{d}_0^{G,1} \\ \mathrm{d}_2^{G,2} \blacktriangle \mathrm{d}_0^{G,1} & = & \mathrm{d}_0^{G,2} \blacktriangle \mathrm{d}_1^{G,1} \\ \mathrm{d}_2^{G,2} \blacktriangle \mathrm{d}_1^{G,1} & = & \mathrm{d}_1^{G,2} \blacktriangle \mathrm{d}_1^{G,1} \\ \mathrm{s}_0^{G,0} \blacktriangle \mathrm{s}_0^{G,1} & = & \mathrm{s}_0^{G,0} \blacktriangle \mathrm{s}_1^{G,1}. \end{array}$$

(2) We have

$$\begin{array}{rcl} [\ker \mathrm{d}_0^{G,2},\, \ker \mathrm{d}_1^{G,2} \cap \ker \mathrm{d}_2^{G,2}] & = & 1 \\ [\ker \mathrm{d}_1^{G,2},\, \ker \mathrm{d}_0^{G,2} \cap \ker \mathrm{d}_2^{G,2}] & = & 1 \\ [\ker \mathrm{d}_2^{G,2},\, \ker \mathrm{d}_0^{G,2} \cap \ker \mathrm{d}_1^{G,2}] & = & 1. \end{array}$$

Then we call

$$G := (G_2,\, G_1,\, G_0,\, \mathrm{d}_0^{G,2},\, \mathrm{d}_1^{G,2},\, \mathrm{d}_2^{G,2},\, \mathrm{s}_0^{G,1},\, \mathrm{s}_1^{G,1},\, \mathrm{d}_0^{G,1},\, \mathrm{d}_1^{G,1},\, \mathrm{s}_0^{G,0})$$

a $\lfloor 2,0\rfloor$-*simplicial group*.

Often, we write the *face morphisms* $\mathrm{d}_i := \mathrm{d}_i^{G,n}$ for $n \in [1,2]$ and $i \in [0,n]$ and the *degeneracy morphisms* $\mathrm{s}_i := \mathrm{s}_i^{G,n}$ for $n \in [0,1]$ and $i \in [0,\, n]$.

Definition 40 Suppose given $\lfloor 2,0\rfloor$-simplicial groups G and H and a tuple of group morphism $\varphi = (\varphi_n)_{n\in\lfloor 2,0\rfloor}$.

Then φ is called a *morphism of* $\lfloor 2,0\rfloor$*-simplicial groups* if

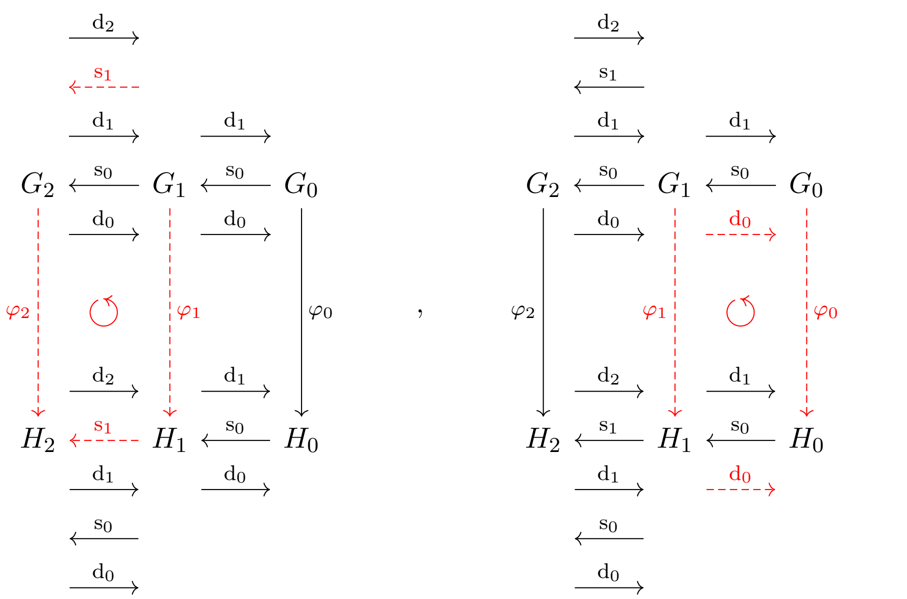
$$\begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & H_n \\ {\scriptstyle \mathrm{d}_i^{G,n}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mathrm{d}_i^{H,n}} \\ G_{n-1} & \xrightarrow[\varphi_{n-1}]{} & H_{n-1} \end{array}$$

for $n \in [1,2]$, $i \in [0,n]$ and

$$\begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & H_n \\ {\scriptstyle \mathrm{s}_j^{G,n}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mathrm{s}_j^{H,n}} \\ G_{n+1} & \xrightarrow[\varphi_{n+1}]{} & H_{n+1} \end{array}$$

for $n \in [0,1]$, $j \in [0,n]$.

Remark 41 By Definition 40, the morphism $\varphi = (\varphi_2,\, \varphi_1,\, \varphi_0) : G \to H$ of $\lfloor 2,0\rfloor$-simplicial groups satisfies several commutative quadrangles, such as the following ones, highlighted in dashed arrows.



, etc.

Remark 42 Suppose given morphisms of $\lfloor 2,0\rfloor$-simplicial groups $\varphi = (\varphi_n)_{n\in\lfloor 2,0\rfloor} : G \to H$ and $\varphi' = (\varphi'_n)_{n\in\lfloor 2,0\rfloor} : H \to K$.

Then the composite

$$\varphi \blacktriangle \varphi' := (\varphi_n \blacktriangle \varphi'_n)_{n\in\lfloor 2,0\rfloor} : G \to K$$

is also a morphism of $\lfloor 2,0\rfloor$-simplicial groups.

Remark 43 Suppose given a $\lfloor 2,0 \rfloor$-simplicial group G.

Then its identity, given by

$$\mathrm{id}_G := (\mathrm{id}_{G_2}, \mathrm{id}_{G_1}, \mathrm{id}_{G_0}) : G \to G,$$

is a morphism of $\lfloor 2,0 \rfloor$-simplicial groups.

Definition 44 We have the *category of* $\lfloor 2,0 \rfloor$*-simplicial groups*, written$\lfloor 2,0 \rfloor$-*SimpGrp*.

It has $\lfloor 2,0 \rfloor$-simplicial groups as objects and $\lfloor 2,0 \rfloor$-simplicial group morphisms as morphisms; cf. Definition 39 and Definition 40.

The composite of morphisms is described in Remark 42.

The identity on an object is described in Remark 43.

3.3 From simplicial groups to $\lfloor 2,0 \rfloor$-simplicial groups

3.3.1 The construction for the object

Some of the arguments in this § 3.3.1 I have learned from **[6]**.

Suppose given a simplicial group G.

Definition 45 Let $G\mathrm{N}_k := \bigcap\limits_{i \in [1,k]} \ker \mathrm{d}_i^{G,k} \trianglelefteq G_k$, for $k \geqslant 0$.

So e.g. $G\mathrm{N}_0 = G_0$ and $G\mathrm{N}_1 = \ker \mathrm{d}_1^{G,1}$.

Note that for $g \in G\mathrm{N}_k$ and $j \in [1,\, k-1]$, we have

$$\begin{array}{rcl} g\,\mathrm{d}_0\,\mathrm{d}_j & = & g\,\mathrm{d}_{j+1}\,\mathrm{d}_0 \\ & = & 1 \cdot \mathrm{d}_0 \\ & = & 1 \end{array}$$

So $g\,\mathrm{d}_0 \in G\mathrm{N}_{k-1}$.

Let $\delta_k = \delta_k^{G\mathrm{N}} := \mathrm{d}_0 \,|_{G\mathrm{N}_k}^{G\mathrm{N}_{k-1}}$. Then

$$\begin{array}{ccccccc} \ldots \to G\mathrm{N}_{k+1} & \xrightarrow{\delta_{k+1}} & G\mathrm{N}_k \xrightarrow{\delta_k} G\mathrm{N}_{k-1} \xrightarrow{\delta_{k-1}} \ldots \xrightarrow{\delta_1} G\mathrm{N}_0 \to 1 \to \ldots \\ x & \longmapsto & x\delta_{k+1} \end{array}$$

is defined to be the *Moore complex* of G.

We have

$$\begin{array}{rcl} x\delta_{k+1}\delta_k & = & x\,\mathrm{d}_0\,\mathrm{d}_0 \\ & = & x\,\mathrm{d}_1\,\mathrm{d}_0 \\ & = & 1. \end{array}$$

for $x \in G\mathrm{N}_{k+1}$, so that $G\mathrm{N}$ is in fact a complex of groups.

Remark 46 Suppose given $g \in G\mathrm{N}_3$ and $x \in G_2$.

Then $(g\,\mathrm{d}_0)^x$ is an element of $(G\mathrm{N}_3)\delta_3$.

So $G\mathrm{NB}_2 := (G\mathrm{N}_3)\delta_3 \trianglelefteq G_2$.

Proof. We have

$$\begin{array}{rcl} (g\,\mathrm{d}_0)^x & = & (g\,\mathrm{d}_0)^{x\,\mathrm{s}_0\,\mathrm{d}_0} \\ & = & (x\,\mathrm{s}_0\,\mathrm{d}_0)^- \cdot (g\,\mathrm{d}_0) \cdot (x\,\mathrm{s}_0\,\mathrm{d}_0) \\ & = & (x^-\,\mathrm{s}_0\,\mathrm{d}_0) \cdot (g\,\mathrm{d}_0) \cdot (x\,\mathrm{s}_0\,\mathrm{d}_0) \\ & = & ((x^-\,\mathrm{s}_0) \cdot g \cdot (x\,\mathrm{s}_0))\,\mathrm{d}_0 \\ & = & ((x\,\mathrm{s}_0)^- \cdot g \cdot (x\,\mathrm{s}_0))\,\mathrm{d}_0 \\ & = & (g^{x\,\mathrm{s}_0})\,\mathrm{d}_0\,. \end{array}$$

We have $(g^{x\,\mathrm{s}_0})\,\mathrm{d}_0 \in (G\mathrm{N}_3)\,\mathrm{d}_0$ because, for $j \in [1,3]$, we have

$$\begin{array}{rcl} (g^{x\,\mathrm{s}_0})\,\mathrm{d}_j &=& ((x\,\mathrm{s}_0)^- \cdot g \cdot (x\,\mathrm{s}_0))\,\mathrm{d}_j \\ &=& (x\,\mathrm{s}_0\,\mathrm{d}_j)^- \cdot (g\,\mathrm{d}_j) \cdot (x\,\mathrm{s}_0\,\mathrm{d}_j) \\ &=& (x s_0\,\mathrm{d}_j)^- \cdot 1 \cdot (x\,\mathrm{s}_0\,\mathrm{d}_j) \\ &=& 1. \end{array}$$

□

Remark 47 Suppose given $i \in [0,2]$.
Let

$$\begin{array}{rcl} \check{\mathrm{d}}_i^{G,2} : G_2/G\mathrm{NB}_2 &\longrightarrow& G_1 \\ gG\mathrm{NB}_2 &\longmapsto& (gG\mathrm{NB}_2)\check{\mathrm{d}}_i^{G,2} := g\,\mathrm{d}_i^{G,2}\,. \end{array}$$

This is a well-defined group morphism since $(G\mathrm{NB}_2)\,\mathrm{d}_i^{G,2} = 1$.

In fact, given $g \in G\mathrm{N}_3$ and thus $g\delta_3 = g\,\mathrm{d}_0^{G,3} \in G\mathrm{NB}_2$, we obtain

$$\begin{array}{rcl} (g\,\mathrm{d}_0^{G,3})\,\mathrm{d}_i^{G,2} &=& g\,\mathrm{d}_{i+1}^{G,3}\,\mathrm{d}_0^{G,2} \\ &=& 1\,\mathrm{d}_0^{G,2} \\ &=& 1. \end{array}$$

Note that $\rho^G \blacktriangle \check{\mathrm{d}}_i^{G,2} = \mathrm{d}_i^{G,2}$, where $\rho^G : G_2 \to G_2/G\mathrm{NB}_2$ is the residue class morphism.

Definition 48 We write

$$G_{3;X} := \bigcap_{i \in X} \ker \mathrm{d}_i^{G_3} \trianglelefteq G_3$$

for $X \subseteq \{0,\ 1,\ 2,\ 3\}$.

We usually omit the set-braces of X when used as an index.

For example we write $G_{3;1} = G_{3;\{1\}}$, $G_{3;1,2,3} = G_{3;\{1,2,3\}}$, etc.

Note that for $Y \subseteq X \subseteq \{0,\ 1,\ 2,\ 3\}$, we have $G_{3,X} \trianglelefteq G_{3,Y}$.

Remark 49 We have

$$\begin{array}{rcl} G\mathrm{NB}_2 &=& (G_{3;1,2,3})\,\mathrm{d}_0^{G,3} \\ &=& (G_{3;0,2,3})\,\mathrm{d}_1^{G,3} \\ &=& (G_{3;0,1,3})\,\mathrm{d}_2^{G,3}\,. \end{array}$$

Proof. We shall make frequent use of Remark 38 and Definition 48.

We write $K_0 := (G_{3;1,2,3})\,\mathrm{d}_0$, $K_1 := (G_{3;0,2,3})\,\mathrm{d}_1$ and $K_2 := (G_{3;0,1,3})\,\mathrm{d}_2$.

We show $K_0 \overset{!}{\leqslant} K_1$.

Suppose given $a \in G_{3;1,2,3} \leqslant G_3$.

So we have $a\,\mathrm{d}_0 \in (G_{3;1,2,3})\,\mathrm{d}_0 = K_0 \leqslant G_2$.

Let $z := a^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0 \in G_3$.

We have

$$\begin{array}{l} z\,\mathrm{d}_0 = (a^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0)\,\mathrm{d}_0 = (a\,\mathrm{d}_0)^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{d}_0 = (a\,\mathrm{d}_0)^- \cdot a\,\mathrm{d}_0 = 1 \\ z\,\mathrm{d}_2 = (a^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0)\,\mathrm{d}_2 = (a\,\mathrm{d}_2)^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{d}_2 = 1 \cdot a\,\mathrm{d}_0\,\mathrm{d}_1\,\mathrm{s}_0 = a\,\mathrm{d}_2\,\mathrm{d}_0\,\mathrm{s}_0 = 1\,\mathrm{d}_0\,\mathrm{s}_0 = 1 \\ z\,\mathrm{d}_3 = (a^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0)\,\mathrm{d}_3 = (a\,\mathrm{d}_3)^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{d}_3 = 1 \cdot a\,\mathrm{d}_0\,\mathrm{d}_2\,\mathrm{s}_0 = a\,\mathrm{d}_3\,\mathrm{d}_0\,\mathrm{s}_0 = 1\,\mathrm{d}_0\,\mathrm{s}_0 = 1. \end{array}$$

So $z \in G_{3;0,2,3}$.

And we also have

$$\begin{aligned} z\,\mathrm{d}_1 &= (a^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0)\,\mathrm{d}_1 \\ &= (a\,\mathrm{d}_1)^- \cdot a\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{d}_1 \\ &= a\,\mathrm{d}_0\,. \end{aligned}$$

This shows $a\,\mathrm{d}_0 = z\,\mathrm{d}_1 \in (G_{3;0,2,3})\,\mathrm{d}_1 = K_1$.

So

$$K_0 \leqslant K_1.$$

We show $K_0 \overset{!}{\geqslant} K_1$.

Suppose given $b \in G_{3;0,2,3} \leqslant G_3$.

So we have $b\,\mathrm{d}_1 \in (G_{3;0,2,3})\,\mathrm{d}_1 = K_1 \leqslant G_2$.

Let $z^* := b^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0 \in G_3$.

We have

$$\begin{aligned} z^*\,\mathrm{d}_1 &= (b^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0)\,\mathrm{d}_1 = (b\,\mathrm{d}_1)^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{d}_1 = (b\,\mathrm{d}_1)^- \cdot b\,\mathrm{d}_1 = 1 \\ z^*\,\mathrm{d}_2 &= (b^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0)\,\mathrm{d}_2 = (b\,\mathrm{d}_2)^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{d}_2 = 1 \cdot b\,\mathrm{d}_1\,\mathrm{d}_1\,\mathrm{s}_0 = b\,\mathrm{d}_2\,\mathrm{d}_1\,\mathrm{s}_0 = 1\,\mathrm{d}_1\,\mathrm{s}_0 = 1 \\ z^*\,\mathrm{d}_3 &= (b^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0)\,\mathrm{d}_3 = (b\,\mathrm{d}_3)^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{d}_3 = 1 \cdot b\,\mathrm{d}_1\,\mathrm{d}_2\,\mathrm{s}_0 = b\,\mathrm{d}_3\,\mathrm{d}_1\,\mathrm{s}_0 = 1\,\mathrm{d}_1\,\mathrm{s}_0 = 1. \end{aligned}$$

So $z^* \in G_{3;1,2,3}$.

And we also have

$$\begin{aligned} z^*\,\mathrm{d}_0 &= (b^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0)\,\mathrm{d}_0 \\ &= (b\,\mathrm{d}_0)^- \cdot b\,\mathrm{d}_1\,\mathrm{s}_0\,\mathrm{d}_0 \\ &= b\,\mathrm{d}_1\,. \end{aligned}$$

This shows $b\,\mathrm{d}_1 = z^*\,\mathrm{d}_0 \in (G_{3;1,2,3})\,\mathrm{d}_0 = K_0$.

So

$$K_0 \geqslant K_1.$$

We show $K_1 \overset{!}{\leqslant} K_2$.

Suppose given $c \in G_{3;0,2,3} \leqslant G_3$.

So we have $c\,\mathrm{d}_1 \in (G_{3;0,2,3})\,\mathrm{d}_1 = K_1 \leqslant G_2$.

Let $z^{**} := c^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1 \in G_3$.

We have

$$\begin{aligned} z^{**}\,\mathrm{d}_0 &= (c^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1)\,\mathrm{d}_0 = (c\,\mathrm{d}_0)^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1\,\mathrm{d}_0 = 1 \cdot c\,\mathrm{d}_1\,\mathrm{d}_0\,\mathrm{s}_0 = c\,\mathrm{d}_0\,\mathrm{d}_0\,\mathrm{s}_0 = 1\,\mathrm{d}_0\,\mathrm{s}_0 = 1 \\ z^{**}\,\mathrm{d}_1 &= (c^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1)\,\mathrm{d}_1 = (c\,\mathrm{d}_1)^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1\,\mathrm{d}_1 = (c\,\mathrm{d}_1)^- \cdot c\,\mathrm{d}_1 = 1 \\ z^{**}\,\mathrm{d}_3 &= (c^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1)\,\mathrm{d}_3 = (c\,\mathrm{d}_3)^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1\,\mathrm{d}_3 = 1 \cdot c\,\mathrm{d}_1\,\mathrm{d}_2\,\mathrm{s}_1 = c\,\mathrm{d}_3\,\mathrm{d}_1\,\mathrm{s}_1 = 1\,\mathrm{d}_1\,\mathrm{s}_1 = 1. \end{aligned}$$

So $z^{**} \in G_{3;0,1,3}$.

And we also have

$$\begin{aligned} z^{**}\,\mathrm{d}_2 &= (c^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1)\,\mathrm{d}_2 \\ &= (c\,\mathrm{d}_2)^- \cdot c\,\mathrm{d}_1\,\mathrm{s}_1\,\mathrm{d}_2 \\ &= c\,\mathrm{d}_1\,. \end{aligned}$$

This shows $c\,\mathrm{d}_1 = z^{**}\,\mathrm{d}_2 \in (G_{3;0,1,3})\,\mathrm{d}_2 = K_2$.

So

$$K_1 \leqslant K_2.$$

We show $K_1 \overset{!}{\geqslant} K_2$.

Suppose given $d \in G_{3;0,1,3} \leqslant G_3$.

So we have $d\,\mathrm{d}_2 \in (G_{3;0,1,3})\,\mathrm{d}_2 = K_2 \leqslant G_2$.

Let $z^{***} := d^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1 \in G_3$.

We have

$$\begin{aligned} z^{***}\,\mathrm{d}_0 &= (d^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1)\,\mathrm{d}_0 = (d\,\mathrm{d}_0)^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1\,\mathrm{d}_0 = 1 \cdot d\,\mathrm{d}_2\,\mathrm{d}_0\,\mathrm{s}_0 = d\,\mathrm{d}_0\,\mathrm{d}_1\,\mathrm{s}_0 = 1\,\mathrm{d}_1\,\mathrm{s}_0 = 1 \\ z^{***}\,\mathrm{d}_2 &= (d^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1)\,\mathrm{d}_2 = (d\,\mathrm{d}_2)^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1\,\mathrm{d}_2 = (d\,\mathrm{d}_2)^- \cdot d\,\mathrm{d}_2 = 1 \\ z^{***}\,\mathrm{d}_3 &= (d^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1)\,\mathrm{d}_3 = (d\,\mathrm{d}_3)^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1\,\mathrm{d}_3 = 1 \cdot d\,\mathrm{d}_2\,\mathrm{d}_2\,\mathrm{s}_1 = d\,\mathrm{d}_3\,\mathrm{d}_2\,\mathrm{s}_1 = 1\,\mathrm{d}_2\,\mathrm{s}_1 = 1. \end{aligned}$$

So $z^{***} \in G_{3;0,2,3}$.

And we also have

$$\begin{aligned} z^{***}\,\mathrm{d}_1 &= (d^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1)\,\mathrm{d}_1 \\ &= (d\,\mathrm{d}_1)^- \cdot d\,\mathrm{d}_2\,\mathrm{s}_1\,\mathrm{d}_1 \\ &= d\,\mathrm{d}_2\,. \end{aligned}$$

This shows $d\,\mathrm{d}_2 = z^{***}\,\mathrm{d}_1 \in (G_{3;0,2,3})\,\mathrm{d}_1 = K_1$.

So

$$K_1 \geqslant K_2.$$

Altogether, we have

$$K_0 = K_1 = K_2.$$

□

Remark 50 The following equations hold in $G_2/G\mathrm{NB}_2$.

(1) $[\ker \check{\mathrm{d}}_0^{G,2},\, \ker \check{\mathrm{d}}_1^{G,2} \cap \ker \check{\mathrm{d}}_2^{G,2}] = 1$

(2) $[\ker \check{\mathrm{d}}_1^{G,2},\, \ker \check{\mathrm{d}}_0^{G,2} \cap \ker \check{\mathrm{d}}_2^{G,2}] = 1$

(3) $[\ker \check{\mathrm{d}}_2^{G,2},\, \ker \check{\mathrm{d}}_0^{G,2} \cap \ker \check{\mathrm{d}}_1^{G,2}] = 1$

Proof.

Ad (1). Suppose given $x = gG\mathrm{NB}_2 \in G_2/G\mathrm{NB}_2$ with $1 = x\check{\mathrm{d}}_0^{G,2} = (gG\mathrm{NB}_2)\check{\mathrm{d}}_0^{G,2} = g\,\mathrm{d}_0^{G,2}$, where $g \in G_2$.

Suppose given $y = hG\mathrm{NB}_2 \in G_2/G\mathrm{NB}_2$ with $1 = y\check{\mathrm{d}}_1^{G,2} = (hG\mathrm{NB}_2)\check{\mathrm{d}}_1^{G,2} = h\,\mathrm{d}_1^{G,2}$ and with $1 = y\check{\mathrm{d}}_2^{G,2} = (hG\mathrm{NB}_2)\check{\mathrm{d}}_2^{G,2} = h\,\mathrm{d}_2^{G,2}$, where $h \in G_2$.

We have to show that $[x, y] = [gG\mathrm{NB}_2,\, hG\mathrm{NB}_2] = [g, h]G\mathrm{NB}_2 \overset{!}{=} 1_{G_2} \cdot G\mathrm{NB}_2$.

This means that we want to show that $[g, h] \overset{!}{\in} G\mathrm{NB}_2 = G\mathrm{N}_3\,\mathrm{d}_0 = (G_{3;1,2,3})\,\mathrm{d}_0 = (G_{3;0,2,3})\,\mathrm{d}_1$; cf. Remark 49.

Let $c := [g\,\mathrm{s}_1,\, h\,\mathrm{s}_0] \in G_3$. We have

$$\begin{aligned} c\,\mathrm{d}_0 &= [g\,\mathrm{s}_1,\, h\,\mathrm{s}_0]\,\mathrm{d}_0 = [g\,\mathrm{s}_1\,\mathrm{d}_0,\, h\,\mathrm{s}_0\,\mathrm{d}_0] = [g\,\mathrm{d}_0\,\mathrm{s}_0,\, h] = [1\,\mathrm{s}_0,\, h] = 1 \\ c\,\mathrm{d}_2 &= [g\,\mathrm{s}_1,\, h\,\mathrm{s}_0]\,\mathrm{d}_2 = [g\,\mathrm{s}_1\,\mathrm{d}_2,\, h\,\mathrm{s}_0\,\mathrm{d}_2] = [g,\, h\,\mathrm{d}_1\,\mathrm{s}_0] = [g,\, 1\,\mathrm{s}_0] = 1 \\ c\,\mathrm{d}_3 &= [g\,\mathrm{s}_1,\, h\,\mathrm{s}_0]\,\mathrm{d}_3 = [g\,\mathrm{s}_1\,\mathrm{d}_3,\, h\,\mathrm{s}_0\,\mathrm{d}_3] = [g\,\mathrm{d}_2\,\mathrm{s}_1,\, h\,\mathrm{d}_2\,\mathrm{s}_0] = [g\,\mathrm{d}_2\,\mathrm{s}_1,\, 1\,\mathrm{s}_0] = 1. \end{aligned}$$

So $c \in G_{3;0,2,3}$.

And we also have

$$\begin{aligned} c\,\mathrm{d}_1 &= [g\,\mathrm{s}_1,\, h\,\mathrm{s}_0]\,\mathrm{d}_1 \\ &= [g\,\mathrm{s}_1\,\mathrm{d}_1,\, h\,\mathrm{s}_0\,\mathrm{d}_1] \\ &= [g, h]. \end{aligned}$$

This shows $[g, h] = c\,\mathrm{d}_1 \in (G_{3;0,2,3})\,\mathrm{d}_1$.

Ad (2). Suppose given $x = g\mathrm{GNB}_2 \in G_2/G\mathrm{NB}_2$ with $1 = x\check{\mathrm{d}}_1^{G,2} = (gG\mathrm{NB}_2)\check{\mathrm{d}}_1^{G,2} = g\,\mathrm{d}_1^{G,2}$, where $g \in G_2$.

Suppose given $y = hG\mathrm{NB}_2 \in G_2/G\mathrm{NB}_2$ with $1 = y\check{\mathrm{d}}_0^{G,2} = (hG\mathrm{NB}_2)\check{\mathrm{d}}_0^{G,2} = h\,\mathrm{d}_0^{G,2}$ and with $1 = y\check{\mathrm{d}}_2^{G,2} = (hG\mathrm{NB}_2)\check{\mathrm{d}}_2^{G,2} = h\,\mathrm{d}_2^{G,2}$, where $h \in G_2$.

We have to show that $[x,y] = [gG\mathrm{NB}_2,\, hG\mathrm{NB}_2] = [g,h]G\mathrm{NB}_2 \overset{!}{=} 1_{G_2} \cdot G\mathrm{NB}_2$.

This means that we want to show that $[g,h] \overset{!}{\in} G\mathrm{NB}_2 = (G_{3;1,2,3})\,\mathrm{d}_0 = (G_{3;0,2,3})\,\mathrm{d}_1$; cf. Remark 49.

Let $c := [g\,\mathrm{s}_0,\, h\,\mathrm{s}_1] \in G_3$. We have

$$\begin{aligned} c\,\mathrm{d}_0 &= [g\,\mathrm{s}_0,\, h\,\mathrm{s}_1]\,\mathrm{d}_0 = [g\,\mathrm{s}_0\,\mathrm{d}_0,\, h\,\mathrm{s}_1\,\mathrm{d}_0] = [g,\, h\,\mathrm{d}_0\,\mathrm{s}_0] = [g,\, 1\,\mathrm{s}_0] = 1 \\ c\,\mathrm{d}_2 &= [g\,\mathrm{s}_0,\, h\,\mathrm{s}_1]\,\mathrm{d}_2 = [g\,\mathrm{s}_0\,\mathrm{d}_2,\, h\,\mathrm{s}_1\,\mathrm{d}_2] = [g\,\mathrm{d}_1\,\mathrm{s}_0,\, h] = [1\,\mathrm{s}_0,\, h] = 1 \\ c\,\mathrm{d}_3 &= [g\,\mathrm{s}_0,\, h\,\mathrm{s}_1]\,\mathrm{d}_3 = [g\,\mathrm{s}_0\,\mathrm{d}_3,\, h\,\mathrm{s}_1\,\mathrm{d}_3] = [g\,\mathrm{d}_2\,\mathrm{s}_0,\, h\,\mathrm{d}_2\,\mathrm{s}_1] = [g\,\mathrm{d}_2\,\mathrm{s}_1,\, 1\,\mathrm{s}_1] = 1. \end{aligned}$$

So $c \in G_{3;1,2,3}$.

And we also have

$$\begin{aligned} c\,\mathrm{d}_1 &= [g\,\mathrm{s}_0,\, h\,\mathrm{s}_1]\,\mathrm{d}_1 \\ &= [g\,\mathrm{s}_0\,\mathrm{d}_1,\, h\,\mathrm{s}_1\,\mathrm{d}_1] \\ &= [g,h]. \end{aligned}$$

This shows $[g,h] = c\,\mathrm{d}_1 \in (G_{3;0,2,3})\,\mathrm{d}_1$.

Ad (3). Suppose given $x = gG\mathrm{NB}_2 \in G_2/G\mathrm{NB}_2$ with $1 = x\check{\mathrm{d}}_2^{G,2} = (gG\mathrm{NB}_2)\check{\mathrm{d}}_2^{G,2} = g\,\mathrm{d}_2^{G,2}$, where $g \in G_2$.

Suppose given $y = hG\mathrm{NB}_2 \in G_2/G\mathrm{NB}_2$ with $1 = y\check{\mathrm{d}}_0^{G,2} = (hG\mathrm{NB}_2)\check{\mathrm{d}}_0^{G,2} = h\,\mathrm{d}_0^{G,2}$ and with $1 = y\check{\mathrm{d}}_1^{G,2} = (hG\mathrm{NB}_2)\check{\mathrm{d}}_1^{G,2} = h\,\mathrm{d}_1^{G,2}$, where $h \in G_2$.

We have to show that $[x,y] = [gG\mathrm{NB}_2,\, hG\mathrm{NB}_2] = [g,h]G\mathrm{NB}_2 \overset{!}{=} 1_{G_2} \cdot G\mathrm{NB}_2$.

This means that we want to show that $[g,h] \overset{!}{\in} G\mathrm{NB}_2 = (G_{3;1,2,3})\,\mathrm{d}_0 = (G_{3;0,1,3})\,\mathrm{d}_2$; cf. Remark 49.

Let $c := [g\,\mathrm{s}_1,\, h\,\mathrm{s}_2] \in G_3$. We have

$$\begin{aligned} c\,\mathrm{d}_0 &= [g\,\mathrm{s}_1,\, h\,\mathrm{s}_2]\,\mathrm{d}_0 = [g\,\mathrm{s}_1\,\mathrm{d}_0,\, h\,\mathrm{s}_2\,\mathrm{d}_0] = [g\,\mathrm{d}_0\,\mathrm{s}_0,\, h\,\mathrm{d}_0\,\mathrm{s}_1] = [g\,\mathrm{d}_0\,\mathrm{s}_0,\, 1\,\mathrm{s}_1] = 1 \\ c\,\mathrm{d}_1 &= [g\,\mathrm{s}_1,\, h\,\mathrm{s}_2]\,\mathrm{d}_1 = [g\,\mathrm{s}_1\,\mathrm{d}_1,\, h\,\mathrm{s}_2\,\mathrm{d}_1] = [g,\, h\,\mathrm{d}_1\,\mathrm{s}_1] = [g,\, 1\,\mathrm{s}_1] = 1 \\ c\,\mathrm{d}_3 &= [g\,\mathrm{s}_1,\, h\,\mathrm{s}_2]\,\mathrm{d}_3 = [g\,\mathrm{s}_1\,\mathrm{d}_3,\, h\,\mathrm{s}_2\,\mathrm{d}_3] = [g\,\mathrm{d}_2\,\mathrm{s}_1,\, h] = [1\,\mathrm{s}_1,\, h] = 1. \end{aligned}$$

So $c \in G_{3;0,1,3}$.

And we also have

$$\begin{aligned} c\,\mathrm{d}_2 &= [g\,\mathrm{s}_1,\, h\,\mathrm{s}_2]\,\mathrm{d}_2 \\ &= [g\,\mathrm{s}_1\,\mathrm{d}_2,\, h\,\mathrm{s}_2\,\mathrm{d}_2] \\ &= [g,h]. \end{aligned}$$

This shows $[g,h] = c\,\mathrm{d}_2 \in (G_{3;0,1,3})\,\mathrm{d}_2$. □

3.3.2 The construction for the morphism

Remark 51 Suppose given a morphism of simplicial groups $\varphi : G \to H$.

Consider the residue class morphisms

$$\begin{aligned} \rho^G : G_2 &\longrightarrow G_2/G\mathrm{NB}_2 \\ g &\longmapsto gG\mathrm{NB}_2 \end{aligned}$$

and

$$\begin{array}{rcl} \rho^H : H_2 & \longrightarrow & H_2/H\mathrm{NB}_2 \\ h & \longmapsto & hH\mathrm{NB}_2. \end{array}$$

Consider the diagram

$$\begin{array}{ccc} G_2 & \xrightarrow{\varphi_2} & H_2 \\ {\scriptstyle \rho^G}\downarrow & \searrow^{\varphi_2 \blacktriangle \rho^H} & \downarrow{\scriptstyle \rho^H} \\ G_2/G\mathrm{NB}_2 & & H_2/H\mathrm{NB}_2\ . \end{array}$$

We have

$$(G\mathrm{NB}_2)(\varphi_2 \blacktriangle \rho^G) = 1,$$

since given $g \in G\mathrm{N}_3$, we obtain, for $j \in [1,3]$,

$$\begin{array}{rcl} (g\varphi_3)\,\mathrm{d}_j & = & g\,\mathrm{d}_j\,\varphi_2 \\ & = & 1 \cdot \varphi_2 \\ & = & 1, \end{array}$$

i.e. $g\varphi_3 \in H\mathrm{N}_3$, hence $g\varphi_3\delta_3 \in H\mathrm{NB}_2$ and thus

$$\begin{array}{rcl} (g\delta_3)(\varphi_2 \blacktriangle \rho^H) & = & g\,\mathrm{d}_0\,\varphi_2\rho^H \\ & = & g\varphi_3\,\mathrm{d}_0\,\rho^H \\ & = & g\varphi_3\delta_3\rho^H \\ & = & 1. \end{array}$$

Therefore, there exists a unique group morphism $\bar{\varphi}_2 : G_2/G\mathrm{NB}_2 \to H_2/H\mathrm{NB}_2$ such that

$$\begin{array}{ccc} G_2 & \xrightarrow{\varphi_2} & H_2 \\ {\scriptstyle \rho^G}\downarrow & \searrow^{\varphi_2 \blacktriangle \rho^H} & \downarrow{\scriptstyle \rho^H} \\ G_2/G\mathrm{NB}_2 & \xrightarrow[\bar{\varphi}_2]{} & H_2/H\mathrm{NB}_2 \end{array}$$

commutes.

In particular, for $g \in G_2$ we have $(gG\mathrm{NB}_2)\bar{\varphi}_2 = (g\varphi_2)H\mathrm{NB}_2$.

3.3.3 The truncation functor

Definition 52 We shall define the following functor.

$$\begin{array}{rcl} \mathrm{Trunc} := \mathrm{Trunc}_{\lfloor 2,0 \rfloor} : \mathit{SimpGrp} & \longrightarrow & \lfloor 2,0 \rfloor\text{-}\mathit{SimpGrp} \\ \left(\begin{array}{c} G \\ \downarrow \varphi \\ H \end{array} \right) & \longmapsto & \left(\begin{array}{c} G\,\mathrm{Trunc} \\ \downarrow \varphi\,\mathrm{Trunc} \\ H\,\mathrm{Trunc} \end{array} \right) \end{array}$$

(1) Suppose given a simplicial group G. Let

$$\begin{array}{rcl} (G\,\mathrm{Trunc})_0 & := & G_0 \\ (G\,\mathrm{Trunc})_1 & := & G_1 \\ (G\,\mathrm{Trunc})_2 & := & G_2/G\mathrm{NB}_2. \end{array}$$

(a) Let $\mathrm{d}_i^{G\,\mathrm{Trunc},1} := \mathrm{d}_i^{G,1} : (G\,\mathrm{Trunc})_1 \to (G\,\mathrm{Trunc})_0$ for $i \in [0,1]$.

(b) Let $\mathrm{s}_0^{G\,\mathrm{Trunc},0} := \mathrm{s}_0^{G,0} : (G\,\mathrm{Trunc})_0 \to (G\,\mathrm{Trunc})_1$.

(c) Suppose given $i \in [0,2]$.

Let

$$\begin{array}{rcl} \mathrm{d}_i^{G\,\mathrm{Trunc},2} := \check{\mathrm{d}}_i^{G,2} : (G\,\mathrm{Trunc})_2 & \longrightarrow & (G\,\mathrm{Trunc})_1 \\ gG\mathrm{NB}_2 & \longmapsto & (gG\mathrm{NB}_2)\,\mathrm{d}_i^{G\,\mathrm{Trunc},2} := g\,\mathrm{d}_i^{G,2}. \end{array}$$

Cf. Remark 47.

(d) Suppose given $j \in [0,1]$.

Let

$$\begin{array}{rcl} \mathrm{s}_j^{G\,\mathrm{Trunc},1} := \mathrm{s}_j^{G,1} \blacktriangle \rho^G : (G\,\mathrm{Trunc})_1 & \longrightarrow & (G\,\mathrm{Trunc})_2 \\ g & \longmapsto & g\,\mathrm{s}_j^{G\,\mathrm{Trunc},1} := (g\,\mathrm{s}_j^{G,1})G\mathrm{NB}_2. \end{array}$$

(e) We verify condition (1) from Definition 39 for the face and degeneracy morphisms of $G\,\mathrm{Trunc}$.

(i) Suppose given $0 \leqslant i < j \leqslant 2$.

We have

$$\begin{array}{rcl} \rho^G \blacktriangle \mathrm{d}_j^{G\,\mathrm{Trunc},2} \blacktriangle \mathrm{d}_i^{G\,\mathrm{Trunc},1} & = & \rho^G \blacktriangle \check{\mathrm{d}}_j^{G,2} \blacktriangle \mathrm{d}_i^{G,1} \\ & = & \mathrm{d}_j^{G,2} \blacktriangle \mathrm{d}_i^{G,1} \\ & = & \mathrm{d}_i^{G,2} \blacktriangle \mathrm{d}_{j-1}^{G,1} \\ & = & \rho^G \blacktriangle \check{\mathrm{d}}_i^{G,2} \blacktriangle \mathrm{d}_{j-1}^{G,1} \\ & = & \rho^G \blacktriangle \mathrm{d}_i^{G\,\mathrm{Trunc},2} \blacktriangle \mathrm{d}_{j-1}^{G\,\mathrm{Trunc},1}. \end{array}$$

Because of the surjectivity of ρ^G, we get

$$\mathrm{d}_j^{G\,\mathrm{Trunc},2} \blacktriangle \mathrm{d}_i^{G\,\mathrm{Trunc},1} = \mathrm{d}_i^{G\,\mathrm{Trunc},2} \blacktriangle \mathrm{d}_{j-1}^{G\,\mathrm{Trunc},1}.$$

(ii) We have

$$\begin{array}{rcl} \mathrm{s}_0^{G\,\mathrm{Trunc},0} \blacktriangle \mathrm{s}_0^{G\,\mathrm{Trunc},1} & = & \mathrm{s}_0^{G,0} \blacktriangle \mathrm{s}_0^{G,1} \blacktriangle \rho^G \\ & = & \mathrm{s}_0^{G,0} \blacktriangle \mathrm{s}_1^{G,1} \blacktriangle \rho^G \\ & = & \mathrm{s}_0^{G\,\mathrm{Trunc},0} \blacktriangle \mathrm{s}_1^{G\,\mathrm{Trunc},1}. \end{array}$$

(iii) Suppose given $i \in [0, 2]$ and $j \in [0, 1]$.

We have

$$\begin{aligned} \mathrm{s}_j^{G\,\mathrm{Trunc},1} \blacktriangle \mathrm{d}_i^{G\,\mathrm{Trunc},2} &= \mathrm{s}_j^{G,1} \blacktriangle \rho^G \blacktriangle \check{\mathrm{d}}_i^{G,2} \\ &= \mathrm{s}_j^{G,1} \blacktriangle \mathrm{d}_i^{G,2} . \end{aligned}$$

So

$$\begin{aligned} \mathrm{s}_0^{G\,\mathrm{Trunc},1} \blacktriangle \mathrm{d}_2^{G\,\mathrm{Trunc},2} &= \mathrm{s}_0^{G,1} \blacktriangle \mathrm{d}_2^{G,2} \\ &= \mathrm{d}_1^{G,1} \blacktriangle \mathrm{s}_0^{G,0} \\ &= \mathrm{d}_1^{G\,\mathrm{Trunc},1} \blacktriangle \mathrm{s}_0^{G\,\mathrm{Trunc},0} \end{aligned}$$

and

$$\begin{aligned} \mathrm{s}_1^{G\,\mathrm{Trunc},1} \blacktriangle \mathrm{d}_0^{G\,\mathrm{Trunc},2} &= \mathrm{s}_1^{G,1} \blacktriangle \mathrm{d}_0^{G,2} \\ &= \mathrm{d}_0^{G,1} \blacktriangle \mathrm{s}_0^{G,0} \\ &= d_0^{G\,\mathrm{Trunc},1} \blacktriangle \mathrm{s}_0^{G\,\mathrm{Trunc},0} . \end{aligned}$$

If $i - j \in \{0, 1\}$, then

$$\begin{aligned} \mathrm{s}_j^{G\,\mathrm{Trunc},1} \blacktriangle \mathrm{d}_i^{G\,\mathrm{Trunc},2} &= \mathrm{s}_j^{G,1} \blacktriangle \mathrm{d}_i^{G,2} \\ &= \mathrm{id}_{G_1} \\ &= \mathrm{id}_{(G\,\mathrm{Trunc})_1} . \end{aligned}$$

(iv) Suppose given $i \in [0, 1]$.

Then we have

$$\begin{aligned} \mathrm{s}_0^{G\,\mathrm{Trunc},0} \blacktriangle \mathrm{d}_i^{G\,\mathrm{Trunc},1} &= \mathrm{s}_0^{G,0} \blacktriangle \mathrm{d}_i^{G,1} \\ &= \mathrm{id}_{G_0} \\ &= \mathrm{id}_{(G\,\mathrm{Trunc})_0} . \end{aligned}$$

Condition (2) of Definition 39 holds by Remark 50.

Because of (e) we may define $G\,\mathrm{Trunc}$ as

$$\begin{array}{ccccc} & \xrightarrow{\check{\mathrm{d}}_2^{G,2}} & & & \\ & \xleftarrow{\mathrm{s}_1^{G,1} \blacktriangle \rho^G} & & \xrightarrow{\mathrm{d}_1^{G,1}} & \\ G_2/G\mathrm{NB}_2 & \xrightarrow{\check{\mathrm{d}}_1^{G,2}} & G_1 & \xleftarrow{\mathrm{s}_0^{G,0}} & G_0. \\ & \xleftarrow{\mathrm{s}_0^{G,1} \blacktriangle \rho^G} & & \xrightarrow{\mathrm{d}_0^{G,1}} & \\ & \xrightarrow{\check{\mathrm{d}}_0^{G,2}} & & & \end{array}$$

(2) Suppose given a morphism of simplicial groups $G \xrightarrow{\varphi} H$.

Let $(\varphi\,\mathrm{Trunc})_0 := \varphi_0 : (G\,\mathrm{Trunc})_0 \to (H\,\mathrm{Trunc})_0$.

Let $(\varphi\,\mathrm{Trunc})_1 := \varphi_1 : (G\,\mathrm{Trunc})_1 \to (H\,\mathrm{Trunc})_1$.

Using Remark 51, we let

$$\begin{array}{rcl} (\varphi\,\mathrm{Trunc})_2 := \bar{\varphi}_2 : (G\,\mathrm{Trunc})_2 & \longrightarrow & (H\,\mathrm{Trunc})_2 \\ gG\mathrm{NB}_2 & \longmapsto & (gG\mathrm{NB}_2)\bar{\varphi}_2 = (g\varphi_2)H\mathrm{NB}_2 . \end{array}$$

For $i \in [0, 2]$ we have

$$\begin{aligned} \rho^G \blacktriangle \bar{\varphi}_2 \blacktriangle \check{\mathrm{d}}_i^{H,2} &= \varphi_2 \blacktriangle \rho^H \blacktriangle \check{\mathrm{d}}_i^{H,2} \\ &= \varphi_2 \blacktriangle \mathrm{d}_i^{H,2} \\ &= \mathrm{d}_i^{G,2} \blacktriangle \varphi_1 \\ &= \rho^G \blacktriangle \check{\mathrm{d}}_i^{G,2} \blacktriangle \varphi_1 . \end{aligned}$$

Because of the surjectivity of ρ^G, we get

$$\bar{\varphi}_2 \blacktriangle \check{\mathrm{d}}_i^{H,2} = \check{\mathrm{d}}_i^{G,2} \blacktriangle \varphi_1 .$$

Suppose given $i \in [0,2]$ and $j \in [0,1]$.

Then we have the following.

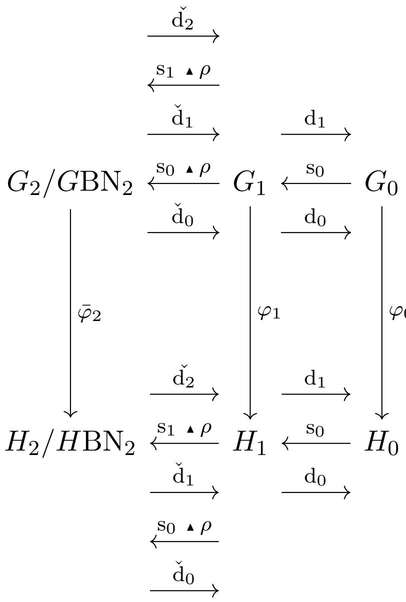
(a) $(\varphi\,\mathrm{Trunc})_2 \blacktriangle \mathrm{d}_i^{H\,\mathrm{Trunc},2} = \bar{\varphi}_2 \blacktriangle \check{\mathrm{d}}_i^{H,2} = \check{\mathrm{d}}_i^{G,2} \blacktriangle \varphi_1 = \mathrm{d}_i^{G\,\mathrm{Trunc},2} \blacktriangle (\varphi\,\mathrm{Trunc})_1$

(b) $(\varphi\,\mathrm{Trunc})_1 \blacktriangle \mathrm{d}_j^{H\,\mathrm{Trunc},1} = \varphi_1 \blacktriangle \mathrm{d}_j^{H,1} = \mathrm{d}_j^{G,1} \blacktriangle \varphi_0 = \mathrm{d}_j^{G\,\mathrm{Trunc},1} \blacktriangle (\varphi\,\mathrm{Trunc})_0$

(c) $(\varphi\,\mathrm{Trunc})_1 \blacktriangle \mathrm{s}_j^{H\,\mathrm{Trunc},1} = \varphi_1 \blacktriangle \mathrm{s}_j^{H,1} \blacktriangle \rho^H = \mathrm{s}_j^{G,1} \blacktriangle \varphi_2 \blacktriangle \rho^H = \mathrm{s}_j^{G,1} \blacktriangle \rho^G \blacktriangle \bar{\varphi}_2 = \mathrm{s}_j^{G\,\mathrm{Trunc},1} \blacktriangle (\varphi\,\mathrm{Trunc})_2$

(d) $(\varphi\,\mathrm{Trunc})_0 \blacktriangle \mathrm{s}_0^{H\,\mathrm{Trunc},0} = \varphi_0 \blacktriangle \mathrm{s}_0^{H,0} = \mathrm{s}_0^{G,0} \blacktriangle \varphi_1 = \mathrm{s}_0^{G\,\mathrm{Trunc},0} \blacktriangle (\varphi\,\mathrm{Trunc})_1$

So $\varphi\,\mathrm{Trunc}$ is a morphism of $\lfloor 2,\,0 \rfloor$-simplicial groups; cf. Definition 40.



(3) Suppose given a simplicial group G and morphisms of simplicial groups $G \xrightarrow{\varphi} H \xrightarrow{\varphi'} K$.

Then we have

(a) $(\mathrm{id}_G)\,\mathrm{Trunc} = \mathrm{id}_{(G\,\mathrm{Trunc})}$

(b) $(\varphi \blacktriangle \varphi')\,\mathrm{Trunc} = \varphi\,\mathrm{Trunc} \blacktriangle \varphi'\,\mathrm{Trunc}\,.$

So $\mathrm{Trunc} : \mathcal{SimpGrp} \to \lfloor 2,0 \rfloor\text{-}\mathcal{SimpGrp}$ is a functor.

Proof.

Ad (3.a). For $n \in [0,1]$ we have

$$\begin{aligned} (\mathrm{id}_G\,\mathrm{Trunc})_n &= (\mathrm{id}_G)_n \\ &= \mathrm{id}_{G_n} \\ &= \mathrm{id}_{(G\,\mathrm{Trunc})_n}\,. \end{aligned}$$

Cf. Remark 43.

Moreover, we have

$$\begin{aligned} (\mathrm{id}_G\,\mathrm{Trunc})_2 &= \overline{(\mathrm{id}_G)_2} \\ &= \overline{\mathrm{id}_{G_2}} \\ &= \mathrm{id}_{G_2/G\mathrm{NB}_2} \\ &= \mathrm{id}_{(G\,\mathrm{Trunc})_2}\,. \end{aligned}$$

Ad (3.b). For $n \in [0,1]$ we have

$$\begin{aligned}((\varphi \blacktriangle \varphi')\,\mathrm{Trunc})_n &= (\varphi \blacktriangle \varphi')_n \\ &= \varphi_n \blacktriangle \varphi'_n \\ &= (\varphi\,\mathrm{Trunc})_n \blacktriangle (\varphi'\,\mathrm{Trunc})_n.\end{aligned}$$

Cf. Remark 42.

Moreover, we have

$$\begin{aligned}((\varphi \blacktriangle \varphi')\,\mathrm{Trunc})_2 &= \overline{(\varphi_2 \blacktriangle \varphi'_2)} \\ &= (\bar{\varphi}_2 \blacktriangle \bar{\varphi}'_2)\end{aligned}$$

because of

$$\begin{aligned}\rho^G \blacktriangle \overline{(\varphi_2 \blacktriangle \varphi'_2)} &= (\varphi_2 \blacktriangle \varphi'_2) \blacktriangle \rho^K \\ &= \varphi_2 \blacktriangle (\varphi'_2 \blacktriangle \rho^K) \\ &= \varphi_2 \blacktriangle (\rho^H \blacktriangle \bar{\varphi}'_2) \\ &= (\varphi_2 \blacktriangle \rho^H) \blacktriangle \bar{\varphi}'_2 \\ &= (\rho^G \blacktriangle \bar{\varphi}_2) \blacktriangle \bar{\varphi}'_2 \\ &= \rho^G \blacktriangle (\bar{\varphi}_2 \blacktriangle \bar{\varphi}'_2)\end{aligned}$$

and the surjectivity of ρ^G. □

4 Decomposing a $\lfloor 2, 0 \rfloor$-simplicial group via semidirect products

Suppose given a $\lfloor 2, 0 \rfloor$-simplicial group G.

Definition 53 We write

(1) $G_{1;X} := \bigcap\limits_{i \in X} \ker \mathrm{d}_i^{G_1} \trianglelefteq G_1$

(2) $G_{2;Y} := \bigcap\limits_{i \in Y} \ker \mathrm{d}_i^{G_2} \trianglelefteq G_2$

for $X \subseteq \{0,\, 1\}$ and $Y \subseteq \{0,\, 1,\, 2\}$.

We usually omit the set-braces of X and Y when used as an index.

For example we write $G_{1;1} = G_{1;\{1\}}$, $G_{2;0,2} = G_{2;\{0,2\}}$, etc.

Note that for $X' \subseteq X \subseteq \{0,\, 1\}$ and $Y' \subseteq Y \subseteq \{0,\, 1,\, 2\}$, we have $G_{1,X} \trianglelefteq G_{1,X'}$ and $G_{2,Y} \trianglelefteq G_{2,Y'}$.

Remark 54 The condition in Definition 39.(2) reads

$$\begin{array}{rcl} [G_{2;0},\, G_{2;1,2}] & = & 1 \\ [G_{2;1},\, G_{2;0,2}] & = & 1 \\ [G_{2;2},\, G_{2;0,1}] & = & 1. \end{array}$$

Lemma 55 Let $\varphi_0 := \mathrm{id}_{G_0} : G_0 \to G_0$.

We have the group morphism

$$\begin{array}{rcl} \gamma_1 : G_0 & \longrightarrow & \mathrm{Aut}(G_{1;1}) \\ g_0 & \longmapsto & (g_0\gamma_1 : g_1 \mapsto g_1^{g_0} := g_1^{g_0\,\mathrm{s}_0}) \end{array}$$

and we consider the semidirect product $G_0 \ltimes_{\gamma_1} G_{1;1}$.

Then we have the group isomorphism

$$\begin{array}{rcl} \varphi_1 : G_0 \ltimes_{\gamma_1} G_{1;1} & \longrightarrow & G_1 \\ (g_0, g_1) & \longmapsto & g_0\,\mathrm{s}_0 \cdot g_1, \end{array}$$

with inverse

$$\begin{array}{rcl} \varphi_1^- : G_1 & \longrightarrow & G_0 \ltimes_{\gamma_1} G_{1;1} \\ g_1 & \longmapsto & (g_1\,\mathrm{d}_1, g_1^-\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g_1). \end{array}$$

We have the group morphism

$$\begin{array}{rcl} \gamma_2'' : G_{1;1} & \longrightarrow & \mathrm{Aut}(G_{2;1,2}) \\ g_1 & \longmapsto & (g_1\gamma_2'' : g_2 \mapsto g_2^{g_1} := g_2^{g_1\,\mathrm{s}_0}) \end{array}$$

and we consider the semidirect product $G_{1;1} \ltimes_{\gamma_2''} \mathrm{Aut}\, G_{2;1,2}$.

We have the group morphism

$$\begin{array}{rcl} \gamma_2 : G_0 \ltimes_{\gamma_1} G_{1;1} & \longrightarrow & \mathrm{Aut}(G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}) \\ (g_0, g_1) & \longmapsto & \left((g_0, g_1)\gamma_2 : (\tilde{g}_1, g_2) \mapsto \left(\begin{array}{l} (\tilde{g}_1, g_2)^{(g_0, g_1)} \\ := (\tilde{g}_1^{g_0\,\mathrm{s}_0 \cdot g_1}, \tilde{g}_1^-\,\mathrm{s}_0^{g_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot g_1\,\mathrm{s}_0} \\ \cdot\, \tilde{g}_1\,\mathrm{s}_0^{g_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot g_1\,\mathrm{s}_1} \cdot g_2^{g_0\,\mathrm{s}_0\,\mathrm{s}_1 \cdot g_1\,\mathrm{s}_1}) \end{array} \right) \right). \end{array}$$

and we consider the semidirect product $(G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2})$.

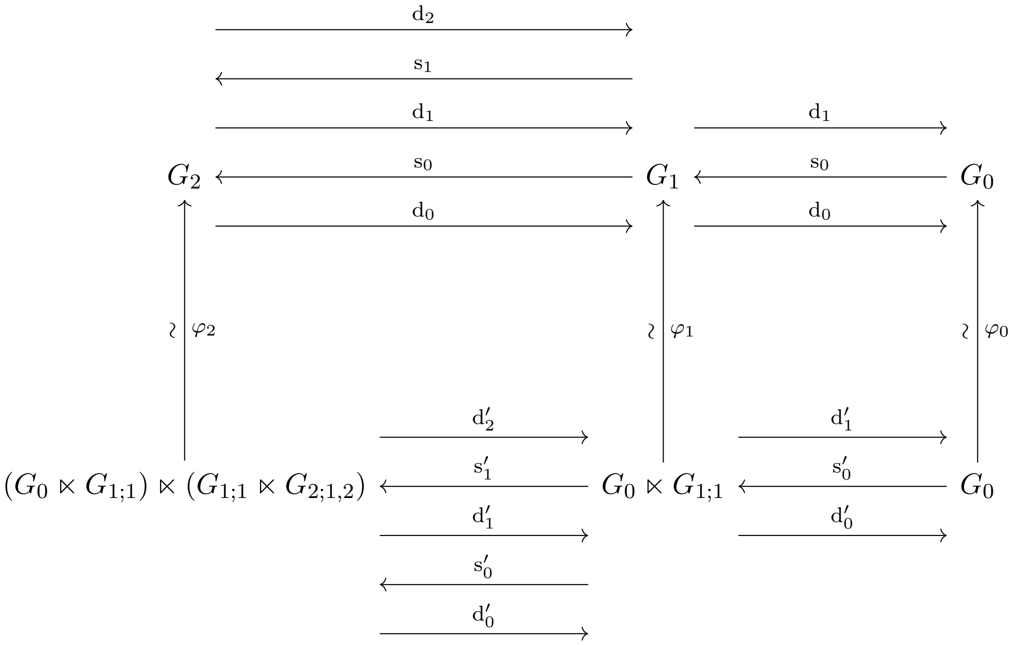
Then we have the group isomorphism

$$\begin{array}{rcl} \varphi_2 : (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}) & \longrightarrow & G_2 \\ ((g_0, g_1), (\tilde{g}_1, g_2)) & \longmapsto & g_0 \,\mathrm{s}_0\, \mathrm{s}_1 \cdot g_1 \,\mathrm{s}_1 \cdot \tilde{g}_1 \,\mathrm{s}_0 \cdot g_2, \end{array}$$

with inverse

$$\begin{array}{rcl} \varphi_2^- : G_2 & \longrightarrow & (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}) \\ g_2 & \longmapsto & ((g_2 \,\mathrm{d}_2\, \mathrm{d}_1, g_2^- \,\mathrm{d}_2\, \mathrm{d}_1\, \mathrm{s}_0 \cdot g_2 \,\mathrm{d}_2), (g_2^- \,\mathrm{d}_2 \cdot g_2 \,\mathrm{d}_1, g_2^- \,\mathrm{d}_1\, \mathrm{s}_0 \cdot g_2 \,\mathrm{d}_2\, \mathrm{s}_0 \cdot g_2^- \,\mathrm{d}_2\, \mathrm{s}_1 \cdot g_2)). \end{array}$$

We will construct the following isomorphism of $\lfloor 2,0\rfloor$-simplicial groups.



We let

$$\begin{array}{rcl} G_0' & := & G_0 \\ G_1' & := & G_0 \ltimes_{\gamma_1} G_{1;1} \\ G_2' & := & (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}). \end{array}$$

We define the following maps.

$$\begin{array}{rcl} \mathrm{d}_1' \overset{\text{abbr.}}{:=} \mathrm{d}_1^{G',1} : G_1' & \longrightarrow & G_0' \\ (g_0, g_1) & \longmapsto & g_0 \end{array}$$

$$\begin{array}{rcl} \mathrm{s}_0' \overset{\text{abbr.}}{:=} \mathrm{s}_0^{G',0} : G_0' & \longrightarrow & G_1' \\ g_0 & \longmapsto & (g_0, 1) \end{array}$$

$$\begin{array}{rcl} \mathrm{d}_0' \overset{\text{abbr.}}{:=} \mathrm{d}_0^{G',1} : G_1' & \longrightarrow & G_0' \\ (g_0, g_1) & \longmapsto & g_0 \cdot \, g_1 \,\mathrm{d}_0 \end{array}$$

$$\begin{array}{rcl} \mathrm{d}_2' \overset{\text{abbr.}}{:=} \mathrm{d}_2^{G',2} : G_2' & \longrightarrow & G_1' \\ ((g_0, g_1), (\tilde{g}_1, g_2)) & \longmapsto & (g_0, g_1) \end{array}$$

$$\begin{array}{rcl} \mathrm{s}_1' \overset{\text{abbr.}}{:=} \mathrm{s}_1^{G',1} : G_1' & \longrightarrow & G_2' \\ (g_0, g_1) & \longmapsto & ((g_0, g_1), (1, 1)) \end{array}$$

$$
\begin{array}{rcl}
\mathrm{d}_1' \overset{\text{abbr.}}{:=} \mathrm{d}_1^{G',2} : G_2' & \longrightarrow & G_1' \\
((g_0, g_1), (\tilde{g}_1, g_2)) & \longmapsto & (g_0, g_1 \cdot \tilde{g}_1)
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{s}_0' \overset{\text{abbr.}}{:=} \mathrm{s}_0^{G',1} : G_1' & \longrightarrow & G_2' \\
(g_0, g_1) & \longmapsto & ((g_0, 1), (g_1, 1))
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{d}_0' \overset{\text{abbr.}}{:=} \mathrm{d}_0^{G',2} : G_2' & \longrightarrow & G_1' \\
((g_0, g_1), (\tilde{g}_1, g_2)) & \longmapsto & (g_0 \cdot g_1 \, \mathrm{d}_0, \tilde{g}_1 \cdot g_2 \, \mathrm{d}_0)
\end{array}
$$

Then the assertions $(1, 2)$ hold.

(1) We have

$$
\begin{aligned}
\mathrm{d}_1^{G',1} &= \varphi_1 \blacktriangle \mathrm{d}_1 \blacktriangle \varphi_0^- \\
\mathrm{s}_0^{G',0} &= \varphi_0 \blacktriangle \mathrm{s}_0 \blacktriangle \varphi_1^- \\
\mathrm{d}_0^{G',1} &= \varphi_1 \blacktriangle \mathrm{d}_0 \blacktriangle \varphi_0^- \\
\mathrm{d}_2^{G',2} &= \varphi_2 \blacktriangle \mathrm{d}_2 \blacktriangle \varphi_1^- \\
\mathrm{s}_1^{G',1} &= \varphi_1 \blacktriangle \mathrm{s}_1 \blacktriangle \varphi_2^- \\
\mathrm{d}_1^{G',2} &= \varphi_2 \blacktriangle \mathrm{d}_1 \blacktriangle \varphi_1^- \\
\mathrm{s}_0^{G',1} &= \varphi_1 \blacktriangle \mathrm{s}_0 \blacktriangle \varphi_2^- \\
\mathrm{d}_0^{G',2} &= \varphi_2 \blacktriangle \mathrm{d}_0 \blacktriangle \varphi_1^- .
\end{aligned}
$$

In particular, $\mathrm{d}_1^{G',1}$, $\mathrm{s}_0^{G',0}$, $\mathrm{d}_0^{G',1}$, $\mathrm{d}_2^{G',2}$, $\mathrm{s}_1^{G',1}$, $\mathrm{d}_1^{G',2}$, $\mathrm{s}_0^{G',1}$, $\mathrm{d}_0^{G',2}$ are group morphisms.

(2) We have the $\lfloor 2,0 \rfloor$-simplicial group

$$G' = (G_2',\, G_1',\, G_0',\, \mathrm{d}_0^{G',2},\, \mathrm{d}_1^{G',2},\, \mathrm{d}_2^{G',2},\, \mathrm{s}_0^{G',1},\, \mathrm{s}_1^{G',1},\, \mathrm{d}_0^{G',1},\, \mathrm{d}_1^{G',1},\, \mathrm{s}_0^{G',0}).$$

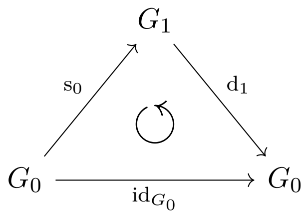
Moreover,

$$\varphi = (\varphi_2,\, \varphi_1,\, \varphi_0) : G' \to G$$

is an isomorphism of $\lfloor 2,0 \rfloor$-simplicial groups.

Proof. We make repeated use of Lemma 21.

We have the following commutative triangle of group morphisms.



We have the group morphism

$$
\begin{array}{rcl}
\gamma_1 : G_0 & \longrightarrow & \mathrm{Aut}(G_{1;1}) \\
g_0 & \longmapsto & (g_0\gamma_1 : g_1 \mapsto g_1^{g_0} := g_1^{g_0 \, \mathrm{s}_0})
\end{array}
$$

and we consider the semidirect product $G_0 \ltimes_{\gamma_1} G_{1;1}$.

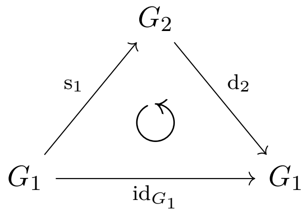
Then we have the group isomorphism

$$
\begin{array}{rcl}
\varphi_1 : G_0 \ltimes_{\gamma_1} G_{1;1} & \longrightarrow & G_1 \\
(g_0, g_1) & \longmapsto & g_0 \, \mathrm{s}_0 \cdot g_1,
\end{array}
$$

with inverse

$$\begin{aligned} \varphi_1^- : G_1 &\longrightarrow G_0 \ltimes_{\gamma_1} G_{1;1} \\ g_1 &\longmapsto (g_1 \, \mathrm{d}_1, g_1^- \, \mathrm{d}_1 \, \mathrm{s}_0 \cdot g_1). \end{aligned}$$

We have the following commutative triangle of group morphisms.



We have the group morphism

$$\begin{aligned} \gamma_2' : G_1 &\longrightarrow \mathrm{Aut}(G_{2;2}) \\ g_1 &\longmapsto (g_1\gamma_2' : g_2 \mapsto g_2^{g_1} := g_2^{g_1 \, \mathrm{s}_1}) \end{aligned}$$

and we consider the semidirect product $G_1 \ltimes_{\gamma_2'} G_{2;2}$.

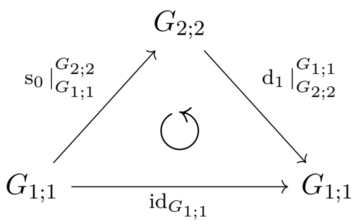
Then we have the group isomorphism

$$\begin{aligned} \varphi_2' : G_1 \ltimes_{\gamma_2'} G_{2;2} &\longrightarrow G_2 \\ (g_1, g_2) &\longmapsto g_1 \, \mathrm{s}_1 \cdot g_2, \end{aligned}$$

with inverse

$$\begin{aligned} \varphi_2'^- : G_2 &\longrightarrow G_1 \ltimes_{\gamma_2'} G_{2;2} \\ g_2 &\longmapsto (g_2 \, \mathrm{d}_2, g_2^- \, \mathrm{d}_2 \, \mathrm{s}_1 \cdot g_2). \end{aligned}$$

We the following commutative triangle of group morphisms.



We have the group morphism

$$\begin{aligned} \gamma_2'' : G_{1;1} &\longrightarrow \mathrm{Aut}(G_{2;1,2}) \\ g_1 &\longmapsto (g_1\gamma_2'' : g_2 \mapsto g_2^{g_1} := g_2^{g_1 \, \mathrm{s}_0}) \end{aligned}$$

and we consider the semidirect product $G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}$.

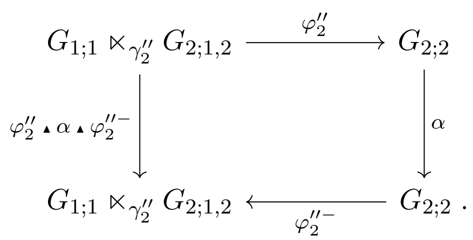
Then we have the group isomorphism

$$\begin{aligned} \varphi_2'' : G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2} &\longrightarrow G_{2;2} \\ (g_1, g_2) &\longmapsto g_1 \, \mathrm{s}_0 \cdot g_2, \end{aligned}$$

with inverse

$$\begin{aligned} \varphi_2''^- : G_{2;2} &\longrightarrow G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2} \\ g_2 &\longmapsto (g_2 \, \mathrm{d}_1, g_2^- \, \mathrm{d}_1 \, \mathrm{s}_0 \cdot g_2). \end{aligned}$$

Given $\alpha \in \mathrm{Aut}(G_{2;2})$, we may compare



So we have

$$\begin{array}{ccccc} G_1 & \xrightarrow{\gamma_2'} & \mathrm{Aut}(G_{2;2}) & \qquad & \alpha \\ \uparrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \check{\varphi}_2''} & & \downarrow \\ G_0 \ltimes_{\gamma_1} G_{1;1} & \xrightarrow[\gamma_2]{} & \mathrm{Aut}(G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}) & & \varphi_2'' \blacktriangle \alpha \blacktriangle \varphi_2''^{-} \end{array}$$

with the group morphism

$$\begin{array}{rcl} \gamma_2 := \varphi_1 \blacktriangle \gamma_2' \blacktriangle \check{\varphi}_2'' : G_0 \ltimes G_{1;1} & \longrightarrow & \mathrm{Aut}(G_{1;1} \ltimes G_{2;1,2}) \\ (g_0, g_1) & \longmapsto & \left(\begin{array}{l} (g_0, g_1)\gamma_2 : (\tilde{g}_1, g_2) \mapsto (\tilde{g}_1, g_2)^{(g_0, g_1)} \\ := (\tilde{g}_1, g_2)((g_0, g_1)\gamma_2) \\ = (\tilde{g}_1, g_2)((g_0, g_1)\varphi_1 \blacktriangle \gamma_2' \blacktriangle \check{\varphi}_2'') \\ = (\tilde{g}_1, g_2)(\varphi_2'' \blacktriangle (g_0 \,\mathrm{s}_0 \cdot g_1)\gamma_2' \blacktriangle \varphi_2''^{-}) \\ = (\tilde{g}_1 \,\mathrm{s}_0 \cdot g_2)((g_0 \,\mathrm{s}_0 \cdot g_1)\gamma_2' \blacktriangle \varphi_2''^{-}) \\ = ((\tilde{g}_1 \,\mathrm{s}_0 \cdot g_2)^{g_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot g_1 \,\mathrm{s}_1})\varphi_2''^{-} \\ = ((\tilde{g}_1 \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot g_2 \,\mathrm{d}_1)^{g_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \,\mathrm{d}_1 \cdot g_1 \,\mathrm{s}_1 \,\mathrm{d}_1}, \\ (g_2^{-} \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot \tilde{g}_1^{-} \,\mathrm{s}_0 \,\mathrm{d}_1 \,\mathrm{s}_0)^{g_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g_1 \,\mathrm{s}_1 \,\mathrm{d}_1 \,\mathrm{s}_0} \cdot (\tilde{g}_1 \,\mathrm{s}_0 \cdot g_2)^{g_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot g_1 \,\mathrm{s}_1}) \\ = (\tilde{g}_1^{g_0 \,\mathrm{s}_0 \cdot g_1}, \tilde{g}_1^{-} \,\mathrm{s}_0^{g_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot g_1 \,\mathrm{s}_0} \cdot \tilde{g}_1 \,\mathrm{s}_0^{g_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot g_1 \,\mathrm{s}_1} \cdot g_2^{g_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot g_1 \,\mathrm{s}_1}) \end{array} \right) \end{array}$$

and we consider the semidirect product $(G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2})$.

We consider the bijective map

$$\begin{array}{rcl} \varphi_1 \ltimes \varphi_2'' : (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}) & \longrightarrow & G_1 \ltimes_{\gamma_2'} G_{2;2} \\ ((g_0, g_1), (\tilde{g}_1, g_2)) & \longmapsto & ((g_0, g_1)\varphi_1, (\tilde{g}_1, g_2)\varphi_2'') = (g_0 \,\mathrm{s}_0 \cdot g_1, \tilde{g}_1 \,\mathrm{s}_0 \cdot g_2). \end{array}$$

Let $(g_0, g_1) \in G_0 \ltimes_{\gamma_1} G_{1;1}$ and $(\tilde{g}_1, g_2) \in G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}$.

We have

$$\begin{array}{rcl} ((g_0, \tilde{g}_1) \cdot (g_1, g_2))(\varphi_1 \ltimes \varphi_2'') & = & (g_0 \cdot g_1, (\tilde{g}_1)(g_1\gamma_2) \cdot g_2)(\varphi_1 \ltimes \varphi_2'') \\ & = & ((g_0 \cdot g_1)\varphi_1, ((\tilde{g}_1)(g_1\gamma_2) \cdot g_2)\varphi_2'') \\ & = & (g_0\varphi_1 \cdot g_1\varphi_1, (\tilde{g}_1)(g_1\gamma_2)\varphi_2'' \cdot g_2\varphi_2'') \end{array}$$

and

$$\begin{array}{rcl} (g_0, \tilde{g}_1)(\varphi_1 \ltimes \varphi_2'') \cdot (g_1, g_2)(\varphi_1 \ltimes \varphi_2'') & = & (g_0\varphi_1, \tilde{g}_1\varphi_2'') \cdot (g_1\varphi_1, g_2\varphi_2'') \\ & = & (g_0\varphi_1 \cdot g_1\varphi_1, (\tilde{g}_1\varphi_2'')(g_1\varphi_1\gamma_2') \cdot g_2\varphi_2''). \end{array}$$

We have

$$(\tilde{g}_1)(g_1\gamma_2)\varphi_2'' = (\tilde{g}_1\varphi_2'')(g_1\varphi_1\gamma_2'),$$

because

$$(g_1\gamma_2) \blacktriangle \varphi_2'' = \varphi_2'' \blacktriangle (g_1\varphi_1\gamma_2'),$$

since this means

$$\begin{array}{rcl} g_1\gamma_2 & = & \varphi_2'' \blacktriangle g_1\varphi_1\gamma_2' \blacktriangle \varphi_2''^{-} \\ & = & g_1\varphi_1\gamma_2'\check{\varphi}_2'' \end{array}$$

and this is how we defined $\gamma_2 = \varphi_1 \blacktriangle \gamma_2' \blacktriangle \check{\varphi}_2''$.

So $\varphi_1 \ltimes \varphi_2''$ is a group morphism.

Moreover $\varphi_1 \ltimes \varphi_2''$ is a group isomorphism, since φ_1 and φ_2'' are bijective.

So we have the group isomorphism

$$\varphi_2 := (\varphi_1 \ltimes \varphi_2'') \blacktriangle \varphi_2',$$

with inverse

$$\varphi_2^- = \varphi_2'^- \blacktriangle (\varphi_1 \ltimes \varphi_2'')^- .$$

We have

$$\begin{array}{ccccc} (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}) & \xrightarrow{\varphi_1 \ltimes \varphi_2''} & G_1 \ltimes_{\gamma_2'} G_{2;2} & \xrightarrow{\varphi_2'} & G_2 \\ ((g_0, g_1), (\tilde{g}_1, g_2)) & \longmapsto & (g_0 \,\mathrm{s}_0 \cdot g_1, \tilde{g}_1 \,\mathrm{s}_0 \cdot g_2) & \longmapsto & \left(\begin{array}{l} g_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \cdot g_1 \,\mathrm{s}_1 \cdot \tilde{g}_1 \,\mathrm{s}_0 \cdot g_2 \\ = ((g_0, g_1), (\tilde{g}_1, g_2))\varphi_2 \end{array} \right) \end{array}$$

and

$$\begin{array}{ccccc} G_2 & \xrightarrow{\varphi_2'^-} & G_1 \ltimes_{\gamma_2'} G_{2;2} & \xrightarrow{(\varphi_1 \ltimes \varphi_2'')^-} & (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2}) \\ g_2 & \longmapsto & (g_2 \,\mathrm{d}_2, g_2^- \,\mathrm{d}_2 \,\mathrm{s}_1 \cdot g_2) & \longmapsto & \left(\begin{array}{l} ((g_2 \,\mathrm{d}_2 \,\mathrm{d}_1, g_2^- \,\mathrm{d}_2 \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g_2 \,\mathrm{d}_2), \\ (g_2^- \,\mathrm{d}_2 \cdot g_2 \,\mathrm{d}_1, g_2^- \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g_2 \,\mathrm{d}_2 \,\mathrm{s}_0 \cdot g_2^- \,\mathrm{d}_2 \,\mathrm{s}_1 \cdot g_2)) \\ = g_2 \varphi_2^- \end{array} \right) . \end{array}$$

Ad (1). We have to show for $i \in [0,2]$ and $j \in [0,1]$ that

$$\begin{array}{rcl} \varphi_2 \blacktriangle \mathrm{d}_i & \stackrel{!}{=} & \mathrm{d}_i' \blacktriangle \varphi_1 \\ \varphi_1 \blacktriangle \mathrm{d}_j & \stackrel{!}{=} & \mathrm{d}_j' \blacktriangle \varphi_0 \\ \varphi_1 \blacktriangle \mathrm{s}_j & \stackrel{!}{=} & \mathrm{s}_j' \blacktriangle \varphi_2 \\ \varphi_0 \blacktriangle \mathrm{s}_0 & \stackrel{!}{=} & \mathrm{s}_0' \blacktriangle \varphi_1 . \end{array}$$

For $(g_0, g_1) \in G_0 \ltimes_{\gamma_1} G_{1;1}$, we have

$$\begin{array}{rcl} (g_0, g_1)(\varphi_1 \blacktriangle \mathrm{d}_1) & = & (g_0 \,\mathrm{s}_0 \cdot g_1) \,\mathrm{d}_1 \\ & = & g_0 \end{array}$$

and

$$\begin{array}{rcl} (g_0, g_1)(\mathrm{d}_1' \blacktriangle \varphi_0) & = & (g_0)\varphi_0 \\ & = & g_0 . \end{array}$$

So

$$\varphi_1 \blacktriangle \mathrm{d}_1 = \mathrm{d}_1' \blacktriangle \varphi_0 .$$

For $g_0 \in G_0$, we have

$$\begin{array}{rcl} (g_0)(\varphi_0 \blacktriangle \mathrm{s}_0) & = & (g_0) \,\mathrm{s}_0 \\ & = & g_0 \,\mathrm{s}_0 \end{array}$$

and

$$\begin{array}{rcl} (g_0)(\mathrm{s}_0' \blacktriangle \varphi_1) & = & (g_0, 1)\varphi_1 \\ & = & g_0 \,\mathrm{s}_0 . \end{array}$$

So

$$\varphi_0 \blacktriangle \mathrm{s}_0 = \mathrm{s}_0' \blacktriangle \varphi_1 .$$

For $(g_0, g_1) \in G_0 \ltimes_{\gamma_1} G_{1;1}$, we have

$$\begin{array}{rcl} (g_0, g_1)(\varphi_1 \blacktriangle \mathrm{d}_0) & = & (g_0 \,\mathrm{s}_0 \cdot g_1) \,\mathrm{d}_0 \\ & = & g_0 \cdot \, g_1 \,\mathrm{d}_0 \end{array}$$

and

$$\begin{array}{rcl} (g_0, g_1)(\mathrm{d}_0' \blacktriangle \varphi_0) & = & (g_0 \cdot \, g_1 \,\mathrm{d}_0)\varphi_0 \\ & = & g_0 \cdot \, g_1 \,\mathrm{d}_0 . \end{array}$$

So

$$\varphi_1 \blacktriangle \mathrm{d}_0 = \mathrm{d}_0' \blacktriangle \varphi_0.$$

For $(g_0, g_1), (\tilde{g}_1, g_2) \in (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2})$, we have

$$\begin{aligned} ((g_0, g_1), (\tilde{g}_1, g_2))(\varphi_2 \blacktriangle \mathrm{d}_2) &= (g_0 \mathrm{s}_0 \mathrm{s}_1 \cdot g_1 \mathrm{s}_1 \cdot \tilde{g}_1 \mathrm{s}_0 \cdot g_2) \mathrm{d}_2 \\ &= g_0 \mathrm{s}_0 \cdot g_1 \end{aligned}$$

and

$$\begin{aligned} ((g_0, g_1), (\tilde{g}_1, g_2))(\mathrm{d}_2' \blacktriangle \varphi_1) &= (g_0, g_1)\varphi_1 \\ &= g_0 \mathrm{s}_0 \cdot g_1. \end{aligned}$$

So

$$\varphi_2 \blacktriangle \mathrm{d}_2 = \mathrm{d}_2' \blacktriangle \varphi_1.$$

For $(g_0, g_1) \in G_0 \ltimes_{\gamma_1} G_{1;1}$, we have

$$\begin{aligned} (g_0, g_1)(\varphi_1 \blacktriangle \mathrm{s}_1) &= (g_0 \mathrm{s}_0 \cdot g_1) \mathrm{s}_1 \\ &= g_0 \mathrm{s}_0 \mathrm{s}_1 \cdot g_1 \mathrm{s}_1 \end{aligned}$$

and

$$\begin{aligned} (g_0, g_1)(\mathrm{s}_1' \blacktriangle \varphi_2) &= ((g_0, g_1), (1, 1))\varphi_2 \\ &= g_0 \mathrm{s}_0 \mathrm{s}_1 \cdot g_1 \mathrm{s}_1 . \end{aligned}$$

So

$$\varphi_1 \blacktriangle \mathrm{s}_1 = \mathrm{s}_1' \blacktriangle \varphi_2.$$

For $(g_0, g_1), (\tilde{g}_1, g_2) \in (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2})$, we have

$$\begin{aligned} ((g_0, g_1), (\tilde{g}_1, g_2))(\varphi_2 \blacktriangle \mathrm{d}_1) &= (g_0 \mathrm{s}_0 \mathrm{s}_1 \cdot g_1 \mathrm{s}_1 \cdot \tilde{g}_1 \mathrm{s}_0 \cdot g_2) \mathrm{d}_1 \\ &= g_0 \mathrm{s}_0 \cdot g_1 \cdot \tilde{g}_1 \end{aligned}$$

and

$$\begin{aligned} ((g_0, g_1), (\tilde{g}_1, g_2))(\mathrm{d}_1' \blacktriangle \varphi_1) &= (g_0, g_1 \cdot \tilde{g}_1)\varphi_1 \\ &= g_0 \mathrm{s}_0 \cdot g_1 \cdot \tilde{g}_1. \end{aligned}$$

So

$$\varphi_2 \blacktriangle \mathrm{d}_1 = \mathrm{d}_1' \blacktriangle \varphi_1.$$

For $(g_0, g_1) \in G_0 \ltimes_{\gamma_1} G_{1;1}$, we have

$$\begin{aligned} (g_0, g_1)(\varphi_1 \blacktriangle \mathrm{s}_0) &= (g_0 \mathrm{s}_0 \cdot g_1) \mathrm{s}_0 \\ &= g_0 \mathrm{s}_0 \mathrm{s}_1 \cdot g_1 \mathrm{s}_0 \end{aligned}$$

and

$$\begin{aligned} (g_0, g_1)(\mathrm{s}_0' \blacktriangle \varphi_2) &= ((g_0, 1), (g_1, 1))\varphi_2 \\ &= g_0 \mathrm{s}_0 \mathrm{s}_1 \cdot g_1 \mathrm{s}_0 . \end{aligned}$$

So

$$\varphi_1 \blacktriangle \mathrm{s}_0 = \mathrm{s}_0' \blacktriangle \varphi_2.$$

For $(g_0, g_1), (\tilde{g}_1, g_2) \in (G_0 \ltimes_{\gamma_1} G_{1;1}) \ltimes_{\gamma_2} (G_{1;1} \ltimes_{\gamma_2''} G_{2;1,2})$, we have

$$\begin{aligned} ((g_0, g_1), (\tilde{g}_1, g_2))(\varphi_2 \blacktriangle \mathrm{d}_0) &= (g_0 \mathrm{s}_0 \mathrm{s}_1 \cdot g_1 \mathrm{s}_1 \cdot \tilde{g}_1 \mathrm{s}_0 \cdot g_2) \mathrm{d}_0 \\ &= g_0 \mathrm{s}_0 \cdot g_1 \mathrm{d}_0 \mathrm{s}_0 \cdot \tilde{g}_1 \cdot g_2 \mathrm{d}_0 \end{aligned}$$

and

$$\begin{aligned} ((g_0, g_1), (\tilde{g}_1, g_2))(\mathrm{d}_0' \blacktriangle \varphi_1) &= (g_0 \cdot g_1 \mathrm{d}_0, \tilde{g}_1 \cdot g_2 \mathrm{d}_0)\varphi_1 \\ &= g_0 \mathrm{s}_0 \cdot g_1 \mathrm{d}_0 \mathrm{s}_0 \cdot \tilde{g}_1 \cdot g_2 \mathrm{d}_0 . \end{aligned}$$

So

$$\varphi_2 \blacktriangle \mathrm{d}_0 = \mathrm{d}'_0 \blacktriangle \varphi_1.$$

Ad (2). We have

$$\begin{aligned} s_0^{G',0} \blacktriangle \mathrm{d}_0^{G',0} &\overset{(1)}{=} (\varphi_0 \blacktriangle \mathrm{s}_0 \blacktriangle \varphi_1^-) \blacktriangle (\varphi_1 \blacktriangle \mathrm{d}_0 \blacktriangle \varphi_0^-) \\ &= \varphi_0 \blacktriangle \mathrm{id}_{G_0} \blacktriangle \varphi_0^- \\ &= \mathrm{id}_{G'_0}. \end{aligned}$$

And so on.

Moreover, we have

$$\begin{aligned} \ker \mathrm{d}_i^{G',2} &\overset{(1)}{=} \ker(\varphi_2 \blacktriangle \mathrm{d}_i^{G,2} \blacktriangle \varphi_1^-) \\ &= \ker(\varphi_2 \blacktriangle \mathrm{d}_i^{G,2}) \\ &= (\ker \mathrm{d}_i^{G,2})\varphi_2^- \end{aligned}$$

for $i \in [0,2]$.

So

$$\begin{aligned} [\ker \mathrm{d}_0^{G',2}, \ker \mathrm{d}_1^{G',2} \cap \ker \mathrm{d}_2^{G',2}] &= [(\ker \mathrm{d}_0^{G,2})\varphi_2^-, (\ker \mathrm{d}_1^{G,2})\varphi_2^- \cap (\ker \mathrm{d}_2^{G,2})\varphi_2^-] \\ &\overset{\varphi_2 \text{ isom.}}{=} ([\ker \mathrm{d}_0^{G,2}, \ker \mathrm{d}_1^{G,2} \cap \ker \mathrm{d}_2^{G,2}])\varphi_2^- \\ &= 1\varphi_2^- \\ &= 1. \end{aligned}$$

And so on.

Finally, by (1), $\varphi = (\varphi_2, \varphi_1, \varphi_0) : G' \to G$ is an isomorphism of $\lfloor 2,0\rfloor$-simplicial groups. □

Remark 56 We abbreviate $\mathrm{N}_k := G\mathrm{N}_k$.

Observing

$$\begin{aligned} \mathrm{N}_0 &= G_0 \\ \mathrm{N}_1 &= G_{1;1} \\ \mathrm{N}_2 &= G_{2;1,2}, \end{aligned}$$

we may write

$$\begin{aligned} G'_0 &= \mathrm{N}_0 \\ G'_1 &= \mathrm{N}_0 \ltimes_{\gamma_1} \mathrm{N}_1 \\ G'_2 &= (\mathrm{N}_0 \ltimes_{\gamma_1} \mathrm{N}_1) \ltimes_{\gamma_2} (\mathrm{N}_1 \ltimes_{\gamma''_2} \mathrm{N}_2). \end{aligned}$$

5 $\lfloor 2, 0 \rfloor$-simplicial groups and crossed squares

5.1 Crossed squares

The following notion of a crossed square is due to Loday [**3,** Def. 5.1], adapted to our context.

Definition 57 Suppose given groups L, M, M', P and group morphisms

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & M' \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\mu'} \\ M & \xrightarrow[\mu]{} & P\,. \end{array}$$

Suppose given group morphisms

$$\begin{array}{lcllcl} M & \xrightarrow{\gamma_{M,L}} & \mathrm{Aut}(L) & : m & \longmapsto & (l \mapsto (l)(m\gamma_{M,L}) =: l^m) \\ M' & \xrightarrow{\gamma_{M',L}} & \mathrm{Aut}(L) & : m' & \longmapsto & (l \mapsto (l)(m'\gamma_{M',L}) =: l^{m'}) \\ P & \xrightarrow{\gamma_{P,L}} & \mathrm{Aut}(L) & : p & \longmapsto & (l \mapsto (l)(p\gamma_{P,L}) =: l^p) \\ P & \xrightarrow{\gamma_{P,M}} & \mathrm{Aut}(M) & : p & \longmapsto & (m \mapsto (m)(p\gamma_{P,M}) =: m^p) \\ P & \xrightarrow{\gamma_{P,M'}} & \mathrm{Aut}(M') & : p & \longmapsto & (m' \mapsto (m')(p\gamma_{P,M'}) =: m'^p) \end{array}$$

and the map

$$\begin{array}{rcl} M \times M' & \xrightarrow{\chi} & L \\ (m, m') & \longmapsto & (m, m')\chi =: \lceil m, m' \rceil. \end{array}$$

Suppose that the following properties (CS 1, 2, 3, 4) hold.

(CS 1) We have $\lambda \blacktriangle \mu = \lambda' \blacktriangle \mu'$.

We write $\kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : L \to P$.

(CS 2) We have the following crossed modules.

(1) $(L,\, M,\, \gamma_{M,L},\, \lambda)$

(2) $(L,\, M',\, \gamma_{M',L},\, \lambda')$

(3) $(L,\, P,\, \gamma_{P,L},\, \kappa)$

(4) $(M,\, P,\, \gamma_{P,M},\, \mu)$

(5) $(M',\, P,\, \gamma_{P,M'},\, \mu')$

(CS 3) We have the following morphisms of crossed modules.

(1) $(\mathrm{id}_L,\, \mu) : (L,\, M,\, \gamma_{M,L},\, \lambda) \to (L,\, P,\, \gamma_{P,L},\, \kappa)$

(2) $(\mathrm{id}_L,\, \mu') : (L,\, M',\, \gamma_{M',L},\, \lambda') \to (L,\, P,\, \gamma_{P,L},\, \kappa)$

(3) $(\lambda,\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M,\, P,\, \gamma_{P,M},\, \mu)$

(4) $(\lambda',\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M',\, P,\, \gamma_{P,M'},\, \mu')$

(CS 4)

(1) $m \cdot \lceil m, m' \rceil \lambda = m^{m'\mu'}$ for $m \in M$ and $m' \in M'$

(2) $m'^{m\mu} \cdot \lceil m, m' \rceil \lambda' = m'$ for $m \in M$ and $m' \in M'$

(3) $l \cdot \lceil l\lambda, m' \rceil = l^{m'}$ for $l \in L$ and $m' \in M'$

(4) $l^m \cdot \lceil m, l\lambda' \rceil = l$ for $m \in M$ and $l \in L$

(5) $\lceil m \cdot m^*, m' \rceil = \lceil m, m' \rceil^{m^*} \cdot \lceil m^*, m' \rceil$ for $m, m^* \in M$ and $m' \in M'$

(6) $\lceil m, m' \cdot m^{*\prime} \rceil = \lceil m, m^{*\prime} \rceil \cdot \lceil m, m' \rceil^{m^{*\prime}}$ for $m \in M$ and $m', m^{*\prime} \in M'$

(7) $\lceil m, m' \rceil^p = \lceil m^p, m'^p \rceil$ for $m \in M$, $m' \in M'$ and $p \in P$

(8) $((l^{m'})^m)^{\lceil m, m' \rceil} = (l^m)^{m'}$ for $m \in M$, $m' \in M'$ and $l \in L$

Then

$$C := (L,\, M,\, M',\, P,\, \gamma_{M,L},\, \gamma_{M',L},\, \gamma_{P,L},\, \gamma_{P,M},\, \gamma_{P,M'},\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

is called a *crossed square.*

Notation. We write

$$\begin{aligned}
C_{1;1} &:= L \\
C_{1;0} &:= M \\
C_{0;1} &:= M' \\
C_{0;0} &:= P \\
\gamma^C_{1,0} &:= \gamma_{M,L} \\
\gamma^C_{0,1} &:= \gamma_{M',L} \\
\gamma^{1,1}_C &:= \gamma_{P,L} \\
\gamma^{1,0}_C &:= \gamma_{P,M} \\
\gamma^{0,1}_C &:= \gamma_{P,M'} \\
\lambda^{1,0}_C &:= \lambda \\
\lambda^{0,1}_C &:= \lambda' \\
\mu^C_{1,0} &:= \mu \\
\mu^C_{0,1} &:= \mu' \\
\chi_C &:= \chi \\
\kappa_C &:= \kappa.
\end{aligned}$$

Notation. We often write just $C = (L,\, M,\, M',\, P)$ to denote this crossed square.

Remark 58 So $(m, m')\chi = \lceil m, m' \rceil$ for $m \in M$ and $m' \in M'$.

This notation is supposed to remind of a commutator bracket.

Definition 59 Suppose given crossed squares

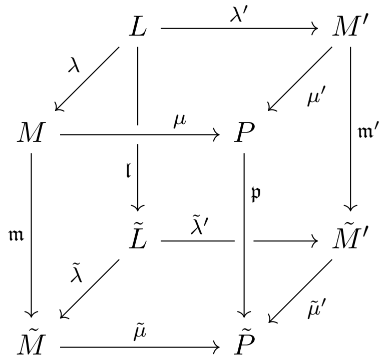
$$(L,\, M,\, M',\, P,\, \gamma_{M,L},\, \gamma_{M',L},\, \gamma_{P,L},\, \gamma_{P,M},\, \gamma_{P,M'},\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

and

$$(\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}},\, \gamma_{\tilde{M}',\tilde{L}},\, \gamma_{\tilde{P},\tilde{L}},\, \gamma_{\tilde{P},\tilde{M}},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi}).$$

We write $\kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : L \to P$ and $\tilde{\kappa} := \tilde{\lambda} \blacktriangle \tilde{\mu} = \tilde{\lambda}' \blacktriangle \tilde{\mu}' : \tilde{L} \to \tilde{P}$.

Suppose given group morphisms $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{m}'$, $\mathfrak{p}$ fitting into the following diagram.



Suppose that the following properties $(1, 2)$ hold.

(1) We have the following morphisms of crossed modules.

(a) $(\mathfrak{l},\, \mathfrak{m}) : (L,\, M,\, \gamma_{L,M},\, \lambda) \to (\tilde{L},\, \tilde{M},\, \gamma_{\tilde{L},\tilde{M}},\, \tilde{\lambda})$

(b) $(\mathfrak{l},\, \mathfrak{m}') : (L,\, M',\, \gamma_{L,M'},\, \lambda') \to (\tilde{L},\, \tilde{M}',\, \gamma_{\tilde{L},\tilde{M}'},\, \tilde{\lambda}')$

(c) $(\mathfrak{m},\, \mathfrak{p}) : (M,\, P,\, \gamma_{P,M},\, \mu) \to (\tilde{M},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}},\, \tilde{\mu})$

(d) $(\mathfrak{m}',\, \mathfrak{p}) : (M',\, P,\, \gamma_{P,M'},\, \mu') \to (\tilde{M}',\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\mu}')$

(e) $(\mathfrak{l},\, \mathfrak{p}) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (\tilde{L},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\kappa})$

(2) We have

$$\lceil m\mathfrak{m},\, m'\mathfrak{m}' \rceil = \lceil m, m' \rceil \mathfrak{l}$$

for $m \in M$ and $m' \in M'$.

Then we call

$$\mathfrak{c} := (\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) : (L,\, M,\, M',\, P) \to (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P})$$

a *morphism of crossed squares.*

Notation. We write

$$\begin{aligned} \mathfrak{c}_{1,1} &:= \mathfrak{l} \\ \mathfrak{c}_{1,0} &:= \mathfrak{m} \\ \mathfrak{c}_{0,1} &:= \mathfrak{m}' \\ \mathfrak{c}_{0,0} &:= \mathfrak{p}. \end{aligned}$$

Remark 60 Suppose given morphisms of crossed squares

$$(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) : (L,\, M,\, M',\, P) \to (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P})$$

and

$$(\tilde{\mathfrak{l}},\, \tilde{\mathfrak{m}},\, \tilde{\mathfrak{m}}',\, \tilde{\mathfrak{p}}) : (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P}) \to (\tilde{\tilde{L}},\, \tilde{\tilde{M}},\, \tilde{\tilde{M}}',\, \tilde{\tilde{P}}).$$

Then the composite

$$(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) \blacktriangle (\tilde{\mathfrak{l}},\, \tilde{\mathfrak{m}},\, \tilde{\mathfrak{m}}',\, \tilde{\mathfrak{p}}) := (\mathfrak{l} \blacktriangle \tilde{\mathfrak{l}},\, \mathfrak{m} \blacktriangle \tilde{\mathfrak{m}},\, \mathfrak{m}' \blacktriangle \tilde{\mathfrak{m}}',\, \mathfrak{p} \blacktriangle \tilde{\mathfrak{p}}) : (L,\, M,\, M',\, P) \to (\tilde{\tilde{L}},\, \tilde{\tilde{M}},\, \tilde{\tilde{M}}',\, \tilde{\tilde{P}})$$

is also a morphism of crossed squares.

Proof. We have to show properties $(1, 2)$ from Definition 59.

Ad (1). The following morphisms are morphisms of crossed squares as composites of crossed modules morphisms; cf. Remark 27.

(a) $(\mathfrak{l} \blacktriangle \tilde{\mathfrak{l}},\, \mathfrak{m} \blacktriangle \tilde{\mathfrak{m}}) : (L,\, M) \to (\tilde{\tilde{L}},\, \tilde{\tilde{M}})$

(b) $(\mathfrak{l}\blacktriangle\tilde{\mathfrak{l}},\ \mathfrak{m}'\blacktriangle\tilde{\mathfrak{m}}') : (L,\ M') \to (\tilde{\tilde{L}},\ \tilde{\tilde{M}}')$

(c) $(\mathfrak{m}\blacktriangle\tilde{\mathfrak{m}},\ \mathfrak{p}\blacktriangle\tilde{\mathfrak{p}}) : (M,\ P) \to (\tilde{\tilde{M}},\ \tilde{\tilde{P}})$

(d) $(\mathfrak{m}'\blacktriangle\tilde{\mathfrak{m}}',\ \mathfrak{p}\blacktriangle\tilde{\mathfrak{p}}) : (M',\ P) \to (\tilde{\tilde{M}}',\ \tilde{\tilde{P}})$

(e) $(\mathfrak{l}\blacktriangle\tilde{\mathfrak{l}},\ \mathfrak{p}\blacktriangle\tilde{\mathfrak{p}}) : (L,\ P) \to (\tilde{\tilde{L}},\ \tilde{\tilde{P}})$

Ad (2). Suppose given $m \in M$ and $m' \in M'$. Then we have

$$\begin{aligned}\lceil m(\mathfrak{m}\blacktriangle\tilde{\mathfrak{m}}), m'(\mathfrak{m}'\blacktriangle\tilde{\mathfrak{m}}')\rceil &= \lceil (m\mathfrak{m})\tilde{\mathfrak{m}}, (m'\mathfrak{m}')\tilde{\mathfrak{m}}'\rceil \\ &= \lceil m\mathfrak{m}, m'\mathfrak{m}'\rceil\tilde{\mathfrak{l}} \\ &= (\lceil m, m'\rceil\mathfrak{l})\tilde{\mathfrak{l}} \\ &= \lceil m, m'\rceil(\mathfrak{l}\blacktriangle\tilde{\mathfrak{l}}).\end{aligned}$$

□

Remark 61 Suppose given a crossed square $(L,\ M,\ M',\ P)$.

Then its identity, given by

$$\mathrm{id}_{(L,M,M',P)} := (\mathrm{id}_L,\ \mathrm{id}_M,\ \mathrm{id}_{M'},\ \mathrm{id}_P) : (L,\ M,\ M',\ P) \to (L,\ M,\ M',\ P),$$

is a morphism of crossed squares.

Definition 62 We have the *category of crossed squares*, written $\mathcal{C}r\mathcal{S}q$.

It has crossed squares as objects and morphisms of crossed squares as morphisms; cf. Definition 57 and Definition 59.

Composition of morphisms is described in Remark 60.

The identity on an object is described in Remark 61.

Remark 63 Suppose given a morphism of crossed squares

$$\mathfrak{c} = (\mathfrak{l},\ \mathfrak{m},\ \mathfrak{m}',\ \mathfrak{p}) : (L,\ M,\ M',\ P) \to (\tilde{L},\ \tilde{M},\ \tilde{M}',\ \tilde{P}),$$

such that $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{m}'$, $\mathfrak{p}$ are isomorphisms of groups.

Then $\mathfrak{c}$ is an isomorphism in $\mathcal{C}r\mathcal{S}q$ with inverse

$$\mathfrak{c}^- = (\mathfrak{l}^-,\ \mathfrak{m}^-,\ \mathfrak{m}'^-,\ \mathfrak{p}^-) : (\tilde{L},\ \tilde{M},\ \tilde{M}',\ \tilde{P}) \to (L,\ M,\ M',\ P).$$

Proof. It suffices to show that $(\mathfrak{l}^-,\ \mathfrak{m}^-,\ \mathfrak{m}'^-,\ \mathfrak{p}^-)$ is a morphism of crossed squares.

We shall verify conditions $(1, 2)$ of Definition 59.

Ad (1). By Remark 30, $(\mathfrak{l}^-,\ \mathfrak{m}^-)$, $(\mathfrak{l}^-,\ \mathfrak{m}'^-)$, $(\mathfrak{m}^-,\ \mathfrak{p}^-)$, $(\mathfrak{m}'^-,\ \mathfrak{p}^-)$, $(\mathfrak{l}^-,\ \mathfrak{p}^-)$ are morphisms of crossed modules.

Ad (2). Suppose given $\tilde{m} \in \tilde{M}$ and $\tilde{m}' \in \tilde{M}'$.

Then we have

$$\begin{aligned}\lceil \tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-\rceil &= \lceil \tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-\rceil\mathfrak{l}\mathfrak{l}^- \\ &= \lceil \tilde{m}\mathfrak{m}^-\mathfrak{m}, \tilde{m}'\mathfrak{m}'^-\mathfrak{m}'\rceil\mathfrak{l}^- \\ &= \lceil \tilde{m}, \tilde{m}'\rceil\mathfrak{l}^-.\end{aligned}$$

□

Example 64 Suppose given a group P and normal subgroups $M,\ M' \trianglelefteq P$.

Let $L := M \cap M' \trianglelefteq P$.

The inclusion morphisms yield the following commutative diagram.

$$\begin{array}{ccc} L & \overset{\lambda'}{\hookrightarrow} & M' \\ {\scriptstyle\lambda}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\mu'} \\ M & \underset{\mu}{\hookrightarrow} & P\,. \end{array}$$

Conjugation in P gives

$$\begin{array}{lclcccl}
M & \xrightarrow{\gamma_{M,L}} & \mathrm{Aut}(L) & : & m & \longmapsto & (l \mapsto l^m) \\
M' & \xrightarrow{\gamma_{M',L}} & \mathrm{Aut}(L) & : & m' & \longmapsto & (l \mapsto l^{m'}) \\
P & \xrightarrow{\gamma_{P,L}} & \mathrm{Aut}(L) & : & p & \longmapsto & (l \mapsto l^p) \\
P & \xrightarrow{\gamma_{P,M}} & \mathrm{Aut}(M) & : & p & \longmapsto & (m \mapsto m^p) \\
P & \xrightarrow{\gamma_{P,M'}} & \mathrm{Aut}(M') & : & p & \longmapsto & (m' \mapsto m'^p).
\end{array}$$

The commutator bracket in P gives

$$\begin{array}{rcl}
M \times M' & \xrightarrow{\chi} & L \\
(m, m') & \longmapsto & \left(\begin{array}{l} \lceil m, m' \rceil := [m, m'] \\ = m^- m'^- m m' \\ = m^- \cdot m^{m'} \\ = (m'^m)^- \cdot m' \end{array}\right).
\end{array}$$

Then

$$(L,\, M,\, M',\, P,\, \gamma_{M,L},\, \gamma_{M',L},\, \gamma_{P,L},\, \gamma_{P,M},\, \gamma_{P,M'},\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

is a crossed square.

Proof.

Ad (CS 1). We have the inclusion morphism

$$\begin{array}{rcl} \kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : L & \to & P \\ l & \mapsto & l. \end{array}$$

Ad (CS 2.1). The quadruple $(L,\, M,\, \gamma_{M,L},\, \lambda)$ is a crossed module; cf. Example 24.

Ad (CS 2.2). The quadruple $(L,\, M',\, \gamma_{M',L},\, \lambda')$ is a crossed module; cf. Example 24.

Ad (CS 2.3). The quadruple $(L,\, P,\, \gamma_{P,L},\, \kappa)$ is a crossed module; cf. Example 24.

Ad (CS 2.4). The quadruple $(M,\, P,\, \gamma_{P,M},\, \mu)$ is a crossed module; cf. Example 24.

Ad (CS 2.5). The quadruple $(M',\, P,\, \gamma_{P,M'},\, \mu')$ is a crossed module; cf. Example 24.

Ad (CS 3.1). Since $\lambda \blacktriangle \mu = \kappa = \mathrm{id}_L \blacktriangle \kappa$, we have the morphism

$$(\mathrm{id}_L,\, \mu) : (L,\, M,\, \gamma_{M,L},\, \lambda) \to (L,\, P,\, \gamma_{P,L},\, \kappa)$$

of crossed modules; cf. Example 31.

Ad (CS 3.2). Since $\lambda' \blacktriangle \mu' = \kappa = \mathrm{id}_L \blacktriangle \kappa$, we have the morphism

$$(\mathrm{id}_L,\, \mu') : (L,\, M',\, \gamma_{M',L},\, \lambda') \to (L,\, P,\, \gamma_{P,L},\, \kappa)$$

of crossed modules; cf. Example 31.

Ad (CS 3.3). Since $\kappa \blacktriangle \mathrm{id}_P = \kappa = \lambda \blacktriangle \mu$, we have the morphism

$$(\lambda,\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M,\, P,\, \gamma_{P,M},\, \mu)$$

of crossed modules; cf. Example 31.

Ad (CS 3.4). Since $\kappa \blacktriangle \mathrm{id}_P = \kappa = \lambda' \blacktriangle \mu'$, we have the morphism

$$(\lambda', \mathrm{id}_P) : (L, P, \gamma_{P,L}, \kappa) \to (M', P, \gamma_{P,M'}, \mu')$$

of crossed modules; cf. Example 31.

Ad (CS 4.1). Suppose given $m \in M$ and $m' \in M'$.

Then we have

$$\begin{aligned} m \cdot \lceil m, m' \rceil \lambda &= m \cdot [m, m'] \\ &= m \cdot m^- m'^- m m' \\ &= m^{m'} \\ &= m^{m'\mu'}. \end{aligned}$$

Ad (CS 4.2). Suppose given $m \in M$ and $m' \in M'$.

Then we have

$$\begin{aligned} m'^{m\mu} \cdot \lceil m, m' \rceil \lambda' &= m'^m \cdot [m, m'] \\ &= m^- m' m \cdot m^- m'^- m m' \\ &= m'. \end{aligned}$$

Ad (CS 4.3). Suppose given $l \in L$ and $m' \in M'$.

Then we have

$$\begin{aligned} l \cdot \lceil l\lambda, m' \rceil &= l \cdot [l, m'] \\ &= l \cdot l^- m'^- l m' \\ &= l^{m'}. \end{aligned}$$

Ad (CS 4.4). Suppose given $l \in L$ and $m \in M$.

Then we have

$$\begin{aligned} l^m \cdot \lceil m, l\lambda' \rceil &= l^m \cdot [m, l] \\ &= m^- l m \cdot m^- l^- m l \\ &= l. \end{aligned}$$

Ad (CS 4.5). Suppose given $m, m^* \in M$ and $m' \in M'$.

Then we have

$$\begin{aligned} \lceil m, m' \rceil^{m^*} \cdot \lceil m^*, m' \rceil &= m^{*-} m^- m'^- m m' m^* m^{*-} m'^- m^* m' \\ &= m^{*-} m^- m'^- m m^* m' \\ &= \lceil m \cdot m^*, m' \rceil. \end{aligned}$$

Ad (CS 4.6). Suppose given $m \in M$ and $m^{*\prime}, m^{*\prime} \in M'$.

Then we have

$$\begin{aligned} \lceil m, m^{*\prime} \rceil \cdot \lceil m, m' \rceil^{m^{*\prime}} &= m^- m^{*\prime -} m m^{*\prime} m^{*\prime -} m^- m'^- m m' m^{*\prime} \\ &= m^- m^{*\prime -} m'^- m m' m^{*\prime} \\ &= \lceil m, m' \cdot m^{*\prime} \rceil. \end{aligned}$$

Ad (CS 4.7). Suppose given $m \in M$, $m' \in M'$ and $p \in P$.

Then we have

$$\begin{aligned} \lceil m, m' \rceil^p &= (m^- m'^- m m')^p \\ &= (m^p)^- (m'^p)^- m^p m'^p \\ &= \lceil m^p, m'^p \rceil. \end{aligned}$$

Ad (CS 4.8). Suppose given $m \in M$, $m' \in M'$ and $l \in L$.

Then we have

$$\begin{array}{rcl} ((l^{m'})^{m})^{\lceil m,m' \rceil} & = & l^{m' \cdot m \cdot \lceil m,m' \rceil} \\ & = & l^{m' \cdot m \cdot m^{-} m'^{-} m m'} \\ & = & (l^{m})^{m'}. \end{array}$$

□

Example 65 Suppose given abelian groups L, M, M', P and commutative quadrangle of group morphisms

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & M' \\ {\scriptstyle \lambda}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mu'} \\ M & \xrightarrow[\mu]{} & P\,. \end{array}$$

We have the group morphisms

$$\begin{array}{lcllcll} M & \xrightarrow{\gamma_{M,L}} & \mathrm{Aut}(L) & : & m & \longmapsto & (l \mapsto l^{m} := l) \\ M' & \xrightarrow{\gamma_{M',L}} & \mathrm{Aut}(L) & : & m' & \longmapsto & (l \mapsto l^{m'} := l) \\ P & \xrightarrow{\gamma_{P,L}} & \mathrm{Aut}(L) & : & p & \longmapsto & (l \mapsto l^{p} := l) \\ P & \xrightarrow{\gamma_{P,M}} & \mathrm{Aut}(M) & : & p & \longmapsto & (m \mapsto m^{p} := m) \\ P & \xrightarrow{\gamma_{P,M'}} & \mathrm{Aut}(M') & : & p & \longmapsto & (m' \mapsto m'^{p} := m') \end{array}$$

and the map

$$\begin{array}{rcl} M \times M' & \xrightarrow{\chi} & L \\ (m, m') & \longmapsto & \lceil m, m' \rceil := 1. \end{array}$$

Then

$$(L,\, M,\, M',\, P,\, \gamma_{M,L},\, \gamma_{M',L},\, \gamma_{P,L},\, \gamma_{P,M},\, \gamma_{P,M'},\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

is a crossed square.

Proof.

Ad (CS 1). The quadrangle above is supposed to be commutative.

So

$$\lambda \blacktriangle \mu = \lambda' \blacktriangle \mu'.$$

We write

$$\begin{array}{rcl} \kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : L & \to & P \\ l & \mapsto & l\kappa = l\lambda\mu = l\lambda'\mu'. \end{array}$$

Ad (CS 2.1). The quadruple $(L,\, M,\, \gamma_{M,L},\, \lambda)$ is a crossed module; cf. Example 25.

Ad (CS 2.2). The quadruple $(L,\, M',\, \gamma_{M',L},\, \lambda')$ is a crossed module; cf. Example 25.

Ad (CS 2.3). The quadruple $(L,\, P,\, \gamma_{P,L},\, \kappa)$ is a crossed module; cf. Example 25.

Ad (CS 2.4). The quadruple $(M,\, P,\, \gamma_{P,M},\, \mu)$ is a crossed module; cf. Example 25.

Ad (CS 2.5). The quadruple $(M',\, P,\, \gamma_{P,M'},\, \mu')$ is a crossed module; cf. Example 25.

Ad (CS 3.1). Since $\lambda \blacktriangle \mu = \kappa = \mathrm{id}_L \blacktriangle \kappa$, we have the morphism

$$(\mathrm{id}_L,\, \mu) : (L,\, M,\, \gamma_{M,L},\, \lambda) \to (L,\, P,\, \gamma_{P,L},\, \kappa)$$

of crossed modules; cf. Example 32.

Ad (CS 3.2). Since $\lambda' \blacktriangle \mu' = \kappa = \mathrm{id}_L \blacktriangle \kappa$, we have the morphism

$$(\mathrm{id}_L,\, \mu') : (L,\, M',\, \gamma_{M',L},\, \lambda') \to (L,\, P,\, \gamma_{P,L},\, \kappa)$$

of crossed modules; cf. Example 32.

Ad (CS 3.3). Since $\kappa \blacktriangle \mathrm{id}_P = \kappa = \lambda \blacktriangle \mu$, we have the morphism

$$(\lambda,\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M,\, P,\, \gamma_{P,M},\, \mu)$$

of crossed modules; cf. Example 32.

Ad (CS 3.4). Since $\kappa \blacktriangle \mathrm{id}_P = \kappa = \lambda' \blacktriangle \mu'$, we have the morphism

$$(\lambda',\, \mathrm{id}_P) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (M',\, P,\, \gamma_{P,M'},\, \mu')$$

of crossed modules; cf. Example 32.

Ad (CS 4.1). Suppose given $m \in M$ and $m' \in M'$.

Then we have

$$\begin{array}{rcl} m \cdot \lceil m, m' \rceil \lambda & = & m \\ & = & m^{m'\mu'} . \end{array}$$

Ad (CS 4.2). Suppose given $m \in M$ and $m' \in M'$.

Then we have

$$\begin{array}{rcl} m'^{m\mu} \cdot \lceil m, m' \rceil \lambda' & = & m'^{m\mu} \\ & = & m' . \end{array}$$

Ad (CS 4.3). Suppose given $l \in L$ and $m' \in M'$.

Then we have

$$\begin{array}{rcl} l \cdot \lceil l\lambda, m' \rceil & = & l \\ & = & l^{m'} . \end{array}$$

Ad (CS 4.4). Suppose given $m \in M$ and $l \in L$.

Then we have

$$\begin{array}{rcl} l^m \cdot \lceil m, l\lambda' \rceil & = & l^m \\ & = & l . \end{array}$$

Ad (CS 4.5, 4.6, 4.7). The required equations amount to the equation $1 = 1$.

Ad (CS 4.8). Suppose given $m \in M$, $m' \in M'$ and $l \in L$.

Then we have

$$\begin{array}{rcl} ((l^{m'})^m)^{\lceil m, m' \rceil} & = & (l^{m'})^m \\ & = & l \\ & = & (l^m)^{m'} . \end{array}$$

$\square$

Example 66 Suppose given an abelian group A with $A \neq 1$.

We have the commutative quadrangle of group morphisms

$$\begin{array}{ccc} A & \xrightarrow{\lambda' = \mathrm{id}_A} & A \\ {\scriptstyle \lambda = \mathrm{id}_A} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \mu'} \\ A & \xrightarrow[\mu]{} & 1 \,. \end{array}$$

As a particular case of the construction in Example 65, we have the crossed square

$$(A,\, A,\, A,\, 1),$$

for which neither μ nor μ' is injective.

Remark 67 Suppose given a crossed square

$$C = (L,\, M,\, M',\, P,\, \gamma_{M,L},\, \gamma_{M',L},\, \gamma_{P,L},\, \gamma_{P,M},\, \gamma_{P,M'},\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi).$$

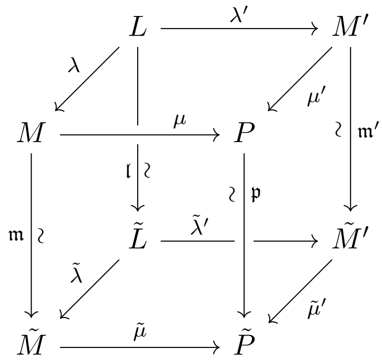
Suppose given group isomorphisms $L \xrightarrow[\sim]{\mathfrak{l}} \tilde{L}$, $M \xrightarrow[\sim]{\mathfrak{m}} \tilde{M}$, $M' \xrightarrow[\sim]{\mathfrak{m}'} \tilde{M}'$ and $P \xrightarrow[\sim]{\mathfrak{p}} \tilde{P}$.

Let $\tilde{\lambda} := \mathfrak{l}^- \blacktriangle \lambda \blacktriangle \mathfrak{m} : \tilde{L} \to \tilde{M}$.

Let $\tilde{\lambda}' := \mathfrak{l}^- \blacktriangle \lambda' \blacktriangle \mathfrak{m}' : \tilde{L} \to \tilde{M}'$.

Let $\tilde{\mu} := \mathfrak{m}^- \blacktriangle \mu \blacktriangle \mathfrak{p} : \tilde{M} \to \tilde{P}$.

Let $\tilde{\mu}' := \mathfrak{m}'^- \blacktriangle \mu' \blacktriangle \mathfrak{p} : \tilde{M}' \to \tilde{P}$.



Recall from Remark 18 that a group isomorphism $\mathfrak{q} : G \xrightarrow{\sim} H$ yields the group isomorphism

$$\begin{array}{rcl} \hat{\mathfrak{q}} : \mathrm{Aut}(G) & \xrightarrow{\sim} & \mathrm{Aut}(H) \\ \alpha & \longmapsto & \mathfrak{q}^- \blacktriangle \alpha \blacktriangle \mathfrak{q}. \end{array}$$

Let $\gamma_{\tilde{M},\tilde{L}} := \mathfrak{m}^- \blacktriangle \gamma_{M,L} \blacktriangle \hat{\mathfrak{l}} : \tilde{M} \to \mathrm{Aut}(\tilde{L})$.

Let $\gamma_{\tilde{M}',\tilde{L}} := \mathfrak{m}'^- \blacktriangle \gamma_{M',L} \blacktriangle \hat{\mathfrak{l}} : \tilde{M}' \to \mathrm{Aut}(\tilde{L})$.

Let $\gamma_{\tilde{P},\tilde{L}} := \mathfrak{p}^- \blacktriangle \gamma_{P,L} \blacktriangle \hat{\mathfrak{l}} : \tilde{P} \to \mathrm{Aut}(\tilde{L})$.

Let $\gamma_{\tilde{P},\tilde{M}} := \mathfrak{p}^- \blacktriangle \gamma_{P,M} \blacktriangle \hat{\mathfrak{m}} : \tilde{P} \to \mathrm{Aut}(\tilde{M})$.

Let $\gamma_{\tilde{P},\tilde{M}'} := \mathfrak{p}^- \blacktriangle \gamma_{P,M'} \blacktriangle \hat{\mathfrak{m}}' : \tilde{P} \to \mathrm{Aut}(\tilde{M}')$.

Let $\tilde{\chi} := (\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l} : \tilde{M} \times \tilde{M}' \to \tilde{L}$.

(1) Then

$$\tilde{C} := (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}},\, \gamma_{\tilde{M}',\tilde{L}},\, \gamma_{\tilde{P},\tilde{L}},\, \gamma_{\tilde{P},\tilde{M}},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi})$$

is a crossed square.

(2) Moreover, $(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p})$ is an isomorphism of crossed squares from C to $\tilde{C}$.

Proof. We write $\kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : L \to P$.

Ad (1). We have to show that $\tilde{C}$ is a crossed square; cf. Definition 57.

Ad (CS 1). We have

$$\begin{array}{rcl} \tilde{\lambda} \blacktriangle \tilde{\mu} & = & (\mathfrak{l}^- \blacktriangle \lambda \blacktriangle \mathfrak{m}) \blacktriangle (\mathfrak{m}^- \blacktriangle \mu \blacktriangle \mathfrak{p}) \\ & = & \mathfrak{l}^- \blacktriangle \lambda \blacktriangle \mu \blacktriangle \mathfrak{p} \\ & = & \mathfrak{l}^- \blacktriangle \lambda' \blacktriangle \mu' \blacktriangle \mathfrak{p} \\ & = & (\mathfrak{l}^- \blacktriangle \lambda' \blacktriangle \mathfrak{m}') \blacktriangle (\mathfrak{m}'^- \blacktriangle \mu' \blacktriangle \mathfrak{p}) \\ & = & \tilde{\lambda}' \blacktriangle \tilde{\mu}'. \end{array}$$

We write $\tilde{\kappa} := \tilde{\lambda} \blacktriangle \tilde{\mu} = \tilde{\lambda}' \blacktriangle \tilde{\mu}' : \tilde{L} \to \tilde{P}$.

Note that we have obtained $\tilde{\kappa} = \mathfrak{l}^- \blacktriangle \kappa \blacktriangle \mathfrak{p}$.

Ad (CS 2). By Remark 33.(1), we have the following crossed modules.

(1) $(\tilde{L},\, \tilde{M},\, \gamma_{\tilde{M},\tilde{L}},\, \tilde{\lambda})$

(2) $(\tilde{L},\, \tilde{M}',\, \gamma_{\tilde{M}',\tilde{L}},\, \tilde{\lambda}')$

(3) $(\tilde{L},\, \tilde{P},\, \gamma_{\tilde{M},\tilde{P}},\, \tilde{\kappa})$

(4) $(\tilde{M},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}},\, \tilde{\mu})$

(5) $(\tilde{M}',\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\mu}')$

Ad (CS 3). By Remark 33.(2), we obtain the following morphisms of crossed modules.

$(\mathfrak{l}, \mathfrak{m}) : (L,\, M,\, \gamma_{M,L},\, \lambda) \xrightarrow{\sim} (\tilde{L},\, \tilde{M},\, \gamma_{\tilde{M},\tilde{L}},\, \tilde{\lambda})$

$(\mathfrak{l}, \mathfrak{m}') : (L,\, M',\, \gamma_{M',L},\, \lambda') \xrightarrow{\sim} (\tilde{L},\, \tilde{M}',\, \gamma_{\tilde{M}',\tilde{L}},\, \tilde{\lambda}')$

$(\mathfrak{l}, \mathfrak{p}) : (L,\, P,\, \gamma_{M,P},\, \kappa) \xrightarrow{\sim} (\tilde{L},\, \tilde{P},\, \gamma_{\tilde{M},\tilde{P}},\, \tilde{\kappa})$

$(\mathfrak{m}, \mathfrak{p}) : (M,\, P,\, \gamma_{P,M},\, \mu) \xrightarrow{\sim} (\tilde{M},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}},\, \tilde{\mu})$

$(\mathfrak{m}', \mathfrak{p}) : (M',\, P,\, \gamma_{P,M'},\, \tilde{\mu}') \xrightarrow{\sim} (\tilde{M}',\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\mu}')$

Thus we have the following morphisms of crossed modules as composites of crossed module morphisms.

(1) $(\mathfrak{l}, \mathfrak{m})^- \blacktriangle (\mathrm{id}_L, \mu) \blacktriangle (\mathfrak{l}, \mathfrak{p}) = (\mathrm{id}_{\tilde{L}}, \tilde{\mu}) : (\tilde{L},\, \tilde{M},\, \gamma_{\tilde{M},\tilde{L}},\, \tilde{\lambda}) \longrightarrow (\tilde{L},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\kappa})$

(2) $(\mathfrak{l}, \mathfrak{m}')^- \blacktriangle (\mathrm{id}_L, \mu') \blacktriangle (\mathfrak{l}, \mathfrak{p}) = (\mathrm{id}_{\tilde{L}}, \tilde{\mu}') : (\tilde{L},\, \tilde{M}',\, \gamma_{\tilde{M}',\tilde{L}},\, \tilde{\lambda}) \longrightarrow (\tilde{L},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\kappa})$

(3) $(\mathfrak{l}, \mathfrak{p})^- \blacktriangle (\lambda, \mathrm{id}_P) \blacktriangle (\mathfrak{m}, \mathfrak{p}) = (\tilde{\lambda}, \mathrm{id}_{\tilde{P}}) : (\tilde{L},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\kappa}) \longrightarrow (\tilde{M},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}},\, \tilde{\mu})$

(4) $(\mathfrak{l}, \mathfrak{p})^- \blacktriangle (\lambda', \mathrm{id}_P) \blacktriangle (\mathfrak{m}', \mathfrak{p}) = (\tilde{\lambda}', \mathrm{id}_{\tilde{P}}) : (\tilde{L},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\kappa}) \longrightarrow (\tilde{M}',\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\mu}')$

Ad (CS 4). We write $\lceil m, m' \rceil = (m, m')\chi$ for $m \in M$ and $m' \in M'$.

We write $\lceil \tilde{m}, \tilde{m}' \rceil = (\tilde{m}, \tilde{m}')\tilde{\chi}$ for $\tilde{m} \in M$ and $\tilde{m}' \in M'$.

Ad (CS 4.1). Suppose given $\tilde{m} \in \tilde{M}$ and $\tilde{m}' \in \tilde{M}'$.

We write $m := \tilde{m}\mathfrak{m}^- \in M$ and $m' := \tilde{m}'\mathfrak{m}'^- \in M'$.

We obtain

$$
\begin{array}{rcl}
\tilde{m} \cdot \lceil \tilde{m}, \tilde{m}' \rceil \tilde{\lambda} & = & \tilde{m} \cdot (\tilde{m}, \tilde{m}')\tilde{\chi}\tilde{\lambda} \\
& = & \tilde{m} \cdot (\tilde{m}, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l})(\mathfrak{l}^- \blacktriangle \lambda \blacktriangle \mathfrak{m}) \\
& = & \tilde{m} \cdot (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\lambda\mathfrak{m} \\
& = & \tilde{m}\mathfrak{m}^-\mathfrak{m} \cdot (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\lambda\mathfrak{m} \\
& = & (\tilde{m}\mathfrak{m}^- \cdot (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\lambda)\mathfrak{m} \\
& = & (m \cdot (m, m')\chi\lambda)\mathfrak{m} \\
& = & (m \cdot \lceil m, m' \rceil \lambda)\mathfrak{m} \\
& \overset{\text{(CS 4.1)}}{\underset{\text{for } C}{=}} & (m^{m'\mu'})\mathfrak{m} \\
& = & (m)((m'\mu')\gamma_{P,M})\mathfrak{m} \\
& = & (\tilde{m}\mathfrak{m}^-)(\tilde{m}'\mathfrak{m}'^-\mu'\gamma_{P,M})\mathfrak{m} \\
& = & (\tilde{m}\mathfrak{m}^-)(\tilde{m}'\mathfrak{m}'^-\mu'\mathfrak{p}\mathfrak{p}^-\gamma_{P,M})\mathfrak{m} \\
& = & (\tilde{m}\mathfrak{m}^-)(\tilde{m}'\tilde{\mu}'\mathfrak{p}^-\gamma_{P,M})\mathfrak{m} \\
& = & (\tilde{m}\mathfrak{m}^-)(\tilde{m}'\tilde{\mu}'\mathfrak{p}^-\gamma_{P,M}\hat{\mathfrak{m}}\hat{\mathfrak{m}}^-)\mathfrak{m}
\end{array}
$$

$$
\begin{aligned}
&= (\tilde{m}\mathfrak{m}^-)(\tilde{m}'\tilde{\mu}'\gamma_{\tilde{P},\tilde{M}}\hat{\mathfrak{m}}^-)\mathfrak{m} \\
&\overset{\text{R.18}}{=} (\tilde{m}\mathfrak{m}^-)(\mathfrak{m} \blacktriangle (\tilde{m}'\tilde{\mu}'\gamma_{\tilde{P},\tilde{M}}) \blacktriangle \mathfrak{m}^-)\mathfrak{m} \\
&= (\tilde{m})(\tilde{m}'\tilde{\mu}'\gamma_{\tilde{P},\tilde{M}}) \\
&= (\tilde{m})^{\tilde{m}'\tilde{\mu}'}.
\end{aligned}
$$

Ad (CS 4.2). Suppose given $\tilde{m} \in \tilde{M}$ and $\tilde{m}' \in \tilde{M}'$.

We write $m := \tilde{m}\mathfrak{m}^- \in M$ and $m' := \tilde{m}'\mathfrak{m}'^- \in M'$.

We obtain

$$
\begin{aligned}
& \tilde{m}'^{\tilde{m}\tilde{\mu}} \cdot \lceil \tilde{m}, \tilde{m}' \rceil \tilde{\lambda}' \\
= \;& \tilde{m}'^{\tilde{m}\tilde{\mu}} \cdot (\tilde{m}, \tilde{m}')\tilde{\chi}\tilde{\lambda}' \\
= \;& (\tilde{m}')(\tilde{m}\tilde{\mu}\gamma_{\tilde{P},\tilde{M}'}) \cdot (\tilde{m}, \tilde{m}')\tilde{\chi}\tilde{\lambda}' \\
= \;& (\tilde{m}')(\tilde{m}(\mathfrak{m}^- \blacktriangle \mu \blacktriangle \mathfrak{p})(\mathfrak{p}^- \blacktriangle \gamma_{P,M'} \blacktriangle \hat{\mathfrak{m}}')) \cdot (\tilde{m}, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l})(\mathfrak{l}^- \blacktriangle \lambda' \blacktriangle \mathfrak{m}') \\
= \;& (\tilde{m}')(\tilde{m}\mathfrak{m}^-\mu\gamma_{P,M'}\hat{\mathfrak{m}}') \cdot (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\lambda'\mathfrak{m}' \\
= \;& (\tilde{m}'\mathfrak{m}'^-\mathfrak{m}')(\tilde{m}\mathfrak{m}^-\mu\gamma_{P,M'}\hat{\mathfrak{m}}') \cdot (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\lambda'\mathfrak{m}' \\
= \;& (m'\mathfrak{m}')(m\mu\gamma_{P,M'}\hat{\mathfrak{m}}') \cdot (m, m')\chi\lambda'\mathfrak{m}' \\
\overset{\text{R.18}}{=} \;& (m'\mathfrak{m}')(\mathfrak{m}'^- \blacktriangle m\mu\gamma_{P,M'} \blacktriangle \mathfrak{m}') \cdot (m, m')\chi\lambda'\mathfrak{m}' \\
= \;& ((m')(m\mu\gamma_{P,M'}) \cdot (m, m')\chi\lambda')\mathfrak{m}' \\
= \;& (m'^{m\mu} \cdot \lceil m, m' \rceil \lambda')\mathfrak{m}' \\
\underset{\text{for } C}{\overset{\text{(CS 4.2)}}{=}} \;& m'\mathfrak{m}' \\
= \;& \tilde{m}'.
\end{aligned}
$$

Ad (CS 4.3). Suppose given $\tilde{l} \in \tilde{L}$ and $\tilde{m}' \in \tilde{M}'$.

We write $l := \tilde{l}\mathfrak{l}^- \in L$ and $m' := \tilde{m}'\mathfrak{m}'^- \in M'$.

We obtain

$$
\begin{aligned}
\tilde{l} \cdot \lceil \tilde{l}\tilde{\lambda}, \tilde{m}' \rceil &= \tilde{l} \cdot (\tilde{l}\tilde{\lambda}, \tilde{m}')\tilde{\chi} \\
&= \tilde{l} \cdot (\tilde{l}(\mathfrak{l}^- \blacktriangle \lambda \blacktriangle \mathfrak{m}), \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \\
&= \tilde{l} \cdot (\tilde{l}\mathfrak{l}^-\lambda\mathfrak{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= \tilde{l} \cdot (\tilde{l}\mathfrak{l}^-\lambda, \tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= \tilde{l}\mathfrak{l}^-\mathfrak{l} \cdot (\tilde{l}\mathfrak{l}^-\lambda, \tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= (\tilde{l}\mathfrak{l}^- \cdot (\tilde{l}\mathfrak{l}^-\lambda, \tilde{m}'\mathfrak{m}'^-)\chi)\mathfrak{l} \\
&= (l \cdot (l\lambda, m')\chi)\mathfrak{l} \\
&= (l \cdot \lceil l\lambda, m' \rceil)\mathfrak{l} \\
&\underset{\text{for } C}{\overset{\text{(CS 4.3)}}{=}} (l^{m'})\mathfrak{l} \\
&= ((l)(m'\gamma_{M',L}))\mathfrak{l} \\
&= (\tilde{l}\mathfrak{l}^-)(\tilde{m}'\mathfrak{m}'^-\gamma_{M',L})\mathfrak{l} \\
&= (\tilde{l}\mathfrak{l}^-)(\tilde{m}'(\mathfrak{m}'^- \blacktriangle \gamma_{M',L} \blacktriangle \hat{\mathfrak{l}})\hat{\mathfrak{l}}^-)\mathfrak{l} \\
&= (\tilde{l}\mathfrak{l}^-)(\tilde{m}'\gamma_{\tilde{M}',\tilde{L}}\hat{\mathfrak{l}}^-)\mathfrak{l} \\
&\overset{\text{R.18}}{=} (\tilde{l}\mathfrak{l}^-)(\mathfrak{l} \blacktriangle \tilde{m}'\gamma_{\tilde{M}',\tilde{L}} \blacktriangle \mathfrak{l}^-)\mathfrak{l} \\
&= (\tilde{l})(\tilde{m}'\gamma_{\tilde{M}',\tilde{L}}) \\
&= \tilde{l}^{\tilde{m}'}.
\end{aligned}
$$

Ad (CS 4.4). Suppose given $\tilde{l} \in \tilde{L}$ and $\tilde{m} \in \tilde{M}$.

We write $l := \tilde{l}\mathfrak{l}^- \in L$ and $m := \tilde{m}\mathfrak{m}^- \in M$.

We obtain

$$
\begin{aligned}
\tilde{l}^{\tilde{m}} \cdot \lceil \tilde{m}, \tilde{l}\tilde{\lambda}' \rceil &= (\tilde{l})(\tilde{m}\gamma_{\tilde{M},\tilde{L}}) \cdot (\tilde{m}, \tilde{l}\tilde{\lambda}')\tilde{\chi} \\
&= (\tilde{l})(\tilde{m}(\mathfrak{m}^- \blacktriangle \gamma_{M,L} \blacktriangle \hat{\mathfrak{l}})) \cdot (\tilde{m}, \tilde{l}(\mathfrak{l}^- \blacktriangle \lambda' \blacktriangle \mathfrak{m}'))((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \\
&= (\tilde{l})(\tilde{m}\mathfrak{m}^-\gamma_{M,L}\hat{\mathfrak{l}})) \cdot (\tilde{m}\mathfrak{m}^-, \tilde{l}\mathfrak{l}^-\lambda'\mathfrak{m}'\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= (\tilde{l})(\tilde{m}\mathfrak{m}^-\gamma_{M,L}\hat{\mathfrak{l}}) \cdot (\tilde{m}\mathfrak{m}^-, \tilde{l}\mathfrak{l}^-\lambda')\chi\mathfrak{l} \\
&\overset{\text{R.18}}{=} (\tilde{l})(\mathfrak{l}^- \blacktriangle \tilde{m}\mathfrak{m}^-\gamma_{M,L} \blacktriangle \mathfrak{l}) \cdot (\tilde{m}\mathfrak{m}^-, \tilde{l}\mathfrak{l}^-\lambda')\chi\mathfrak{l} \\
&= (l)(m\gamma_{M,L})\mathfrak{l} \cdot (m, l\lambda')\chi\mathfrak{l} \\
&= ((l)(m\gamma_{M,L}) \cdot (m, l\lambda')\chi)\mathfrak{l} \\
&= (l^m \cdot \lceil m, l\lambda' \rceil)\mathfrak{l} \\
&\underset{\text{for } C}{\overset{\text{(CS 4.4)}}{=}} l\mathfrak{l} \\
&= \tilde{l}.
\end{aligned}
$$

Ad (CS 4.5). Suppose given $\tilde{m}$, $\tilde{m}^* \in \tilde{M}$ and $\tilde{m}' \in \tilde{M}'$.

We write $m := \tilde{m}\mathfrak{m}^- \in M$, $m^* := \tilde{m}^*\mathfrak{m}^- \in M$ and $m' := \tilde{m}'\mathfrak{m}'^- \in M'$.

We obtain

$$
\begin{aligned}
\lceil \tilde{m} \cdot \tilde{m}^*, \tilde{m}' \rceil &= (\tilde{m} \cdot \tilde{m}^*, \tilde{m}')\tilde{\chi} \\
&= (\tilde{m} \cdot \tilde{m}^*, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \\
&= ((\tilde{m} \cdot \tilde{m}^*)\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= (\tilde{m}\mathfrak{m}^- \cdot \tilde{m}^*\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= (m \cdot m^*, m')\chi\mathfrak{l} \\
&= \lceil m \cdot m^*, m' \rceil\mathfrak{l} \\
&\underset{\text{for } C}{\overset{\text{(CS 4.5)}}{=}} (\lceil m, m' \rceil^{m^*} \cdot \lceil m^*, m' \rceil)\mathfrak{l} \\
&= \lceil m, m' \rceil^{m^*}\mathfrak{l} \cdot \lceil m^*, m' \rceil\mathfrak{l} \\
&= \lceil m, m' \rceil(m^*\gamma_{M,L})\mathfrak{l} \cdot \lceil m^*, m' \rceil\mathfrak{l} \\
&= (m, m')\chi(m^*\gamma_{M,L})\mathfrak{l} \cdot (m^*, m')\chi\mathfrak{l} \\
&= (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi(m^*\gamma_{M,L})\mathfrak{l} \cdot (\tilde{m}^*\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= (\tilde{m}, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi)(m^*\gamma_{M,L} \blacktriangle \mathfrak{l}) \cdot (\tilde{m}^*, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \\
&= (\tilde{m}, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l})(\mathfrak{l}^- \blacktriangle m^*\gamma_{M,L} \blacktriangle \mathfrak{l}) \cdot (\tilde{m}^*, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \\
&= (\tilde{m}, \tilde{m}')\tilde{\chi}(\mathfrak{l}^- \blacktriangle m^*\gamma_{M,L} \blacktriangle \mathfrak{l}) \cdot (\tilde{m}^*, \tilde{m}')\tilde{\chi} \\
&\overset{\text{R.18}}{=} (\tilde{m}, \tilde{m}')\tilde{\chi}(m^*\gamma_{M,L}\hat{\mathfrak{l}}) \cdot (\tilde{m}^*, \tilde{m}')\tilde{\chi} \\
&= (\tilde{m}, \tilde{m}')\tilde{\chi}(m^*\mathfrak{m}(\mathfrak{m}^- \blacktriangle \gamma_{M,L} \blacktriangle \hat{\mathfrak{l}})) \cdot (\tilde{m}^*, \tilde{m}')\tilde{\chi} \\
&= (\tilde{m}, \tilde{m}')\tilde{\chi}(m^*\mathfrak{m}\gamma_{\tilde{M},\tilde{L}}) \cdot (\tilde{m}^*, \tilde{m}')\tilde{\chi} \\
&= (\tilde{m}, \tilde{m}')\tilde{\chi}(\tilde{m}^*\gamma_{\tilde{M},\tilde{L}}) \cdot (\tilde{m}^*, \tilde{m}')\tilde{\chi} \\
&= \lceil \tilde{m}, \tilde{m}' \rceil(\tilde{m}^*\gamma_{\tilde{M},\tilde{L}}) \cdot \lceil \tilde{m}^*, \tilde{m}' \rceil \\
&= \lceil \tilde{m}, \tilde{m}' \rceil^{\tilde{m}^*} \cdot \lceil \tilde{m}^*, \tilde{m}' \rceil.
\end{aligned}
$$

Ad (CS 4.6). Suppose given $\tilde{m} \in \tilde{M}$ and $\tilde{m}'$, $\tilde{m}^{*\prime} \in \tilde{M}'$.

We write $m := \tilde{m}\mathfrak{m}^- \in M$, $m' := \tilde{m}'\mathfrak{m}'^- \in M'$ and $m^{*\prime} := \tilde{m}^{*\prime}\mathfrak{m}'^- \in M'$.

We obtain

$$
\begin{aligned}
\lceil \tilde{m}, \tilde{m}' \cdot \tilde{m}^{*\prime} \rceil &= (\tilde{m}, \tilde{m}' \cdot \tilde{m}^{*\prime})\tilde{\chi} \\
&= (\tilde{m}, \tilde{m}' \cdot \tilde{m}^{*\prime})((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \\
&= (\tilde{m}\mathfrak{m}^-, (\tilde{m}' \cdot \tilde{m}^{*\prime})\mathfrak{m}'^-)\chi\mathfrak{l}
\end{aligned}
$$

$$
\begin{aligned}
&= (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^- \cdot \tilde{m}^{*\prime}\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= (m, m' \cdot m^{*\prime})\chi\mathfrak{l} \\
&= \lceil m, m' \cdot m^{*\prime} \rceil \mathfrak{l} \\
&\overset{\substack{\text{(CS 4.6)}\\ \text{for } C}}{=} (\lceil m, m^{*\prime} \rceil \cdot \lceil m, m' \rceil^{m^{*\prime}})\mathfrak{l} \\
&= (\lceil m, m^{*\prime} \rceil \cdot \lceil m, m' \rceil (m^{*\prime}\gamma_{M',L}))\mathfrak{l} \\
&= \lceil m, m^{*\prime} \rceil \mathfrak{l} \cdot \lceil m, m' \rceil (m^{*\prime}\gamma_{M',L})\mathfrak{l} \\
&= (m, m^{*\prime})\chi\mathfrak{l} \cdot (m, m')\chi(m^{*\prime}\gamma_{M',L})\mathfrak{l} \\
&= (\tilde{m}\mathfrak{m}^-, \tilde{m}^{*\prime}\mathfrak{m}'^-)\chi\mathfrak{l} \cdot (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi(\tilde{m}^{*\prime}\mathfrak{m}'^-\gamma_{M',L})\mathfrak{l} \\
&= (\tilde{m}, \tilde{m}^{*\prime})((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \cdot (\tilde{m}, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi)(\tilde{m}^{*\prime}\mathfrak{m}'^-\gamma_{M',L} \blacktriangle \mathfrak{l}) \\
&= (\tilde{m}, \tilde{m}^{*\prime})((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \cdot (\tilde{m}, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l})(\mathfrak{l}^- \blacktriangle \tilde{m}^{*\prime}\mathfrak{m}'^-\gamma_{M',L} \blacktriangle \mathfrak{l}) \\
&= (\tilde{m}, \tilde{m}^{*\prime})\tilde{\chi} \cdot (\tilde{m}, \tilde{m}')\tilde{\chi}(\mathfrak{l}^- \blacktriangle \tilde{m}^{*\prime}\mathfrak{m}'^-\gamma_{M',L} \blacktriangle \mathfrak{l}) \\
&\overset{\text{R.18}}{=} (\tilde{m}, \tilde{m}^{*\prime})\tilde{\chi} \cdot (\tilde{m}, \tilde{m}')\tilde{\chi}(\tilde{m}^{*\prime}\mathfrak{m}'^-\gamma_{M',L}\hat{\mathfrak{l}}) \\
&= (\tilde{m}, \tilde{m}^{*\prime})\tilde{\chi} \cdot (\tilde{m}, \tilde{m}')\tilde{\chi}(\tilde{m}^{*\prime}(\mathfrak{m}'^- \blacktriangle \gamma_{M',L} \blacktriangle \hat{\mathfrak{l}})) \\
&= (\tilde{m}, \tilde{m}^{*\prime})\tilde{\chi} \cdot (\tilde{m}, \tilde{m}')\tilde{\chi}(\tilde{m}^{*\prime}\gamma_{\tilde{M}',\tilde{L}}) \\
&= \lceil \tilde{m}, \tilde{m}^{*\prime} \rceil \cdot \lceil \tilde{m}, \tilde{m}' \rceil (\tilde{m}^{*\prime}\gamma_{\tilde{M}',\tilde{L}}) \\
&= \lceil \tilde{m}, \tilde{m}^{*\prime} \rceil \cdot \lceil \tilde{m}, \tilde{m}' \rceil^{\tilde{m}^{*\prime}}.
\end{aligned}
$$

Ad (CS 4.7). Suppose given $\tilde{m} \in \tilde{M}$, $\tilde{m}' \in \tilde{M}'$ and $\tilde{p} \in P$.

We write $m := \tilde{m}\mathfrak{m}^- \in M$, $m' := \tilde{m}'\mathfrak{m}'^- \in M'$ and $p := \tilde{p}\mathfrak{p}^- \in P$.

We obtain

$$
\begin{aligned}
\lceil \tilde{m}, \tilde{m}' \rceil^{\tilde{p}} &= \lceil \tilde{m}, \tilde{m}' \rceil (\tilde{p}\gamma_{\tilde{P},\tilde{L}}) \\
&= (\tilde{m}, \tilde{m}')\tilde{\chi}(\tilde{p}\gamma_{\tilde{P},\tilde{L}}) \\
&= (\tilde{m}, \tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l})(\tilde{p}(\mathfrak{p}^- \blacktriangle \gamma_{P,L} \blacktriangle \hat{\mathfrak{l}})) \\
&= (\tilde{m}\mathfrak{m}^-, \tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l}(\tilde{p}\mathfrak{p}^-\gamma_{P,L}\hat{\mathfrak{l}}) \\
&= (m, m')\chi\mathfrak{l}(p\gamma_{P,L}\hat{\mathfrak{l}}) \\
&\overset{\text{R.18}}{=} (m, m')\chi\mathfrak{l}(\mathfrak{l}^- \blacktriangle p\gamma_{P,L} \blacktriangle \mathfrak{l}) \\
&= (m, m')\chi(p\gamma_{P,L})\mathfrak{l} \\
&= \lceil m, m' \rceil (p\gamma_{P,L})\mathfrak{l} \\
&= \lceil m, m' \rceil^p \mathfrak{l} \\
&\overset{\substack{\text{(CS 4.7)}\\ \text{for } C}}{=} \lceil m^p, m'^p \rceil \mathfrak{l} \\
&= \lceil (m)(p\gamma_{P,M}), m'(p\gamma_{P,M'}) \rceil \mathfrak{l} \\
&= (m(p\gamma_{P,M}), m'(p\gamma_{P,M'}))\chi\mathfrak{l} \\
&= (\tilde{m}\mathfrak{m}^-(p\gamma_{P,M}), \tilde{m}'\mathfrak{m}'^-(p\gamma_{P,M'}))\chi\mathfrak{l} \\
&= (\tilde{m}(\mathfrak{m}^- \blacktriangle (p\gamma_{P,M})), \tilde{m}'(\mathfrak{m}'^- \blacktriangle (p\gamma_{P,M'})))\chi\mathfrak{l} \\
&= (\tilde{m}(\mathfrak{m}^- \blacktriangle (p\gamma_{P,M}) \blacktriangle \mathfrak{m})\mathfrak{m}^-, \tilde{m}'(\mathfrak{m}'^- \blacktriangle (p\gamma_{P,M'}) \blacktriangle \mathfrak{m}'^-)\mathfrak{m}')\chi\mathfrak{l} \\
&\overset{\text{R.18}}{=} (\tilde{m}(p\gamma_{P,M}\hat{\mathfrak{m}})\mathfrak{m}^-, \tilde{m}'(p\gamma_{P,M'}\hat{\mathfrak{m}}'^-)\mathfrak{m}')\chi\mathfrak{l} \\
&= (\tilde{m}(p\gamma_{P,M}\hat{\mathfrak{m}}), \tilde{m}'(p\gamma_{P,M'}\hat{\mathfrak{m}}'^-))((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \\
&= (\tilde{m}(p\gamma_{P,M}\hat{\mathfrak{m}}), \tilde{m}'(p\gamma_{P,M'}\hat{\mathfrak{m}}'^-))\tilde{\chi} \\
&= (\tilde{m}(\tilde{p}\mathfrak{p}^-\gamma_{P,M}\hat{\mathfrak{m}}), \tilde{m}'(\tilde{p}\mathfrak{p}^-\gamma_{P,M'}\hat{\mathfrak{m}}'^-))\tilde{\chi} \\
&= (\tilde{m}(\tilde{p}(\mathfrak{p}^- \blacktriangle \gamma_{P,M} \blacktriangle \hat{\mathfrak{m}})), \tilde{m}'(\tilde{p}(\mathfrak{p}^- \blacktriangle \gamma_{P,M'} \blacktriangle \hat{\mathfrak{m}}'^-)))\tilde{\chi} \\
&= (\tilde{m}(\tilde{p}\gamma_{\tilde{P},\tilde{M}}), \tilde{m}'(\tilde{p}\gamma_{\tilde{P},\tilde{M}'}))\tilde{\chi} \\
&= \lceil \tilde{m}(\tilde{p}\gamma_{\tilde{P},\tilde{M}}), \tilde{m}'(\tilde{p}\gamma_{\tilde{P},\tilde{M}'}) \rceil \\
&= \lceil \tilde{m}^{\tilde{p}}, \tilde{m}'^{\tilde{p}} \rceil.
\end{aligned}
$$

Ad (CS 4.8). Suppose given $\tilde{l} \in \tilde{L}$, $\tilde{m} \in \tilde{M}$ and $\tilde{m}' \in \tilde{M}'$.

We write $l := \tilde{l}\mathfrak{l}^- \in L$, $m := \tilde{m}\mathfrak{m}^- \in M$ and $m' := \tilde{m}'\mathfrak{m}'^- \in M'$.

We obtain

$$\begin{aligned}
((\tilde{l}^{\tilde{m}'})^{\tilde{m}})^{\lceil \tilde{m},\tilde{m}' \rceil} &= ((\tilde{l}^{\tilde{m}'})^{\tilde{m}})^{(\tilde{m},\tilde{m}')\tilde{\chi}} \\
&= (\tilde{l}(\tilde{m}'\gamma_{\tilde{M}',\tilde{L}})(\tilde{m}\gamma_{\tilde{M},\tilde{L}}))^{(\tilde{m},\tilde{m}')\tilde{\chi}} \\
&= (\tilde{l}(\tilde{m}'(\mathfrak{m}'^- \blacktriangle \gamma_{M',L} \blacktriangle \hat{\mathfrak{l}}))(\tilde{m}(\mathfrak{m}^- \blacktriangle \gamma_{M,L} \blacktriangle \hat{\mathfrak{l}})))^{(\tilde{m},\tilde{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l})} \\
&= (\tilde{l}(\tilde{m}'\mathfrak{m}'^-\gamma_{M',L}\hat{\mathfrak{l}})(\tilde{m}\mathfrak{m}^-\gamma_{M,L}\hat{\mathfrak{l}}))^{(\tilde{m}\mathfrak{m}^-,\tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l}} \\
&\overset{\text{R.18}}{=} (\tilde{l}(\mathfrak{l}^- \blacktriangle \tilde{m}'\mathfrak{m}'^-\gamma_{M',L} \blacktriangle \mathfrak{l})(\mathfrak{l}^- \blacktriangle \tilde{m}\mathfrak{m}^-\gamma_{M,L} \blacktriangle \mathfrak{l}))^{(\tilde{m}\mathfrak{m}^-,\tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l}} \\
&= (\tilde{l}\mathfrak{l}^-(\tilde{m}'\mathfrak{m}'^-\gamma_{M',L})(\tilde{m}\mathfrak{m}^-\gamma_{M,L})\mathfrak{l})^{(\tilde{m}\mathfrak{m}^-,\tilde{m}'\mathfrak{m}'^-)\chi\mathfrak{l}} \\
&= (l(m'\gamma_{M',L})(m\gamma_{M,L})\mathfrak{l})^{(m,m')\chi\mathfrak{l}} \\
&= ((l(m'\gamma_{M',L})(m\gamma_{M,L}))^{(m,m')\chi})\mathfrak{l} \\
&= ((l(m'\gamma_{M',L})(m\gamma_{M,L}))^{\lceil m,m' \rceil})\mathfrak{l} \\
&= (((l^{m'})^{m})^{\lceil m,m' \rceil})\mathfrak{l} \\
&\overset{\substack{\text{(CS 4.8)} \\ \text{for } C}}{=} ((l^{m})^{m'})\mathfrak{l} \\
&= l(m\gamma_{M,L})(m'\gamma_{M',L})\mathfrak{l} \\
&= \tilde{l}\mathfrak{l}^-(\tilde{m}\mathfrak{m}^-\gamma_{M,L})(\tilde{m}'\mathfrak{m}'^-\gamma_{M',L})\mathfrak{l} \\
&= \tilde{l}(\mathfrak{l}^- \blacktriangle \tilde{m}\mathfrak{m}^-\gamma_{M,L} \blacktriangle \mathfrak{l})(\mathfrak{l}^- \blacktriangle \tilde{m}'\mathfrak{m}'^-\gamma_{M',L} \blacktriangle \mathfrak{l}) \\
&\overset{\text{R.18}}{=} \tilde{l}(\tilde{m}\mathfrak{m}^-\gamma_{M,L}\hat{\mathfrak{l}})(\tilde{m}'\mathfrak{m}'^-\gamma_{M',L}\hat{\mathfrak{l}}) \\
&= \tilde{l}(\tilde{m}(\mathfrak{m}^- \blacktriangle \gamma_{M,L} \blacktriangle \hat{\mathfrak{l}}))(\tilde{m}'(\mathfrak{m}'^- \blacktriangle \gamma_{M',L} \blacktriangle \hat{\mathfrak{l}})) \\
&= \tilde{l}(\tilde{m}\gamma_{\tilde{M},\tilde{L}})(\tilde{m}'\gamma_{\tilde{M}',\tilde{L}}) \\
&= (\tilde{l}^{\tilde{m}})^{\tilde{m}'}.
\end{aligned}$$

Ad (2). By Remark 63, it suffices to show that $(\mathfrak{l},\, \mathfrak{m},\, \mathfrak{m}',\, \mathfrak{p}) : C \to \tilde{C}$ is a morphism of crossed squares; cf. Definition 59.

Ad (1 of Definition 59). By Remark 33.(2), we have the following morphisms of crossed modules.

(1) $(\mathfrak{l},\, \mathfrak{m}) : (L,\, M,\, \gamma_{L,M},\, \lambda) \to (\tilde{L},\, \tilde{M},\, \gamma_{\tilde{L},\tilde{M}},\, \tilde{\lambda})$

(2) $(\mathfrak{l},\, \mathfrak{m}') : (L,\, M',\, \gamma_{L,M'},\, \lambda') \to (\tilde{L},\, \tilde{M}',\, \gamma_{\tilde{L},\tilde{M}'},\, \tilde{\lambda}')$

(3) $(\mathfrak{m},\, \mathfrak{p}) : (M,\, P,\, \gamma_{P,M},\, \mu) \to (\tilde{M},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}},\, \tilde{\mu})$

(4) $(\mathfrak{m}',\, \mathfrak{p}) : (M',\, P,\, \gamma_{P,M'},\, \mu') \to (\tilde{M}',\, \tilde{P},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\mu}')$

(5) $(\mathfrak{l},\, \mathfrak{p}) : (L,\, P,\, \gamma_{P,L},\, \kappa) \to (\tilde{L},\, \tilde{P},\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\kappa})$

Ad (2 of Definition 59). Suppose given $m \in M$ and $m' \in M'$. We obtain

$$\begin{aligned}
\lceil m\mathfrak{m}, m'\mathfrak{m}' \rceil &= (m\mathfrak{m}, m'\mathfrak{m}')\tilde{\chi} \\
&= (m\mathfrak{m}, m'\mathfrak{m}')((\mathfrak{m}^- \times \mathfrak{m}'^-) \blacktriangle \chi \blacktriangle \mathfrak{l}) \\
&= (m\mathfrak{m}\mathfrak{m}^-, m'\mathfrak{m}'\mathfrak{m}'^-)\chi\mathfrak{l} \\
&= (m, m')\chi\mathfrak{l} \\
&= \lceil m, m' \rceil\mathfrak{l}.
\end{aligned}$$

□

5.2 From $\lfloor 2,0 \rfloor$-simplicial groups to crossed squares

5.2.1 The construction for the objects

Suppose given a $\lfloor 2,0 \rfloor$-simplicial group G.

Remark 68 The following statements hold.

(1) $[G_{2;0},\, G_{2;1,2}] = 1$

(2) $[G_{2;0,1},\, G_{2;1,2}] = 1$

(3) $[G_{2;0,2},\, G_{2;1,2}] = 1$

(4) $[G_{2;0},\, G_{2;1}] \leqslant G_{2;0,1}$

(5) $[G_{2;0},\, G_{2;2}] \leqslant G_{2;0,2}$

(6) $[G_{2;1},\, G_{2;2}] \leqslant G_{2;1,2}$

Proof.

Ad $(1,2,3)$. The required equations hold due to Remark 54.

Ad (4). Suppose given $z \in G_{2;0}$ and $g \in G_{2;1}$.

Then we have

$$\begin{aligned}[z,g]\,\mathrm{d}_0 &= [z\,\mathrm{d}_0,\, g\,\mathrm{d}_0] = [1,\, g\,\mathrm{d}_0] = 1\\ [z,g]\,\mathrm{d}_1 &= [z\,\mathrm{d}_1,\, g\,\mathrm{d}_1] = [z\,\mathrm{d}_1,\, 1] = 1.\end{aligned}$$

So $[z,g] \in G_{2;0,1}$.

Ad (5). Suppose given $z \in G_{2;0}$ and $h \in G_{2;2}$.

Then we have

$$\begin{aligned}[z,h]\,\mathrm{d}_0 &= [z\,\mathrm{d}_0,\, h\,\mathrm{d}_0] = [1,\, h\,\mathrm{d}_0] = 1\\ [z,h]\,\mathrm{d}_2 &= [z\,\mathrm{d}_2,\, h\,\mathrm{d}_2] = [z\,\mathrm{d}_2,\, 1] = 1.\end{aligned}$$

So $[z,h] \in G_{2;0,2}$.

Ad (6). Suppose given $g \in G_{2;1}$ and $h \in G_{2;2}$.

Then we have

$$\begin{aligned}[g,h]\,\mathrm{d}_1 &= [g\,\mathrm{d}_1,\, h\,\mathrm{d}_1] = [1,\, h\,\mathrm{d}_1] = 1\\ [g,h]\,\mathrm{d}_2 &= [g\,\mathrm{d}_2,\, h\,\mathrm{d}_2] = [g\,\mathrm{d}_2,\, 1] = 1.\end{aligned}$$

So $[g,h] \in G_{2;1,2}$. □

Remark 69 We have $[g,h] = [gx,\, yh]$ for $g \in G_{2;1}$, $h \in G_{2;2}$, $x \in G_{2;0,1}$ and $y \in G_{2;0,2}$.

Proof.

We have

$$\begin{aligned}[g,h][gx,\, yh]^- &= [g,h][yh,\, gx]\\ &= g^-h^-ghh^-y^-x^-g^-yhgx\\ &\overset{[x,y]=1}{=} g^-h^-gx^-y^-g^-yhgx\\ &\overset{[g,y]=1}{=} g^-h^-gx^-g^-y^-yhgx\\ &= g^-h^-gx^-h^gx\\ &\overset{[h^g,x]=1}{=} g^-h^-gh^gx^-x\\ &= 1.\end{aligned}$$

So $[g,h] = [gx,\, yh]$. □

Lemma 70 Consider the groups

$$\begin{array}{lcl} L & := & G_{2;1,2} \\ M & := & G_{2;1}/G_{2;0,1} \\ M' & := & G_{2;2}/G_{2;0,2} \\ P & := & G_2/G_{2;0} \end{array}$$

and the group morphisms

$$\begin{array}{lcllcll} G_{2;1,2} & \overset{\lambda}{\longrightarrow} & G_{2;1}/G_{2;0,1} & : & k & \longmapsto & kG_{2;0,1} \\ G_{2;1}/G_{2;0,1} & \overset{\mu}{\longrightarrow} & G_2/G_{2;0} & : & gG_{2;0,1} & \longmapsto & gG_{2;0} \\ G_{2;1,2} & \overset{\lambda'}{\longrightarrow} & G_{2;2}/G_{2;0,2} & : & k & \longmapsto & kG_{2;0,2} \\ G_{2;2}/G_{2;0,2} & \overset{\mu'}{\longrightarrow} & G_2/G_{2;0} & : & hG_{2;0,2} & \longmapsto & hG_{2;0}. \end{array}$$

The existence of λ is ensured, because $G_{2;1,2} \leqslant G_{2;1}$.

The existence of μ is ensured, because $G_{2;1} \leqslant G_2$ and $G_{2;0,1} \leqslant G_{2;0}$.

The existence of λ' is ensured, because $G_{2;1,2} \leqslant G_{2;2}$.

The existence of μ' is ensured, because $G_{2;2} \leqslant G_2$ and $G_{2;0,2} \leqslant G_{2;0}$.

$$\begin{array}{ccc} G_{2;1,2} & \xrightarrow{\quad \lambda' \quad} & G_{2;2}/G_{2;0,2} \\ \Big\downarrow {\scriptstyle \lambda} & & \Big\downarrow {\scriptstyle \mu'} \\ G_{2;1}/G_{2;0,1} & \xrightarrow[\quad \mu \quad]{} & G_2/G_{2;0} \end{array}$$

With the help of Remark 68 and Lemma 17, we have the group morphisms

$$\begin{array}{lcllcll} G_{2;1}/G_{2;0,1} & \overset{\gamma_{M,L}}{\longrightarrow} & \mathrm{Aut}(G_{2;1,2}) & : & gG_{2;0,1} & \longmapsto & (k \mapsto k^{gG_{2;0,1}} := k^g) \\ G_{2;2}/G_{2;0,2} & \overset{\gamma_{M',L}}{\longrightarrow} & \mathrm{Aut}(G_{2;1,2}) & : & hG_{2;0,2} & \longmapsto & (k \mapsto k^{hG_{2;0,2}} := k^h) \\ G_2/G_{2;0} & \overset{\gamma_{P,L}}{\longrightarrow} & \mathrm{Aut}(G_{2;1,2}) & : & lG_{2;0} & \longmapsto & (k \mapsto k^{lG_{2;0}} := k^l) \\ G_2/G_{2;0} & \overset{\gamma_{P,M}}{\longrightarrow} & \mathrm{Aut}(G_{2;1}/G_{2;0,1}) & : & lG_{2;0} & \longmapsto & (gG_{2;0,1} \mapsto (gG_{2;0,1})^{lG_{2;0}} := g^l G_{2;0,1}) \\ G_2/G_{2;0} & \overset{\gamma_{P,M'}}{\longrightarrow} & \mathrm{Aut}(G_{2;2}/G_{2;0,2}) & : & lG_{2;0} & \longmapsto & (hG_{2;0,2} \mapsto (hG_{2;0,2})^{lG_{2;0}} := h^l G_{2;0,2}). \end{array}$$

Here $g \in G_{2;1}$, $h \in G_{2;2}$, $l \in G_2$ and $k \in G_{2;1,2}$.

With the help of Remark 68.(6) and Remark 69, we have the map

$$\begin{array}{rcl} G_{2;1}/G_{2;0,1} \times G_{2;2}/G_{2;0,2} & \overset{\chi}{\longrightarrow} & G_{2;1,2} \\ (gG_{2;0,1},\, hG_{2;0,2}) & \longmapsto & [g,h] =: \lceil gG_{2;0,1},\, hG_{2;0,2} \rceil, \end{array}$$

where $g \in G_{2;1}$ and $h \in G_{2;2}$.

Then

$$G\,\mathrm{Sq} := (G_{2;1,2},\, G_{2;1}/G_{2;0,1},\, G_{2;2}/G_{2;0,2},\, G_2/G_{2;0},\, \gamma_{M,L},\, \gamma_{M',L},\, \gamma_{P,L},\, \gamma_{P,M},\, \gamma_{P,M'},\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi)$$

is a crossed square.

Proof.

Ad (CS 1). For $k \in G_{2;1,2}$, we have

$$\begin{aligned} (k)(\lambda \blacktriangle \mu) &= (kG_{2;0,1})\mu \\ &= kG_{2;0} \\ &= (kG_{2;0,2})\mu' \\ &= (k)(\lambda' \blacktriangle \mu'). \end{aligned}$$

So $\lambda \blacktriangle \mu = \lambda' \blacktriangle \mu'$ and we write

$$\begin{aligned} \kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : G_{2;1,2} &\longrightarrow G_2/G_{2;0} \\ k &\longmapsto kG_{2;0}. \end{aligned}$$

Ad (CS 2.1). We have to show that $(G_{2;1,2},\, G_{2;1}/G_{2;0,1},\, \gamma_{M,L},\, \lambda)$ is a crossed module.

Ad (CM 1). Suppose given $k \in G_{2;1,2}$ and $g \in G_{2;1}$.

Then we have

$$\begin{aligned} (k^{gG_{2;0,1}})\lambda &= (k^g)\lambda \\ &= k^g G_{2;0,1} \\ &= (kG_{2;0,1})^{gG_{2;0,1}} \\ &= (k\lambda)^{gG_{2;0,1}}. \end{aligned}$$

Ad (CM 2). Suppose given $k,\, \tilde{k} \in G_{2;1,2}$.

Then we have

$$\begin{aligned} k^{\tilde{k}} &= k^{\tilde{k}G_{2;0,1}} \\ &= k^{\tilde{k}\lambda}. \end{aligned}$$

Ad (CS 2.2). We have to show that $(G_{2;1,2},\, G_{2;2}/G_{2;0,2},\, \gamma_{M',L},\, \lambda')$ is a crossed module.

Ad (CM 1). Suppose given $k \in G_{2;1,2}$ and $h \in G_{2;2}$.

Then we have

$$\begin{aligned} (k^{hG_{2;0,2}})\lambda' &= (k^h)\lambda' \\ &= k^h G_{2;0,2} \\ &= (kG_{2;0,2})^{hG_{2;0,2}} \\ &= (k\lambda')^{hG_{2;0,2}}. \end{aligned}$$

Ad (CM 2). Suppose given $k,\, \tilde{k} \in G_{2;1,2}$.

Then we have

$$\begin{aligned} k^{\tilde{k}} &= k^{\tilde{k}G_{2;0,2}} \\ &= k^{\tilde{k}\lambda'}. \end{aligned}$$

Ad (CS 2.3). We have to show that $(G_{2;1,2},\, G_2/G_{2;0},\, \gamma_{P,L},\, \kappa)$ is a crossed module.

Ad (CM 1). Suppose given $k \in G_{2;1,2}$ and $l \in G_2$.

Then we have

$$\begin{aligned} (k^{lG_{2;0}})\kappa &= (k^l)\kappa \\ &= k^l G_{2;0} \\ &= (kG_{2;0})^{lG_{2;0}} \\ &= (k\kappa)^{lG_{2;0}}. \end{aligned}$$

Ad (CM 2). Suppose given $k,\, \tilde{k} \in G_{2;1,2}$.

Then we have

$$\begin{aligned} k^{\tilde{k}} &= k^{\tilde{k}G_{2;0}} \\ &= k^{\tilde{k}\kappa}. \end{aligned}$$

Ad (CS 2.4). We have to show that $(G_{2;1}/G_{2;0,1},\, G_2/G_{2;0},\, \gamma_{P,M},\, \mu)$ is a crossed module.

Ad (CM 1). Suppose given $g \in G_{2;1}$ and $l \in G_2$.

Then we have

$$\begin{aligned} ((gG_{2;0,1})^{lG_{2;0}})\mu &= ((g^l)G_{2;0,1})\mu \\ &= g^l G_{2;0} \\ &= (gG_{2;0})^{lG_{2;0}} \\ &= ((gG_{2;0,1})\mu)^{lG_{2;0}}. \end{aligned}$$

Ad (CM 2). Suppose given $g,\, \tilde{g} \in G_{2;1}$.

Then we have

$$\begin{aligned} (gG_{2;0,1})^{\tilde{g}G_{2;0,1}} &= g^{\tilde{g}}G_{2;0,1} \\ &= (gG_{2;0,1})^{\tilde{g}G_{2;0}} \\ &= (gG_{2;0,1})^{(\tilde{g}G_{2;0,1})\mu}. \end{aligned}$$

Ad (CS 2.5). We have to show that $(G_{2;2}/G_{2;0,2},\, G_2/G_{2;0},\, \gamma_{P,M'},\, \mu')$ is a crossed module.

Ad (CM 1). Suppose given $h \in G_{2;2}$ and $l \in G_2$.

Then we have

$$\begin{aligned} ((hG_{2;0,2})^{lG_{2;0}})\mu' &= ((h^l)G_{2;0,2})\mu' \\ &= h^l G_{2;0} \\ &= (hG_{2;0})^{lG_{2;0}} \\ &= ((hG_{2;0,2})\mu')^{lG_{2;0}}. \end{aligned}$$

Ad (CM 2). Suppose given $h,\, \tilde{h} \in G_{2;2}$.

Then we have

$$\begin{aligned} (hG_{2;0,2})^{\tilde{h}G_{2;0,2}} &= h^{\tilde{h}}G_{2;0,2} \\ &= (hG_{2;0,2})^{\tilde{h}G_{2;0}} \\ &= (hG_{2;0,2})^{(\tilde{h}G_{2;0,2})\mu'}. \end{aligned}$$

Ad (CS 3.1). We have to show that

$$(\mathrm{id}_{G_{2;1,2}},\, \mu) : (G_{2;1,2},\, G_{2;1}/G_{2;0,1},\, \gamma_{M,L},\, \lambda) \to (G_{2;1,2},\, G_2/G_{2;0},\, \gamma_{P,L},\, \kappa)$$

is a morphism of crossed modules.

Suppose given $k \in G_{2;1,2}$ and $g \in G_{2;1}$.

We have

$$\begin{aligned} k\lambda\mu &= k\kappa \\ &= k\,\mathrm{id}_{G_{2;1,2}}\,\kappa \end{aligned}$$

and

$$\begin{aligned} (k^{gG_{2;0,1}})\,\mathrm{id}_{G_{2;1,2}} &= k^g \\ &= (k\,\mathrm{id}_{G_{2;1,2}})^{gG_{2;0}} \\ &= (k\,\mathrm{id}_{G_{2;1,2}})^{gG_{2;0,1}\mu}. \end{aligned}$$

Ad (CS 3.2). We have to show that

$$(\mathrm{id}_{G_{2;1,2}},\, \mu') : (G_{2;1,2},\, G_{2;2}/G_{2;0,2},\, \gamma_{M',L},\, \lambda') \to (G_{2;1,2},\, G_2/G_{2;0},\, \gamma_{P,L},\, \kappa)$$

is a morphism of crossed modules.

Suppose given $k \in G_{2;1,2}$ and $h \in G_{2;2}$.

We have

$$\begin{aligned} k\lambda'\mu' &= k\kappa \\ &= k \,\mathrm{id}_{G_{2;1,2}}\, \kappa \end{aligned}$$

and

$$\begin{aligned} (k^{hG_{2;0,2}})\,\mathrm{id}_{G_{2;1,2}} &= k^h \\ &= (k\,\mathrm{id}_{G_{2;1,2}})^{hG_{2;0}} \\ &= (k\,\mathrm{id}_{G_{2;1,2}})^{hG_{2;0,2}\mu'}. \end{aligned}$$

Ad (CS 3.3). We have to show that

$$(\lambda,\, \mathrm{id}_{G_2/G_{2;0}}) : (G_{2;1,2},\, G_2/G_{2;0},\, \gamma_{P,L},\, \kappa) \to (G_{2;1}/G_{2;0,1},\, G_2/G_{2;0},\, \gamma_{P,M},\, \mu)$$

is a morphism of crossed modules.

Suppose given $k \in G_{2;1,2}$ and $l \in G_2$.

We have

$$\begin{aligned} k\kappa\,\mathrm{id}_{G_{2;0}} &= k\kappa \\ &= k\lambda\mu \end{aligned}$$

and

$$\begin{aligned} (k^{lG_{2;0}})\lambda &= (k^l)G_{2;0,1} \\ &= (kG_{2;0,1})^{lG_{2;0}} \\ &= (k\lambda)^{lG_{2;0}\,\mathrm{id}_{G_2/G_{2;0}}}. \end{aligned}$$

Ad (CS 3.4). We have to show that

$$(\lambda',\, \mathrm{id}_{G_2/G_{2;0}}) : (G_{2;1,2},\, G_2/G_{2;0},\, \gamma_{P,L},\, \kappa) \to (G_{2;2}/G_{2;0,2},\, G_2/G_{2;0},\, \gamma_{P,M'},\, \mu')$$

is a morphism of crossed modules.

Suppose given $k \in G_{2;1,2}$ and $l \in G_2$.

We have

$$\begin{aligned} k\kappa\,\mathrm{id}_{G_{2;0}} &= k\kappa \\ &= k\lambda'\mu' \end{aligned}$$

and

$$\begin{aligned} (k^{lG_{2;0}})\lambda' &= (k^l)G_{2;0,2} \\ &= (kG_{2;0,2})^{lG_{2;0}} \\ &= (k\lambda')^{lG_{2;0}\,\mathrm{id}_{G_2/G_{2;0}}}. \end{aligned}$$

Ad (CS 4.1). Suppose given $m \in M = G_{2;1}/G_{2;0,1}$ and $m' \in M' = G_{2;2}/G_{2;0,2}$.

We write $m = gG_{2;0,1}$, where $g \in G_{2;1}$, and $m' = hG_{2;0,2}$, where $h \in G_{2;2}$.

We have

$$\begin{aligned} m \cdot \lceil m, m' \rceil \lambda &= gG_{2;0,1} \cdot \lceil gG_{2;0,1},\, hG_{2;0,2} \rceil \lambda \\ &= gG_{2;0,1} \cdot [g, h]\lambda \\ &= gG_{2;0,1} \cdot [g, h]G_{2;0,1} \\ &= (g \cdot [g, h])G_{2;0,1} \\ &= (gg^- h^- gh)G_{2;0,1} \\ &= (h^- gh)G_{2;0,1} \\ &= g^h G_{2;0,1} \\ &= (gG_{2;0,1})^{hG_{2;0}} \\ &= (gG_{2;0,1})^{hG_{2;0,2}\mu'} \\ &= m^{m'\mu'}. \end{aligned}$$

Ad (CS 4.2). Suppose given $m \in M = G_{2;1}/G_{2;0,1}$ and $m' \in M' = G_{2;2}/G_{2;0,2}$.

We write $m = gG_{2;0,1}$, where $g \in G_{2;1}$, and $m' = hG_{2;0,2}$, where $h \in G_{2;2}$.

We have

$$\begin{aligned} m'^{m\mu} \cdot \lceil m, m' \rceil \lambda' &= (hG_{2;0,2})^{gG_{2;0,1}\mu} \cdot \lceil gG_{2;0,1},\, hG_{2;0,2} \rceil \lambda' \\ &= (hG_{2;0,2})^{gG_{2;0}} \cdot [g,h] G_{2;0,2} \\ &= h^g G_{2;0,2} \cdot [g,h] G_{2;0,2} \\ &= (h^g [g,h]) G_{2;0,2} \\ &= (g^- h g g^- h^- g h) G_{2;0,2} \\ &= h G_{2;0,2} \\ &= m'. \end{aligned}$$

Ad (CS 4.3). Suppose given $m' \in M' = G_{2;2}/G_{2;0,2}$ and $k \in G_{2;1,2}$.

We write $m' = hG_{2;0,2}$, where $h \in G_{2;2}$.

We have

$$\begin{aligned} k \cdot \lceil k\lambda, m' \rceil &= k \cdot \lceil k\lambda,\, hG_{2;0,2} \rceil \\ &= k \cdot \lceil kG_{2;0,1},\, hG_{2;0,2} \rceil \\ &= k \cdot [k,h] \\ &= k k^- h^- k h \\ &= h^- k h \\ &= k^h \\ &= k^{hG_{2;0,2}} \\ &= k^{m'}. \end{aligned}$$

Ad (CS 4.4). Suppose given $m \in M = G_{2;1}/G_{2;0,1}$ and $k \in G_{2;1,2}$.

We write $m = gG_{2;0,1}$, where $g \in G_{2;1}$.

We have

$$\begin{aligned} k^m \cdot \lceil m, k\lambda' \rceil &= k^{gG_{2;0,1}} \cdot \lceil gG_{2;0,1},\, k\lambda' \rceil \\ &= k^{gG_{2;0,1}} \cdot \lceil gG_{2;0,1},\, kG_{2;0,2} \rceil \\ &= k^g [g,k] \\ &= g^- k g g^- k^- g k \\ &= k. \end{aligned}$$

Ad (CS 4.5). Suppose given $m,\, m^* \in M = G_{2;1}/G_{2;0,1}$ and $m' \in M' = G_{2;2}/G_{2;0,2}$.

We write $m = gG_{2;0,1}$ and $m^* = g^* G_{2;0,1}$, where $g,\, g^* \in G_{2;1}$ and $m' = hG_{2;0,2}$, where $h \in G_{2;2}$.

We have

$$\begin{aligned} \lceil m, m' \rceil^{m^*} \cdot \lceil m^*, m' \rceil &= \lceil gG_{2;0,1},\, hG_{2;0,2} \rceil^{g^* G_{2;0,1}} \cdot \lceil g^* G_{2;0,1},\, hG_{2;0,2} \rceil \\ &= [g,h]^{g^*} \cdot [g^*, h] \\ &= g^{*-} g^- h^- g h g^* g^{*-} h^- g^* h \\ &= g^{*-} g^- h^- g g^* h \\ &= [g \cdot g^*,\, h] \\ &= \lceil gG_{2;0,1} \cdot g^* G_{2;0,1},\, hG_{2;0,2} \rceil \\ &= \lceil m \cdot m^*, m' \rceil. \end{aligned}$$

Ad (CS 4.6). Suppose given $m \in M = G_{2;1}/G_{2;0,1}$ and $m',\, m^{*\prime} \in M' = G_{2;2}/G_{2;0,2}$.

We write $m = gG_{2;0,1}$, where $g \in G_{2;1}$, $m' = hG_{2;0,2}$, and $m^{*\prime} = h^* G_{2;0,2}$, where $h,\, h^* \in G_{2;2}$.

We have

$$\begin{aligned} \lceil m', m^{*\prime} \rceil \cdot \lceil m, m' \rceil^{m^{*\prime}} &= \lceil gG_{2;0,1},\, h^* G_{2;0,2} \rceil \cdot \lceil gG_{2;0,1},\, hG_{2;0,2} \rceil^{h^* G_{2;0,2}} \\ &= [g, h^*] \cdot [g,h]^{h^*} \\ &= g^- h^{*-} g h^* h^{*-} g^- h^- g h h^* \\ &= g^- h^{*-} h^- g h h^* \\ &= [g,\, h \cdot h^*] \\ &= \lceil gG_{2;0,1},\, hG_{2;0,2} \cdot h^* G_{2;0,2} \rceil \\ &= \lceil m, m' \cdot m^{*\prime} \rceil. \end{aligned}$$

Ad (CS 4.7). Suppose given $m \in M = G_{2;1}/G_{2;0,1}$, $m' \in M' = G_{2;2}/G_{2;0,2}$ and $p \in P = G_2/G_{2;0}$.

We write $m = gG_{2;0,1}$, where $g \in G_{2;1}$, $m' = hG_{2;0,2}$, where $h \in G_{2;2}$, and $p = lG_{2;0}$, where $l \in G_2$.

We have

$$\begin{aligned} \lceil m^p, m'^p \rceil &= \lceil (gG_{2;0,1})^{lG_{2;0}},\, (hG_{2;0,2})^{lG_{2;0}} \rceil \\ &= \lceil g^l G_{2;0,1},\, h^l G_{2;0,2} \rceil \\ &= [g^l, h^l] \\ &= [l^- gl, l^- hl] \\ &= l^- g^- ll^- h^- ll^- gll^- hl \\ &= l^- g^- h^- ghl \\ &= [g,h]^l \\ &= \lceil gG_{2;0,1},\, hG_{2;0,2} \rceil^{lG_{2;0}} \\ &= \lceil m, m' \rceil^p. \end{aligned}$$

Ad (CS 4.8). Suppose given $m \in M = G_{2;1}/G_{2;0,1}$, $m' \in M' = G_{2;2}/G_{2;0,2}$ and $k \in G_{2;1,2}$.

We write $m = gG_{2;0,1}$, where $g \in G_{2;1}$, and $m' = hG_{2;0,2}$, where $h \in G_{2;2}$.

We have

$$\begin{aligned} ((k^{m'})^m)^{\lceil m,m' \rceil} &= ((k^{hG_{2;0,2}})^{gG_{2;0,1}})^{\lceil gG_{2;0,1}, hG_{2;0,2} \rceil} \\ &= ((k^h)^g)^{[g,h]} \\ &= h^- g^- hgg^- h^- khgg^- h^- gh \\ &= h^- g^- kgh \\ &= (k^g)^h \\ &= (k^{gG_{2;0,1}})^{hG_{2;0,2}} \\ &= (k^m)^{m'}. \end{aligned}$$

□

5.2.2 The construction for the morphisms

Suppose given a morphism $\varphi : G \to H$ of $\lfloor 2,0 \rfloor$-simplicial groups.

Remark 71 There exists the group morphism

$$\begin{aligned} \varphi_{2;A} := \varphi_2|_{G_{2;A}}^{H_{2;A}} : G_{2;A} &\to H_{2;A} \\ g_2 &\mapsto g_2\varphi_2 \end{aligned}$$

for $A \subseteq \{0,1,2\}$.

Cf. Definition 53.(2).

Proof. Suppose given $x \in G_{2;A}$.

This means $x\,\mathrm{d}_i = 1$ for $i \in A$.

Therefore we have

$$\begin{aligned} x\varphi_2\,\mathrm{d}_i &= x\,\mathrm{d}_i\,\varphi_1 \\ &= 1 \end{aligned}$$

for $i \in A$.

So we have $x\varphi_2 \in H_{2;A}$.

Altogether,

$$(G_{2;A})\varphi_2 \subseteq H_{2;A}.$$

□

Remark 72 Suppose given $B \subseteq A \subseteq \{0,1,2\}$.

Then there exists the group morphism

$$\begin{array}{rcl} \bar{\varphi}_{2;B,A} : G_{2;B}/G_{2;A} & \longrightarrow & H_{2;B}/H_{2;A} \\ g_2 G_{2;A} & \longmapsto & g_2 \varphi_2 H_{2;A}. \end{array}$$

Cf. Definition 53.(2).

Proof. The following quadrangle of group morphisms commutes,

$$\begin{array}{ccc} G_{2;A} & \xrightarrow{\varphi_{2;A}} & H_{2;A} \\ \downarrow & \circlearrowleft & \downarrow \\ G_{2;B} & \xrightarrow[\varphi_{2;B}]{} & H_{2;B} \\ \downarrow{\scriptstyle \rho^G} & & \downarrow{\scriptstyle \rho^H} \\ G_{2;B}/G_{2;A} & & H_{2;B}/H_{2;A} \end{array},$$

where ρ^G and ρ^H are the residue class morphisms.

Cf. Remark 71.

Since we have

$$(G_{2;A})(\varphi_{2;B} \blacktriangle \rho^H) = (G_{2;A}\varphi_{2;A})\rho^H = 1,$$

we may complete the commutative diagram.

$$\begin{array}{ccc} G_{2;A} & \xrightarrow{\varphi_{2;A}} & H_{2;A} \\ \downarrow & \circlearrowleft & \downarrow \\ G_{2;B} & \xrightarrow[\varphi_{2;B}]{} & H_{2;B} \\ \downarrow{\scriptstyle \rho^G} & \circlearrowleft & \downarrow{\scriptstyle \rho^H} \\ G_{2;B}/G_{2;A} & \xrightarrow[\bar{\varphi}_{2;B,A}]{} & H_{2;B}/H_{2;A} \end{array}$$

For $g_2 \in G_{2;B}$, we get

$$\begin{array}{rcl} g_2 G_{2;A} \bar{\varphi}_{2;B,A} & = & g_2 \rho^G \bar{\varphi}_{2;B,A} \\ & = & g_2 \varphi_{2;B} \rho^H \\ & = & g_2 \varphi_{2;B} H_{2;A}. \end{array}$$

□

Remark 73 We abbreviate

$$\begin{array}{rcl} \varphi_{2;1,2} := \varphi_{2;\{1,2\}} : G_{2;1,2} & \longrightarrow & H_{2;1,2} \\ g_2 & \longmapsto & g_2 \varphi_2 \end{array}$$

$$\begin{array}{rcl} \bar{\varphi}_{2;\emptyset} := \varphi_{2;\{\emptyset\},\{0\}} : G_2/G_{2;0} & \longrightarrow & H_2/H_{2;0} \\ g_2 G_{2;0} & \longmapsto & g_2 \varphi_2 H_{2;0} \end{array}$$

$$\begin{array}{rcl} \bar{\varphi}_{2;1} := \varphi_{2;\{1\},\{0,1\}} : G_{2;1}/G_{2;0,1} & \longrightarrow & H_{2;1}/H_{2;0,1} \\ g_2 G_{2;0,1} & \longmapsto & g_2\varphi_2 H_{2;0,1} \end{array}$$

$$\begin{array}{rcl} \bar{\varphi}_{2;2} := \varphi_{2;\{2\},\{0,2\}} : G_{2;2}/G_{2;0,2} & \longrightarrow & H_{2;2}/H_{2;0,2} \\ g_2 G_{2;0,2} & \longmapsto & g_2\varphi_2 H_{2;0,2}. \end{array}$$

Lemma 74 We have the crossed squares

$$\begin{array}{rcl} G\,\mathrm{Sq} & = & (L,\, M,\, M',\, P,\, \gamma_{M,L},\, \gamma_{M',L},\, \gamma_{P,L},\, \gamma_{P,M},\, \gamma_{P,M'},\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi) \\ & = & (G_{2;1,2},\, G_{2;1}/G_{2;0,1},\, G_{2;2}/G_{2;0,2},\, G_2/G_{2;0},\, \gamma_{M,L},\, \gamma_{M',L},\, \gamma_{P,L},\, \gamma_{P,M},\, \gamma_{P,M'},\, \lambda,\, \lambda',\, \mu,\, \mu',\, \chi) \end{array}$$

and

$$\begin{array}{rcl} H\,\mathrm{Sq} & = & (\tilde{L},\, \tilde{M},\, \tilde{M}',\, \tilde{P},\, \gamma_{\tilde{M},\tilde{L}},\, \gamma_{\tilde{M}',\tilde{L}},\, \gamma_{\tilde{P},\tilde{L}},\, \gamma_{\tilde{P},\tilde{M}},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi}) \\ & = & (H_{2;1,2},\, H_{2;1}/H_{2;0,1},\, H_{2;2}/H_{2;0,2},\, H_2/H_{2;0},\, \gamma_{\tilde{M},\tilde{L}},\, \gamma_{\tilde{M}',\tilde{L}},\, \gamma_{\tilde{P},\tilde{L}},\, \gamma_{\tilde{P},\tilde{M}},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\lambda},\, \tilde{\lambda}',\, \tilde{\mu},\, \tilde{\mu}',\, \tilde{\chi}). \end{array}$$

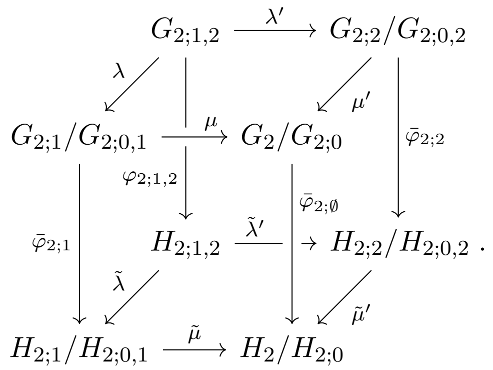
Cf. Lemma 70.

Let $\varphi\,\mathrm{Sq} := (\varphi_{2;1,2},\, \bar{\varphi}_{2;1},\, \bar{\varphi}_{2;2},\, \bar{\varphi}_{2;\emptyset})$, cf. Remark 71 and Remark 72.

Then

$$\varphi\,\mathrm{Sq} : G\,\mathrm{Sq} \to H\,\mathrm{Sq}$$

is a morphism of crossed squares.



Proof. We have to show the properties $(1,2)$ from Definition 59.

We write $\kappa := \lambda \blacktriangle \mu = \lambda' \blacktriangle \mu' : G_{2;1,2} \to G_2/G_{2;0}$ and $\tilde{\kappa} := \tilde{\lambda} \blacktriangle \tilde{\mu} = \tilde{\lambda}' \blacktriangle \tilde{\mu}' : H_{2;1,2} \to H_2/H_{2;0}$.

Ad (1.a). We have to show that

$$(\varphi_{2;1,2},\, \bar{\varphi}_{2;1}) : (G_{2;1,2},\, G_{2;1}/G_{2;0,1},\, \gamma_{L,M},\, \lambda) \to (H_{2;1,2},\, H_{2;1}/H_{2;0,1},\, \gamma_{\tilde{L},\tilde{M}},\, \tilde{\lambda})$$

is a morphism of crossed modules.

Suppose given $k \in G_{2;1,2}$ and $g \in G_{2;1}$.

Then we have

$$(k)(\lambda \blacktriangle \bar{\varphi}_{2;1}) = (kG_{2;0,1})\bar{\varphi}_{2;1} = k\varphi_2 H_{2;0,1}$$

and

$$(k)(\varphi_{2;1,2} \blacktriangle \tilde{\lambda}) = (k\varphi_2)\tilde{\lambda} = k\varphi_2 H_{2;0,1},$$

which is the same.

So

$$\lambda \blacktriangle \bar{\varphi}_{2;1} = \varphi_{2;1,2} \blacktriangle \tilde{\lambda}.$$

Furthermore, we have

$$\begin{aligned} (k^{gG_{2;0,1}})\varphi_{2;1,2} &= (k^g)\varphi_{2;1,2} \\ &= (k^g)\varphi_2 \\ &= (k\varphi_2)^{g\varphi_2} \\ &= (k\varphi_{2;1,2})^{g\varphi_2 H_{2;0,1}} \\ &= (k\varphi_{2;1,2})^{gG_{2;0,1}\bar{\varphi}_{2;1}}. \end{aligned}$$

Ad (1.b). We have to show that

$$(\varphi_{2;1,2},\, \bar{\varphi}_{2;2}) : (G_{2;1,2},\, G_{2;2}/G_{2;0,2},\, \gamma_{L,M'},\, \lambda') \to (H_{2;1,2},\, H_{2;2}/H_{2;0,2},\, \gamma_{\tilde{L},\tilde{M}'},\, \tilde{\lambda}')$$

is a morphism of crossed modules.

Suppose given $k \in G_{2;1,2}$ and $h \in G_{2;2}$.

Then we have

$$(k)(\lambda' \blacktriangle \bar{\varphi}_{2;2}) = (kG_{2;0,2})\bar{\varphi}_{2;2} = k\varphi_2 H_{2;0,2}$$

and

$$(k)(\varphi_{2;1,2} \blacktriangle \tilde{\lambda}') = (k\varphi_2)\tilde{\lambda}' = k\varphi_2 H_{2;0,2},$$

which is the same.

So

$$\lambda' \blacktriangle \bar{\varphi}_{2;2} = \varphi_{2;1,2} \blacktriangle \tilde{\lambda}'.$$

Furthermore, we have

$$\begin{aligned} (k^{hG_{2;0,2}})\varphi_{2;1,2} &= (k^h)\varphi_{2;1,2} \\ &= (k^h)\varphi_2 \\ &= (k\varphi_2)^{h\varphi_2} \\ &= (k\varphi_{2;1,2})^{h\varphi_2 H_{2;0,2}} \\ &= (k\varphi_{2;1,2})^{hG_{2;0,2}\bar{\varphi}_{2;2}}. \end{aligned}$$

Ad (1.c). We have to show that

$$(\bar{\varphi}_{2;1},\, \bar{\varphi}_{2;\emptyset}) : (G_{2;1}/G_{2;0,1},\, G_2/G_{2;0},\, \gamma_{P,M},\, \mu) \to (H_{2;1}/H_{2;0,1},\, H_2/H_{2;0},\, \gamma_{\tilde{P},\tilde{M}},\, \tilde{\mu})$$

is a morphism of crossed modules.

Suppose given $g \in G_{2;1}$ and $l \in G_2$.

Then we have

$$(gG_{2;0,1})(\mu \blacktriangle \bar{\varphi}_{2;\emptyset}) = (gG_{2;0})\bar{\varphi}_{2;\emptyset} = g\varphi_2 H_{2;0}$$

and

$$(gG_{2;0,1})(\bar{\varphi}_{2;1} \blacktriangle \tilde{\mu}) = (g\varphi_2 H_{2;0,1})\tilde{\mu} = g\varphi_2 H_{2;0},$$

which is the same.

So

$$\mu \blacktriangle \bar{\varphi}_{2;\emptyset} = \bar{\varphi}_{2;1} \blacktriangle \tilde{\mu}.$$

Furthermore, we have

$$\begin{aligned} ((gG_{2;0,1})^{lG_{2;0}})\bar{\varphi}_{2;1} &= ((g^l)G_{2;0,1})\bar{\varphi}_{2;1} \\ &= ((g^l)\varphi_2)H_{2;0,1} \\ &= (g\varphi_2)^{l\varphi_2} H_{2;0,1} \\ &= (g\varphi_2 H_{2;0,1})^{l\varphi_2 H_{2;0}} \\ &= (gG_{2;0,1}\bar{\varphi}_{2;1})^{lG_{2;0}\bar{\varphi}_{2;\emptyset}}. \end{aligned}$$

Ad (1.d). We have to show that

$$(\bar{\varphi}_{2;2},\, \bar{\varphi}_{2;\emptyset}) : (G_{2;2}/G_{2;0,2},\, G_2/G_{2;0},\, \gamma_{P,M'},\, \mu') \to (H_{2;2}/H_{2;0,2},\, H_2/H_{2;0},\, \gamma_{\tilde{P},\tilde{M}'},\, \tilde{\mu}')$$

is a morphism of crossed modules.

Suppose given $h \in G_{2;2}$ and $l \in G_2$.

Then we have

$$(hG_{2;0,2})(\mu' \blacktriangle \bar{\varphi}_{2;\emptyset}) = (hG_{2;0})\bar{\varphi}_{2;\emptyset} = h\varphi_2 H_{2;0}$$

and

$$(hG_{2;0,2})(\bar{\varphi}_{2;2} \blacktriangle \tilde{\mu}') = (h\varphi_2 H_{2;0,2})\tilde{\mu}' = h\varphi_2 H_{2;0},$$

which is the same.

So

$$\mu' \blacktriangle \bar{\varphi}_{2;\emptyset} = \bar{\varphi}_{2;2} \blacktriangle \tilde{\mu}'.$$

Furthermore, we have

$$\begin{aligned} ((hG_{2;0,2})^{lG_{2;0}})\bar{\varphi}_{2;2} &= ((h^l)G_{2;0,2})\bar{\varphi}_{2;2} \\ &= ((h^l)\varphi_2)H_{2;0,2} \\ &= (h\varphi_2)^{l\varphi_2} H_{2;0,2} \\ &= (h\varphi_2 H_{2;0,2})^{l\varphi_2 H_{2;0}} \\ &= (hG_{2;0,2}\bar{\varphi}_{2;2})^{lG_{2;0}\bar{\varphi}_{2;\emptyset}}. \end{aligned}$$

Ad (1.e). We have to show that

$$(\varphi_{2;1,2},\, \bar{\varphi}_{2;\emptyset}) : (G_{2;1,2},\, G_2/G_{2;0},\, \gamma_{P,L},\, \kappa) \to (H_{2;1,2},\, H_2/H_{2;0},\, \gamma_{\tilde{P},\tilde{L}},\, \tilde{\kappa})$$

is a morphism of crossed modules.

Suppose given $k \in G_{2;1,2}$ and $l \in G_2$.

Then we have

$$(k)(\kappa \blacktriangle \bar{\varphi}_{2;\emptyset}) = (kG_{2;0})\bar{\varphi}_{2;\emptyset} = k\varphi_2 H_{2;0}$$

and

$$(k)(\varphi_{2;1,2} \blacktriangle \tilde{\kappa}) = (k\varphi_2)\tilde{\kappa} = k\varphi_2 H_{2;0},$$

which is the same.

So

$$\kappa \blacktriangle \bar{\varphi}_{2;\emptyset} = \varphi_{2;1,2} \blacktriangle \tilde{\kappa}.$$

Furthermore, we have

$$\begin{aligned} (k^{lG_{2;0}})\varphi_{2;1,2} &= (k^l)\varphi_{2;1,2} \\ &= (k^l)\varphi_2 \\ &= (k\varphi_2)^{l\varphi_2} \\ &= (k\varphi_{2;1,2})^{l\varphi_2 H_{2;0}} \\ &= (k\varphi_{2;1,2})^{lG_{2;0}\bar{\varphi}_{2;\emptyset}}. \end{aligned}$$

Ad (2). Suppose given $g \in G_{2;1}$ and $h \in G_{2;2}$.

Then we have

$$\begin{aligned} \lceil gG_{2;0,1}\bar{\varphi}_{2;1}, hG_{2;0,2}\bar{\varphi}_{2;2} \rceil &= \lceil g\varphi_2 H_{2;0,1}, h\varphi_2 H_{2;0,2} \rceil \\ &= [g\varphi_2, h\varphi_2] \\ &= [g, h]\varphi_2 \\ &= \lceil gG_{2;0,1}, hG_{2;0,2} \rceil \varphi_2 \\ &= \lceil gG_{2;0,1}, hG_{2;0,2} \rceil \varphi_{2;1,2}. \end{aligned}$$

□

5.2.3 The functor Sq

Definition 75 We shall define the following functor.

$$\begin{array}{rcl} \mathrm{Sq} : \lfloor 2,0 \rfloor\text{-}\mathit{SimpGrp} & \longrightarrow & \mathit{CrSq} \\ \left(\begin{array}{c} G \\ \downarrow \varphi \\ H \end{array} \right) & \longmapsto & \left(\begin{array}{c} G\,\mathrm{Sq} \\ \downarrow \varphi\,\mathrm{Sq} \\ H\,\mathrm{Sq} \end{array} \right) \end{array}$$

(1) Suppose given a $\lfloor 2,0 \rfloor$-simplicial group G.

The crossed square $G\,\mathrm{Sq}$ has been defined in Lemma 70.

$$G\,\mathrm{Sq} = \left(\begin{array}{ccc} G_{2;1,2} & \xrightarrow{\lambda'} & G_{2;2}/G_{2;0,2} \\ \lambda \downarrow & & \downarrow \mu' \\ G_{2;1}/G_{2;0,1} & \xrightarrow[\mu]{} & G_2/G_{2;0} \end{array} \right)$$

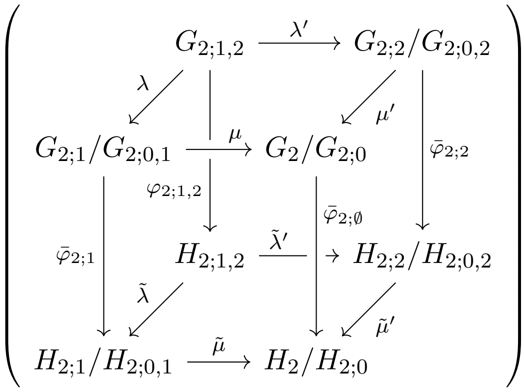
In the notation of Lemma 70, we have

$$\begin{array}{rcl} (G\,\mathrm{Sq})_{1,1} & = & G_{2;1,2} \\ (G\,\mathrm{Sq})_{1,0} & = & G_{2;1}/G_{2;0,1} \\ (G\,\mathrm{Sq})_{0,1} & = & G_{2;2}/G_{2;0,2} \\ (G\,\mathrm{Sq})_{0,0} & = & G_2/G_{2;0} \\ \gamma_{1,0}^{G\,\mathrm{Sq}} & = & \gamma_{M,L} \\ \gamma_{0,1}^{G\,\mathrm{Sq}} & = & \gamma_{M',L} \\ \gamma_{G\,\mathrm{Sq}}^{1,1} & = & \gamma_{P,L} \\ \gamma_{G\,\mathrm{Sq}}^{1,0} & = & \gamma_{P,M} \\ \gamma_{G\,\mathrm{Sq}}^{0,1} & = & \gamma_{P,M'} \\ \lambda_{G\,\mathrm{Sq}}^{1,0} & = & \lambda \\ \lambda_{G\,\mathrm{Sq}}^{0,1} & = & \lambda' \\ \mu_{1,0}^{G\,\mathrm{Sq}} & = & \mu \\ \mu_{0,1}^{G\,\mathrm{Sq}} & = & \mu' \\ \chi_{G\,\mathrm{Sq}} & = & \chi \\ \kappa_{G\,\mathrm{Sq}} & = & \kappa. \end{array}$$

(2) Suppose given a morphism of $\lfloor 2,0 \rfloor$-simplicial groups $G \xrightarrow{\varphi} H$.

The following diagram morphism is a morphism of crossed squares, by Lemma 74.

$$\begin{pmatrix} G \\ \varphi \big\downarrow \\ H \end{pmatrix} \mathrm{Sq} :=$$



So we have

$$\begin{array}{rcl}
(\varphi\,\mathrm{Sq})_{1,1} & = & \varphi_{2;1,2} \\
(\varphi\,\mathrm{Sq})_{1,0} & = & \bar{\varphi}_{2;1} \\
(\varphi\,\mathrm{Sq})_{0,1} & = & \bar{\varphi}_{2;2} \\
(\varphi\,\mathrm{Sq})_{0,0} & = & \bar{\varphi}_{2;\emptyset} \\
\lambda^{1,0}_{G\,\mathrm{Sq}} & = & \lambda \\
\lambda^{0,1}_{G\,\mathrm{Sq}} & = & \lambda' \\
\mu^{G\,\mathrm{Sq}}_{1,0} & = & \mu \\
\mu^{G\,\mathrm{Sq}}_{0,1} & = & \mu' \\
\chi_{G\,\mathrm{Sq}} & = & \chi \\
\kappa_{G\,\mathrm{Sq}} & = & \tilde{\kappa} \\
\lambda^{1,0}_{H\,\mathrm{Sq}} & = & \tilde{\lambda} \\
\lambda^{0,1}_{H\,\mathrm{Sq}} & = & \tilde{\lambda}' \\
\mu^{H\,\mathrm{Sq}}_{1,0} & = & \tilde{\mu} \\
\mu^{H\,\mathrm{Sq}}_{0,1} & = & \tilde{\mu}' \\
\chi_{H\,\mathrm{Sq}} & = & \tilde{\chi} \\
\kappa_{H\,\mathrm{Sq}} & = & \tilde{\kappa}
\end{array}$$

(3) Suppose given morphisms of $\lfloor 2,0 \rfloor$-simplicial groups $G \xrightarrow{\varphi} H \xrightarrow{\varphi'} K$.

Then we have

(a) $(\mathrm{id}_G)\,\mathrm{Sq} = \mathrm{id}_{(G\,\mathrm{Sq})}$

(b) $(\varphi \blacktriangle \varphi')\,\mathrm{Sq} = \varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq}$.

In particular, $\mathrm{Sq} : \lfloor 2,0 \rfloor$-$\mathcal{SimpGrp} \to \mathcal{CrSq}$ is a functor.

Proof.

Ad (3.a). Suppose given $k \in G_{2;1,2}$.

Then we have

$$\begin{array}{rcl}
k(\mathrm{id}_G\,\mathrm{Sq})_{1,1} & = & k(\mathrm{id}_G)_{2;1,2} \\
& = & k\,\mathrm{id}_{G_{2;1,2}} \\
& = & k\,\mathrm{id}_{(G\,\mathrm{Sq})_{1,1}} \\
& = & k(\mathrm{id}_{G\,\mathrm{Sq}})_{1,1}.
\end{array}$$

Suppose given $g \in G_{2;1}$.

Then we have

$$\begin{array}{rcl}(gG_{2;0,1})(\mathrm{id}_G \mathrm{Sq})_{1,0} & = & gG_{2;0,1}\overline{(\mathrm{id}_G)}_{2;1} \\ & = & g(\mathrm{id}_G)_2 G_{2;0,1} \\ & = & g\,\mathrm{id}_{G_2}\, G_{2;0,1} \\ & = & gG_{2;0,1} \\ & = & gG_{2;0,1}\,\mathrm{id}_{G_{2;1}/G_{2;0,1}} \\ & = & gG_{2;0,1}\,\mathrm{id}_{(G\,\mathrm{Sq})_{1,0}} \\ & = & gG_{2;0,1}(\mathrm{id}_{G\,\mathrm{Sq}})_{1,0}.\end{array}$$

Suppose given $h \in G_{2;2}$.

Then we have

$$\begin{array}{rcl}(hG_{2;0,2})(\mathrm{id}_G \mathrm{Sq})_{0,1} & = & hG_{2;0,2}\overline{(\mathrm{id}_G)}_{2;2} \\ & = & h(\mathrm{id}_G)_2 G_{2;0,2} \\ & = & h\,\mathrm{id}_{G_2}\, G_{2;0,2} \\ & = & hG_{2;0,2} \\ & = & hG_{2;0,2}\,\mathrm{id}_{G_{2;2}/G_{2;0,2}} \\ & = & hG_{2;0,2}\,\mathrm{id}_{(G\,\mathrm{Sq})_{0,1}} \\ & = & hG_{2;0,2}(\mathrm{id}_{G\,\mathrm{Sq}})_{0,1}.\end{array}$$

Suppose given $l \in G_2$.

Then we have

$$\begin{array}{rcl}(lG_{2;0})(\mathrm{id}_G \mathrm{Sq})_{0,0} & = & lG_{2;0}\overline{(\mathrm{id}_G)}_{2;\emptyset} \\ & = & l(\mathrm{id}_G)_2 G_{2;0} \\ & = & l\,\mathrm{id}_{G_2}\, G_{2;0} \\ & = & lG_{2;0} \\ & = & lG_{2;0}\,\mathrm{id}_{G_2/G_{2;0}} \\ & = & lG_{2;0}\,\mathrm{id}_{(G\,\mathrm{Sq})_{0,0}} \\ & = & lG_{2;0}(\mathrm{id}_{G\,\mathrm{Sq}})_{0,0}.\end{array}$$

Altogether, we have

$$\begin{array}{rcl}\mathrm{id}_G \mathrm{Sq} & = & ((\mathrm{id}_G \mathrm{Sq})_{1,1}, (\mathrm{id}_G \mathrm{Sq})_{1,0}, (\mathrm{id}_G \mathrm{Sq})_{0,1}, (\mathrm{id}_G \mathrm{Sq})_{0,0}) \\ & = & ((\mathrm{id}_{G\,\mathrm{Sq}})_{1,1}, (\mathrm{id}_{G\,\mathrm{Sq}})_{1,0}, (\mathrm{id}_{G\,\mathrm{Sq}})_{0,1}, (\mathrm{id}_{G\,\mathrm{Sq}})_{0,0}) \\ & = & \mathrm{id}_{G\,\mathrm{Sq}}\,.\end{array}$$

Ad (3.b). Suppose given $k \in G_{2;1,2}$.

Then we have

$$\begin{array}{rcl}k((\varphi \blacktriangle \varphi')\,\mathrm{Sq})_{1,1} & = & k(\varphi \blacktriangle \varphi')_{2;1,2} \\ & = & k(\varphi \blacktriangle \varphi')_2 \\ & = & k(\varphi_2 \blacktriangle \varphi'_2) \\ & = & (k\varphi_2)\varphi'_2 \\ & = & (k\varphi_{2;1,2})\varphi'_{2;1,2} \\ & = & k(\varphi_{2;1,2} \blacktriangle \varphi'_{2;1,2}) \\ & = & k((\varphi\,\mathrm{Sq})_{1,1} \blacktriangle (\varphi'\,\mathrm{Sq})_{1,1}) \\ & = & k(\varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq})_{1,1}.\end{array}$$

Suppose given $g \in G_{2;1}$.

Then we have

$$\begin{aligned}
(gG_{2;0,1})((\varphi \blacktriangle \varphi')\,\mathrm{Sq})_{1,0} &= gG_{2;0,1}\overline{(\varphi \blacktriangle \varphi')}_{2;1} \\
&= g(\varphi \blacktriangle \varphi')_2 K_{2;0,1} \\
&= g(\varphi_2 \blacktriangle \varphi'_2) K_{2;0,1} \\
&= (g\varphi_2)\varphi'_2 K_{2;0,1} \\
&= (g\varphi_2 H_{2;0,1})\bar{\varphi}'_{2;1} \\
&= (gG_{2;0,1}\bar{\varphi}_{2;1})\bar{\varphi}'_{2;1} \\
&= (gG_{2;0,1})(\bar{\varphi}_{2;1} \blacktriangle \bar{\varphi}'_{2;1}) \\
&= (gG_{2;0,1})((\varphi\,\mathrm{Sq})_{1,0} \blacktriangle (\varphi'\,\mathrm{Sq})_{1,0}) \\
&= (gG_{2;0,1})(\varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq})_{1,0}.
\end{aligned}$$

Suppose given $h \in G_{2;2}$.

Then we have

$$\begin{aligned}
(hG_{2;0,2})((\varphi \blacktriangle \varphi')\,\mathrm{Sq})_{0,1} &= hG_{2;0,2}\overline{(\varphi \blacktriangle \varphi')}_{2;2} \\
&= h(\varphi \blacktriangle \varphi')_2 K_{2;0,2} \\
&= h(\varphi_2 \blacktriangle \varphi'_2) K_{2;0,2} \\
&= (h\varphi_2)\varphi'_2 K_{2;0,2} \\
&= (h\varphi_2 H_{2;0,2})\bar{\varphi}'_{2;2} \\
&= (hG_{2;0,2}\bar{\varphi}_{2;2})\bar{\varphi}'_{2;2} \\
&= (hG_{2;0,2})(\bar{\varphi}_{2;2} \blacktriangle \bar{\varphi}'_{2;2}) \\
&= (hG_{2;0,2})((\varphi\,\mathrm{Sq})_{0,1} \blacktriangle (\varphi'\,\mathrm{Sq})_{0,1}) \\
&= (hG_{2;0,2})(\varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq})_{0,1}.
\end{aligned}$$

Suppose given $l \in G_2$.

Then we have

$$\begin{aligned}
(lG_{2;0})((\varphi \blacktriangle \varphi')\,\mathrm{Sq})_{0,0} &= lG_{2;0}\overline{(\varphi \blacktriangle \varphi')}_{2;\emptyset} \\
&= l(\varphi \blacktriangle \varphi')_2 K_{2;0} \\
&= l(\varphi_2 \blacktriangle \varphi'_2) K_{2;0} \\
&= (l\varphi_2)\varphi'_2 K_{2;0} \\
&= (l\varphi_2 H_{2;0})\bar{\varphi}'_{2;\emptyset} \\
&= (lG_{2;0}\bar{\varphi}_{2;\emptyset})\bar{\varphi}'_{2;\emptyset} \\
&= (lG_{2;0})(\bar{\varphi}_{2;\emptyset} \blacktriangle \bar{\varphi}'_{2;\emptyset}) \\
&= (lG_{2;0})((\varphi\,\mathrm{Sq})_{0,0} \blacktriangle (\varphi'\,\mathrm{Sq})_{0,0}) \\
&= (lG_{2;0})(\varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq})_{0,0}.
\end{aligned}$$

Altogether, we have

$$\begin{aligned}
(\varphi \blacktriangle \varphi')\,\mathrm{Sq} &= (((\varphi \blacktriangle \varphi')\,\mathrm{Sq})_{1,1}, ((\varphi \blacktriangle \varphi')\,\mathrm{Sq})_{1,0}, ((\varphi \blacktriangle \varphi')\,\mathrm{Sq})_{0,1}, ((\varphi \blacktriangle \varphi')\,\mathrm{Sq})_{0,0}) \\
&= ((\varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq})_{1,1}, (\varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq})_{1,0}, (\varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq})_{0,1}, (\varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq})_{0,0}) \\
&= \varphi\,\mathrm{Sq} \blacktriangle \varphi'\,\mathrm{Sq}\,.
\end{aligned}$$

□

5.3 An isomorphic copy $G\,\mathrm{Sq}^!$ of $G\,\mathrm{Sq}$ avoiding factor groups

We shall make frequent use of Remark 56.

Remark 76 We have the following group isomorphisms.

(1)

$$\begin{array}{rrcl} \psi_{1,0}: & G_{2;1}/G_{2;0,1} & \xrightarrow{\sim} & G_{1;0} \\ & g_2 G_{2;0,1} & \longmapsto & g_2\,\mathrm{d}_0 \\ & (g_1\,\mathrm{s}_0 \cdot g_1^-\,\mathrm{s}_1)G_{2;0,1} & \longleftarrow & g_1 \end{array}$$

(2)

$$\begin{array}{rrcl} \psi_{0,1}: & G_{2;2}/G_{2;0,2} & \xrightarrow{\sim} & G_{1;1} \\ & h_2 G_{2;0,2} & \longmapsto & h_2\,\mathrm{d}_0 \\ & h_1\,\mathrm{s}_0\, G_{2;0,2} & \longleftarrow & h_1 \end{array}$$

(3)

$$\begin{array}{rrcl} \psi_{0,0}: & G_2/G_{2;0} & \xrightarrow{\sim} & G_1 \\ & k_2 G_{2;0} & \longmapsto & k_2\,\mathrm{d}_0 \\ & k_1\,\mathrm{s}_0\, G_{2;0} & \longleftarrow & k_1 \end{array}$$

Proof.

Ad (1). We have $g_2\,\mathrm{d}_0 \in G_{1;0}$ for $g_2 \in G_{2;1}$ and $g_1\,\mathrm{s}_0 \cdot g_1^-\,\mathrm{s}_1 \in G_{2;1}$ for $g_1 \in G_{1;0}$, because

$$\begin{array}{rcl} g_2\,\mathrm{d}_0\,\mathrm{d}_0 & = & g_2\,\mathrm{d}_1\,\mathrm{d}_0 \\ & = & 1\,\mathrm{d}_0 \\ & = & 1 \end{array}$$

and

$$\begin{array}{rcl} (g_1\,\mathrm{s}_0 \cdot g_1^-\,\mathrm{s}_1)\,\mathrm{d}_1 & = & g_1\,\mathrm{s}_0\,\mathrm{d}_1 \cdot g_1^-\,\mathrm{s}_1\,\mathrm{d}_1 \\ & = & g_1 g_1^- \\ & = & 1. \end{array}$$

Moreover, we have

$$\ker(\mathrm{d}_0|_{G_{2;1}}^{G_{1;0}}) = G_{2;1} \cap \ker(\mathrm{d}_0) = G_{2;0,1} \trianglelefteq G_{2;1}$$

and, for $g_1 \in G_{1;0}$,

$$\begin{array}{rcl} (g_1\,\mathrm{s}_0 \cdot g_1^-\,\mathrm{s}_1)\,\mathrm{d}_0 & = & g_1\,\mathrm{s}_0\,\mathrm{d}_0 \cdot g_1^-\,\mathrm{s}_1\,\mathrm{d}_0 \\ & = & g_1 \cdot g_1^-\,\mathrm{d}_0\,\mathrm{s}_0 \\ & = & g_1. \end{array}$$

So the injective group morphism

$$\psi_{1,0}: G_{2;1}/G_{2;0,1} \to G_{1;0}$$

is surjective.

Suppose given $g_1 \in G_{1;0}$.

The claimed inverse $\psi_{1,0}^-$ yields

$$\begin{array}{rcl} (g_1)(\psi_{1,0}^- \blacktriangle \psi_{1,0}) & = & (g_1\psi_{1,0}^-)\psi_{1,0} \\ & = & (g_1\,\mathrm{s}_0 \cdot g_1^-\,\mathrm{s}_1)G_{2;0,1}\psi_{1,0} \\ & = & (g_1\,\mathrm{s}_0 \cdot g_1^-\,\mathrm{s}_1)\,\mathrm{d}_0 \\ & = & g_1\,\mathrm{s}_0\,\mathrm{d}_0 \cdot g_1^-\,\mathrm{s}_1\,\mathrm{d}_0 \\ & = & g_1 \cdot g_1^-\,\mathrm{d}_0\,\mathrm{s}_0 \\ & = & g_1 \\ & = & g_1\,\mathrm{id}_{G_{1;0}}. \end{array}$$

So $\psi_{1,0}^{-} \blacktriangle \psi_{1,0} = \mathrm{id}_{G_{1;0}}$ and thus $\psi_{1,0}^{-}$ is in fact the inverse to $\psi_{1,0}$.

Therefore

$$\psi_{1,0} : G_{2;1}/G_{2;0,1} \to G_{1;0}$$

is an isomorphism, with inverse as claimed.

Ad (2). We have $h_2 \,\mathrm{d}_0 \in G_{1;1}$ for $h_2 \in G_{2;2}$ and $h_1 \,\mathrm{s}_0 \in G_{2;2}$ for $h_1 \in G_{1;1}$, because

$$\begin{aligned} h_2 \,\mathrm{d}_0 \,\mathrm{d}_1 &= h_2 \,\mathrm{d}_2 \,\mathrm{d}_0 \\ &= 1 \,\mathrm{d}_0 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} h_1 \,\mathrm{s}_0 \,\mathrm{d}_2 &= h_1 \,\mathrm{d}_1 \,\mathrm{s}_0 \\ &= 1 \,\mathrm{s}_0 \\ &= 1. \end{aligned}$$

Moreover we have

$$\ker(\mathrm{d}_0 |_{G_{2;2}}^{G_{1;1}}) = G_{2;2} \cap \ker(\mathrm{d}_0) = G_{2;0,2} \trianglelefteq G_{2;2}$$

and

$$\mathrm{s}_0 |_{G_{1;1}}^{G_{2;2}} \blacktriangle \mathrm{d}_0 |_{G_{2;2}}^{G_{1;1}} = \mathrm{id}_{G_{1;1}} .$$

So

$$\psi_{0,1} : G_{2;2}/G_{2;0,2} \to G_{1;1}$$

is an isomorphism, with inverse as claimed; cf. Lemma 22.

Ad (3). We have $\mathrm{s}_0 \blacktriangle \mathrm{d}_0 = \mathrm{id}_{G_1}$.

Moreover we have $\ker(\mathrm{d}_0) = G_{2;0} \trianglelefteq G_2$.

So

$$\psi_{0,} : G_2/G_{2;0} \to G_1$$

is an isomorphism, with inverse as claimed; cf. Lemma 22. □

Remark 77 We have the group isomorphism

$$\begin{aligned} \varphi_1 : \mathrm{N}_0 \ltimes \mathrm{N}_1 &\longrightarrow G_1 \\ (n_0, n_1) &\longmapsto n_0 \,\mathrm{s}_0 \cdot n_1, \end{aligned}$$

with inverse

$$\begin{aligned} \varphi_1^{-} : G_1 &\longrightarrow \mathrm{N}_0 \ltimes \mathrm{N}_1 \\ g &\longmapsto (g \,\mathrm{d}_1, g^{-} \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g); \end{aligned}$$

cf. Lemma 55.

Remark 78 We define the subgroup

$$\mathrm{U}_1 := G_{1;0}\varphi_1^{-} \leqslant \mathrm{N}_0 \ltimes \mathrm{N}_1.$$

Then

$$\mathrm{U}_1 = \{(n_1^{-} \,\mathrm{d}_0, n_1) : n_1 \in \mathrm{N}_1\}.$$

Furthermore, we have the group isomorphism

$$\begin{aligned} \check{\varphi}_1 := \varphi_1|_{\mathrm{U}_1}^{G_{1;0}} : \quad \mathrm{U}_1 &\longrightarrow G_{1;0} \\ (n_1^{-} \,\mathrm{d}_0, n_1) &\longmapsto n_1^{-} \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_1, \end{aligned}$$

with inverse

$$\begin{aligned} \check{\varphi}_1^{-} : \quad G_{1;0} &\longrightarrow \mathrm{U}_1 \\ g &\longmapsto (g \,\mathrm{d}_1, g^{-} \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g). \end{aligned}$$

$$\begin{array}{ccc} G_{1;0} & \xleftarrow[\sim]{\check{\varphi}_1} & \mathrm{U}_1 \\ \downarrow & & \downarrow \\ G_1 & \xleftarrow[\sim]{\varphi_1} & \mathrm{N}_0 \ltimes \mathrm{N}_1 \end{array}$$

Proof. We show $\mathrm{U}_1 \overset{!}{\subseteq} \{(n_1^- \,\mathrm{d}_0, n_1) : n_1 \in \mathrm{N}_1\}$.

Suppose given $u_1 \in \mathrm{U}_1$.

Let $g_1 := u_1\varphi_1 \in G_{1;0}$.

Then $u_1 = g_1\varphi_1^- = (g_1 \,\mathrm{d}_1, g_1^- \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g_1)$.

Let $\tilde{n}_1 := g_1^- \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g_1$.

We have

$$\begin{array}{rcl} \tilde{n}_1 \,\mathrm{d}_1 & = & (g_1^- \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g_1) \,\mathrm{d}_1 \\ & = & g_1^- \,\mathrm{d}_1 \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot g_1 \,\mathrm{d}_1 \\ & = & g_1^- \,\mathrm{d}_1 \cdot g_1 \,\mathrm{d}_1 \\ & = & 1. \end{array}$$

So $\tilde{n}_1 \in \mathrm{N}_1$. Which also follows from Remark 77.

We have to show that $(\tilde{n}_1^- \,\mathrm{d}_0, \tilde{n}_1) \overset{!}{=} (g_1 \,\mathrm{d}_1, g_1^- \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot g_1) = u_1$.

In fact, we have

$$\begin{array}{rcl} \tilde{n}_1^- \,\mathrm{d}_0 & = & (g_1^- \cdot g_1 \,\mathrm{d}_1 \,\mathrm{s}_0) \,\mathrm{d}_0 \\ & = & g_1^- \,\mathrm{d}_0 \cdot g_1 \,\mathrm{d}_1 \,\mathrm{s}_0 \,\mathrm{d}_0 \\ & = & g_1^- \,\mathrm{d}_0 \cdot g_1 \,\mathrm{d}_1 \\ & = & g_1 \,\mathrm{d}_1 \,. \end{array}$$

So $u_1 \in \{(n_1^- \,\mathrm{d}_0, n_1) : n_1 \in \mathrm{N}_1\}$.

We show $\mathrm{U}_1 \overset{!}{\supseteq} \{(n_1^- \,\mathrm{d}_0, n_1) : n_1 \in \mathrm{N}_1\}$.

For $n_1 \in \mathrm{N}_1$, we have

$$\begin{array}{rcl} ((n_1^- \,\mathrm{d}_0, n_1)\varphi_1) \,\mathrm{d}_0 & = & (n_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_1) \,\mathrm{d}_0 \\ & = & n_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{d}_0 \cdot n_1 \,\mathrm{d}_0 \\ & = & n_1^- \,\mathrm{d}_0 \cdot n_1 \,\mathrm{d}_0 \\ & = & 1. \end{array}$$

So we have $(n_1^- \,\mathrm{d}_0, n_1)\varphi_1 \in G_{1;0}$.

Hence $(n_1^- \,\mathrm{d}_0, n_1) \in \mathrm{U}_1$.

Altogether, we have

$$\mathrm{U}_1 = \{(n_1^- \,\mathrm{d}_0, n_1) : n_1 \in \mathrm{N}_1\}.$$

□

Remark 79 We have the following group isomorphisms.

(1)

$$\begin{array}{rcl} \mathfrak{q}_{1,1} := \mathrm{id} : G_{2;1,2} & \overset{\sim}{\longrightarrow} & \mathrm{N}_2 \\ k & \longmapsto & k \\ k & \longleftarrow & k \end{array}$$

(2)

$$\begin{array}{rcl} \mathfrak{q}_{0,1} := \psi_{0,1} : G_{2;2}/G_{2;0,2} & \xrightarrow{\sim} & \mathrm{N}_1 \\ hG_{2;0,2} & \longmapsto & h\,\mathrm{d}_0 \\ (n_1\,\mathrm{s}_0)G_{2;0,2} & \longleftarrow & n_1 \end{array}$$

(3)

$$\begin{array}{rcl} \mathfrak{q}_{0,0} := \psi_{0,0} \blacktriangle \varphi_1^- : G_2/G_{2;0} & \xrightarrow{\sim} & \mathrm{N}_0 \ltimes \mathrm{N}_1 \\ lG_{2;0} & \longmapsto & \Big((l\,\mathrm{d}_0)\varphi_1^- = (l\,\mathrm{d}_0\,\mathrm{d}_1, l^-\,\mathrm{d}_0\,\mathrm{d}_1\,\mathrm{s}_0 \cdot l\,\mathrm{d}_0)\Big) \\ \left(\begin{array}{l} (n_0\,\mathrm{s}_0 \cdot n_1)\psi_{0,0}^- \\ = (n_0\,\mathrm{s}_0 \cdot n_1)\,\mathrm{s}_0\,G_{2;0} \\ = (n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_0)G_{2;0} \end{array}\right) & \longleftarrow & (n_0, n_1) \end{array}$$

(4)

$$\begin{array}{rcl} \mathfrak{q}_{1,0} := \psi_{1,0} \blacktriangle \check{\varphi}_1^- : G_{2;1}/G_{2;0,1} & \xrightarrow{\sim} & \mathrm{U}_1 \\ gG_{2;0,1} & \longmapsto & \Big((g\,\mathrm{d}_0)\check{\varphi}_1^- = (g\,\mathrm{d}_0\,\mathrm{d}_1, g^-\,\mathrm{d}_0\,\mathrm{d}_1\,\mathrm{s}_0 \cdot g\,\mathrm{d}_0)\Big) \\ \left(\begin{array}{l} (n_1^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_1)\psi_{1,0}^- \\ = ((n_1^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_1)\,\mathrm{s}_0 \cdot (n_1^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot n_1)^-\,\mathrm{s}_1)G_{2;0,1} \\ = ((n_1^-\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_0) \cdot (n_1^-\,\mathrm{s}_1 \cdot n_1\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_1))G_{2;0,1} \\ = (n_1^-\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_0 \cdot n_1^-\,\mathrm{s}_1 \cdot n_1\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0)G_{2;0,1} \\ = (n_1\,\mathrm{s}_0 \cdot n_1^-\,\mathrm{s}_1)^{(n_1\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0)}G_{2;0,1} \end{array}\right) & \longleftarrow & (n_1^-\,\mathrm{d}_0, n_1) \end{array}$$

Remark 80 Using Remark 18, we obtain the following group isomorphisms.

(1) We have the group isomorphism

$$\begin{array}{rcl} \mathrm{id} := \hat{\mathfrak{q}}_{1,1} : \mathrm{Aut}(G_{2;1,2}) & \longrightarrow & \mathrm{Aut}(\mathrm{N}_2) \\ \alpha & \longmapsto & \mathfrak{q}_{1,1}^- \blacktriangle \alpha \blacktriangle \mathfrak{q}_{1,1} = \alpha. \end{array}$$

Cf. Remark 79.(1).

(2) We have the group isomorphism

$$\begin{array}{rcl} \hat{\mathfrak{q}}_{1,0} : \mathrm{Aut}(G_{2;1}/G_{2;0,1}) & \longmapsto & \mathrm{Aut}(\mathrm{U}_1) \\ \alpha & \longmapsto & \mathfrak{q}_{1,0}^- \blacktriangle \alpha \blacktriangle \mathfrak{q}_{1,0}. \end{array}$$

Cf. Remark 79.(4).

(3) We have the group isomorphism

$$\begin{array}{rcl} \hat{\mathfrak{q}}_{0,1} : \mathrm{Aut}(G_{2;2}/H_{2;0,2}) & \longrightarrow & \mathrm{Aut}(\mathrm{N}_1) \\ \alpha & \longmapsto & \mathfrak{q}_{0,1}^- \blacktriangle \alpha \blacktriangle \mathfrak{q}_{0,1}. \end{array}$$

Cf. Remark 79.(2).

Lemma 81 We shall construct the crossed square $G\,\mathrm{Sq}^!$ and the isomorphism of crossed squares $\varphi : G\,\mathrm{Sq} \to G\,\mathrm{Sq}^!$.

We shall make use of Remark 67 to substitute isomorphically.

Consider the groups

$$\begin{array}{rcl} G\,\mathrm{Sq}^!_{1,1} & := & \mathrm{N}_2 \\ G\,\mathrm{Sq}^!_{1,0} & := & \mathrm{U}_1 \\ G\,\mathrm{Sq}^!_{0,1} & := & \mathrm{N}_1 \\ G\,\mathrm{Sq}^!_{0,0} & := & \mathrm{N}_0 \ltimes_{\gamma_1} \mathrm{N}_1 = \mathrm{N}_0 \ltimes \mathrm{N}_1. \end{array}$$

Concerning U_1, cf. Lemma 55 and Remark 78.

Concerning $\mathrm{N}_0 \ltimes_{\gamma_1} \mathrm{N}_1$, cf. Remark 56.

We have the group morphisms

$$
\begin{array}{lrcl}
\lambda^{1,0}_{G\,\mathrm{Sq}^!} = \mathfrak{q}^-_{1,1} \blacktriangle \lambda^{1,0}_{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{1,0} : & \mathrm{N}_2 & \longrightarrow & \mathrm{U}_1 \\
 & n_2 & \longmapsto & (1, n_2\,\mathrm{d}_0) \\[1em]
\mu^{G\,\mathrm{Sq}^!}_{1,0} = \mathfrak{q}^-_{1,0} \blacktriangle \mu^{G\,\mathrm{Sq}}_{1,0} \blacktriangle \mathfrak{q}_{0,0} : & \mathrm{U}_1 & \longrightarrow & \mathrm{N}_0 \ltimes \mathrm{N}_1 \\
 & (n_1^-\,\mathrm{d}_0, n_1) & \longmapsto & (n_1^-\,\mathrm{d}_0, n_1) \\[1em]
\lambda^{0,1}_{G\,\mathrm{Sq}^!} = \mathfrak{q}^-_{1,1} \blacktriangle \lambda^{0,1}_{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{0,1} : & \mathrm{N}_2 & \longrightarrow & \mathrm{N}_1 \\
 & n_2 & \longmapsto & n_2\,\mathrm{d}_0 \\[1em]
\mu^{G\,\mathrm{Sq}^!}_{0,1} = \mathfrak{q}^-_{0,1} \blacktriangle \mu^{G\,\mathrm{Sq}}_{0,1} \blacktriangle \mathfrak{q}_{0,0} : & \mathrm{N}_1 & \longrightarrow & \mathrm{N}_0 \ltimes \mathrm{N}_1 \\
 & n_1 & \longmapsto & (1, n_1).
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{N}_2 & \xrightarrow{\lambda^{0,1}_{G\,\mathrm{Sq}^!}} & \mathrm{N}_1 \\
{\scriptstyle \lambda^{1,0}_{G\,\mathrm{Sq}^!}} \downarrow & & \downarrow {\scriptstyle \mu^{G\,\mathrm{Sq}^!}_{0,1}} \\
\mathrm{U}_1 & \xrightarrow[\mu^{G\,\mathrm{Sq}^!}_{1,0}]{} & \mathrm{N}_0 \ltimes \mathrm{N}_1
\end{array}
$$

We have the group morphisms

$$
\begin{array}{lrcl}
\gamma^{G\,\mathrm{Sq}^!}_{1,0} = \mathfrak{q}^-_{1,0} \blacktriangle \gamma^{G\,\mathrm{Sq}}_{1,0} \blacktriangle \hat{\mathfrak{q}}_{1,1} : & \mathrm{U}_1 & \longrightarrow & \mathrm{Aut}(\mathrm{N}_2) \\
 & (n_1^-\,\mathrm{d}_0, n_1) & \longmapsto & \Big(n_2 \mapsto n_2^{(n_1^-\,\mathrm{d}_0, n_1)} = n_2^{(n_1\,\mathrm{s}_0 \cdot n_1^-\,\mathrm{s}_1)^{n_1\,\mathrm{d}_0\,\mathrm{s}_0\,\mathrm{s}_0}} \Big) \\[1em]
\gamma^{G\,\mathrm{Sq}^!}_{0,1} = \mathfrak{q}^-_{0,1} \blacktriangle \gamma^{G\,\mathrm{Sq}}_{0,1} \blacktriangle \hat{\mathfrak{q}}_{1,1} : & \mathrm{N}_1 & \longrightarrow & \mathrm{Aut}(\mathrm{N}_2) \\
 & n_1 & \longmapsto & (n_2 \mapsto n_2^{n_1} = n_2^{n_1\,\mathrm{s}_0}) \\[1em]
\gamma^{1,1}_{G\,\mathrm{Sq}^!} = \mathfrak{q}^-_{0,0} \blacktriangle \gamma^{1,1}_{G\,\mathrm{Sq}} \blacktriangle \hat{\mathfrak{q}}_{1,1} : & \mathrm{N}_0 \ltimes \mathrm{N}_1 & \longrightarrow & \mathrm{Aut}(\mathrm{N}_2) \\
 & (n_0, n_1) & \longmapsto & \Big(n_2 \mapsto n_2^{(n_0, n_1)} = n_2^{n_0\,\mathrm{s}_0\,\mathrm{s}_0 \cdot n_1\,\mathrm{s}_0} \Big) \\[1em]
\gamma^{1,0}_{G\,\mathrm{Sq}^!} = \mathfrak{q}^-_{0,0} \blacktriangle \gamma^{1,0}_{G\,\mathrm{Sq}} \blacktriangle \hat{\mathfrak{q}}_{1,0} : & \mathrm{N}_0 \ltimes \mathrm{N}_1 & \longrightarrow & \mathrm{Aut}(\mathrm{U}_1) \\
 & (n_0, n_1) & \longmapsto & \left((\tilde{n}_1^-\,\mathrm{d}_0, \tilde{n}_1) \mapsto \left(\begin{array}{l} (\tilde{n}_1^-\,\mathrm{d}_0, \tilde{n}_1)^{(n_0, n_1)} \\ = \Big((\tilde{n}_1^-\,\mathrm{d}_0)^{n_0}, (\tilde{n}_1\,\mathrm{d}_0\,\mathrm{s}_0)^{n_0\,\mathrm{s}_0} \\ \cdot\, (\tilde{n}_1^-\,\mathrm{d}_0\,\mathrm{s}_0 \cdot \tilde{n}_1)^{n_0\,\mathrm{s}_0 \cdot n_1} \Big) \end{array} \right) \right) \\[1em]
\gamma^{0,1}_{G\,\mathrm{Sq}^!} = \mathfrak{q}^-_{0,0} \blacktriangle \gamma^{0,1}_{G\,\mathrm{Sq}} \blacktriangle \hat{\mathfrak{q}}_{0,1} : & \mathrm{N}_0 \ltimes \mathrm{N}_1 & \longrightarrow & \mathrm{Aut}(\mathrm{N}_1) \\
 & (n_0, n_1) & \longmapsto & \Big(\tilde{n}_1 \mapsto \tilde{n}_1^{(n_0, n_1)} = \tilde{n}_1^{n_0\,\mathrm{s}_0 \cdot n_1} \Big).
\end{array}
$$

In each case, the operation on the left hand side of the equation is given by the formula on the right hand side that uses conjugation on N_2, N_1 and N_0.

We have the map

$$\chi_{G\,\mathrm{Sq}^!} = (\mathfrak{q}_{1,0}^- \times \mathfrak{q}_{0,1}^-) \blacktriangle \chi_{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{1,1} : \quad \begin{array}{rcl} \mathrm{U}_1 \times \mathrm{N}_1 & \longrightarrow & \mathrm{N}_2 \\ ((n_1^- \,\mathrm{d}_0, n_1),\, \tilde{n}_1) & \longmapsto & \begin{pmatrix} [(n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0}, \tilde{n}_1 \,\mathrm{s}_0] \\ =: \lceil (n_1^- \,\mathrm{d}_0, n_1),\, \tilde{n}_1 \rceil \end{pmatrix}. \end{array}$$

Then we have that

$$G\,\mathrm{Sq}^! := (\mathrm{N}_2,\, \mathrm{U}_1,\, \mathrm{N}_1,\, \mathrm{N}_0 \ltimes \mathrm{N}_1,\, \gamma_{1,0}^{G\,\mathrm{Sq}^!},\, \gamma_{0,1}^{G\,\mathrm{Sq}^!},\, \gamma_{G\,\mathrm{Sq}^!}^{1,1},\, \gamma_{G\,\mathrm{Sq}^!}^{1,0},\, \gamma_{G\,\mathrm{Sq}^!}^{0,1},\, \lambda_{G\,\mathrm{Sq}^!}^{1,0},\, \lambda_{G\,\mathrm{Sq}^!}^{0,1},\, \mu_{1,0}^{G\,\mathrm{Sq}^!},\, \mu_{0,1}^{G\,\mathrm{Sq}^!},\, \chi_{G\,\mathrm{Sq}^!})$$

is a crossed square and

$$\mathfrak{q} := (\mathfrak{q}_{1,1},\, \mathfrak{q}_{1,0},\, \mathfrak{q}_{0,1},\, \mathfrak{q}_{0,0}) : G\,\mathrm{Sq} \xrightarrow{\sim} G\,\mathrm{Sq}^!$$

is an isomorphism of crossed squares.

Note that we have $\kappa_{G\,\mathrm{Sq}^!} := \lambda_{G\,\mathrm{Sq}^!}^{1,0} \blacktriangle \mu_{1,0}^{G\,\mathrm{Sq}^!} = \lambda_{G\,\mathrm{Sq}^!}^{0,1} \blacktriangle \mu_{0,1}^{G\,\mathrm{Sq}^!} : \mathrm{N}_2 \to \mathrm{N}_0 \ltimes \mathrm{N}_1$.

Proof. First, we verify the claimed mapping rules.

We have $\lambda_{G\,\mathrm{Sq}^!}^{1,0} = \mathfrak{q}_{1,1}^- \blacktriangle \lambda_{G\,\mathrm{Sq}}^{1,0} \blacktriangle \mathfrak{q}_{1,0}$.

For $n_2 \in \mathrm{N}_2$, we have

$$\begin{array}{rcl} n_2 \lambda_{G\,\mathrm{Sq}^!}^{1,0} & = & n_2(\mathfrak{q}_{1,1}^- \blacktriangle \lambda_{G\,\mathrm{Sq}}^{1,0} \blacktriangle \mathfrak{q}_{1,0}) \\ & = & ((n_2 \mathfrak{q}_{1,1}^-)\lambda_{G\,\mathrm{Sq}}^{1,0})\mathfrak{q}_{1,0} \\ & = & (n_2 \lambda_{G\,\mathrm{Sq}}^{1,0})\mathfrak{q}_{1,0} \\ & = & (n_2 G_{2;0,1})\mathfrak{q}_{1,0} \\ & = & (n_2 \,\mathrm{d}_0 \,\mathrm{d}_1, n_2^- \,\mathrm{d}_0 \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot n_2 \,\mathrm{d}_0) \\ & = & (n_2 \,\mathrm{d}_2 \,\mathrm{d}_0, n_2^- \,\mathrm{d}_2 \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_2 \,\mathrm{d}_0) \\ & = & (1, n_2 \,\mathrm{d}_0). \end{array}$$

We have $\mu_{1,0}^{G\,\mathrm{Sq}^!} = \mathfrak{q}_{1,0}^- \blacktriangle \mu_{1,0}^{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{0,0}$.

For $(n_1^- \,\mathrm{d}_0, n_1) \in \mathrm{U}_1$, we have

$$\begin{array}{rl} & (n_1^- \,\mathrm{d}_0, n_1)\mu_{1,0}^{G\,\mathrm{Sq}^!} \\ = & (n_1^- \,\mathrm{d}_0, n_1)(\mathfrak{q}_{1,0}^- \blacktriangle \mu_{1,0}^{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{0,0}) \\ = & (((n_1^- \,\mathrm{d}_0, n_1)\mathfrak{q}_{1,0}^-)\mu_{1,0}^{G\,\mathrm{Sq}})\mathfrak{q}_{0,0} \\ = & (((n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{(n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0)} G_{2;0,1})\mu_{1,0}^{G\,\mathrm{Sq}})\mathfrak{q}_{0,0} \\ = & (((n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{(n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0)} G_{2;0})\mathfrak{q}_{0,0} \\ = & ((n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{(n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0)} \,\mathrm{d}_0 \,\mathrm{d}_1, (n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{(n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0)})^- \,\mathrm{d}_0 \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot (n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{(n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0)} \,\mathrm{d}_0). \end{array}$$

First we calculate

$$\begin{array}{rcl} (n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{(n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0)} \,\mathrm{d}_0 & = & (n_1 \,\mathrm{s}_0 \,\mathrm{d}_0 \cdot n_1^- \,\mathrm{s}_1 \,\mathrm{d}_0)^{(n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \,\mathrm{d}_0)} \\ & = & (n_1 \cdot n_1^- \,\mathrm{s}_1 \,\mathrm{d}_0)^{(n_1 \,\mathrm{d}_0 \,\mathrm{s}_0)} \\ & = & n_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_1. \end{array}$$

Then we calculate

$$\begin{array}{rcl} (n_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_1) \,\mathrm{d}_1 & = & n_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot n_1 \,\mathrm{d}_1 \\ & = & n_1^- \,\mathrm{d}_0\,. \end{array}$$

Then we have

$$(n_1^- \,\mathrm{d}_0, n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot (n_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot n_1)) = (n_1^- \,\mathrm{d}_0, n_1).$$

We have $\lambda_{G\,\mathrm{Sq}^!}^{0,1} = \mathfrak{q}_{1,1}^- \blacktriangle \lambda_{G\,\mathrm{Sq}}^{0,1} \blacktriangle \mathfrak{q}_{0,1}$.

For $n_2 \in \mathrm{N}_2$, we have

$$\begin{aligned} n_2 \lambda^{0,1}_{G\,\mathrm{Sq}^!} &= n_2(\mathfrak{q}^-_{1,1} \blacktriangle \lambda^{0,1}_{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{0,1}) \\ &= ((n_2 \mathfrak{q}^-_{1,1})\lambda^{0,1}_{G\,\mathrm{Sq}})\mathfrak{q}_{0,1} \\ &= (n_2 \lambda^{0,1}_{G\,\mathrm{Sq}})\mathfrak{q}_{0,1} \\ &= (n_2 G_{2;0,2})\mathfrak{q}_{0,1} \\ &= n_2 \, \mathrm{d}_0 \, . \end{aligned}$$

We have $\mu^{G\,\mathrm{Sq}^!}_{0,1} = \mathfrak{q}^-_{0,1} \blacktriangle \mu^{G\,\mathrm{Sq}}_{0,1} \blacktriangle \mathfrak{q}_{0,0}$.

For $n_1 \in \mathrm{N}_1$, we have

$$\begin{aligned} n_1 \mu^{G\,\mathrm{Sq}^!}_{0,1} &= n_1(\mathfrak{q}^-_{0,1} \blacktriangle \mu^{G\,\mathrm{Sq}}_{0,1} \blacktriangle \mathfrak{q}_{0,0}) \\ &= ((n_1 \mathfrak{q}^-_{0,1})\mu^{G\,\mathrm{Sq}}_{0,1})\mathfrak{q}_{0,0}) \\ &= ((n_1 \, \mathrm{s}_0 \, G_{2;0,2})\mu^{G\,\mathrm{Sq}}_{0,1})\mathfrak{q}_{0,0}) \\ &= (n_1 \, \mathrm{s}_0 \, G_{2;0})\mathfrak{q}_{0,0}) \\ &= (n_1 \, \mathrm{s}_0 \, \mathrm{d}_0 \, \mathrm{d}_1, n_1^- \, \mathrm{s}_0 \, \mathrm{d}_0 \, \mathrm{d}_1 \, \mathrm{s}_0 \cdot n_1 \, \mathrm{s}_0 \, \mathrm{d}_0) \\ &= (n_1 \, \mathrm{d}_1, n_1^- \, \mathrm{d}_1 \, \mathrm{s}_0 \cdot n_1) \\ &= (1, n_1). \end{aligned}$$

We have $\gamma^{G\,\mathrm{Sq}^!}_{1,0} = \mathfrak{q}^-_{1,0} \blacktriangle \gamma^{G\,\mathrm{Sq}}_{1,0} \blacktriangle \hat{\mathfrak{q}}_{1,1} = \mathfrak{q}^-_{1,0} \blacktriangle \gamma^{G\,\mathrm{Sq}}_{1,0}$.

For $n_2 \in \mathrm{N}_2$ and $n_1 \in \mathrm{N}_1$, we have

$$\begin{aligned} n_2^{(n_1^- \, \mathrm{d}_0, n_1)} &= (n_2)((n_1^- \, \mathrm{d}_0, n_1)(\mathfrak{q}^-_{1,0} \blacktriangle \gamma^{G\,\mathrm{Sq}}_{1,0})) \\ &= (n_2)((n_1^- \, \mathrm{d}_0, n_1)\mathfrak{q}^-_{1,0})\gamma^{G\,\mathrm{Sq}}_{1,0} \\ &= (n_2)((n_1 \, \mathrm{s}_0 \cdot n_1^- \, \mathrm{s}_1)^{(n_1 \, \mathrm{d}_0 \, \mathrm{s}_0 \, \mathrm{s}_0)} G_{2;0,1})\gamma^{G\,\mathrm{Sq}}_{1,0} \\ &= n_2^{(n_1 \, \mathrm{s}_0 \cdot n_1^- \, \mathrm{s}_1)^{n_1 \, \mathrm{d}_0 \, \mathrm{s}_0 \, \mathrm{s}_0}}. \end{aligned}$$

We have $\gamma^{G\,\mathrm{Sq}^!}_{0,1} = \mathfrak{q}^-_{0,1} \blacktriangle \gamma^{G\,\mathrm{Sq}}_{0,1} \blacktriangle \hat{\mathfrak{q}}_{1,1} = \mathfrak{q}^-_{0,1} \blacktriangle \gamma^{G\,\mathrm{Sq}}_{0,1}$.

For $n_2 \in \mathrm{N}_2$ and $n_1 \in \mathrm{N}_1$, we have

$$\begin{aligned} n_2^{n_1} &= (n_2)((n_1)(\mathfrak{q}^-_{0,1} \blacktriangle \gamma^{G\,\mathrm{Sq}}_{0,1})) \\ &= (n_2)((n_1)\mathfrak{q}^-_{0,1})\gamma^{G\,\mathrm{Sq}}_{0,1} \\ &= (n_2)((n_1 \, \mathrm{s}_0)G_{2;0,2})\gamma^{G\,\mathrm{Sq}}_{0,1} \\ &= n_2^{n_1 \, \mathrm{s}_0}. \end{aligned}$$

We have $\gamma^{1,1}_{G\,\mathrm{Sq}^!} = \mathfrak{q}^-_{0,0} \blacktriangle \gamma^{1,1}_{G\,\mathrm{Sq}} \blacktriangle \hat{\mathfrak{q}}_{1,1} = \mathfrak{q}^-_{0,0} \blacktriangle \gamma^{1,1}_{G\,\mathrm{Sq}}$.

For $n_2 \in \mathrm{N}_2$, $n_1 \in \mathrm{N}_1$ and $n_0 \in \mathrm{N}_0$, we have

$$\begin{aligned} n_2^{(n_0, n_1)} &= (n_2)((n_0, n_1)(\mathfrak{q}^-_{0,0} \blacktriangle \gamma^{1,1}_{G\,\mathrm{Sq}})) \\ &= (n_2)((n_0, n_1)\mathfrak{q}^-_{0,0})\gamma^{1,1}_{G\,\mathrm{Sq}} \\ &= (n_2)((n_0 \, \mathrm{s}_0 \, \mathrm{s}_0 \cdot n_1 \, \mathrm{s}_0)G_{2;0})\gamma^{1,1}_{G\,\mathrm{Sq}} \\ &= n_2^{n_0 \, \mathrm{s}_0 \, \mathrm{s}_0 \cdot n_1 \, \mathrm{s}_0}. \end{aligned}$$

We have $\gamma^{1,0}_{G\,\mathrm{Sq}^!} = \mathfrak{q}^-_{0,0} \blacktriangle \gamma^{1,0}_{G\,\mathrm{Sq}} \blacktriangle \hat{\mathfrak{q}}_{1,0}$.

For $n_1,\ \tilde{n}_1 \in \mathrm{N}_1$ and $n_0 \in \mathrm{N}_0$, we have

$$\begin{aligned}
(\tilde{n}_1^- \,\mathrm{d}_0, \tilde{n}_1)^{(n_0,n_1)} &= (\tilde{n}_1^- \,\mathrm{d}_0, \tilde{n}_1)((n_0,n_1)(\mathfrak{q}_{0,0}^- \blacktriangle \gamma^{1,0}_{G\,\mathrm{Sq}} \blacktriangle \hat{\mathfrak{q}}_{1,0}))\\
&= (\tilde{n}_1^- \,\mathrm{d}_0, \tilde{n}_1)(((n_0,n_1)\mathfrak{q}_{0,0}^-)\gamma^{1,0}_{G\,\mathrm{Sq}})\hat{\mathfrak{q}}_{1,0})\\
&= (\tilde{n}_1^- \,\mathrm{d}_0, \tilde{n}_1)(((n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0)G_{2;0})\gamma^{1,0}_{G\,\mathrm{Sq}})\hat{\mathfrak{q}}_{1,0}\\
&= (\tilde{n}_1^- \,\mathrm{d}_0, \tilde{n}_1)(\mathfrak{q}_{1,0}^- \blacktriangle ((n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0)G_{2;0})\gamma^{1,0}_{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{1,0}).
\end{aligned}$$

Moreover we have

$$(\tilde{n}_1^- \,\mathrm{d}_0, \tilde{n}_1)\mathfrak{q}_{1,0}^- = (\tilde{n}_1 \,\mathrm{s}_0 \cdot \tilde{n}_1^- \,\mathrm{s}_1)^{\tilde{n}_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0} G_{2;0,1}.$$

First we calculate

$$\begin{aligned}
&(\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1 \,\mathrm{s}_0 \cdot \tilde{n}_1^- \,\mathrm{s}_1 \cdot \tilde{n}_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_1)^{(n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0)} \,\mathrm{d}_0\\
= {}&(\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \,\mathrm{d}_0 \cdot \tilde{n}_1 \,\mathrm{s}_0 \,\mathrm{d}_0 \cdot \tilde{n}_1^- \,\mathrm{s}_1 \,\mathrm{d}_0 \cdot \tilde{n}_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_1 \,\mathrm{d}_0)^{(n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \,\mathrm{d}_0 \cdot n_1 \,\mathrm{s}_0 \,\mathrm{d}_0)}\\
= {}&(\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1 \cdot \ \tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{d}_0 \,\mathrm{s}_0)^{(n_0 \,\mathrm{s}_0 \cdot n_1)}\\
= {}&(\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1 \cdot \ \tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1 \,\mathrm{d}_0 \,\mathrm{s}_0)^{(n_0 \,\mathrm{s}_0 \cdot n_1)}\\
= {}&(\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1)^{(n_0 \,\mathrm{s}_0 \cdot n_1)}.
\end{aligned}$$

Then we calculate

$$\begin{aligned}
(\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1)^{(n_0 \,\mathrm{s}_0 \cdot n_1)} \,\mathrm{d}_1 &= (\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot \tilde{n}_1 \,\mathrm{d}_1)^{n_0 \,\mathrm{s}_0 \,\mathrm{d}_1 \cdot n_1 \,\mathrm{d}_1}\\
&= (\tilde{n}_1^- \,\mathrm{d}_0)^{n_0}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&((\tilde{n}_1 \,\mathrm{s}_0 \cdot \tilde{n}_1^- \,\mathrm{s}_1)^{\tilde{n}_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0} G_{2;0,1})\mathfrak{q}_{1,0}\\
= {}&((\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1 \,\mathrm{s}_0 \cdot \tilde{n}_1^- \,\mathrm{s}_1 \cdot \tilde{n}_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_1)^{n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0} G_{2;0,1})\mathfrak{q}_{1,0}\\
= {}&((\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1)^{(n_0 \,\mathrm{s}_0 \cdot n_1)} \,\mathrm{d}_1, ((\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1)^{(n_0 \,\mathrm{s}_0 \cdot n_1)})^- \,\mathrm{d}_1 \,\mathrm{s}_0 \cdot (\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1)^{(n_0 \,\mathrm{s}_0 \cdot n_1)})\\
= {}&((\tilde{n}_1^- \,\mathrm{d}_0)^{n_0}, (\tilde{n}_1 \,\mathrm{d}_0 \,\mathrm{s}_0)^{n_0 \,\mathrm{s}_0} \cdot \ (\tilde{n}_1^- \,\mathrm{d}_0 \,\mathrm{s}_0 \cdot \tilde{n}_1)^{(n_0 \,\mathrm{s}_0 \cdot \tilde{n}_1)}).
\end{aligned}$$

We have $\gamma^{0,1}_{G\,\mathrm{Sq}^!} = \mathfrak{q}_{0,0}^- \blacktriangle \gamma^{0,1}_{G\,\mathrm{Sq}} \blacktriangle \hat{\mathfrak{q}}_{0,1}$.

For $n_1,\ \tilde{n}_1 \in \mathrm{N}_1$ and $n_0 \in \mathrm{N}_0$, we have

$$\begin{aligned}
\tilde{n}_1^{(n_0,n_1)} &= (\tilde{n}_1)((n_0,n_1)(\mathfrak{q}_{0,0}^- \blacktriangle \gamma^{0,1}_{G\,\mathrm{Sq}} \blacktriangle \hat{\mathfrak{q}}_{0,1}))\\
&= (\tilde{n}_1)(((n_0,n_1)\mathfrak{q}_{0,0}^-)\gamma^{0,1}_{G\,\mathrm{Sq}})\hat{\mathfrak{q}}_{0,1}\\
&= (\tilde{n}_1)(((n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0)G_{2;0})\gamma^{0,1}_{G\,\mathrm{Sq}})\hat{\mathfrak{q}}_{0,1}\\
&= (\tilde{n}_1)(\mathfrak{q}_{0,1}^- \blacktriangle ((n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0)G_{2;0})\gamma^{0,1}_{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{0,1}).
\end{aligned}$$

Moreover we have

$$\tilde{n}_1\mathfrak{q}_{0,1}^- = (\tilde{n}_1 \,\mathrm{s}_0)G_{2;0,2}.$$

So we have

$$\begin{aligned}
((\tilde{n}_1 \,\mathrm{s}_0)^{(n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0)} G_{2;0,2})\mathfrak{q}_{0,1} &= (\tilde{n}_1 \,\mathrm{s}_0)^{(n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \cdot n_1 \,\mathrm{s}_0)} \,\mathrm{d}_0\\
&= (\tilde{n}_1 \,\mathrm{s}_0 \,\mathrm{d}_0)^{(n_0 \,\mathrm{s}_0 \,\mathrm{s}_0 \,\mathrm{d}_0 \cdot n_1 \,\mathrm{s}_0 \,\mathrm{d}_0)}\\
&= \tilde{n}_1^{n_0 \,\mathrm{s}_0 \cdot n_1}.
\end{aligned}$$

We have $\chi_{G\,\mathrm{Sq}^!} = (\mathfrak{q}_{1,0} \ltimes \mathfrak{q}_{0,1})^- \blacktriangle \chi_{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{1,1}$.

For $n_1,\ \tilde{n}_1 \in \mathrm{N}_1$, we have

$$\begin{aligned}
\lceil (n_1^- \,\mathrm{d}_0, n_1), \tilde{n}_1 \rceil &= ((n_1^- \,\mathrm{d}_0, n_1), \tilde{n}_1)((\mathfrak{q}_{1,0}^- \times \mathfrak{q}_{0,1}^-) \blacktriangle \chi_{G\,\mathrm{Sq}} \blacktriangle \mathfrak{q}_{1,1})\\
&= (((n_1^- \,\mathrm{d}_0, n_1), \tilde{n}_1)(\mathfrak{q}_{1,0}^- \times \mathfrak{q}_{0,1}^-))\chi_{G\,\mathrm{Sq}})\mathfrak{q}_{1,1}\\
&= (((n_1^- \,\mathrm{d}_0, n_1)\mathfrak{q}_{1,0}^-, \tilde{n}_1\mathfrak{q}_{0,1}^-)\chi_{G\,\mathrm{Sq}})\mathfrak{q}_{1,1}\\
&= (((n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0} G_{2;0,1}, \tilde{n}_1 \,\mathrm{s}_0 \, G_{2;0,2})\chi_{G\,\mathrm{Sq}})\mathfrak{q}_{1,1}\\
&= [(n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0}, \tilde{n}_1 \,\mathrm{s}_0]\mathfrak{q}_{1,1}\\
&= [(n_1 \,\mathrm{s}_0 \cdot n_1^- \,\mathrm{s}_1)^{n_1 \,\mathrm{d}_0 \,\mathrm{s}_0 \,\mathrm{s}_0}, \tilde{n}_1 \,\mathrm{s}_0].
\end{aligned}$$

By Remark 67.(1), we have that

$$G\,\mathrm{Sq}^! := (\mathrm{N}_2,\, \mathrm{U}_1,\, \mathrm{N}_1,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{1,0}^{G\,\mathrm{Sq}^!},\, \gamma_{0,1}^{G\,\mathrm{Sq}^!},\, \gamma_{G\,\mathrm{Sq}^!}^{1,1},\, \gamma_{G\,\mathrm{Sq}^!}^{1,0},\, \gamma_{G\,\mathrm{Sq}^!}^{0,1},\, \lambda_{G\,\mathrm{Sq}^!}^{1,0},\, \lambda_{G\,\mathrm{Sq}^!}^{0,1},\, \mu_{1,0}^{G\,\mathrm{Sq}^!},\, \mu_{0,1}^{G\,\mathrm{Sq}^!},\, \chi_{G\,\mathrm{Sq}^!})$$

is a crossed square.

By Remark 67.(2), we have that

$$\mathfrak{q} := (\mathfrak{q}_{1,1},\, \mathfrak{q}_{1,0},\, \mathfrak{q}_{0,1},\, \mathfrak{q}_{0,0}) : G\,\mathrm{Sq} \to G\,\mathrm{Sq}^!$$

is an isomorphism of crossed squares. □

Remark 82 The defining properties of the crossed square $G\,\mathrm{Sq}^!$, shown in Lemma 81 as a consequence, read as follows.

(CS 1) We have $\kappa_{G\,\mathrm{Sq}^!} = \lambda_{G\,\mathrm{Sq}^!}^{1,0} \blacktriangle \mu_{1,0}^{G\,\mathrm{Sq}^!} = \lambda_{G\,\mathrm{Sq}^!}^{0,1} \blacktriangle \mu_{0,1}^{G\,\mathrm{Sq}^!}$.

(CS 2) We have the following crossed modules.

(1) $(\mathrm{N}_2,\, \mathrm{U}_1,\, \gamma_{1,0}^{G\,\mathrm{Sq}^!},\, \lambda_{G\,\mathrm{Sq}^!}^{1,0})$

(2) $(\mathrm{N}_2,\, \mathrm{N}_1,\, \gamma_{0,1}^{G\,\mathrm{Sq}^!},\, \lambda_{G\,\mathrm{Sq}^!}^{0,1})$

(3) $(\mathrm{N}_2,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{1,1},\, \kappa_{G\,\mathrm{Sq}^!})$

(4) $(\mathrm{U}_1,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{1,0},\, \mu_{1,0}^{G\,\mathrm{Sq}^!})$

(5) $(\mathrm{N}_1,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{0,1},\, \mu_{0,1}^{G\,\mathrm{Sq}^!})$

(CS 3) We have the following morphisms of crossed modules.

(1) $(\mathrm{id}_{\mathrm{N}_2}, \mu_{1,0}^{G\,\mathrm{Sq}^!}) : (\mathrm{N}_2,\, \mathrm{U}_1,\, \gamma_{1,0}^{G\,\mathrm{Sq}^!},\, \lambda_{G\,\mathrm{Sq}^!}^{1,0}) \to (\mathrm{N}_2,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{1,1},\, \kappa_{G\,\mathrm{Sq}^!})$

(2) $(\mathrm{id}_{\mathrm{N}_2}, \mu_{0,1}^{G\,\mathrm{Sq}^!}) : (\mathrm{N}_2,\, \mathrm{N}_1,\, \gamma_{0,1}^{G\,\mathrm{Sq}^!},\, \lambda_{G\,\mathrm{Sq}^!}^{0,1}) \to (\mathrm{N}_2,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{1,1},\, \kappa_{G\,\mathrm{Sq}^!})$

(3) $(\lambda_{G\,\mathrm{Sq}^!}^{1,0}, \mathrm{id}_{\mathrm{N}_0\ltimes\mathrm{N}_1}) : (\mathrm{N}_2,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{1,1},\, \kappa_{G\,\mathrm{Sq}^!}) \to (\mathrm{U}_1,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{1,0},\, \mu_{1,0}^{G\,\mathrm{Sq}^!})$

(4) $(\lambda_{G\,\mathrm{Sq}^!}^{0,1}, \mathrm{id}_{\mathrm{N}_0\ltimes\mathrm{N}_1}) : (\mathrm{N}_2,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{1,1},\, \kappa_{G\,\mathrm{Sq}^!}) \to (\mathrm{N}_1,\, \mathrm{N}_0\ltimes\mathrm{N}_1,\, \gamma_{G\,\mathrm{Sq}^!}^{0,1},\, \mu_{0,1}^{G\,\mathrm{Sq}^!})$

(CS 4.1) Suppose given $n_1,\, \tilde{n}_1 \in \mathrm{N}_1$.

Then we have

$$(n_1^-\,\mathrm{d}_0, n_1)\cdot \lceil (n_1^-\,\mathrm{d}_0, n_1), \tilde{n}_1\rceil \lambda_{G\,\mathrm{Sq}^!}^{1,0} = (n_1^-\,\mathrm{d}_0, n_1)^{\tilde{n}_1 \mu_{0,1}^{G\,\mathrm{Sq}^!}}.$$

(CS 4.2) Suppose given $n_1,\, \tilde{n}_1 \in \mathrm{N}_1$.

Then we have

$$\tilde{n}_1^{(n_1^-\,\mathrm{d}_0, n_1)\mu_{1,0}^{G\,\mathrm{Sq}^!}} \cdot \lceil (n_1^-\,\mathrm{d}_0 \cdot n_1), \tilde{n}_1\rceil \lambda_{G\,\mathrm{Sq}^!}^{0,1} = \tilde{n}_1.$$

(CS 4.3) Suppose given $n_2 \in \mathrm{N}_2$ and $n_1,\, \tilde{n}_1 \in \mathrm{N}_1$.

Then we have

$$n_2 \cdot \lceil n_2 \lambda_{G\,\mathrm{Sq}^!}^{1,0}, \tilde{n}_1\rceil = n_2^{\tilde{n}_1}.$$

(CS 4.4) Suppose given $n_2 \in \mathrm{N}_2$ and $n_1 \in \mathrm{N}_1$.

Then we have

$$n_2^{(n_1^-\,\mathrm{d}_0, n_1)} \cdot \lceil (n_1^-\,\mathrm{d}_0, n_1), n_2 \lambda_{G\,\mathrm{Sq}^!}^{0,1}\rceil = n_2.$$

(CS 4.5) Suppose given $n_1,\, n_1^*,\, \tilde{n}_1 \in \mathrm{N}_1$.

Then we have

$$\lceil (n_1^-\,\mathrm{d}_0, n_1)\cdot (n_1^{*-}\,\mathrm{d}_0, n_1^*), \tilde{n}_1\rceil = \lceil (n_1^-\,\mathrm{d}_0, n_1), \tilde{n}_1\rceil^{(n_1^{*-}\,\mathrm{d}_0, n_1^*)} \cdot \lceil (n_1^{*-}\,\mathrm{d}_0, n_1^*), \tilde{n}_1\rceil.$$

(CS 4.6) Suppose given $n_1,\ \tilde{n}_1,\ \tilde{n}_1^* \in \mathrm{N}_1$.

Then we have

$$\lceil (n_1^- \, \mathrm{d}_0, n_1), \tilde{n}_1 \cdot \tilde{n}_1^* \rceil = \lceil (n_1^- \, \mathrm{d}_0, n_1), \tilde{n}_1^* \rceil \cdot \lceil (n_1^- \, \mathrm{d}_0, n_1), \tilde{n}_1 \rceil^{\tilde{n}_1^*}.$$

(CS 4.7) Suppose given $n_0 \in \mathrm{N}_0$ and $n_1,\ \tilde{n}_1,\ \tilde{\tilde{n}}_1 \in \mathrm{N}_1$.

Then we have

$$\lceil (\tilde{n}_1^- \, \mathrm{d}_0, \tilde{n}_1), \tilde{\tilde{n}}_1 \rceil^{(n_0, n_1)} = \lceil (\tilde{n}_1^- \, \mathrm{d}_0, \tilde{n}_1)^{(n_0, n_1)}, \tilde{\tilde{n}}_1^{(n_0, n_1)} \rceil.$$

(CS 4.8) Suppose given $n_1,\ \tilde{n}_1 \in \mathrm{N}_1$ and $n_2 \in \mathrm{N}_2$.

Then we have

$$((n_2^{\tilde{n}_1})^{(n_1^- \, \mathrm{d}_0, n_1)})^{\lceil (n_1^- \, \mathrm{d}_0, n_1), \tilde{n}_1 \rceil} = (n_2^{(n_1^- \, \mathrm{d}_0, n_1)})^{\tilde{n}_1}.$$

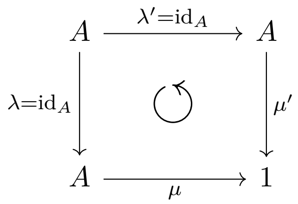
Remark 83 The functor $\mathrm{Sq} : \lfloor 2, 0 \rfloor$-$\mathcal{SimpGrp} \to \mathcal{CrSq}$ is not an equivalence.

More precisely Sq is not dense.

Proof.

Let A be an abelian group with $A \neq 1$.

Consider the crossed square $C := (A,\ A,\ A,\ 1)$ from Example 66.

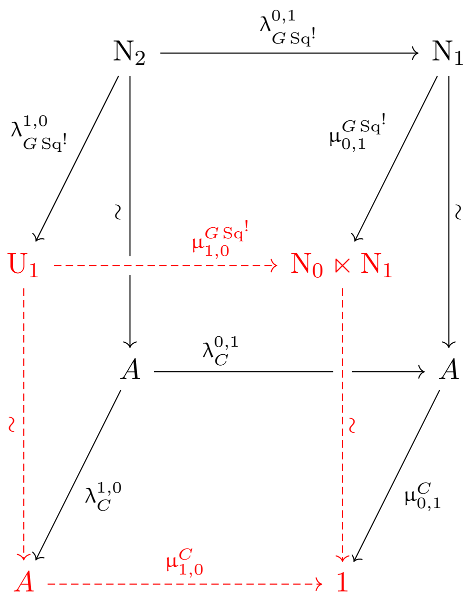


It suffices to show that C is not isomorphic to a crossed square of the form $G\,\mathrm{Sq}$ for a simplicial group G.

Assume that $G\,\mathrm{Sq} \simeq C$.

Then $G\,\mathrm{Sq}^! \simeq G\,\mathrm{Sq} \simeq C$.

So we have an isomorphism as follows.



The morphism $\mu_{1,0}^{G\,\mathrm{Sq}^!}$ is injective; cf. Lemma 81. But the morphism $\mu_{1,0}^{C}$ is not injective.

So the commutative quadrangle in front yields a *contradiction*.

So C is not isomorphic to $G\,\mathrm{Sq}^!$ and therefore not isomorphic to $G\,\mathrm{Sq}$. □

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Zusammenfassung

Die Simplex-Kategorie $\triangle$ ist die Kategorie der totalgeordneten Mengen der Form $[0,n]$ und ihrer monotonen Abbildungen. Eine simpliziale Gruppe ist ein Funktor von $\triangle^{\mathrm{op}}$ in die Kategorie der Gruppen. Also ist eine simpliziale Gruppe G eine Folge von Gruppen $\dots$, G_3, G_2, G_1, G_0, zusammen mit Randmorphismen $\mathrm{d}_i^{G,n}$ und Ausartungsmorphismen $s_j^{G,n}$, die gewisse Relationen erfüllen. Simpliziale Gruppen modellieren topologische Räume auf algebraische Weise.

Eine $\lfloor 2,0\rfloor$-simpliziale Gruppe G besteht aus Gruppen G_2, G_1, G_0, zusammen mit Randmorphismen $\mathrm{d}_i^{G,n}$ und Ausartungsmorphismen $\mathrm{s}_j^{G,n}$, die gewisse Relationen erfüllen. Dabei soll noch die Conduché-Bedingung gelten, die besagt, dass gewisse Untergruppen von G_2 kommutieren.

Mit Hilfe der Abschneideoperation Trunc bekommen wir aus einer simplizialen Gruppe G eine $\lfloor 2,0\rfloor$-simpliziale Gruppe $G\,\mathrm{Trunc}$.

$$G\,\mathrm{Trunc} = \left(G_2/G\mathrm{NB}_2 \; \begin{array}{c} \xrightarrow{\check{\mathrm{d}}_2^{G,2}} \\ \xleftarrow{\mathrm{s}_1^{G,1}\,\blacktriangle\,\rho^G} \\ \xrightarrow{\check{\mathrm{d}}_1^{G,2}} \\ \xleftarrow{\mathrm{s}_0^{G,1}\,\blacktriangle\,\rho^G} \\ \xrightarrow{\check{\mathrm{d}}_0^{G,2}} \end{array} \; G_1 \; \begin{array}{c} \xrightarrow{\mathrm{d}_1^{G,1}} \\ \xleftarrow{\mathrm{s}_0^{G,0}} \\ \xrightarrow{\mathrm{d}_0^{G,1}} \end{array} \; G_0 \right)$$

Der Begriff des verschränkten Quadrats ist eine Verallgemeinerung des Begriffs des verschränkten Moduls. Es ist ein kommutatives Viereck von Gruppen mit Zusatzdaten, die bestimmte Voraussetzungen erfüllen.

Wir konstruieren den Funktor Sq von der Kategorie der $\lfloor 2,0\rfloor$-simplizialen Gruppen in die Kategorie der verschränkten Quadrate.

Sei G eine $\lfloor 2,0\rfloor$-simpliziale Gruppe. Dann hat $G\,\mathrm{Sq}$ die Gestalt

$$G\,\mathrm{Sq} = \left(\begin{array}{ccc} G_{2;1,2} & \longrightarrow & G_{2;2}/G_{2;0,2} \\ \downarrow & & \downarrow \\ G_{2;1}/G_{2;0,1} & \longrightarrow & G_2/G_{2;0} \end{array} \right).$$

Hierbei ist $G_{2;1,2} = \ker(\mathrm{d}_1^{G,2}) \cap \ker(\mathrm{d}_2^{G,2})$, etc.

Insgesamt haben wir Funktoren

$$\mathit{SimpGrp} \xrightarrow{\mathrm{Trunc}} \lfloor 2,0\rfloor\text{-}\mathit{SimpGrp} \xrightarrow{\mathrm{Sq}} \mathit{CrSq}$$

erhalten.

Wir können das erhaltene verschränkte Quadrat $G\,\mathrm{Sq}$ isomorph ersetzen durch ein verschränktes Quadrat $G\,\mathrm{Sq}^!$, welches keine Faktorgruppen mehr enthält.

$$G\,\mathrm{Sq}^! = \left(\begin{array}{ccc} \mathrm{N}_2 & \longrightarrow & \mathrm{N}_1 \\ \downarrow & & \downarrow \\ \mathrm{U}_1 & \longrightarrow & \mathrm{N}_0 \ltimes \mathrm{N}_1 \end{array} \right)$$

Hierbei sind N_0, N_1, N_2 die Terme aus dem Moorekomplex von G, und U_1 ist eine gewisse Untergruppe von $\mathrm{N}_0 \ltimes \mathrm{N}_1$.

Es ist also $G\,\mathrm{Sq} \simeq G\,\mathrm{Sq}^!$.

Wir können schließlich noch nachweisen, dass der Funktor Sq keine Äquivalenz ist.

$$\lfloor 2,0 \rfloor\text{-}\mathit{SimpGrp} \xrightarrow{\not\simeq} \mathit{CrSq}.$$

Dies steht im Kontrast zur Äquivalenz von den $\lfloor 1,0 \rfloor$-simplizialen Gruppen zu den verschränkten Moduln.

$$\lfloor 1,0 \rfloor\text{-}\mathit{SimpGrp} \xrightarrow{\sim} \mathit{CrMod}\,.$$

Erklärung

Ich versichere hiermit, dass ich die Arbeit selbstständig und nur mit den angegebenen Hilfsmitteln angefertigt habe und dass alle Stellen, die dem Wortlaut oder dem Sinne nach anderen Werken entnommen sind, durch Angabe der Quellen als Entlehnungen kenntlich gemacht worden sind. Die eingereichte Arbeit ist weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen. Das elektronische Exemplar stimmt mit den anderen Exemplaren überein.

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