Kadeishvili by induction:

$(A, (m_k)_{k\in[1,n]})$: A_n - alg. $/ \mathbb{Z}$

$(HA, (\tilde{m}_k)_{k\in[1,n-1]})$: A_{n-1} - alg $/ \mathbb{Z}$

$q = (q_k)_{k\in[1,n-1]}$: A_{n-1} - quasiiso morphism

Need: • $1_X \rho$ neutral w.r.t. $\tilde{m}_2$

follows by Ex. 34:

$$(1_X\rho \otimes b)\tilde{m}_2 = (1_X\rho \otimes a\rho)\tilde{m}_2 = (1_X \otimes a) m_2 \rho = a\rho = b$$

• $0 \overset{!}{=} \psi_n m_1$

$$= \sum_{r\in[2,n]} \sum_{\substack{(i_j)_j \geq (1)_j \\ \sum_{j\in[1,r]} i_j = n}} L((-i_j)_j, (i_j)_j) \Big(\bigotimes_j q_{i_j}\Big) m_r m_1$$

$$- \sum_{\substack{(r,s,t)\geq(0,2,0) \\ r+s+t=n \\ (r,t)\neq(0,0)}} (-1)^{r+st} (\mathrm{id}^{\otimes r} \otimes \tilde{m}_s \otimes \mathrm{id}^{\otimes t}) q_{r+1+t} m_1$$

Plug in:

• $$m_r m_1 = - \sum_{\substack{(u,v,w)\geq(0,1,0) \\ u+v+w=r \\ (u,w)\neq(0,0)}} (-1)^{u+vw} (\mathrm{id}^{\otimes u} \otimes m_v \otimes \mathrm{id}^{\otimes w}) m_{u+1+w}$$

• $q_{r+1+t} m_1 = \dots$

Plug in:

$$\underbrace{q_{r+1+t}\, m_1}\ = \underbrace{\sum_{\substack{(u,v,w)\geq(0,1,0)\\ u+v+w=r+1+t}} (-1)^{u+vw}\, (id^{\otimes u}\otimes \tilde{m}_v \otimes id^{\otimes w})\, q_{u+1+w}}$$

$$- \underbrace{\sum_{x\in[1,r+1+t]} \sum_{\substack{(i_j)_{j\in[1,x]}\geq(1)_j\\ \sum_j i_j = r+1+t}}} \underbrace{L((1-i_j)_j, (i_j)_j)\,(\bigotimes_j q_{i_j})\, m_x}$$

Need:

$$0 \overset{!}{=} \underbrace{- \sum_{r\in[2,u]} \sum_{\substack{(i_j)_{j\in[1,r]}\geq(1)_j\\ \sum_j i_j = u}} \sum_{\substack{(u,v,w)\geq(0,1,0)\\ u+v+w=r\\ (u,w)\neq(0,0)}} L((1-i_j)_j, (i_j)_j)\, (-1)^{u+vw} \cdot (\bigotimes_j q_{i_j}) \cdot (id^{\otimes u}\otimes m_v \otimes id^{\otimes w})\, m_{u+1+w}}_{=:\ \mathrm{I}}$$

$$\underbrace{- \sum_{\substack{(r,s,t)\geq(0,2,0)\\ r+s+t=n\\ (r,t)\neq(0,0)}} \sum_{\substack{(u,v,w)\geq(0,1,0)\\ u+v+w=r+1+t}} (-1)^{r+st+u+vw}\, (id^{\otimes r}\otimes \tilde{m}_s \otimes id^{\otimes t})\, (id^{\otimes u}\otimes \tilde{m}_v \otimes id^{\otimes w})\, q_{u+1+w}}_{=:\ \mathrm{II}}$$

$$\underbrace{+ \sum_{\substack{(r,s,t)\geq(0,2,0)\\ r+s+t=n\\ (r,t)\neq(0,0)}} \sum_{x\in[1,r+1+t]} \sum_{\substack{(i_j)_{j\in[1,x]}\geq(1)_j\\ \sum_j i_j = r+1+t}} \overset{(-1)^{r+st}}{\vee}\, L((1-i_j)_j, (i_j)_j)\, (id^{\otimes r}\otimes \tilde{m}_s \otimes id^{\otimes t}) \cdot (\bigotimes_j q_{i_j}) \cdot m_x}_{=:\ \mathrm{III}}$$

Claim 2: $\mathrm{I} \overset{!}{=} \mathrm{III}$ Claim 3: $\mathrm{II} \overset{!}{=} 0$

S2

Ad case 2

$$i^{\Sigma}_{[a,b]} := \sum_{j\in[a,b]} i_j \;, \qquad y_{[a,b]} := y_a \otimes \cdots \otimes y_b$$

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$$I = \sum_{r\in[2,n]} \; \sum_{\substack{(i_j)_{j\in[1,r]} \geq (1)_j \\ i^{\Sigma}_{[1,r]} = n}} \; \sum_{\substack{(u,v,w)\geq(0,1,0) \\ u+v+w=r \\ (u,w)\neq(0,0)}} (-1)^{(\cancel{w} - i^{\Sigma}_{[u+v+1,r]})\underbrace{(2-v)}_{\equiv_2 v}} \underbrace{\lfloor (1-i_j)_j , (i_j)_j \rfloor}\, (-1)^{u+\cancel{w}} \left(q_{i_{[1,u]}} \otimes q_{i_{[u+1,u+v]}}{}^{m_v} \otimes q_{i_{[r-w+1,r]}} \right) \cdot m_{u+1+w}$$

$v = r-u-w$

with middle part $(\gamma_j)_j$ taken out

$$= \sum_{r\in[2,n]} \; \sum_{\substack{(u,w)>(0,0) \\ u+w\leq r-1}} \; \sum_{\substack{(i_j)_{j\in[1,u+w]} \\ i^{\Sigma}_{[1,u+w]} \leq n-1}} \; \sum_{\substack{(\gamma_j)_{j\in[1,r-u-w]} \geq (1)_j \\ \gamma^{\Sigma}_{[1,r-u-w]} = \underbrace{n - i^{\Sigma}_{[1,u+w]}}_{=: n'}}} \underbrace{\lfloor (1-i_j)_j , (i_j)_j \rfloor \lfloor (1-\gamma_j)_j , (\gamma_j)_j \rfloor (-1)^{(n - i^{\Sigma}_{[1,u]})\overbrace{\gamma^{\Sigma}_{[1,r-u-w]}}^{n'} + (r-\cancel{u}-w-\overbrace{\gamma^{\Sigma}_{[1,r-u-w]}}^{n'}) \cdot i^{\Sigma}_{[r-w+1,r]}}}$$

$$\cdot (-1)^{u + i^{\Sigma}_{[u+v+1,r]}\,\cancel{(r-u-w)}} \left(q_{i_{[1,u]}} \otimes q_{\gamma_{[1,r-u-w]}}{}^{m_v} \otimes q_{i_{[u+1,u+w]}} \right) \cdot m_{u+1+w}$$

$$= \sum_{\substack{(u,w)>(0,0) \\ u+w\leq n-1}} \; \sum_{\substack{(i_j)_{j\in[1,u+w]} \geq (1)_j \\ i^{\Sigma}_{[1,u+w]} \leq n-1}} \; \sum_{v\in[1,n-u-w]} \; \sum_{\substack{(\gamma_j)_{j\in[1,v]} \geq (1)_j \\ \gamma^{\Sigma}_{[1,v]} = n - i^{\Sigma}_{[1,u+w]} = n'}} \lfloor (1-i_j)_j , (i_j)_j \rfloor \lfloor (1-\gamma_j)_j , (\gamma_j)_j \rfloor (-1)^{un' - i^{\Sigma}_{[1,u+w]} n' + u}$$

$$\cdot \left(q_{[1,u]} \otimes q_{\gamma_{[1,v]}}{}^{m_v} \otimes q_{i_{[u+1,u+w]}} \right) m_{u+1+w}$$

Koszul

$$\boxed{(-1)^{\left(\ell-1-i^{\Sigma}_{[1,\ell-1]}\right)(2-s)}}$$

$r = i^{\Sigma}_{[1,\ell-1]} + u$

$t = i^{\Sigma}_{[\ell+1,x]} + w$

$$\text{III} = \sum_{x\in[1,\cancel{u-1}\,u]} \; \sum_{\ell\in[1,x]} \; \sum_{\substack{(i_j)_{j\in[1,x]\setminus\{\ell\}} \geq (1)_j \\ i^{\Sigma}_{[1,\ell-1]} + i^{\Sigma}_{[\ell+1,x]} \leq u-1}} \; \sum_{\substack{(u,s,w) \geq (0,2,0) \\ u+s+w = u - i^{\Sigma}_{[1,\ell-1]} - i^{\Sigma}_{[\ell+1,x]} =: u'}}$$

or 1, for $\tilde{m}_1 = 0$ anyhow

$$\cdot (-1)^{\left(i^{\Sigma}_{[1,\ell-1]}+u\right) + s\left(i^{\Sigma}_{[\ell+1,x]}+w\right)} \left\lfloor (1-i_j)_{j\neq\ell} \,,\, (i_j)_{j\neq\ell} \right\rfloor$$

replacing $\lfloor (1-i_j)_j \,,\, (i_j)_j \rfloor$

$$\cdot (-1)^{(1-(u+1+w))\, i^{\Sigma}_{[\ell+1,x]} + \left(\ell-1-i_{[1,\ell-1]}\right)(u+1+w)}$$

$$\cdot \left(q_{i_{[1,\ell-1]}} \otimes \left(\mathrm{id}^{\otimes u} \otimes \hat{m}_s \otimes \mathrm{id}^{\otimes w}\right) q_{u+1+w} \otimes q_{i_{[\ell+1,x]}} \right) m_x$$

$$= \sum_{x\in[1,u]} \; \sum_{\ell\in[1,x]} \; \sum_{\substack{(i_j)_{j\neq\ell} \geq (1)_j \\ i^{\Sigma}_{[1,\ell-1]} + i^{\Sigma}_{[\ell+1,x]} \leq u-1}} \; \sum_{\substack{(u,s,w) \geq (0,1,0) \\ u+s+w = u'}} (-1)^{\left(\ell-1-i^{\Sigma}_{[1,\ell-1]}\right)s + i^{\Sigma}_{[1,\ell-1]} + s\, i^{\Sigma}_{[1,\ell-1]}} \lfloor (1-i_j)_j \,,\, (i_j)_j \rfloor$$

$$\cdot (-1)^{\left(i_{[1,\ell-1]} + i_{[\ell+1,x]}\right)(u+1+w) + (\ell-1)(u+1+w)} \cdot (-1)^{i^{\Sigma}_{[\ell+1,x]}}$$

$$(-1)^{u+sw} \left(q_{i_{[1,\ell-1]}} \otimes \left(\mathrm{id}^{\otimes u} \otimes \hat{m}_s \otimes \mathrm{id}^{\otimes w}\right) q_{u+1+w} \otimes q_{i_{[\ell+1,x]}} \right)$$

Sign comparison :

From $\underline{\text{III}}$:

$$s\,(\cancel{\ell-1} + \cancel{i^{\Sigma}_{[1,\ell-1]}} + \cancel{i^{\Sigma}_{[\ell+1,x]}}) + \cancel{i^{\Sigma}_{[1,\ell-1]}} + \cancel{i^{\Sigma}_{[\ell+1,x]}}$$

$$+ \left(i^{\Sigma}_{[1,\ell-1]} + i^{\Sigma}_{[\ell+1,x]}\right)(n' \cancel{-s} \cancel{+1}) + (\ell-1)(n' \cancel{-s} + 1)$$

$$\equiv_2 \quad i^{\Sigma}_{[1,x]\setminus\{\ell\}} \cdot n' + (\ell-1)(n'+1)$$

From I : $\quad u n' + i^{\Sigma}_{[1,u+w]} \cdot n' + u$

Put $u = \ell - 1 \quad \rightsquigarrow$ ok

$w = x - \ell$

$x = u + w + 1$

Ad Claim 3

$$\circledast \overset{!}{=} \sum_{\substack{(r,s,t)\geq(0,2,0) \\ r+s+t=n \\ (r,t)\neq(0,0)}} \sum_{\substack{(u,v,w)\geq(0,1,0) \\ u+v+w=r+1+t}} (-1)^{r+st+u+vw} \big(\mathrm{id}^{\otimes r}\otimes \underset{\text{position } r+1}{\tilde{m}_s} \otimes \mathrm{id}^{\otimes t}\big)\cdot\big(\mathrm{id}^{\otimes u}\otimes\tilde{m}_v\otimes \mathrm{id}^{\otimes w}\big)$$

or 1, $\tilde{m}_1 = 0$ anyhow

signs disregarded

$$= \underbrace{\sum_{\substack{(r,s,t)\geq(0,1,0) \\ r+s+t=n \\ (r,t)\neq(0,0)}} \sum_{\substack{(u,v,w)\geq(0,1,0) \\ u+v+w=r+1+t \\ r+1\in[1,u]}} (-1)^{r+st+u+vw} \big(\mathrm{id}^{\otimes r}\otimes\tilde{m}_s\otimes \mathrm{id}^{\otimes u-r-1}\otimes\tilde{m}_v\otimes \mathrm{id}^{\otimes w}\big)}$$

$$+ \sum_{\substack{(r,s,t)\geq(0,1,0) \\ r+s+t=n \\ (r,t)\neq(0,0)}} \sum_{\substack{(u,v,w)\geq(0,1,0) \\ u+v+w=r+1+t \\ r+1\in[u+v+1,u+v+w]}} (-1)^{r+st+u+vw+sv} \big(\mathrm{id}^{\otimes u}\otimes\tilde{m}_v\otimes \mathrm{id}^{\otimes w-t-1}\otimes\tilde{m}_s\otimes \mathrm{id}^{\otimes t}\big)$$

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$$\underbrace{+ \sum_{\substack{(r,s,t)\geq(0,1,0) \\ r+s+t=n \\ (r,t)\neq(0,0)}} \sum_{\substack{(u,v,w)\geq(0,1,0) \\ u+v+w=r+1+t \\ r+1\in[u+1,u+v]}} (-1)^{r+st+u+vw} \big(\mathrm{id}^{\otimes u}\otimes\big(\mathrm{id}^{\otimes r-u}\otimes\tilde{m}_s\otimes \mathrm{id}^{\otimes t-w}\big)\tilde{m}_v \otimes \mathrm{id}^{\otimes w}\big)}_{=0 \text{ by Stasheff for } (\tilde{m}_k)_k}$$

$$= \ldots$$

$r=a,\ s=b,\ v=d,\ w=e,\ u-r-1=c,\ t=n-a-b=\underbrace{c+d+e}_{\substack{>0 \\ \text{anyhow}}}$

$u=c+r+1=c+a+1$

$$= \sum_{(a,b,c,d,e)\geq(0,1,0,1,0)} (-1)^{a+b(c+d+e)+c+a+1+de}$$

$$\cdot \left(id^{\otimes a} \otimes \tilde{m}_b \otimes id^{\otimes c} \otimes \tilde{m}_d \otimes id^{\otimes e}\right)$$

$u=a,\ v=b,\ w-t-1=c,\ s=d,\ t=e,\ r=n-s-t=n-d-e$
$=\underbrace{a+b+c}_{\substack{>0 \\ \text{anyhow}}}$

$w=c+t+1=c+e+1$

$$+ \sum_{(a,b,c,d,e)\geq(0,1,0,1,0)} (-1)^{(a+b+c)+de+a+b+b(c+e+1)+db}$$

$$\cdot \left(id^{\otimes a} \otimes \tilde{m}_b \otimes id^{\otimes c} \otimes \tilde{m}_d \otimes id^{\otimes e}\right)$$

$$= 0$$

sign comparison

$$a+bc+bd+be+c+a+de$$

$$\overset{!}{\equiv}_2\ a+b+c+de+a+b+bc+be+b+db$$