Problem 7 Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category. Suppose given $1 \leq \ell \leq n$ and $\mathcal{Z}$-shift-graded linear maps $L_i \xrightarrow{(f_i, k_i)} M_i$ for $i \in [1, n]$. Suppose given $\mathcal{Z}$-shift-graded linear maps $L \xrightarrow{(f, k)} M$ and $\tilde{L} \xrightarrow{(\tilde{f}, \tilde{k})} \tilde{M}$.

1. Show that 
\[(M_1 \otimes \cdots \otimes M_\ell) \otimes (M_{\ell+1} \otimes \cdots \otimes M_n) = M_1 \otimes \cdots \otimes M_n.\]

2. Show that 
\[((f_1, k_1) \otimes \cdots \otimes (f_\ell, k_\ell)) \otimes ((f_{\ell+1}, k_{\ell+1}) \otimes \cdots \otimes (f_n, k_n)) = (f_1, k_1) \otimes \cdots \otimes (f_n, k_n).\]

3. Construct a $\mathcal{Z}$-graded module $\hat{R}$ such that $(f, k) \otimes (\text{id}_{\hat{R}}, 0) = (f, k)$ and $(\text{id}_{\hat{R}}, 0) \otimes (f, k) = (f, k)$.

4. Construct an isomorphism $L \otimes \tilde{L} \xrightarrow{\tau_{L, \tilde{L}}} L \otimes \tilde{L}$ in $\mathcal{Z}$-grad, and likewise $\tau_{M, \tilde{M}}$, such that the following quadrangle commutes.

Problem 8 Let $B$ be an algebra.

1. Let $\mathcal{A}$ be a linear additive category. Let $\mathcal{N} \subseteq \mathcal{A}$ be a full additive subcategory. Write
\[\text{Null}_{\mathcal{A}, \mathcal{N}}(X, Y) := \{ X \xrightarrow{f} Y : \text{there exists } N \in \text{Ob(\mathcal{N}) and morphisms } X \xrightarrow{u} N \xrightarrow{v} Y \text{ such that } f = uv \}.\]

Let $\mathcal{A}/\mathcal{N}$ be the category that has
\[\text{Ob}(\mathcal{A}/\mathcal{N}) := \text{Ob}(\mathcal{A}) \]
\[\mathcal{A}/\mathcal{N}(X, Y) := \mathcal{A}(X, Y)/\text{Null}_{\mathcal{A}, \mathcal{N}}(X, Y) \text{ for } X, Y \in \text{Ob}(\mathcal{A}/\mathcal{N}).\]

For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, we define composition of the respective residue classes in $\mathcal{A}/\mathcal{N}$ by
\[(f + \text{Null}_{\mathcal{A}, \mathcal{N}}(X, Y)) \cdot (g + \text{Null}_{\mathcal{A}, \mathcal{N}}(Y, Z)) = f \cdot g + \text{Null}_{\mathcal{A}, \mathcal{N}}(X, Z).\]
Show that $\mathcal{A}/\mathcal{N}$ is a linear additive category. Show that $\mathcal{A} \xrightarrow{R} \mathcal{A}/\mathcal{N}$ is a linear functor with $RN \simeq 0$ for $N \in \text{Ob}(\mathcal{N})$.

We often write $\bar{f} := f + \text{Null}_{\mathcal{A}/\mathcal{N}}(X,Y)$.

Given a linear additive category $\mathcal{B}$ and a linear functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ with $FN \simeq 0$ for $N \in \text{Ob}(\mathcal{N})$, show that there exists a unique linear functor $\mathcal{A}/\mathcal{N} \xrightarrow{\bar{F}} \mathcal{B}$ such that $F = \bar{F} \circ R$.

(2) Let $\mathcal{A} := C(B\text{-Mod})$ be the category of complexes of $B$-modules. Let the differential of a complex $X \in \text{Ob}(\mathcal{A})$ be denoted by $d = d_X$. Let $\mathcal{N} \subseteq \mathcal{A}$ be the full additive subcategory of split acyclic complexes, i.e. those isomorphic to a complex of the form
\[ \ldots \to U_{i-1} \oplus U_i \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} U_i \oplus U_{i+1} \to \ldots, \] where $U_i \in \text{Ob} \mathcal{A}$ for $i \in \mathbb{Z}$.

Show that $\text{Null}_{\mathcal{A}/\mathcal{N}}(X,Y)$ consists of those morphisms of complexes $X \xrightarrow{f} Y$ for which there exists a tuple of morphisms $(X^i \xrightarrow{h^i} Y^{i-1})_{i \in \mathbb{Z}}$ such that
\[ f^i = h^i d_{X}^{i-1} + d_X^i h^{i+1} \quad \text{for } i \in \mathbb{Z}. \]

Define $K(B\text{-Mod}) := \mathcal{A}/\mathcal{N}$ to be the homotopy category of complexes of $B$-modules. Write shorthand $K(X,Y) := K(B\text{-Mod})(X,Y)$ for $X, Y \in \text{Ob}(K(B\text{-Mod})) = \text{Ob}(C(B\text{-Mod}))$.

(3) Let $M$ be a $B$-module. Let $P$ be a projective resolution of $M$ with augmentation $\varepsilon : P_0 \to M$. Let $\text{Conc}(M) \in \text{Ob}(C(B\text{-Mod}))$ have $M$ at position 0, and 0 elsewhere. Let $\tilde{\varepsilon} : P \to \text{Conc}(M)$ be the morphism of complexes having entry $\varepsilon$ at position 0.

Let $Q$ be a complex consisting of projective $B$-modules, bounded above. Show that $\text{k}(Q, \tilde{\varepsilon}) : \text{k}(Q, P) \to \text{k}(Q, \text{Conc}(M))$ is an isomorphism.

(4) Using the universal property from (1), construct a shift functor $S$ on $K(B\text{-Mod})$ such that $(SX)^i = X^{i+1}$ and such that $d_{SX}^i = -d_X^{i+1}$ for $i \in \mathbb{Z}$. Show that $S$ is an automorphism.

We also write $S^k = (-)^k$ for $k \in \mathbb{Z}$.

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