M. Künzer

Sheet 2

Problem 4 Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category. Show.

(1) The shift S is an automorphism of $\mathcal{Z}$-grad_0.

(2) $\mathcal{Z}$-grad is a category.

(3) By $S(f,k) := (Sf, k)$ for $(f,k) \in \mathrm{Mor}(\mathcal{Z}\text{-grad})$, an automorphism S on $\mathcal{Z}$-grad is defined.

(4) $\mathcal{Z}$-grad_0 is additive.

(5) $\mathcal{Z}$-grad_0 is isomorphic to a subcategory of $\mathcal{Z}$-grad.
Is this subcategory full? Does $\mathcal{Z}$-grad have a zero object?

Problem 5 Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ and $\tilde{\mathcal{Z}} = (\tilde{\mathcal{Z}}, \tilde{S}, \tilde{\deg})$ be grading categories.
A *(1-)morphism of grading categories* from $\mathcal{Z}$ to $\tilde{\mathcal{Z}}$ is a functor $F : \mathcal{Z} \to \tilde{\mathcal{Z}}$ such that

$$\begin{aligned} F(zS) &= (Fz)\tilde{S} \\ (z)\deg &= (Fz)\tilde{\deg} \end{aligned}$$

for $z \in \mathrm{Mor}(\mathcal{Z})$.

(1) Show that grading categories, together with morphisms of such, form a category Grad.

(2) Show that $(\mathcal{Z}, S^-, -\deg)$ is a grading category, where $(S^-)_{X,Y} := (S_{X,Y})^-$ for $X, Y \in \mathrm{Ob}(\mathcal{Z})$ and $z(-\deg) := -(z\deg)$ for $z \in \mathrm{Mor}(\mathcal{Z})$. Construct an automorphism of order 2 on the category of grading categories.

(3) Show that $(\mathrm{id}_X)\deg = 0$ for $X \in \mathrm{Ob}(\mathcal{Z})$.

(4) Show that there exists exactly one morphism of grading categories from $\mathcal{Z}$ to $\mathbf{Z}$, i.e. that $\mathbf{Z}$ is the terminal grading category.

(5) Show that there is a bijection from the set of morphisms of grading categories from $\mathbf{Z}$ to $\mathcal{Z}$ to the set of endomorphisms of $\mathcal{Z}$ of degree 0.

(6) Suppose given a morphism of grading categories $\mathcal{Z} \xrightarrow{F} \tilde{\mathcal{Z}}$.
Show that there exist functors

$$\mathcal{Z}\text{-grad} \underset{F^{\&}}{\overset{F_{\&}}{\rightleftarrows}} \tilde{\mathcal{Z}}\text{-grad}$$

having $(F^{\&}\tilde{M})^z = \tilde{M}^{Fz}$ for $\tilde{M} \in \mathrm{Ob}(\tilde{\mathcal{Z}}\text{-grad})$ and $z \in \mathrm{Mor}(\mathcal{Z})$, having $(F_{\&}M)^{\tilde{z}} = \bigoplus_{\substack{z \in \mathrm{Mor}(\mathcal{Z}) \\ Fz = \tilde{z}}} M^z$ for $M \in \mathrm{Ob}(\mathcal{Z}\text{-grad})$ and $\tilde{z} \in \mathrm{Mor}(\tilde{\mathcal{Z}})$ and having $F_{\&} \dashv F^{\&}$.

Problem 6 Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category.
Define a category $(\mathcal{Z}\text{-grad})^{\times n, \pm}$ such that we have a functor

$$\begin{array}{ccc} (\mathcal{Z}\text{-grad})^{\times n,\pm} & \xrightarrow{\bigotimes_{i\in[1,n]}} & \mathcal{Z}\text{-grad} \\ (L_i \xrightarrow{(f_i,k_i)} M_i)_{i\in[1,n]} & \longmapsto & (\bigotimes_{i\in[1,n]} L_i \xrightarrow{\bigotimes_{i\in[1,n]}(f_i,k_i)} \bigotimes_{i\in[1,n]} M_i)\ . \end{array}$$