Problem 4 Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category. Show.

1. The shift $S$ is an automorphism of $\mathcal{Z}_{\text{grad}}_0$.
2. $\mathcal{Z}_{\text{grad}}$ is a category.
3. By $S(f, k) := (S(f), k)$ for $(f, k) \in \text{Mor}(\mathcal{Z}_{\text{grad}})$, an automorphism $S$ on $\mathcal{Z}_{\text{grad}}$ is defined.
4. $\mathcal{Z}_{\text{grad}}_0$ is additive.
5. $\mathcal{Z}_{\text{grad}}_0$ is isomorphic to a subcategory of $\mathcal{Z}_{\text{grad}}$.
   Is this subcategory full? Does $\mathcal{Z}_{\text{grad}}$ have a zero object?

Problem 5 Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ and $\tilde{\mathcal{Z}} = (\tilde{\mathcal{Z}}, \tilde{S}, \tilde{\deg})$ be grading categories.

A morphism of grading categories from $\mathcal{Z}$ to $\tilde{\mathcal{Z}}$ is a functor $F: \mathcal{Z} \to \tilde{\mathcal{Z}}$ such that

$$F(zS) = (Fz)\tilde{S}$$

$$F(z)\deg = (Fz)\tilde{\deg}$$

for $z \in \text{Mor}(\mathcal{Z})$.

1. Show that grading categories, together with morphisms of such, form a category $\text{Grad}$.
2. Show that $(\mathcal{Z}, S^{-}, -\deg)$ is a grading category, where $(S^{-})_{X,Y} := (S_{X,Y})^{-}$ for $X, Y \in \text{Ob}(\mathcal{Z})$ and $z(-\deg) := -(z\deg)$ for $z \in \text{Mor}(\mathcal{Z})$. Construct an automorphism of order 2 on the category of grading categories.
3. Show that $(\text{id}_X)\deg = 0$ for $X \in \text{Ob}(\mathcal{Z})$.
4. Show that there exists exactly one morphism of grading categories from $\mathcal{Z}$ to $\mathcal{Z}$, i.e. that $\mathcal{Z}$ is the terminal grading category.
5. Show that there is a bijection from the set of morphisms of grading categories from $\mathcal{Z}$ to $\mathcal{Z}$ to the set of endomorphisms of $\mathcal{Z}$ of degree 0.
6. Suppose given a morphism of grading categories $\mathcal{Z} \xrightarrow{F} \tilde{\mathcal{Z}}$.
   Show that there exist functors

$$\mathcal{Z}_{\text{grad}} \xrightarrow{F_{\tilde{k}}} \tilde{\mathcal{Z}}_{\text{grad}}$$

having $(F_{\tilde{k}}\tilde{M})^z = \tilde{M}^{Fz}$ for $\tilde{M} \in \text{Ob}(\tilde{\mathcal{Z}}_{\text{grad}})$ and $z \in \text{Mor}(\mathcal{Z})$, having

$$(F_{\tilde{k}}M)^{\tilde{z}} = \bigoplus_{z \in \text{Mor}(\mathcal{Z})} M^z$$

for $M \in \text{Ob}(\mathcal{Z}_{\text{grad}})$ and $\tilde{z} \in \text{Mor}(\tilde{\mathcal{Z}})$ and having $F_{\tilde{k}} \dashv F_{\tilde{k}}$.

Problem 6 Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category.

Define a category $(\mathcal{Z}_{\text{grad}})^{\times n, \pm}$ such that we have a functor

$$(\mathcal{Z}_{\text{grad}})^{\times n, \pm} \xrightarrow{\otimes_{i \in [1,n]}} \mathcal{Z}_{\text{grad}}$$

$$(L_i \xrightarrow{(f_i, k_i)} M_i)_{i \in [1,n]} \xrightarrow{\otimes_{i \in [1,n]}} \left( \bigotimes_{i \in [1,n]} L_i \xrightarrow{\otimes_{i \in [1,n]} (f_i, k_i)} \bigotimes_{i \in [1,n]} M_i \right).$$