Problem 21 Let $\mathcal{Z}$ be a grading category. Let $n \in [1, \infty]$. Let $(A, (m_\ell)_\ell)$ be a pre-$A_n$-algebra. Write $m := ((\omega m_\ell)_\ell \in [1,n] \cap \mathbb{Z}) \beta_{\text{Coder}, n, A[1]}$.

(1) Suppose $n \in \mathbb{Z}_{\geq 1}$. Suppose that $(m_\ell)_\ell$ satisfies the Stasheff equation at each $k \in [1, n-1]$. Suppose given $z \in \text{Mor}(\mathcal{Z})$ and $a \in ((A[1])^\otimes n)^z$. Consider the following assertions.

(i) We have

$$a \left( \sum_{(r,s,t) \geq (0,1,0)} (\text{id}^\otimes r \otimes \omega m_s \otimes \text{id}^\otimes t) \cdot \omega m_{r+1+t} \right) = 0$$

(ii) We have $am^2 = 0$.

Show that (i) and (ii) are equivalent.

(2) Suppose given $p \in [1, n]$. Write $m' := ((\omega m_\ell)_\ell \in [1,p] \cap \mathbb{Z}) \beta_{\text{Coder}, p, A[1]}$.

Show that $m' = m|_{T_{\leq p}(A[1])}$.

Problem 22 Let $\mathcal{Z}$ be a grading category. Let $n \in [1, \infty]$. Let $(\tilde{A}, (\tilde{m}_\ell)_\ell)$ and $(A, (m_\ell)_\ell)$ be pre-$A_n$-algebras. Let $f = (f_\ell)_\ell$ be a pre-$A_n$-morphism from $\tilde{A}$ to $A$.

Write

$$\tilde{m} := ((\omega \tilde{m}_\ell)_\ell) \beta_{\text{Coder}, n, \tilde{A}[1]}$$
$$m := ((\omega m_\ell)_\ell) \beta_{\text{Coder}, n, A[1]}$$
$$f := ((\omega f_\ell)_\ell) \beta_{\text{Coalg}, n, \tilde{A}[1], A[1]}$$

(1) Suppose $n \in \mathbb{Z}_{\geq 1}$. Suppose that $(f_\ell)_\ell$ satisfies the Stasheff equation for morphisms at each $k \in [1, n-1]$.

Suppose given $z \in \text{Mor}(\mathcal{Z})$ and $\tilde{a} \in ((\tilde{A}[1])^\otimes n)^z$. Consider the following assertions.

(i) We have

$$\tilde{a} \left( \sum_{(r,s,t) \geq (0,1,0)} (\text{id}^\otimes r \otimes \omega \tilde{m}_s \otimes \text{id}^\otimes t) \cdot \omega f_{r+1+t} \right) \cdot \omega m_r$$

(ii) We have $\tilde{a}(\tilde{m}f - fm) = 0$.

Show that (i) and (ii) are equivalent.

(2) Suppose given $p \in [1, n]$. Write $f' := ((\omega f_\ell)_\ell \in [1,p] \cap \mathbb{Z}) \beta_{\text{Coalg}, p, \tilde{A}[1], A[1]}$.

Show that $f' = f|_{T_{\leq p}(A[1])}$.