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Sheet 8

Problem 19 Let $\mathcal{Z}$ be a grading category.

(1) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta})$ and $V = (V, \Delta)$ be coalgebras over $\mathcal{Z}$. Let $\tilde{V} \xrightarrow{f} V$ be a morphism of coalgebras.

Suppose that f is piecewise bijective.

Show that f is an *isomorphism* of coalgebras, i.e. that there exists a morphism of coalgebras $\tilde{V} \xleftarrow{g} V$ such that $fg = \mathrm{id}_{\tilde{V}}$ and $gf = \mathrm{id}_V$.

Then g is uniquely determined and written $f^- := g$.

(2) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta}, \tilde{\delta})$ and $V = (V, \Delta, \delta)$ be coalgebras with differential over $\mathcal{Z}$. Let $\tilde{V} \xrightarrow{f} V$ be a morphism of coalgebras with differential.

Suppose that f is piecewise bijective.

Show that f is an *isomorphism* of coalgebras with codifferential, i.e. that there exists a morphism of coalgebras with codifferential $\tilde{V} \xleftarrow{g} V$ such that $fg = \mathrm{id}_{\tilde{V}}$ and $gf = \mathrm{id}_V$.

Then g is uniquely determined and written $f^- := g$.

(3) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta})$ and $V = (V, \Delta)$ be coalgebras over $\mathcal{Z}$.

Let $\tilde{V} \xrightarrow{f} V$ be an isomorphism of coalgebras.

Suppose given a codifferential δ on V. Show that $f\delta f^-$ is a codifferential on $\tilde{V}$.

(4) Let $V = (V, \Delta)$ be a coalgebra over $\mathcal{Z}$. Let $\lambda : V \longrightarrow \dot{R}$ be a shift-graded linear map of degree 1; cf. Problem 7.(3). Recall that $\dot{R} \otimes V = V = V \otimes \dot{R}$ by identification.

Let $\delta_\lambda := \Delta(\mathrm{id} \otimes \lambda) - \Delta(\lambda \otimes \mathrm{id})$. Show that δ_λ is a coderivation.

Coderivations of this form are called *inner*.

Problem 20 Let $\mathcal{Z}$ be a grading category.

Let I be a finite set. Let V_i be a $\mathcal{Z}$-graded module for $i \in I$. Recall that the $\mathcal{Z}$-graded module $\bigoplus_{i\in I} V_i$ is defined by letting $(\bigoplus_{i\in I} V_i)^z = \bigoplus_{i\in I} V_i^z$ for $z \in \mathrm{Mor}(\mathcal{Z})$.

(1) Given $j \in I$, construct a shift-graded linear *inclusion* map $\iota_j : V_j \to \bigoplus_{i\in I} V_i$ of degree 0 and a shift-graded linear *projection* map $\pi_j : \bigoplus_{i\in I} V_i \to V_j$ of degree 0.

(2) Suppose given a $\mathcal{Z}$-graded module S. Suppose given $d \in \mathbf{Z}$. Suppose given a shift-graded linear map $s_j : S \to V_j$ of degree d for $j \in I$.

Show that there exists a unique shift-graded linear map $s : S \to \bigoplus_{i\in I} V_i$ of degree d such that $s\pi_j = s_j$ for $j \in I$.

(3) Suppose given a $\mathcal{Z}$-graded module T. Suppose given $d \in \mathbf{Z}$. Suppose given shift-graded linear maps $t_j : V_j \to T$ of degree d for $j \in I$.

Show that there exists a unique shift-graded linear map $t : \bigoplus_{i\in I} V_i \to T$ of degree d such that $\iota_j t = t_j$ for $j \in I$.