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A$_\infty$-categories, WS 16/17

Sheet 3

Problem 7 Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category.

Suppose given $1 \leqslant \ell \leqslant n$ and $\mathcal{Z}$-shift-graded linear maps $L_i \xrightarrow{(f_i,k_i)} M_i$ for $i \in [1,n]$.

Suppose given $\mathcal{Z}$-shift-graded linear maps $L \xrightarrow{(f,k)} M$ and $\tilde{L} \xrightarrow{(\tilde{f},\tilde{k})} \tilde{M}$.

(1) Show that

$$(M_1 \otimes \ldots \otimes M_\ell) \otimes (M_{\ell+1} \otimes \ldots \otimes M_n) \;=\; M_1 \otimes \ldots \otimes M_n \;.$$

(2) Show that

$$((f_1,k_1) \otimes \ldots \otimes (f_\ell,k_\ell)) \otimes ((f_{\ell+1},k_{\ell+1}) \otimes \ldots \otimes (f_n,k_n)) \;=\; (f_1,k_1) \otimes \ldots \otimes (f_n,k_n) \;.$$

(3) Construct a $\mathcal{Z}$-graded module $\dot{R}$ such that $(f,k) \otimes (\mathrm{id}_{\dot{R}}, 0) = (f,k)$ and $(\mathrm{id}_{\dot{R}}, 0) \otimes (f,k) = (f,k)$.

(4) Construct an isomorphism $L \otimes \tilde{L} \xrightarrow[\sim]{\tau_{L,\tilde{L}}} \tilde{L} \otimes L$ in $\mathcal{Z}$-grad, and likewise $\tau_{M,\tilde{M}}$, such that the following quadrangle commutes.

$$\begin{tikzcd}
L \otimes \tilde{L} \arrow[r, "\tau_{L,\tilde{L}}", "\sim"'] \arrow[d, "(f,k)\otimes(\tilde{f},\tilde{k})"'] & \tilde{L} \otimes L \arrow[d, "(-1)^{k\tilde{k}}(\tilde{f},\tilde{k})\otimes(f,k)"] \\
M \otimes \tilde{M} \arrow[r, "\tau_{M,\tilde{M}}", "\sim"'] & \tilde{M} \otimes M
\end{tikzcd}$$

Problem 8 Let B be an algebra.

(1) Let $\mathcal{A}$ be a linear additive category. Let $\mathcal{N} \subseteq \mathcal{A}$ be a full additive subcategory. Write

$$\begin{aligned}
&\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y) \\
&:= \{\, X \xrightarrow{f} Y \;:\; \text{there exists } N \in \mathrm{Ob}(\mathcal{N}) \text{ and morphisms } X \xrightarrow{u} N \xrightarrow{v} Y \text{ such that } f = uv \,\} \;.
\end{aligned}$$

Let $\mathcal{A}/\mathcal{N}$ be the category that has

$$\begin{aligned}
\mathrm{Ob}(\mathcal{A}/\mathcal{N}) &:= \mathrm{Ob}(\mathcal{A}) \\
{}_{\mathcal{A}/\mathcal{N}}(X,Y) &:= {}_{\mathcal{A}}(X,Y) / \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y) \quad \text{for } X,\, Y \in \mathrm{Ob}(\mathcal{A}/\mathcal{N}) \;.
\end{aligned}$$

For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, we define composition of the respective residue classes in $\mathcal{A}/\mathcal{N}$ by

$$(f + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)) \cdot (g + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(Y,Z)) \;=\; f \cdot g + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Z) \;.$$

Show that $\mathcal{A}/\mathcal{N}$ is a linear additive category. Show that $\mathcal{A} \xrightarrow{R} \mathcal{A}/\mathcal{N}$ is a linear functor with $RN \simeq 0$ for $N \in \mathrm{Ob}(\mathcal{N})$.

We often write $\bar{f} := f + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$.

Given a linear additive category $\mathcal{B}$ and a linear functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ with $FN \simeq 0$ for $N \in \mathrm{Ob}(\mathcal{N})$, show that there exists a unique linear functor $\mathcal{A}/\mathcal{N} \xrightarrow{\bar{F}} \mathcal{B}$ such that $F = \bar{F} \circ R$.

(2) Let $\mathcal{A} := \mathrm{C}(B\text{-Mod})$ be the category of complexes of B-modules. Let the differential of a complex $X \in \mathrm{Ob}(\mathcal{A})$ be denoted by $d = d_X$. Let $\mathcal{N} \subseteq \mathcal{A}$ be the full additive subcategory of split acyclic complexes, i.e. those isomorphic to a complex of the form $\cdots \to U^{i-1} \oplus U^i \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} U^i \oplus U^{i+1} \to \ldots$, where $U^i \in \mathrm{Ob}\,\mathcal{A}$ for $i \in \mathbf{Z}$.

Show that $\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$ consists of those morphisms of complexes $X \xrightarrow{f} Y$ for which there exists a tuple of morphisms $(X^i \xrightarrow{h^i} Y^{i-1})_{i \in \mathbf{Z}}$ such that

$$f^i \;=\; h^i d_Y^{i-1} + d_X^i h^{i+1} \qquad \text{for } i \in \mathbf{Z}.$$

Define $\mathrm{K}(B\text{-Mod}) := \mathcal{A}/\mathcal{N}$ to be the *homotopy category* of complexes of B-modules. Write shorthand ${}_{\mathrm{K}}(X,Y) := {}_{\mathrm{K}(B\text{-Mod})}(X,Y)$ for $X,\, Y \in \mathrm{Ob}(\mathrm{K}(B\text{-Mod})) = \mathrm{Ob}(\mathrm{C}(B\text{-Mod}))$.

(3) Let M be a B-module. Let P be a projective resolution of M with augmentation $\varepsilon : P_0 \to M$. Let $\mathrm{Conc}(M) \in \mathrm{Ob}(\mathrm{C}(B\text{-Mod}))$ have M at position 0, and 0 elsewhere. Let $\hat{\varepsilon} : P \to \mathrm{Conc}(M)$ be the morphism of complexes having entry ε at position 0.

Let Q be a complex consisting of projective B-modules, bounded above. Show that ${}_{\mathrm{K}}(Q, \bar{\hat{\varepsilon}}) : {}_{\mathrm{K}}(Q,P) \to {}_{\mathrm{K}}(Q, \mathrm{Conc}(M))$ is an isomorphism.

(4) Using the universal property from (1), construct a shift functor S on $\mathrm{K}(B\text{-Mod})$ such that $(SX)^i = X^{i+1}$ and such that $d_{SX}^i = -d_X^{i+1}$ for $i \in \mathbf{Z}$. Show that S is an automorphism.

We also write $S^k =: (-)^{[k]}$ for $k \in \mathbf{Z}$.