M. Künzer

A_∞-categories, WS 16/17

Sheet 6

Problem 15 Let $\mathcal{Z}$ be a grading category.
Let L and M be $\mathcal{Z}$-graded modules.
Let $L \xrightarrow{f} M$ be a shift-graded linear map of degree $a \in \mathbf{Z}$.

(1) Let K be the $\mathcal{Z}$-graded module defined by $K^z := \mathrm{Kern}(L^z \xrightarrow{f} M^{z[a]})$ for $z \in \mathrm{Mor}(\mathcal{Z})$.
Let $K \xrightarrow{i} L$ be the shift-graded linear map given by inclusion, which is of degree 0.

Suppose given a $\mathcal{Z}$-graded module T and a shift-graded linear map $T \xrightarrow{t} L$ of degree d such that $tf = 0$.

Show that there exist a unique shift-graded linear map $T \xrightarrow{\check{t}} K$ of degree d such that $\check{t}i = t$.

(2) Suppose R to be a field. Suppose f to be piecewise surjective.

Show that there exists a piecewise injective shift-graded linear map $L \xleftarrow{g} M$ of degree $-a$ such that $gf = \mathrm{id}_M$.

Problem 16 Let $\mathcal{Z}$ be a grading category. Let $n \in \mathbf{Z}_{\geqslant 1}$.
Let K_i, L_i and M_i be $\mathcal{Z}$-graded modules for $i \in [1, n]$.
Let $K_i \xrightarrow{u_i} L_i$ be a shift-graded linear map of degree $c_i \in \mathbf{Z}$ for $i \in [1, n]$.
Let $L_i \xrightarrow{f_i} M_i$ be a piecewise surjective shift-graded linear map of degree $a_i \in \mathbf{Z}$ for $i \in [1, n]$.

(1) Show that $\bigotimes_{i\in[1,n]} f_i$ is piecewise surjective.

(2) For $i \in [1, n]$, suppose that $K_i \xrightarrow{u_i} L_i \xrightarrow{f_i} M_i$ to be *exact at* L_i,
i.e. suppose $K_i^{z[-c_i]} \xrightarrow{u_i} L_i^z \xrightarrow{f_i} M_i^{z[a_i]}$ to be exact at L_i^z for $z \in \mathrm{Mor}(\mathcal{Z})$.

Suppose given a $\mathcal{Z}$-graded module T and a shift-graded linear map $\bigotimes_{i\in[1,n]} L_i \xrightarrow{t} T$ of degree d.

Suppose that $(\mathrm{id}^{\otimes j-1} \otimes u_j \otimes \mathrm{id}^{\otimes n-j})t = 0$ for $j \in [1, n]$.

Show that there exists a unique shift-graded linear map $\bigotimes_{i\in[1,n]} M_i \xrightarrow{\tilde{t}} T$ of degree $d - \sum_{i\in[1,n]} a_i$ such that $(\bigotimes_{i\in[1,n]} f_i)\tilde{t} = t$.