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A$_\infty$-categories, WS 16/17

Sheet 5

Problem 13 Let B be an algebra.

Suppose given a diagram $X' \xrightarrow{i} X \xrightarrow{r} X''$ in C(B-Mod) such that $X'^k \xrightarrow{i^k} X^k \xrightarrow{r^k} X''^k$ is short exact for $k \in \mathbf{Z}$. Such a diagram is called a short exact sequence of complexes in B.

(1) Suppose given $T \xrightarrow{f} X$ in C(B-Mod) such that $fr = 0$. Show that there exists a unique morphism $T \xrightarrow{f'} X'$ such that $f'i = f$.

(2) Suppose given $X \xrightarrow{g} T$ in C(B-Mod) such that $ig = 0$. Show that there exists a unique morphism $X'' \xrightarrow{g''} T$ such that $rg'' = g$.

(3) A $\mathbf{Z}$-graded B-module M is a tuple $M = (M^z)_{z\in\mathbf{Z}}$ of B-modules M^z. A graded B-linear map $f : L \to M$ between $\mathbf{Z}$-graded B-modules is a tuple $f = (f^z)_{z\in\mathbf{Z}}$ of B-linear maps f^z. Write B-$\mathbf{Z}$-grad for the category of $\mathbf{Z}$-graded B-modules and graded B-linear maps.

Construct an additive functor $\mathrm{H} : \mathrm{C}(B\text{-Mod}) \to B\text{-}\mathbf{Z}\text{-grad}$ having

$$(\mathrm{H}X)^k \;=\; \mathrm{Kern}(d^k)/\,\mathrm{Im}(d^{k-1})$$

for a complex X with differential $d = (X^k \xrightarrow{d^k} X^{k+1})_k$.

For $Y \xrightarrow{f} Z$ in C(B-Mod), we often write $((\mathrm{H}Y)^k \xrightarrow{(\mathrm{H}f)^k} (\mathrm{H}Z)^k) =: (\mathrm{H}^k Y \xrightarrow{\mathrm{H}^k f} \mathrm{H}^k Z)$.

(4) Construct a B-linear map $\mathrm{H}^k X'' \xrightarrow{\gamma^k_{(i,r)}} \mathrm{H}^{k+1} X'$ for $k \in \mathbf{Z}$, called *connector* of the given short exact sequence $X' \xrightarrow{i} X \xrightarrow{r} X''$, subject to the following conditions (i, ii).

(i) The sequence

$$\ldots \to \mathrm{H}^k X' \xrightarrow{\mathrm{H}^k i} \mathrm{H}^k X \xrightarrow{\mathrm{H}^k r} \mathrm{H}^k X'' \xrightarrow{\gamma^k_{(i,r)}} \mathrm{H}^{k+1} X' \xrightarrow{\mathrm{H}^{k+1} i} \mathrm{H}^{k+1} X \xrightarrow{\mathrm{H}^{k+1} r} \mathrm{H}^{k+1} X'' \to \ldots$$

is exact at each position.

(ii) Given a morphism of short exact sequences, i.e. a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i} & X & \xrightarrow{r} & X'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ Y' & \xrightarrow{j} & Y & \xrightarrow{s} & Y'' \end{array}$$

in C(B-Mod) with (i, r) and (j, s) short exact, we get, for $k \in \mathbf{Z}$, the commutative quadrangle

$$\begin{array}{ccc} \mathrm{H}^k X'' & \xrightarrow{\gamma^k_{(i,r)}} & \mathrm{H}^{k+1} X' \\ \downarrow{\scriptstyle \mathrm{H}^k f''} & & \downarrow{\scriptstyle \mathrm{H}^{k+1} f'} \\ \mathrm{H}^k Y'' & \xrightarrow{\gamma^k_{(j,s)}} & \mathrm{H}^{k+1} Y' \end{array} .$$

p.t.o.

Problem 14 Suppose given an algebra B. Suppose given $n \geqslant 1$.
Suppose given $X_s \in \operatorname{Ob} \mathrm{C}(B\text{-Mod})$ for $s \in [1,n]$. Abbreviate $\underline{X} := (X_s)_{s\in[1,n]}$.
Abbreviate $\mathcal{Z} := \mathbf{Z} \times [1,n]^{\times 2}$, $\mathrm{C} := \mathrm{C}(B\text{-Mod})$ and $\mathrm{K} := \mathrm{K}(B\text{-Mod})$.
Consider the $\mathcal{Z}$-graded module $\mathrm{Z}\operatorname{Hom}_B(\underline{X})$ having, for $z \in \operatorname{Mor}(\mathcal{Z})$,

$$(\mathrm{Z}\operatorname{Hom}_B(\underline{X}))^z \;:=\; \operatorname{Kern}((m_1^{\operatorname{Hom}_B(\underline{X})})^z)\ .$$

Consider the $\mathcal{Z}$-graded module $\mathrm{H}\operatorname{Hom}_B(\underline{X})$, having, for $z \in \operatorname{Mor}(\mathcal{Z})$,

$$(\mathrm{H}\operatorname{Hom}_B(\underline{X}))^z := \operatorname{Kern}((m_1^{\operatorname{Hom}_B(\underline{X})})^z)/\operatorname{Im}((m_1^{\operatorname{Hom}_B(\underline{X})})^{z[-1]})\ .$$

(1) Show that $(\mathrm{Z}\operatorname{Hom}_B(\underline{X}))^{(j,(s,t))} = {}_{\mathrm{C}}(X_s, X_t^{[j]})$ for $(j,(s,t)) \in \operatorname{Mor}(\mathcal{Z})$.

(2) Show that $(\mathrm{H}\operatorname{Hom}_B(\underline{X}))^{(j,(s,t))} = {}_{\mathrm{K}}(X_s, X_t^{[j]})$ for $(j,(s,t)) \in \operatorname{Mor}(\mathcal{Z})$.

(3) Show that $m_2^{\operatorname{Hom}_B(\underline{X})}$ induces a map $m_2^{\mathrm{H}\operatorname{Hom}_B(\underline{X}))} : \mathrm{H}\operatorname{Hom}_B(\underline{X})^{\otimes 2} \to \mathrm{H}\operatorname{Hom}_B(\underline{X})$ that maps $[f] \otimes [g]$ to $[f \cdot g]$ for each composable pair of morphisms (f,g) in C, where we use brackets to denote residue classes of morphisms of C in K.