Problem 19 Let $Z$ be a grading category.

(1) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta})$ and $V = (V, \Delta)$ be coalgebras over $Z$. Let $\tilde{V} \xrightarrow{f} V$ be a morphism of coalgebras.

Suppose that $f$ is piecewise bijective.

Show that $f$ is an isomorphism of coalgebras, i.e. that there exists a morphism of coalgebras $\tilde{V} \xleftarrow{g} V$ such that $fg = \text{id}_{\tilde{V}}$ and $gf = \text{id}_V$.

Then $g$ is uniquely determined and written $f^{-} := g$.

(2) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta}, \tilde{\delta})$ and $V = (V, \Delta, \delta)$ be coalgebras with differential over $Z$. Let $\tilde{V} \xrightarrow{f} V$ be a morphism of coalgebras with differential.

Suppose that $f$ is piecewise bijective.

Show that $f$ is an isomorphism of coalgebras with codifferential, i.e. that there exists a morphism of coalgebras with codifferential $\tilde{V} \xleftarrow{g} V$ such that $fg = \text{id}_{\tilde{V}}$ and $gf = \text{id}_V$.

Then $g$ is uniquely determined and written $f^{-} := g$.

(3) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta})$ and $V = (V, \Delta)$ be coalgebras over $Z$.

Let $\tilde{V} \xrightarrow{f} V$ be an isomorphism of coalgebras.

Suppose given a codifferential $\delta$ on $V$. Show that $f\delta f^{-}$ is a codifferential on $\tilde{V}$.

(4) Let $V = (V, \Delta)$ be a coalgebra over $Z$. Let $\lambda : V \xrightarrow{\rho} \bar{R}$ be a shift-graded linear map of degree 1; cf. Problem 7.(3). Recall that $\bar{R} \otimes V = V \otimes \bar{R}$ by identification.

Let $\delta_{\lambda} := \Delta(\text{id} \otimes \lambda) - \Delta(\lambda \otimes \text{id})$. Show that $\delta_{\lambda}$ is a coderivation.

Coderivations of this form are called inner.

Problem 20 Let $Z$ be a grading category.

Let $I$ be a finite set. Let $V_i$ be a $Z$-graded module for $i \in I$. Recall that the $Z$-graded module $\bigoplus_{i \in I} V_i$ is defined by letting $(\bigoplus_{i \in I} V_i)^z = \bigoplus_{i \in I} V_i^z$ for $z \in \text{Mor}(Z)$.

(1) Given $j \in I$, construct a shift-graded linear inclusion map $\iota_j : V_j \to \bigoplus_{i \in I} V_i^0$ of degree 0 and a shift-graded linear projection map $\pi_j : \bigoplus_{i \in I} V_i \to V_j$ of degree 0.

(2) Suppose given a $Z$-graded module $S$. Suppose given $d \in Z$. Suppose given a shift-graded linear map $s_j : S \to V_j$ of degree $d$ for $j \in I$.

Show that there exists a unique shift-graded linear map $s : S \to \bigoplus_{i \in I} V_i$ of degree $d$ such that $s \pi_j = s_j$ for $j \in I$.

(3) Suppose given a $Z$-graded module $T$. Suppose given $d \in Z$. Suppose given shift-graded linear maps $t_j : V_j \to T$ of degree $d$ for $j \in I$.

Show that there exists a unique shift-graded linear map $t : \bigoplus_{i \in I} V_i \to T$ of degree $d$ such that $\iota_j t = t_j$ for $j \in I$.

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