# $A_\infty$-categories

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## Inhalt

# Introduction

## Problem

Suppose given a commutative ring $R$. Suppose given a finite group $G$.

Suppose given $RG$-modules $X$, $Y$, $Z$.

What $RG$-modules $M$ have a filtration with subfactors $X$, $Y$ and $Z$ ?

I.e. we ask for $RG$-modules $M$ having a chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 = 0$ of submodules such that $M_0/M_1 \simeq X$ and $M_1/M_2 \simeq Y$ and $M_2/M_3 \simeq Z$.

$$\begin{array}{rcl} M & = & M_0 \\ & & \big|\, {\scriptstyle X} \\ & & M_1 \\ & & \big|\, {\scriptstyle Y} \\ & & M_2 \\ & & \big|\, {\scriptstyle Z} \\ 0 & = & M_3 \end{array}$$

The short exact sequences of the form $Z \to M_1 \to Y$ are controlled by $\mathrm{Ext}^1_{RG}(Y, Z)$.

The short exact sequences of the form $M_1 \to M \to X$ are controlled by $\mathrm{Ext}^1_{RG}(X, M_1)$.

The latter $R$-module might be difficult to cope with, because we need to use $M_1$ as input.

It would be preferable to make do with $\mathrm{Ext}^1_{RG}(X, X)$, $\mathrm{Ext}^1_{RG}(X, Y)$, $\mathrm{Ext}^1_{RG}(Y, X)$, ..., $\mathrm{Ext}^1_{RG}(Z, Z)$.

The Yoneda product maps e.g.

$$m_2 \; : \; \mathrm{Ext}^1_{RG}(X, Y) \times \mathrm{Ext}^1_{RG}(Y, Z) \longrightarrow \mathrm{Ext}^1_{RG}(X, Z) \; .$$

But to reconstruct modules such as $M$, we also need higher multiplication maps such as e.g.

$$m_3 \; : \; \mathrm{Ext}^1_{RG}(Z, X) \times \mathrm{Ext}^1_{RG}(X, X) \times \mathrm{Ext}^1_{RG}(X, Y) \longrightarrow \mathrm{Ext}^1_{RG}(Z, Y) \; ,$$

and similarly $m_4$ , $m_5$ , ... These data form an $\mathrm{A}_\infty$-category.

The aimed-for reconstruction will be achieved with KELLER's filt-construction.

## Kadeishvili

### A cohomology algebra

Given a finite group $G$, we can consider its cohomology algebra $\mathrm{H}(G;R)$ with coefficients in a commutative ring $R$. ([1])

This cohomology algebra can be calculated as follows. Let $P$ be a projective resolution of $R$ over $RG$. Let $P^{[k]}$ arise from $P$ by a shift of $k$ steps to the left, where $k \in \mathbf{Z}_{\geqslant 0}$, and by multiplying each differential by $(-1)^k$. Let $\dot{P}$ denote the graded $RG$-module underlying $P$, forgetting the differentials. Form the graded algebra $\mathrm{DG}(G;R)$ having

$$\mathrm{DG}^k(G;R) \;:=\; {}_{RG\text{-grad}}(\dot{P},\dot{P}^{[k]})\;,$$

where the latter stands for the $R$-module of morphisms in the category $RG$-grad of graded $RG$-modules.

Remembering the differentials of $P$ again, $\mathrm{DG}(G;R)$ becomes a differential graded algebra. Its cohomology algebra is $\mathrm{H}(\mathrm{DG}(G;R)) = \mathrm{H}(G;R)$.

### If $R$ is a field: A quasiisomorphism of $\mathrm{A}_\infty$-algebras

Consider the case that $R$ is a field. Suppose given a differential graded algebra $D$ over the field $R$, with differential $d = m_1^D : D = D^{\otimes 1} \longrightarrow D$ and multiplication $m_2^D : D^{\otimes 2} \longrightarrow D$.

Then its cohomology algebra $\mathrm{H}(D)$ carries not only a multiplication map

$$m_2^{\mathrm{H}(D)} \;:\; \mathrm{H}(D)^{\otimes 2} \;\longrightarrow\; \mathrm{H}(D)\;,$$

but also higher multiplication maps

$$m_n^{\mathrm{H}(D)} \;:\; \mathrm{H}(D)^{\otimes n} \;\longrightarrow\; \mathrm{H}(D) \qquad \text{for } n \geqslant 3$$

and

$$m_1^{\mathrm{H}(D)} \;:=\; 0 \;:\; \mathrm{H}(D)^{\otimes 1} \;\longrightarrow\; \mathrm{H}(D)\;,$$

fitting together to turn $\mathrm{H}(D)$ into an $\mathrm{A}_\infty$-algebra. An $\mathrm{A}_\infty$-algebra with $m_1 = 0$ is called minimal.

But also $D$ can be viewed as a $\mathrm{A}_\infty$-algebra by letting $m_n^D := 0 : D^{\otimes n} \longrightarrow D$ for $n \geqslant 3$.

Kadeishvili's Theorem states that there is a quasiisomorphism from $D$ to $\mathrm{H}(D)$, i.e. a morphism of $\mathrm{A}_\infty$-algebras

$$D \longrightarrow \mathrm{H}(D)$$

that induces an isomorphism on cohomology [**1,** Th. 1]. More precisely, it states that the $\mathrm{A}_\infty$-structure on $\mathrm{H}(D)$ can be chosen in such a way that such a quasiisomorphism emerges. The resulting $\mathrm{A}_\infty$-algebra is, of course, determined uniquely up to quasiisomorphism.

[1] In the literature, $\mathrm{H}(G;R)$ is often written $\mathrm{H}^*(G;R)$.

The assumption of $R$ being a field is used in this process to ensure that every surjective $R$-linear map is a retraction. This prevents us from directly generalising to $R$ being a discrete valuation ring, say.

In particular, KADEISHVILI's Theorem can be applied to $D = \mathrm{DG}(G; R)$ and $\mathrm{H}(D) = \mathrm{H}(G; R)$ in the case of $R = \mathbf{F}_p$, where $p$ is a prime divisor of $|G|$, but not in the case of $R = \mathrm{Z}_{(p)}$.

### Generalisation to arbitrary ground rings

To generalise to an arbitrary commutative ring $R$, we replace the cohomology modules by projective resolutions over $R$. I.e. given a differential graded algebra $D$, we choose an augmented projective resolution

$$\ldots \longrightarrow P_2^i \longrightarrow P_1^i \longrightarrow P_0^i \longrightarrow \mathrm{H}^i(D) \longrightarrow 0\,,$$

as suggested by KELLER.

SCHMID's Theorem states, roughly put, that there exists a minimal $\mathrm{eA}_\infty$-algebra structure on $\bigoplus_{i,j} P_j^i$ such that there exists a quasiisomorphism to $D$ [**5,** Th. 90]. Here, on the one hand, an $\mathrm{eA}_\infty$-algebra structure is a refinement of an $\mathrm{A}_\infty$-algebra structure; on the other hand, $\mathrm{eA}_\infty$-minimality is a weakening of $\mathrm{A}_\infty$-minimality.

In particular, SCHMID's Theorem can be applied to $D = \mathrm{DG}(G; R)$ and $\mathrm{H}(D) = \mathrm{H}(G; R)$ in the case of $R = \mathbf{Z}_{(p)}$.

SCHMID's procedure is similar to that of SAGAVE [**3**], one of the differences being that Sagave resolves once more in the process, while Schmid sticks to the initially chosen projective resolutions; cf. [**3,** Th. 1.1, Rem. 4.14].

## Modules

### From $\mathrm{A}_\infty$-algebras to $\mathrm{A}_\infty$-categories

To fix ideas, we consider $RG$-modules again.

Note that $\mathrm{H}(G; R) = \mathrm{Ext}_{RG}(R, R)$, where $R$ is the trivial $RG$-module.

Suppose given $RG$-modules $S_1, \ldots, S_n$. To take these several objects into account, we refine the notion of an $\mathrm{A}_\infty$-algebra to that of an $\mathrm{A}_\infty$-category, in that we endow an $\mathrm{A}_\infty$-algebra with a categorical grading, which is, in a sense, a fixed Peirce decomposition.

If $R$ is a field, the categorical version of KADEISHVILI's Theorem establishes the structure of a minimal $\mathrm{A}_\infty$-category on $\bigoplus_{\alpha,\beta\in[1,n]} \mathrm{Ext}_{RG}(S_\alpha, S_\beta)$.

Over arbitrary $R$, the categorical version of SCHMID's Theorem establishes the structure of a minimal $\mathrm{eA}_\infty$-category on $\bigoplus_{\alpha,\beta\in[1,n]} P_{\alpha,\beta}$, where $P_{\alpha,\beta}$ is a projective resolution of $\mathrm{Ext}_{RG}(S_\alpha, S_\beta)$ over $R$.

### The filt-construction

A finitely generated $RG$-module is called $(S_\alpha)_\alpha$-filtered, if it has a filtration whose subfactors occur in $(S_\alpha)_\alpha$, up to isomorphy, repetition allowed, ordered arbitrarily.

If $R$ is a field, KELLER's filt-construction recovers the full subcategory of $(S_\alpha)_\alpha$-filtered modules in $RG$-mod in terms of the $\mathrm{A}_\infty$-category $\bigoplus_{\alpha,\,\beta\,\in\,[1,n]} \mathrm{Ext}_{RG}(S_\alpha, S_\beta)$; cf. [**2,** §7.7, Theorem]. In particular, if the modules $S_\alpha$ represent the isoclasses of simple modules, we recover the whole category $RG$-mod.

SCHMID generalised this to arbitrary $R$, using the $\mathrm{eA}_\infty$-category $\bigoplus_{\alpha,\,\beta\,\in\,[1,n]} P_{\alpha,\beta}$; cf. [**5,** Th. 131]. ([2])

### A desirable future application

We can e.g. take $G = \mathrm{S}_n$ and $R = \mathbf{Z}_{(p)}$ for some prime divisor of $n!$ and let $S_\alpha$ run through Specht modules, or certain submodules thereof. One might ask whether, in small examples, the shape of the indecomposable projective modules can be explained through the said $\mathrm{eA}_\infty$-category; just as for $\mathbf{Z}_{(5)}\mathrm{S}_5$, the shape of certain indecomposable projective modules can be explained as being glued from two Specht modules via an element of $\mathrm{Ext}^1$; cf. [**4,** Ex. 7].

## Organisatorial matters

We essentially follow the master thesis of STEPHAN SCHMID [**5**]. Responsibility for mistakes and obscurities in this script remains with me. I will be thankful for any hints on this matter.

We presuppose Algebra and some basic knowledge from Homological Algebra, which will be recalled in the exercises if necessary.

Sometimes we refer to exercises and solutions, so they are to be viewed as part of the script.

Because of running modifications, it is recommended to work with the file during the semester and to print a paper copy only afterwards.

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[2]Actually, conditions on $R$ do not play a role in this assertion on the equivalence from the filt-construction to the category of filtered modules; SCHMID gives a somewhat straightened proof and ensures that the equivalence from the filt-construction to filtered modules can be applied to his $\mathrm{eA}_\infty$-category $\bigoplus_{\alpha,\,\beta\,\in\,[1,n]} P_{\alpha,\beta}$.

## Conventions

Let $\mathcal{C}$ be a category.

- Given $a,\, b \in \mathbf{Z}$, we write

  $$[a,b] \;:=\; \{z \in \mathbf{Z} \;:\; a \leqslant z \leqslant b\}$$

  for the integral interval.

- We stipulate that $-\infty < a < \infty$ for $a \in \mathbf{Z}$. We write

  $$\begin{aligned} [a,\infty] \;&:=\; \{z \in \mathbf{Z} \;:\; a \leqslant z\} \cup \{\infty\} \\ [-\infty,a] \;&:=\; \{-\infty\} \cup \{z \in \mathbf{Z} \;:\; z \leqslant a\}\ . \end{aligned}$$

- XXX $\mathbf{Z}_{\geqslant n}$, $\mathbf{Z}_{<n}$

- Given a set $X$, "for $x \in X$" means "for all $x \in X$".

- We use the symbol $\sqcup$ for the (interior and exterior) disjoint union of sets and for the concatenation of tuples.

- All categories are suppose to be small (with respect to a given universe). I.e. we have the sets $\operatorname{Mor}(\mathcal{C})$ and $\operatorname{Ob}(\mathcal{C})$.

- We have source and target maps, $\mathrm{s}_{\mathcal{C}}\,, \mathrm{t}_{\mathcal{C}} : \operatorname{Mor}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{C})$, respectively, mapping a morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ to $f\mathrm{s}_{\mathcal{C}} = X$ and to $f\mathrm{t}_{\mathcal{C}} = Y$, respectively.

- For $X \in \operatorname{Ob}(\mathcal{C})$, we write $\mathrm{id} = \mathrm{id}_X$ for the identity morphism on $X$. In some contexts, we also write $1 = 1_X = \mathrm{id}_X$.

- Given a category $\mathcal{C}$ and $X, Y \in \operatorname{Ob}(\mathcal{C})$, we write ${}_{\mathcal{C}}(X,Y) = \{\, f \in \operatorname{Mor}(\mathcal{C}) : f\mathrm{s}_{\mathcal{C}} = X \text{ and } f\mathrm{t}_{\mathcal{C}} = Y \,\}$.

- Given $k \geqslant 0$, we write $X^{\times k} := \prod_{i \in [1,k]} X$ for the $k$-fold cartesian product. We identify along $X^{\times 1} \to X$, $(x) \mapsto x$. Moreover, $X^{\times 0} = \{\, () \,\}$ is a one-element set.

- XXX tensor product $\bigotimes_{i\in I} M_i$, elementary tensors $(m_i)^{\otimes}_{i\in I}$; if $I = [a,b]$ interval, then $\bigotimes_{i\in[a,b]} M_i =: M_a \otimes M_{a+1} \otimes \cdots \otimes M_b$ and $(m_i)^{\otimes}_{i\in I} =: m_a \otimes m_{a+1} \otimes \cdots \otimes m_b$ XXX as far as possible just as for cartesian products

- XXX tensor product of $R$-modules associative via identification, additive via identification, $R \otimes M = M$ via identification

- XXX concerning tensor products, we freely use [Ritter]

- XXX abbreviate $v_1 \otimes \ldots \otimes v_k = v^{\otimes}_{[1,k]}$ XXX

- XXX$R$-linear preadditive categoryXXX plus Example: Peirce decomposition [Ritter]
- XXX composition of morphisms naturally, composition of functors traditionally, with some exceptions, e.g. for certain standard maps, for maps written in index notation or for shift functors $z \mapsto z[i]$ XXX
- XXX Let Cat be the (1-)category of categories. Let Set be the category of set
- XXX poset: partially ordered set, category of posets and monotone maps: Poset, (full) subposet (full may be omitted), $X_{\leqslant \xi}$
- XXX category of functors $[\![\mathcal{C}, \mathcal{D}]\!]$.
- XXX terminal category !
- XXX inverse often $f^-$
- XXX automorphism of a category
- XXX complex
- XXX exact, short exact in $B$-Mod
- XXX augmented projective resolution

## Fixing the ground ring $R$

Let $R$ be a commutative ring.

By a *module* we understand an $R$-module.

By a *linear map* we understand an $R$-linear map.

We write $\otimes := \underset{R}{\otimes}$ .

By an *algebra* we understand an $R$-algebra.

By a *linear category* we understand an $R$-linear preadditive category.

By a *linear additive category* we understand an $R$-linear additive category.

By a *linear functor* we understand an $R$-linear additive functor.

# Chapter 1

# Kadeishvili

## 1.1 Gradings

### 1.1.1 Grading categories

Let $\mathcal{Z}$ be a category.

**Example 1**

(1) Let $G$ be a group. Then, by abuse of notation, $G$ can be considered as a category with $\mathrm{Ob}(G) = \{G\}$, $\mathrm{Mor}(G) = G$ and composition given by multiplication.

(2) We may specialise (1) to $G = \mathbf{Z}$. So $\mathrm{Ob}(\mathbf{Z}) = \{\mathbf{Z}\}$ and $\mathrm{Mor}(\mathbf{Z}) = \mathbf{Z}$, composition being given by addition. E.g. we get the commutative triangle

$$\begin{array}{ccc} \mathbf{Z} & \xrightarrow{\;8\;} & \mathbf{Z} \\ {\scriptstyle 3}\searrow & & \nearrow{\scriptstyle 5} \\ & \mathbf{Z} & \end{array}$$

(3) Conversely, if $|\,\mathrm{Ob}(\mathbf{Z})| = 1$ and each morphism in $\mathcal{Z}$ is an isomorphism, we may consider $\mathcal{Z}$ as a group. More precisely, $\mathrm{Mor}(\mathcal{Z})$ is a group with multiplication given by composition.

More generally, if $|\,\mathrm{Ob}(\mathcal{Z})| = 1$, we may consider $\mathcal{Z}$ to be a monoid.

(4) Let $I$ be a set. By abuse of notation, let $I^{\times 2}$ denote the *pair category* on $I$, having $\mathrm{Ob}(I^{\times 2}) = I$ and $\mathrm{Mor}(I^{\times 2}) = I^{\times 2}$.

A morphism $(i, j) \in \mathrm{Mor}(X^{\times 2})$ has source $(i, j)\mathrm{s}_{I^{\times 2}} = i$ and target $(i, j)\mathrm{t}_{I^{\times 2}} = j$.

The composite of the morphisms

$$x \xrightarrow{(x,y)} y \xrightarrow{(y,z)} z$$

is

$$x \xrightarrow{(x,z)} z \; .$$

So the identity on $x \in \mathrm{Ob}(X^{\times 2})$ is $\mathrm{id}_x = (x, x)$.

(5) Write $\mathrm{Ob}^{\times 2}(\mathcal{Z}) := (\mathrm{Ob}(\mathcal{Z}))^{\times 2}$ for the pair category on the set $\mathrm{Ob}(\mathcal{Z})$.

**Definition 2** A *shift* $S$ on $\mathcal{Z}$ is a tuple of maps

$$S \;=\; \big( {}_{\mathcal{Z}}(X,Y) \xrightarrow{S_{X,Y}} {}_{\mathcal{Z}}(X,Y) \big)_{X,\,Y \,\in\, \mathrm{Ob}(\mathcal{Z})}$$

such that properties $(1,2)$ hold.

(1) The map $S_{X,Y}$ is bijective for $X,\ Y \in \mathrm{Ob}(\mathcal{Z})$.

(2) Given $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{Z}$, we have

$$(a \cdot b) S_{X,Z} \;=\; a S_{X,Y} \cdot b \;=\; a \cdot b S_{Y,Z} \; .$$

We often write $aS := aS_{X,Y}$ for $X \xrightarrow{a} Y$ in $\mathcal{Z}$.

We often write $a[k] := aS^k$ for $X \xrightarrow{a} Y$ in $\mathcal{Z}$ and $k \in \mathbf{Z}$.

Note that $S$ is not required to be a functor.

**Example 3** We have the identical shift $(\mathrm{id}_{\,{}_{\mathcal{Z}}(X,Y)})_{X,\,Y \,\in\, \mathrm{Ob}(\mathcal{Z})}$ on $\mathcal{Z}$.

**Definition 4** Suppose given a shift $S$ on $\mathcal{Z}$.

A *degree function* on $(\mathcal{Z}, S)$ is a map $\deg : \mathrm{Mor}(\mathcal{Z}) \to \mathbf{Z}$ such that properties $(1,2)$ hold.

(1) Given $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{Z}$, we have

$$(a \cdot b) \deg \;=\; a \deg + b \deg \; .$$

(2) Given $X \xrightarrow{a} Y$ in $\mathcal{Z}$, we have

$$(aS) \deg \;=\; a \deg + 1 \; .$$

So $(a[k]) \deg = a \deg + k$ for $X \xrightarrow{a} Y$ in $\mathcal{Z}$ and $k \in \mathbf{Z}$.

**Definition 5** The category $\mathcal{Z}$, together with a shift $S$ on $\mathcal{Z}$ and a degree function deg on $(\mathcal{Z}, S)$, is called a *grading category*. Cf. Definitions 2 and 4.

We often abbreviate $\mathcal{Z} = (\mathcal{Z}, S, \deg)$.

**Example 6**

(1) The category $\mathcal{Z} := \mathbf{Z}$ carries the shift $(\mathbf{Z} \overset{i}{\to} \mathbf{Z})S := (\mathbf{Z} \overset{i+1}{\longrightarrow} \mathbf{Z})$ for $i \in \mathbf{Z}$, whence $i[k] = i + k$ for $i,\ k\ \in\ \mathbf{Z}$.

Then $(\mathbf{Z}, S)$ carries the degree function $\deg = \mathrm{id}_{\mathbf{Z}}$. So $i \deg = i$ for $i \in \mathbf{Z}$.

(2) We generalise (1). Let $\mathcal{C}$ be a category. Consider the category $\mathcal{Z} := \mathbf{Z} \times \mathcal{C}$.

The example in (1) can be considered as the particular case $\mathcal{C} = !$.

The category $\mathcal{Z}$ carries the shift

$$(\mathbf{Z} \overset{i}{\to} \mathbf{Z},\ X \overset{a}{\to} Y)S \ :=\ (\mathbf{Z} \overset{i+1}{\longrightarrow} \mathbf{Z},\ X \overset{a}{\to} Y)$$

for $(i, a) = (\mathbf{Z} \overset{i}{\to} \mathbf{Z},\ X \overset{a}{\to} Y)$ in $\mathrm{Mor}(\mathcal{Z}) = \mathrm{Mor}(\mathbf{Z}) \times \mathrm{Mor}(\mathcal{C})$.

So $(i, a)[k] = (i + k, a)$ for $i,\ k\ \in\ \mathbf{Z}$.

For $(i, a),\ (j, b)\ \in\ \mathrm{Mor}(\mathcal{Z})$ with $X \overset{a}{\to} Y \overset{b}{\to} Z$ in $\mathcal{C}$, we obtain in fact the following.

$$\begin{array}{rcl} ((i,a)\cdot(j,b))S &=& (i+j, a\cdot b)S &=& (i+j+1, a\cdot b) \\ (i,a)S\cdot(j,b) &=& (i+1,a)\cdot(j,b) &=& (i+j+1, a\cdot b) \\ (i,a)\cdot(j,b)S &=& (i,a)\cdot(j+1,b) &=& (i+j+1, a\cdot b) \end{array}$$

Then $(\mathcal{Z}, S)$ carries the degree function

$$\begin{array}{rcl} \mathrm{Mor}(\mathcal{Z}) & \overset{\deg}{\longrightarrow} & \mathbf{Z} \\ (i,a) & \longmapsto & (i,a)\deg \ :=\ i\ . \end{array}$$

For $(i, a),\ (j, b)\ \in\ \mathbf{Z} \times \mathcal{C}$ with $X \overset{a}{\to} Y \overset{b}{\to} Z$ in $\mathcal{C}$, we obtain in fact the following.

$$\begin{array}{rcl} ((i,a)\cdot(j,b))\deg &=& (i+j, a\cdot b)\deg &=& i+j &=& (i,a)\deg + (j,b)\deg \\ ((i,a)S)\deg &=& (i+1,a)\deg &=& i+1 &=& (i,a)\deg + 1 \end{array}$$

**Definition 7** Suppose given $n \in \mathbf{Z}_{\geqslant 1}$.

A tuple $(y_i)_{i\in[1,n]} \in \mathrm{Mor}(\mathcal{Z})^{\times n}$ is called *composable* if $y_i \mathrm{t}_{\mathcal{Z}} = y_{i+1} \mathrm{s}_{\mathcal{Z}}$ for $i \in [1, n-1]$.

We often abbreviate $\underline{y} = (y_i)_{i\in[1,n]}$.

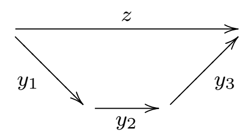
**Definition 8** Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $n \in \mathbf{Z}_{\geqslant 1}$. Let

$$\mathrm{fact}_n(z) \ :=\ \{\, (y_i)_{i\in[1,n]} \in \mathrm{Mor}(\mathcal{Z})^{\times n} \ :\ (y_i)_{i\in[1,n]} \text{ composable and } z = y_1 \cdot y_2 \cdots y_n \,\}$$

be the set of *factorisations* of $z$ of length $n$.

**Example 9**

(1) Let $z \in \mathrm{Mor}(\mathcal{Z})$. Then $\mathrm{fact}_3(z)$ consists of the diagrams $\xrightarrow{y_1}\xrightarrow{y_2}\xrightarrow{y_3}$ with $y_1 \cdot y_2 \cdot y_3 = a$.




(2) Let $z \in \mathrm{Mor}(\mathcal{Z})$. We have $\mathrm{fact}_1(z) = \{z\}$.

(3) For $z \in \mathbf{Z}$, we have $\mathrm{fact}_2(z) = \{\, (y_1, y_2) \in \mathbf{Z}^{\times 2} \;:\; y_1 + y_2 = z \,\}$.

### 1.1.2 Graded modules

Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category; cf. Definition 5.

**Definition 10**

(1) A *$\mathcal{Z}$-graded module* is a map $M : \mathrm{Mor}(\mathcal{Z}) \to \mathrm{Ob}(R\text{-Mod})$, $z \mapsto M^z$, often written $(M^z)_{z\in\mathrm{Mor}(\mathcal{Z})}$ or just $(M^z)_z$.

(2) Suppose given a $\mathcal{Z}$-graded module $M$. Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $m \in M^z$. We write

$$m \deg \;:=\; z \deg$$

for the *degree* of $m$.

(3) Suppose given $\mathcal{Z}$-graded modules $L$ and $M$. A *($\mathcal{Z}$-)graded linear map* $f$ from $L$ to $M$ is a tuple of linear maps $(L^z \xrightarrow{f^z} M^z)_{z\in\mathrm{Mor}\,\mathcal{Z}}$, often written just $(L^z \xrightarrow{f^z} M^z)_z$ or $(f^z)_z$. So

$$(L \xrightarrow{f} M) \;=\; (L^z \xrightarrow{f^z} M^z)_z\,.$$

(4) The category

$$\mathcal{Z}\text{-grad}_0$$

has the $\mathcal{Z}$-graded modules as objects and the graded linear maps as morphisms.

The composite of the $\mathcal{Z}$-graded linear maps $L \xrightarrow{f} M \xrightarrow{g} N$ is given by

$$f \cdot g \;=\; ((f \cdot g)^z)_{z\in\mathrm{Mor}(\mathcal{Z})} \;:=\; (f^z \cdot g^z)_{z\in\mathrm{Mor}(\mathcal{Z})}\,.$$

We have the identity $\mathrm{id}_M := (\mathrm{id}_{M^z})_{z\in\mathrm{Mor}(\mathcal{Z})}$.

(5) Suppose given a set $I$ and $\mathcal{Z}$-graded modules $M_i$ for $i \in I$. Define the *(external) direct sum* of the tuple $(M_i)_{i\in I}$ as the $\mathcal{Z}$-graded module

$$\bigoplus_{i\in I} M_i \;:=\; \Big(\bigoplus_{i\in I} M_i^z\Big)_{z\in\mathrm{Mor}(\mathcal{Z})}\,.$$

for $z \in \mathrm{Mor}(\mathcal{Z})$.

**Example 11** Let $\mathcal{B}$ be a linear category. Let $\mathcal{Z} := \mathbf{Z} \times \mathrm{Ob}^{\times 2}(\mathcal{B})$.

Then $\mathrm{Mor}(\mathcal{Z}) = \mathbf{Z} \times \mathrm{Mor}(\mathrm{Ob}^{\times 2}(\mathcal{B}))$ and so

$$\left( \left\{ \begin{array}{ll} {}_{\mathcal{B}}(X,Y) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{array} \right\} \right)_{(i,(X,Y)) \,\in\, \mathbf{Z} \times \mathrm{Mor}(\mathrm{Ob}^{\times 2}(\mathcal{B}))}$$

is a $(\mathbf{Z} \times \mathrm{Ob}^{\times 2}(\mathcal{B}))$-graded module.

In particular, given an algebra $B$, we obtain a linear category, abusively again denoted by $B$, having $\mathrm{Ob}(B) = \{B\}$ and $\mathrm{Mor}(B) = B$, composition given by multiplication. Then

$$\left( \left\{ \begin{array}{ll} B & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{array} \right\} \right)_{i \,\in\, \mathbf{Z}}$$

is a $\mathbf{Z}$-graded module, concentrated in degree 0.

**Definition 12** Given a $\mathcal{Z}$-graded module $M$, we let $SM$ be the $\mathcal{Z}$-graded module defined by

$$(SM)^z \;:=\; M^{zS}$$

for $z \in \mathrm{Mor}(\mathcal{Z})$.

Given a $\mathcal{Z}$-graded linear map $L \xrightarrow{f} M$, we let $SL \xrightarrow{Sf} SM$ be the $\mathcal{Z}$-graded linear map defined by

$$((SL)^z \xrightarrow{(Sf)^z} (SM)^z) \;:=\; (L^{zS} \xrightarrow{f^{zS}} M^{zS})$$

for $z \in \mathrm{Mor}(\mathcal{Z})$.

We have a functor

$$\begin{array}{rcl} \mathcal{Z}\text{-grad}_0 & \xrightarrow{S} & \mathcal{Z}\text{-grad}_0 \\ (L \xrightarrow{f} M) & \mapsto & (SL \xrightarrow{Sf} SM) \end{array}$$

This functor is an automorphism of $\mathcal{Z}$-grad$_0$ ; cf. Problem4.(1).

We often write $(L^{[k]} \xrightarrow{f^{[k]}} M^{[k]}) := (S^k L \xrightarrow{S^k f} S^k M)$ for $L \xrightarrow{f} M$ in $\mathcal{Z}$-grad$_0$ and $k \in \mathbf{Z}$.

So $(M^{[k]})^z = M^{z[k]}$ and $(f^{[k]})^z = f^{z[k]}$ for $z \in \mathrm{Mor}(\mathcal{Z})$.

**Definition 13** Suppose given $\mathcal{Z}$-graded modules $L$, $M$ and $N$.

A *shift-graded linear map* of *degree* $k$ from $L$ to $M$ is a pair $(f,k)$, where $f : L \to M^{[k]}$ is a graded linear map.

So $f = (L^z \xrightarrow{f^z} M^{z[k]})_{z \in \mathrm{Mor}(\mathcal{Z})}$ .

Write $(f,k)\,\mathrm{deg} := k$.

Suppose given shift-graded linear maps $L \xrightarrow{(f,k)} M \xrightarrow{(g,\ell)} N$. Then $L \xrightarrow{f} M[k]$ and $M[k] \xrightarrow{g[k]} N[k{+}\ell]$ in $\mathcal{Z}$-grad$_0$ , i.e. as graded linear maps. The composite of $(f,k)$ and $(g,\ell)$ is defined by

$$(f,k) \cdot (g,\ell) \;:=\; (f \cdot g[k], k+\ell) \;:\; L \longrightarrow N[k+\ell] \;.$$

We have the identity $(\mathrm{id}_M\,, 0)$ on $M$.

We call $(f,k)$ *piecewise injective* if $f^z$ is injective for $z \in \mathrm{Mor}(\mathcal{Z})$.

We call $(f,k)$ *piecewise surjective* if $f^z$ is surjective for $z \in \mathrm{Mor}(\mathcal{Z})$.

The category

$$\mathcal{Z}\text{-grad}$$

has the $\mathcal{Z}$-graded modules as objects and the shift-graded linear maps as morphisms. Cf. Problem 4.(2).

By abuse of notation, we let

$$(SL \xrightarrow{S(f,k)} SM) := (SL \xrightarrow{(Sf,k)} SM)(Sf,k)$$

for $L \xrightarrow{(f,k)} M$ in $\mathcal{Z}$-grad. Then $S$ is an automorphism on $\mathcal{Z}$-grad; cf. Problem 4.(3). Accordingly, we write

$$(L^{[t]} \xrightarrow{(f,k)^{[t]}} M^{[t]}) := (L^{[t]} \xrightarrow{(f^{[t]},k)} M^{[t]})$$

for $t \in \mathbf{Z}$.

Finally, given morphisms $(f,k)$, $(g,k) \in \mathrm{Mor}(\mathcal{Z}\text{-grad})$ of the same degree and $r$, $s \in R$, we let

$$r(f,k) + s(g,k) \;:=\; (rf + sg, k)\;;$$

cf. Problem 4.(4).

**Definition.**

(1) Suppose given a $\mathcal{Z}$-graded module $M$.

Suppose given a submodule $\tilde{M}^z \subseteq M^z$ for each $z \in \mathrm{Mor}(\mathcal{Z})$. Then $\tilde{M} := (\tilde{M}^z)_{z\in\mathrm{Mor}(\mathcal{Z})}$ is called a $\mathcal{Z}$-graded submodule of $M$. We write $\tilde{M} \subseteq M$. We have the shift-graded linear inclusion map of degree 0

$$\begin{array}{rrcl} & \tilde{M} & \to & M \\ \text{at } z \in \mathrm{Mor}(\mathcal{Z}): & \tilde{M}^z & \to & M^z \\ & \tilde{m} & \mapsto & \tilde{m}\,. \end{array}$$

We may form the factor module $M^z/\tilde{M}^z$ for each $z \in \mathrm{Mor}(\mathcal{Z})$. Then $M/\tilde{M} := (M^z/\tilde{M}^z)_{z\in\mathrm{Mor}(\mathcal{Z})}$ is called the $\mathcal{Z}$-graded factor module of $M$ modulo $\tilde{M}$. We have the shift-graded linear residue-class map of degree 0

$$\begin{array}{rrcl} & M & \to & M/\tilde{M} \\ \text{at } z \in \mathrm{Mor}(\mathcal{Z}): & M^z & \to & M^z/\tilde{M}^z \\ & m & \mapsto & m + \tilde{M}^z\,. \end{array}$$

(2) Suppose given $\mathcal{Z}$-graded modules $L$ and $M$. Suppose given a shift-graded linear map $L \xrightarrow{f} M$ of degree $d$.

Let $\operatorname{Kern}(f) := (\operatorname{Kern}(f^z))_{z \in \operatorname{Mor}(\mathcal{Z})}$. Then $\operatorname{Kern}(f)$ is a $\mathcal{Z}$-graded submodule of $L$.

Let $\operatorname{Im}(f) := (\operatorname{Im}(f^z))_{z \in \operatorname{Mor}(\mathcal{Z})}$. Then $\operatorname{Im}(f)$ is a $\mathcal{Z}$-graded submodule of $M$.

Let $\operatorname{Cokern}(f) := (M^z / \operatorname{Im}(f^z))_{z \in \operatorname{Mor}(\mathcal{Z})}$. Then $\operatorname{Cokern}(f) = M / \operatorname{Im}(f)$.

Suppose given $\mathcal{Z}$-graded submodules $\tilde{L} \subseteq L$ and $\tilde{M} \subseteq M$. Write the inclusions $\tilde{L} \xrightarrow{i} L$ and $\tilde{M} \xrightarrow{j} M$. Suppose that $\operatorname{Im}(i \cdot f) \subseteq \tilde{M}$.

There exists a unique shift graded linear map $\tilde{L} \xrightarrow{f|_{\tilde{L}}^{\tilde{M}}} \tilde{M}$, called the *restriction* of $f$ to $\tilde{L}$ in the source and $\tilde{M}$ in the target, making the diagram

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ \tilde{L} & \xrightarrow{f|_{\tilde{L}}^{\tilde{M}}} & \tilde{M} \end{array}$$

commutative. In particular, $f|_{\tilde{L}}^{\tilde{M}}$ is of degree $d$.

We also write $\tilde{L}f := \operatorname{Im}(f|_{\tilde{L}}) \subseteq \tilde{M}$.

If $\tilde{L} = L$, we also write $f|^{\tilde{M}} := f|_L^{\tilde{M}}$.

If $\tilde{M} = M$, we also write $f|_{\tilde{L}} := f|_{\tilde{L}}^M$.

(3) Suppose given a $\mathcal{Z}$-graded module $M$. Suppose given a set $I$ and $\mathcal{Z}$-graded submodules $M_i \subseteq M$ for $i \in I$.

Write $\sum_{i \in I} M_i := \big(\sum_{i \in I} M_i^z\big)_z$ for the *(inner) sum* of the tuple $(M_i)_{i \in I}$ of submodules, which is a graded submodule of $M$.

Consider the following shift-graded linear map of degree 0.

$$\begin{array}{rccc} & \bigoplus_{i \in I} M_i & \xrightarrow{\varphi_{(M_i)_{i \in I}}} & \sum_{i \in I} M_i \\ \text{at } z \in \operatorname{Mor}(\mathcal{Z}) \colon & \bigoplus_{i \in I} M_i^z & \to & \sum_{i \in I} M_i^z \\ & (m_i)_{i \in I} & \mapsto & \sum_{i \in I} m_i \end{array}$$

We say that $\sum_{i \in I} M_i$ is a(n) *(inner) direct sum* of $(M_i)_{i \in I}$ if $\varphi_{(M_i)_{i \in I}}$ is an isomorphism. In this case, we also write, by abuse of notation, $\bigoplus_{i \in I} M_i := \sum_{i \in I} M_i$.

So the sum $\sum_{i \in I} M_i$ is direct if and only if the sum $\sum_{i \in I} M_i^z$ is direct for $z \in \operatorname{Mor}(\mathcal{Z})$.

(4) Suppose given a $\mathcal{Z}$-graded module $M$. Suppose given a set $I$ and $\mathcal{Z}$-graded submodules $M_i \subseteq M$ for $i \in I$.

Write $\bigcap_{i\in I} M_i := \big(\bigcap_{i\in I} M_i^z\big)_z$ , which is a graded submodule of $M$.

**Remark 14** Let $k \in \mathbf{Z}$.

Given $(f, k) \in \mathrm{Mor}(\mathcal{Z}\text{-grad})$, we often write just $f$ instead of $(f, k)$ if $k$ is known from context. Then $f \deg = k$.

In particular, we often write 0 instead of $(0, k)$ by abuse of notation.

Given a shift-graded linear map $f : L \to M$ of degree $k$, given $a \in \mathrm{Mor}(\mathcal{Z})$ and given $\ell \in L^a$, we often write $\ell f$ instead of $\ell f^a$.

**Example 15** We make use of Remark 14.

Suppose that $\mathcal{Z} = \mathbf{Z}$. A *complex* is a $\mathbf{Z}$-graded module $M$, together with a shift-graded linear map $d : M \to M$ of degree 1 such that $d^2 = 0$.

Removing the abusive language of Remark 14 again, we should write $(d, 1)$ in place of $d$.

So $d = (M^i \xrightarrow{d^i} M^{i+1})_{i\in\mathbf{Z}}$ .

Moreover, we should write $(d, 1)^2 = (d \cdot d^{[1]}, 2)$ in place of $d^2$. So in fact, we require $(d \cdot d^{[1]}, 2) = (0, 2)$, i.e. $0 = d \cdot d^{[1]} = (M^i \xrightarrow{d^i \cdot d^{i+1}} M^{i+2})_{i\in\mathbf{Z}}$ , i.e. $d^i \cdot d^{i+1} = 0$ for $i \in \mathbf{Z}$.

### 1.1.3 Tensor products

Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category.

We will not make use of Remark 14 in this §1.1.3.

**Definition 16** Suppose given $n \in \mathbf{Z}_{\geqslant 1}$ .

(1) Suppose given a $\mathcal{Z}$-graded module $M_i$ for $i \in [1, n]$.

Let $\bigotimes_{i\in[1,n]} M_i$ be the $\mathcal{Z}$-graded module defined by

$$\Big(\bigotimes_{i\in[1,n]} M_i\Big)^z := \bigoplus_{\underline{y}\,\in\,\mathrm{fact}_n(z)} \ \bigotimes_{i\,\in\,[1,n]} M_i^{y_i}$$

for $z \in \mathrm{Mor}(\mathcal{Z})$.

We often write $M_1 \otimes \ldots \otimes M_n := \bigotimes_{i\in[1,n]} M_i$ .

(2) Suppose given $(u_i)_{i\in[1,n]}$ , $(v_i)_{i\in[1,n]} \in \mathbf{Z}^{\times n}$. Let

$$\lfloor (u_i)_i\,, (v_i)_i \rfloor \ := \ (-1)^{\sum_{1\leqslant i<j\leqslant n} u_i v_j} \ .$$

(3) Suppose given $\mathcal{Z}$-graded modules $L_i$ and $M_i$ for $i \in [1,n]$.

Suppose given shift-graded linear maps $L_i \xrightarrow{(f_i\,,\,k_i)} M_i$ for $i \in [1,n]$.

Write $k := \sum_{i\in[1,n]} k_i$. Define the shift-graded linear map

$$\bigotimes_{i\in[1,n]} L_i \xrightarrow{\bigotimes_{i\in[1,n]}(f_i\,,\,k_i) := \big(\bigotimes_{i\in[1,n]} f_i\,,\,k\big)} \bigotimes_{i\in[1,n]} M_i$$

at $z \in \operatorname{Mor}(\mathcal{Z})$ by

$$(\bigotimes_{i\in[1,n]} L_i)^z \;=\; \bigoplus_{\underline{y}\,\in\,\mathrm{fact}_n(z)} \;\bigotimes_{i\,\in\,[1,n]} L_i^{y_i} \xrightarrow{\big(\bigotimes_{i\in[1,n]} f_i\big)^z} \bigoplus_{\underline{\tilde{y}}\,\in\,\mathrm{fact}_n(z[k])} \;\bigotimes_{i\,\in\,[1,n]} M_i^{\tilde{y}_i} \;=\; (\bigotimes_{i\in[1,n]} M_i)^{z[k]}\,,$$

mapping an elementary tensor

$$(\ell_i)^{\otimes}_{i\in[1,n]} \;\in\; \bigotimes_{i\,\in\,[1,n]} L_i^{y_i}$$

to

$$\big((\ell_i)^{\otimes}_{i\in[1,n]}\big)\big(\bigotimes_{i\in[1,n]} f_i\big)^z \;:=\; \lfloor (k_i)_i\,,(y_i \deg)_i \rfloor (\ell_i f_i^{y_i})^{\otimes}_{i\in[1,n]} \;\in\; \bigotimes_{i\,\in\,[1,n]} M_i^{y_i[k_i]}\,.$$

The sign $\lfloor (k_i)_i\,,(\ell_i \deg)_i \rfloor \in \{-1,+1\}$ is called the *Koszul sign*; cf. (2). Note that $y_i \deg = \ell_i \deg$. Note that in fact, $y_1[k_1]\cdot y_2[k_2]\cdots y_n[k_n] = (y_1\cdot y_2\cdots y_n)[k] = z[k]$.

We often write $(f_1\,,k_1)\otimes\ldots\otimes(f_n\,,k_n) := \bigotimes_{i\in[1,n]}(f_i\,,k_i)$.

(3) In Problem 7.(3), we construct a $\mathcal{Z}$-graded module $\dot{R}$ such that

$$\dot{R}\otimes M \;=\; M \;=\; M\otimes\dot{R}$$

for a $\mathcal{Z}$-graded module $M$ and, more precisely,

$$(f,k)\otimes(\mathrm{id}_{\dot{R}}\,,0) \;=\; (f,k) \;=\; (\mathrm{id}_{\dot{R}}\,,0)\otimes(f,k)$$

for a shift-graded linear map $M \xrightarrow{(f,k)} N$ between $\mathcal{Z}$-graded modules $M$ and $N$.

We stipulate that $\bigotimes_{i\in[1,0]} M_i := \dot{R}$ and that $\bigotimes_{i\in[1,0]}(f_i\,,\,k_i) = \mathrm{id}_{\dot{R}}$, in the context of $(1,2)$. In particular,

$$M^{\otimes k} \;:=\; \bigotimes_{i\in[1,k]} M$$

and

$$(f,k)^{\otimes k} \;:=\; \bigotimes_{i\in[1,k]}(f,k)$$

are defined for $k \in \mathbf{Z}_{\geqslant 0}$, where $M^{\otimes 0} = \dot{R}$ and $(f,k)^{\otimes 0} = \mathrm{id}_{\dot{R}}$.

**Example 17** Suppose given $n \in \mathbf{Z}_{\geqslant 1}$ and $(L_i \xrightarrow{(f_i\,,\,k_i)} M_i)_{i\in[1,n]} \in \mathrm{Mor}((\mathcal{Z}\text{-grad})^{\times n})$.

(0) We have $\lfloor (1,4,5)\,,(-7,2,3)\rfloor = (-1)^{1\cdot 2+1\cdot 3+4\cdot 3} = -1$.

(1) Suppose $n = 1$. We have $\big(\bigotimes_{i\in[1,1]} L_i \xrightarrow{\bigotimes_{i\in[1,1]}(f_i\,,\,k_i)} \bigotimes_{i\in[1,1]} M_i\big) = \big(L_1 \xrightarrow{(f_1\,,\,k_1)} M_1\big)$.

Note that $\mathrm{fact}_1(z) = \{z\}$ for $z \in \mathrm{Mor}(\mathcal{Z})$, cf. Example 9.(2), and that the Koszul sign is $+1$ if $n = 1$.

(2) Suppose $n = 2$. The shift-graded linear map

$$\big(\bigotimes_{i\in[1,2]} L_i \xrightarrow{\bigotimes_{i\in[1,2]}(f_i\,,\,k_i)} \bigotimes_{i\in[1,2]} M_i\big) \;=\; \big(L_1\otimes L_2 \xrightarrow{(f_1\otimes f_2\,,\,k)} M_1\otimes M_2\big)$$

of degree $k := k_1 + k_2$ has at $z \in \mathrm{Mor}(\mathcal{Z})$ the entry

$$(L_1\otimes L_2)^z \;=\; \bigoplus_{\underline{y}\,\in\,\mathrm{fact}_2(z)} L_1^{y_1}\otimes L_2^{y_2} \;\xrightarrow{(f_1\otimes f_2)^z}\; \bigoplus_{\underline{\tilde{y}}\,\in\,\mathrm{fact}_2(z[k])} M_1^{\tilde{y}_1}\otimes M_2^{\tilde{y}_2} \;=\; (M_1\otimes M_2)^{z[k]}\;,$$

mapping an elementary tensor

$$\ell_1\otimes\ell_2 \;\in\; L_1^{y_1}\otimes L_2^{y_2}$$

to

$$(\ell_1\otimes\ell_2)(f_1\otimes f_2)^z \;=\; (-1)^{k_1\cdot(\ell_2\,\mathrm{deg})}(\ell_1 f_1^{y_1}\otimes \ell_2 f_2^{y_2}) \;\in\; M_1^{y_1[k_1]}\otimes M_2^{y_2[k_2]}\;.$$

Here, the Koszul sign

$$\lfloor (k_1,k_2)\,,(\ell_1\,\mathrm{deg},\ell_2\,\mathrm{deg})\rfloor \;=\; (-1)^{k_1\cdot(\ell_2\,\mathrm{deg})}$$

can be interpreted as being caused by pulling $f_1$, of degree $k_1$, across $\ell_2$, of degree $\ell_2\,\mathrm{deg}$.

Consider the case $\mathcal{Z} = \mathbf{Z}$. Then $z \in \mathbf{Z}$. The map

$$(L_1\otimes L_2)^z \;=\; \bigoplus_{\substack{y_1\,,\,y_2\,\in\,\mathbf{Z},\\ y_1+y_2\,=\,z}} L_1^{y_1}\otimes L_2^{y_2} \;\xrightarrow{(f_1\otimes f_2)^z}\; \bigoplus_{\substack{\tilde{y}_1\,,\,\tilde{y}_2\,\in\,\mathbf{Z},\\ \tilde{y}_1+\tilde{y}_2\,=\,z+k}} M_1^{\tilde{y}_1}\otimes M_2^{\tilde{y}_2} \;=\; (M_1\otimes M_2)^{z+k}$$

maps

$$\ell_1\otimes\ell_2 \;\in\; L_1^{y_1}\otimes L_2^{y_2}$$

to

$$(\ell_1\otimes\ell_2)(f_1\otimes f_2)^z \;=\; (-1)^{k_1\cdot y_2}(\ell_1 f_1\otimes \ell_2 f_2) \;\in\; M_1^{y_1+k_1}\otimes M_2^{y_2+k_2}\;.$$

(3) Suppose $n = 3$. The shift-graded linear map

$$\Big(\bigotimes_{i\in[1,3]} L_i \xrightarrow{\bigotimes_{i\in[1,3]}(f_i\,,k_i)} \bigotimes_{i\in[1,3]} M_i\Big) \;=\; \big(L_1\otimes L_2\otimes L_3 \xrightarrow{(f_1\otimes f_2\otimes f_3\,,k)} M_1\otimes M_2\otimes M_3\big)$$

of degree $k := k_1 + k_2 + k_3$ has at $z \in \operatorname{Mor}(\mathcal{Z})$ the entry

$$(L_1\otimes L_2\otimes L_3)^z = \bigoplus_{\underline{y}\,\in\,\mathrm{fact}_3(z)} L_1^{y_1}\otimes L_2^{y_2}\otimes L_3^{y_3} \xrightarrow{(f_1\otimes f_2\otimes f_3)^z} \bigoplus_{\underline{\tilde y}\,\in\,\mathrm{fact}_3(z[k])} M_1^{\tilde y_1}\otimes M_2^{\tilde y_2}\otimes M_3^{\tilde y_3} = (M_1\otimes M_2\otimes M_3)^{z[k]}\,,$$

mapping an elementary tensor

$$\ell_1\otimes\ell_2\otimes\ell_3 \;\in\; L_1^{y_1}\otimes L_2^{y_2}\otimes L_3^{y_3}$$

to

$$\begin{aligned}(\ell_1\otimes\ell_2\otimes\ell_3)(f_1\otimes f_2\otimes f_3)^z \;&=\; (-1)^{k_1\cdot(\ell_2\deg+\ell_3\deg)+k_2\cdot(\ell_3\deg)}(\ell_1 f_1^{y_1}\otimes\ell_2 f_2^{y_2}\otimes\ell_3 f_3^{y_3})\\ &\in\;\; M_1^{y_1[k_1]}\otimes M_2^{y_2[k_2]}\otimes M_3^{y_3[k_3]}\end{aligned}$$

Here, the Koszul sign

$$\lfloor (k_1,k_2,k_3)\,,(\ell_1\deg,\ell_2\deg,\ell_3\deg)\rfloor \;=\; (-1)^{k_1\cdot(\ell_2\deg+\ell_3\deg)+k_2\cdot(\ell_3\deg)}$$

can be interpreted as being caused by pulling $f_1$, of degree $k_1$, across $\ell_2\otimes\ell_3$, of degree $\ell_2\deg+\ell_3\deg$, and $f_2$, of degree $k_2$, across $\ell_3$, of degree $\ell_3\deg$.

Consider the case $\mathcal{Z} = \mathbf{Z}$. Then $z\in\mathbf{Z}$. The map

$$(L_1\otimes L_2\otimes L_3)^z \;=\; \bigoplus_{\substack{y_1,y_2,y_3\,\in\,\mathbf{Z},\\ y_1+y_2+y_3=z}} L_1^{y_1}\otimes L_2^{y_2}\otimes L_3^{y_3} \xrightarrow{(f_1\otimes f_2\otimes f_3)^z} \bigoplus_{\substack{\tilde y_1,\tilde y_2,\tilde y_3\,\in\,\mathbf{Z},\\ \tilde y_1+\tilde y_2+\tilde y_3=z+k}} M_1^{\tilde y_1}\otimes M_2^{\tilde y_2}\otimes M_3^{\tilde y_3} \;=\; (M_1\otimes M_2\otimes M_3)^{z+k}$$

maps

$$\ell_1\otimes\ell_2\otimes\ell_3 \;\in\; L_1^{y_1}\otimes L_2^{y_2}\otimes L_3^{y_3}$$

to

$$(\ell_1\otimes\ell_2\otimes\ell_3)(f_1\otimes f_2\otimes f_3)^z \;=\; (-1)^{k_1\cdot(y_2+y_3)+k_2\cdot y_3}(\ell_1 f_1\otimes\ell_2 f_2\otimes\ell_3 f_3) \;\in\; M_1^{y_1+k_1}\otimes M_2^{y_2+k_2}\otimes M_3^{y_3+k_3}\,.$$

**Example 18** We consider the tensor product of two complexes; cf. Example 15.

Let $M$ be a complex with differential $(d,1)$. Let $\tilde M$ be a complex with differential $(\tilde d,1)$.

Then the $\mathbf{Z}$-graded module $M\otimes\tilde M$ has at position $z\in\mathbf{Z}$ the entry

$$\bigoplus_{i,j\,\in\,\mathbf{Z},\;i+j=z} M^i\otimes\tilde M^j\,.$$

The entry at $i+j$ of the graded linear map $(D,1) := (d,1) \otimes (\mathrm{id},0) + (\mathrm{id},0) \otimes (\tilde{d},1)$ of degree 1 maps the elementary tensor

$$m \otimes \tilde{m} \ \in \ M^i \otimes \tilde{M}^j$$

to

$$\begin{aligned} (m \otimes \tilde{m})D &= (-1)^{1\cdot j} m d^i \otimes \tilde{m}\,\mathrm{id} + (-1)^{0\cdot j} m\,\mathrm{id} \otimes \tilde{m}\tilde{d}^j \\ &= (-1)^j m d^i \otimes \tilde{m} + m \otimes \tilde{m}\tilde{d}^j \ . \end{aligned}$$

So entry at $i+j$ of the graded linear map $(D,1)^2 = (D \cdot D^[1], 2)$ of degree 2 maps it to

$$\begin{aligned} & ((m \otimes \tilde{m})D)D^{[1]} \\ = \ & ((-1)^j m d^i \otimes \tilde{m} + m \otimes \tilde{m}\tilde{d}^j)D^[1] \\ = \ & (-1)^{j+j} m d^i d^{i+1} \otimes \tilde{m} + (-1)^j m d^i \otimes \tilde{m}\tilde{d}^j + (-1)^{j+1} m d^i \otimes \tilde{m}\tilde{d}^j + m \otimes \tilde{m}\tilde{d}^j \tilde{d}^{j+1} \\ = \ & 0 + (-1)^j m d^i \otimes \tilde{m}\tilde{d}^j (1-1) + 0 \\ = \ & 0 \ . \end{aligned}$$

So $(D,1)^2 = 0$. Hence $M \otimes \tilde{M}$, with differential $(D,1)$, is a complex.

This would not have been the case without inserting a sign such as the Koszul sign.

## 1.2 $\mathrm{A}_\infty$-algebras and $\mathrm{A}_\infty$-categories

Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category; cf. Definition 5.

Henceforth, we make use of Remark 14.

**Definition 19** Suppose given $n \in [0,\infty]$.

(1) Suppose given a $\mathcal{Z}$-graded module $A$.

Suppose given a shift-graded linear map $m_k^A = (m_k^A\,,\, 2-k) : A^{\otimes k} \to A$ of degree $2-k$ for $k \in [1,n] \cap \mathbf{Z}$.

Then $A = (A, (m_k^A)_{k\in[1,n]\cap\mathbf{Z}})$ is a *pre-*$\mathrm{A}_n$*-algebra (over* $\mathcal{Z}$*)*.

A pre-$\mathrm{A}_n$-algebra $A = (A, (m_k^A)_{k\in[1,n]\cap\mathbf{Z}})$ is an $\mathrm{A}_n$*-algebra (over* $\mathcal{Z}$*)* if the *Stasheff equation*

$$0 \ = \sum_{\substack{(r,s,t)\in \mathbf{Z}_{\geqslant 0}\times\mathbf{Z}_{\geqslant 1}\times\mathbf{Z}_{\geqslant 0} \\ r+s+t=k}} (-1)^{r+st} (\mathrm{id}^{\otimes r} \otimes m_s^A \otimes \mathrm{id}^{\otimes t}) \cdot m_{r+1+t}^A$$

holds for $k \in [1,n] \cap \mathbf{Z}$.

Note that each summand of the right-hand side is a shift-graded linear map from $A^{\otimes k}$ to $A$ of degree $3-k$.

We often abbreviate $A = (A, (m_k)_k) = (A, (m_k^A)_k) := (A, (m_k^A)_{k \in [1,n] \cap \mathbf{Z}})$.

Sometimes, the tuple $(m_k)_k$ is referred to as an $\mathrm{A}_n$-*structure* on the $\mathcal{Z}$-graded module $A$. An entry $m_k$ of this tuple is sometimes referred to as $k$th shift-graded linear *multiplication map*.

We often abbreviate the condition $(r, s, t) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 1} \times \mathbf{Z}_{\geqslant 0}$ on the indexing triples to $(r, s, t) \geqslant (0, 1, 0)$.

(2) Suppose given $\mathcal{Z}$-graded modules $\tilde{A}$ and $A$.

Suppose given a shift-graded linear map $f_k = (f_k\,, 1-k) : \tilde{A}^{\otimes k} \to A$ of degree $1-k$ for $k \in [1, n] \cap \mathbf{Z}$.

Then $f = (f_k)_{k \in [1,n] \cap \mathbf{Z}} : \tilde{A} \to A$ is a *pre-*$\mathrm{A}_n$*-morphism (over* $\mathcal{Z}$*)*.

Suppose given $\mathrm{A}_n$-algebras $\tilde{A} = (\tilde{A}, (m_k^{\tilde{A}})_{k \in [1,n] \cap \mathbf{Z}})$ and $A = (A, (m_k^A)_{k \in [1,n] \cap \mathbf{Z}})$.

A pre-$\mathrm{A}_n$-morphism $f = (f_k)_{k \in [1,n] \cap \mathbf{Z}} : \tilde{A} \to A$ is an $\mathrm{A}_n$*-morphism* or a *morphism of* $\mathrm{A}_n$*-algebras (over* $\mathcal{Z}$*)* if the *Stasheff equation for morphisms*

$$\sum_{\substack{(r,s,t) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 1} \times \mathbf{Z}_{\geqslant 0} \\ r+s+t = k}} (-1)^{r+st} (\mathrm{id}^{\otimes r} \otimes m_s^{\tilde{A}} \otimes \mathrm{id}^{\otimes t}) \cdot f_{r+1+t} \;=\; \sum_{r \in [1,k]} \; \sum_{\substack{(i_j)_{j \in [1,r]} \in \mathbf{Z}_{\geqslant 1}^{\times r} \\ \sum_{j \in [1,r]} i_j = k}} \lfloor (1-i_j)_j \,, (i_j)_j \rfloor \Big( \bigotimes_{j \in [1,r]} f_{i_j} \Big) \cdot m_r^A$$

holds for $k \in [1, n] \cap \mathbf{Z}$.

Note that each summand of the left- and of the right-hand side is a shift-graded linear map from $\tilde{A}^{\otimes k}$ to $A$ of degree $2-k$.

We often abbreviate $(f_k)_k = (f_k)_{k \in [1,n] \cap \mathbf{Z}}$ .

We often abbreviate the condition $(r, s, t) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 1} \times \mathbf{Z}_{\geqslant 0}$ on the indexing triples to $(r, s, t) \geqslant (0, 1, 0)$ and the condition $(i_j)_{j \in [1,r]} \in \mathbf{Z}_{\geqslant 1}^{\times r}$ on the indexing tuples to $(i_j)_j \geqslant (1)_j$ .

**Remark 20** Suppose given $1 \leqslant \ell \leqslant n \leqslant \infty$.

(1) Given an $\mathrm{A}_n$-algebra $(A, (m_k^A)_{k \in [1,n] \cap \mathbf{Z}})$, we get an $\mathrm{A}_\ell$-algebra $(A, (m_k^A)_{k \in [1,\ell] \cap \mathbf{Z}})$.

(2) Given an $\mathrm{A}_n$-morphism $(f_k)_{k \in [1,n] \cap \mathbf{Z}}$ from an $\mathrm{A}_n$-algebra $(\tilde{A}, (m_k^{\tilde{A}})_{k \in [1,n] \cap \mathbf{Z}})$ to an $\mathrm{A}_n$-algebra $(A, (m_k^A)_{k \in [1,n] \cap \mathbf{Z}})$, we get an $\mathrm{A}_\ell$-morphism $(f_k)_{k \in [1,\ell] \cap \mathbf{Z}}$ from $(\tilde{A}, (m_k^{\tilde{A}})_{k \in [1,\ell] \cap \mathbf{Z}})$ to $(A, (m_k^A)_{k \in [1,\ell] \cap \mathbf{Z}})$ .

**Example 21**

We consider the Stasheff equation from Definition 19.(1) for an $\mathrm{A}_n$-algebra $A = (A, (m_k^A)_k) = (A, (m_k)_k)$ for $k \in [1, 3]$, supposing $n \geqslant k$.

(1) For $k = 1$, the Stasheff equation reads

$$0 \;=\; m_1 \cdot m_1 \; .$$

So in case $\mathcal{Z} = \mathbf{Z}$, the graded module $A$ is a complex with differential $m_1$ .

(2) For $k = 2$, the Stasheff equation reads

$$0 \;=\; -(\mathrm{id} \otimes m_1) \cdot m_2 - (m_1 \otimes \mathrm{id}) \cdot m_2 + m_2 \cdot m_1 \; .$$

In case $\mathcal{Z} = \mathbf{Z}$, we obtain for $a,\, b \,\in\, A^0$

$$((a \otimes b)m_2)m_1 \;=\; (a \otimes bm_1)m_2 + (am_1 \otimes b)m_2 \; .$$

Interpreting $m_1$ as differential and $m_2$ as multiplication, this is a product rule for the differential, often called the *Leibniz rule*.

(3) For $k = 3$, the Stasheff equation reads

$$0 \;=\; (m_1 \otimes \mathrm{id}^{\otimes 2}) \cdot m_3 + (\mathrm{id} \otimes m_1 \otimes \mathrm{id}) \cdot m_3 + (\mathrm{id}^{\otimes 2} \otimes m_1) \cdot m_3 + (m_2 \otimes \mathrm{id}) \cdot m_2 - (\mathrm{id} \otimes m_2) \cdot m_2 + m_3 \cdot m_1 \; .$$

In case $\mathcal{Z} = \mathbf{Z}$, we obtain for $a,\, b,\, c \,\in\, A^0$

$$\begin{aligned} & (a \otimes (b \otimes c)m_2)m_2 - ((a \otimes b)m_2 \otimes c) \cdot m_2 \\ & = \; (am_1 \otimes b \otimes c)m_3 + (a \otimes bm_1 \otimes c)m_3 + (a \otimes b \otimes cm_1)m_3 + ((a \otimes b \otimes c)m_3)m_1 \end{aligned}$$

Interpreting $m_2$ as multiplication, we observe that this multiplication is associative if $m_3 = 0$ or if $m_1 = 0$.

We do not claim that associativity of $m_2$ entails $m_3 = 0$.

**Example 22**

We consider the Stasheff equation for morphisms from Definition 19.(2) for a morphism $f = (f_k)_k : \tilde{A} \to A$ of $\mathrm{A}_n$-algebras for $k \in [1, 3]$, supposing $n \geqslant k$.

We consider the conditions of Definition 19.(2) for $k \in [1, 3]$, supposing $n \geqslant k$.

(1) For $k = 1$, we obtain the condition

$$m_1^{\tilde{A}} \cdot f_1 \;=\; f_1 \cdot m_1^A \; .$$

In case $\mathcal{Z} = \mathbf{Z}$, we obtain that $f_1$ is a morphism of complexes from $\tilde{A}$, having differential $m_1^{\tilde{A}}$, to $A$, having differential $m_1^A$ .

(2) For $k = 2$, we obtain the condition

$$\begin{aligned} & -(\mathrm{id} \otimes m_1^{\tilde{A}}) \cdot f_2 - (m_1^{\tilde{A}} \otimes \mathrm{id}) \cdot f_2 + m_2^{\tilde{A}} \cdot f_1 \\ & = \; f_2 \cdot m_1^A + (f_1 \otimes f_1) \cdot m_2^A \; . \end{aligned}$$

(3) For $k = 3$, we obtain the condition

$$\begin{aligned}&(m_1^{\tilde{A}} \otimes \mathrm{id}^{\otimes 2}) \cdot f_3 + (\mathrm{id} \otimes m_1^{\tilde{A}} \otimes \mathrm{id}) \cdot f_3 + (\mathrm{id}^{\otimes 2} \otimes m_1^{\tilde{A}}) \cdot f_3 + (m_2^{\tilde{A}} \otimes \mathrm{id}) \cdot f_2 - (\mathrm{id} \otimes m_2^{\tilde{A}}) \cdot f_2 + m_3^{\tilde{A}} \cdot f_1 \\ &= \; f_3 \cdot m_1^A + (f_1 \otimes f_2) \cdot m_2^A - (f_2 \otimes f_1) \cdot m_2^A + (f_1 \otimes f_1 \otimes f_1) \cdot m_3^A \, .\end{aligned}$$

**Definition 23** Let $n \in [2, \infty]$. Suppose given an $\mathrm{A}_n$-algebra $A = (A, (m_k^A)_{k \in [1,n] \cap \mathbf{Z}})$.

Then $A$ is called *unital*, if for $X \in \mathrm{Ob}\,\mathcal{Z}$, there exists a *neutral* element $1_{A,X} = 1_X \in A^{\mathrm{id}_X}$ such that $(1, 2)$ hold.

(1) We have $(a \otimes 1_X) m_2^A = a$ for $z \in \mathrm{Mor}(\mathcal{Z})$ with $z \mathrm{t}_{\mathcal{Z}} = X$ and $a \in A^z$.

(2) We have $(1_X \otimes b) m_2^A = b$ for $w \in \mathrm{Mor}(\mathcal{Z})$ with $w \mathrm{s}_{\mathcal{Z}} = X$ and $b \in A^w$.

Then $1_X$ is uniquely determined, for given an element $c \in A^{\mathrm{id}_X}$ having properties $(1, 2)$, then $1_X = (1_X \otimes c) m_2^A = c$.

Note that $(1_X \otimes 1_X) m_2^A m_1^A \overset{1.}{=} 1_X m_1^A \overset{2.}{=} (1_X m_1^A \otimes 1_X) m_2^A + (1_X \otimes 1_X m_1^A) m_2^A = 1_X m_1^A + 1_X m_1^A$ for $X \in \mathrm{Ob}\,\mathcal{Z}$, whence $1_X m_1^A = 0$; cf. Example 21.(2).

**Definition 24** Let $n \in [2, \infty]$.

Suppose given unital $\mathrm{A}_n$-algebras $\tilde{A}$ and $A$.

Suppose given an $\mathrm{A}_n$-morphism $\tilde{A} \xrightarrow{f} A$, so $f = (f_k)_{k \in [1,n] \cap \mathbf{Z}}$.

Then $f$ is called *unital*, if for $X \in \mathrm{Ob}\,\mathcal{Z}$, we have $1_{\tilde{A},X} f_1 = 1_{A,X}$.

**Example 25** Suppose given $n \in [3, \infty]$.

Suppose given a unital $\mathrm{A}_n$-algebra $A = (A, (m_k)_k)$.

Suppose that $A^z = 0$ for $z \in \mathrm{Mor}(\mathcal{Z})$ with $\deg z \in \mathbf{Z} \smallsetminus \{0\}$.

Then $m_k = 0$ for $k \in [1, n] \smallsetminus \{2\}$. In fact, the shift-graded linear map $m_k = (m_k, 2-k)$ actually maps from $A^{\otimes k}$ to $A^{[2-k]}$. So given $z \in \mathrm{Mor}(\mathcal{Z})$ and $\underline{y} \in \mathrm{fact}_k(z)$ and

$$a_1 \otimes \ldots \otimes a_k \;\in\; A^{y_1} \otimes \ldots \otimes A^{y_k} \, ,$$

its image is

$$(a_1 \otimes \ldots \otimes a_k) m_k \;\in\; A^{z[2-k]} \, .$$

In order that this image be nonzero, we need that on the one hand, $a_i$ is nonzero for $i \in [1, k]$, so necessarily $y_i \deg = 0$ for $i \in [1, k]$. On the other hand, we must necessarily have $z[2-k] \deg = 0$. But

$$0 \;=\; z[2-k] \deg \;=\; z \deg + 2 - k \;=\; \Big( \sum_{i \in [1,k]} y_i \Big) + 2 - k \;=\; 2 - k \, ,$$

so $k = 2$.

In particular, interpreting $m_2$ as multiplication, it is associative; cf. Example 21.(3).

*Case* $\mathcal{Z} = \mathbf{Z}$. Then $A^0$, together with the multiplication

$$\begin{array}{ccccccc} A^0 & \otimes & A^0 & \to & A^0 & & \\ a & \otimes & b & \mapsto & a \cdot b & := & (a \otimes b)m_2 \end{array}$$

is an algebra. The element $1_{\mathbf{Z}} \in A^{\mathrm{id}_{\mathbf{Z}}} = A^0$ is neutral with respect to multiplication; cf. Definition 23.

*Case* $\mathcal{Z} = \mathbf{Z} \times I^{\times 2}$ *for a set* $I$ *; cf. Example* 1.(4). Then we have a linear category $A^0$ with $\mathrm{Ob}(A^0) = I$ and ${}_{A^0}(i, j) = A^{(0,(i,j))}$ for $i,\, j \in I$. Its composition is given by

$$\begin{array}{ccccccc} A^{(0,(i,j))} & \otimes & A^{(0,(j,k))} & \to & A^{(0,(i,k))} & & \\ a & \otimes & b & \mapsto & a \cdot b & := & (a \otimes b)m_2 \end{array}$$

for $i,\, j,\, k \in I$. Given $(\mathbf{Z}, i) \in \{\mathbf{Z}\} \times I = \mathrm{Ob}(\mathbf{Z} \times I^{\times 2})$, the element $1_{(\mathbf{Z},i)} \in A^{\mathrm{id}_{(\mathbf{Z},i)}} = A^{(0,(i,i))}$ is neutral with respect to composition; cf. Definition 23.

**Example 26** Suppose given $n \in [3, \infty]$.

Suppose given unital $\mathrm{A}_n$-algebras $\tilde{A}$ and $A$.

Suppose given a unital $\mathrm{A}_n$-morphism $\tilde{A} \xrightarrow{f} A$, so $f = (f_k)_{k \in [1,n] \cap \mathbf{Z}}$ .

Suppose that $\tilde{A}^z = 0$ and $A^z = 0$ for $z \in \mathrm{Mor}(\mathcal{Z})$ with $\deg z \in \mathbf{Z} \smallsetminus \{0\}$. So $m_k^{\tilde{A}} = 0$ and $m_k^A = 0$ for $k \in [1, n] \smallsetminus \{2\}$ ; cf. Example 25.

Moreover, $f_k = 0$ for $k \in [1, n] \smallsetminus \{1\}$. In fact, the shift-graded linear map $f_k = (f_k, 1-k)$ actually maps from $\tilde{A}^{\otimes k}$ to $A^{[1-k]}$. So given $z \in \mathrm{Mor}(\mathcal{Z})$ and $\underline{y} \in \mathrm{fact}_k(z)$ and

$$a_1 \otimes \ldots \otimes a_k \ \in \ A^{y_1} \otimes \ldots \otimes A^{y_k} \ ,$$

its image is

$$(a_1 \otimes \ldots \otimes a_k) f_k \ \in \ A^{z[1-k]} \ .$$

In order that this image be nonzero, we need that on the one hand, $a_i$ is nonzero for $i \in [1, k]$, so necessarily $y_i \deg = 0$ for $i \in [1, k]$. On the other hand, we must necessarily have $z[1-k] \deg = 0$. But

$$0 \ = \ z[1-k] \deg \ = \ z \deg + 1 - k \ = \ \Big( \sum_{i \in [1,k]} y_i \Big) + 1 - k \ = \ 1 - k \ ,$$

so $k = 1$.

Now $\tilde{A} \xrightarrow{f_1} A$ satisfies $m_2^{\tilde{A}} \cdot f_1 = (f_1 \otimes f_1) \cdot m_2^A$ ; cf. Example 22.(2).

*Case* $\mathcal{Z} = \mathbf{Z}$. Then $\tilde{A}^0$ and $A^0$ are algebras; cf. Example 25. Using the multiplication notation from there, we obtain

$$(\tilde{a} \cdot \tilde{b}) f_1 \ = \ (\tilde{a} \otimes \tilde{b})(m_2^{\tilde{A}} \cdot f_1) \ = \ (\tilde{a} \otimes \tilde{b})(f_1 \otimes f_1) \cdot m_2^A \ = \ \tilde{a} f_1 \cdot \tilde{b} f_1$$

for $\tilde{a}$, $\tilde{b} \in \tilde{A}^0$. Moreover, we have

$$1_{\tilde{A},\mathbf{Z}} f_1 \;=\; 1_{A,\mathbf{Z}} \;.$$

So $f_1$ is a morphism of algebras from $\tilde{A}^0$ to $A^0$.

*Case $\mathcal{Z} = \mathbf{Z} \times I^{\times 2}$ for a set $I$ ; cf. Example* 1.(4). Then $\tilde{A}^0$ and $A^0$ are linear categories; cf. Example 25. Using the composition notation from there, we obtain

$$(\tilde{a} \cdot \tilde{b}) f_1 \;=\; (\tilde{a} \otimes \tilde{b})(m_2^{\tilde{A}} \cdot f_1) \;=\; (\tilde{a} \otimes \tilde{b})(f_1 \otimes f_1) \cdot m_2^{A} \;=\; \tilde{a} f_1 \cdot \tilde{b} f_1$$

for $i$, $j$, $k \in I$, for $a \in \tilde{A}^{(0,(i,j))}$ and $a \in \tilde{A}^{(0,(j,k))}$. Moreover, we have

$$1_{\tilde{A}} f_1 \;=\; 1_A \;.$$

So $f_1$ is a linear functor from $\tilde{A}^0$ to $A^0$.

Exceptionally, $f_1$ is written on the right, i.e. naturally.

**Definition 27** Recall that $\mathcal{Z}$ is a grading category.

(1) An $\mathrm{A}_\infty$-algebra over $\mathbf{Z}$ is called a *classical* $\mathrm{A}_\infty$*-algebra.*

(2) Suppose given a set $I$.

A unital $\mathrm{A}_\infty$-algebra $A$ over $\mathbf{Z} \times I^{\times 2}$ is called an $\mathrm{A}_\infty$*-category* with set of *objects* $\mathrm{Ob}(A) = I$ ; cf. Example 1.(4).

(3) A unital $\mathrm{A}_\infty$-algebra $A = (A, (m_k)_{k \in \mathbf{Z}_{\geqslant 1}})$ over $\mathcal{Z}$ with $m_k = 0$ for $k \geqslant 3$ is called a *differential graded algebra (over $\mathcal{Z}$).* Cf. Problem 11.

(4) A unital $\mathrm{A}_\infty$-algebra $A = (A, (m_k)_{k \in \mathbf{Z}_{\geqslant 1}})$ over $\mathbf{Z}$ with $m_k = 0$ for $k \geqslant 3$ is called a *classical differential graded algebra.*

(5) Suppose given a set $I$.

A unital $\mathrm{A}_\infty$-algebra $A = (A, (m_k)_{k \in \mathbf{Z}_{\geqslant 1}})$ over $\mathbf{Z} \times I^{\times 2}$ with $m_k = 0$ for $k \geqslant 3$ is called a *differential graded category* with set of objects $I$ ; cf. Example 1.(4).

(6) A unital $\mathrm{A}_\infty$-algebra $A = (A, (m_k)_{k \in \mathbf{Z}_{\geqslant 1}})$ over $\mathcal{Z}$ with $m_1 = 0$ is called *minimal.*

## 1.3 The regular differential graded category for complexes

Suppose given an algebra $B$.

Suppose given $N \in \mathbf{Z}_{\geqslant 1}$. Suppose given complexes $X_s \in \mathrm{Ob}(\mathrm{C}(B\text{-Mod}))$ for $s \in [1,N]$, where $X_s$ carries the differential $d_s = (X_s^i \xrightarrow{d_s^i} X_s^{i+1})_i$. Write $\underline{X} := (X_s)_{s\in[1,N]}$.
Let $\mathcal{Z} := \mathbf{Z} \times [1,N]^{\times 2}$.
Write

$$\mathrm{Hom}_B^j(X_s\,,X_t) := \{\, (f^i)_{i\in\mathbf{Z}} \,:\, X_s^i \xrightarrow{f^i} X_t^{i+j} \text{ is a } B\text{-linear map for } i \in \mathbf{Z}\,\}$$

for $s,\, t \in [1,N]$ and $j \in \mathbf{Z}$.

In general, the inclusion ${}_{\mathrm{C}(B\text{-Mod})}(X_s\,,X_t) \subseteq \mathrm{Hom}_B^0(X_s\,,X_t)$ is strict. The definition of $\mathrm{Hom}_B^j(X_s\,,X_t)$ does not involve the differentials of $X_s$ and of $X_t$.

We shall construct the *regular* differential graded algebra $\mathrm{Hom}_B(\underline{X})$ of $\underline{X}$ over $\mathcal{Z}$ on the set of objects $[1,N]$.

As a $\mathcal{Z}$-graded module, define $\mathrm{Hom}_B(\underline{X})$ by letting

$$\mathrm{Hom}_B(\underline{X})^{(j,(s,t))} \;:=\; \mathrm{Hom}_B^j(X_s\,,X_t)$$

for $(j,(s,t)) \in \mathrm{Mor}(\mathcal{Z})$.
Let

$$\mathrm{Hom}_B(\underline{X}) \xrightarrow{m_1^{\mathrm{Hom}_B(\underline{X})}} \mathrm{Hom}_B(\underline{X})$$

be defined at $(j,(s,t)) \in \mathrm{Mor}(\mathcal{Z})$ by

$$\begin{array}{ccc} \mathrm{Hom}_B^j(X_s\,,X_t) & \xrightarrow{m_1^{\mathrm{Hom}_B(\underline{X})}} & \mathrm{Hom}_B^{j+1}(X_s\,,X_t) \\ (f^i)_i & \mapsto & (f^i d_t^{i+j} - (-1)^j d_s^i f^{i+1})_i\,. \end{array}$$

$$\begin{array}{ccc} X_s^i & \xrightarrow{f^i} & X_t^{i+j} \\ \downarrow{\scriptstyle d_s^i} & & \downarrow{\scriptstyle d_t^{i+j}} \\ X_s^{i+1} & \xrightarrow{f^{i+1}} & X_t^{i+j+1} \end{array}$$

Let

$$\mathrm{Hom}_B(\underline{X})^{\otimes 2} \xrightarrow{m_2^{\mathrm{Hom}_B(\underline{X})}} \mathrm{Hom}_B(\underline{X})$$

be defined at $(j,(s,t)) \in \mathrm{Mor}(\mathcal{Z})$, on the summand belonging to

$$\big((k,(s,u)),(\ell,(u,t))\big) \in \mathrm{fact}_2\,\big((j,(s,t))\big)\,,$$

i.e. $k+\ell = j$ and $u \in [1,N]$, by

$$\begin{array}{ccccc} \mathrm{Hom}_B^k(X_s\,,X_u) & \otimes & \mathrm{Hom}_B^\ell(X_u\,,X_t) & \xrightarrow{m_2^{\mathrm{Hom}_B(\underline{X})}} & \mathrm{Hom}_B^j(X_s\,,X_t) \\ (f^i)_i & \otimes & (g^i)_i & \mapsto & (f^i g^{i+k})_i\,. \end{array}$$

$$X_s^i \xrightarrow{f^i} X_u^{i+k} \xrightarrow{g^{i+k}} X_t^{i+k+\ell}$$

**Lemma 28** *Recall that* $\mathcal{Z} = \mathbf{Z} \times [1, N]^{\times 2}$. *Recall that* $X_s \in \mathrm{C}(B\text{-Mod})$.

*Consider the* $\mathcal{Z}$*-graded module* $\mathrm{Hom}_B(\underline{X})$ *and the shift-graded morphisms*

$$\mathrm{Hom}_B(\underline{X}) \xrightarrow{m_1^{\mathrm{Hom}_B(\underline{X})}} \mathrm{Hom}_B(\underline{X})$$

*of degree* 1 *and*

$$\mathrm{Hom}_B(\underline{X})^{\otimes 2} \xrightarrow{m_2^{\mathrm{Hom}_B(\underline{X})}} \mathrm{Hom}_B(\underline{X})$$

*of degree* 0 *constructed above. Let* $m_k^{\mathrm{Hom}_B(\underline{X})} := 0$, *as a shift-graded linear map of degree* $2-k$ *from* $\mathrm{Hom}_B(\underline{X})^{\otimes k}$ *to* $\mathrm{Hom}_B(\underline{X})$, *for* $k \in \mathbf{Z}_{\geqslant 3}$ .

*Then* $(\mathrm{Hom}_B(\underline{X}), (m_k^{\mathrm{Hom}_B(\underline{X})})_{k\in\mathbf{Z}_{\geqslant 1}})$ *is a differential graded category on the set of objects* $[1, N]$ *; cf. Definition* 27.(5).

*Proof.* We have to show the Stasheff equation for $k \in [1, 3]$ and the existence of neutral elements; cf. Problem 11. Write $m_k := m_k^{\mathrm{Hom}_B(\underline{X})}$ for $k \in \mathbf{Z}_{\geqslant 1}$ .

*Case* $k = 1$. We have to show that $m_1 \cdot m_1 \overset{!}{=} 0$ ; cf. Example 21.(1).

Given $(j, (s, t)) \in \mathrm{Mor}(\mathcal{Z})$ and $(f_i)_i \in \mathrm{Hom}_B^j(X_s\,, X_t)$, we obtain

$$\begin{array}{cl} & ((f_i)_i)(m_1 \cdot m_1) \\ = & ((f^i d_t^{i+j} - (-1)^j d_s^i f^{i+1})_i)m_1 \\ = & ((f^i d_t^{i+j} - (-1)^j d_s^i f^{i+1}) d_t^{i+j+1} - (-1)^{j+1} d_s^i (f^{i+1} d_t^{i+j+1} - (-1)^j d_s^{i+1} f^{i+2}))_i \\ = & (f^i d_t^{i+j} d_t^{i+j+1} - (-1)^j d_s^i f^{i+1} d_t^{i+j+1} - (-1)^{j+1} d_s^i f^{i+1} d_t^{i+j+1} + (-1)^{j+1}(-1)^j d_s^i d_s^{i+1} f^{i+2}))_i \\ = & (0)_i\,. \end{array}$$

*Case* $k = 2$. We have to show that $m_2 \otimes m_1 \overset{!}{=} (\mathrm{id} \otimes m_1) \cdot m_2 + (m_1 \otimes \mathrm{id}) \cdot m_2$ ; cf. Example 21.(2).

Given $(j, (s, t)) \in \mathrm{Mor}(\mathcal{Z})$ and $\big((k, (s, u)), (\ell, (u, t))\big) \in \mathrm{fact}_2\big((j, (s, t))\big)$, i.e. $k + \ell = j$ and $u \in [1, N]$, and $(f^i)_i \in \mathrm{Hom}_B^k(X_s\,, X_u)$ and $(g^i)_i \in \mathrm{Hom}_B^\ell(X_u\,, X_t)$, we obtain

$$\begin{array}{cl} & ((f^i)_i \otimes (g^i)_i)(m_2 \otimes m_1) \\ = & ((f^i g^{i+k})_i)m_1 \\ = & (f^i g^{i+k} d_t^{i+j} - (-1)^j d_s^i f^{i+1} g^{i+1+k})_i \end{array}$$

and

$$\begin{array}{cl} & ((f^i)_i \otimes (g^i)_i)((\mathrm{id} \otimes m_1) \cdot m_2 + (m_1 \otimes \mathrm{id}) \cdot m_2) \\ \overset{\text{Koszul}}{=} & ((f^i)_i \otimes ((g^i)_i)m_1)m_2 + (-1)^\ell (((f^i)_i)m_1 \otimes (g^i)_i)m_2 \\ = & ((f^i)_i \otimes (g^i d_t^{i+\ell} - (-1)^\ell d_u^i g^{i+1})_i)m_2 + (-1)^\ell((f^i d_u^{i+k} - (-1)^k d_s^i f^{i+1})_i \otimes (g^i)_i)m_2 \\ = & (f^i g^{i+k} d_t^{i+k+\ell} - (-1)^\ell f^i d_u^{i+k} g^{i+k+1})_i + (-1)^\ell (f^i d_u^{i+k} g^{i+k+1} - (-1)^{k+\ell} d_s^i f^{i+1} g^{i+k+1})_i \\ = & (f^i g^{i+k} d_t^{i+k+\ell} - (-1)^{k+\ell} d_s^i f^{i+1} g^{i+k+1})_i\,, \end{array}$$

which is the same.

*Case* $k = 3$. We have to show that $(m_2 \otimes \mathrm{id}) \cdot m_2 \overset{!}{=} (\mathrm{id} \otimes m_2) \cdot m_2$ ; cf. Example 21.(3).

Given $(j,(s,t)) \in \mathrm{Mor}(\mathcal{Z})$ and $\big((k,(s,u)),(\ell,(u,v)),(p,(v,t))\big) \in \mathrm{fact}_3\big((j,(s,t))\big)$, i.e. $k+\ell+p = j$ and $u,\, v \in [1,N]$, and $(f^i)_i \in \mathrm{Hom}_B^k(X_s\,,X_u)$ and $(g^i)_i \in \mathrm{Hom}_B^\ell(X_u\,,X_v)$ and $(h^i)_i \in \mathrm{Hom}_B^\ell(X_v\,,X_t)$, we obtain

$$\begin{array}{cl} & ((f^i)_i \otimes (g^i)_i \otimes (h^i)_i)((m_2 \otimes \mathrm{id}) \cdot m_2) \\ \overset{\text{Koszul}}{=} & (((f^i)_i \otimes (g^i)_i)m_2 \otimes (h^i)_i)m_2 \\ = & ((f^i g^{i+k})_i \otimes (h^i)_i)m_2 \\ = & (f^i g^{i+k} h^{i+k+\ell})_i \end{array}$$

and

$$\begin{array}{cl} & ((f^i)_i \otimes (g^i)_i \otimes (h^i)_i)((\mathrm{id} \otimes m_2) \cdot m_2) \\ \overset{\text{Koszul}}{=} & ((f^i)_i \otimes ((g^i)_i \otimes (h^i)_i)m_2)m_2 \\ = & ((f^i)_i \otimes (g^i h^{i+\ell})_i)m_2 \\ = & (f^i g^{i+k} h^{i+k+\ell})_i\ , \end{array}$$

which is the same.

We have to show the existence of neutral elements, i.e. that $\mathrm{Hom}_B(\underline{X})$ is unital; cf. Definition 23. Given $(\mathbf{Z},s) \in \mathrm{Ob}(\mathcal{Z})$, let

$$1_s \;:=\; 1_{(\mathbf{Z},s)} \;:=\; (\mathrm{id}_{X_s^i})_i\ .$$

Given $(j,(s,t)) \in \mathrm{Mor}(\mathcal{Z})$ and $(f_i)_i \in \mathrm{Hom}_B^j(X_s\,,X_t)$, we obtain $(j,(s,t))\mathrm{t}_{\mathcal{Z}} = (\mathbf{Z},t)$ and

$$((f_i)_i \otimes 1_{(\mathbf{Z},t)})m_2 \;=\; (f_i \cdot \mathrm{id}_{X_t^{i+j}})_i \;=\; (f_i)_i\ ,$$

and we obtain $(j,(s,t))\mathrm{s}_{\mathcal{Z}} = (\mathbf{Z},s)$ and

$$(1_{(\mathbf{Z},s)} \otimes (f_i)_i)m_2 \;=\; (\mathrm{id}_{X_s^i} \cdot f^i)_i \;=\; (f^i)_i\ .$$

□

## 1.4 Cohomology

Let $\mathcal{Z}$ be a grading category.

**Definition 29** Let $n \in [1,\infty]$.

(1) Suppose given an $\mathrm{A}_n$-algebra $A$ over $\mathcal{Z}$.

Let $\mathrm{Z}A := \mathrm{Kern}(m_1)$ be the $\mathcal{Z}$-graded module of *cycles*.

Let $\mathrm{B}A := \mathrm{Im}(m_1)$ be the $\mathcal{Z}$-graded module of *boundaries*.

Note that $\mathrm{B}A \subseteq \mathrm{Z}A$ since $m_1^2 = 0$ ; cf. Example 21.(1).

Let $\mathrm{H}A := (\mathrm{Z}A)/(\mathrm{B}A)$ be the $\mathcal{Z}$-graded *cohomology module* of $\mathcal{A}$.

Specifically, we have, at $z \in \mathrm{Mor}(\mathcal{Z})$,

$$\begin{array}{rcl} (\mathrm{Z}A)^z & = & \mathrm{Kern}(A^z \xrightarrow{m_1^A} A^{z[1]}) \\ (\mathrm{B}A)^z & = & \mathrm{Im}(A^{z[-1]} \xrightarrow{m_1^A} A^z)\ . \end{array}$$

Note that $(\mathrm{B}A)^z \subseteq (\mathrm{Z}A)^z$ ; cf. Example 21.(1).

(2) Suppose given a morphism $\tilde{A} \xrightarrow{f} A$ of $\mathrm{A}_n$-algebras. We shall define a shift-graded linear map

$$\mathrm{H}\tilde{A} \xrightarrow{\mathrm{H}f} \mathrm{H}A$$

of degree 0. At $z \in \mathrm{Mor}(\mathcal{Z})$, it is given by

$$\begin{array}{rcl} (\mathrm{H}\tilde{A})^z & \xrightarrow{(\mathrm{H}f)^z} & (\mathrm{H}\tilde{A})^z \\ a + (\mathrm{B}\tilde{A})^z & \mapsto & af_1 + (\mathrm{B}A)^z \end{array}$$

This is a welldefined linear map, since $f_1$ maps $(\mathrm{B}\tilde{A})^z$ to $(\mathrm{B}A)^z$ as well as $(\mathrm{Z}\tilde{A})^z$ to $(\mathrm{Z}A)^z$, because given $a' \in \tilde{A}^{z[-1]}$, we get

$$a' m_1^{\tilde{A}} f_1 \;=\; a' f_1 m_1^A \;\in\; (\mathrm{B}A)^z\ ;$$

cf. Example 22.(1).

Sometimes, we also write $\mathrm{H}f_1 := \mathrm{H}f$.

(3) A morphism $\tilde{A} \xrightarrow{f} A$ of $\mathrm{A}_n$-algebras is called a *quasiisomorphism* if $\mathrm{H}f$ is an isomorphism.

Since we do not know yet how to compose $\mathrm{A}_n$-morphisms, we do not have a category of $\mathrm{A}_n$-algebras at our disposal. Hence, at this point, we cannot decide whether $\mathrm{H}$ is a functor from the category of $\mathrm{A}_n$-algebras over $\mathcal{Z}$ to $\mathcal{Z}$-grad. Cf. Problem 23.(7) below.

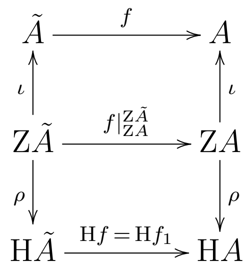
Our aim is to show the Theorem of Kadeishvili, Theorem 50 below, which, in case $R$ is a field, will establish the existence of a minimal $\mathrm{A}_n$-structure on $A$ and at the same time a quasiisomorphism from $\mathrm{H}A$ to $A$. This theorem seems to be hard to obtain by a direct calculation, though. We will make a detour, reinterpret Stasheff equations as a codifferential condition on a tensor coalgebra, in order to obtain an understandable proof. In the following Remark 30, we illustrate the first two steps towards Kadeishvili.

**Remark 30** Let $n \in [1, \infty]$. Suppose given an $\mathrm{A}_n$-algebra $A$ over $\mathcal{Z}$.

Denote by $\mathrm{B}A \xrightarrow{\tilde{\iota}} \mathrm{Z}A \xrightarrow{\iota} A$ the inclusion morphisms.

Denote by $\mathrm{Z}A \xrightarrow{\rho} \mathrm{H}A$ the residue class morphism. Note that $\mathrm{B}A = \mathrm{Kern}(\rho)$.

In particular, given a morphism $\tilde{A} \xrightarrow{f} A$ of $\mathrm{A}_n$-algebras, we get the following commutative diagram.



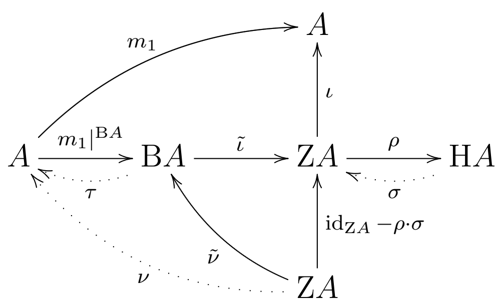
If $R$ is a field, we may choose a shift-graded linear map $\mathrm{Z}A \xleftarrow{\sigma} \mathrm{H}A$ of degree 0 such that $\sigma\rho = \mathrm{id}_{\mathrm{H}A}$ ; cf. Problem 15.(2).

If $R$ is a field, we may choose a shift-graded linear map $A \xleftarrow{\tau} \mathrm{B}A$ of degree $-1$ such that $\tau(m_1|^{\mathrm{B}A}) = \mathrm{id}_{\mathrm{H}A}$ ; cf. Problem 15.(2).

Since $(\mathrm{id}_{\mathrm{Z}A} - \rho \cdot \sigma)\rho = \rho - \rho \cdot \sigma\rho = \rho - \rho = 0$, there exists a unique shift-graded linear map $\mathrm{Z}A \xrightarrow{\tilde{\nu}} \mathrm{B}A$ of degree 0 such that $\tilde{\nu} \cdot \tilde{\iota} = \mathrm{id}_{\mathrm{Z}A} - \rho \cdot \sigma$ ; cf. Problem 15.(1). Write $\nu := \tilde{\nu}\tau$.

So

$$\nu \cdot m_1 \; = \; \tilde{\nu} \cdot \tau \cdot (m_1|^{\mathrm{B}A}) \cdot \tilde{\iota} \cdot \iota \; = \; \tilde{\nu} \cdot \tilde{\iota} \cdot \iota \; = \; (\mathrm{id}_{\mathrm{Z}A} - \rho \cdot \sigma) \cdot \rho \,.$$



Here the existence of the shift graded linear maps written with dotted arrows is only ensured if $R$ is a field.

**Remark 31** Suppose $R$ to be a field.

Suppose given an $\mathrm{A}_3$-algebra $A$ over $\mathcal{Z}$.

We will construct a minimal $\mathrm{A}_3$-structure $(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$ on $\mathrm{H}A$ and a quasiisomorphism $(q_1, q_2, q_3)$ of $\mathrm{A}_3$-algebras from $\mathrm{H}A$ to $A$.

*Step* 1. For this step, we will only need $A$ as an $\mathrm{A}_1$-algebra.

Let $\tilde{m}_1 = 0$. Let $q_1 := \sigma \cdot \iota$.

We have

$$\tilde{m}_1 \cdot \tilde{m}_1 \; = \; 0.$$

Hence the Stasheff equation at $k = 1$ holds; cf. Example 21.(1).

We have

$$\tilde{m}_1 \cdot q_1 \;=\; 0 \;=\; q_1 \cdot m_1 \,.$$

Hence the Stasheff equation for morphisms at $k = 1$ holds; cf. Example 22.(1).

Since we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}A & \xrightarrow{q_1} & A \\
\| & & \uparrow{\scriptstyle \iota} \\
\mathrm{ZH}A & \xrightarrow{\sigma} & \mathrm{Z}A \\
\| & & \downarrow{\scriptstyle \rho} \\
\mathrm{HH}A & = & \mathrm{H}A\,,
\end{array}$$

we have $\mathrm{H}q_1 = \mathrm{id}_{\mathrm{H}A}$, which is an isomorphism. So $q = (q_1, q_2, q_3)$, once constructed as a morphism of $\mathrm{A}_3$-algebras, will be a quasiisomorphism; cf. Definition 29.(2, 3).

*Step* 2. For this step, we will only need $A$ as an $\mathrm{A}_2$-algebra.

By Example 21.(2), we get

$$\begin{array}{rcl}
\iota^{\otimes 2} \cdot m_2 \cdot m_1 & = & \iota^{\otimes 2}(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1) \\
& = & (\iota m_1 \otimes \iota + \iota \otimes \iota m_1) \\
& = & 0\,.
\end{array}$$

Thus by Problem 15.(1), we get a unique shift-graded linear map $\check{m}_2 : (\mathrm{Z}A)^{\otimes 2} \to \mathrm{Z}A$ of degree 0 such that the following quadrangle commutes.

$$\begin{array}{ccc}
A^{\otimes 2} & \xrightarrow{m_2} & A \\
{\scriptstyle \iota^{\otimes 2}}\uparrow & & \uparrow{\scriptstyle \iota} \\
(\mathrm{Z}A)^{\otimes 2} & \xrightarrow{\check{m}_2} & \mathrm{Z}A
\end{array}$$

We *claim* that $((m_1|^{\mathrm{Z}A}) \otimes \mathrm{id}_{\mathrm{Z}A}) \cdot \check{m}_2 \cdot \rho \overset{!}{=} 0$ and that $(\mathrm{id}_{\mathrm{Z}A} \otimes (m_1|^{\mathrm{Z}A})) \cdot \check{m}_2 \cdot \rho \overset{!}{=} 0$.

We prove the first equation. The second then follows by an analogous reasoning.

Given $z \in \mathrm{Mor}(\mathcal{Z})$ and $(u, v) \in \mathrm{fact}_2(z[-1])$ and $a \in A^u$ and $\tilde{a} \in (\mathrm{Z}A)^v$, we have $a \otimes \tilde{a} \in (A^{\otimes 2})^{z[-1]}$ and obtain

$$\begin{array}{rcl}
(a \otimes \tilde{a})((m_1|^{\mathrm{Z}A}) \otimes \mathrm{id}_{\mathrm{Z}A})\check{m}_2 \cdot \rho & = & (a \otimes \tilde{a})(m_1 \otimes \mathrm{id})m_2 + (\mathrm{B}A)^z \\
& \overset{\text{Ex. 21.(2)}}{=} & -(a \otimes \tilde{a})(\mathrm{id} \otimes m_1)m_2 + (a \otimes \tilde{a})m_2 \cdot m_1 + (\mathrm{B}A)^z \\
& = & -(a \otimes \tilde{a}m_1) + (a \otimes \tilde{a})m_2 \cdot m_1 + (\mathrm{B}A)^z \\
& = & \big((a \otimes \tilde{a})m_2\big)m_1 + (\mathrm{B}A)^z \\
& = & 0\,.
\end{array}$$

This proves the *claim*. So by Problem 16.(2), we obtain a unique shift-graded linear map $\hat{m}_2 : (\mathrm{H}A)^{\otimes 2} \to \mathrm{H}A$ of degree 0 such that the following quadrangle commutes.

$$\begin{array}{ccc} (\mathrm{Z}A)^{\otimes 2} & \xrightarrow{\check{m}_2} & \mathrm{Z}A \\ {\scriptstyle \rho^{\otimes 2}} \downarrow & & \downarrow {\scriptstyle \rho} \\ (\mathrm{H}A)^{\otimes 2} & \xrightarrow{\hat{m}_2} & \mathrm{H}A \end{array}$$

The shift-graded linear map $\hat{m}_2$ can be obtained in a second way still. We will call the one resulting from the second construction $\tilde{m}_2$, with the aim of showing $\hat{m}_2 \overset{!}{=} \tilde{m}_2$.

Writing

$$\Psi_2 \ := \ (q_1 \otimes q_1) \cdot m_2 \ ,$$

taking under consideration that $\tilde{m}_1 = 0$, the Stasheff equation for morphisms the shift-graded linear maps $\tilde{m}_2$ and $q_2$ are to satisfy reads

$$\tilde{m}_2 \cdot q_1 - q_2 \cdot m_1 \ \overset{!}{=} \ \Psi_2 \ ;$$

cf. Example 22.(2).

We *claim* that $\Psi_2$ factors over $\iota$ as $\Psi_2 = \check{\Psi}_2 \cdot \iota$. We have to show that $\Psi_2 \cdot m_1 = 0$; cf. Problem 15.(1). In fact,

$$\begin{array}{rcl} (q_1 \otimes q_1) \cdot m_2 \cdot m_1 & \overset{\text{Ex. } 21.(2)}{=} & (q_1 \otimes q_1) \cdot (m_1 \otimes \mathrm{id}) \cdot m_2 + (q_1 \otimes q_1) \cdot (\mathrm{id} \otimes m_1) \cdot m_2 \\ & = & (\underbrace{q_1 m_1}_{=0} \otimes q_1) \cdot m_2 + (q_1 \otimes \underbrace{q_1 m_1}_{=0}) \cdot m_2 \\ & = & 0 \ . \end{array}$$

This proves the *claim*.

Letting

$$\begin{array}{rcl} q_2 & := & -\check{\Psi}_2 \cdot \nu \\ \tilde{m}_2 & := & \check{\Psi}_2 \cdot \rho \ , \end{array}$$

we obtain

$$\begin{array}{rcl} \tilde{m}_2 \cdot q_1 - q_2 \cdot m_1 & = & \check{\Psi}_2 \cdot \rho \cdot q_1 + \check{\Psi}_2 \cdot \nu \cdot m_1 \\ & = & \check{\Psi}_2 \cdot \rho \cdot \sigma \cdot \iota + \check{\Psi}_2 \cdot (\mathrm{id} - \rho \cdot \sigma) \cdot \iota \\ & = & \check{\Psi}_2 \cdot \iota \\ & = & \Psi_2 \ , \end{array}$$

as required.

It remains to show

$$\hat{m}_2 \ \overset{!}{=} \ \tilde{m}_2 \ .$$

It suffices to show that $\rho^{\otimes 2} \hat{m}_2 \overset{!}{=} \rho^{\otimes 2} \tilde{m}_2$, since $\rho^{\otimes 2}$ is piecewise surjective; cf. Problem 16.(1). So we have to show that $\rho^{\otimes 2} \check{\Psi}_2 \rho \overset{!}{=} \check{m}_2 \rho$. It suffices to find a shift-graded linear map $\xi : (\mathrm{Z}A)^{\otimes 2} \to A$ of degree $-1$ such that

$$\rho^{\otimes 2} \check{\Psi}_2 - \check{m}_2 \ \overset{!}{=} \ \xi m_1|^{\mathrm{Z}A} \ ,$$

i.e. such that

$$(\rho^{\otimes 2}\check{\Psi}_2 - \check{m}_2)\iota \;\stackrel{!}{=}\; \xi m_1|^{ZA}\iota \;=\; \xi m_1 \; .$$

But

$$\begin{array}{rl} & (\rho^{\otimes 2}\check{\Psi}_2 - \check{m}_2)\iota \\ = & \rho^{\otimes 2}\check{\Psi}_2\iota - \check{m}_2\iota \\ = & \rho^{\otimes 2}\Psi_2 - \iota^{\otimes 2}m_2 \\ = & \rho^{\otimes 2}(q_1 \otimes q_1)m_2 - \iota^{\otimes 2}m_2 \\ = & (\rho\cdot\sigma\cdot\iota \otimes \rho\cdot\sigma\cdot\iota - \iota\otimes\iota)m_2 \\ = & ((\iota - \nu m_1)\otimes(\iota - \nu m_1) - \iota\otimes\iota)m_2 \\ = & (-\nu m_1 \otimes \iota - \iota \otimes \nu m_1 + \nu m_1 \otimes \nu m_1)m_2 \\ = & -(\nu\otimes\iota)(m_2 m_1 - \mathrm{id}\otimes m_1) - (\iota\otimes\nu)(m_2 m_1 - (m_1\otimes\mathrm{id})m_2) + (\nu m_1 \otimes \nu)(m_2 m_1 - m_1 \otimes \mathrm{id}) \\ = & (-(\nu\otimes\iota)m_2 - (\iota\otimes\nu)m_2 + (\nu m_1\otimes\nu)m_2)m_1 \end{array}$$

since $\iota m_1 = 0$.

*An associativity.* We *claim*

$$(\tilde{m}_2 \otimes \mathrm{id} - \mathrm{id}\otimes\tilde{m}_2)\tilde{m}_2 \;\stackrel{!}{=}\; 0 \; .$$

It suffices to show that $\rho^{\otimes 3}(\tilde{m}_2 \otimes \mathrm{id} - \mathrm{id}\otimes\tilde{m}_2)\tilde{m}_2 \stackrel{!}{=} 0$, since $\rho^{\otimes 3}$ is piecewise surjective; cf. Problem 16.(1). Now

$$\begin{array}{rcl} \rho^{\otimes 3}(\tilde{m}_2 \otimes \mathrm{id} - \mathrm{id}\otimes\tilde{m}_2)\tilde{m}_2 & = & (\rho^{\otimes 2}\tilde{m}_2 \otimes \rho - \rho\otimes\rho^{\otimes 2}\tilde{m}_2)\tilde{m}_2 \\ & = & (\check{m}_2\rho\otimes\rho - \rho\otimes\check{m}_2\rho)\tilde{m}_2 \\ & = & (\check{m}_2\otimes\mathrm{id} - \mathrm{id}\otimes\check{m}_2)\rho^{\otimes 2}\tilde{m}_2 \\ & = & (\check{m}_2\otimes\mathrm{id} - \mathrm{id}\otimes\check{m}_2)\check{m}_2\rho \; . \end{array}$$

So it suffices to find a shift-graded linear map $\eta : (\mathrm{Z}A)^{\otimes 3} \to A$ of degree $-1$ such that $(\check{m}_2\otimes\mathrm{id} - \mathrm{id}\otimes\check{m}_2)\check{m}_2 \stackrel{!}{=} \eta m_1|^{\mathrm{Z}A}$, i.e. such that $(\check{m}_2\otimes\mathrm{id} - \mathrm{id}\otimes\check{m}_2)\check{m}_2\iota \stackrel{!}{=} \eta m_1|^{\mathrm{Z}A}\iota = \eta m_1$. We obtain

$$\begin{array}{rl} & (\check{m}_2\otimes\mathrm{id} - \mathrm{id}\otimes\check{m}_2)\check{m}_2\cdot\iota \\ = & (\check{m}_2\otimes\mathrm{id} - \mathrm{id}\otimes\check{m}_2)\iota^{\otimes 2}m_2 \\ = & (\check{m}_2\cdot\iota\otimes\iota - \iota\otimes\check{m}_2\cdot\iota)m_2 \\ = & (\iota^{\otimes 2}m_2\otimes\iota - \iota\otimes\iota^{\otimes 2}m_2)m_2 \\ = & \iota^{\otimes 3}(m_2\otimes\mathrm{id} - \mathrm{id}\otimes m_2)m_2 \\ \stackrel{\text{Ex. 21.(3)}}{=} & \iota^{\otimes 3}(-(m_1\otimes\mathrm{id}^{\otimes 2})\cdot m_3 - (\mathrm{id}\otimes m_1\otimes\mathrm{id})\cdot m_3 - (\mathrm{id}^{\otimes 2}\otimes m_1)\cdot m_3 - m_3\cdot m_1) \\ = & -\iota^{\otimes 3}\cdot m_3\cdot m_1 \end{array}$$

since $\iota m_1 = 0$.

*Step* 3. Write

$$\Psi_3 \;:=\; (-\tilde{m}_2\otimes\mathrm{id} + \mathrm{id}\otimes\tilde{m}_2)q_2 + (q_1\otimes q_2)\cdot m_2 - (q_2\otimes q_1)\cdot m_2 + (q_1\otimes q_1\otimes q_1)\cdot m_3 \; .$$

We have to find $\tilde{m}_3$ and $q_3$ such that

$$\tilde{m}_3 \cdot q_1 - q_3 \cdot m_1 \stackrel{!}{=} \Psi_3 \, ;$$

cf. Example 22.(3).

Provided we can show that $\Psi_3 \cdot m_1 \stackrel{!}{=} 0$, then we can write $\Psi_3 = \check{\Psi}_3 \cdot \iota$; cf. Problem 15.(1). Then letting

$$\begin{array}{rcl} q_3 & := & -\check{\Psi}_3 \cdot \nu \\ \tilde{m}_3 & := & \check{\Psi}_3 \cdot \rho \, , \end{array}$$

we obtain

$$\begin{array}{rcl} \tilde{m}_3 \cdot q_1 - q_3 \cdot m_1 & = & \check{\Psi}_3 \cdot \rho \cdot q_1 + \check{\Psi}_3 \cdot \nu \cdot m_1 \\ & = & \check{\Psi}_3 \cdot \rho \cdot \sigma \cdot \iota + \check{\Psi}_3 \cdot (\mathrm{id} - \rho \cdot \sigma) \cdot \iota \\ & = & \check{\Psi}_3 \cdot \rho \cdot \sigma \cdot \iota + \check{\Psi}_3 \cdot (\mathrm{id} - \rho \cdot \sigma) \cdot \iota \\ & = & \check{\Psi}_3 \cdot \iota \\ & = & \Psi_3 \, , \end{array}$$

as required.

So it remains to show $\Psi_3 \cdot m_1 \stackrel{!}{=} 0$. Plugging in $q_2 \cdot m_1 = \tilde{m}_2 \cdot q_1 - (q_1 \otimes q_1) m_2$ from Example 22.(2) and $m_2 \cdot m_1 = (m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1) m_2$ from Example 21.(2) and $m_3 m_1 = -(m_1 \otimes \mathrm{id}^{\otimes 2}) \cdot m_3 - (\mathrm{id} \otimes m_1 \otimes \mathrm{id}) \cdot m_3 - (\mathrm{id}^{\otimes 2} \otimes m_1) \cdot m_3 - (m_2 \otimes \mathrm{id}) \cdot m_2 + (\mathrm{id} \otimes m_2) \cdot m_2$ from Example 21.(3), using $q_1 \cdot m_1 = \sigma \cdot \iota \cdot m_1 = 0$ as well as associativity of $\tilde{m}_2$, we obtain

$$\begin{array}{rl} & \Psi_3 \cdot m_1 \\ = & -(\tilde{m}_2 \otimes \mathrm{id}) q_2 \cdot m_1 + (\mathrm{id} \otimes \tilde{m}_2) q_2 \cdot m_1 \\ & +(q_1 \otimes q_2) \cdot m_2 \cdot m_1 - (q_2 \otimes q_1) \cdot m_2 \cdot m_1 + (q_1 \otimes q_1 \otimes q_1) \cdot m_3 \cdot m_1 \\ = & -(\tilde{m}_2 \otimes \mathrm{id}) \tilde{m}_2 \cdot q_1 + (\tilde{m}_2 \otimes \mathrm{id})(q_1 \otimes q_1) m_2 \\ & +(\mathrm{id} \otimes \tilde{m}_2) \tilde{m}_2 \cdot q_1 - (\mathrm{id} \otimes \tilde{m}_2)(q_1 \otimes q_1) m_2 \\ & +(q_1 \otimes q_2)(m_1 \otimes \mathrm{id}) m_2 + (q_1 \otimes q_2)(\mathrm{id} \otimes m_1) m_2 \\ & -(q_2 \otimes q_1)(m_1 \otimes \mathrm{id}) m_2 - (q_2 \otimes q_1)(\mathrm{id} \otimes m_1) m_2 \\ & -(q_1 \otimes q_1 \otimes q_1)(m_1 \otimes \mathrm{id}^{\otimes 2}) \cdot m_3 \\ & -(q_1 \otimes q_1 \otimes q_1)(\mathrm{id} \otimes m_1 \otimes \mathrm{id}) \cdot m_3 \\ & -(q_1 \otimes q_1 \otimes q_1)(\mathrm{id}^{\otimes 2} \otimes m_1) \cdot m_3 \\ & -(q_1 \otimes q_1 \otimes q_1)(m_2 \otimes \mathrm{id}) \cdot m_2 \\ & +(q_1 \otimes q_1 \otimes q_1)(\mathrm{id} \otimes m_2) \cdot m_2 \\ = & (\tilde{m}_2 \cdot q_1 \otimes q_1) m_2 \\ & -(q_1 \otimes \tilde{m}_2 \cdot q_1) m_2 \\ & +(q_1 \otimes q_2 \cdot m_1) m_2 \\ & -(q_2 \cdot m_1 \otimes q_1) m_2 \\ & -((q_1 \otimes q_1) m_2 \otimes q_1) m_2 \\ & +(q_1 \otimes (q_1 \otimes q_1) m_2) \cdot m_2 \\ = & ((\tilde{m}_2 \cdot q_1 - q_2 \cdot m_1 - (q_1 \otimes q_1) m_2) \otimes q_1) m_2 \\ & (q_1 \otimes (-\tilde{m}_2 \cdot q_1 + q_2 \cdot m_1 + (q_1 \otimes q_1) m_2)) m_2 \\ = & 0 \, . \end{array}$$

Note that the only nontrivial Stasheff equation for $(\tilde m_3, \tilde m_2, \tilde m_1)$ takes place at $k = 2$, where it reads $(\tilde m_2 \otimes \mathrm{id} - \mathrm{id} \otimes \tilde m_2) \cdot \tilde m_2 = 0$, whose validity we have verified.

To directly proceed in this way, i.e. to construct $\Psi_n$ analogously for $n \geqslant 4$ and to prove $\Psi_n \cdot m_1 \overset{!}{=} 0$ directly, seems to be involved. We will take a conceptual detour to prove the Theorem of Kadeishvili; cf. Theorem 50 below.

## 1.5 Getting rid of signs by conjugation

Let $\mathcal{Z}$ be a grading category.

**Definition 35** Let $A$ be a $\mathcal{Z}$-graded module. Recall that $A^{[1]}$ is the $\mathcal{Z}$-graded module having

$$(A^{[1]})^z \;=\; A^{z[1]}$$

for $z \in \mathrm{Mor}(\mathcal{Z})$; cf. Definition 12.

Define the shift-graded linear map

$$\omega \;=\; \omega_A \;:\; A^{[1]} \;\to\; A$$

at $z \in \mathrm{Mor}(\mathcal{Z})$ by

$$\begin{array}{rcl} (A^{[1]})^z & \overset{\omega}{\to} & A^{z[1]} \\ a & \mapsto & a \;. \end{array}$$

**Lemma 36** *Let $n \in [0, \infty]$.*

*Let $(A, (m_\ell)_\ell)$ be a pre-$\mathrm{A}_n$-algebra over $\mathcal{Z}$.*

*Given $\ell \in [1, n] \cap \mathbf{Z}$, we write*

$${}^{\omega}m_\ell \;:=\; \omega^{\otimes \ell} \cdot m_\ell \cdot \omega^- \;:\; (A^{[1]})^{\otimes \ell} \;\to\; A^{[1]} \;,$$

*called the $\omega$-*conjugate *of $m_\ell$. Note that ${}^{\omega}m_\ell$ is of degree $1$, independent of $\ell$.*

*Suppose given $k \in [1, n] \cap \mathbf{Z}$.*

*Given $(r, s, t) \geqslant (0, 1, 0)$ with $r + s + t = k$, we have*

$$(\mathrm{id}^{\otimes r} \otimes {}^{\omega}m_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^{\omega}m_{r+1+t} \;=\; \omega^{\otimes k}\big((-1)^{r+st}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \cdot m_{r+1+t}\big)\omega^- \;.$$

*In particular, the Stasheff equation at $k$, viz.*

$$0 \;=\; \sum_{\substack{(r,s,t) \geqslant (0,1,0) \\ r+s+t = k}} (-1)^{r+st}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \cdot m_{r+1+t} \;,$$

*holds if and only if*

$$0 \;=\; \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t\,=\,k}} (\mathrm{id}^{\otimes r}\otimes {}^{\omega}m_s\otimes \mathrm{id}^{\otimes t})\cdot {}^{\omega}m_{r+1+t}$$

*holds.*

*Proof.* Given $(r,s,t)\geqslant(0,1,0)$ with $r+s+t=k$, we have

$$\begin{array}{rl} & (\mathrm{id}^{\otimes r}\otimes {}^{\omega}m_s\otimes \mathrm{id}^{\otimes t})\cdot {}^{\omega}m_{r+1+t} \\ = & (\mathrm{id}^{\otimes r}\otimes {}^{\omega}m_s\otimes \mathrm{id}^{\otimes t})\cdot \omega^{\otimes r+1+t}\cdot m_{r+1+t}\cdot\omega^{-} \\ = & (-1)^r(\omega^{\otimes r}\otimes(\omega^{\otimes s}\cdot m_s)\otimes\omega^{\otimes t})\cdot m_{r+1+t}\cdot\omega^{-} \\ = & \omega^{\otimes k}\big((-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes m_s\otimes \mathrm{id}^{\otimes t})\cdot m_{r+1+t}\big)\omega^{-}\ . \end{array}$$

So

$$0 \;=\; \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t\,=\,k}} (\mathrm{id}^{\otimes r}\otimes {}^{\omega}m_s\otimes \mathrm{id}^{\otimes t})\cdot {}^{\omega}m_{r+1+t}$$

holds if and only if

$$0 \;=\; \omega^{\otimes k}\Big(\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t\,=\,k}} (-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes m_s\otimes \mathrm{id}^{\otimes t})\cdot m_{r+1+t}\Big)\omega^{-}$$

holds, i.e. if and only if

$$0 \;=\; \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t\,=\,k}} (-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes m_s\otimes \mathrm{id}^{\otimes t})\cdot m_{r+1+t}$$

holds. □

**Lemma 37** *Let $n\in[0,\infty]$.*

*Let $\tilde{A}=(\tilde{A},(\tilde{m}_\ell)_\ell)$ and $A=(A,(m_\ell)_\ell)$ be pre-$\mathrm{A}_n$-algebras over $\mathcal{Z}$.*

*Let $f=(f_\ell)_\ell$ be a pre-$\mathrm{A}_n$-morphism from $\tilde{A}$ to $A$.*

*Given $\ell\in[1,n]\cap\mathbf{Z}$, we write*

$${}^{\omega}f_\ell \;:=\; \omega^{\otimes\ell}\cdot f_\ell\cdot\omega^{-}\;:\;(\tilde{A}^{[1]})^{\otimes\ell}\;\to\;A^{[1]}\ ,$$

*called the $\omega$-*conjugate *of $f_\ell$. Note that ${}^{\omega}f_\ell$ is of degree $0$, independent of $\ell$.*

*Suppose given $k\in[1,n]\cap\mathbf{Z}$.*

*Given $(r,s,t)\geqslant(0,1,0)$ with $r+s+t=k$, we have*

$$(\mathrm{id}^{\otimes r}\otimes {}^{\omega}\tilde{m}_s\otimes \mathrm{id}^{\otimes t})\cdot {}^{\omega}f_{r+1+t} \;=\; \omega^{\otimes k}\big((-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes\tilde{m}_s\otimes \mathrm{id}^{\otimes t})\cdot f_{r+1+t}\big)\omega^{-}\ .$$

*Given $r \in [1,k]$ and $(i_j)_{j\in[1,r]} \geqslant 0$, we have*

$$\Big(\bigotimes_{j\in[1,r]} {}^{\omega}f_{i_j}\Big)\cdot {}^{\omega}m_r \;=\; \omega^{\otimes k}\Big(\lfloor(1-i_j)_j\,,(i_j)_j\rfloor\Big(\bigotimes_{j\in[1,r]} f_{i_j}\Big)\cdot m_r\Big)\omega^- \,.$$

*In particular, the Stasheff equation for morphisms at $k$, viz.*

$$\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes\tilde m_s\otimes\mathrm{id}^{\otimes t})\cdot f_{r+1+t} \;=\; \sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=k}} \lfloor(1-i_j)_j\,,(i_j)_j\rfloor\Big(\bigotimes_{j\in[1,r]} f_{i_j}\Big)\cdot m_r$$

*holds if and only if*

$$\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (\mathrm{id}^{\otimes r}\otimes{}^{\omega}\tilde m_s\otimes\mathrm{id}^{\otimes t})\cdot {}^{\omega}f_{r+1+t} \;=\; \sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=k}} \Big(\bigotimes_{j\in[1,r]} {}^{\omega}f_{i_j}\Big)\cdot {}^{\omega}m_r$$

*holds.*

*Proof.* Given $(r,s,t)\geqslant(0,1,0)$ with $r+s+t=k$, we have

$$\begin{aligned}
&(\mathrm{id}^{\otimes r}\otimes{}^{\omega}\tilde m_s\otimes\mathrm{id}^{\otimes t})\cdot {}^{\omega}f_{r+1+t}\\
=\;&(\mathrm{id}^{\otimes r}\otimes{}^{\omega}\tilde m_s\otimes\mathrm{id}^{\otimes t})\cdot\omega^{\otimes r+1+t}\cdot f_{r+1+t}\cdot\omega^-\\
=\;&(-1)^r(\omega^{\otimes r}\otimes(\omega^{\otimes s}\cdot\tilde m_s)\otimes\omega^{\otimes t})\cdot f_{r+1+t}\cdot\omega^-\\
=\;&\omega^{\otimes k}\big((-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes\tilde m_s\otimes\mathrm{id}^{\otimes t})\cdot f_{r+1+t}\big)\omega^- \,.
\end{aligned}$$

Given $r\in[1,k]$ and $(i_j)_{j\in[1,r]}\geqslant 0$, we have

$$\begin{aligned}
&\big(\textstyle\bigotimes_{j\in[1,r]} {}^{\omega}f_{i_j}\big)\cdot{}^{\omega}m_r\\
=\;&\big(\textstyle\bigotimes_{j\in[1,r]} {}^{\omega}f_{i_j}\big)\cdot\omega^{\otimes r}\cdot m_r\cdot\omega^-\\
=\;&\big(\textstyle\bigotimes_{j\in[1,r]} {}^{\omega}f_{i_j}\omega\big)\cdot m_r\cdot\omega^-\\
=\;&\big(\textstyle\bigotimes_{j\in[1,r]} \omega^{\otimes i_j}f_{i_j}\big)\cdot m_r\cdot\omega^-\\
=\;&\omega^{\otimes k}\big(\lfloor(1-i_j)_j\,,(i_j)_j\rfloor\big(\textstyle\bigotimes_{j\in[1,r]} f_{i_j}\big)\cdot m_r\big)\omega^-
\end{aligned}$$

So

$$\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (\mathrm{id}^{\otimes r}\otimes{}^{\omega}\tilde m_s\otimes\mathrm{id}^{\otimes t})\cdot {}^{\omega}f_{r+1+t} \;=\; \sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ k=\sum_j i_j=k}} \Big(\bigotimes_{j\in[1,r]} {}^{\omega}f_{i_j}\Big)\cdot {}^{\omega}m_r$$

holds if and only if

$$\omega^{\otimes k}\Big(\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes\tilde m_s\otimes\mathrm{id}^{\otimes t})\cdot f_{r+1+t}\Big)\omega^- \;=\; \omega^{\otimes k}\Big(\sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=k}} \lfloor(1-i_j)_j\,,(i_j)_j\rfloor\Big(\bigotimes_{j\in[1,r]} f_{i_j}\Big)\cdot m_r\Big)\omega^-$$

holds, i.e. if and only if

$$\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes\tilde m_s\otimes\mathrm{id}^{\otimes t})\cdot f_{r+1+t} \;=\; \sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=k}} \lfloor(1-i_j)_j\,,(i_j)_j\rfloor\Big(\bigotimes_{j\in[1,r]} f_{i_j}\Big)\cdot m_r$$

holds. □

## 1.6 A tensor coalgebra interpretation

Let $\mathcal{Z}$ be a grading category.

Given a $\mathcal{Z}$-graded module $V$ and $a,\, b \in \mathbf{Z}_{\geqslant 1}$, we often abbreviate an elementary tensor as follows. Given $v_i \in V$ for $i \in [a,b]$, we write

$$v^{\otimes}_{[a,b]} := v_a \otimes v_{a+1} \otimes \ldots v_{b-1} \otimes v_b \;.$$

**Definition 38**

(1) A *coalgebra* over $\mathcal{Z}$ is a $\mathcal{Z}$-graded module $T$, equipped with a shift-graded linear map $\Delta : T \to T \otimes T$ of degree 0, called *comultiplication*, that is *coassociative*, i.e. that satisfies

$$\Delta(\Delta \otimes \mathrm{id}) \;=\; \Delta(\mathrm{id} \otimes \Delta) \;.$$

$$\begin{array}{ccc} T \otimes T & \xrightarrow{\mathrm{id} \otimes \Delta} & T \otimes T \otimes T \\ \uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle \Delta \otimes \mathrm{id}} \\ T & \xrightarrow{\Delta} & T \otimes T \end{array}$$

Often, we just write $T = (T, \Delta)$.

(2) Suppose given coalgebras $T = (T, \Delta)$ and $\tilde{T} = (\tilde{T}, \tilde{\Delta})$ over $\mathcal{Z}$. A *morphism of coalgebras*, also called *coalgebra morphism*, (over $\mathcal{Z}$) from $T$ to $\tilde{T}$ is a shift-graded linear map $\psi : T \to \tilde{T}$ of degree 0 such that

$$\psi \tilde{\Delta} \;=\; \Delta(\psi \otimes \psi) \;.$$

$$\begin{array}{ccc} T \otimes T & \xrightarrow{\psi \otimes \psi} & T \otimes T \\ \uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle \Delta} \\ T & \xrightarrow{\psi} & T \end{array}$$

(3) Suppose given a coalgebra $T = (T, \Delta)$ over $\mathcal{Z}$. A *coderivation* on $T$ is a shift-graded linear map $T \xrightarrow{\delta} T$ of degree 1 such that the *co-Leibniz-rule*

$$\delta \Delta \;=\; \Delta(\mathrm{id} \otimes \delta + \delta \otimes \mathrm{id})$$

holds.

Note that both sides are linear in $\delta$, so that a linear combination of coderivations on $T$ is again a coderivation on $T$.

A coderivation $\delta$ on $T$ is called a *codifferential* if $\delta^2 = 0$.

(4) A *coalgebra with codifferential* over $\mathcal{Z}$ is a coalgebra $T$ over $\mathcal{Z}$, equipped with a codifferential $\delta$ on $T$.

Often, we just write $T = (T, \Delta, \delta)$.

(5) Suppose given coalgebras $T = (T, \Delta, \delta)$ and $\tilde{T} = (\tilde{T}, \tilde{\Delta}, \tilde{\delta})$ over $\mathcal{Z}$. A *morphism of coalgebras with codifferential* (over $\mathcal{Z}$) is a coalgebra morphism $T \xrightarrow{\psi} \tilde{T}$ such that

$$\psi\delta \;=\; \delta\psi \;.$$

**Lemma 39 (and Definition)** Let $V$ be a $\mathcal{Z}$-graded module. Let $n,\, \tilde{n} \,\in\, [0, \infty]$.

Consider the $\mathcal{Z}$-graded module

$$\mathrm{T}_{\leqslant n}(V) \;:=\; \bigoplus_{k\in[1,n]\cap\mathbf{Z}} V^{\otimes k} \;.$$

In particular, we often write

$$\mathrm{T}(V) \;:= \mathrm{T}_{\leqslant\infty}(V) \;=\; \bigoplus_{k\in\mathbf{Z}_{\geqslant 1}} V^{\otimes k} \;.$$

Moreover, we identify

$$V \;=\; \mathrm{T}_{\leqslant 1}(V) \;.$$

(1) Let the shift-graded linear map $\Delta = \Delta_{n,V} : \mathrm{T}_{\leqslant n}(V) \to \mathrm{T}_{\leqslant n}(V) \otimes \mathrm{T}_{\leqslant n}(V)$ of degree 0 be defined at $z \in \mathrm{Mor}(\mathcal{Z})$ on the summand for $k \in [1, n] \cap \mathbf{Z}$, viz.

$$(V^{\otimes k})^z \;=\; \bigoplus_{(y_1,\dots,y_k)\in\mathrm{fact}_k(z)} \; \bigotimes_{i\in[1,k]} V^{y_i} \;,$$

by defining it on its summand at $(y_1, \dots, y_k) \in \mathrm{fact}_k(z)$ by

$$\begin{array}{rcl} \bigotimes_{i\in[1,k]} V^{y_i} & \to & (\mathrm{T}_{\leqslant n}(V) \otimes \mathrm{T}_{\leqslant n}(V))^z \\ v^{\otimes}_{[1,k]} \;=\; v_1 \otimes \dots \otimes v_k & \mapsto & \displaystyle\sum_{\substack{(i,j)\in\mathbf{Z}_{\geqslant 1}\times\mathbf{Z}_{\geqslant 1}\\ i+j=k}} v_1 \otimes \dots \otimes v_i \boldsymbol{\otimes} v_{i+1} \otimes \dots \otimes v_{i+j} \;=\; \sum_{\substack{(i,j)\in\mathbf{Z}_{\geqslant 1}\times\mathbf{Z}_{\geqslant 1}\\ i+j=k}} v^{\otimes}_{[1,i]} \boldsymbol{\otimes} v^{\otimes}_{[i+1,i+j]} \;. \end{array}$$

Here the boldfaced tensor product symbol $\boldsymbol{\otimes}$ merely indicates the summand the term is mapped to, that is $v_1 \otimes \dots \otimes v_i \boldsymbol{\otimes} v_{i+1} \otimes \dots \otimes v_{i+j} \in (V^{\otimes i} \otimes V^{\otimes j})^z$.

Note that

$$v^{\otimes}_{[1,k]}\Delta \;\in\; (\mathrm{T}_{\leqslant n-1}(V) \otimes \mathrm{T}_{\leqslant n-1}(V))^z \;.$$

So also the restricted shift-graded linear map $\Delta|^{\mathrm{T}_{\leqslant n-1}(V)\otimes\mathrm{T}_{\leqslant n-1}(V)}$ of degree 0 exists.

*Then* $\mathrm{T}_{\leqslant n}(V) \;=\; (\mathrm{T}_{\leqslant n}(V), \Delta_{n,V})$ *is a coalgebra, called the* tensor coalgebra *of* $V$ *bounded by* $n$.

If $n = \infty$, we usually omit to mention that $\mathrm{T}(V)$ is bounded by $\infty$.

Recall that for $k \in [1, n] \cap \mathbf{Z}$, we have shift-graded inclusion and projection maps $V^{\otimes k} \xrightarrow{\iota_k} \mathrm{T}_{\leqslant n}(V) \xrightarrow{\pi_k} V^{\otimes k}$ of degree 0 ; cf. Problem 20.(1).

(2) *Suppose given* $k \in [1,n] \cap \mathbf{Z}$*. The image of* $\iota_k \Delta : V^{\otimes k} \to \mathrm{T}_{\leqslant n}(V) \otimes \mathrm{T}_{\leqslant n}(V)$ *is contained in* $\mathrm{T}_{\leqslant k-1}(V) \otimes \mathrm{T}_{\leqslant k-1}(V)$*.*

(3) Let the shift-graded linear map

$$\mathrm{T}_{\leqslant n}(V) \otimes \mathrm{T}_{\leqslant \tilde{n}}(V) \xrightarrow{\mu_{n,\tilde{n},V}} \mathrm{T}_{\leqslant n+\tilde{n}}(V)$$

of degree 0 be defined at $z \in \mathrm{Mor}(\mathcal{Z})$ for $(u, \tilde{u}) \in \mathrm{fact}_2(z)$, i.e. $z = u\tilde{u}$, by

$$\mathrm{T}_{\leqslant n}(V)^u \otimes \mathrm{T}_{\leqslant \tilde{n}}(V)^{\tilde{u}} \xrightarrow{\mu_{n,\tilde{n},V}} \mathrm{T}_{\leqslant n+\tilde{n}}(V)^{u\tilde{u}} \ ,$$

which in turn on

$$(V^{\otimes k})^u \otimes (V^{\otimes \tilde{k}})^{\tilde{u}} \xrightarrow{\mu_{n,\tilde{n},V}} (V^{\otimes k+\tilde{k}})^{u\tilde{u}}$$

for $k \in [1,n]$ and $\tilde{k} \in [1,\tilde{n}]$ is defined on the summand belonging to

$$\begin{array}{rcl} (y_1, \ldots, y_k) & \in & \mathrm{fact}_k(u) \\ (\tilde{y}_1, \ldots, \tilde{y}_{\tilde{k}}) & \in & \mathrm{fact}_{\tilde{k}}(\tilde{u}) \end{array}$$

by

$$\begin{array}{rclcl} (V^{y_1} \otimes \ldots \otimes V^{y_k}) & \otimes & (V^{\tilde{y}_1} \otimes \ldots \otimes V^{\tilde{y}_{\tilde{k}}}) & \xrightarrow{\mu_{n,\tilde{n},V}} & V^{\otimes k+\tilde{k}} \\ v^{\otimes}_{[1,k]} & \otimes & \tilde{v}^{\otimes}_{[1,\tilde{k}]} & \mapsto & v^{\otimes}_{[1,k]} \otimes \tilde{v}^{\otimes}_{[1,\tilde{k}]} \ . \end{array}$$

(4) *We have* $\mathrm{Kern}\,\Delta = V$*.*

*Proof.*

*Ad* (1). We have to show coassociativity of $\Delta$. Let $z \in \mathrm{Mor}(\mathcal{Z})$. Let $k \in [1,n] \cap \mathbf{Z}$. Let $(y_1, \ldots, y_k) \in \mathrm{fact}_k(z)$. Let $v_i \in V^{y_i}$ for $i \in [1,k]$. Recall that we may abbreviate $v^{\otimes}_{[1,k]} = v_1 \otimes \ldots v_k$ .

On the one hand, we obtain

$$\begin{aligned} v^{\otimes}_{[1,k]}\Delta(\mathrm{id} \otimes \Delta) &= \Big( \sum_{\substack{(i,j) \geqslant (1,1) \\ i+j=k}} v^{\otimes}_{[1,i]} \otimes v^{\otimes}_{[i+1,i+j]} \Big)(\mathrm{id} \otimes \Delta) \\ &= \sum_{\substack{(i,j) \geqslant (1,1) \\ i+j=k}} v^{\otimes}_{[1,i]} \otimes \Big( \sum_{\substack{(u,w) \geqslant (1,1) \\ u+w=j}} v^{\otimes}_{[i+1,i+u]} \otimes v^{\otimes}_{[i+u+1,i+u+w]} \Big) \\ &= \sum_{\substack{(i,u,w) \geqslant (1,1,1) \\ i+u+w=k}} v^{\otimes}_{[1,i]} \otimes v^{\otimes}_{[i+1,i+u]} \otimes v^{\otimes}_{[i+u+1,i+u+w]} \ . \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} v^{\otimes}_{[1,k]}\Delta(\Delta\otimes\mathrm{id}) &= \Big(\sum_{\substack{(i,j)\geqslant(1,1)\\ i+j=k}} v^{\otimes}_{[1,i]}\otimes v^{\otimes}_{[i+1,i+j]}\Big)(\Delta\otimes\mathrm{id}) \\ &= \sum_{\substack{(i,j)\geqslant(1,1)\\ i+j=k}} \Big(\sum_{\substack{(u,w)\geqslant(1,1)\\ u+w=i}} v^{\otimes}_{[1,u]}\otimes v^{\otimes}_{[u+1,u+w]}\Big)\otimes v^{\otimes}_{[i+1,i+j]} \\ &= \sum_{\substack{(u,w,j)\geqslant(1,1,1)\\ u+w+j=k}} v^{\otimes}_{[1,u]}\otimes v^{\otimes}_{[u+1,u+w]}\otimes v^{\otimes}_{[u+w+1,u+w+j]}\ . \end{aligned}$$

So both results coincide. Hence $\Delta(\mathrm{id}\otimes\Delta) = \Delta(\Delta\otimes\mathrm{id})$.

*Ad* (4). We have to show that $(\mathrm{Kern}\,\Delta)^z \overset{!}{=} V^z$ for $z \in \mathrm{Mor}(\mathcal{Z})$.

*Ad* $\supseteq$. Suppose given $v_1 \in V^z$. Then $v_1\Delta = v^{\otimes}_{[1,1]}\Delta = \sum_{\substack{(i,j)\geqslant(1,1)\\ i+j=1}} v^{\otimes}_{[1,i]}\otimes v^{\otimes}_{[i+1,i+j]} = 0$ as an empty sum.

*Ad* $\subseteq$. Write $\Delta' := \Delta|^{\mathrm{T}_{\leqslant n-1}(V)\otimes\mathrm{T}_{\leqslant n-1}(V)}$ ; cf. (1). We have to show that $(\mathrm{Kern}\,\Delta')^z \overset{!}{\subseteq} V^z$. Note that we have the shift-graded linear projection map $\mathrm{T}_{\leqslant n-1}(V) \xrightarrow{\pi_1} V^{\otimes 1} = V$ ; cf. Problem 20.(1). So we have

$$\mathrm{T}_{\leqslant n}(V) \xrightarrow{\Delta'} \mathrm{T}_{\leqslant n-1}(V)\otimes\mathrm{T}_{\leqslant n-1}(V) \xrightarrow{\pi_1\otimes\mathrm{id}} V\otimes\mathrm{T}_{\leqslant n-1}(V) \xrightarrow{\mu_{1,n-1,V}} V\ .$$

Let $k \in [1,n]\cap\mathbf{Z}$. Let $(y_1,\dots,y_k) \in \mathrm{fact}_k(z)$. Let $v_i \in V^{y_i}$ for $i \in [1,k]$. If $k \geqslant 2$, then we obtain

$$\begin{aligned} v^{\otimes}_{[1,k]}\Delta'(\pi_1\otimes\mathrm{id})\mu_{1,n-1,V} &= \Big(\sum_{\substack{(i,j)\geqslant(1,1)\\ i+j=k}} v^{\otimes}_{[1,i]}\otimes v^{\otimes}_{[i+1,i+j]}\Big)(\pi_1\otimes\mathrm{id})\mu_{1,n-1,V} \\ &= (v_1\otimes v^{\otimes}_{[2,k]})\mu_{1,n-1,V} \qquad \text{(using } k\geqslant 2\text{)} \\ &= v^{\otimes}_{[1,k]}\ . \end{aligned}$$

So given $\xi_k \in (V^{\otimes k})^z$ for $k \in [1,n]\cap\mathbf{Z}$, with support $\{\, k \in [1,n]\cap\mathbf{Z} \,:\, k \not 0 \,\}$ being finite, we let $\xi := (\xi_k)_{k\in[1,n]\cap\mathbf{Z}}$ and obtain

$$\xi\Delta'(\pi_1\otimes\mathrm{id})\mu_{1,n-1,V} = (\xi_k)_{k\in[1,n]\cap\mathbf{Z}}\Delta'(\pi_1\otimes\mathrm{id})\mu_{1,n-1,V} = (0)\sqcup(\xi_k)_{k\in[2,n]\cap\mathbf{Z}}\ .$$

So if $\xi \in (\mathrm{Kern}\,\Delta')^z$, we obtain $\xi_k = 0$ for $k \in [2,n]\cap\mathbf{Z}$, i.e. $\xi \in V^z$. ▫

**Corollary 41** *Let* $n \in [1,\infty]$.

*Suppose given a* $\mathcal{Z}$*-graded module* $V$.

*Suppose given a* $\mathcal{Z}$*-graded module* $U$*, an integer* $d \in \mathbf{Z}$ *and a shift-graded linear map* $U \xrightarrow{u} \mathrm{T}_{\leqslant n}(V)$ *of degree* $d$.

*Recall that we have shift-graded inclusion and projection maps* $V = V^{\otimes 1} \xrightarrow{\iota_1} \mathrm{T}_{\leqslant n}(V) \xrightarrow{\pi_1} V^{\otimes 1} = V$ *of degree* 0 *; cf. Problem* 20.(1).

*If* $u\cdot\Delta = 0$*, then* $u = u\cdot\pi_1\cdot\iota_1$ .

*Proof.* By Lemma 39.(4), now $V \xrightarrow{\iota_1} \mathrm{T}_{\leqslant n}(V)$ is the shift-graded linear inclusion map of the kernel in the sense of Problem 15.(1). Moreover, $\iota_1 \cdot \pi_1 = \mathrm{id}_V$ .

If $u\Delta = 0$, then there exists a shift-graded linear map $U \xrightarrow{\check{u}} V$ of degree $d$ such that $u = \check{u} \cdot \iota_1$ by loc. cit. Therefore $u \cdot \pi_1 \cdot \iota_1 = \check{u} \cdot \iota_1 \cdot \pi_1 \cdot \iota_1 = \check{u} \cdot \iota_1 = u$. ▫

**Proposition 42 (Lifting to coderivations)**

*Let $n \in [1, \infty]$. Let $V$ be a $\mathcal{Z}$-graded module.*

*Let*

$$\begin{array}{rcl} \mathrm{Coder}_n(V) & := & \{\, \mathrm{T}_{\leqslant n}(V) \xrightarrow{\delta} \mathrm{T}_{\leqslant n}(V) \;:\; \delta \text{ is a coderivation} \,\} \\ \mathrm{Coder}_n^{\mathrm{red}}(V) & := & \{\, (V^{\otimes k} \xrightarrow{\mu_k} V)_{k \in [1,n] \cap \mathbf{Z}} \;:\; \mu_k \text{ is a shift-graded linear map of degree 1 for } k \in [1,n] \,\} \end{array}$$

*So* $\mathrm{Coder}_n(V)$ *is a submodule of the module of all shift-graded linear map maps of degree* 1 *from* $\mathrm{T}_{\leqslant n}(V)$ *to* $\mathrm{T}_{\leqslant n}(V)$. *And* $\mathrm{Coder}_n^{\mathrm{red}}(V)$ *is a module with linear combinations being formed entrywise.*

*We have the mutually inverse module morphisms*

$$\begin{array}{rcl} \mathrm{Coder}_n(V) & \overset{\sim}{\longleftrightarrow} & \mathrm{Coder}_n^{\mathrm{red}}(V) \\ \delta & \xmapsto{\alpha = \alpha_{\mathrm{Coder},n,V}} & (\iota_k \cdot \delta \cdot \pi_1)_{k \in [1,n] \cap \mathbf{Z}} \\ \mu\beta & \xleftarrow[\phantom{xx}]{\beta = \beta_{\mathrm{Coder},n,V}} & \mu = (\mu_k)_{k \in [1,n] \cap \mathbf{Z}} \;, \end{array}$$

*where $\mu\beta$ is determined by*

$$\iota_k \cdot (\mu\beta) \;:=\; \sum_{\substack{(r,s,t) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 1} \times \mathbf{Z}_{\geqslant 0} \\ r+s+t = k}} (\mathrm{id}^{\otimes r} \otimes \mu_s \otimes \mathrm{id}^{\otimes t}) \cdot \iota_{r+1+t} \;:\; V^{\otimes k} \;\to\; \mathrm{T}_{\leqslant n}(V) \;.$$

*for $k \in [1, n] \cap \mathbf{Z}$.*

*Proof.*

*Welldefinedness of $\beta$.* Suppose given $\mu = (\mu_k)_k \in \mathrm{Coder}_n^{\mathrm{red}}(V)$. First, $\mu\beta$ is a shift-graded linear map of degree 1.

We need to show that $\mu\beta$ is a coderivation. Suppose given $k \in [1, n] \cap \mathbf{Z}$. Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $(y_1, \ldots, y_k) \in \mathrm{fact}_k(z)$. Write $y_i \deg =: d_i$ for $i \in [1, k]$. Write $d_{[a,b]} := \sum_{i \in [a,b]} d_i$ for $a$, $b \in [1, k]$. Suppose given $v_i \in V^{y_i}$ for $i \in [1, k]$. We have to show that

$$v_{[1,k]}^{\otimes}(\mu\beta)\Delta \overset{!}{=} v_{[1,k]}^{\otimes}\Delta(\mathrm{id} \otimes (\mu\beta) + (\mu\beta) \otimes \mathrm{id}) \;.$$

In fact, we obtain

$$
\begin{aligned}
& v^{\otimes}_{[1,k]}(\mu\beta)\Delta \\
= \; & v^{\otimes}_{[1,k]}\Big( \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (\mathrm{id}^{\otimes r}\otimes\mu_s\otimes\mathrm{id}^{\otimes t})\cdot\iota_{r+1+t}\Big)\Delta \\
= \; & \Big( \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (-1)^{d_{[r+s+1,r+s+t]}}\; v^{\otimes}_{[1,r]}\otimes v^{\otimes}_{[r+1,r+s]}\mu_s\otimes v^{\otimes}_{[r+s+1,r+s+t]}\Big)\Delta \\
= \; & \sum_{\substack{(r',r'',s,t)\geqslant(1,0,1,0)\\ r'+r''+s+t=k}} (-1)^{d_{[r'+r''+s+1,r'+r''+s+t]}}\; v^{\otimes}_{[1,r']}\bigotimes v^{\otimes}_{[r'+1,r'+r'']}\otimes v^{\otimes}_{[r'+r''+1,r'+r''+s]}\mu_s\otimes v^{\otimes}_{[r'+r''+s+1,r'+r''+s+t]} \\
+ \; & \sum_{\substack{(r,s,t',t'')\geqslant(0,1,0,1)\\ r+s+t'+t''=k}} (-1)^{d_{[r+s+1,r+s+t'+t'']}}\; v^{\otimes}_{[1,r]}\otimes v^{\otimes}_{[r+1,r+s]}\mu_s\otimes v^{\otimes}_{[r+s+1,r+s+t']}\bigotimes v^{\otimes}_{[r+s+t'+1,r+s+t'+t'']} \\
= \; & \sum_{\substack{(p,q)\geqslant(1,1)\\ p+q=k}} \Big( \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=q}} (-1)^{d_{[p+r+s+1,\,p+r+s+t]}}\; v^{\otimes}_{[1,\,p]}\bigotimes v^{\otimes}_{[p+1,\,p+r]}\otimes v^{\otimes}_{[p+r+1,\,p+r+s]}\mu_s\otimes v^{\otimes}_{[p+r+s+1,\,p+r+s+t]} \\
& \qquad + \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=p}} (-1)^{d_{[r+s+1,r+s+t+q]}}\; v^{\otimes}_{[1,r]}\otimes v^{\otimes}_{[r+1,r+s]}\mu_s\otimes v^{\otimes}_{[r+s+1,r+s+t]}\bigotimes v^{\otimes}_{[r+s+t+1,r+s+t+q]}\Big) \\
= \; & \sum_{\substack{(p,q)\geqslant(1,1)\\ p+q=k}} \Big( v^{\otimes}_{[1,\,p]}\bigotimes\Big( \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=q}} (-1)^{d_{[p+r+s+1,\,p+r+s+t]}}\; v^{\otimes}_{[p+1,\,p+r]}\otimes v^{\otimes}_{[p+r+1,\,p+r+s]}\mu_s\otimes v^{\otimes}_{[p+r+s+1,\,p+r+s+t]}\Big) \\
& \qquad +(-1)^{d_{[p+1,p+q]}}\Big( \sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=p}} (-1)^{d_{[r+s+1,r+s+t]}}\; v^{\otimes}_{[1,r]}\otimes v^{\otimes}_{[r+1,r+s]}\mu_s\otimes v^{\otimes}_{[r+s+1,r+s+t]}\Big)\bigotimes v^{\otimes}_{[p+1,\,p+q]}\Big) \\
= \; & \sum_{\substack{(p,q)\geqslant(1,1)\\ p+q=k}} \Big( v^{\otimes}_{[1,\,p]}\bigotimes\Big( v^{\otimes}_{[p+1,\,p+q]}(\mu\beta)\Big) \\
& \qquad +(-1)^{d_{[p+1,p+q]}}\Big( v^{\otimes}_{[1,\,p]}(\mu\beta)\Big)\bigotimes v^{\otimes}_{[p+1,\,p+q]}\Big) \\
= \; & \sum_{\substack{(p,q)\geqslant(1,1)\\ p+q=k}} \Big( v^{\otimes}_{[1,\,p]}\bigotimes v^{\otimes}_{[p+1,\,p+q]}\Big)(\mathrm{id}\otimes(\mu\beta)+(\mu\beta)\otimes\mathrm{id}) \\
= \; & v^{\otimes}_{[1,k]}\Delta(\mathrm{id}\otimes(\mu\beta)+(\mu\beta)\otimes\mathrm{id})\;.
\end{aligned}
$$

*Composite* $\beta \cdot \alpha \overset{!}{=} \mathrm{id}$. Suppose given $\mu = (\mu_k)_k \in \mathrm{Coder}_n^{\mathrm{red}}(V)$. We obtain

$$\begin{array}{rcl} \mu\beta\alpha & = & (\iota_k \cdot (\mu\beta) \cdot \pi_1)_k \\ & = & \Big( \sum\limits_{\substack{(r,s,t) \geqslant (0,1,0) \\ r+s+t = k}} (\mathrm{id}^{\otimes r} \otimes \mu_s \otimes \mathrm{id}^{\otimes t}) \cdot \iota_{r+1+t} \cdot \pi_1 \Big)_k \\ & = & (\mathrm{id}^{\otimes 0} \otimes \mu_k \otimes \mathrm{id}^{\otimes 0})_k \\ & = & \mu \, . \end{array}$$

*Injectivity of* $\alpha$. Suppose given $\delta \in \mathrm{Coder}_n V$ such that $\delta\alpha = 0$. We have to show that $\delta \overset{!}{=} 0$.

By induction, we show that $\delta|_{\mathrm{T}_{\leqslant \ell}(V)} \overset{!}{=} 0$ for $\ell \in [0, n] \cap \mathbf{Z}$.

*Base of the induction.* We have $\mathrm{T}_{\leqslant 0}(V) = 0$, whence $\delta|_{\mathrm{T}_{\leqslant 0}(V)} = 0$.

*Step of the induction.* Suppose given $\ell \in [0, n-1] \cap \mathbf{Z}$. Suppose that $\delta|_{\mathrm{T}_{\leqslant \ell}(V)} = 0$. We have to show that $\iota_{\ell+1} \cdot \delta \overset{!}{=} 0$.

Since $\iota_{\ell+1} \cdot \Delta$ restricts to $\mathrm{T}_{\leqslant \ell}(V) \otimes \mathrm{T}_{\leqslant \ell}(V)$ in the target and since $(\mathrm{id} \otimes \delta)|_{\mathrm{T}_{\leqslant \ell}(V) \otimes \mathrm{T}_{\leqslant \ell}(V)} = 0$ and $(\delta \otimes \mathrm{id})|_{\mathrm{T}_{\leqslant \ell}(V) \otimes \mathrm{T}_{\leqslant \ell}(V)} = 0$, we have

$$\iota_{\ell+1} \cdot \delta \cdot \Delta \;=\; \iota_{\ell+1} \cdot \Delta \cdot (\mathrm{id} \otimes \delta + \delta \otimes \mathrm{id}) \;=\; 0 \, .$$

By Corollary 41, we conclude that

$$\iota_{\ell+1} \cdot \delta \;=\; \iota_{\ell+1} \cdot \delta \cdot \pi_1 \cdot \iota_1 \;=\; 0 \, ,$$

the latter since $\delta\alpha = (\iota_k \cdot \delta \cdot \pi_1)_k = 0$.

This concludes the *induction*.

If $n \in \mathbf{Z}_{\geqslant 1}$, then letting $\ell = n$, this shows $\delta = 0$.

If $n = \infty$, then $\iota_\ell \cdot \delta = 0$ for $\ell \in \mathbf{Z}_{\geqslant 1}$, whence $\delta = 0$. ▫

**Proposition 43 (Lifting to coalgebra morphisms)**

*Let* $n \in [1, \infty]$. *Let* $\tilde{V}$ *and* $V$ *be* $\mathcal{Z}$*-graded modules.*

*Let*

$$\begin{array}{rcl} \mathrm{Coalg}_n(\tilde{V}, V) & := & \{ \mathrm{T}_{\leqslant n}(\tilde{V}) \xrightarrow{\psi} \mathrm{T}_{\leqslant n}(V) \,:\, \psi \text{ is a coalgebra morphism} \} \\ \mathrm{Coalg}_n^{\mathrm{red}}(\tilde{V}, V) & := & \{ (\tilde{V}^{\otimes k} \xrightarrow{\varphi_k} V)_{k \in [1,n] \cap \mathbf{Z}} \,:\, \varphi_k \text{ is a shift-graded linear map of degree 0 for } k \in [1, n] \} \end{array}$$

*So* $\mathrm{Coalg}_n(\tilde{V}, V)$ *and* $\mathrm{Coalg}_n^{\mathrm{red}}(\tilde{V}, V)$ *are sets.*

*We have the mutually inverse bijections*

$$\begin{array}{rcl} \mathrm{Coalg}_n V & \overset{\sim}{\longleftrightarrow} & \mathrm{Coalg}_n^{\mathrm{red}} \\ \psi & \xmapsto{\alpha = \alpha_{\mathrm{Coalg}, n, \tilde{V}, V}} & (\iota_k \cdot \psi \cdot \pi_1)_{k \in [1,n] \cap \mathbf{Z}} \\ \varphi\beta & \xleftarrow[]{\beta = \beta_{\mathrm{Coalg}, n, \tilde{V}, V}} & \varphi = (\varphi_k)_{k \in [1,n] \cap \mathbf{Z}} \, , \end{array}$$

*where $\varphi\beta$ is determined by*

$$\iota_k \cdot (\varphi\beta) \;:=\; \sum_{r\in[1,k]} \;\sum_{\substack{(i_j)_{j\in[1,r]}\in\mathbf{Z}_{\geqslant 1}^{\times r} \\ \sum_{j\in[1,r]} i_j = k}} \Big(\bigotimes_{j\in[1,r]} \varphi_{i_j}\Big)\cdot \iota_r \;:\; \tilde{V}^{\otimes k} \;\to\; \mathrm{T}_{\leqslant n}(V)\;.$$

*for $k \in [1,n]\cap\mathbf{Z}$.*

*Proof.*

*Welldefinedness of $\beta$.* Suppose given $\varphi = (\varphi_k)_k \in \mathrm{Coalg}_n^{\mathrm{red}}(\tilde{V}, V)$. First, $\varphi\beta$ is a shift-graded linear map of degree 0.

We need to show that $\varphi\beta$ is a coalgebra morphism. Suppose given $k \in [1,n]\cap\mathbf{Z}$. Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $(y_1,\dots,y_k)\in \mathrm{fact}_k(z)$. Suppose given $\tilde{v}_i \in \tilde{V}^{y_i}$ for $i\in[1,k]$. We have to show that

$$\tilde{v}^{\otimes}_{[1,k]}(\varphi\beta)\Delta \;\overset{!}{=}\; \tilde{v}^{\otimes}_{[1,k]}\Delta((\varphi\beta)\otimes(\mu\beta))\;.$$

Given $r\in[1,k]$ and $(i_j)_{j\in[1,r]} \geqslant (1)_j$ such that $\sum_j i_j = k$ and given $s\in[1,r]$, we write

$$[i_s] \;:=\; \Big[1 + \sum_{j\in[1,s-1]} i_j\,,\; \sum_{j\in[1,s]} i_j\Big]\;.$$

We obtain

$$\begin{aligned}
&\tilde{v}^{\otimes}_{[1,k]}(\varphi\beta)\Delta \\
=\;& \tilde{v}^{\otimes}_{[1,k]}\Big(\sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j = k}} \Big(\textstyle\bigotimes_{j\in[1,r]}\varphi_{i_j}\Big)\cdot\iota_r\Big)\Delta \\
=\;& \Big(\sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j = k}} \tilde{v}^{\otimes}_{[i_1]}\varphi_{i_1}\otimes\dots\otimes\tilde{v}^{\otimes}_{[i_r]}\varphi_{i_r}\Big)\Delta \\
=\;& \sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j = k}}\;\sum_{\substack{(s,t)\geqslant(1,1)\\ s+t=r}} \tilde{v}^{\otimes}_{[i_1]}\varphi_{i_1}\otimes\dots\otimes\tilde{v}^{\otimes}_{[i_s]}\varphi_{i_s}\otimes\tilde{v}^{\otimes}_{[i_{s+1}]}\varphi_{i_{s+1}}\otimes\dots\otimes\tilde{v}^{\otimes}_{[i_{s+t}]}\varphi_{i_{s+t}} \\
=\;& \sum_{\substack{(p,q)\geqslant(1,1)\\ p+q=k}}\Big(\sum_{s\in[1,p]}\;\sum_{\substack{(i_j)_{j\in[1,s]}\geqslant(1)_j\\ \sum_j i_j = p}} \tilde{v}^{\otimes}_{[i_1]}\varphi_{i_1}\otimes\dots\otimes\tilde{v}^{\otimes}_{[i_s]}\varphi_{i_s}\Big)\otimes\Big(\sum_{t\in[1,q]}\;\sum_{\substack{(i_j)_{j\in[1,t]}\geqslant(1)_j\\ \sum_j i_j = q}} \tilde{v}^{\otimes}_{p+[i_{s+1}]}\varphi_{i_{s+1}}\otimes\dots\otimes\tilde{v}^{\otimes}_{p+[i_{s+t}]}\varphi_{i_{s+t}}\Big) \\
=\;& \Big(\sum_{\substack{(p,q)\geqslant(1,1)\\ p+q=k}} \tilde{v}^{\otimes}_{[1,\,p]}(\varphi\beta)\otimes\tilde{v}^{\otimes}_{[p+1,\,p+q]}(\varphi\beta)\Big) \\
=\;& \Big(\sum_{\substack{(p,q)\geqslant(1,1)\\ p+q=k}} \tilde{v}^{\otimes}_{[1,\,p]}\otimes\tilde{v}^{\otimes}_{[p+1,\,p+q]}\Big)((\varphi\beta)\otimes(\varphi\beta)) \\
=\;& \tilde{v}^{\otimes}_{[1,k]}\Delta((\varphi\beta)\otimes(\mu\beta))\;.
\end{aligned}$$

*Composite* $\beta\cdot\alpha \overset{!}{=} \mathrm{id}$. Suppose given $\varphi = (\varphi_k)_k \in \mathrm{Coalg}_n^{\mathrm{red}}(\tilde{V}, V)$. We obtain

$$\begin{array}{rcl} \varphi\beta\alpha & = & (\iota_k \cdot (\varphi\beta) \cdot \pi_1)_k \\ & = & \Big( \sum\limits_{r\in[1,k]} \sum\limits_{\substack{(i_j)_{j\in[1,r]} \geqslant (1)_j \\ \sum_j i_j = k}} \Big( \bigotimes_{j\in[1,r]} \varphi_{i_j} \Big) \cdot \iota_r \cdot \pi_1 \Big)_k \\ & = & (\bigotimes_{j\in[1,1]} \varphi_k)_k \\ & = & \varphi \,. \end{array}$$

*Injectivity of* $\alpha$. Suppose given $\psi,\, \psi' \in \mathrm{Coalg}_n(\tilde{V}, V)$ such that $\psi\alpha = \psi'\alpha$. We have to show that $\psi \overset{!}{=} \psi'$.

By induction, we show that $\psi|_{\mathrm{T}_{\leqslant \ell}(\tilde{V})} \overset{!}{=} \psi'|_{\mathrm{T}_{\leqslant \ell}(\tilde{V})}$ for $\ell \in [0,n] \cap \mathbf{Z}$.

*Base of the induction.* We have $\mathrm{T}_{\leqslant 0}(\tilde{V}) = 0$, whence $\psi|_{\mathrm{T}_{\leqslant 0}(\tilde{V})} = 0 = \psi'|_{\mathrm{T}_{\leqslant 0}(\tilde{V})}$ .

*Step of the induction.* Suppose given $\ell \in [0, n-1] \cap \mathbf{Z}$. Suppose that $\psi|_{\mathrm{T}_{\leqslant \ell}(\tilde{V})} = \psi'|_{\mathrm{T}_{\leqslant \ell}(\tilde{V})}$ . We have to show that $\iota_{\ell+1} \cdot (\psi - \psi') \overset{!}{=} 0$. Note that $\psi - \psi'$ is only a shift-graded linear map of degree 0.

We have

$$\begin{array}{rcl} \iota_{\ell+1} \cdot (\psi - \psi') \cdot \Delta & = & \iota_{\ell+1} \cdot \psi \cdot \Delta - \iota_{\ell+1} \cdot \psi' \cdot \Delta \\ & = & \iota_{\ell+1} \cdot \Delta \cdot (\psi \otimes \psi) - \iota_{\ell+1} \cdot \Delta \cdot (\psi' \otimes \psi') \\ & = & \iota_{\ell+1} \cdot \Delta \cdot (\psi \otimes (\psi - \psi') + (\psi - \psi') \otimes \psi') \\ & = & 0 \,, \end{array}$$

since $\iota_{\ell+1} \cdot \Delta$ restricts in the target to $\mathrm{T}_{\leqslant \ell}(\tilde{V}) \otimes \mathrm{T}_{\leqslant \ell}(\tilde{V})$ and since $\psi - \psi'$ vanishes on $\mathrm{T}_{\leqslant \ell}(\tilde{V})$. By Corollary 41, we conclude that

$$\iota_{\ell+1} \cdot (\psi - \psi') \;=\; \iota_{\ell+1} \cdot (\psi - \psi') \cdot \pi_1 \cdot \iota_1 \;=\; 0 \,,$$

the latter since $(\iota_k \cdot \delta \cdot \pi_1)_k = \psi\alpha = \psi'\alpha = (\iota_k \cdot \psi' \cdot \pi_1)_k$ .

This concludes the *induction.*

If $n \in \mathbf{Z}_{\geqslant 1}$ , then letting $\ell = n$, this shows $\psi = \psi'$.

If $n = \infty$, then $\iota_\ell \cdot \psi = \iota_\ell \cdot \psi'$ for $\ell \in \mathbf{Z}_{\geqslant 1}$ , whence $\psi = \psi'$. ▫

**Corollary 44** *Let* $n \in [1, \infty]$. *Let* $\tilde{V}$ *and* $V$ *be* $\mathcal{Z}$*-graded modules.*

*Suppose given* $k \in [1, n]$.

(1) *Let* $\delta : \mathrm{T}_{\leqslant n}(V) \to \mathrm{T}_{\leqslant n}(V)$ *be a coderivation. Then* $\delta|_{\mathrm{T}_{\leqslant k}(V)}^{\mathrm{T}_{\leqslant k}(V)}$ *exists.*

(2) *Let* $\psi : \mathrm{T}_{\leqslant n}(\tilde{V}) \to \mathrm{T}_{\leqslant n}(V)$ *be a coalgebra morphism. Then* $\psi|_{\mathrm{T}_{\leqslant k}(\tilde{V})}^{\mathrm{T}_{\leqslant k}(V)}$ *exists.*

*Proof.*

*Ad* (1). We have $\delta = \delta\alpha\beta$, and

$$\iota_\ell \cdot (\delta\alpha\beta) \;=\; \sum_{\substack{(r,s,t)\in \mathbf{Z}_{\geqslant 0}\times\mathbf{Z}_{\geqslant 1}\times\mathbf{Z}_{\geqslant 0} \\ r+s+t=\ell}} (\mathrm{id}^{\otimes r} \otimes(\iota_s \cdot \delta \cdot \pi_1) \otimes \mathrm{id}^{\otimes t}) \cdot \iota_{r+1+t}$$

maps to $\mathrm{T}_{\leqslant k}(V)$ for $\ell \in [1,k]\cap\mathbf{Z}$ ; cf. Proposition 42.

*Ad* (2). We have $\psi = \psi\alpha\beta$, and

$$\iota_\ell \cdot (\psi\alpha\beta) \;=\; \sum_{r\in[1,k]} \; \sum_{\substack{(i_j)_{j\in[1,r]}\in\mathbf{Z}_{\geqslant 1}^{\times r} \\ \sum_{j\in[1,r]} i_j = \ell}} \Big( \bigotimes_{j\in[1,r]} (\iota_{i_j} \cdot \psi \cdot \pi_1) \Big) \cdot \iota_r$$

maps to $\mathrm{T}_{\leqslant k}(V)$ for $\ell \in [1,k]\cap\mathbf{Z}$ ; cf. Proposition 43. ▫

**Lemma 45** *Let* $n \in [1,\infty]$. *Let* $\tilde{V}$ *and* $V$ *be* $\mathcal{Z}$*-graded modules.*
*Suppose given* $k \in [0, n-1]\cap\mathbf{Z}$.

(1) *Suppose given a coderivation* $\mathrm{T}_{\leqslant n}(V) \xrightarrow{\delta} \mathrm{T}_{\leqslant n}(V)$.
*Suppose that* $\delta^2|_{\mathrm{T}_{\leqslant k}(V)} = 0$.
*Then* $\iota_{k+1} \cdot \delta^2 = \iota_{k+1} \cdot \delta^2 \cdot \pi_1 \cdot \iota_1$ .

(2) *Suppose given coderivations* $\mathrm{T}_{\leqslant n}(\tilde{V}) \xrightarrow{\tilde{\delta}} \mathrm{T}_{\leqslant n}(\tilde{V})$ *and* $\mathrm{T}_{\leqslant n}(V) \xrightarrow{\delta} \mathrm{T}_{\leqslant n}(V)$.
*Suppose given a coalgebra morphism* $\mathrm{T}_{\leqslant n}(\tilde{V}) \xrightarrow{\psi} \mathrm{T}_{\leqslant n}(V)$.
*Suppose that* $(\tilde{\delta} \cdot \psi - \psi \cdot \delta)|_{\mathrm{T}_{\leqslant k}(V)} = 0$.
*Then* $\iota_{k+1} \cdot (\tilde{\delta} \cdot \psi - \psi \cdot \delta) = \iota_{k+1} \cdot (\tilde{\delta} \cdot \psi - \psi \cdot \delta) \cdot \pi_1 \cdot \iota_1$ .

*Proof.*

*Ad* (1). By Corollary 41, we need to show that $\iota_{k+1} \cdot \delta^2 \cdot \Delta \overset{!}{=} 0$. In fact, we get

$$\begin{aligned}
\iota_{k+1} \cdot \delta^2 \cdot \Delta &= \iota_{k+1} \cdot \delta \cdot \delta \cdot \Delta \\
&= \iota_{k+1} \cdot \delta \cdot \Delta \cdot (\mathrm{id} \otimes\delta + \delta \otimes \mathrm{id}) \\
&= \iota_{k+1} \cdot \Delta \cdot (\mathrm{id} \otimes\delta + \delta \otimes \mathrm{id}) \cdot (\mathrm{id} \otimes\delta + \delta \otimes \mathrm{id}) \\
&= \iota_{k+1} \cdot \Delta \cdot (\mathrm{id} \otimes\delta^2 - \delta \otimes \delta + \delta \otimes \delta + \delta^2 \otimes \mathrm{id}) \\
&= \iota_{k+1} \cdot \Delta \cdot (\mathrm{id} \otimes\delta^2 + \delta^2 \otimes \mathrm{id}) \; ;
\end{aligned}$$

cf. Problem 6. Now $\iota_{k+1} \cdot \Delta$ restricts in the target to $\mathrm{T}_{\leqslant k}(V) \otimes \mathrm{T}_{\leqslant k}(V)$, so that we may conclude from $\delta^2|_{\mathrm{T}_{\leqslant k}(V)} = 0$ that $\iota_{k+1} \cdot \delta^2 \cdot \Delta = 0$.

*Ad* (2). By Corollary 41, we need to show that $\iota_{k+1} \cdot (\tilde{\delta} \cdot \psi - \psi \cdot \delta) \cdot \Delta \overset{!}{=} 0$. In fact, we get

$$\begin{aligned} & \iota_{k+1} \cdot (\tilde{\delta} \cdot \psi - \psi \cdot \delta) \cdot \Delta \\ = \; & \iota_{k+1} \cdot \tilde{\delta} \cdot \psi \cdot \Delta - \iota_{k+1} \cdot \psi \cdot \delta \cdot \Delta \\ = \; & \iota_{k+1} \cdot \tilde{\delta} \cdot \Delta \cdot (\psi \otimes \psi) - \iota_{k+1} \cdot \psi \cdot \Delta \cdot (\mathrm{id} \otimes \delta + \delta \otimes \mathrm{id}) \\ = \; & \iota_{k+1} \cdot \Delta \cdot (\mathrm{id} \otimes \tilde{\delta} + \tilde{\delta} \otimes \mathrm{id}) \cdot (\psi \otimes \psi) - \iota_{k+1} \cdot \Delta \cdot (\psi \otimes \psi) \cdot (\mathrm{id} \otimes \delta + \delta \otimes \mathrm{id}) \\ = \; & \iota_{k+1} \cdot \Delta \cdot (\psi \otimes (\tilde{\delta} \cdot \psi) + (\tilde{\delta} \cdot \psi) \otimes \psi - \psi \otimes (\psi \cdot \delta) - (\psi \cdot \delta) \otimes \psi) \\ = \; & \iota_{k+1} \cdot \Delta \cdot (\psi \otimes (\tilde{\delta} \cdot \psi - \psi \cdot \delta) + (\tilde{\delta} \cdot \psi - \psi \cdot \delta) \otimes \psi \; . \end{aligned}$$

Now $\iota_{k+1} \cdot \Delta$ restricts in the target to $\mathrm{T}_{\leqslant k}(V) \otimes \mathrm{T}_{\leqslant k}(V)$, so that we may conclude from $(\tilde{\delta} \cdot \psi - \psi \cdot \delta)|_{\mathrm{T}_{\leqslant k}(V)} = 0$ that $\iota_{k+1} \cdot (\tilde{\delta} \cdot \psi - \psi \cdot \delta) \cdot \Delta = 0$. ▫

**Proposition 46** *Let* $n \in [1, \infty]$.

*Suppose given a pre-*$\mathrm{A}_n$*-algebra* $(A, (m_\ell)_\ell)$ *over* $\mathcal{Z}$. *Write*

$$\mathfrak{m} \;:=\; ((^\omega m_\ell)_\ell)\beta_{\mathrm{Coder},n,A^{[1]}} \; ,$$

*which is a coderivation on* $\mathrm{T}_{\leqslant n}(A^{[1]})$ *; cf. Proposition* 42.

*The following assertions* (1) *and* (2) *are equivalent.*

(1) *The tuple* $(m_\ell)_\ell$ *satisfies the Stasheff equation at* $k \in [1, n] \cap \mathbf{Z}$ *; cf. Definition* 19.(1).

(2) *The coderivation* $\mathfrak{m}$ *is a codifferential, i.e.* $\mathfrak{m}^2 = 0$.

*Proof.* Suppose given $u \in [0, n] \cap \mathbf{Z}$. We *claim* equivalence of the following assertions $(1_u)$ and $(2_u)$.

$(1_u)$ We have $\displaystyle\sum_{\substack{(r,s,t) \geqslant (0,1,0) \\ r+s+t = k}} (\mathrm{id}^{\otimes r} \otimes {}^\omega m_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^\omega m_{r+1+t} = 0$ for $k \in [1, u]$.

$(2_u)$ We have $\mathfrak{m}^2|_{\mathrm{T}_{\leqslant u}(A^{[1]})} = 0$.

We proceed by induction on $u$. For $u = 0$, both assertions $(1_0)$ and $(2_0)$ hold.

Suppose given $u \in [0, n-1] \cap \mathbf{Z}$. By induction, we suppose that the assertions $(1_{u-1})$ and $(2_{u-1})$ are equivalent.

Consider the following assertions (i) and (ii).

(i) We have $\sum\limits_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=u+1}} (\mathrm{id}^{\otimes r}\otimes {}^{\omega}m_s\otimes \mathrm{id}^{\otimes t})\cdot {}^{\omega}m_{r+1+t} = 0$.

(ii) We have $\iota_{u+1}\cdot\mathfrak{m}^2 = 0$.

We have to show that $(2_u)\wedge(\mathrm{i}) \overset{!}{\Leftrightarrow} (2_u)\wedge(\mathrm{ii})$, for then

$$(1_{u+1}) \;\Leftrightarrow\; ((1_u)\wedge(\mathrm{i})) \;\Leftrightarrow\; ((2_u)\wedge(\mathrm{i})) \;\Leftrightarrow\; ((2_u)\wedge(\mathrm{ii})) \;\Leftrightarrow\; (2_{u+1})\ .$$

We have

$$\begin{array}{rcl}
\iota_{u+1}\cdot\mathfrak{m}^2 & \overset{\text{L. 45.(1)}}{=} & \iota_{u+1}\cdot\mathfrak{m}^2\cdot\pi_1\cdot\iota_1 \\
& \overset{\text{P. 42}}{=} & \sum\limits_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (\mathrm{id}^{\otimes r}\otimes {}^{\omega}m_s\otimes \mathrm{id}^{\otimes t})\cdot\iota_{r+1+t}\cdot\mathfrak{m}\cdot\pi_1\cdot\iota_1 \\
& \overset{\text{P. 42}}{=} & \sum\limits_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (\mathrm{id}^{\otimes r}\otimes {}^{\omega}m_s\otimes \mathrm{id}^{\otimes t})\cdot {}^{\omega}m_{r+1+t}\cdot\iota_1\ .
\end{array}$$

The needed equivalence now follows from $\iota_1$ being piecewise injective. This concludes the induction.

This proves the *claim.*

*Case* $n\in\mathbf{Z}$. Letting $u=n$, the assertion of the Proposition follows by Lemma 36.

*Case* $n=\infty$. We conclude as follows.

$$\begin{array}{rl}
& \text{The tuple } (m_\ell)_\ell \text{ satisfies the Stasheff equation at } k\in[1,\infty]\cap\mathbf{Z}\,. \\
\Leftrightarrow & \text{The tuple } (m_\ell)_\ell \text{ satisfies the Stasheff equation at } k\in[1,u] \text{ for } u\in[0,\infty]\cap\mathbf{Z}. \\
\overset{\text{L. 36}}{\Leftrightarrow} & \text{We have } \sum\limits_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (\mathrm{id}^{\otimes r}\otimes {}^{\omega}m_s\otimes \mathrm{id}^{\otimes t})\cdot {}^{\omega}m_{r+1+t} = 0 \text{ for } k\in[1,u]. \\
\Leftrightarrow & \text{We have } \mathfrak{m}^2|_{\mathrm{T}_{\leqslant u}(A^{[1]})} = 0 \text{ for } u\in\mathbf{Z}_{\geqslant 0}. \\
\Leftrightarrow & \text{We have } \iota_\ell\cdot\mathfrak{m}^2 = 0 \text{ for } \ell\in\mathbf{Z}_{\geqslant 1}. \\
\Leftrightarrow & \text{We have } \mathfrak{m}^2 = 0.
\end{array}$$

□

**Proposition 47** *Let* $n\in[1,\infty]$.

*Suppose given pre-*$\mathrm{A}_n$*-algebras* $\tilde{A}=(\tilde{A},(\tilde{m}_\ell)_\ell)$ *and* $A=(A,(m_\ell)_\ell)$ *over* $\mathcal{Z}$.

*Suppose given a pre-*$\mathrm{A}_n$*-morphism* $f=(f_\ell)_\ell$ *from* $\tilde{A}$ *to* $A$.

*Write*

$$\begin{array}{rcl}
\tilde{\mathfrak{m}} & := & (({}^{\omega}\tilde{m}_\ell)_\ell)\beta_{\mathrm{Coder},n,\tilde{A}^{[1]}} \\
\mathfrak{m} & := & (({}^{\omega}m_\ell)_\ell)\beta_{\mathrm{Coder},n,A^{[1]}}\ ,
\end{array}$$

*which are coderivations on* $\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})$ *resp. on* $\mathrm{T}_{\leqslant n}(A^{[1]})$ *; cf. Proposition* 42.

*Write*

$$\mathfrak{f} \;:=\; ((\,^{\omega}\! f_\ell)_\ell)\beta_{\mathrm{Coalg},n,\tilde{A}^{[1]},A^{[1]}}\ ,$$

*which is a coalgebra morphism from* $\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})$ *to* $\mathrm{T}_{\leqslant n}(A^{[1]})$ *; cf. Proposition* 43.

*The following assertions* (1) *and* (2) *are equivalent.*

(1) *The tuple* $(f_\ell)_\ell$ *satisfies the Stasheff equation for morphisms at* $k \in [1,n]\cap\mathbf{Z}$ *; cf. Definition* 19.(1).

(2) *The coalgebra morphism* $\mathfrak{f}$ *satisfies* $\tilde{\mathfrak{m}}\cdot\mathfrak{f} = \mathfrak{f}\cdot\mathfrak{m}$.

*If* $\tilde{A}$ *and* $A$ *are* $\mathrm{A}_n$*-algebras,* (1) *means that* $f$ *is an* $\mathrm{A}_n$*-morphism, whereas* (2) *means, using Proposition* 46*, that* $\mathfrak{f}$ *is a morphism of coalgebras with codifferential.*

*Proof.* Suppose given $u \in [0,n]\cap\mathbf{Z}$. We *claim* equivalence of the following assertions $(1_u)$ and $(2_u)$.

$(1_u)$ We have

$$\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=k}} (\mathrm{id}^{\otimes r}\otimes{}^{\omega}\tilde{m}_s\otimes\mathrm{id}^{\otimes t})\cdot{}^{\omega}\! f_{r+1+t} \;=\; \sum_{r\in[1,k]}\;\sum_{\substack{(i_j)_{i\in[1,r]}\geqslant(1)_j\\ \sum_j i_j = k}} (\,^{\omega}\! f_{i_1}\otimes\ldots\otimes{}^{\omega}\! f_{i_r})\cdot{}^{\omega} m_{r+1+t}$$

for $k\in[1,u]$.

$(2_u)$ We have $(\tilde{\mathfrak{m}}\cdot\mathfrak{f}-\mathfrak{f}\cdot\mathfrak{m})|_{\mathrm{T}_{\leqslant u}(\tilde{A}^{[1]})} = 0$.

We proceed by induction on $u$. For $u=0$, both assertions $(1_0)$ and $(2_0)$ hold.

Suppose given $u\in[0,n-1]\cap\mathbf{Z}$. By induction, we suppose that the assertions $(1_{u-1})$ and $(2_{u-1})$ are equivalent.

Consider the following assertions (i) and (ii).

(i) We have

$$\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=u+1}} (\mathrm{id}^{\otimes r}\otimes{}^{\omega}\tilde{m}_s\otimes\mathrm{id}^{\otimes t})\cdot{}^{\omega}\! f_{r+1+t} \;=\; \sum_{r\in[1,u+1]}\;\sum_{\substack{(i_j)_{i\in[1,r]}\geqslant(1)_j\\ \sum_j i_j = u+1}} (\,^{\omega}\! f_{i_1}\otimes\ldots\otimes{}^{\omega}\! f_{i_r})\cdot{}^{\omega} m_r\ .$$

(ii) We have $\iota_{u+1}\cdot(\tilde{\mathfrak{m}}\cdot\mathfrak{f}-\mathfrak{f}\cdot\mathfrak{m}) = 0$.

We have to show that $(2_u) \wedge \text{(i)} \overset{!}{\Leftrightarrow} (2_u) \wedge \text{(ii)}$, for then

$$(1_{u+1}) \;\Leftrightarrow\; ((1_u) \wedge \text{(i)}) \;\Leftrightarrow\; ((2_u) \wedge \text{(i)}) \;\Leftrightarrow\; ((2_u) \wedge \text{(ii)}) \;\Leftrightarrow\; (2_{u+1})\ .$$

We have

$$\begin{aligned}
& \iota_{u+1} \cdot (\tilde{\mathfrak{m}} \cdot \mathfrak{f} - \mathfrak{f} \cdot \mathfrak{m}) \\
\overset{\text{L. 45.(2)}}{=} \; & \iota_{u+1} \cdot (\tilde{\mathfrak{m}} \cdot \mathfrak{f} - \mathfrak{f} \cdot \mathfrak{m}) \cdot \pi_1 \cdot \iota_1 \\
\overset{\text{P. 42, P. 43}}{=} \; & \sum_{\substack{(r,s,t) \geqslant (0,1,0) \\ r+s+t=k}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot \iota_{r+1+t} \cdot \mathfrak{f} \cdot \pi_1 \cdot \iota_1 - \sum_{r \in [1,u+1]} \; \sum_{\substack{(i_j)_{i \in [1,r]} \geqslant (1)_j \\ \sum_j i_j = u+1}} ({}^{\omega}f_{i_1} \otimes \ldots \otimes {}^{\omega}f_{i_r}) \cdot \iota_r \cdot \mathfrak{m} \cdot \pi_1 \cdot \iota_1 \\
\overset{\text{P. 42, P. 43}}{=} \; & \Bigg( \sum_{\substack{(r,s,t) \geqslant (0,1,0) \\ r+s+t=k}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^{\omega}f_{r+1+t} - \sum_{r \in [1,u+1]} \; \sum_{\substack{(i_j)_{i \in [1,r]} \geqslant (1)_j \\ \sum_j i_j = u+1}} ({}^{\omega}f_{i_1} \otimes \ldots \otimes {}^{\omega}f_{i_r}) \cdot {}^{\omega}m_r \Bigg) \cdot \iota_1\ .
\end{aligned}$$

The needed equivalence now follows from $\iota_1$ being piecewise injective. This concludes the induction.

This proves the *claim.*

*Case* $n \in \mathbf{Z}$. Letting $u = n$, the assertion of the Proposition follows by Lemma 37.

*Case* $n = \infty$. We conclude as follows.

The tuple $(f_\ell)_\ell$ satisfies the Stasheff equation for morphisms at $k \in [1, \infty] \cap \mathbf{Z}$

$\Leftrightarrow$ The tuple $(f_\ell)_\ell$ satisfies the Stasheff equation for morphisms at $k \in [1, u]$ for $u \in [0, \infty] \cap \mathbf{Z}$

$\overset{\text{L. 36}}{\Leftrightarrow}$ We have

$$\sum_{\substack{(r,s,t) \geqslant (0,1,0) \\ r+s+t=k}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^{\omega}f_{r+1+t} \;=\; \sum_{r \in [1,k]} \; \sum_{\substack{(i_j)_{i \in [1,r]} \geqslant (1)_j \\ \sum_j i_j = k}} ({}^{\omega}f_{i_1} \otimes \ldots \otimes {}^{\omega}f_{i_r}) \cdot {}^{\omega}m_{r+1+t}$$

for $k \in [1, u]$.

$\Leftrightarrow$ We have $(\tilde{\mathfrak{m}} \cdot \mathfrak{f} - \mathfrak{f} \cdot \mathfrak{m})|_{\mathrm{T}_{\leqslant u}(\tilde{A}^{[1]})} = 0$ for $u \in \mathbf{Z}_{\geqslant 0}$

$\Leftrightarrow$ We have $\iota_\ell \cdot (\tilde{\mathfrak{m}} \cdot \mathfrak{f} - \mathfrak{f} \cdot \mathfrak{m}) = 0$ for $\ell \in \mathbf{Z}_{\geqslant 1}$

$\Leftrightarrow$ We have $\tilde{\mathfrak{m}} \cdot \mathfrak{f} = \mathfrak{f} \cdot \mathfrak{m}$ .

□

## 1.7 Kadeishvili's theorem

With the coalgebra reinterpretation of §1.6 at hand, we can complete the task tentatively begun in Remark 30, which is to prove Kadeishvili's theorem.

Let $\mathcal{Z}$ be a grading category.

**Lemma 48** *Let* $n \in [1, \infty]$.

*Let* $\tilde{A} = (\tilde{A}, (\tilde{m}_\ell)_\ell)$ *be a pre-*$\mathrm{A}_n$*-algebra over* $\mathcal{Z}$.

*Let* $A = (A, (m_\ell)_\ell)$ *be an* $\mathrm{A}_n$*-algebra over* $\mathcal{Z}$.

*Let* $f = (f_\ell)_\ell$ *be a pre-*$\mathrm{A}_n$*-morphisms from* $\tilde{A}$ *to* $A$ *that satisfies the Stasheff equation for morphisms at* $k \in [1, n] \cap \mathbf{Z}$.

*Suppose that* $f_1$ *is piecewise injective.*

*Then* $\tilde{A}$ *is an* $\mathrm{A}_n$*-algebra. So then* $f : \tilde{A} \to A$ *is a morphism of* $\mathrm{A}_n$*-algebras.*

*Proof.* Using Propositions 42 and 43, we write

$$\begin{array}{rcll} \tilde{\mathfrak{m}} & := & (({}^\omega\tilde{m}_\ell)_\ell)\beta_{\mathrm{Coder},n,\tilde{A}^{[1]}} & \text{(coderivation on } \mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})\text{ )} \\ \mathfrak{m} & := & (({}^\omega m_\ell)_\ell)\beta_{\mathrm{Coder},n,A^{[1]}} & \text{(coderivation on } \mathrm{T}_{\leqslant n}(A^{[1]})\text{ )} \\ \mathfrak{f} & := & (({}^\omega f_\ell)_\ell)\beta_{\mathrm{Coalg},n,\tilde{A}^{[1]},A^{[1]}} & \text{(coalgebra morphism from } \mathrm{T}_{\leqslant n}(\tilde{A}^{[1]}) \text{ to } \mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})\text{ )}\ . \end{array}$$

We have to show that $\tilde{\mathfrak{m}}^2 \overset{!}{=} 0$ ; cf. Proposition 46.

We *claim* that $\tilde{\mathfrak{m}}^2|_{\mathrm{T}_{\leqslant k}(\tilde{A}^{[1]})} \overset{!}{=} 0$ for $k \in [0, n] \cap \mathbf{Z}$.

We proceed by induction on $k$. For $k = 0$, we get $\mathrm{T}_{\leqslant 0}(\tilde{A}^{[1]})$, whence the assertion.

Suppose given $k \in [0, n-1] \cap \mathbf{Z}$. By induction, we have $\tilde{\mathfrak{m}}^2|_{\mathrm{T}_{\leqslant k}(\tilde{A}^{[1]})} = 0$. We need to show that $\tilde{\mathfrak{m}}^2|_{\mathrm{T}_{\leqslant k+1}(\tilde{A}^{[1]})} \overset{!}{=} 0$. It suffices to show that $\iota_{k+1} \cdot \tilde{\mathfrak{m}}^2 \overset{!}{=} 0$.

By Lemma 45.(1), we have $\iota_{k+1} \cdot \tilde{\mathfrak{m}}^2 \cdot \pi_1 \cdot \iota_1$ . Hence

$$\begin{array}{rcl} 0 & \overset{\text{P. 46}}{=} & \iota_{k+1} \cdot \mathfrak{f} \cdot \mathfrak{m}^2 \cdot \pi_1 \\ & \overset{\text{P. 47}}{=} & \iota_{k+1} \cdot \tilde{\mathfrak{m}} \cdot \mathfrak{f} \cdot \mathfrak{m} \cdot \pi_1 \\ & \overset{\text{P. 47}}{=} & \iota_{k+1} \cdot \tilde{\mathfrak{m}}^2 \cdot \mathfrak{f} \cdot \pi_1 \\ & \overset{\text{P. 47}}{=} & \iota_{k+1} \cdot \tilde{\mathfrak{m}}^2 \cdot \pi_1 \cdot \iota_1 \cdot \mathfrak{f} \cdot \pi_1 \\ & \overset{\text{P. 43}}{=} & \iota_{k+1} \cdot \tilde{\mathfrak{m}}^2 \cdot \pi_1 \cdot {}^\omega f_1\ . \end{array}$$

Since $f_1$ is piecewise injective, so is ${}^\omega f_1$. Hence $\iota_{k+1} \cdot \tilde{\mathfrak{m}}^2 \cdot \pi_1 = 0$. Thus

$$\iota_{k+1} \cdot \tilde{\mathfrak{m}}^2 \;=\; \iota_{k+1} \cdot \tilde{\mathfrak{m}}^2 \cdot \pi_1 \cdot \iota_1 \;=\; 0\ .$$

This proves the *claim*.

If $n \in \mathbf{Z}_{\geqslant 1}$ , then letting $k = n$, the claim gives $\tilde{\mathfrak{m}}^2 = 0$.

If $n = \infty$, then $\iota_k \cdot \tilde{\mathfrak{m}}^2 = 0$ for $k \in \mathbf{Z}_{\geqslant 1}$ , whence $\tilde{\mathfrak{m}}^2 = 0$. ▫

**Lemma 49** *Let* $n \in \mathbf{Z}_{\geqslant 1}$ .

*Let* $\tilde{A} = (\tilde{A}, (\tilde{m}_\ell)_{\ell\in[1,n+1]})$ *and* $A = (A, (m_\ell)_{\ell\in[1,n+1]})$ *be pre-*$\mathrm{A}_{n+1}$*-algebras over* $\mathcal{Z}$.

*Let* $f = (f_\ell)_{\ell\in[1,n+1]}$ *be a pre-*$\mathrm{A}_{n+1}$*-morphism from* $\tilde{A}$ *to* $A$.

*Suppose that the following assertions* (i, ii, iii) *hold.*

(i) $(\tilde{A}, (\tilde{m}_\ell)_{\ell\in[1,n]})$ *is an* $\mathrm{A}_n$*-algebra,* $(A, (m_\ell)_{\ell\in[1,n+1]})$ *is an* $\mathrm{A}_{n+1}$*-algebra and* $(f_\ell)_{\ell\in[1,n]}$ *is an* $\mathrm{A}_n$*-morphism from* $(\tilde{A}, (\tilde{m}_\ell)_{\ell\in[1,n]})$ *to* $(A, (m_\ell)_{\ell\in[1,n]})$.

(ii) $\tilde{m}_1 = 0$.

(iii) $f_1$ *is a quasiisomorphism.*

*Write*

$$\begin{aligned}\Psi_{n+1} \;\; := \;\; & - \sum_{\substack{(r,s,t)\geqslant(0,2,0)\\ r+s+t=n+1\\ (r,t)>(0,0)}} (-1)^{r+st}(\mathrm{id}^{\otimes r}\otimes\tilde{m}_s\otimes\mathrm{id}^{\otimes t})\cdot f_{r+1+t}\\ & + \sum_{r\in[2,n+1]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=n+1}} \lfloor(1-i_j)_j\,,(i_j)_j\rfloor(f_{i_1}\otimes\ldots\otimes f_{i_r})\cdot m_r\,.\end{aligned}$$

*Then*

$$\Psi_{n+1}\cdot m_1 \;=\; 0\,.$$

*Proof.* Write

$$\begin{aligned}{}^{\omega}\Psi_{n+1} \;\; & := \;\; \omega^{\otimes n+1}\cdot\Psi_n\cdot\omega^{-}\\ & \overset{\text{L. }37}{=} \;\; - \sum_{\substack{(r,s,t)\geqslant(0,2,0)\\ r+s+t=n+1\\ (r,t)>(0,0)}} (\mathrm{id}^{\otimes r}\otimes{}^{\omega}\tilde{m}_s\otimes\mathrm{id}^{\otimes t})\cdot{}^{\omega}f_{r+1+t}\\ & \qquad + \sum_{r\in[2,n+1]}\;\sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=n+1}} ({}^{\omega}f_{i_1}\otimes\ldots\otimes{}^{\omega}f_{i_r})\cdot{}^{\omega}m_r\,.\end{aligned}$$

Using Propositions 42 and 43, we write

$$\begin{array}{lll}\tilde{\mathfrak{m}} & := & (({}^{\omega}\tilde{m}_\ell)_{\ell\in[1,n+1]})\beta_{\mathrm{Coder},n+1,\tilde{A}^{[1]}} \quad \text{(coderivation on } \mathrm{T}_{\leqslant n+1}(\tilde{A}^{[1]})\,)\\ \mathfrak{m} & := & (({}^{\omega}m_\ell)_{\ell\in[1,n+1]})\beta_{\mathrm{Coder},n+1,A^{[1]}} \quad \text{(coderivation on } \mathrm{T}_{\leqslant n+1}(A^{[1]})\,)\\ \mathfrak{f} & := & (({}^{\omega}f_\ell)_{\ell\in[1,n+1]})\beta_{\mathrm{Coalg},n+1,\tilde{A}^{[1]},A^{[1]}} \quad \text{(coalgebra morphism from } \mathrm{T}_{\leqslant n+1}(\tilde{A}^{[1]}) \text{ to } \mathrm{T}_{\leqslant n+1}(\tilde{A}^{[1]})\,)\,.\end{array}$$

By Problem 22, we have

$$\begin{array}{lll}\tilde{\mathfrak{m}}|^{\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})}_{\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})} & := & (({}^{\omega}\tilde{m}_\ell)_{\ell\in[1,n]})\beta_{\mathrm{Coder},n+1,\tilde{A}^{[1]}} \quad \text{(coderivation on } \mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})\,)\\ \mathfrak{m}|^{\mathrm{T}_{\leqslant n}(A^{[1]})}_{\mathrm{T}_{\leqslant n}(A^{[1]})} & := & (({}^{\omega}m_\ell)_{\ell\in[1,n]})\beta_{\mathrm{Coder},n+1,A^{[1]}} \quad \text{(coderivation on } \mathrm{T}_{\leqslant n}(A^{[1]})\,)\\ \mathfrak{f}|^{\mathrm{T}_{\leqslant n}(A^{[1]})}_{\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})} & := & (({}^{\omega}f_\ell)_{\ell\in[1,n]})\beta_{\mathrm{Coalg},n+1,\tilde{A}^{[1]},A^{[1]}} \quad \text{(coalgebra morphism from } \mathrm{T}_{\leqslant n}(\tilde{A}^{[1]}) \text{ to } \mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})\,)\,.\end{array}$$

So by (i), we have

$$\begin{array}{rcl} \tilde{\mathfrak{m}}^2|_{\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})} & = & 0 \\ \mathfrak{m}^2 & = & 0 \\ (\tilde{\mathfrak{m}}\cdot\mathfrak{f} - \mathfrak{f}\cdot\mathfrak{m})|_{\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})} & = & 0\,; \end{array}$$

cf. Propositions 46 and 47.

Note that

$$\begin{array}{rcl} \iota_{n+1}\cdot\tilde{\mathfrak{m}} & = & \sum\limits_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=n+1}} (\mathrm{id}^{\otimes r}\otimes{}^\omega\tilde{m}_s\otimes\mathrm{id}^{\otimes t})\cdot\iota_{r+1+t} \\ & \overset{\text{(ii)}}{=} & \sum\limits_{\substack{(r,s,t)\geqslant(0,2,0)\\ r+s+t=n+1}} (\mathrm{id}^{\otimes r}\otimes{}^\omega\tilde{m}_s\otimes\mathrm{id}^{\otimes t})\cdot\iota_{r+1+t}\ . \end{array}$$

In particular, $(\iota_{n+1}\cdot\tilde{\mathfrak{m}})|^{\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})}$ exists.

We obtain

$$\begin{array}{rl} & \iota_{n+1}\cdot(\tilde{\mathfrak{m}}\cdot\mathfrak{f} - \mathfrak{f}\cdot\mathfrak{m})\cdot\pi_1 \\ = & \iota_{n+1}\cdot\tilde{\mathfrak{m}}\cdot\mathfrak{f}\cdot\pi_1 - \iota_{n+1}\mathfrak{f}\cdot\mathfrak{m}\cdot\pi_1 \\ \overset{\text{P. 42, P. 43}}{=} & \sum\limits_{\substack{(r,s,t)\geqslant(0,2,0)\\ r+s+t=n+1}} (\mathrm{id}^{\otimes r}\otimes{}^\omega\tilde{m}_s\otimes\mathrm{id}^{\otimes t})\cdot\iota_{r+1+t}\cdot\mathfrak{f}\cdot\pi_1 \\ & -\sum\limits_{r\in[1,n+1]}\ \sum\limits_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=n+1}} (\,{}^\omega f_{i_1}\otimes\ldots\otimes{}^\omega f_{i_r})\cdot\iota_r\cdot\mathfrak{m}\cdot\pi_1 \\ \overset{\text{P. 42, P. 43}}{=} & \sum\limits_{\substack{(r,s,t)\geqslant(0,2,0)\\ r+s+t=n+1}} (\mathrm{id}^{\otimes r}\otimes{}^\omega\tilde{m}_s\otimes\mathrm{id}^{\otimes t})\cdot{}^\omega f_{r+1+t} \\ & -\sum\limits_{r\in[1,n+1]}\ \sum\limits_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=n+1}} (\,{}^\omega f_{i_1}\otimes\ldots\otimes{}^\omega f_{i_r})\cdot{}^\omega m_r \\ = & {}^\omega\tilde{m}_{n+1}\cdot{}^\omega f_1 + \sum\limits_{\substack{(r,s,t)\geqslant(0,2,0)\\ r+s+t=n+1\\ (r,t)>0}} (\mathrm{id}^{\otimes r}\otimes{}^\omega m_s\otimes\mathrm{id}^{\otimes t})\cdot{}^\omega f_{r+1+t} \\ & -{}^\omega f_{n+1}\cdot{}^\omega m_1 - \sum\limits_{r\in[2,n+1]}\ \sum\limits_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j=n+1}} (\,{}^\omega f_{i_1}\otimes\ldots\otimes{}^\omega f_{i_r})\cdot{}^\omega m_r \\ = & -{}^\omega\Psi_{n+1} + {}^\omega\tilde{m}_{n+1}\cdot{}^\omega f_1 - {}^\omega f_{n+1}\cdot{}^\omega m_1\ . \end{array}$$

Thus

$$
\begin{array}{cl}
 & -\,{}^{\omega}\Psi_{n+1}\cdot{}^{\omega}m_1 \\
= & \iota_{n+1}\cdot(\tilde{\mathfrak{m}}\cdot\mathfrak{f}-\mathfrak{f}\cdot\mathfrak{m})\cdot\pi_1\cdot{}^{\omega}m_1 - {}^{\omega}\tilde{m}_{n+1}\cdot{}^{\omega}f_1\cdot{}^{\omega}m_1 + {}^{\omega}f_{n+1}\cdot{}^{\omega}m_1\cdot{}^{\omega}m_1 \\
\overset{\text{(i)}}{=} & \iota_{n+1}\cdot(\tilde{\mathfrak{m}}\cdot\mathfrak{f}-\mathfrak{f}\cdot\mathfrak{m})\cdot\pi_1\cdot{}^{\omega}m_1 - {}^{\omega}\tilde{m}_{n+1}\cdot{}^{\omega}\tilde{m}_1\,{}^{\omega}f_1 + {}^{\omega}f_{n+1}\cdot 0 \\
\overset{\text{(ii)}}{=} & \iota_{n+1}\cdot(\tilde{\mathfrak{m}}\cdot\mathfrak{f}-\mathfrak{f}\cdot\mathfrak{m})\cdot\pi_1\cdot{}^{\omega}m_1 \\
\overset{\text{P. 42}}{=} & \iota_{n+1}\cdot(\tilde{\mathfrak{m}}\cdot\mathfrak{f}-\mathfrak{f}\cdot\mathfrak{m})\cdot\pi_1\cdot\iota_1\cdot\mathfrak{m}\cdot\pi_1 \\
\overset{\text{L. 45.(2)}}{=} & \iota_{n+1}\cdot(\tilde{\mathfrak{m}}\cdot\mathfrak{f}-\mathfrak{f}\cdot\mathfrak{m})\cdot\mathfrak{m}\cdot\pi_1 \\
\overset{\mathfrak{m}^2=0}{=} & \iota_{n+1}\cdot\tilde{\mathfrak{m}}\cdot\mathfrak{f}\cdot\mathfrak{m}\cdot\pi_1 \\
\overset{(\iota_{n+1}\cdot\tilde{\mathfrak{m}})|^{\mathrm{T}_{\leqslant n}(\tilde{A}^{[1]})}}{=} & \iota_{n+1}\cdot\tilde{\mathfrak{m}}\cdot\tilde{\mathfrak{m}}\cdot\mathfrak{f}\cdot\pi_1 \\
\overset{\text{L. 45.(1)}}{=} & \iota_{n+1}\cdot\tilde{\mathfrak{m}}^2\cdot\pi_1\cdot\iota_1\cdot\mathfrak{f}\cdot\pi_1 \\
\overset{\text{P. 43}}{=} & \iota_{n+1}\cdot\tilde{\mathfrak{m}}^2\cdot\pi_1\cdot{}^{\omega}f_1\ .
\end{array}
$$

Therefore, we obtain

$$
\begin{array}{rcl}
\Psi_{n+1}\cdot m_1 & = & -\omega^{\otimes n+1}\cdot({}^{\omega}\Psi_{n+1}\cdot{}^{\omega}m_1)\cdot\omega \\
 & = & \omega^{\otimes n+1}\cdot\iota_{n+1}\cdot\tilde{\mathfrak{m}}^2\cdot\pi_1\cdot{}^{\omega}f_1\cdot\omega \\
 & = & \Big(\omega^{\otimes n+1}\cdot\iota_{n+1}\cdot\tilde{\mathfrak{m}}^2\cdot\pi_1\cdot\omega\Big)\cdot f_1\ .
\end{array}
$$

Hence it suffices to show the following claim.

*Claim.* Suppose given $\mathcal{Z}$-graded module $T$, an element $d\in\mathbf{Z}$, a shift-graded linear map $T\xrightarrow{\xi}A$ of degree $d-1$ and a shift-graded linear map $T\xrightarrow{\eta}\tilde{A}$ of degree $d$ such that

$$\xi\cdot m_1 \;=\; \eta\cdot f_1\ .$$

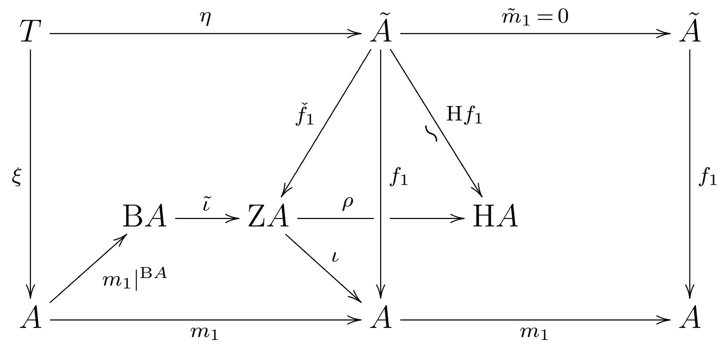
Then $\eta=0$ and $\xi\cdot m_1=0$.

It suffices to show $\eta\overset{!}{=}0$. We use the notation of Remark 30.

We have $f_1\cdot m_1=\tilde{m}_1\cdot f_1\overset{\text{(ii)}}{=}0$. By Problem 15.(1), this yields the commutative diagram

$$
\begin{array}{ccc}
\tilde{A} & \xrightarrow{f_1} & A \\
\| & & \uparrow{\scriptstyle\iota} \\
\mathrm{Z}\tilde{A} & \xrightarrow{\check{f}_1} & \mathrm{Z}A \\
\| & & \downarrow{\scriptstyle\rho} \\
\mathrm{H}\tilde{A} & \underset{\mathrm{H}f_1}{\xrightarrow{\sim}} & \mathrm{H}A\ .
\end{array}
$$

So we get the following commutative diagram.




We have

$$\xi \cdot m_1|^{\mathrm{B}A} \cdot \tilde{\iota} \cdot \iota \;=\; \xi \cdot m_1 \;=\; \eta \cdot f_1 \;=\; \eta \cdot \check{f}_1 \cdot \iota ,$$

by pointwise injectivity of $\iota$ thus

$$\xi \cdot m_1|^{\mathrm{B}A} \cdot \tilde{\iota} \;=\; \eta \cdot \check{f}_1 \; .$$

So

$$\eta \cdot \mathrm{H}f_1 \;=\; \eta \cdot \check{f}_1 \cdot \rho \;=\; \xi \cdot m_1|^{\mathrm{B}A} \cdot \tilde{\iota} \cdot \rho \;=\; 0 \; .$$

Since $\mathrm{H}f_1$ is an isomorphism by (iii), we conclude that $\eta = 0$. This proves the *claim*. ▫

In Lemma 49, it would have been sufficient to require $\mathrm{H}f_1$ to be pointwise injective, for this suffices to prove the Claim.

**Theorem 50 (Kadeishvili)** *Suppose that $R$ is a* field*.*

*Let $n \in [1, \infty]$. Recall that $\mathcal{Z}$ is a grading category.*

*Let $A = (A, (m_\ell)_\ell)$ be an $\mathrm{A}_n$-algebra over $\mathcal{Z}$.*

*There exist tuples of shift-graded linear maps $(\tilde{m}_\ell)_\ell$ and $(q_\ell)_\ell$ such that*

$$\mathrm{H}A \;=\; (\mathrm{H}A, (\tilde{m}_\ell)_\ell)$$

*is a minimal $\mathrm{A}_n$-algebra over $\mathcal{Z}$ and such that*

$$q \;:=\; (q_\ell)_\ell \;:\; \mathrm{H}A \;\to\; A$$

*is a quasiisomorphism. Cf. Definitions* 27.(6) *and* 29.(3)*.*

*If $A$ is unital and $n \geqslant 2$, then $(\mathrm{H}A, (\tilde{m}_\ell)_\ell)$ and $q = (q_\ell)_\ell$ can be chosen to be unital; cf. Definitions* 23 *and* 24*.*

When writing $(m_\ell^{\mathrm{H}A})_\ell$ instead of $(\tilde{m}_\ell)_\ell$, no uniqueness is implied of this structure of an $\mathrm{A}_n$-algebra on $A$ with said properties.

*Proof.* We use the notation of Remark 30. Where necessary, we shall briefly recall arguments of Remark 31.

*First*, we do not suppose $A$ to be unital.

We proceed by induction on $n$.

*Base.* Suppose $n = 1$. Let $\tilde{m}_1 := 0$. Let

$$q_1 \;:=\; \sigma \cdot \iota \,.$$

Note that $q_1$ is piecewise injective since $\sigma$ and $\iota$ are.

Then $\tilde{m}_1 \cdot q_1 = 0 = \sigma \cdot \iota \cdot m_1 = q_1 \cdot m_1$ ; i.e. the Stasheff equation for morphisms holds at $1$ ; cf. Example 22.(1).

We have $\tilde{m}_1^2 = 0^2 = 0$ ; i.e. the Stasheff equation for $\mathrm{H}A$ holds at $1$ ; cf. also Lemma 48.

Since we have the commutative diagram

$$\begin{array}{ccc} \mathrm{H}A & \xrightarrow{\;q_1\;} & A \\ \| & & \uparrow{\scriptstyle \iota} \\ \mathrm{ZH}A & \xrightarrow{\;\sigma\;} & \mathrm{Z}A \\ \| & & \downarrow{\scriptstyle \rho} \\ \mathrm{HH}A & = & \mathrm{H}A \end{array} ,$$

we have $\mathrm{H}q_1 = \mathrm{id}_{\mathrm{H}A}$, which is an isomorphism.

*Step.* Suppose the assertion to be known for $n \in \mathbf{Z}_{\geqslant 1}$ . We have to show the assertion for $n+1$. We have to show that there exists a shift-graded linear map $q_{n+1} : (\mathrm{H}A)^{\otimes n+1} \to A$ of degree $-n$ and a shift-graded linear map $\tilde{m}_{n+1} : (\mathrm{H}A)^{\otimes n+1} \to \mathrm{H}A$ of degree $1-n$ such that $(\tilde{m}_\ell)_{\ell \in [1,n+1]}$ satisfies the Stasheff equation at $n+1$ and such that $(q_\ell)_{\ell \in [1,n+1]}$ satisfies the Stasheff equation for morphisms at $n+1$, with respect to $(\tilde{m}_\ell)_{\ell \in [1,n+1]}$ and $(m_\ell)_{\ell \in [1,n+1]}$ .

Since $q_1$ is piecewise injective, it suffices, by Lemma 48, to show the Stasheff equation for morphisms for $(q_\ell)_{\ell \in [1,n+1]}$ at $n+1$.

As in Lemma 49, we write

$$\begin{aligned} \Psi_{n+1} \;\;:=\;\; & - \sum_{\substack{(r,s,t) \geqslant (0,2,0) \\ r+s+t = n+1 \\ (r,t) > (0,0)}} (-1)^{r+st} (\mathrm{id}^{\otimes r} \otimes \tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot q_{r+1+t} \\ & + \sum_{r \in [2,n+1]} \; \sum_{\substack{(i_j)_{j \in [1,r]} \geqslant (1)_j \\ \sum_j i_j = n+1}} \lfloor (1-i_j)_j \,, (i_j)_j \rfloor (q_{i_1} \otimes \ldots \otimes q_{i_r}) \cdot m_r \;, \end{aligned}$$

which is a shift-graded linear map from $(\mathrm{H}A)^{\otimes n+1}$ to $A$ of degree $1-n$.

In this defining expression for $\Psi_{n+1}$ , in fact $\tilde{m}_i$ and $q_i$ are involved only for $i \in [1, n]$.

By Lemma 49, letting for the moment $\tilde{m}_{n+1}$ and $q_{n+1}$ be arbitrary, e.g. zero, we have

$$\Psi_{n+1} \cdot m_1 \;=\; 0 \,.$$

So

$$\Psi_{n+1} \;=\; \check{\Psi}_{n+1} \cdot \iota \,,$$

where $\check{\Psi}_{n+1}$ is a shift-graded linear map from $(\mathrm{H}A)^{\otimes n+1}$ to $\mathrm{Z}A$ of degree $1 - n$ ; cf. Problem 15.(1).

Taking into account that $\tilde{m}_1 = 0$, the Stasheff equation for morphisms at $n + 1$, which we have to show, writes

$$\Psi_{n+1} \overset{!}{=} \tilde{m}_{n+1} \cdot q_1 - q_{n+1} \cdot m_1 \,.$$

Let

$$\begin{array}{rcr} q_{n+1} & := & -\check{\Psi}_{n+1} \cdot \nu \\ \tilde{m}_{n+1} & := & \check{\Psi}_{n+1} \cdot \rho \,. \end{array}$$

Then

$$\begin{array}{rcl} \tilde{m}_{n+1} \cdot q_1 - q_{n+1} \cdot m_1 & = & \check{\Psi}_{n+1} \cdot \rho \cdot \sigma \cdot \iota + \check{\Psi}_{n+1} \cdot \nu \cdot m_1 \\ & \overset{\text{R. 30}}{=} & \check{\Psi}_{n+1} \cdot \rho \cdot \sigma \cdot \iota + \check{\Psi}_{n+1} \cdot (\mathrm{id}_{\mathrm{Z}A} - \rho \cdot \sigma) \cdot \iota \\ & = & \Psi_{n+1} \,. \end{array}$$

*Second,* we suppose $A$ to be unital and $n \geqslant 2$.

As in Remark 31, we obtain the following commutative diagram.

$$\begin{array}{ccc} A^{\otimes 2} & \xrightarrow{m_2} & A \\ {\scriptstyle \iota^{\otimes 2}}\uparrow & & \uparrow{\scriptstyle \iota} \\ (\mathrm{Z}A)^{\otimes 2} & \xrightarrow{\check{m}_2} & \mathrm{Z}A \\ {\scriptstyle \rho^{\otimes 2}}\downarrow & & \downarrow{\scriptstyle \rho} \\ (\mathrm{H}A)^{\otimes 2} & \xrightarrow{\hat{m}_2} & \mathrm{H}A \end{array}$$

Moreover, we get $\Psi_2 = (q_1 \otimes q_1) \otimes m_2$ . Letting, as before, $\Psi_2 \;=\; \check{\Psi}_2 \cdot \iota$ and $q_2 := -\check{\Psi}_2 \cdot \nu$ and $\tilde{m}_2 := \check{\Psi}_2 \cdot \rho$, we have

$$\hat{m}_2 \;=\; \tilde{m}_2 \,;$$

cf. Remark 31.

For $X \in \mathrm{Ob}(\mathcal{Z})$, we have $1_{A,X} \in \mathrm{Z}A$ ; cf. Definition 23.

So for $X \xrightarrow{x} Y \xrightarrow{y} Z$ in $\mathcal{Z}$, for $a \in (\mathrm{Z}A)^x$ and $b \in (\mathrm{Z}A)^y$, we get

$$(1_{A,Y}\rho \otimes b\rho)\tilde{m}_2 \;=\; (1_{A,Y} \otimes b)\rho^{\otimes 2}\tilde{m}_2 \;=\; (1_{A,Y} \otimes b)\check{m}_2\rho = b\rho \,,$$

since

$$(1_{A,Y} \otimes b)\check{m}_2 \;=\; (1_{A,Y} \otimes b)\check{m}_2\iota \;=\; (1_{A,Y} \otimes b)\iota^{\otimes 2} m_2 \;=\; (1_{A,Y} \otimes b)m_2 \;=\; b\ .$$

Likewise, we get

$$(a\rho \otimes 1_{A,Y}\rho)\tilde{m}_2 \;=\; a\rho\ .$$

So the element $1_{A,Y}\rho \in (\mathrm{H}A)^{\mathrm{id}_Y}$ is neutral, i.e. $1_{\mathrm{H}A,Y} = 1_{A,Y}\rho$.

Hence the $\mathrm{A}_n$-algebra $\mathrm{H}A = (\mathrm{H}A, (\tilde{m}_\ell)_\ell)$ is unital.

By Problem 18, the choice of $\sigma$ made in Remark 30 can be made in such a way that $1_{\mathrm{H}A,X}\sigma = 1_{A,X}\rho\sigma = 1_{A,X}$ for $X \in \mathrm{Ob}(\mathcal{Z})$, whence

$$1_{\mathrm{H}A,X}q_1 \;=\; 1_{\mathrm{H}A,X}\sigma\,\iota \;=\; 1_{A,X}\iota = 1_{A,X}\ .$$

Therefore the $\mathrm{A}_n$-morphism $q = (q_\ell)_\ell$ is unital. □

In the induction step of Theorem 50, we could have used an arbitrary shift-graded linear map $\Psi_{n+1}$ from $(\mathrm{H}A)^{\otimes n+1}$ to $A$ of degree $1-n$ that satisfies $\Psi_{n+1}\cdot m_1 = 0$ and define $q_{n+1} := -\check{\Psi}_{n+1}\cdot\nu$ and $\tilde{m}_{n+1} := \check{\Psi}_{n+1}\cdot\rho$. Lemma 49 merely guarantees the existence of such a shift-graded linear map.

**Remark 51** Let $G$ be a finite group. Let $N \in \mathbf{Z}_{\geqslant 1}$. Let $M_1\,,\,\ldots,\,M_N$ be $RG$-modules. Suppose $M_1 = R$ to carry the trivial $RG$-module structure, i.e. $gr = r$ for $g \in G$ and $r \in R = M_1$.

Let $P_s$ be a projective resolution of $M_s$ for $s \in [1,N]$. Write $\underline{P} := (P_s)_{s\in[1,N]}$.

Let $A := \mathrm{Hom}_{RG}(\underline{P})$ be the regular differential graded category of $\underline{P}$; cf. Lemma 28.

So $A$ is a unital $\mathrm{A}_\infty$-algebra over

$$\mathcal{Z} \;:=\; \mathbf{Z} \times [1,N]^{\times 2}\ .$$

For $(j,(s,t)) \in \mathrm{Mor}(\mathcal{Z})$, we get

$$(\mathrm{H}A)^{j,(s,t)} \;=\; {}_{\mathrm{K}}(\,P_s\,,\,P_t^{[j]}\,) \;=:\; \mathrm{Ext}^j_{RG}(M_s,M_t)\ ,$$

where we have written $\mathrm{K} := \mathrm{K}(RG\text{-Mod})$; cf. Problem 14.(2). In particular,

$$(\mathrm{H}A)^{j,(1,1)} \;=\; {}_{\mathrm{K}}(\,P_1\,,\,P_1^{[j]}\,) \;=:\; \mathrm{Ext}^j_{RG}(M_1,M_1) \;=\; \mathrm{Ext}^j_{RG}(R,R) \;=\; \mathrm{H}^j(G;R)\ ,$$

the group cohomology of $G$ over the ground ring $R$.

Now suppose $R$ to be a field.

Kadeishvili's Theorem 50 yields the structure $(\tilde{m}_\ell)_\ell$ of a minimal $\mathrm{A}_\infty$-algebra over $\mathcal{Z}$ on $\mathrm{H}A$ and a unital quasiisomorphism

$$\mathrm{H}A \;\to\; A\ .$$

In particular, $\tilde{m}_1 = 0$. Moreover,

$$\begin{array}{rcllcl} \mathrm{Ext}^j_{RG}(M_s, M_t) & \otimes & \mathrm{Ext}^k_{RG}(M_t, M_u) & \stackrel{\tilde{m}_2}{\longrightarrow} & \mathrm{Ext}^{j+k}_{RG}(M_s, M_u) \\ [f] & \otimes & [g] & \mapsto & [f \cdot g^{[j]}] \end{array}$$

is the *Yoneda* product, where $(j, (s,t))$, $(k, (t,u)) \in \mathrm{Mor}(\mathcal{Z})$. Cf. Lemma 28, Remark 31.

In particular,

$$\mathrm{H}^j(G; R) \quad \otimes \quad \mathrm{H}^j(G; R) \quad \stackrel{\tilde{m}_2}{\longrightarrow} \quad \mathrm{H}^j(G; R)$$

is also known as *cup* product.

In that sense, $\tilde{m}_n$ for $n \in \mathbf{Z}_{\geqslant 3}$ are sometimes referred to as "higher" cup products on the cohomology ring of $G$ over the ground field $R$.

# Chapter 2

# Schmid's extension of Kadeishvili

The purpose of the extra machinery in this §2.1 is to remove the restriction on $R$ to be a field from Theorem 50.

Let $\mathcal{Z}$ be a grading category.

## 2.1 Split-filtered $\mathcal{Z}$-graded modules

**Definition 52** A *split-filtered $\mathcal{Z}$-graded module* is a $\mathcal{Z}$-graded module $M$, together with a tuple $(M^{\langle i\rangle})_{i\in\mathbf{Z}}$ of $\mathcal{Z}$-graded submodules of $M$ such that the following conditions $(1,2)$ hold.

(1) We have $M^{\langle i\rangle} = 0$ for $i \in \mathbf{Z}_{<0}$.

(2) We have $M = \bigoplus_{i\in\mathbf{Z}_{\geqslant 0}} M^{\langle i\rangle}$.

We often abbreviate $M = (M, (M^{\langle i\rangle})_i)$.

Write $M^{\leqslant k} := \bigoplus_{i\in[0,k]} M^{\langle i\rangle}$ for $k \in \mathbf{Z}$. So $M^{\leqslant k}$ is a $\mathcal{Z}$-graded submodule of $M$.

We have shift-graded linear inclusion and projection maps

$$M^{\langle i\rangle} \xrightarrow{\iota_M^{\langle i\rangle}} M \xrightarrow{\pi_M^{\langle i\rangle}} M^{\langle i\rangle}$$

of degree 0 for $i \in \mathbf{Z}$. We often abbreviate $\iota^{\langle i\rangle} = \iota_M^{\langle i\rangle}$ and $\pi^{\langle i\rangle} = \pi_M^{\langle i\rangle}$.

So $\iota^{\langle i\rangle}\pi^{\langle i\rangle} = \mathrm{id}_{M^{\langle i\rangle}}$ for $i \in \mathbf{Z}$ and $\iota^{\langle i\rangle}\pi^{\langle j\rangle} = 0$ for $i,\ j \in \mathbf{Z}$ with $i \neq j$.

With a similar abuse of notation, we also have shift-graded linear inclusion and projection maps

$$M^{\langle i\rangle} \xrightarrow{\iota^{\langle i\rangle}} M^{\leqslant k} \xrightarrow{\pi^{\langle i\rangle}} M^{\langle i\rangle}$$

of degree 0 for $i \in [0, k]$.

Given $k, \ell \in \mathbf{Z}$ such that $\ell \leqslant k$, we also have the shift-graded linear inclusion and projection maps

$$M^{\leqslant \ell} \xrightarrow{\iota^{\leqslant \ell}} M^{\leqslant k} \xrightarrow{\pi^{\leqslant \ell}} M^{\leqslant \ell}$$

of degree 0.

I do not know whether a variant of the theory can be carried through with filtered $\mathcal{Z}$-graded modules instead of split-filtered $\mathcal{Z}$-graded modules.

**Example 53** Let $X$ be a $\mathcal{Z}$-graded module. For $z \in \mathrm{Mor}(\mathcal{Z})$, choose

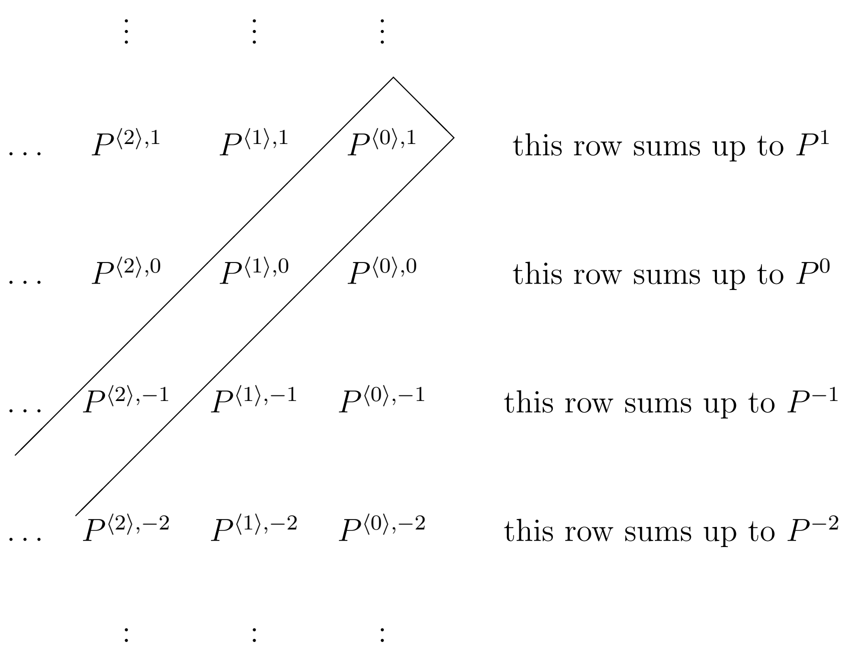
$$\ldots \to P^{\langle 2\rangle, z[-2]} \to P^{\langle 1\rangle, z[-1]} \to P^{\langle 0\rangle, z[0]} \to X^z \to 0$$

to be an augmented projective resolution of $X^z$ (over $R$), i.e. $P^{\langle k\rangle, z[-k]}$ is projective for $k \in \mathbf{Z}_{\geqslant 0}$ and the sequence is exact at each position.

Write $P^z := \bigoplus_{i \in \mathbf{Z}_{\geqslant 0}} P^{\langle i\rangle, z}$ for $z \in \mathrm{Mor}(\mathcal{Z})$.

Then $P$ is a split-filtered $\mathcal{Z}$-graded module with $P^{\langle i\rangle} := (P^{z,\langle i\rangle})_{z \in \mathrm{Mor}(\mathcal{Z})}$ for $i \in \mathbf{Z}$.

If $\mathcal{Z} = \mathbf{Z}$, we can picture the components of $P$ as follows.




Note that the objects of the respectively chosen projective resolutions can be found in the diagonals of this diagram, such as the boxed one, whose objects belong to a projective resolution of $X^1$.

## 2.2 $\mathrm{eA}_\infty$-algebras and $\mathrm{eA}_\infty$-categories

**Definition 54** Let $n \in [0, \infty]$.

An $\mathrm{eA}_n$-algebra over $\mathcal{Z}$ is a split-filtered $\mathcal{Z}$-graded module $A = (A, (A^{\langle i \rangle})_i)$, together with the structure of an $\mathrm{A}_n$-algebra $(m_\ell)_\ell$ on the $\mathcal{Z}$-graded module $A$, such that the *Schmid condition*

$$(\bigotimes_{j\in[1,k]} A^{\langle i_j \rangle})m_k \;\subseteq\; A^{\leqslant 2k-2+\sum_{j\in[1,k]} i_j}$$

holds for $k \in [1, n] \cap \mathbf{Z}$ and $(i_j)_{j\in[1,k]} \in \mathbf{Z}_{\geqslant 0}^{\times k}$.

We often abbreviate $A = (A, (m_\ell)_{\ell\in[1,n]}, (A^{\langle i \rangle})_{i\in\mathbf{Z}})$.

We often write $(A^{\langle i \rangle})^z =: A^{\langle i \rangle, z}$ for $i \in \mathbf{Z}$ and $z \in \mathrm{Mor}(\mathcal{Z})$.

We often write $(A^{\leqslant k})^z =: A^{\leqslant k, z}$ for $k \in \mathbf{Z}$ and $z \in \mathrm{Mor}(\mathcal{Z})$.

The "e" in "$\mathrm{eA}_n$-algebra" stands for "extended".

Schmid states that the Schmid condition was motivated by Sagave; cf. [**5,** Def. 76, (EA 3)], [**3,** Def. 2.1]. It is a bit weaker than Sagave's implicitly stated condition.

**Definition 55** Let $n \in [0, \infty]$.

An $\mathrm{eA}_n$-algebra $A = (A, (m_\ell)_{\ell\in[1,n]}, (A^{\langle i \rangle})_{i\in\mathbf{Z}})$ over $\mathcal{Z}$ is called *minimal* if the *strong Schmid condition*

$$(\bigotimes_{j\in[1,k]} A^{\langle i_j \rangle})m_k \;\subseteq\; A^{\leqslant 2k-3+\sum_{j\in[1,k]} i_j}$$

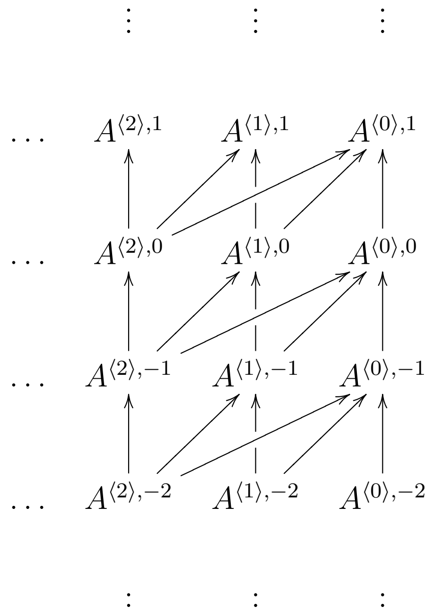
holds for $k \in [1, n] \cap \mathbf{Z}$ and $(i_j)_{j\in[1,k]} \in \mathbf{Z}_{\geqslant 0}^{\times k}$.

**Remark 56** Let $n \in [1, \infty]$.

Let $A = (A, (m_\ell)_{\ell\in[1,n]}, (A^{\langle i \rangle})_{i\in\mathbf{Z}})$ be an $\mathrm{eA}_n$-algebra.

(1) For $k = 1$, the Schmid condition reads $A^{\langle i \rangle} m_1 \subseteq A^{\leqslant i}$ for $i \in \mathbf{Z}_{\geqslant 0}$.

So if $\mathcal{Z} = \mathbf{Z}$, taking into account that $m_1$ is of degree 1, the (possibly nonvanishing) components of $m_1$ can be visualised as follows.



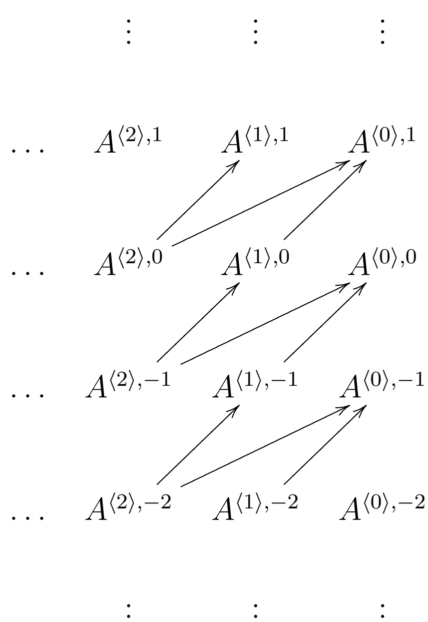
By this, we mean that $\iota^{\langle i \rangle} m_1 \pi^{\langle j \rangle} = 0$ unless $j \in [0, i]$.

E.g. on $A^{\langle 2 \rangle}$, the shift-graded linear map $m_1$ of degree 1 has the components

$$\iota^{\langle 2 \rangle} \cdot m_1 \cdot \pi^{\langle 2 \rangle} \; , \quad \iota^{\langle 2 \rangle} \cdot m_1 \cdot \pi^{\langle 1 \rangle} \; , \quad \iota^{\langle 2 \rangle} \cdot m_1 \cdot \pi^{\langle 0 \rangle} \; ,$$

all others vanish.

The strong Schmid condition reads $A^{\langle i \rangle} m_1 \subseteq A^{\leqslant i-1}$ for $i \in \mathbf{Z}_{\geqslant 0}$. So in case $A$ is minimal, the components of $m_1$ can be visualised as follows.



(2) Suppose that $n \geqslant 2$. For $k = 2$, the Schmid condition reads

$$(A^{\langle i_1 \rangle} \otimes A^{\langle i_2 \rangle})m_2 \ \subseteq \ A^{\leqslant 2+i_1+i_2}$$

for $i_1\,,\, i_2 \,\in\, \mathbf{Z}_{\geqslant 0}\,$.

In case of $A$ being minimal, the strong Schmid condition reads

$$(A^{\langle i_1 \rangle} \otimes A^{\langle i_2 \rangle})m_2 \ \subseteq \ A^{\leqslant 1+i_1+i_2}$$

for $i_1\,,\, i_2 \,\in\, \mathbf{Z}_{\geqslant 0}\,$.

**Remark 57** Let $n \in [0, \infty]$.

Suppose given an $\mathrm{A}_n$-algebra $A' = (A', (m'_\ell)_\ell)$.

Define an $\mathrm{eA}_n$-algebra $A = (A, (m_\ell)_\ell, (A^{\langle i \rangle})_i)$ by letting $A = A'$ as $\mathcal{Z}$-graded modules, by letting $m_\ell := m'_\ell$ for $\ell \in \mathbf{Z}_{\geqslant 1}$ and by letting

$$A^{\langle i \rangle} \ := \ \begin{cases} A' & \text{if } i = 0 \\ 0 & \text{if } i \in \mathbf{Z} \smallsetminus \{0\} \end{cases}$$

for $i \in \mathbf{Z}$.

In fact, we have $\bigoplus_{i \in \mathbf{Z}} A^{\langle i \rangle} = A^{\langle 0 \rangle} = A' = A$.

We have to verify the Schmid condition. For $k \in [1, n] \cap \mathbf{Z}$ and $i_1\,,\, \dots\,,\, i_k \,\in\, \mathbf{Z}_{\geqslant 0}\,$, we obtain

$$(A^{\langle i_1 \rangle} \otimes \ldots \otimes A^{\langle i_k \rangle})m_k \begin{cases} = \ 0 & \text{if there exists } j \in [1, k] \text{ with } i_j \geqslant 1 \\ \subseteq \ A^{\langle 0 \rangle} = A^{\leqslant 2k-2+\sum_j i_j} & \text{if } i_1 = \cdots = i_k = 0\,, \end{cases}$$

since in the second case, we have $2k - 2 \geqslant 0$ and $\sum_j i_j = 0$.

Now the $\mathrm{eA}_n$-algebra $A$ is minimal if and only if $(A^{\langle 0 \rangle} \otimes \ldots \otimes A^{\langle 0 \rangle})m_k \subseteq A^{\leqslant 2k-3}$ for $k \in \mathbf{Z}_{\geqslant 1}$. Since $2k - 3 \geqslant 0$ and thus $A^{\leqslant 2k-3} = A^{\langle 0 \rangle}$ if $k \geqslant 2$, this condition is equivalent to $A^{\langle 0 \rangle}m_1 \subseteq A^{\leqslant -1} = 0$, i.e. to $A'm'_1 = 0$, i.e. to $A'$ being a minimal $\mathrm{A}_n$-algebra.

For short,

$$A \text{ minimal} \ \Leftrightarrow \ A' \text{ minimal}$$

## 2.3 A base of an induction

**Remark 58 (and definition)** Let $A = (A, (m_1), (A^{\langle i \rangle})_i)$ be a minimal $\mathrm{eA}_1$-algebra.

We have $(A^{\langle i \rangle})m_1 \subseteq A^{\leqslant i-1}$ for $i \in \mathbf{Z}_{\geqslant 0}\,$; cf. Remark 56.(1). That is, we have

$$\iota^{\langle i \rangle} \cdot m_1 \ = \sum_{j \in [0, i-1]} \iota^{\langle i \rangle} \cdot m_1 \cdot \pi^{\langle j \rangle} \cdot \iota^{\langle j \rangle}$$

for $i \in \mathbf{Z}_{\geqslant 0}$ .

For $i \in \mathbf{Z}_{\geqslant}0$, we consider the shift-graded linear map

$$m_1^{\langle i \rangle} \;:=\; \iota^{\langle i \rangle} \cdot m_1 \cdot \pi^{\langle i-1 \rangle} \;:\; A^{\langle i \rangle} \;\to\; A^{\langle i-1 \rangle}$$

of degree 1.

For $i \in \mathbf{Z}_{\geqslant 1}$, we have

$$\begin{array}{rcl} m_1^{\langle i \rangle} \cdot m_1^{\langle i-1 \rangle} & = & \iota^{\langle i \rangle} \cdot m_1 \cdot \pi^{\langle i-1 \rangle} \cdot \iota^{\langle i-1 \rangle} \cdot m_1 \cdot \pi^{\langle i-2 \rangle} \\ & = & \iota^{\langle i \rangle} \cdot \overbrace{m_1 \cdot m_1}^{= 0 \text{ by Stasheff}} \cdot \pi^{\langle i-2 \rangle} \\ & & - \sum_{j \in [0,i-2]} \iota^{\langle i \rangle} \cdot m_1 \cdot \pi^{\langle j \rangle} \cdot \underbrace{\iota^{\langle j \rangle} \cdot m_1 \cdot \pi^{\langle i-2 \rangle}}_{= 0 \text{ since } j \leqslant i-2} \\ & = & 0 \, . \end{array}$$

If $\mathrm{Im}(m_1^{\langle i \rangle}) = \mathrm{Kern}(m_1^{\langle i-1 \rangle})$ for $i \in \mathbf{Z}_{\geqslant 2}$ , then $A$ is called *diagonally resolving*.

Cf. Example 53.

**Lemma 59** *Let* $A = (A, (m_1), (A^{\langle i \rangle})_i)$ *be a diagonally resolving minimal* $\mathrm{eA}_1$*-algebra.*

(1) *We have* $\mathrm{Z}A = A^{\langle 0 \rangle} + \mathrm{B}A$.

(2) *We have* $A^{\leqslant k} m_1 = \mathrm{B}A \cap A^{\leqslant k-1}$ *for* $k \in \mathbf{Z}$.

*Proof.*

*Ad* (1).

*Ad* $\supseteq$. We have $A^{\langle 0 \rangle} m_1 \subseteq A^{\leqslant -1} = 0$, whence $A^{\langle 0 \rangle} \subseteq \mathrm{Z}A$. So $A^{\langle 0 \rangle} + \mathrm{B}A \subseteq \mathrm{Z}A$.

*Ad* $\subseteq$. We *claim* that $(A^{\leqslant j} \cap \mathrm{Z}A) + \mathrm{B}A \overset{!}{\subseteq} (A^{\leqslant j-1} \cap \mathrm{Z}A) + \mathrm{B}A$ for $j \in \mathbf{Z}_{\geqslant 1}$ .

Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$. Suppose given $a \in (A^{\leqslant j} \cap \mathrm{Z}A)^z = A^{\leqslant j,z} \cap (\mathrm{Z}A)^z$. We have to show that $a \overset{!}{\in} ((A^{\leqslant j-1} \cap \mathrm{Z}A) + \mathrm{B}A)^z = (A^{\leqslant j-1,z} \cap (\mathrm{Z}A)^z) + (\mathrm{B}A)^z$.

Since $a \in (\mathrm{Z}A)^z$ and since $(\mathrm{B}A)^z \subseteq (\mathrm{Z}A)^z$, it suffices to show that $a \overset{!}{\in} A^{\leqslant j-1,z} + (\mathrm{B}A)^z$.

We have

$$\begin{array}{rcl} (a\pi^{\langle j \rangle}) m_1^{\langle j \rangle} & = & a\pi^{\langle j \rangle} \iota^{\langle j \rangle} m_1 \pi^{\langle j-1 \rangle} \\ & = & (a - \sum_{i \in [0,j-1]} a\pi^{\langle i \rangle} \iota^{\langle i \rangle}) m_1 \pi^{\langle j-1 \rangle} \\ & = & - \sum_{i \in [0,j-1]} a\pi^{\langle i \rangle} \iota^{\langle i \rangle} m_1 \pi^{\langle j-1 \rangle} \\ & = & 0 \, , \end{array}$$

since $i \leqslant j-1$ for $i \in [0, j-1]$, i.e. $a\pi^{\langle j\rangle}\iota^{\langle j\rangle} \in \mathrm{Kern}(m_1^{\langle j\rangle})^z$. Since $A$ is diagonally resolving and since $j \geqslant 1$, we conclude that $a\pi^{\langle j\rangle} \in \mathrm{Im}(m_1^{\langle j+1\rangle})^{z[-1]}$. So there exists $a' \in A^{\langle j+1\rangle, z[-1]}$ such that

$$a\pi^{\langle j\rangle} \;=\; a' m_1^{\langle j+1\rangle} \;=\; a'\iota^{\langle j+1\rangle} m_1 \pi^{\langle j\rangle} \;.$$

Now

$$\begin{aligned} a &= a\pi^{\langle j\rangle}\iota^{\langle j\rangle} + \Big(\textstyle\sum_{i\in[0,j-1]} a\pi^{\langle i\rangle}\iota^{\langle i\rangle}\Big) \\ &= a'\iota^{\langle j+1\rangle} m_1 \pi^{\langle j\rangle}\iota^{\langle j\rangle} + \Big(\textstyle\sum_{i\in[0,j-1]} a\pi^{\langle i\rangle}\iota^{\langle i\rangle}\Big) \\ &= \underbrace{a'\iota^{\langle j+1\rangle} m_1}_{\in (\mathrm{B}A)^z} - \underbrace{\Big(\textstyle\sum_{i\in[0,j-1]} a'\iota^{\langle j+1\rangle} m_1 \pi^{\langle i\rangle}\iota^{\langle i\rangle}\Big)}_{\in A^{\leqslant j-1,z}} + \underbrace{\Big(\textstyle\sum_{i\in[0,j-1]} a\pi^{\langle i\rangle}\iota^{\langle i\rangle}\Big)}_{\in A^{\leqslant j-1,z}} \;. \end{aligned}$$

This proves the *claim.*

Given $z \in \mathrm{Mor}(\mathcal{Z})$ and $a \in (\mathrm{Z}A)^z$, we have to show that $z \overset{!}{\in} (A^{\langle 0\rangle} + \mathrm{B}A)^z = A^{\langle 0\rangle,z} + (\mathrm{B}A)^z$. There exists $j \in \mathbf{Z}_{\geqslant 1}$ such that $a \in A^{\leqslant j,z}$. So

$$\begin{aligned} a &\in (A^{\leqslant j,z} \cap (\mathrm{Z}A)^z) + (\mathrm{B}A)^z \\ &\overset{\text{Claim}}{\subseteq} (A^{\leqslant j-1,z} \cap (\mathrm{Z}A)^z) + (\mathrm{B}A)^z \\ &\overset{\text{Claim}}{\subseteq} \dots \\ &\overset{\text{Claim}}{\subseteq} (A^{\leqslant 0,z} \cap (\mathrm{Z}A)^z) + (\mathrm{B}A)^z \\ &= A^{\langle 0\rangle,z} + (\mathrm{B}A)^z \;. \end{aligned}$$

*Ad* (2).

*Ad* $\subseteq$. We have $A^{\leqslant k} m_1 \subseteq \mathrm{B}A$. We have $A^{\leqslant k} m_1 \subseteq A^{\leqslant k-1}$ by minimality of $A$.

*Ad* $\supseteq$. *Claim.* Given $j \in \mathbf{Z}_{\geqslant 0}$ and $\ell \in \mathbf{Z}_{\geqslant j+1}$, we have $A^{\leqslant \ell} m_1 \cap A^{\leqslant j-1} \overset{!}{\subseteq} A^{\leqslant \ell-1} m_1 \cap A^{\leqslant j-1}$. Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $a \in A^{\leqslant \ell, z[-1]}$ such that $am_1 \in A^{\leqslant j-1,z}$. We have to show that $am_1 \overset{!}{\in} A^{\leqslant \ell-1,z[-1]} m_1$.

Note that $am_1 \in A^{\leqslant j-1,z} \subseteq A^{\leqslant \ell-2,z}$. So we have

$$\begin{aligned} a\pi^{\langle \ell\rangle} m_1^{\langle \ell\rangle} &= a\pi^{\langle \ell\rangle}\iota^{\langle \ell\rangle} m_1 \pi^{\langle \ell-1\rangle} \\ &= (a - \textstyle\sum_{i\in[0,\ell-1]} a\pi^{\langle i\rangle}\iota^{\langle i\rangle}) m_1 \pi^{\langle \ell-1\rangle} \\ &= \underbrace{am_1\pi^{\langle \ell-1\rangle}}_{= 0 \text{ since } am_1 \in A^{\leqslant \ell-2,z}} - \textstyle\sum_{i\in[0,\ell-1]} a\pi^{\langle i\rangle} \underbrace{\iota^{\langle i\rangle} m_1 \pi^{\langle \ell-1\rangle}}_{= 0 \text{ since } i \leqslant \ell-1} \\ &= 0 \;. \end{aligned}$$

Since $A$ is diagonally resolving and since $\ell \geqslant 1$, there exists $a' \in A^{\langle \ell+1\rangle, z[-2]}$ such that

$$a\pi^{\langle \ell\rangle} \;=\; a' m_1^{\langle \ell+1\rangle} \;=\; a'\iota^{\langle \ell+1\rangle} m_1 \pi^{\langle \ell\rangle} \;.$$

So

$$\begin{array}{rcl} am_1 & = & a\pi^{\langle \ell \rangle}\iota^{\langle \ell \rangle}m_1 + (a - a\pi^{\langle \ell \rangle}\iota^{\langle \ell \rangle})m_1 \\ & = & a'\iota^{\langle \ell+1 \rangle}m_1\pi^{\langle \ell \rangle}\iota^{\langle \ell \rangle}m_1 + (a - a\pi^{\langle \ell \rangle}\iota^{\langle \ell \rangle})m_1 \\ & = & a'\iota^{\langle \ell+1 \rangle}\underbrace{m_1 m_1}_{=0} - \big(\sum_{i\in[0,\ell-1]} \underbrace{a'\iota^{\langle \ell+1 \rangle}m_1\pi^{\langle i \rangle}\iota^{\langle i \rangle}m_1}_{\in A^{\leqslant \ell-1,z[-1]}m_1}\big) + (\underbrace{a - a\pi^{\langle \ell \rangle}\iota^{\langle \ell \rangle}}_{\in A^{\leqslant \ell-1,z[-1]}m_1})m_1 \\ & \in & A^{\leqslant \ell-1,z[-1]}m_1\ . \end{array}$$

This proves the *claim*.

Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $a \in A^{z[-1]}$ such that $am_1 \in A^{\leqslant k-1,z}$.

We have to show that $am_1 \overset{!}{\in} A^{\leqslant k,z[-1]}m_1$ .

There exists $\ell \in \mathbf{Z}$ such that $a \in A^{\leqslant \ell,z[-1]}$.

If $\ell \leqslant k$, we have $am_1 \in A^{\leqslant \ell,z[-1]}m_1 \subseteq A^{\leqslant k,z[-1]}m_1$ .

If $\ell \geqslant k+1$, we obtain

$$\begin{array}{rcl} am_1 & \in & A^{\leqslant \ell,z[-1]}m_1 \cap A^{\leqslant k-1,z} \\ & = & (A^{\leqslant \ell}m_1 \cap A^{\leqslant k-1})^z \\ & \overset{\text{Claim}}{\subseteq} & (A^{\leqslant \ell-1}m_1 \cap A^{\leqslant k-1})^z \\ & \overset{\text{Claim}}{\subseteq} & \dots \\ & \overset{\text{Claim}}{\subseteq} & (A^{\leqslant k}m_1 \cap A^{\leqslant k-1})^z \\ & = & (A^{\leqslant k}m_1)^z \\ & = & A^{\leqslant k,z[-1]}m_1\ . \end{array}$$

□

**Proposition 60** *Suppose given an* $\mathrm{A}_1$*-algebra* $(A,(m_1))$ *over* $\mathcal{Z}$*.*

*For* $z \in \mathrm{Mor}(\mathcal{Z})$*, suppose given an augmented projective resolution of the module* $(\mathrm{H}A)^z$ *(over* $R$*), written as follows.*

$$\cdots \to \tilde{A}^{\langle 2 \rangle,z[-2]} \xrightarrow{d^{\langle 2 \rangle,z[-2]}} \tilde{A}^{\langle 1 \rangle,z[-1]} \xrightarrow{d^{\langle 1 \rangle,z[-1]}} \tilde{A}^{\langle 0 \rangle,z[0]} \xrightarrow{\varepsilon^z} (\mathrm{H}A)^z \to 0$$

*These linear maps assemble to shift-graded linear maps between* $\mathcal{Z}$*-graded modules as follows.*

$$\cdots \to \tilde{A}^{\langle 2 \rangle} \xrightarrow{d^{\langle 2 \rangle}} \tilde{A}^{\langle 1 \rangle} \xrightarrow{d^{\langle 1 \rangle}} \tilde{A}^{\langle 0 \rangle} \xrightarrow{\varepsilon} \mathrm{H}A \to 0$$

*Here* $d^{\langle i \rangle}$ *is of degree* $1$ *for* $i \in \mathbf{Z}_{\geqslant 1}$ *. Moreover,* $\varepsilon$ *is of degree* $0$*.*

*Let* $\tilde{A}^{\langle i \rangle} := 0$ *for* $i \in \mathbf{Z}_{<0}$ *. Let* $d^{\langle 0 \rangle} := 0 : \tilde{A}^{\langle 0 \rangle} \to \tilde{A}^{\langle -1 \rangle} = 0$*, as shift-graded linear map of degree* $1$*.*

*Let* $\tilde{A} := \bigoplus_{i\in\mathbf{Z}} \tilde{A}^{\langle i \rangle}$*. So* $\tilde{A} = (\tilde{A},(\tilde{A}^{\langle i \rangle})_i)$ *is a split-filtered* $\mathcal{Z}$*-graded module.*

*Then there exist*

- *a shift-graded linear map* $e^{\langle k\rangle} : \tilde{A}^{\langle k\rangle} \to \tilde{A}^{\leqslant k-2}$ *of degree* 1 *for* $k \in \mathbf{Z}_{\geqslant 0}$ ,
- *a shift-graded linear map* $q_1 : \tilde{A} \to A$ *of degree* 0

*such that, letting the shift-graded linear map* $\tilde{m}_1 : \tilde{A} \to \tilde{A}$ *of degree* 1 *be defined by*

$$\iota^{\langle i\rangle} \cdot \tilde{m}_1 \;:=\; d^{\langle i\rangle} \cdot \iota^{\langle i-1\rangle} + e^{\langle i\rangle} \cdot \iota^{\leqslant i-2}$$

*for* $i \in \mathbf{Z}_{\geqslant 0}$ , *then the following assertions* $(1, 2, 3, 4, 5, 6)$ *hold.*

(1) $\tilde{A} = (\tilde{A}, (\tilde{m}_1), (\tilde{A}^{\langle i\rangle})_i)$ *is a minimal* $\mathrm{eA}_1$*-algebra over* $\mathcal{Z}$*.*

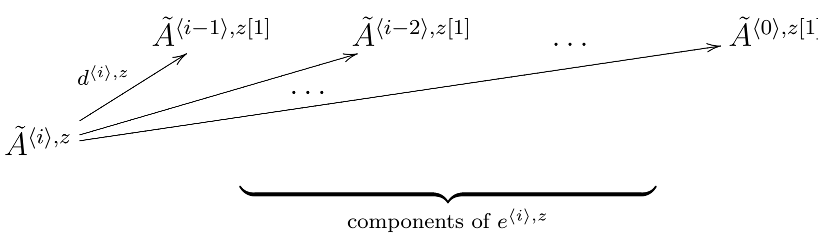
(2) $(q_1)$ *is a quasiisomorphism of* $\mathrm{A}_1$*-algebras from* $\tilde{A}$ *to* $A$*.*

(3) $\tilde{A}^{\langle i\rangle, z}$ *is a projective module (over* $R$*) for* $i \in \mathbf{Z}$ *and* $z \in \mathrm{Mor}(\mathcal{Z})$*.*

(4) $\tilde{A}$ *is diagonally resolving.*

(5) $q_1|^{\mathrm{Z}A}_{\tilde{A}^{\langle 0\rangle}} \cdot \rho$ *exists and is piecewise surjective.*

(6) *We have* $\tilde{A}^{\leqslant j}\tilde{m}_1 = \mathrm{B}\tilde{A} \cap \tilde{A}^{\leqslant j-1}$ *for* $j \in \mathbf{Z}$*.*




In Kadeishvili's Theorem 50, an $\mathrm{A}_\infty$-structure was constructed on the cohomology $\mathrm{H}A$ of a given $\mathrm{A}_\infty$-algebra $A$, in case $R$ is a field.

We will construct a minimal $\mathrm{eA}_\infty$-structure – so in particular, an $\mathrm{A}_\infty$-structure – on an arbitrarily chosen projective resolution $\tilde{A}$, in the sense of Proposition 60, of the cohomology $\mathrm{H}A$ of a given $\mathrm{A}_\infty$-algebra $A$ ; cf. Theorem refXXX below.

Note that in case $R$ is a field, one may choose the trivial projective resolution, where $\varepsilon = \mathrm{id}_A$ and where $A^{\langle i\rangle} = 0$ for $i \in \mathbf{Z}_{\geqslant 1}$ , so that $\tilde{A} = \mathrm{H}A$. So in this case the assertions of said Theorems coincide; cf. Remark 57.

XXXAAAXXX

# Appendix A

# Problems and solutions

## A.1 Problems

**Problem 1 (Introduction)**

Consider the commutative ring $\mathbf{Z}$. Consider the $\mathbf{Z}$-algebra $\mathbf{Z}$.

Determine the isoclasses of the $\mathbf{Z}$-modules $M$ that have a chain of submodules

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 = 0$$

such that

$$\begin{array}{rcl} M_0/M_1 & \simeq & \mathbf{Z}/(2) \\ M_1/M_2 & \simeq & \mathbf{Z}/(4) \\ M_2/M_3 & \simeq & \mathbf{Z}/(2)\,. \end{array}$$

**Problem 2 (§1.1.1)**

Let Cat denote the (1-)category of categories, (1-)morphisms being functors.

Let Set denote the category of sets, morphisms being maps.

(1) Given a set $X$, how many isoclasses does the pair category $X^{\times 2}$ have?

(2) Construct a full and faithful functor $P : \mathrm{Set} \to \mathrm{Cat}$ sending $X$ to $X^{\times 2}$.

(3) Show that the functor $\mathrm{Ob} : \mathrm{Cat} \to \mathrm{Set}$ has $P$ as a right adjoint, i.e. $\mathrm{Ob} \dashv P$.

(4) Determine unit and counit of the adjunction in (3).

**Problem 3 (§1.1.1)** Let Poset denote the category of posets and monotone maps.

(1) Suppose given a poset $X$. Show that we have a subcategory $CX$ of the pair category $X^{\times 2}$ with $\mathrm{Ob}(CX) = X$ and $\mathrm{Mor}(CX) = \{\, (x, y) \in X^{\times 2} \,:\, x \leqslant y \,\}$.

(2) Construct a functor $C : \text{Poset} \longrightarrow \text{Cat}$.

(3) Given $n \in \mathbf{Z}_{\geqslant 0}$ , we write $\Delta_n := C[0,n]$.

We have the monotone map $\omega : [0,1] \to [0,n]$ , $0 \mapsto 0$, $1 \mapsto n$.

Suppose given a category $\mathcal{Z}$ and $z \in \text{Mor}(\mathcal{Z})$. Let $F_z : \Delta_1 \to \mathcal{Z}$, $(0,1) \mapsto z$.

Let $n \geqslant 1$. Show that $\text{fact}_n(z)$ is in bijection to

$$\{\, \Delta_n \overset{G}{\to} \mathcal{Z} \;:\; G \text{ is a functor such that } G \circ (C\omega) = F_z \,\} \,.$$

**Problem 4 (§1.1.2)** Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category. Show.

(1) The shift $S$ is an automorphism of $\mathcal{Z}\text{-grad}_0$ .

(2) $\mathcal{Z}$-grad is a category.

(3) By $S(f,k) := (Sf,k)$ for $(f,k) \in \text{Mor}(\mathcal{Z}\text{-grad})$, an automorphism $S$ on $\mathcal{Z}$-grad is defined.

(4) $\mathcal{Z}\text{-grad}_0$ is additive.

(5) $\mathcal{Z}\text{-grad}_0$ is isomorphic to a subcategory of $\mathcal{Z}$-grad.
Is this subcategory full? Does $\mathcal{Z}$-grad have a zero object?

**Problem 5 (§1.1.1)** Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ and $\tilde{\mathcal{Z}} = (\tilde{\mathcal{Z}}, \tilde{S}, \tilde{\deg})$ be grading categories.

A *(1-)morphism of grading categories* from $\mathcal{Z}$ to $\tilde{\mathcal{Z}}$ is a functor $F : \mathcal{Z} \to \tilde{\mathcal{Z}}$ such that

$$\begin{array}{rcl} F(zS) & = & (Fz)\tilde{S} \\ (z)\deg & = & (Fz)\tilde{\deg} \end{array}$$

for $z \in \text{Mor}(\mathcal{Z})$.

(1) Show that grading categories, together with morphisms of such, form a category Grad.

(2) Show that $(\mathcal{Z}, S^-, -\deg)$ is a grading category, where $(S^-)_{X,Y} := (S_{X,Y})^-$ for $X, Y \in \text{Ob}(\mathcal{Z})$ and $z(-\deg) := -(z\deg)$ for $z \in \text{Mor}(\mathcal{Z})$. Construct an automorphism of order 2 on the category of grading categories.

(3) Show that $(\text{id}_X)\deg = 0$ for $X \in \text{Ob}(\mathcal{Z})$.

(4) Show that there exists exactly one morphism of grading categories from $\mathcal{Z}$ to $\mathbf{Z}$, i.e. that $\mathbf{Z}$ is the terminal grading category.

(5) Show that there is a bijection from the set of morphisms of grading categories from $\mathbf{Z}$ to $\mathcal{Z}$ to the set of endomorphisms of $\mathcal{Z}$ of degree 0.

(6) Suppose given a morphism of grading categories $\mathcal{Z} \xrightarrow{F} \tilde{\mathcal{Z}}$.

Show that there exist functors

$$\mathcal{Z}\text{-grad} \underset{F^{\&}}{\overset{F_{\&}}{\rightleftarrows}} \tilde{\mathcal{Z}}\text{-grad}$$

having $(F^{\&}\tilde{M})^z = \tilde{M}^{Fz}$ for $\tilde{M} \in \mathrm{Ob}(\tilde{\mathcal{Z}}\text{-grad})$ and $z \in \mathrm{Mor}(\mathcal{Z})$, having $(F_{\&}M)^{\tilde{z}} = \bigoplus_{\substack{z \in \mathrm{Mor}(\mathcal{Z}) \\ Fz = \tilde{z}}} M^z$ for $M \in \mathrm{Ob}(\mathcal{Z}\text{-grad})$ and $\tilde{z} \in \mathrm{Mor}(\tilde{\mathcal{Z}})$ and having $F_{\&} \dashv F^{\&}$.

**Problem 6 (§1.1.3)** Let $(\mathcal{Z}, S, \deg)$ be a grading category.

Define a category $(\mathcal{Z}\text{-grad})^{\times n, \pm}$ such that we have a functor

$$\begin{array}{ccc} (\mathcal{Z}\text{-grad})^{\times n,\pm} & \xrightarrow{\bigotimes_{i\in[1,n]}} & \mathcal{Z}\text{-grad} \\ (L_i \xrightarrow{(f_i,k_i)} M_i)_{i\in[1,n]} & \longmapsto & (\bigotimes_{i\in[1,n]} L_i \xrightarrow{\bigotimes_{i\in[1,n]}(f_i,k_i)} \bigotimes_{i\in[1,n]} M_i)\ . \end{array}$$

**Problem 7 (§1.1.3)** Let $\mathcal{Z} = (\mathcal{Z}, S, \deg)$ be a grading category.

Suppose given $1 \leqslant \ell \leqslant n$ and $\mathcal{Z}$-shift-graded linear maps $L_i \xrightarrow{(f_i,k_i)} M_i$ for $i \in [1, n]$.

Suppose given $\mathcal{Z}$-shift-graded linear maps $L \xrightarrow{(f,k)} M$ and $\tilde{L} \xrightarrow{(\tilde{f},\tilde{k})} \tilde{M}$.

(1) Show that

$$(M_1 \otimes \ldots \otimes M_\ell) \otimes (M_{\ell+1} \otimes \ldots \otimes M_n) \;=\; M_1 \otimes \ldots \otimes M_n\ .$$

(2) Show that

$$((f_1, k_1) \otimes \ldots \otimes (f_\ell, k_\ell)) \otimes ((f_{\ell+1}, k_{\ell+1}) \otimes \ldots \otimes (f_n, k_n)) \;=\; (f_1, k_1) \otimes \ldots \otimes (f_n, k_n)\ .$$

(3) Construct a $\mathcal{Z}$-graded module $\dot{R}$ such that $(f, k) \otimes (\mathrm{id}_{\dot{R}}, 0) = (f, k)$ and $(\mathrm{id}_{\dot{R}}, 0) \otimes (f, k) = (f, k)$.

(4) Construct an isomorphism $L \otimes \tilde{L} \xrightarrow[\sim]{\tau_{L,\tilde{L}}} \tilde{L} \otimes L$ in $\mathcal{Z}$-grad, and likewise $\tau_{M,\tilde{M}}$, such that the following quadrangle commutes.

$$\begin{array}{ccc} L \otimes \tilde{L} & \xrightarrow[\sim]{\tau_{L,\tilde{L}}} & \tilde{L} \otimes L \\ {\scriptstyle (f,k)\otimes(\tilde{f},\tilde{k})} \downarrow & & \downarrow {\scriptstyle (-1)^{k\tilde{k}}(\tilde{f},\tilde{k})\otimes(f,k)} \\ M \otimes \tilde{M} & \xrightarrow[\sim]{\tau_{M,\tilde{M}}} & \tilde{M} \otimes M \end{array}$$

**Problem 8 (Problem 17)** Let $B$ be an algebra.

(1) Let $\mathcal{A}$ be a linear additive category. Let $\mathcal{N} \subseteq \mathcal{A}$ be a full additive subcategory. Write

$$\begin{array}{l}\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y) \\ := \{ X \xrightarrow{f} Y \;:\; \text{there exists } N \in \mathrm{Ob}(\mathcal{N}) \text{ and morphisms } X \xrightarrow{u} N \xrightarrow{v} Y \text{ such that } f = uv \} .\end{array}$$

Let $\mathcal{A}/\mathcal{N}$ be the category that has

$$\begin{array}{rcl}\mathrm{Ob}(\mathcal{A}/\mathcal{N}) & := & \mathrm{Ob}(\mathcal{A}) \\ {}_{\mathcal{A}/\mathcal{N}}(X,Y) & := & {}_{\mathcal{A}}(X,Y)/\,\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y) \quad \text{for } X,\, Y \,\in\, \mathrm{Ob}(\mathcal{A}/\mathcal{N}) .\end{array}$$

For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, we define composition of the respective residue classes in $\mathcal{A}/\mathcal{N}$ by

$$(f + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)) \cdot (g + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(Y,Z)) \;=\; f \cdot g + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Z) .$$

Show that $\mathcal{A}/\mathcal{N}$ is a linear additive category. Show that $\mathcal{A} \xrightarrow{R} \mathcal{A}/\mathcal{N}$ is a linear functor with $RN \simeq 0$ for $N \in \mathrm{Ob}(\mathcal{N})$.

We often write $\bar{f} := f + \mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$.

Given a linear additive category $\mathcal{B}$ and a linear functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ with $FN \simeq 0$ for $N \in \mathrm{Ob}(\mathcal{N})$, show that there exists a unique linear functor $\mathcal{A}/\mathcal{N} \xrightarrow{\bar{F}} \mathcal{B}$ such that $F = \bar{F} \circ R$.

(2) Let $\mathcal{A} := \mathrm{C}(B\text{-Mod})$ be the category of complexes of $B$-modules. Let the differential of a complex $X \in \mathrm{Ob}(\mathcal{A})$ be denoted by $d = d_X$ . Let $\mathcal{N} \subseteq \mathcal{A}$ be the full additive subcategory of split acyclic complexes, i.e. those isomorphic to a complex of the form $\cdots \to U^{i-1} \oplus U^i \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} U^i \oplus U^{i+1} \to \ldots$, where $U^i \in \mathrm{Ob}\,\mathcal{A}$ for $i \in \mathbf{Z}$.

Show that $\mathrm{Null}_{\mathcal{A},\mathcal{N}}(X,Y)$ consists of those morphisms of complexes $X \xrightarrow{f} Y$ for which there exists a tuple of morphisms $(X^i \xrightarrow{h^i} Y^{i-1})_{i \in \mathbf{Z}}$ such that

$$f^i \;=\; h^i d_Y^{i-1} + d_X^i h^{i+1} \qquad \text{for } i \in \mathbf{Z}.$$

Define $\mathrm{K}(B\text{-Mod}) := \mathcal{A}/\mathcal{N}$ to be the *homotopy category* of complexes of $B$-modules. Write shorthand ${}_{\mathrm{K}}(X,Y) := {}_{\mathrm{K}(B\text{-Mod})}(X,Y)$ for $X,\, Y \in \mathrm{Ob}(\mathrm{K}(B\text{-Mod})) = \mathrm{Ob}(\mathrm{C}(B\text{-Mod}))$.

(3) Let $M$ be a $B$-module. Let $P$ be a projective resolution of $M$ with augmentation $\varepsilon : P_0 \to M$. Let $\mathrm{Conc}(M) \in \mathrm{Ob}(\mathrm{C}(B\text{-Mod}))$ have $M$ at position 0, and 0 elsewhere. Let $\hat{\varepsilon} : P \to \mathrm{Conc}(M)$ be the morphism of complexes having entry $\varepsilon$ at position 0.

Let $Q$ be a complex consisting of projective $B$-modules, bounded above. Show that ${}_{\mathrm{K}}(Q, \bar{\hat{\varepsilon}}) : {}_{\mathrm{K}}(Q,P) \to {}_{\mathrm{K}}(Q, \mathrm{Conc}(M))$ is an isomorphism.

(4) Using the universal property from (1), construct a shift functor $S$ on $\mathrm{K}(B\text{-Mod})$ such that $(SX)^i = X^{i+1}$ and such that $d^i_{SX} = -d^{i+1}_X$ for $i \in \mathbf{Z}$. Show that $S$ is an automorphism.

We also write $S^k =: (-)^{[k]}$ for $k \in \mathbf{Z}$.

**Problem 9 (Problem 17)** Let $q \in \mathbf{Z}_{\geqslant 1}$. Consider the cyclic group $\mathrm{C}_q = \langle\, c \,:\, c^q \,\rangle$.

Abbreviate $\mathrm{K} := \mathrm{K}(RC_q\text{-Mod})$.

(1) Construct a projective resolution $P$ of the trivial $R\mathrm{C}_q$-module $R$ that is periodic of period length 2.

(2) Calculate ${}_{\mathrm{K}}(P, \mathrm{Conc}(R)^{[i]})$ for $i \in \mathbf{Z}$.

(3) Calculate ${}_{\mathrm{K}}(P, P^{[i]})$ for $i \in \mathbf{Z}$.

(4) Calculate the composition map

$$ {}_{\mathrm{K}}(P, P^{[i]}) \otimes {}_{\mathrm{K}}(P^{[i]}, P^{[i+j]}) \;\to\; {}_{\mathrm{K}}(P, P^{[i+j]}) $$

for $i,\ j \in \mathbf{Z}$.

**Problem 10 (§1.1.1)** Let $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ be grading categories.

Let $\mathcal{Z} \xrightarrow{F} \tilde{\mathcal{Z}}$ be a morphism of grading categories; cf. Problem 5.

Let $n \in \mathbf{Z}_{\geqslant 1}$. Let $M_i$ be a $\mathcal{Z}$-graded module for $i \in [1,n]$. Write $\underline{M} := (M_i)_{i\in[1,n]}$.

(1) Construct an isomorphism $F_{\&}(\bigotimes_{i\in[1,n]} M_i) \xrightarrow[\sim]{\sigma_{\underline{M}}} \bigotimes_{i\in[1,n]} F_{\&} M_i$ in $\tilde{\mathcal{Z}}$-grad.

(2) Show that the following quadrangle commutes.

$$\begin{array}{ccc}
F_{\&}(\bigotimes_{i\in[1,n]} M_i) & \xrightarrow{\sigma_{\underline{M}}} & \bigotimes_{i\in[1,n]} F_{\&} M_i \\
{\scriptstyle F_{\&}(\bigotimes_{i\in[1,n]}(f_i\,,\,k_i))}\downarrow & & \downarrow{\scriptstyle \bigotimes_{i\in[1,n]} F_{\&}(f_i\,,\,k_i)} \\
F_{\&}(\bigotimes_{i\in[1,n]} M'_i) & \xrightarrow{\sigma_{\underline{M}'}} & \bigotimes_{i\in[1,n]} F_{\&} M'_i
\end{array}$$

**Problem 11 (§1.2)** Let $\mathcal{Z}$ be a grading category.

Let $A$ be a $\mathcal{Z}$-graded module.

Suppose given shift-graded maps $m_1 : A \to A$ of degree 1 and $m_2 : A^{\otimes 2} \to A$ of degree 0. For $n \in \mathbf{Z}_{\geqslant 3}$, we let $m_n := 0$, as shift-graded linear map from $A^{\otimes n}$ to $A$ of degree $2-n$.

Suppose that $(m_n)_{n\in\mathbf{Z}_{\geqslant 1}}$ satisfies the Stasheff equations for $k \in [1,3]$.

Suppose that for each $X \in \mathrm{Ob}(\mathcal{Z})$, there exists an element $1_X \in A^{\mathrm{id}_X}$ such that for $z,\ w \in \mathrm{Mor}(\mathcal{Z})$ such that $z\mathrm{t}_{\mathcal{Z}} = X = w\mathrm{s}_{\mathcal{Z}}$ and for $a \in A^z$ and $b \in A^w$, we have $(a \otimes 1_X)m_2 = a$ and $(1_X \otimes b)m_2 = b$.

Show that $(A, (m_n)_{n\in\mathbf{Z}_{\geqslant 1}})$ is a differential graded algebra over $\mathcal{Z}$.

**Problem 12 (§1.2)** Suppose given a grading category $\mathcal{Z}$.

Suppose given $\mathrm{A}_\infty$-algebras $\tilde{A}$ and $A$.

Suppose given a shift-graded linear map $f_1 : \tilde{A} \to A$ of degree 0.

Suppose that $f_1^{\otimes k} \cdot m_k^A = m_k^{\tilde{A}} \cdot f_1$ for $k \in \mathbf{Z}_{\geqslant 1}$.

Let $f_k = 0$ for $k \in \mathbf{Z}_{\geqslant 2}$, as shift-graded linear map from $\tilde{A}^{\otimes k}$ to $A$ of degree $1 - k$.

Show that $(f_k)_{k \in \mathbf{Z}_{\geqslant 1}}$ is a morphism of $\mathrm{A}_\infty$-algebras.

**Problem 13 (§XXX)** Let $B$ be an algebra.

Suppose given a diagram $X' \xrightarrow{i} X \xrightarrow{r} X''$ in $\mathrm{C}(B\text{-Mod})$ such that $X'^k \xrightarrow{i^k} X^k \xrightarrow{r^k} X''^k$ is short exact for $k \in \mathbf{Z}$. Such a diagram is called a short exact sequence of complexes in $B$.

(1) Suppose given $T \xrightarrow{f} X$ in $\mathrm{C}(B\text{-Mod})$ such that $fr = 0$. Show that there exists a unique morphism $T \xrightarrow{f'} X'$ such that $f'i = f$.

(2) Suppose given $X \xrightarrow{g} T$ in $\mathrm{C}(B\text{-Mod})$ such that $ig = 0$. Show that there exists a unique morphism $X'' \xrightarrow{g''} T$ such that $rg'' = g$.

(3) A $\mathbf{Z}$-graded $B$-module $M$ is a tuple $M = (M^z)_{z \in \mathbf{Z}}$ of $B$-modules $M^z$. A graded $B$-linear map $f : L \to M$ between $\mathbf{Z}$-graded $B$-modules is a tuple $f = (f^z)_{z \in \mathbf{Z}}$ of $B$-linear maps $f^z$. Write $B$-$\mathbf{Z}$-grad for the category of $\mathbf{Z}$-graded $B$-modules and graded $B$-linear maps.

Construct an additive functor $\mathrm{H} : \mathrm{C}(B\text{-Mod}) \to B\text{-}\mathbf{Z}\text{-grad}$ having

$$(\mathrm{H}X)^k \;=\; \mathrm{Kern}(d^k)/\,\mathrm{Im}(d^{k-1})$$

for a complex $X$ with differential $d = (X^k \xrightarrow{d^k} X^{k+1})_k$.

For $Y \xrightarrow{f} Z$ in $\mathrm{C}(B\text{-Mod})$, we often write $((\mathrm{H}Y)^k \xrightarrow{(\mathrm{H}f)^k} (\mathrm{H}Z)^k) =: (\mathrm{H}^k Y \xrightarrow{\mathrm{H}^k f} \mathrm{H}^k Z)$.

(4) Construct a $B$-linear map $\mathrm{H}^k X'' \xrightarrow{\gamma^k_{(i,r)}} \mathrm{H}^{k+1} X'$ for $k \in \mathbf{Z}$, called *connector* of the given short exact sequence $X' \xrightarrow{i} X \xrightarrow{r} X''$, subject to the following conditions (i, ii).

(i) The sequence

$$\ldots \to \mathrm{H}^k X' \xrightarrow{\mathrm{H}^k i} \mathrm{H}^k X \xrightarrow{\mathrm{H}^k r} \mathrm{H}^k X'' \xrightarrow{\gamma^k_{(i,r)}} \mathrm{H}^{k+1} X' \xrightarrow{\mathrm{H}^{k+1} i} \mathrm{H}^{k+1} X \xrightarrow{\mathrm{H}^{k+1} r} \mathrm{H}^{k+1} X'' \to \ldots$$

is exact at each position.

(ii) Given a morphism of short exact sequences, i.e. a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i} & X & \xrightarrow{r} & X'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ Y' & \xrightarrow{j} & Y & \xrightarrow{s} & Y'' \end{array}$$

in $\mathrm{C}(B\text{-Mod})$ with $(i, r)$ and $(j, s)$ short exact, we get, for $k \in \mathbf{Z}$, the commutative quadrangle

$$\begin{array}{ccc} \mathrm{H}^k X'' & \xrightarrow{\gamma^k_{(i,r)}} & \mathrm{H}^{k+1} X' \\ \downarrow{\scriptstyle \mathrm{H}^k f''} & & \downarrow{\scriptstyle \mathrm{H}^{k+1} f'} \\ \mathrm{H}^k Y'' & \xrightarrow{\gamma^k_{(j,s)}} & \mathrm{H}^{k+1} Y' \end{array} .$$

**Problem 14 (§1.3, §1.4)** Suppose given an algebra $B$. Suppose given $n \geqslant 1$.

Suppose given $X_s \in \mathrm{Ob}\,\mathrm{C}(B\text{-Mod})$ for $s \in [1, n]$. Abbreviate $\underline{X} := (X_s)_{s \in [1,n]}$ .

Abbreviate $\mathcal{Z} := \mathbf{Z} \times [1, n]^{\times 2}$, $\mathrm{C} := \mathrm{C}(B\text{-Mod})$ and $\mathrm{K} := \mathrm{K}(B\text{-Mod})$.

(1) Show that $(\mathrm{Z}\,\mathrm{Hom}_B(\underline{X}))^{(j,(s,t))} = {}_{\mathrm{C}}(X_s, X_t^{[j]})$ for $(j, (s, t)) \in \mathrm{Mor}(\mathcal{Z})$.

(2) Show that $(\mathrm{H}\,\mathrm{Hom}_B(\underline{X}))^{(j,(s,t))} = {}_{\mathrm{K}}(X_s, X_t^{[j]})$ for $(j, (s, t)) \in \mathrm{Mor}(\mathcal{Z})$.

(3) Show that $m_2^{\mathrm{Hom}_B(\underline{X})}$ induces a map $m_2^{\mathrm{H}\,\mathrm{Hom}_B(\underline{X})} : \mathrm{H}\,\mathrm{Hom}_B(\underline{X})^{\otimes 2} \to \mathrm{H}\,\mathrm{Hom}_B(\underline{X})$ that maps $[f] \otimes [g]$ to $[f \cdot g]$ for each composable pair of morphisms $(f, g)$ in C, where we use brackets to denote residue classes of morphisms of C in K.

**Problem 15 (§1.4)** Let $\mathcal{Z}$ be a grading category.

Let $L$ and $M$ be $\mathcal{Z}$-graded modules.

Let $L \xrightarrow{f} M$ be a shift-graded linear map of degree $a \in \mathbf{Z}$.

(1) Let $K := \mathrm{Kern}(f)$, i.e. $K^z := \mathrm{Kern}(L^z \xrightarrow{f} M^{z[a]})$ for $z \in \mathrm{Mor}(\mathcal{Z})$. Let $K \xrightarrow{i} L$ be the shift-graded inclusion map of degree 0.

Suppose given a $\mathcal{Z}$-graded module $T$ and a shift-graded linear map $T \xrightarrow{t} L$ of degree $d$ such that $tf = 0$.

Show that there exist a unique shift-graded linear map $T \xrightarrow{\check{t}} K$ of degree $d$ such that $\check{t}i = t$.

(2) Suppose $R$ to be a field. Suppose $f$ to be piecewise surjective.

Show that there exists a piecewise injective shift-graded linear map $L \xleftarrow{g} M$ of degree $-a$ such that $gf = \mathrm{id}_M$ .

**Problem 16 (§1.4)** Let $\mathcal{Z}$ be a grading category. Let $n \in \mathbf{Z}_{\geqslant 1}$ .

Let $K_i$ , $L_i$ and $M_i$ be $\mathcal{Z}$-graded modules for $i \in [1, n]$.

Let $K_i \xrightarrow{u_i} L_i$ be a shift-graded linear map of degree $c_i \in \mathbf{Z}$ for $i \in [1, n]$.

Let $L_i \xrightarrow{f_i} M_i$ be a piecewise surjective shift-graded linear map of degree $a_i \in \mathbf{Z}$ for $i \in [1, n]$.

(1) Show that $\bigotimes_{i\in[1,n]} f_i$ is piecewise surjective.

(2) For $i \in [1, n]$, suppose that $K_i \xrightarrow{u_i} L_i \xrightarrow{f_i} M_i$ to be *exact at* $L_i$ , i.e. suppose $K_i^{z[-c_i]} \xrightarrow{u_i} L_i^z \xrightarrow{f_i} M_i^{z[a_i]}$ to be exact at $L_i^z$ for $z \in \mathrm{Mor}(\mathcal{Z})$.

Suppose given a $\mathcal{Z}$-graded module $T$ and a shift-graded linear map $\bigotimes_{i\in[1,n]} L_i \xrightarrow{t} T$ of degree $d$.

Suppose that $(\mathrm{id}^{\otimes j-1} \otimes u_j \otimes \mathrm{id}^{\otimes n-j})t = 0$ for $j \in [1, n]$.

Show that there exists a unique shift-graded linear map $\bigotimes_{i\in[1,n]} M_i \xrightarrow{\tilde{t}} T$ of degree $d - \sum_{i\in[1,n]} a_i$ such that $(\bigotimes_{i\in[1,n]} f_i)\tilde{t} = t$.

**Problem 17 (§1.4)** Let $p > 0$ be a prime.

Let $P \in \mathrm{Ob}\,\mathrm{C}(\mathbf{F}_p\mathrm{C}_p\text{-Mod})$ be the projective resolution of the trivial $\mathbf{F}_p\mathrm{C}_p$-module as found in Problem 9.(1).

Let $\underline{X} := (P)$, so that $\underline{X}$ has $P$ as its only tuple entry.

Let $A := \mathrm{Hom}_{\mathbf{F}_p\mathrm{C}_p}(\underline{X})$ be the regular differential graded category, i.e. differential graded algebra over $\mathbf{Z} = \mathbf{Z} \times [1, 1]^{\times 2}$.

Recall from Problem 9 and Problem 14 that we have calculated the $\mathbf{Z}$-graded module $\mathrm{H}A$, i.e. that we know $\mathbf{F}_p$-linear generators for its graded pieces.

Find a minimal $\mathrm{A}_3$-structure $(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$ on $\mathrm{H}A$ and a quasiisomorphism $(q_1, q_2, q_3)$ : $\mathrm{H}A \to A$ of $\mathrm{A}_3$-algebras.

**Problem 18 (§1.4)** Suppose $R$ to be a field.

Let $\mathcal{Z}$ be a grading category. Let $n \in [1, \infty]$. Let $A$ be a unital $\mathrm{A}_n$-algebra over $\mathcal{Z}$.

Consider the shift-graded linear residue class map $\mathrm{Z}A \xrightarrow{\rho} \mathrm{H}A$ of degree 0.

Show that there exists a shift graded linear map $\mathrm{Z}A \xleftarrow{\sigma} \mathrm{H}A$ of degree 0 such that $\sigma\rho = \mathrm{id}_{\mathrm{H}A}$ and such that $(1_X\rho)\sigma = 1_X$ for $X \in \mathrm{Ob}(\mathcal{Z})$.

**Problem 19 (§1.6)** Let $\mathcal{Z}$ be a grading category.

(1) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta})$ and $V = (V, \Delta)$ be coalgebras over $\mathcal{Z}$. Let $\tilde{V} \xrightarrow{f} V$ be a morphism of coalgebras.

Suppose that $f$ is piecewise bijective.

Show that $f$ is an *isomorphism* of coalgebras, i.e. that there exists a morphism of coalgebras $\tilde{V} \xleftarrow{g} V$ such that $fg = \mathrm{id}_{\tilde{V}}$ and $gf = \mathrm{id}_V$ .

Then $g$ is uniquely determined and written $f^- := g$.

(2) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta}, \tilde{\delta})$ and $V = (V, \Delta, \delta)$ be coalgebras with differential over $\mathcal{Z}$. Let $\tilde{V} \xrightarrow{f} V$ be a morphism of coalgebras with differential.

Suppose that $f$ is piecewise bijective.

Show that $f$ is an *isomorphism* of coalgebras with codifferential, i.e. that there exists a morphism of coalgebras with codifferential $\tilde{V} \xleftarrow{g} V$ such that $fg = \mathrm{id}_{\tilde{V}}$ and $gf = \mathrm{id}_V$ .

Then $g$ is uniquely determined and written $f^- := g$.

(3) Let $\tilde{V} = (\tilde{V}, \tilde{\Delta})$ and $V = (V, \Delta)$ be coalgebras over $\mathcal{Z}$.

Let $\tilde{V} \xrightarrow{f} V$ be an isomorphism of coalgebras.

Suppose given a codifferential $\delta$ on $V$. Show that $f\delta f^-$ is a codifferential on $\tilde{V}$.

(4) Let $V = (V, \Delta)$ be a coalgebra over $\mathcal{Z}$. Let $\lambda : V \longrightarrow \dot{R}$ be a shift-graded linear map of degree $0$; cf. Problem 7.(3). Recall that $\dot{R} \otimes V = V = V \otimes \dot{R}$ by identification.

Let $\delta_\lambda := \Delta(\mathrm{id} \otimes \lambda) - \Delta(\lambda \otimes \mathrm{id})$. Show that $\delta_\lambda$ is a coderivation.

Coderivations of this form are called *inner*.

**Problem 20 (§1.1.2)** Let $\mathcal{Z}$ be a grading category.

Let $I$ be a set. Let $V_i$ be a $\mathcal{Z}$-graded module for $i \in I$. Recall that the $\mathcal{Z}$-graded module $\bigoplus_{i \in I} V_i$ is defined by letting $(\bigoplus_{i \in I} V_i)^z = \bigoplus_{i \in I} V_i^z$ for $z \in \mathrm{Mor}(\mathcal{Z})$.

(1) Given $j \in I$, construct a shift-graded linear *inclusion* map $\iota_j : V_j \to \bigoplus_{i \in I} V_i$ of degree 0 and a shift-graded linear *projection* map $\pi_j : \bigoplus_{i \in I} V_i \to V_j$ of degree 0.

(2) Suppose given a $\mathcal{Z}$-graded module $S$. Suppose given $d \in \mathbf{Z}$. Suppose given a shift-graded linear map $s_j : S \to V_j$ of degree $d$ for $j \in I$.

Show that there exists a unique shift-graded linear map $s : S \to \bigoplus_{i \in I} V_i$ of degree $d$ such that $s\pi_j = s_j$ for $j \in I$.

(3) Suppose $I$ to be finite.

Suppose given a $\mathcal{Z}$-graded module $T$. Suppose given $d \in \mathbf{Z}$. Suppose given shift-graded linear maps $t_j : V_j \to T$ of degree $d$ for $j \in I$.

Show that there exists a unique shift-graded linear map $t : \bigoplus_{i \in I} V_i \to T$ of degree $d$ such that $\iota_j t = t_j$ for $j \in I$.

**Problem 21 (§1.6)** Let $\mathcal{Z}$ be a grading category. Let $n \in [1, \infty]$. Let $(A, (m_\ell)_\ell)$ be a pre-$\mathrm{A}_n$-algebra.

Write $\mathfrak{m} := (({}^\omega m_\ell)_{\ell \in [1,n] \cap \mathbf{Z}})\beta_{\mathrm{Coder},n,A^{[1]}}$ .

(1) Suppose given $p \in [1, n]$. Write $\mathfrak{m}' := (({}^\omega m_\ell)_{\ell \in [1,p] \cap \mathbf{Z}})\beta_{\mathrm{Coder},p,A^{[1]}}$ .

Show that $\mathfrak{m}' = \mathfrak{m}|^{\mathrm{T}_{\leqslant p}(A^{[1]})}_{\mathrm{T}_{\leqslant p}(A^{[1]})}$ .

(2) Suppose $n \in \mathbf{Z}_{\geqslant 1}$ . Suppose that $(m_\ell)_\ell$ satisfies the Stasheff equation at each $k \in [1, n-1]$.

Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $a \in ((A^{[1]})^{\otimes n})^z$. Consider the following assertions.

(i) We have
$$a \left( \sum_{\substack{(r,s,t) \geqslant (0,1,0) \\ r+s+t=n}} (\mathrm{id}^{\otimes r} \otimes {}^\omega m_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^\omega m_{r+1+t} \right) = 0$$

(ii) We have $a\mathfrak{m}^2 = 0$.

Show that (i) and (ii) are equivalent.

**Problem 22 (§1.6)** Let $\mathcal{Z}$ be a grading category. Let $n \in [1, \infty]$. Let $(\tilde{A}, (\tilde{m}_\ell)_{\ell \in [1,n] \cap \mathbf{Z}})$ and $(A, (m_\ell)_{\ell \in [1,n] \cap \mathbf{Z}})$ be pre-$\mathrm{A}_n$-algebras. Let $f = (f_\ell)_{\ell \in [1,n] \cap \mathbf{Z}}$ be a pre-$\mathrm{A}_n$-morphism from $\tilde{A}$ to $A$.

Write
$$\begin{aligned} \tilde{\mathfrak{m}} &:= (({}^\omega \tilde{m}_\ell)_{\ell \in [1,n] \cap \mathbf{Z}})\beta_{\mathrm{Coder},n,\tilde{A}^{[1]}} \\ \mathfrak{m} &:= (({}^\omega m_\ell)_{\ell \in [1,n] \cap \mathbf{Z}})\beta_{\mathrm{Coder},n,A^{[1]}} \\ \mathfrak{f} &:= (({}^\omega f_\ell)_{\ell \in [1,n] \cap \mathbf{Z}})\beta_{\mathrm{Coalg},n,\tilde{A}^{[1]},A^{[1]}} \ . \end{aligned}$$

(1) Suppose given $p \in [1, n]$. Write
$$\begin{aligned} \tilde{\mathfrak{m}}' &:= (({}^\omega \tilde{m}_\ell)_{\ell \in [1,p] \cap \mathbf{Z}})\beta_{\mathrm{Coder},p,\tilde{A}^{[1]}} \\ \mathfrak{m}' &:= (({}^\omega m_\ell)_{\ell \in [1,p] \cap \mathbf{Z}})\beta_{\mathrm{Coder},p,A^{[1]}} \\ \mathfrak{f}' &:= (({}^\omega f_\ell)_{\ell \in [1,p] \cap \mathbf{Z}})\beta_{\mathrm{Coalg},p,\tilde{A}^{[1]},A^{[1]}} \end{aligned}$$

Show that
$$\begin{aligned} \tilde{\mathfrak{m}}' &= \mathfrak{f}|^{\mathrm{T}_{\leqslant p}(\tilde{A}^{[1]})}_{\mathrm{T}_{\leqslant p}(\tilde{A}^{[1]})} \\ \mathfrak{m}' &= \mathfrak{m}|^{\mathrm{T}_{\leqslant p}(A^{[1]})}_{\mathrm{T}_{\leqslant p}(A^{[1]})} \\ \mathfrak{f}' &= \mathfrak{f}|^{\mathrm{T}_{\leqslant p}(A^{[1]})}_{\mathrm{T}_{\leqslant p}(\tilde{A}^{[1]})} \ . \end{aligned}$$

(2) Suppose $n \in \mathbf{Z}_{\geqslant 1}$. Suppose that $(f_\ell)_\ell$ satisfies the Stasheff equation for morphisms at each $k \in [1, n-1]$.

Suppose given $z \in \operatorname{Mor}(\mathcal{Z})$ and $\tilde{a} \in ((\tilde{A}^{[1]})^{\otimes n})^z$. Consider the following assertions.

(i) We have

$$\tilde{a}\left(\sum_{\substack{(r,s,t)\geqslant(0,1,0)\\ r+s+t=n}} (\mathrm{id}^{\otimes r} \otimes {}^{\omega}\tilde{m}_s \otimes \mathrm{id}^{\otimes t}) \cdot {}^{\omega}f_{r+1+t}\right) = \tilde{a}\left(\sum_{r\in[1,k]} \sum_{\substack{(i_j)_{j\in[1,r]}\geqslant(1)_j\\ \sum_j i_j = n}} \Big(\bigotimes_{j\in[1,r]} {}^{\omega}f_{i_j}\Big) \cdot {}^{\omega}m_r\right)$$

(ii) We have $\tilde{a}(\tilde{\mathfrak{m}}\mathfrak{f} - \mathfrak{f}\mathfrak{m}) = 0$.

Show that (i) and (ii) are equivalent.

**Problem 23 (§XXX)** Let $n \in [1, \infty]$. Let $\mathcal{Z}$ be a grading category.

Let $(A, (m_\ell)_\ell)$, $(A', (m'_\ell)_\ell)$, $(A'', (m''_\ell)_\ell)$, $(A''', (m'''_\ell)_\ell)$ be $\mathrm{A}_n$-algebras over $\mathcal{Z}$.

Let $f = (f_\ell)_\ell : A \to A'$, $f' = (f'_\ell)_\ell : A' \to A''$, $f'' = (f''_\ell)_\ell : A'' \to A'''$ be $\mathrm{A}_n$-morphisms.

Write ${}^{\omega}f := ({}^{\omega}f_\ell)_\ell$.

Define

$$f \cdot f' := {}^{\omega^-}\big((({}^{\omega}f)\beta \cdot ({}^{\omega}f')\beta)\alpha\big).$$

(1) Show that $f \cdot f'$ is a morphism of $\mathrm{A}_n$-algebras from $A$ to $A''$.

(2) Write $f \cdot f'$ in terms of $(f_\ell)_\ell$ and $(f'_\ell)_\ell$. What is the entry of $f \cdot f'$ at $\ell = 1$?

(3) Show that $(f \cdot f') \cdot f'' = f \cdot (f' \cdot f'')$.

(4) Suppose given shift-graded linear maps $g : A \to A'$ and $g' : A' \to A''$ of degree 0. Define $\mathrm{strict}_n(g) := (g, 0, 0, \ldots)$.
When is $\mathrm{strict}_n(g) : A \to A'$ a morphism of $\mathrm{A}_n$-algebras? Is $\mathrm{strict}_n(\mathrm{id}_A)$ a morphism of $\mathrm{A}_n$-algebras? If $\mathrm{strict}_n(g)$ and $\mathrm{strict}_n(g')$ are morphisms of $\mathrm{A}_n$-algebras, show that $\mathrm{strict}_n(gg') = \mathrm{strict}_n(g) \cdot \mathrm{strict}_n(g')$.

(5) Show that $f \cdot \mathrm{strict}_n(\mathrm{id}_{A'}) = f$ and that $\mathrm{strict}_n(\mathrm{id}_{A'}) \cdot f' = f'$.

(6) Define the category $\mathrm{A}_n$-$\mathcal{Z}$-alg of $\mathrm{A}_n$-algebras over $\mathcal{Z}$ and $\mathrm{A}_n$-morphisms.
Therein, define the subcategory strict-$\mathrm{A}_n$-$\mathcal{Z}$-alg of $\mathrm{A}_n$-algebras over $\mathcal{Z}$ and strict $\mathrm{A}_n$-morphisms.

(7) Show that $\mathrm{H}$ is a functor from $\mathrm{A}_n$-$\mathcal{Z}$-alg to $\mathcal{Z}$-$\mathrm{grad}_0$.

**Problem 24 (§XXX)** Let $n \in [1,\infty]$. Let $\mathcal{Z} \xrightarrow{F} \tilde{\mathcal{Z}}$ be a morphism of grading categories; cf. Problem 5. Let $(A,(m_k)_k)$ be an $\mathrm{A}_n$-algebra over $\mathcal{Z}$.

(1) Show that $F_\& A = (F_\& A, (\sigma^- \cdot F_\& m_\ell)_\ell)$ is an $\mathrm{A}_n$-algebra over $\tilde{\mathcal{Z}}$, where $\sigma = \sigma_{(A,\ldots,A)}$; cf. Problem 10.

(2) Consider the case $n = \infty$, $u \in \mathbf{Z}_{\geqslant 1}$, $\mathcal{Z} = \mathbf{Z} \times [1,u]^{\times 2}$, $\tilde{\mathcal{Z}} = \mathbf{Z}$ and $P$ being the projection, mapping a morphism $(j,(s,t))$ to $j$.

Given a unital $\mathbf{Z} \times [1,u]^{\times 2}$-algebra $A$, i.e. an $\mathrm{A}_\infty$-category with set of objects $[1,u]$, show that its *total* $\mathrm{A}_\infty$*-algebra* $P_\& A$ is unital.

**Problem 25 (§XXX)** Let $\mathcal{Z}$ be a grading category.

Let $A = (A,(m_1),(A^{\langle i\rangle})_i)$ be a minimal $\mathrm{eA}_1$-algebra over $\mathcal{Z}$.

Suppose that there exist shift-graded linear map $d^{\langle i\rangle} : A^{\langle i\rangle} \to A^{\langle i-1\rangle}$ of degree 1 and shift-graded linear map $e^{\langle i\rangle} : A^{\langle i\rangle} \to A^{\leqslant i-2}$ of degree 1 for $i \in \mathbf{Z}_{\geqslant 0}$ such that

$$\iota^{\langle i\rangle} \cdot m_1 \;=\; d^{\langle i\rangle} \cdot \iota^{\langle i-1\rangle} + e^{\langle i\rangle} \cdot \iota^{\leqslant i-2} .$$

holds for $i \in \mathbf{Z}_{\geqslant 0}$ .

(1) Express the Stasheff equation at 1 in terms of $d^{\langle i\rangle}$ and $e^{\langle i\rangle}$, where $i \in \mathbf{Z}_{\geqslant 0}$ .

(2) Show that $A$ is diagonally resolving if and only if $\mathrm{Kern}(d^{\langle i\rangle}) = \mathrm{Im}(d^{\langle i+1\rangle})$ for $i \in \mathbf{Z}_{\geqslant 1}$ .

**Problem 26 (§XXX)** Let $\mathcal{Z}$ be a grading category.

Suppose given an $\mathrm{eA}_\infty$-algebra $(A,(m_k)_k,(A^{\langle i\rangle})_i)$ over $\mathcal{Z}$. Suppose that $A^{\langle i\rangle} = 0$ for $i \in \mathbf{Z} \smallsetminus [0,\ell]$.

For which integers $k \in \mathbf{Z}_{\geqslant 1}$ is the Schmid condition on $m_k$ not void?

For which integers $k \in \mathbf{Z}_{\geqslant 1}$ is the strong Schmid condition on $m_k$ not void?

(1) Consider the case $\ell = 1$.

(2) Consider the case $\ell = 2$.

(3) Consider the case $\ell = 3$.

**Problem 27 (§XXX)** Let $\mathcal{Z}$ be a grading category.

Suppose given an $\mathrm{eA}_\infty$-algebra $(A,(m_k)_k,(A^{\langle i\rangle})_i)$ over $\mathcal{Z}$. Let $k \geqslant 1$. Let $(j_1,\ldots,j_k) \in \mathbf{Z}_{\geqslant 0}^{\times k}$ .

What bound results from the Schmid condition for the image of $A^{\langle j_1\rangle} \otimes \ldots \otimes A^{\langle j_k\rangle}$ under a summand of the Stasheff equation at $k$ ?

**Problem 28** Let $X = (X, \leqslant)$ be a poset. We call $X$ *artinian* if it does not contain a strictly descending chain. We call $X$ *superartinian* if $X_{\leqslant \xi}$ is finite for all $\xi$. We call $X$ *discrete* if $(\leqslant) = (=)$. We call $X$ *narrow* if each discrete subposet of $X$ is finite.

Suppose given $k \in \mathbf{Z}_{\geqslant 1}$ and posets $Y_1, \ldots, Y_k$.

(1) Show that $X$ is artinian if and only if each nonempty subposet of $X$ has a minimal element.

(2) If $X$ is superartinian, show that $X$ is artinian. Does the converse hold?

(3) Construct the product $\prod_{i \in [1,k]} Y_i$ in Poset, which is to be equipped with monotone maps $\prod_{i \in [1,k]} Y_i \xrightarrow{\pi_j} Y_j$ for $j \in [1, k]$ such that for each poset $T$ and each tuple $(T \xrightarrow{t_i} Y_i)_i$ of monotone maps, there exists a unique monotone map $T \xrightarrow{t} \prod_{i \in [1,k]} Y_i$ such that $t \cdot \pi_j = t_j$ for $j \in [1, k]$.

(4) If $Y_i$ is artinian for $i \in [1, k]$, show that $\prod_{i \in [1,k]} Y_i$ is artinian.

(5) If $Y_i$ is superartinian for $i \in [1, k]$, show that $\prod_{i \in [1,k]} Y_i$ is superartinian.

(6) Show that $\mathbf{Z}_{\geqslant 0}^{\times k} := \prod_{i \in [1,k]} \mathbf{Z}_{\geqslant 0}$ is superartinian and narrow.

## A.2 Solutions

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