$\mathcal{Z}$: category

S : shift on $\mathcal{Z}$, i.e.

$$S_{X,Y} = S : {}_{\mathcal{Z}}(X,Y) \longrightarrow {}_{\mathcal{Z}}(X,Y) \quad \text{for } X, Y \in \mathrm{Ob}\,\mathcal{Z}$$

(1) bijective

(2) $(a \cdot b)S = aS \cdot b = a \cdot bS$ for $X \xrightarrow{a} Y \xrightarrow{b} Z$

deg : degree function,

deg : $\mathrm{Mor}(\mathcal{Z}) \longrightarrow \mathbb{Z}$ such that

(1) $(a \cdot b)\deg = a\deg + b\deg$ for $X \xrightarrow{a} Y \xrightarrow{b} Z$

(2) $(aS)\deg = a\deg + 1$ for $X \xrightarrow{a} Y$

$$\mathcal{Z} = (\mathcal{Z}, S, \deg) \quad \text{grading category}$$

Ob$(\mathbb{Z}) = \{\mathbb{Z}\}$, Mor$(\mathbb{Z}) = \mathbb{Z}$, composition = addition

<u>Examples</u>

(1) $\mathcal{Z} = \mathbb{Z}$.

$iS = i+1$

$i\deg = i$

later needed for A_∞-algebras

composition entrywise

(2) $\mathcal{Z} = \mathbb{Z} \times \mathcal{C}$ ($\mathcal{C}$: category)

$(i,a)S = (i+1, a)$

$(i,a)\deg = i$

later needed for A_∞-categories

$\mathcal{Z} = (\mathcal{Z}, S, \deg)$: grading category

$z \in \mathrm{Mor}(\mathcal{Z}) \rightsquigarrow \mathrm{fact}_3(z) = \left\{ (y_1, y_2, y_3) \in \left(\mathrm{Mor}(\mathcal{Z})\right)^{\times 3} : \right.$ factorisations of z of length 3 $\left[\begin{tikzcd} \bullet \arrow[r, "y_1"] \arrow[rrr, bend left, "z"] & \bullet \arrow[r, "y_2"] \arrow[r, phantom, "C", shift left=3] & \bullet \arrow[r, "y_3"] & \bullet \end{tikzcd} \right] \left. \right\}$

- $\mathcal{Z}$-graded module: $M = (M^z)_{z \in \mathrm{Mor}(\mathcal{Z})}$ module
- Graded linear map:

linear map

$$\left(L \xrightarrow{f} M\right) = \left(L^z \xrightarrow{f^z} M^z\right)_{z \in \mathrm{Mor}(\mathcal{Z})}$$

$\mathcal{Z}$-graded modules   modules

category $\mathcal{Z}\text{-grad}_0$

- Shift: $\left(L \xrightarrow{f} M\right)^{[k]} = \left(L^{z[k]} \xrightarrow{f^{z[k]}} M^{z[k]}\right)_{z \in \mathrm{Mor}(\mathcal{Z})}$,

i.e. $(f^{[k]})^z = f^{z[k]}$.

- Shift-graded linear map of degree k:

$L \xrightarrow{(f,k)} M$ with $k \in \mathbb{Z}$, $\underbrace{L \xrightarrow{f} M^{[k]}}_{\text{graded linear map}}$

$$\begin{tikzcd} L \arrow[r, "(f,k)"] \arrow[rr, bend right] & M \arrow[r, "(g,l)"] & N \end{tikzcd}$$

$$(f,k) \cdot (g,l) = (f \cdot g^{[k]}, k+l)$$

category $\mathcal{Z}\text{-grad}$

Note: $L \xrightarrow{f} M^{[k]} \xrightarrow{g^{[k]}} N^{[k+l]}$

$\mathcal{Z} = (Z, S, \deg)$: grading category $\quad$ (Ex: $Z = \mathbb{Z}$)

L_i, M_i for $i \in [1,n]$: Z-graded modules

$L_i \xrightarrow{(f_i, k_i)} M_i$ for $i \in [1,n]$: shift-graded linear maps

(so $L_i \xrightarrow{f_i} M_i^{[k_i]}$ graded linear map)

Define the Z-graded module :

$$\bigotimes_{i\in[1,n]} M_i := \left(\bigoplus_{y \in \mathrm{fact}_n(z)} \bigotimes_{i \in [1,n]} M_i^{y_i} \right)_{z \in \mathrm{Mor}(Z)}$$

Define the shift-graded linear map

$$\left(\bigotimes_{i\in[1,n]} L_i \xrightarrow[\left(\bigotimes_{i\in[1,n]} f_i,\, k\right)]{\bigotimes_{i\in[1,n]}(f_i, k_i)} \bigotimes_{i\in[1,n]} M_i \right), \quad \text{where } k := \sum_{i\in[1,n]} k_i$$

by $\left(\bigotimes_{i\in[1,n]} L_i \xrightarrow{\bigotimes_{i\in[1,n]} f_i} \left(\bigotimes_{i\in[1,n]} M_i \right)^{[k]} \right)$,

having at $z \in \mathrm{Mor}(Z)$

$$\left(\left(\bigotimes_{i\in[1,n]} L_i \right)^{z} \xrightarrow{\bigotimes_{i\in[1,n]} f_i} \left(\bigotimes_{i\in[1,n]} M_i \right)^{z[k]} \right)$$

$$\left(\bigoplus_{y \in \mathrm{fact}_n(z)} \bigotimes_{i\in[1,n]} L_i^{y_i} \;\; \bigoplus_{\tilde{y} \in \mathrm{fact}_n(z[k])} \bigotimes_{i\in[1,n]} M_i^{\tilde{y}_i} \right),$$

mapping $(l_i)^{\otimes}_{i\in[1,n]} \in \bigotimes_{i\in[1,n]} L_i^{y_i}$

to $\underbrace{\lfloor (k_i)_i , (y_i \deg)_i \rfloor}_{\text{Koszul sign : } (-1)^{\sum_{1\le i<j\le n} k_i \cdot (y_j \deg)}} (l_i f_i^{y_i})^{\otimes}_{i\in[1,n]} \in \bigotimes_{i\in[1,n]} M_i^{y_i[k_i]}$

$\mathcal{Z} = (\mathcal{Z}, S, \deg)$: grading category. Ex. $\mathcal{Z} = \mathbb{Z}$.

Def 19 S.g. $n \in [0, \infty]$.

S.g. a $\mathcal{Z}$-graded module A.

(1) S.g. a shift-graded linear map

abuse, cf. Rem. 14

$$m_k \overset{\downarrow}{=} (m_k, 2-k) : A^{\otimes k} \longrightarrow A$$

for $k \in [1, n] \cap \mathbb{Z}$.

Then $A = (A, (m_k)_{k \in [1,n] \cap \mathbb{Z}})$ is a

pre-A_n-algebra (over $\mathcal{Z}$).

A pre-A_n-algebra $A = \left(A, (m_k)_{k \in [1,n] \cap \mathbb{Z}}\right)$

is an A_n-algebra (over $\mathcal{Z}$)

if the Stasheff equation

$$0 = \sum_{\substack{r,s,t \in \mathbb{Z}_{\geq 0},\, s \geq 1 \\ r+s+t=k}} (-1)^{r+st} \, (\mathrm{id}^{\otimes r} \otimes \underset{\substack{\uparrow \\ \text{i.e.} \\ (m_s,\, 2-s)}}{m_s} \otimes \mathrm{id}^{\otimes t}) \cdot \underset{\substack{\uparrow \\ \text{i.e.} \\ (m_{r+1+t},\, 1-r-t)}}{m_{r+1+t}}$$

holds for $k \in [1, n] \cap \mathbb{Z}$.

cf Rem. 14

$\mathcal{Z}$: grading category

$n \in [0, \infty]$

$$A = \Big(\underset{\mathcal{Z}\text{-graded module}}{A}, \; (\overset{\text{aka } m_k^A}{m_k})_{k \in [1,n] \cap \mathbb{Z}} \Big) \quad A_n\text{-Algebra over } \mathcal{Z}$$

$A^{\otimes k} \xrightarrow{m_k} A$ is a shift-graded lin. map

$m_k \overset{\text{R.14}}{=} (m_k, 2-k)$

Stasheff equations: for $k \in [1,n] \cap \mathbb{Z}$, we have

$$0 = \sum_{\substack{(r,s,t) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \\ r+s+t = k}} (-1)^{r+st} \, (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \cdot m_{r+1+t}$$

$$\Big(\tilde{A} \xrightarrow{f} A \Big)$$

$$= \Big(\underset{A_n\text{-alg.}}{(\tilde{A}, (\tilde{m}_k)_k)} \xrightarrow{(f_k)_{k \in [1,n] \cap \mathbb{Z}}} \underset{A_n\text{-alg.}}{(A, (m_k)_k)} \Big)$$

morphism of A_n-algebras over $\mathcal{Z}$.

$\tilde{A}^{\otimes k} \xrightarrow{f_k} A$ is a shift-graded lin. map,

$f_k \overset{\text{R.14}}{=} (f_k, 1-k)$

Stasheff eq. for morphisms: for $k \in [1,n] \cap \mathbb{Z}$, we have

$$\sum_{\substack{(r,s,t) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \\ r+s+t = k}} (-1)^{r+st} (\mathrm{id}^{\otimes r} \otimes \tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot f_{r+1+t}$$

$$= \sum_{r \in [1,k]} \; \sum_{\substack{(i_j)_j \in \mathbb{Z}_{\geq 1}^{\times r} \\ \sum_{j \in [1,r]} i_j = k}} [(1-i_j)_j, (i_j)_j] \Big(\bigotimes_{j \in [1,r]} f_{i_j} \Big) \cdot m_r$$

<u>Differential graded category</u> on set of objects I:

unital A_∞-category $\mathcal{A} = (\mathcal{A}, (m_k)_{k \in \mathbb{Z}_{\geq 1}})$

$(1_x \otimes a) m_2 = a$, $(a \otimes 1_y) m_2 = a$

over $\mathbb{Z} \times I^{\times 2}$ such that $m_k = 0$ for $k \geq 3$

Has to fulfill Stasheff equation

for $k = 1$: $m_1 \cdot m_1 = 0$

for $k = 2$: $m_2 \cdot m_1 = (\mathrm{id} \otimes m_1) \cdot m_2 + (m_1 \otimes \mathrm{id}) \cdot m_2$

for $k = 3$: $(m_2 \otimes \mathrm{id}) \cdot m_2 = (\mathrm{id} \otimes m_2) \cdot m_2$

Suppose given an algebra B. ordinary associative algebra

Suppose given $n \in \mathbb{Z}_{\geq 1}$,

$$X_s = (\dots \longrightarrow X_s^i \xrightarrow{d_s^i} X_s^{i+1} \xrightarrow{d_s^{i+1}} X_s^{i+2} \longrightarrow \dots)$$

$\in C(B\text{-Mod})$ for $s \in [1, n]$.

Let $Z := \mathbb{Z} \times [1, n]^{\times 2}$.

differentials not involved here

Write $\mathrm{Hom}_B^j(X_s, X_t) := \{(f^i)_{i \in \mathbb{Z}} : X_s^i \xrightarrow{f^i} X_t^{i+j}$ is a B-linear map for $i \in \mathbb{Z}\}$

for $s, t \in [1, n]$, $j \in \mathbb{Z}$.

Abbreviate $\underline{X} := (X_s)_{s \in [1, n]}$.

Aim: construct differential graded category on set of objects $[1, n]$ with $\mathrm{Hom}_B^j(X_s, X_t)$ as graded pieces

$\mathcal{B}$: algebra ← e.g. $\mathcal{B} = R G$

$n \in \mathbb{Z}_{\geq 1}$

$X_s \in \mathrm{Ob}\,C(\mathcal{B}\text{-Mod})$ for $s \in [1,n]$

$\underline{X} = (X_s)_{s \in [1,n]}$

The regular differential graded category $\mathrm{Hom}_{\mathcal{B}}(\underline{X})$ ← unital A_∞-algebra over $\mathbb{Z} \times [1,n]^{\times 2}$ such that $m_k = 0$ for $k \geq 3$

on the set of objects $[1,n]$ is:

- $\mathbb{Z} \times [1,n]^{\times 2}$-graded module $\mathrm{Hom}_{\mathcal{B}}(\underline{X})$

  having

  $$\mathrm{Hom}_{\mathcal{B}}(\underline{X})^{(j,(s,t))} := \mathrm{Hom}^{j}_{\mathcal{B}}(X_s, X_t)$$

  ↑ contains all tuples $(f^i : X_s^i \to X_t^{i+j})_i$ of $\mathcal{B}$-linear maps

  at $(j,(s,t)) \in \mathrm{Mor}\,(\mathbb{Z} \times [1,n]^{\times 2})$

- $m_1^{\mathrm{Hom}_{\mathcal{B}}(\underline{X})} : \mathrm{Hom}_{\mathcal{B}}(\underline{X}) \to \mathrm{Hom}_{\mathcal{B}}(\underline{X})$

  mapping

  $$\underbrace{(f^i)_i}_{\deg = j} \longmapsto \underbrace{(f^i d_t^{i+j} - (-1)^j d_s^i f^{i+1})_i}_{\deg = j+1}$$

- $m_2^{\mathrm{Hom}_{\mathcal{B}}(\underline{X})} : \mathrm{Hom}_{\mathcal{B}}(\underline{X})^{\otimes 2} \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\underline{X})$

  mapping

  $$\underbrace{(f^i)_i}_{\deg = k} \otimes \underbrace{(g^i)_i}_{\deg = l} \longmapsto \underbrace{(f^i \cdot g^{i+k})_i}_{\deg = k+l}$$

$\mathcal{Z}$: grading cat. $n \in [1, \infty]$.

<u>Lemma 29</u> $(A, (m_i)_i)$: A_n-algebra $/ \mathcal{Z}$

$(\tilde{A}, (\tilde{m}_i)_i)$: pre-A_n-algebra $/ \mathcal{Z}$

$(f_i)_i : \tilde{A} \to A$: pre-A_n-morphism,

satisfying Stasheff for morphisms

for $k \in [1, n]$.

Suppose that f_1 is piecewise injective.

Then $(\tilde{A}, (\tilde{m}_i)_i)$ is an A_n-algebra.

"automatism"

<u>Proof</u> Induction on $n \geq 1$. makes sense, since the numbers of the Stasheff equations are finite

Have: $(m_i)_i$: Stasheff for $k \in [1, n]$.

$(\tilde{m}_i)_i$: Stasheff for $k \in [1, n-1]$

$(\tilde{m}_i)_i, (f_i)_i, (m_i)_i$: Stasheff for morphisms for $k \in [1, n]$.

Need: $(\tilde{m}_i)_i$: Stasheff for $k = n$.

Present state: Remains to show

$$0 \overset{!}{=} \sum_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t = n}} \; \sum_{u \in [1, r+1+t]} \; \sum_{\substack{(i_j)_j \geq (1)_j \\ \sum_j i_j = r+1+t}} (-1)^{r+st} \, \lfloor (1-i_j)_j, (i_j)_j \rfloor \left\{ \begin{array}{l} \cdot (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \\ \cdot \left(\bigotimes_{j \in [1, u]} f_{i_j} \right) \cdot m_u \end{array} \right.$$

$\mathcal{Z}$: grading cat.

$n \in [1, \infty]$

(1) $A = (A, (m_k)_k)$: A_n-alg. $/\mathcal{Z}$

ZA : $\mathcal{Z}$-graded module of cycles

$$(ZA)^z := \mathrm{Ker}\,(A^z \xrightarrow{m_1} A^{z[1]}) \quad \text{at } z \in \mathrm{Mor}(\mathcal{Z})$$

BA : $\mathcal{Z}$-graded module of boundaries

$$(BA)^z := \mathrm{Im}\,(A^{z[-1]} \xrightarrow{m_1} A^z) \quad \text{at } z \in \mathrm{Mor}(\mathcal{Z})$$

$(BA)^z \subseteq (ZA)^z$, since $m_1 \cdot m_1 = 0$

(2) HA : $\mathcal{Z}$-graded cohomology module

$$(HA)^z = (ZA)^z / (BA)^z \quad \text{at } z \in \mathrm{Mor}(\mathcal{Z})$$

Ex 34

A : A_n - alg. / $\mathbb{Z}$

$n \in [3,\infty]$ grading cat

$$\begin{tikzcd}
 & & A & \\
A \arrow[urr, bend left, "m_1"] \arrow[r, "m_1|^{ZA}"] & ZA \arrow[r, "\beta"] & ZA \arrow[u, "\iota"'] \arrow[r, "\rho"] \arrow[ul, phantom, "\circlearrowright"] & HA \arrow[l, bend right, "\sigma_1"'] \\
 & & ZA \arrow[u, "\mathrm{id}_{ZA} - \rho\sigma_1"'] \arrow[ull, red, "\exists\nu"] \arrow[ul, phantom, "\circlearrowright"] &
\end{tikzcd}$$

$$\begin{tikzcd}
A^{\otimes 2} \arrow[r, "m_2"] & A \\
(ZA)^{\otimes 2} \arrow[u, "\iota^{\otimes 2}"] \arrow[r, "\check{m}_2"] \arrow[ur, phantom, "\circlearrowright"] & ZA \arrow[u, "\iota"'] \arrow[d, "\rho"] \\
(HA)^{\otimes 2} \arrow[r, "\tilde{m}_2"] \arrow[from=u, "\rho^{\otimes 2}"] \arrow[ur, phantom, "\circlearrowright"] & HA
\end{tikzcd}$$

associative (properly understood)

Kadeishvili by induction:

$(A, (m_k)_{k\in[1,n]})$: A_n-alg. $/\ \mathbb{Z}$

$(HA, (\tilde{m}_k)_{k\in[1,n-1]})$: A_{n-1}-alg $/\ \mathbb{Z}$

$q = (q_k)_{k\in[1,n-1]}$: A_{n-1} - quasiiso morphism

Need: • $1_X\rho$ neutral w.r.t. $\tilde{m}_2$

follows by Ex. 34:

$$(1_X\rho \otimes b)\tilde{m}_2 = (1_X\rho \otimes a\rho)\tilde{m}_2 = (1_X \otimes a)\tilde{m}_2\,\rho = a\rho = b$$

• $0 \overset{!}{=} \Psi_n m_1$

$$= \sum_{r\in[2,n]} \sum_{\substack{(i_j)_j \geq (1)_j \\ \sum_{j\in[1,r]} i_j = n}} L((1-i_j)_j, (i_j)_j)\Big(\bigotimes_j q_{i_j}\Big)\underbrace{m_r m_1}$$

$$- \sum_{\substack{(r,s,t)\geq(0,2,0) \\ r+s+t=n \\ (r,t)\neq(0,0)}} (-1)^{r+st}\,(id^{\otimes r}\otimes \tilde{m}_s \otimes id^{\otimes t})\,\underbrace{q_{r+1+t}\, m_1}$$

Plug in:

• $$\underbrace{m_r m_1} = -\underbrace{\sum_{\substack{(u,v,w)\geq(0,1,0) \\ u+v+w=r \\ (u,w)\neq(0,0)}} (-1)^{u+vw}\,(id^{\otimes u}\otimes m_v \otimes id^{\otimes w})\,m_{u+1+w}}$$

• $\underbrace{q_{r+1+t}\, m_1} = \dots$

$\mathcal{Z}$: grading category

A : $\mathcal{Z}$-graded module

$A^{[1]} \xrightarrow{\omega} A$: shift-graded linear map of degree 1,

at $z \in \mathrm{Mor}(\mathcal{Z})$:

$$\begin{array}{ccc} (A^{[1]})^z & \longrightarrow & (A^{[1]})^z \\ a & \longmapsto & a\omega =: a \end{array}$$

Lemma 36 $n \in [0,\infty]$

$(A,(m_\ell)_\ell)$: pre-A_n-alg. / $\mathcal{Z}$

$${}^{\omega}m_\ell := \omega^{\otimes \ell} \cdot m_\ell \cdot \omega^-$$

$$\begin{tikzcd} A^{\otimes \ell} \arrow[r, "\underbrace{m_\ell}_{\deg = 2-\ell}"] & A \arrow[d, "2", "\omega^-"'] \\ (A^{[1]})^{\otimes \ell} \arrow[u, "\omega^{\otimes \ell}", "2"'] \arrow[r, "\underbrace{{}^{\omega}m_\ell}_{\deg = 1}"'] \arrow[ur, phantom, "G"] & A^{[1]} \end{tikzcd}$$

Then

$$\sum_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t = k}} \boxed{(-1)^{r+st}} (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t})\, m_{r+1+t} = 0$$

(Stasheff at $k \in [1,n] \cap \mathbb{Z}$)

$$\Updownarrow$$

$$\sum_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t = k}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}m_s \otimes \mathrm{id}^{\otimes t})\, {}^{\omega}m_{r+1+t} = 0$$

Lemma 37 $n \in [0,\infty]$, $(\tilde{A},(\tilde{m}_\ell)_\ell)$, $(A,(m_\ell)_\ell)$ pre-A_n-alg. / $\mathcal{Z}$

$f = (f_\ell)_\ell : \tilde{A} \to A$: pre-A_n-morphism.

$${}^{\omega}f_\ell := \omega^{\otimes \ell} \cdot f_\ell \cdot \omega^-$$

Then

$$\sum_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t = k}} \boxed{(-1)^{r+st}} (\mathrm{id}^{\otimes r} \otimes \tilde{m}_s \otimes \mathrm{id}^{\otimes t})\, f_{r+1+t} = \sum_{r \in [1,k]} \sum_{\substack{(i_j)_{j \in [1,r]} \geq (1)_j \\ \sum_j i_j = k}} \boxed{[(-i_j)_j, (i_j)_j]} \Big(\bigotimes_j f_{i_j}\Big) m_r$$

$$\Updownarrow$$

$$\sum_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t = k}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t})\, {}^{\omega}f_{r+1+t} = \sum_{r \in [1,k]} \sum_{\substack{(i_j)_{j \in [1,r]} \geq (1)_j \\ \sum_j i_j = k}} \Big(\bigotimes_j {}^{\omega}f_{i_j}\Big)\, {}^{\omega}m_r$$

$\mathcal{Z}$: grading cat.

- **coalgebra** over $\mathcal{Z}$: $\quad T = (T, \Delta)$

  T: $\mathcal{Z}$-graded module

  shift-graded linear map

  $\Delta : T \to T \otimes T$ sglm of deg 0

  s.th. $\Delta(1 \otimes \Delta) = \Delta(\Delta \otimes 1)$

- **morphism** of coalg.: $\quad T \xrightarrow{f} \tilde{T}$ sglm of deg. 0

  s.th. $f \tilde{\Delta} = \Delta(f \otimes f)$

- **coderivation** on T : $\quad T \xrightarrow{\delta} T$ sglm of deg. 1

  s.th. $\delta \Delta = \Delta(\mathrm{id} \otimes \delta + \delta \otimes \mathrm{id})$

  co-Leibniz-rule

  δ is a **codifferential**, if $\delta^2 = 0$.

- **coalgebra with codifferential** : $T = (\underbrace{T, \Delta}_{\text{coalg.}}, \delta)$ δ: codiff.

- **morphism** of coalg. with codiff. :

  $T \xrightarrow{f} \tilde{T}$ morph. of coalg.

  s.th. $f \tilde{\delta} = \delta f$.

<u>Lemma 39</u> $\quad V$: $\mathcal{Z}$-graded module

$n \in [1, \infty]$

$$T_{\leq n} V := \bigoplus_{k \in [1,n] \cap \mathbb{Z}} V^{\otimes k}$$

(1) $\quad \Delta : T_{\leq n} V \longrightarrow T_{\leq n} V \otimes T_{\leq n} V$

$$v_1 \otimes \cdots \otimes v_k =: v^{\otimes}_{[1,k]} \longmapsto \sum_{\substack{(i,j) \geq (1,1) \\ i+j=k}} v^{\otimes}_{[1,i]} \otimes v^{\otimes}_{[i+1,i+j]}$$

$\in V^{y_1} \otimes \cdots \otimes V^{y_k}$

for $(y_1, \dots, y_k) \in \mathrm{fact}_k(z)$,

$z \in \mathrm{Mor}(\mathcal{Z})$, $k \in [1,n] \cap \mathbb{Z}$

(2) $\quad \pi_1 : T_{\leq n} V \longrightarrow V \quad , \quad (\xi_k)_k \longmapsto \xi_1$

(3) $\quad \mu : T_{\leq n} V \otimes T_{\leq \tilde{n}} V \to T_{\leq n+\tilde{n}} V \quad , \quad \underbrace{v^{\otimes}_{[1,k]}}_{\in (V^{\otimes k})^{u}} \otimes \underbrace{\tilde{v}^{\otimes}_{[1,\tilde{k}]}}_{\in (V^{\otimes \tilde{k}})^{\tilde{u}}} \mapsto \underbrace{v^{\otimes}_{[1,k]} \otimes \tilde{v}^{\otimes}_{[1,\tilde{k}]}}_{\in (V^{\otimes k+\tilde{k}})^{u\tilde{u}}}$

(4) $\quad \mathrm{Kern}(\Delta) = V$

$\mathcal{Z}$: grading category

Cor 41 V, W : $\mathcal{Z}$-graded modules, $n \in [1, \infty]$

$W \xrightarrow{u} T_{\leq n} V$ sglm s.th. $u \Delta = 0$

$\Rightarrow u = u \pi_1 \iota_1$

Prop 42 (lifting to coderivations)

V : $\mathcal{Z}$-graded module

$$\mathrm{Coder}_n(V) := \{ T_{\leq n} V \xrightarrow{\delta} T_{\leq n} V : \delta \text{ is a coderivation} \}$$

$$\mathrm{Coder}_n^{\mathrm{red}}(V) := \{ (V^{\otimes k} \xrightarrow{\mu_k} V)_{k \in [1,n] \cap \mathbb{Z}} : \mu_k \text{ is a sglm of deg. } 1 \ \forall k \}$$

Module isomorphism :

$$\begin{array}{ccc} \mathrm{Coder}_n(V) & \longleftrightarrow & \mathrm{Coder}_n^{\mathrm{red}}(V) \\ \delta & \xrightarrow{\alpha = \alpha_{\mathrm{Coder}, n, V}} & (\iota_k \delta \pi_1)_k \\ \mu\beta & \xleftarrow{\beta = \tilde{\beta}_{\mathrm{Coder}, n, V}} & \mu = (\mu_k)_k \end{array}$$

Input e.g. $({}^{\omega}\mathrm{m}_k)_k$

where $\iota_k (\mu\beta) := \sum_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t = k}} \mathrm{id}^{\otimes r} \otimes \mu_s \otimes \mathrm{id}^{\otimes t}$

Prop 43 (lifting to morphisms of coalgebras).

$\tilde{V}, V$: $\mathcal{Z}$-graded modules, $n \in [1, \infty]$

$$\mathrm{Coalg}_n(\tilde{V}, V) = \{ T_{\leq n} \tilde{V} \xrightarrow{\varphi} T_{\leq n} V : \varphi \text{ is a morph. of coalg} \}$$

Input e.g. $({}^{\omega}f_k)_k$

$$\mathrm{Coalg}_n^{\mathrm{red}}(\tilde{V}, V) = \{ (\tilde{V}^{\otimes k} \xrightarrow{\varphi_k} V)_{k \in [1,n] \cap \mathbb{Z}} : \varphi_k \text{ is a sglm of deg. } 0 \ \forall k \}$$

$\mathcal{Z}$: grading cat.

<u>Prop. 4.3</u> $n \in [1, \infty]$, $\tilde{V}, V$: $\mathcal{Z}$-graded modules

$$\mathrm{Coalg}_n(\tilde{V}, V) := \{ T_{\le n}\tilde{V} \xrightarrow{\psi} T_{\le n} V \ : \ \psi \text{ morph. of coalg} \}$$

$$\mathrm{Coalg}_n^{\mathrm{red}}(\tilde{V}, V) := \{ (\tilde{V}^{\otimes k} \xrightarrow{\varphi_k} V)_{k \in [1,n] \cap \mathbb{Z}} : \varphi_k \text{ sglin of deg } 0 \}$$

We have a bijection:

$$\begin{array}{ccc} \mathrm{Coalg}_n(\tilde{V}, V) & \longrightarrow & \mathrm{Coalg}_n^{\mathrm{red}}(\tilde{V}, V) \\ \psi & \xmapsto{\alpha = \alpha_{\mathrm{Coalg}, n, V, \tilde{V}}} & (\iota_k \psi \pi_1)_k \\ \varphi\beta & \xleftarrow{\beta = \beta_{\mathrm{Coalg}, n, V, \tilde{V}}} & \varphi = (\varphi_k)_k \end{array}$$

where $$\iota_k(\varphi\beta) = \sum_{r \in [1,k]} \ \sum_{\substack{(i_j)_{j \in [1,r]} \geq (1)_j \\ \sum_j i_j = k}} \underbrace{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_r}}_{\tilde{V}^{\otimes k} \to V^{\otimes r}}$$

for $k \in [1,n] \cap \mathbb{Z}$.

<u>Lemma 4.5</u>: $n \in [1, \infty]$, $k \in [0, n-1] \cap \mathbb{Z}$

$\tilde{V}, V$: $\mathcal{Z}$-graded modules

(1) $T_{\le n} V \xrightarrow{\delta} T_{\le n} V$: coderivation

$$\delta^2|_{T_{\le k} V} = 0 \implies \iota_{k+1}\delta^2 = \iota_{k+1}\delta^2 \pi_1 \iota_1$$

(2) $T_{\le n}\tilde{V} \xrightarrow{\tilde{\delta}} T_{\le n}\tilde{V}$, $T_{\le n} V \xrightarrow{\delta} T_{\le n} V$: coderivations

$T_{\le n}\tilde{V} \xrightarrow{\psi} T_{\le n} V$: morphism of coalgebras

$$(\tilde{\delta}\psi - \psi\delta)|_{T_{\le k}\tilde{V}} = 0 \implies \iota_{k+1}(\tilde{\delta}\psi - \psi\delta)$$
$$= \iota_{k+1}(\tilde{\delta}\psi - \psi\delta)\pi_1\iota_1$$

Prop. 46 $\quad n \in [1, \infty]$

$(A, (m_\ell)_\ell)$: pre-A_n-alg. / $\mathbb{Z}$

$m := \left(({}^\omega m_\ell)_\ell \right) \beta_{\mathrm{Coder}, n, A^{[1]}}$ ← Coderivation on $T_{\leq n}(A^{[1]})$

Then:

(1) $(m_\ell)_\ell$ satisfies Stasheff at $k \in [1, n] \cap \mathbb{Z}$

$\Updownarrow$

(2) $m^2 = 0$

Prop. 47 $\quad n \in [1, \infty]$

$(\tilde{A}, (\tilde{m}_\ell)_\ell)$, $(A, (m_\ell)_\ell)$: pre-A_n-alg / $\mathbb{Z}$

$f = (f_\ell)_\ell : \tilde{A} \to A$: pre-A_n-morphism

$\tilde{m} := \left(({}^\omega \tilde{m}_\ell)_\ell \right) \beta_{\mathrm{Coder}, n, \tilde{A}^{[1]}}$ ← Coderivation on $T_{\leq n}(\tilde{A}^{[1]})$

$m := \left(({}^\omega m_\ell)_\ell \right) \beta_{\mathrm{Coder}, n, A^{[1]}}$ ← Coderivation on $T_{\leq n}(A^{[1]})$

$\mathfrak{f} := \left(({}^\omega f_\ell)_\ell \right) \beta_{\mathrm{Coalg}, n, \tilde{A}^{[1]}, A^{[1]}}$: $T_{\leq n}(\tilde{A}^{[1]}) \to T_{\leq n}(A^{[1]})$

morphism of coalgebras

Then:

(1) $(f_\ell)_\ell$ satisfies Stasheff for morphisms at $k \in [1, n] \cap \mathbb{Z}$.

$\Updownarrow$

(2) $\tilde{m} \cdot \mathfrak{f} = \mathfrak{f} \cdot m$

Lemma 48 (L29 with shorter proof)

$n \in [1, \infty]$, $(\tilde{A}, (\tilde{m}_\ell)_\ell)$: pre-A_n-alg./$\mathbb{Z}$, $(A, (m_\ell)_\ell)$: A_n-alg /$\mathbb{Z}$

$f = (f_\ell)_\ell : \tilde{A} \to A$: pre-A_n-morph. satisfying Stasheff f.m.

Suppose that f_1 is piecewise injective.

Then $\tilde{A} = (\tilde{A}, (\tilde{m}_\ell)_\ell)$ is an A_n-algebra.

Th 50 (Kadeishvili)

R : field , $n \in [1, \infty]$, $\mathcal{Z}$: grading cat.

A : A_n - alg. / $\mathcal{Z}$

There exist $(\tilde{m}_e)_e$ and $(q_e)_e$ such that :

- $(HA, (\tilde{m}_e)_e)$ is a minimal ($\tilde{m}_1 = 0$) A_n - algebra / $\mathcal{Z}$
- $q = (q_e)_e : HA \longrightarrow A$ is a qis

If A is unital , then HA and q can be chosen to be unital .

to do

$\mathcal{Z}$: grading category

Split-filtered $\mathcal{Z}$-graded module:

- $\mathcal{Z}$-graded module Π
- tuple of $\mathcal{Z}$-graded submodules $(\Pi^{<i>})_{i \in \mathbb{Z}}$

such that $\Pi^{<i>} = 0$ for $i < 0$ and $\Pi = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \Pi^{<i>}$

$n \in [0, \infty]$

Write $\Pi^{\leq k} := \bigoplus_{i \in [0,k]} \Pi^{<i>}$

e A_n-algebra over $\mathcal{Z}$:

$$A = \left(A, (m_\ell)_{\ell \in [1,n] \cap \mathbb{Z}}, (A^{<i>})_{i \in \mathbb{Z}} \right)$$

s.th.
- $(A, (m_\ell)_\ell)$ is an A_n-alg./$\mathcal{Z}$
- $(A, (A^{<i>})_i)$ is a split-filtered $\mathcal{Z}$-graded module
- $\left(A^{<j_1>} \otimes \dots \otimes A^{<j_k>} \right) m_k \subseteq A^{\leq 2k - 2 + \sum_{i \in [1,k]} j_i}$

minimal: $-2 \rightsquigarrow -3$

action of $\boxed{m}$

general case

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
\cdots\ A^{<2>,1} & A^{<1>,1} & A^{<0>,1} \\
\cdots\ A^{<2>,0} \arrow[u] \arrow[ur] \arrow[urr] & A^{<1>,0} \arrow[u] \arrow[ur] & A^{<0>,0} \arrow[u] \\
\cdots\ A^{<2>,-1} \arrow[u] \arrow[ur] \arrow[urr] & A^{<1>,-1} \arrow[u] \arrow[ur] & A^{<0>,-1} \arrow[u] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

minimal case

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
\cdots\ A^{<2>,1} & A^{<1>,1} & A^{<0>,1} \\
\cdots\ A^{<2>,0} \arrow[ur] \arrow[urr] & A^{<1>,0} \arrow[ur] & A^{<0>,0} \\
\cdots\ A^{<2>,-1} \arrow[ur] \arrow[urr] & A^{<1>,-1} \arrow[ur] & A^{<0>,-1} \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

No vertical components!

$\mathcal{Z}$: grading cat.

$$A = \left(A, (m_1), (A^{\langle i \rangle})_i \right) \text{ : minimal } eA_1\text{-algebra} / \mathcal{Z}$$

$$m_1^{\langle j \rangle} := {}_1^{\langle j \rangle} m_1 \, {}_{\pi}^{\langle j+1 \rangle} : A^{\langle j \rangle} \longrightarrow A^{\langle j+1 \rangle}$$

A is diagonally resolving

iff $\operatorname{Ker} m_1^{\langle j \rangle} = \operatorname{Im} m_1^{\langle j+1 \rangle}$ for $j \in \mathbb{Z}_{\geq 1}$.

Lemma 59 $A = \left(A, (m_1), (A^{\langle i \rangle})_i \right)$ diagonally resolving eA_1-algebra$/\mathcal{Z}$.

(1) $\quad ZA = A^{\langle 0 \rangle} + BA$

(2) $\quad (A^{\leq k})_{m_1} = BA \cap A^{\leq k-1}$ for $k \in \mathbb{Z}$.

to do

Prop. 60

$(A, (m_i))$: A-alg. $/ \mathbb{Z}$

i.e. we may choose

For $z \in \mathrm{Mor}(\mathcal{Z})$, suppose given an augmented proj. res.

proj. res. augmentation

$$\cdots \tilde{A}^{\langle 2\rangle, z[-2]} \xrightarrow{d^{\langle 2\rangle, z[-2]}} \tilde{A}^{\langle 1\rangle, z[-1]} \xrightarrow{d^{\langle 1\rangle, z[-1]}} \tilde{A}^{\langle 0\rangle, z[0]} \xrightarrow{\varepsilon^z} (\mathrm{H}A)^z \to 0$$

resolved module

of (R-)modules

Assemble to:

$$\cdots \longrightarrow \tilde{A}^{\langle 2\rangle} \xrightarrow{d^{\langle 2\rangle}} \tilde{A}^{\langle 1\rangle} \xrightarrow{d^{\langle 1\rangle}} \tilde{A}^{\langle 0\rangle} \xrightarrow{\varepsilon} \mathrm{H}A \to 0$$

sglm of deg. 0

sglm. of deg 1

Let $\tilde{A} := \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \tilde{A}^{\langle i\rangle}$.

So $(\tilde{A}, (\tilde{A}^{\langle i\rangle})_i)$ is a split-filtered $\mathbb{Z}$-graded module.

Then there exist

- $e^{\langle k\rangle} : \tilde{A}^{\langle k\rangle} \longrightarrow \tilde{A}^{\leq k-2}$: sglm of deg. 1 f. $k \in \mathbb{Z}_{\geq 0}$
- $q_1 : \tilde{A} \longrightarrow A$: sglm of deg. 0

such that, letting the sglm $\tilde{m}_1 : \tilde{A} \to \tilde{A}$ of degree 1 be defined by

where $d^{\langle 0\rangle} := 0$

$$\iota^{\langle i\rangle} \cdot \tilde{m}_1 := d^{\langle i\rangle} \cdot \iota^{\langle i-1\rangle} + e^{\langle i\rangle} \cdot \iota^{\leq i-2}$$

for $i \in \mathbb{Z}_{\geq 0}$, then (1, 2, 3, 4, 5, 6) hold.

(1) $\tilde{A} = (\tilde{A}, (\tilde{m}_1))$ is a minimal eA_1-alg $/Z$.

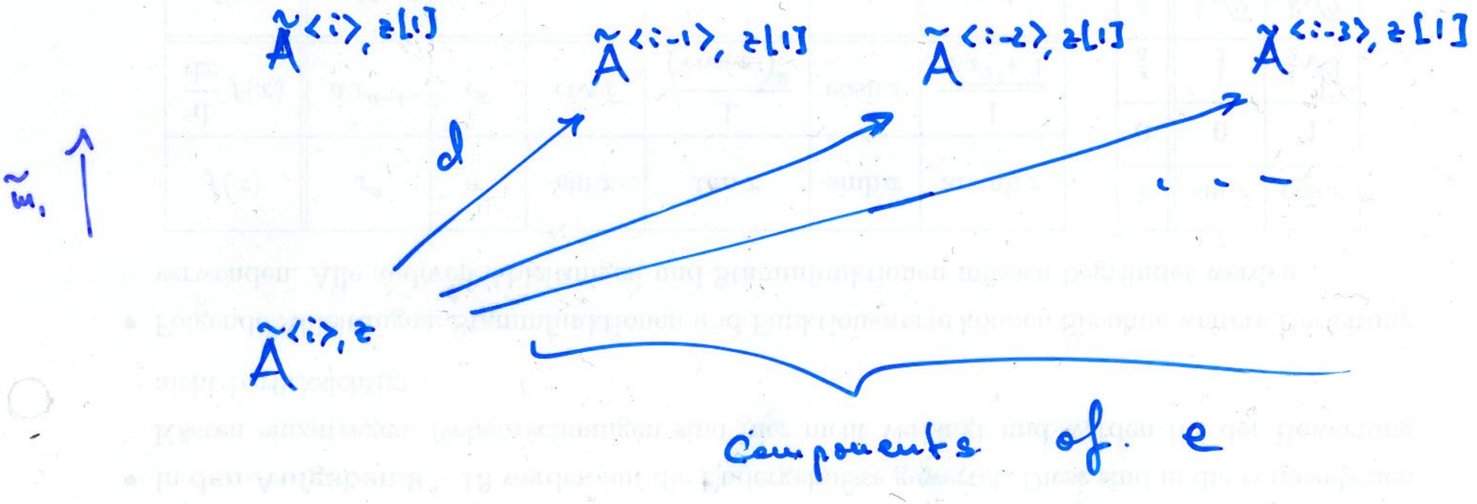
(2) (q_1) is a q.i.s of A-alg. from $\tilde{A}$ to A.

(3) ✓ $\tilde{A}^{\langle i\rangle, z}$ is projective for $i \in \mathbb{Z}_{\geq 0}$ and $z \in \Gamma_{tor}(Z)$.

(4) $\tilde{A}$ is diagonally resolving

(5) $q_1\big|^{ZA}_{\tilde{A}^{\langle 0\rangle}\cdot e}$ exists and is piecewise surjective

(6) $\tilde{A}^{\leq j}\tilde{m}_1 = \tilde{A}^{\leq j-1} + B\tilde{A}$ for $j \in \mathbb{Z}_{\geq 0}$.



"d is complemented by e"

We search for $k \geq 0$:

$$e^{\langle k\rangle} : \tilde{A}^{\langle k\rangle} \longrightarrow \tilde{A}^{\leq k-2} \qquad \text{sglm of deg } 1$$

$$q^{\langle k\rangle} : \tilde{A}^{\langle k\rangle} \longrightarrow A \qquad \text{sglm of deg } 0$$

Letting $\tilde{m}_1^{\leq j} : \tilde{A}^{\leq j} \longrightarrow \tilde{A}^{\leq j-1}$ be def. by

$$\iota^{\langle i\rangle}\, \tilde{m}_1^{\leq j} := d^{\langle i\rangle}\, \iota^{\langle i-1\rangle} + e^{\langle i\rangle}\, \iota^{\leq i-2} \quad \text{f. } i \in [0, j]$$

provided these are already constructed

$$\iota^{\langle i\rangle}\, q^{\leq j} := q^{\langle i\rangle}, \quad \text{for } i \in [0, j]$$

we consider the conditions:

$(N^{\langle j\rangle})$ $\quad (d^{\langle j\rangle}\, \iota^{\langle j-1\rangle} + e^{\langle j\rangle}\, \iota^{\leq j-2}) \cdot \tilde{m}_1^{\leq j-1} = 0$

$(N^{\leq j})$ $\quad \tilde{m}_1^{\leq j} \cdot \tilde{m}_1^{\leq j-1} = 0$

$(C^{\langle j\rangle})$ $\quad q^{\langle j\rangle} \cdot m_1 = (d^{\langle j\rangle}\, \iota^{\langle j-1\rangle} + e^{\langle j\rangle}\, \iota^{\leq j-2})\, q^{\leq j-1}$

$(C^{\leq j})$ $\quad q^{\leq j} \cdot m_1 = \tilde{m}_1^{\leq j} \cdot q^{\leq j-1}$

- Let $e^{\langle 0\rangle} = 0$. Let

$$\begin{tikzcd}
\tilde{A}^{\langle 0\rangle} \arrow[r, "\varepsilon"] \arrow[dr, "\varepsilon'"] \arrow[ddr, "q^{\langle 0\rangle}"'] & HA \\
& ZA \arrow[u, "\rho"'] \arrow[d, "\iota"] \\
& A
\end{tikzcd}$$

Then $(N^{\langle 0\rangle}, N^{\leq 0}, C^{\langle 0\rangle}, C^{\leq 0})$ hold.

- S.g. $k \geq 1$, $e^{\langle 0\rangle}, \dots, e^{\langle k-1\rangle}, q^{\langle 0\rangle}, \dots, q^{\langle k-1\rangle}$ s.th. $(N^{\langle j\rangle}, N^{\leq j}, C^{\langle j\rangle}, C^{\leq j})$ hold for $j \in [0, k-1]$

Search: $e^{\langle k\rangle}, q^{\langle k\rangle}$ s.th. $(N^{\langle k\rangle}, N^{\leq k}, C^{\langle k\rangle}, C^{\leq k})$ holds.

Claim $\quad \tilde{A}^{\langle k\rangle}\, d^{\langle k\rangle}\, \iota^{\langle k-1\rangle}\, \tilde{m}_1^{\leq k-1} \overset{!}{\subseteq} \tilde{A}^{\leq k-2}\, \tilde{m}_1^{\leq k-2}\, \iota^{\leq k-3}$

S.g. $z \in \mathrm{Mor}(Z)$. S.g. $x \in \tilde{A}^{\langle k\rangle, z}$.

Need: $x d^{\langle k\rangle} \iota^{\langle k-1\rangle} \tilde{m}_1^{\le k-1} \overset{!}{\in} \left(\tilde{A}^{\le k-2} \tilde{m}_1^{\le k-2} \iota^{\le k-3}\right)^{\mathbb{Z}[2]}$

$\|$

$x d^{\langle k\rangle} e^{\langle k-1\rangle} \iota^{\le k-3}$

Write $y := x d^{\langle k\rangle} e^{\langle k-1\rangle} \in \tilde{A}^{\le k-3, \mathbb{Z}[2]}$

Need $y \overset{!}{\in} \tilde{A}^{\le k-2, \mathbb{Z}[2]} \tilde{m}_1^{\le k-2} =: I_{k-2}$

Have: $y \in \left(\mathrm{Kern}(\tilde{m}_1^{\le k-3})\right)^{\mathbb{Z}[2]} =: K_{k-3}$

Subclaim: For $i \in [0, k-3]$, there exist

$y_{k-3-i} \in \left(\mathrm{Kern}(\tilde{m}_1^{\le k-3-i})\right)^{\mathbb{Z}[2]} =: K_{k-3-i}$

$y'_{k-3-i} \in \left(\mathrm{Im}(\tilde{m}_1^{\le k-2})\right)^{\mathbb{Z}[2]} = I_{k-2}$

such that $y = y_{k-3-i}\, \iota^{\le k-3-i} + y'_{k-3-i}$

Have constructed:

$$e^{\langle 0\rangle}, \dots, e^{\langle k\rangle}, q^{\langle 0\rangle}, \dots, q^{\langle k\rangle}$$

s.th. $(N^{\langle j\rangle})$ for $j\in[0,k]$, $(N^{\le j})$ for $j\in[0,k]$, $(C^{\langle j\rangle})$ for $j\in[0,k]$, $(C^{\le j})$ for $j\in[0,k-1]$ hold

Need:

$\tilde{m}_1 : \tilde{A} \to \tilde{A}$ sglm of deg 1, $\tilde{m}_1^2 \overset{!}{=} 0$

$q_1 : \tilde{A} \to A$ sglm of deg 0, $q_1 m_1 \overset{!}{=} \tilde{m}_1 q_1$

q_1 quasiisomorphism

$(\tilde{A}, (\tilde{m}_1), (\tilde{A}^{\langle i\rangle})_i)$: minimal eA_1-algebra $/\mathbb{Z}$,

so strong Schmid cond. should hold

$q_1 \big|_{A^{\langle 0\rangle}}^{ZA} \cdot \rho$: piecewise surjective

$$\tilde{A}^{\le j}\, \tilde{m}_1 = B\tilde{A} \cap \tilde{A}^{\le j-1}$$

Z : grading category

$n \in \mathbb{Z}_{\geq 2}$

$\left(A, (m_k)_{k \in [1,n]}\right)$: A_n-algebra$/Z$

$\left(\tilde{A}, (\tilde{m}_k)_{k \in [1,n-1]}, (\tilde{A}^{\langle i \rangle})_i\right)$: minimal eA_{n-1}-alg.$/Z$

$(q_k)_{k \in [1,n-1]}$: $\tilde{A} \longrightarrow A$: q's of A_{n-1}-alg.

such that

- $\tilde{A}$ is piecewise proj.
- $(\tilde{A}^{\leq j})\tilde{m}_1 = Z\tilde{A} \cap \tilde{A}^{\leq j-1}$ for $j \in \mathbb{Z}$
- $q_1|^{ZA}_{\tilde{A}^{\langle 0 \rangle}} : \rho$ exists and is piecewise surjective

Need : $^{\omega}\tilde{m}_n : (\tilde{A}^{[1]})^{\otimes n} \longrightarrow \tilde{A}^{[1]}$, sglm of deg 1,

$^{\omega}q_n : (\tilde{A}^{[1]})^{\otimes n} \longrightarrow A^{[1]}$, sglm of deg 0,

such that

- $\sum\limits_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t=n}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^{\omega}\tilde{m}_{r+1+t} \overset{!}{=} 0$ (Stasheff)
- $\sum\limits_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t=n}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^{\omega}q_{r+1+t}$ (Stasheff for morphisms)

$\overset{!}{=} \sum\limits_{r \in [1,n]} \sum\limits_{\substack{(i_j)_{j \in [1,r]} \geq (1)_j \\ \sum_j i_j = n}} ({}^{\omega}q_{i_1} \otimes \cdots \otimes {}^{\omega}q_{i_r})\, {}^{\omega}m_r$

- $(\tilde{A}^{\langle i_1 \rangle,[1]} \otimes \cdots \otimes \tilde{A}^{\langle i_n \rangle,[1]})\, {}^{\omega}\tilde{m}_n \overset{!}{\subseteq} \tilde{A}^{\leq 2n-3+i_1+\cdots+i_n,[1]}$ (strong Schmid)

for $n \geq 1$ and $(i_1, \ldots, i_n) \in \mathbb{Z}^{\times n}_{\geq 0}$.

Write: • for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^{\times n}$

$$\tilde{A}^{\langle\alpha\rangle, [1]} := \tilde{A}^{\langle\alpha_1\rangle, [1]} \otimes \dots \otimes \tilde{A}^{\langle\alpha_n\rangle, [1]}$$

• for $X \subseteq \mathbb{Z}^{\times n}$

$$\tilde{A}^{\langle X\rangle, [1]} := \bigoplus_{\alpha \in X} \tilde{A}^{\langle\alpha\rangle, [1]}$$

• for $\alpha \in \mathbb{Z}_{\geq 0}^{\times n}$

$$I^\alpha := \{\beta \in \mathbb{Z}_{\geq 0}^{\times n} : \beta < \alpha\}$$

• $({}^{\omega}\tilde{m}_1)^{(n)} := \sum_{r \in [0, n-1]} \mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_1 \otimes \mathrm{id}^{\otimes n-1-r}$

sglm from $(\tilde{A}^{[1]})^{\otimes n}$ to $(\tilde{A}^{[1]})^{\otimes n}$ of deg. n

<u>Rem 62</u> For $\alpha \in \mathbb{Z}_{\geq 0}^{\times n}$, we have

$$\left(\tilde{A}^{\langle\alpha\rangle, [1]}\right)\left({}^{\omega}\tilde{m}_1\right)^{(n)} \subseteq \tilde{A}^{\langle I^\alpha\rangle, [1]}$$

An **admissible triple** (L, Π^L, Q^L) consists of

• $L \subseteq \mathbb{Z}_{\geq 0}^{\times n}$ with: for $\alpha \in L$, we have $I^\alpha \subseteq L$

• $\Pi^L : \tilde{A}^{\langle L\rangle, [1]} \to \tilde{A}^{[1]}$ sglm of deg. 1

• $Q^L : \tilde{A}^{\langle L\rangle, [1]} \to A^{[1]}$ sglm of deg 0

such that

• $0 = ({}^{\omega}\tilde{m}_1)^{(n)} \cdot \Pi^L + \Pi^L \cdot {}^{\omega}\tilde{m} + \sum_{\substack{(r,s,t) \geq (0,2,0) \\ (r,t) > (0,0) \\ r+s+t = n}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^{\omega}\tilde{m}_{r+1+t}$ holds on $\tilde{A}^{\langle L\rangle, [1]}$.

• $({}^{\omega}\tilde{m}_1)^{n} \cdot Q^L + \Pi^L \cdot {}^{\omega}q_1 + \sum_{\substack{(r,s,t) \geq (0,2,0) \\ (r,t) > (0,0) \\ r+s+t = n}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^{\omega}q_{r+1+t}$

$= Q^L \cdot {}^{\omega}m_1 + \sum_{r \in [2,n]} \sum_{\substack{(i_j)_{j \in [1,r]} \geq (1)_j \\ \sum_j i_j = n}} ({}^{\omega}q_{i_1} \otimes \dots \otimes {}^{\omega}q_{i_r}) \cdot {}^{\omega}m_r$ holds on $\tilde{A}^{\langle L\rangle, [1]}$

...

...

- $\left(\tilde{A}^{\langle\alpha\rangle,[1]}\right)\Pi^L \subseteq \tilde{A}^{\leq 2n-3+\sum_i \alpha_i,[1]}$ for $\alpha \in L$

$\mathcal{O}\!\mathcal{l}$: set of admissible triples

- $(\emptyset, 0, 0) \in \mathcal{O}\!\mathcal{l}$

- $(L, \Pi^L, Q^L) \leq (L', \Pi^{L'}, Q^{L'}) \;:\!\iff$

$$L \subseteq L' \wedge \Pi^{L'}\big|_{\tilde{A}^{\langle L\rangle,[1]}} = \Pi^L \wedge Q^{L'}\big|_{\tilde{A}^{\langle L\rangle,[1]}} = Q^L$$

Lemma 64 S.g. $(L, \Pi^L, Q^L) \in \mathcal{O}\!\mathcal{l}$.

S.g. $\alpha \in \mathbb{Z}_{\geq 0}^{\times n}$ s.th. $I^\alpha \subseteq L$.

Write $L' := L \cup \{\alpha\}$.

Then there exist sglm $\Pi^{L'} : \tilde{A}^{\langle L'\rangle,[1]} \longrightarrow \tilde{A}^{[1]}$

and $Q^{L'} : \tilde{A}^{\langle L'\rangle,[1]} \longrightarrow A^{[1]}$ such that

$(L', \Pi^{L'}, Q^{L'}) \in \mathcal{O}\!\mathcal{l}$ and $(L, \Pi^L, Q^L) \leq (L', \Pi^{L'}, Q^{L'})$

Pf. Wlog $\alpha \notin L$.

For $\Pi : (\tilde{A}^{[1]})^{\otimes n} \longrightarrow \tilde{A}^{[1]}$ sglm of deg 1

s.th. $\Pi\big|_{\tilde{A}^{\langle L\rangle,[1]}} = \Pi^L$ and $\tilde{A}^{\langle\alpha\rangle,[1]}\,\Pi \subseteq \tilde{A}^{\leq 2n-3+\sum_i \alpha_i,[1]}$

write $\tilde{m}_\Pi := (\omega\tilde{m}_1, \dots, \omega\tilde{m}_{n-1}, \Pi)$ Coder$_n(\tilde{A}^{[1]})$

coderivation on $T_{\leq n}(\tilde{A}^{[1]})$.

So $\iota_k \tilde{m}_{\Pi} \pi_1 = \begin{cases} \omega_{\tilde{m}_k} & \text{if } k \in [1, n-1] \\ \Pi & \text{if } k = n \end{cases}$

Have projection $(\tilde{A}^{[1]})^{\otimes n} \xrightarrow{p} \tilde{A}^{\langle n \rangle, [1]}$.

so e.g. $\Pi = p \cdot \bar{\Pi}^L$.

$m := (\omega_{m_1}, \dots, \omega_{m_n})\big|_{\mathrm{Coder}, n, A^{[1]}}$

Codifferential on $T_{\le n}(\tilde{A}^{[1]})$

For $Q : (\tilde{A}^{[1]})^{\otimes n} \longrightarrow A^{[1]}$ sym. of deg 0

s.th. $Q\big|_{\tilde{A}^{\langle n \rangle, [1]}} = Q^L$, we write

$q_Q := (\omega_{q_1}, \dots, \omega_{q_{n-1}}, Q)\big|_{\mathrm{Coalg}, n, \tilde{A}^{[1]}, A^{[1]}}$

being a coalg. morph. from $T_{\le n}(\tilde{A}^{[1]})$ to $T_{\le n}(A^{[1]})$.

So

$$\iota_k q_Q \pi_1 = \begin{cases} \omega_{q_k} & \text{if } k \in [1, n-1] \\ Q & \text{if } k = n \end{cases}$$

Have: • $\tilde{m}_\sqcap = \tilde{m}_\sqcap \underbrace{\pi_{\leq n-1}\, \iota_{\leq n-1}}_{} + \pi_n \left({}^{\omega}\tilde{m}_1\right)^{(n)} \iota_n$ (*)

$$T_{\leq n-1}(\tilde{A}^{[1]}) \underset{\pi_{\leq n-1}}{\overset{\iota_{\leq n-1}}{\rightleftarrows}} T_{\leq n}(\tilde{A}^{[1]})$$

• $\left(\tilde{A}^{\langle\alpha\rangle,[1]}\right)\left({}^{\omega}\tilde{m}_1\right)^{(n)} \subseteq \tilde{A}^{\langle I^\alpha\rangle,[1]} \subseteq \tilde{A}^{\langle L\rangle,[1]}$ (***)

• $\left(\tilde{m}_\sqcap\right)^2 \Big|_{\tilde{A}^{\langle L\rangle,[1]}} = 0$ (***')

• $\left(\tilde{m}_\sqcap q_a - q_a m\right)\Big|_{\tilde{A}^{\langle L\rangle,[1]}} = 0$ (5*)

• $\left(\tilde{m}_\sqcap\right)^2 \Big|_{T_{\leq n-1}(\tilde{A}^{[1]})} = 0$ (**)

• $\left(\tilde{m}_\sqcap q_a - q_a m\right)\Big|_{T_{\leq n-1}(\tilde{A}^{[1]})} = 0$ (4*)

(**) $\Rightarrow$ $\left(\tilde{A}^{[1]}\right)^{\otimes n} \tilde{m}_\sqcap^2 \subseteq \tilde{A}^{[1]}$

(4*) $\Rightarrow$ $\left(\tilde{A}^{[1]}\right)^{\otimes n}\left(\tilde{m}_\sqcap q_a - q_a m\right) \subseteq A^{[1]}$

<u>Claim</u>: $\Big(\underbrace{\tilde{A}^{\langle\alpha\rangle,[1]}}_{\text{"critical summand"}}\Big) m_\sqcap^2 \overset{!}{\subseteq} \left(\tilde{A}^{[1]}\right) {}^{\omega}\tilde{m}_1\, \iota_1$

Have:

• $\left(\tilde{A}^{\langle\alpha\rangle,[1]}\right)\tilde{m}_\sqcap^2 \subseteq \tilde{A}^{[1]}$

• $\left(\tilde{A}^{\langle\alpha\rangle,[1]}\right)\tilde{m}_\sqcap^2\, \pi_1\, {}^{\omega}\tilde{m}_1 = 0$

• $\left(\left(\tilde{m}_\sqcap q_a - q_a m\right)\pi_1 \cdot {}^{\omega}m_1 \cdot \iota_1\right)\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}}$

$= \cdots = \left(\hat{m}_\sqcap q_a m\right)\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}} = \cdots$

$$\omega_n \tilde{m}^2_{\Pi} \pi_1 = \Pi \cdot \omega\tilde{m}_1 + (\omega\tilde{m}_1)^{(n)} \cdot \Pi$$

$$+ \sum_{k \in [2, n-1]} \sum_{\substack{(r,s,t) \geq (0,1,0) \\ r+s+t = n \\ r+1+t = k}} (id^{\otimes r} \otimes \omega\tilde{m}_s \otimes id^{\otimes t}) \cdot \omega\tilde{m}_k$$

6 *

↑ general Π

↓: Choose Π' and Q'

Idea:

$\Pi' \overset{\text{correct}}{\rightsquigarrow} \Pi'' \overset{\text{correct}}{\rightsquigarrow} \Pi'''$, $\Pi^{L'} := \Pi'''|_{\tilde{A}^{\langle L' \rangle, [1]}}$

$Q' \xrightarrow{\text{correct}} Q''$, $Q^{L'} := Q''|_{\tilde{A}^{\langle L' \rangle, [1]}}$

Solution: $(L', \Pi^{L'}, Q^{L'})$

$$\tau := 2n - 3 + \sum_i \alpha_i$$

$$\left(\tilde{A}^{\langle \alpha \rangle, [1]}\right) \tilde{m}^2_{\Pi'} \pi_1 \subseteq \tilde{A}^{\leq \tau - 1, [1]}$$

$\Pi'' : (\tilde{A}^{[1]})^{\otimes n} \longrightarrow \tilde{A}^{[1]}$ sqlm of deg 1

constructed with $\tilde{m}^2_{\Pi''}\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}} \pi_1 = 0$

Then $w' : \tilde{A}^{\langle\alpha\rangle,[1]} \longrightarrow \tilde{A}^{\langle 0\rangle,[1]}$ sqlm of deg 1

$v : \tilde{A}^{\langle\alpha\rangle,[1]} \longrightarrow A^{[1]}$ sqlm of deg 0

constructed such that

$$\left(\tilde{m}_{\Pi''} \cdot \mathfrak{q}_{Q'} \cdot \pi_1\right)\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}} - w' \cdot \iota^{\langle 0\rangle} \cdot \omega_{\mathfrak{q}_1}$$

$$= \left(\mathfrak{q}_{Q'} \cdot m \cdot \pi_1\right)\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}} - v \cdot \omega_{m_1}$$

Let $\Pi''' : (\tilde{A}^{[1]})^{\otimes n} \longrightarrow \tilde{A}^{[1]}$ be def. by

$$\Pi'''\Big|_{\tilde{A}^{\langle L\rangle,[1]}} := \Pi''\Big|_{\tilde{A}^{\langle L\rangle,[1]}} = \Pi'\Big|_{\tilde{A}^{\langle L\rangle,[1]}} = \Pi^L$$

$$\Pi'''\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}} := \Pi''\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}} - w' \cdot \iota^{\langle 0\rangle}$$

dummy $\rightarrow$ $\Pi'''\Big|_{\tilde{A}^{\langle\beta\rangle,[1]}} := \Pi''\Big|_{\tilde{A}^{\langle\beta\rangle,[1]}}$ for $\beta \in \mathbb{Z}_{\geq 0}^{\times n} \setminus L'$

Let $Q'' : (\tilde{A}^{[1]})^{\otimes n} \longrightarrow A^{[1]}$ be def. by

$$Q''\Big|_{\tilde{A}^{\langle L\rangle,[1]}} := Q'\Big|_{\tilde{A}^{\langle L\rangle,[1]}} = Q^L$$

$$Q''\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}} := Q'\Big|_{\tilde{A}^{\langle\alpha\rangle,[1]}} - v$$

dummy $\rightarrow$ $Q''\Big|_{\tilde{A}^{\langle\beta\rangle,[1]}} := Q'\Big|_{\tilde{A}^{\langle\beta\rangle,[1]}}$ for $\beta \in \mathbb{Z}_{\geq 0}^{\times n} \setminus L'$

Solution of § 2.4 (induction step)

Prop. 65 There exist:

$\tilde{m}_n : \tilde{A}^{\otimes n} \longrightarrow \tilde{A}$ sglm of degree $2-n$

$q_n : \tilde{A}^{\otimes n} \longrightarrow A$ sglm of degree $1-n$

such that

- $\left(\tilde{A}, (\tilde{m}_k)_{k\in[1,n]}, (\tilde{A}^{(i)})_i\right)$ is a minimal eA_n-algebra
- $(q_k)_{k\in[1,n]} : \tilde{A} \longrightarrow A$ is a qis of A_n-algebras

I : set

$\mathcal{Z} := \mathbb{Z} \times I^{\times 2}$ ← pair category

So $\mathrm{Mor}(\mathcal{Z}) = \{ (z, (i,j)) : z \in \mathbb{Z},\ i,j \in I \}$

$\hat{I} := \{ (k, \underbrace{(i_1, \dots, i_k)}_{\underline{i}}) : k \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_k \in I \}$

$\hat{\mathcal{Z}} := \mathbb{Z} \times \hat{I}^{\times 2}$

So $\mathrm{Mor}(\hat{\mathcal{Z}}) = \{ (z, ((k, \underline{i}), (l, \underline{j}))) : z \in \mathbb{Z},\ k, l \geq 0,\ \underline{i} \in I^{\times k},\ \underline{j} \in I^{\times l} \}$

M, N : $\mathcal{Z}$-graded modules

$M^{(z,(i,j))} = M^{z,i,j}$

$\hat{M}$: $\hat{\mathcal{Z}}$-graded module with graded pieces :

$$\hat{M}^{(z,((k,\underline{i}),(l,\underline{j})))} = \hat{M}^{z,(k,\underline{i}),(l,\underline{j})}$$

$$:= \bigoplus_{\substack{s \in [1,k] \\ t \in [1,l]}} M^{z, i_s, j_t} = \begin{pmatrix} M^{z,i_1,j_1} & M^{z,i_1,j_2} & \cdots \\ M^{z,i_2,j_1} & M^{z,i_2,j_2} & \cdots \\ \vdots & & \vdots \\ & \cdots & M^{z,i_k,j_l} \end{pmatrix}$$

S.g. $u \geq 1$, $M^{\otimes u} \xrightarrow{f} N$ sglm of degree d.

Define sglm $\hat{M}^{\otimes u} \xrightarrow{\hat{f}} \hat{N}$

of degree d : at $(z, (k^0, \underline{i}^0), (k^u, \underline{i}^u)) =: \hat{z}$

and at the factorisation consisting of $\hat{y}_v := (y_v, (k^{v-1}, \underline{i}^{v-1}), (k^v, \underline{i}^v))$ for $v \in [1,u]$,

↑ so $z = y_1 + \dots + y_u$

of maps :

$$\hat{\Pi}^{\hat{y}_1} \otimes \cdots \otimes \hat{\Pi}^{\hat{y}_n} \xrightarrow{\hat{f}} \hat{N}^{\hat{z}[d]}$$

$$\left(m^1_{p,q}\right)_{\substack{p\in[1,k^0]\\ q\in[1,k^1]}} \otimes \cdots \otimes \left(m^n_{p,q}\right)_{\substack{p\in[1,k^{n-1}]\\ q\in[1,k^n]}}$$

$$\longmapsto \left(\sum_{\substack{r_1\in[1,k^1]\\ r_2\in[1,k^2]\\ \vdots \\ r_{n-1}\in[1,k^{n-1}]}} \left(m^1_{p,r_1} \otimes m^2_{r_1,r_2} \otimes \cdots \otimes m^n_{r_{n-1},q}\right) f \right)_{\substack{p\in[1,k^0]\\ q\in[1,k^n]}}$$

Example:

A : A_∞-alg / $\mathbb{Z}$

$m_2 : A^{\otimes 2} \longrightarrow A$ sglm of deg 0

$\hat{m}_2 : \hat{A}^{\otimes 2} \longrightarrow \hat{A}$

$\mathrm{Mor}(\hat{\mathbb{Z}}) \overset{\text{identify}}{=} \{(z,k,l) : z\in\mathbb{Z},\ k,l\in\mathbb{Z}_{\geq 0}\}$

S.g. $\hat{z} = (z, \underset{\substack{\shortparallel\\ 2}}{k^0}, \underset{\substack{\shortparallel\\ 2}}{k^2}) \in \mathrm{Mor}(\hat{\mathbb{Z}})$ e.g.

and a factorisation into $\hat{y}_1 = (y_1, \underset{\substack{\shortparallel\\ 2}}{k^0}, \underset{\substack{\shortparallel\\ 2}}{k^1})$

and $\hat{y}_2 = (y_2, \underset{\substack{\shortparallel\\ 2}}{k^1}, \underset{\substack{\shortparallel\\ 2}}{k^2})$ (so $z = y_1 + y_2$)

Then $$\Bigg(\overbrace{\underbrace{\begin{pmatrix} a^1_{1,1} & a^1_{1,2} \\ a^1_{2,1} & a^1_{2,2} \end{pmatrix}}_{\substack{2\times 2\\ k^0 = \ = k^1}}}^{\in\ \hat{A}^{y_1, k^0, k^1}} \otimes \overbrace{\underbrace{\begin{pmatrix} a^2_{1,1} & a^2_{1,2} \\ a^2_{2,1} & a^2_{2,2} \end{pmatrix}}_{\substack{2\times 2\\ k^1 = \ = k^2}}}^{\in\ \hat{A}^{y_2, k^1, k^2}} \Bigg) \hat{m}_2 = \cdots$$