A generalized MIT Bag operator on spin manifolds in the non-relativistic limit

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Abstract. We consider Dirac-like operators with piecewise constant mass terms on spin manifolds, and we study the behaviour of their spectra when the mass parameters become large. In several asymptotic regimes, effective operators appear: the extrinsic Dirac operator and a generalized MIT Bag Dirac operator. This extends some results previously known for the Euclidean spaces to the case of general spin manifolds.

Contents

1. Introduction

The MIT Bag model was developed by the physicists in order to describe the behaviour of quarks fields inside hadrons. Mathematically, the hadron is seen as a compact region $\mathcal{K}$ with smooth boundary of the ambient space, where the quarks are supposed to be confined. This could be quantified by saying that the quantum flux through the border of $\mathcal{K}$ is null, a condition which is satisfied if we add the so-called MIT Bag condition on the boundary of $\mathcal{K}$ (see [11] for the details). Moreover, the quarks fields inside the hadron are Dirac fields, which means they are governed by the Dirac equation.

A Dirac field in the case of the space of dimension 3 is a $\mathbb{C}^4$-valued function ψ also depending on time, and the Dirac equation takes the form

$$H_m\psi := \left(-i\sum_{k=1}^{3}\alpha_k\partial_k + m\beta\right)\psi = i\,\frac{\partial}{\partial t}\psi \tag{1.1}$$

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where $\alpha_1, \alpha_2, \alpha_3, \beta \in M_4(\mathbb{C})$ are four Hermitian matrices satisfying the conditions $\alpha_k\alpha_l+\alpha_l\alpha_k = 2\delta_k^l \mathrm{I}_4$, $\beta^2 = \mathrm{I}_4$ and α_k anti-commutes with β for all $k, l \in \{1, 2, 3\}$. In view of this equation, the Dirac operator H_m can be interpreted as a Hamiltonian, and the description of its spectrum is a natural question. Thus, in the context of the MIT Bag model, we are interested in the operator resulting from the combination of H_m restricted to the region $\mathcal{K}$ together with the MIT Bag boundary condition, namely

$$H_m^{\mathcal{K}}\psi := H_m\psi, \quad \mathrm{dom}(H_m^{\mathcal{K}}) = \{\psi \in H^1(\mathcal{K}, \mathbb{C}^4),\ -i\,\beta(\alpha\cdot\mathbf{n})\,\psi_{|\partial\mathcal{K}} = \psi_{|\partial\mathcal{K}}\}, \quad (1.2)$$

where $\mathbf{n}$ is the outer normal vector field along $\partial\mathcal{K}$. The spectrum of this operator has been investigated in [2], where the non-relativistic limit was considered, i.e. the asymptotic regime where the mass goes to infinity. From a physical point of view, this last fact means that the speed of light becomes large, since this constant is hidden in the mass term in (1.1). It was shown that if we denote by $(\mu_j)_{j\geq 1}$ the non-decreasing sequence of positive eigenvalues of $H_m^{\mathcal{K}}$, one has the asymptotic

$$\mu_j \underset{m\to-\infty}{=} \widetilde{\mu}_j^{\frac{1}{2}} + \mathcal{O}(m^{-\frac{1}{2}}) \quad (1.3)$$

where $(\widetilde{\mu}_j)$ is the non-decreasing sequence of eigenvalues of an effective operator acting on the boundary of $\mathcal{K}$.

In the same framework, the MIT Bag Dirac operator was interpreted as the limit of a Dirac-type operator with a potential corresponding to two masses m and M in the regions $\mathcal{K}$ and $\mathcal{K}^c$ respectively [1]. More precisely, if we define the operator

$$H_{m,M} := H_m + (M-m)\mathbf{1}_{\mathcal{K}^c}, \quad \mathrm{dom}(H_{m,M}) := H^1(\mathbb{R}^3, \mathbb{C}^4), \quad (1.4)$$

then the eigenvalues of $H_{m,M}$ converge to the corresponding ones of $H_m^{\mathcal{K}}$ when $M \to +\infty$.

In the recent article [13], the case of Euclidean spaces was studied in order to enlarge the precedent results. The expression of the operator in dimension 3 given by (1.2) was generalized to dimension n by considering $n+1$ Hermitian matrices $\alpha_1, \ldots, \alpha_{n+1} \in M_N(\mathbb{C})$ ($N := 2^{\lfloor \frac{n+1}{2} \rfloor}$) satisfying the Clifford conditions $\alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_k^l \mathrm{I}_N$ and by setting

$$D_m\psi := \left(-i\sum_{k=1}^{n+1}\alpha_k\partial_k + m\alpha_{n+1}\right)\psi, \mathrm{dom}(D_m) = H^1(\mathbb{R}^n, \mathbb{C}^N). \quad (1.5)$$

This last operator is not the intrinsic Dirac operator in $\mathbb{R}^n$ but it can be interpreted like in (1.1) as the Hamiltonian appearing in the Dirac equation of a Lorentzian space of dimension $n+1$. From these considerations, the MIT Bag Dirac operator A_m can be defined by

$$A_m := D_m, \mathrm{dom}(A_m) := \{\psi \in H^1(\mathcal{K}, \mathbb{C}^4),\ -i\,\alpha_{n+1}\sum_{k=1}^{n}\mathbf{n}_k\alpha_k\,\psi_{|\partial\mathcal{K}} = \psi_{|\partial\mathcal{K}}\}. \quad (1.6)$$

With this definition, the result on the convergence of the eigenvalues of A_m still holds, and the effective operator on the boundary can be explicited. Namely, the eigenvalues of A_m^2 converge to the eigenvalues of the square of the intrinsic Dirac operator on $\partial\mathcal{K}$. Moreover, if $n \notin 4\mathbb{Z}$, the spectra of the operators are symmetric with respect to the origin, and we recover the result stated in dimension 3.

As for the Minkowski space, the operator A_m can be viewed as the limit of an operator with two masses [13, Theorem 1.2]. This operator is defined in the same way as before:

$$B_{m,M} := D_m + (M-m)\mathbf{1}_{\mathcal{K}}\alpha_{n+1}, \quad \mathrm{dom}(B_{m,M}) := H^1(\mathbb{R}^n, \mathbb{C}^N), \quad (1.7)$$

and the eigenvalues of $B_{m,M}^2$ converge to the eigenvalues of A_m^2 when $M \to +\infty$. In addition, a combination of the two previous asymptotic behaviours is also true [13, Theorem 1.3]: in the asymptotic regime $m \to -\infty$ and $M \to +\infty$ with $\frac{m}{M} \to 0$, one has that the eigenvalues of $B_{m,M}^2$ converge to the corresponding ones of the intrinsic Dirac operator on the boundary $\partial\mathcal{K}$.

In the precedent discussion, the spaces considered where always flat, but the Dirac operator can be defined in a more general setting, for example over a manifold admitting a Spin-structure. Consequently, our aim in the present text is to extend the results of [13] to this more general framework. In order to do so, the first step is to understand the geometrical meaning of the operator considered in the MIT Bag model, because we recall that the Dirac operator considered in [13] is not the intrinsic Dirac operator of the Euclidean space. Indeed, the operator D_m is the so-called Dirac-Witten operator on $\mathbb{R}^n$ seen as an hypersurface of $\mathbb{R}^{n+1}$, plus a mass term which is actually the Clifford multiplication by the vector $i\, m x_{n+1}$ in $\mathbb{R}^{n+1}$.

Nevertheless, even if the expression (1.6) is a direct generalization of equation (1.2), the Dirac-Witten operator is not the operator we obtain from the physical model [11]. Indeed, in (1.1) we used the alpha matrices, but the Dirac equation is more often written using the gamma matrices defined by

$$\gamma^0 := \beta, \quad \gamma^k := -i\,\gamma^0\alpha_k, \qquad\qquad k = 1, 2, 3.$$

If one rewrites (1.1) with the γ matrices, one obtains

$$H_m\psi = \left(\sum_{k=1}^{3} \gamma^0\gamma^k\partial_k + m\gamma^0\right)\psi, \tag{1.8}$$

and this last operator is, up to a change of sign, the extrinsic Dirac operator on the hypersurface $\mathbb{R}^3$ plus the mass term. Moreover, the boundary condition defined in [2] by $-i\,\beta(\alpha\cdot\mathbf{n})\psi = \psi$ reads $i\,(\gamma\cdot\mathbf{n})\psi = \psi$ and this last boundary condition is the MIT Bag boundary condition as introduced in [11].

All together, we have two natural ways of setting the problem in the case of a complete spin manifold $\mathcal{N}$. In both cases, we have to see $\mathcal{N}$ as an hypersurface of the Riemannian product $\mathcal{C} := \mathcal{N}\times\mathbb{R}$, and we denote by ν the outer normal vector field over $\mathcal{N}$. In addition, the region $\mathcal{K}$ is now a compact submanifold of $\mathcal{N}$ with boundary. The theory of Spin-structures restricted to hypersurfaces gives that $\mathcal{C}$ and $\partial\mathcal{K}$ are also spin manifolds. Consequently, we can define the spinor bundle $\Sigma\mathcal{C}$ over $\mathcal{C}$, and the extrinsic Dirac operator $\mathcal{D}^{\mathcal{N}}$, which acts on spinors of $\mathcal{C}$ restricted to $\mathcal{N}$.

From the previous discussion, the obvious generalization of the MIT Bag Dirac operator in the Euclidean spaces (1.6) is defined as the Dirac-Witten operator on $\mathcal{N}$ plus a mass term, and we add the boundary condition $i\,\nu\cdot\mathbf{n}\cdot\Psi = \Psi$ on $\partial\mathcal{K}$. This last condition is not the MIT Bag boundary condition, but the condition associated with a chirality operator, and it is consistent with the condition imposed in (1.6). Namely, we have

$$A_m := \nu\cdot\mathcal{D}^{\mathcal{N}} + i\,m\,\nu\cdot,\ \mathrm{dom}(A_m) = \left\{\Psi\in H^1(\Sigma\mathcal{C}_{|\mathcal{K}}), i\,\nu\cdot\mathbf{n}\cdot\Psi = \Psi \text{ on } \partial\mathcal{K}\right\}. \tag{1.9}$$

Furthermore, the cylinder $\mathcal{C}$ can be endowed with a Lorentzian metric such that ν is a time-like vector, and in this case, solving the Dirac equation in $\mathcal{C}$ in the same way as for dimension 3 lets us with the study of the extrinsic Dirac operator on $\mathcal{N}$ plus the mass term. The boundary condition imposed in this case is the original MIT Bag boundary condition $i\,\mathbf{n}\cdot\Psi = \Psi$.

Actually, the two operators we defined this way are unitarily equivalent since the manifold $\mathcal{N}$ is totally geodesic in $\mathcal{C}$. This last result explains how the operator studied in [13] is obtained from the physical model, and the two definitions we gave above are equivalent.

In the same way as before, the two-masses operator is obtained by adding a potential corresponding to two masses in $\mathcal{K}$ and $\mathcal{K}^c$ in the expression of the operator A_m. Since in our framework the manifold $\mathcal{N}$ is complete but not necessarily compact, $B_{m,M}$ is defined as the closure of the operator

$$\widetilde{B}_{m,M} := \nu \cdot \mathcal{D}^{\mathcal{N}} + i\,(m\mathbf{1}_{\mathcal{K}} + M\mathbf{1}_{\mathcal{K}^c})\nu\cdot, \tag{1.10}$$

whose domain is the set of smooth sections with compact support in $\Sigma\mathcal{C}_{|\mathcal{N}}$. This definition is consistent with (1.7) because it was shown in [13] that the two-masses operator is essentially self-adjoint on the smooth functions with compact support.

The operators A_m and $B_{m,M}$ are self-adjoint and we are interested in the behaviour of the spectrum of A_m when $m \to -\infty$ and the spectrum of $B_{m,M}$ in the asymptotic regime $M \to +\infty$ and $\min(-m, M) \to +\infty$. These limits are the ones studied in [13], and the three main theorems we state below are the counterparts of [13, Theorems 1.1, 1.2, 1.3].

From now on, we use for $j \in \mathbb{N}$ and a lower semibounded operator T the notation $E_j(T)$, which stands for the j-th eigenvalue of T when counted with multiplicity in the non-decreasing order.

First of all, one has the convergence of the eigenvalues of A_m^2 to the eigenvalues of the square of the Dirac operator on $\partial\mathcal{K}$:

Theorem 1.1. *For any $j \in \mathbb{N}$, one has* $E_j(A_m^2) \underset{m\to-\infty}{\longrightarrow} E_j\left((\not{D}^{\partial\mathcal{K}})^2\right)$.

The two operators A_m^2 and $B_{m,M}^2$ are surprisingly related in the asymptotic regime $M \to +\infty$:

Theorem 1.2. *For any $j \in \mathbb{N}$, there is $M_0 \in \mathbb{R}$ such that for all $M \geq M_0$, $B_{m,M}^2$ has at least j eigenvalues, and one has* $E_j(B_{m,M}^2) \underset{M\to+\infty}{\longrightarrow} E_j(A_m^2)$.

In addition, one has a combination of these two results:

Theorem 1.3. *For any $j \in \mathbb{N}$, there is $\tau_j \in \mathbb{R}$ such that for all $M \geq \tau_j$ and $m \leq -\tau_j$, the operator $B_{m,M}^2$ has at least j eigenvalues, and one has* $E_j(B_{m,M}^2) \underset{\min(M,-m)\to+\infty}{\longrightarrow} E_j\left((\not{D}^{\partial\mathcal{K}})^2\right)$.

Note that Theorem 1.3 is an improvement of [13, Theorem 1.3] since we drop the assumption $\frac{m}{M} \to 0$.

Remark 1.4. We can also look at the operator A_m^2 when $m \to +\infty$ and the operator $B_{m,M}^2$ when $m, M \to +\infty$ (or $m, M \to -\infty$). We can prove that in these two cases, the spectrum escapes to infinity (see Remarks 7.2 and 9.1 below).

Organization of the paper. The proofs of the three theorems are really close to the ones written in [13] once we have stated the correct geometrical context. The global strategy is thus to compute sesquilinear forms for the operators A_m^2 and $B_{m,M}^2$ in order to find lower and upper bounds for the limits of the eigenvalues by use of the Min-Max principle.

In section 2 we first recall some fundamental results in spectral theory on the correspondence between self-adjoint operator and sesquilinear forms on Hilbert space. The Min-Max principle, which is the key point of our proof, is stated, and we also give a quick review on the monotone convergence theorem in the case of sesquilinear forms. This last theorem is helpful to find the lower bounds for the limits of the eigenvalues, since it gives a description of the asymptotic domain of the operators. After these preliminaries on operators theory, we introduce the basic tools needed to understand the geometrical context. Indeed, the theory of restriction of the spin structure of spin manifolds to oriented hypersurfaces plays a significant role in the understanding of the generalized MIT Bag operator.

Section 3 is devoted to the construction of the operators. We develop here the discussion about the two equivalent ways of defining A_m. We also define the operator $B_{m,M}$ and we show that it is self-adjoint as a direct consequence of the completeness of $\mathcal{N}$. The self-adjointness of A_m is more difficult to prove, and we need to compute the sesquilinear form for A_m^2 in order to understand its graph norm and its domain. The computations for the forms of square operators are done in Section 4 and the main tool used to this aim is the Schrödinger-Lichnerowicz formula, which gives the expression of the square of the Dirac operator on a spin manifold. Once we get the sesquilinear forms, the graph norm of A_m is shown to be equivalent to the H^1 norm on its domain, and we can use the analysis done in [8] to conclude on self-adjointness.

An important idea to prove the main results is that we can restrict the analysis to a tubular neighbourhood of the boundary of $\mathcal{K}$. Thanks to this restriction of domain, we only have to understand the operators on a generalized cylinder $\partial\mathcal{K} \times (-\delta, \delta)$ with $\delta > 0$. However, there is an additional difficulty since we cannot compare the covariant derivatives on the different slices of the cylinder as it is done in [13]. Thus, we prove some comparison lemmas in section 5, where we express the operators in tubular coordinates.

The aim of this restriction is to be able to separate the variables in the generalized cylinder previously introduced. Thus, some one-dimensional operators will appear later in the analysis, and we devote section 6 to the spectral analysis of these operators, even if a large part of this work has already been done in [13, Section 3].

In section 7 we prove Theorem 1.1. The geometrical context is well-defined, and it remains to follow the lines of [13, Section 4]. The proof is done by restricting the analysis to the tubular neighbourhood of $\partial\mathcal{K}$ intersected with the interior of $\mathcal{K}$ thanks to the Min-Max principle. Next, an upper bound can be found for the limit by choosing good test functions which are tensorial products between eigenspinors of a model operator on $\partial\mathcal{K}$ and the first eigenfunction of a one-dimensional operator. The proof of the lower bound relies on the monotone convergence theorem after operating a transformation on the operator in tubular coordinates.

The result stated in Theorem 1.2 is proved in section 8. We find an appropriate extension operator which sends eigenspinors of A_m^2 into $\mathrm{dom}(B_{m,M})$, and this gives the upper bound. The lower bound is once again a consequence of the monotone convergence theorem together with the Min-Max principle.

Finally, we prove Theorem 1.3 in section 9 using a combination of the precedent arguments. After restricting the problem to the tubular neighbourhood of $\partial\mathcal{K}$, the upper bound is found in the same way as for Theorem 1.1 by choosing good test functions in the Min-Max principle, and the lower bound is a consequence of the monotone convergence theorem.

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2. Notations and preliminaries.

2.1. **About spectral theory.** Let $\mathbf{H}$ be an infinite-dimensional Hilbert space endowed with the inner product $(\cdot,\cdot)_{\mathbf{H}}$. For a self-adjoint and lower semibounded operator T on $\mathbf{H}$, we denote by $\operatorname{dom} T$ its domain, and for any $j \in \mathbb{N}$, $E_j(T)$ is the jth eigenvalue of T, counted with multiplicity in the non-decreasing order. We also note $\sigma(T)$, $\sigma_{ess}(T)$ and $\sigma_d(T)$ the spectrum, the essential spectrum and the discrete spectrum of T respectively.

We denote the adjoint of an operator T by T^* and its closure by $\overline{T}$.

For a sesquilinear form t in $\mathbf{H}$, we denote its domain by $\mathcal{Q}(t)$. There is a one-to-one correspondence between densely defined, closed, symmetric, lower semibounded forms and lower semibounded self-adjoint operators (see [12, VI, Theorem 2.1] for details). For a lower semi-bounded self-adjoint operator T, we will denote by $\mathcal{Q}(T)$ the domain of the associated form. If T and T' are two such operators, and t, t' are the associated forms, we write $T \leq T'$ if $\mathcal{Q}(T') \subset \mathcal{Q}(T)$ and $t(u,u) \leq t'(u,u)$ for all $u \in \mathcal{Q}(T')$.

For $j \in \mathbb{N}$, we define the jth Rayleigh quotient of the form t by

$$\Lambda_j(t) := \inf_{\substack{V \subset \mathcal{Q}(t) \\ \dim V = j}} \sup_{u \in V \setminus \{0\}} \frac{t(u,u)}{\|u\|^2_{\mathcal{H}}}. \tag{2.1}$$

We recall that if t and t' are two semibounded from below bilinear forms, we write $t \leq t'$ if $\mathcal{Q}(t') \subset \mathcal{Q}(t)$ and $t(u,u) \leq t'(u,u)$ for all $u \in \mathcal{Q}(t')$.

Let t be a closed symmetric lower semibounded form, and T its associated operator. The well-known Min-Max principle gives a link between the Rayleigh quotients of t and the eigenvalues of T. More precisely, we have the following theorem:

Theorem 2.1 (Min-Max principle)**.** *Let $\Sigma := \inf \sigma_{ess} T$. We are in one of the following cases:*

- *(a) $\Lambda_j(t) < \Sigma$ for all j, $\lim_{m \to +\infty} \Lambda_m(t) = \Sigma$ and $E_j(T) = \Lambda_j(t)$ for all j.*
- *(b) $\sigma_{ess} T < +\infty$ and there is $N < +\infty$ such that the interval $(-\infty, \Sigma)$ contains exactly N eigenvalues of T counted with multiplicity and for all $j \leq N$, one has $\Lambda_j(t) = E_j(T)$ and $\Lambda_m(t) = \Sigma$ for all $m > N$.*

The proofs of the spectral part of this text will use monotone convergence of operators. The result stated below is a reformulation of [4, Theorem 4.2].

Theorem 2.2. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of lower semibounded self-adjoint operators in closed subspaces $(\mathbf{H}_n)_{n\in\mathbb{N}}$ of $\mathbf{H}$, and let $(t_n)_{n\in\mathbb{N}}$ be the sequence of associated forms. Assume there exists $\gamma \in \mathbb{R}$ such that $t_n \geq \gamma$ for all n and suppose moreover that the sequence (t_n) (or equivalently (T_n)) is non-decreasing. Then, the form t_∞ defined by*

$$\mathcal{Q}(t_\infty) = \left\{ h \in \bigcap_{n \in \mathbb{N}} \mathcal{Q}(t_n),\ \lim_{n \to \infty} t_n(h,h) < \infty \right\} \tag{2.2}$$

and $t_\infty(h,h) = \lim_{n\to\infty} t_n(h,h)$ for all $h \in \mathcal{Q}(t_\infty)$ is closed, symmetric, and $t_\infty \geq \gamma$. Moreover, if $\mathbf{H}_\infty := \overline{\mathcal{Q}(t_\infty)}$, one can define the self-adjoint operator T_∞ on $\mathbf{H}_\infty$ associated with t_∞, and the sequence (T_n) strongly converges to T_∞ in the generalized resolvent sense, i.e. for all $\lambda < \gamma$, one has

$$((T_n - \lambda)^{-1} \oplus 0_{\mathbf{H}_n^\perp})h \underset{n\to\infty}{\longrightarrow} ((T_\infty - \lambda)^{-1} \oplus 0_{\mathbf{H}_\infty^\perp})h,\ \forall h \in \mathbf{H}. \tag{2.3}$$

Since we are interested in the behaviour of the spectrum, we claim that in the framework of Theorem 2.2, one has actually the convergence of the eigenvalues of T_n

to the corresponding eigenvalues of T_∞. To show this, we first recall [15, Theorem 2.1]:

Theorem 2.3. *Let (T_n) be a sequence of self-adjoint operators which are bounded from below with $T_n \leq T_{n+1}$, strongly converging to T in the generalized resolvent sense. Assume that the essential spectrum of T_n is contained in $[0,+\infty)$ for all $n \in \mathbb{N}$. Suppose that T has j_0 negative eigenvalues (j_0 might be infinite). Then,*

$$E_j(T_n) \underset{n\to+\infty}{\longrightarrow} E_j(T) \text{ for all } j \leq j_0$$
$$\lim_{n\to+\infty} E_j(T_n) \geq \eta \text{ for all } j > j_0.$$

Moreover,

$$\|\mathbf{1}_{(-\infty,\lambda)}(T_n) - \mathbf{1}_{(-\infty,\lambda)}(T)\| \underset{n\to+\infty}{\longrightarrow} 0 \text{ for all } \lambda < 0.$$

From Theorem 2.2 and Theorem 2.3 we deduce the following corollary:

Corollary 2.4. *Let $(T_n)_{n\in\mathbb{N}}$ and T_∞ be like in Theorem 2.2. Assume moreover that $\sigma_{ess}(T_{n_0}) \subset [\eta,+\infty)$ for some $n_0 \in \mathbb{N}$ and that T_∞ has j_0 eigenvalues below η (j_0 might be infinite). Then, one has*

$$E_j(T_n) \underset{n\to+\infty}{\longrightarrow} E_j(T) \text{ for all } j \leq j_0 \tag{2.4}$$

and

$$\|\mathbf{1}_{(-\infty,\lambda)}(T_n) - \mathbf{1}_{(-\infty,\lambda)}(T_\infty)\| \underset{n\to+\infty}{\longrightarrow} 0, \quad \forall \lambda < \eta. \tag{2.5}$$

Proof. We consider for $n \geq n_0$ large enough the bounded self-adjoint operators in $\mathbf{H}$

$$B_n := \frac{1}{\eta-\gamma} - ((T_n-\gamma)^{-1} \oplus 0_{\mathbf{H}_n^\perp})$$
$$B_\infty := \frac{1}{\eta-\gamma} - ((T_\infty-\gamma)^{-1} \oplus 0_{\mathbf{H}_\infty^\perp}).$$

From [4, Proposition 2.2], it comes that for all $n \geq n_0$, one has $B_n \leq B_{n+1} \leq B_\infty$. In addition, $\sigma_{ess}(B_n) \subset [0, \frac{1}{\eta-\gamma}]$, $\sigma_{ess}(B_\infty) \subset [0, \frac{1}{\eta-\gamma}]$, and (B_n) converges strongly to B_∞. Thus, Theorem 2.3 gives that for all $j \in \mathbb{N}$ such that $E_j(B_\infty) < 0$ one has

$$E_j(B_n) \underset{n\to+\infty}{\longrightarrow} E_j(B_\infty) \tag{2.6}$$

and that for all $t < 0$, there holds

$$\|\mathbf{1}_{(-\infty,t)}(B_n) - \mathbf{1}_{(-\infty,t)}(B_\infty)\| \underset{n\to\infty}{\longrightarrow} 0. \tag{2.7}$$

For $\lambda > \gamma$, we define the strictly increasing function $f(\lambda) := \frac{1}{\eta-\gamma} - \frac{1}{\lambda-\gamma}$. One has $B_n = f(T_n)$ and $B_\infty = f(T_\infty)$ and we deduce that for all $j \leq j_0$

$$E_j(T_n) \underset{n\to+\infty}{\longrightarrow} E_j(T) \text{ for all } j \leq j_0$$

and from

$$\mathbf{1}_{(-\infty,f(\lambda))}(B_n) = \mathbf{1}_{(-\infty,\lambda)}(T_n), \quad \mathbf{1}_{(-\infty,f(\lambda))}(B_\infty) = \mathbf{1}_{(-\infty,\lambda)}(T_\infty),$$

we deduce that for all $\lambda < \eta$

$$\|\mathbf{1}_{(-\infty,\lambda)}(T_n) - \mathbf{1}_{(-\infty,\lambda)}(T_\infty)\| \underset{n\to\infty}{\longrightarrow} 0. \qquad \square$$

2.2. **Clifford algebra.** We recall here the basic facts about Clifford algebra, and we refer to [5] for the details. For any $d \in \mathbb{N}$, the real Clifford algebra Cl_d is the quotient of the tensorial algebra over $\mathbb{R}^d$ by the two-sided ideal generated by the elements $x \otimes x + \|x\|^2 1$. The induced product on the quotient algebra is called the Clifford product, and is denoted by "$\cdot$". The complex Clifford algebra is defined by $\mathbb{C}\mathrm{l}_d := \mathrm{Cl}_d \otimes_{\mathbb{R}} \mathbb{C}$. The spin group is the subgroup of Cl_d given by

$$\mathrm{Spin}_d := \{x_1 \cdot \ldots \cdot x_{2k} \in \mathrm{Cl}_d,\ k \in \mathbb{N} \text{ and } x_j \in \mathbb{R}^d,\ \|x_j\| = 1 \text{ for all } 1 \leq j \leq 2k\}.$$

We define the complex volume form as the element of $\mathbb{C}\mathrm{l}_d$

$$\omega_d^{\mathbb{C}} := i^{\lfloor \frac{d+1}{2} \rfloor} e_1 \cdot \ldots \cdot e_d \tag{2.8}$$

where $(e_1, \ldots, e_d)$ is any positively-oriented orthonormal frame of $\mathbb{R}^d$, canonically identified with a basis of $\mathbb{C}^d$.

If d is even, $\mathbb{C}\mathrm{l}_d$ admits an unique irreducible complex representation (ρ_d, Σ_d) where Σ_d is a complex vector space of dimension $2^{\frac{d}{2}}$. When restricted to the Spin group, this Clifford module decomposes into $\Sigma_d = \Sigma_d^+ \oplus \Sigma_d^-$ and the representation splits in two irreducible inequivalent representations $(\rho_d^\pm, \Sigma_d^\pm)$. These submodules are characterized by the action of the complex volume form, namely $\omega_d^{\mathbb{C}}$ acts as $\pm\mathrm{Id}$ on $\Sigma_d^\pm$.

When d is odd, $\mathbb{C}\mathrm{l}_d$ admits two irreducible inequivalent representations over complex vector spaces of dimension $2^{\frac{d-1}{2}}$. They are characterized by the action of the complex volume form which acts as $\pm\mathrm{Id}$. We denote by (ρ_d, Σ_d) the representation on which $\omega_d^{\mathbb{C}}$ acts as the identity.

2.3. **Notations for manifolds and bundles.** In all this text, the manifolds will be considered smooth and paracompact.

Let $(\mathcal{M}, g)$ be a Riemannian manifold of dimension $d+1$, with boundary $\partial\mathcal{M}$ (possibly empty). If $\mathcal{M}$ is oriented, we denote by $v_{\mathcal{M}}$ the volume form on $\mathcal{M}$ compatible with the metric. Throughout this article, integrations will be done with respect to the Riemannian measure, which coincides with the integration with respect to the volume form $v_{\mathcal{M}}$ in the oriented case.

We denote by $\nabla^{\mathcal{M}}$ the Levi-Civita connection of $(\mathcal{M}, g)$ and by $\mathrm{R}^{\mathcal{M}}$, $\mathrm{Ric}^{\mathcal{M}}$, $\mathrm{Scal}^{\mathcal{M}}$ the Riemann curvature tensor, the Ricci tensor, and the scalar curvature of $\mathcal{M}$ respectively.

If E is a vector bundle over $\mathcal{M}$, we denote respectively by $\Gamma(E)$, $\Gamma_c(E)$ and $\Gamma_{cc}(E)$ the smooth sections of E, the smooth sections of E with compact support in $\mathcal{M}$, and the smooth sections of E with compact support in $\mathcal{M} \setminus \partial\mathcal{M}$. If moreover E is a Hermitian bundle, we note $L^2(E)$ the space of square integrable sections of E. If it is necessary, we will write $L^2(E, v_{\mathcal{M}})$ to specify the measure used for the integration.

We now assume that $\mathcal{M}$ is oriented. The manifold $\mathcal{M}$ admits a spin structure if there exists a map χ and a principal bundle $P_{\mathrm{Spin}_{d+1}}\mathcal{M}$ over $\mathcal{M}$ such that for every $u \in P_{\mathrm{Spin}_{d+1}}\mathcal{M}$ we have the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Spin}_{d+1} & \xrightarrow{s \mapsto us} & P_{\mathrm{Spin}_{d+1}}\mathcal{M} & & \\
\big\downarrow & & \big\downarrow \chi & \searrow & \\
 & & & & \mathcal{M} \\
 & & & \nearrow & \\
\mathrm{SO}_{d+1} & \xrightarrow{g \mapsto \chi(u) g} & P_{\mathrm{SO}_{d+1}}\mathcal{M} & &
\end{array} \tag{2.9}$$

Given a spin structure on $\mathcal{M}$, we define the associated complex spinor bundle by $\Sigma\mathcal{M} := P_{\mathrm{Spin}_{d+1}}\mathcal{M} \times_{\rho_{d+1}} \Sigma_{d+1}$ where we recall that $(\rho_{d+1}, \Sigma_{d+1})$ is an irreducible representation of the Clifford algebra $\mathbb{C}\mathrm{l}_{d+1}$ as defined in section 2.2.

There is a natural action of the Clifford bundle $\mathbb{C}\mathcal{M} := P_{\mathrm{SO}_{d+1}} \times_r \mathbb{C}\mathrm{l}_{d+1}$ (where r is the action of SO_{d+1} on $\mathbb{R}^d$ extended to a representation on $\mathbb{C}\mathrm{l}_d$) defined by:

$$[\chi(u), v]([u, \psi]) := [u, \rho_{d+1}(v)\psi] \tag{2.10}$$

for all $u \in P_{\mathrm{Spin}_{d+1}}\mathcal{M}$, $v \in \mathbb{C}l_{d+1}$ and $\psi \in \Sigma_{d+1}$. This action is called the Clifford product and will be denoted by "$\cdot$".

One has a canonical Hermitian product $\langle \cdot, \cdot \rangle$ on $\Sigma\mathcal{M}$ for which the Clifford product by a unit vector is unitary. Moreover, one obtains a metric connection on $\Sigma\mathcal{M}$ by lifting the Levi-Civita connection on the orthonormal frame bundle of $\mathcal{M}$ through the map χ. The covariant derivative obtained this way will still be denoted by $\nabla^{\mathcal{M}}$.

We define the intrinsic Dirac operator $\not{D}^{\mathcal{M}}$ on $\mathcal{M}$, by its pointwise expression

$$\not{D}^{\mathcal{M}}\Psi = \sum_{k=1}^{d+1} e_k \cdot \nabla^{\mathcal{M}}_{e_k}\Psi, \ \mathrm{dom}(\not{D}^{\mathcal{M}}) = \Gamma_c(\Sigma\mathcal{M}), \tag{2.11}$$

where $(e_1, \ldots, e_{d+1})$ is an orthonormal frame. This definition does not depend on the choice of the frame.

Finally, we remind the Schrödinger-Lichnerowicz formula, which will be a fundamental tool to compute sesquilinear forms of operators. A proof can be found in [6, Theorem 1.3.8].

Theorem 2.5 (Schrödinger-Lichnerowicz formula)**.** *The Dirac operator $\not{D}^{\mathcal{M}}$ satisfies the formula*

$$(\not{D}^{\mathcal{M}})^2 = \left(\nabla^{\mathcal{M}}\right)^* \nabla^{\mathcal{M}} + \frac{\mathrm{Scal}^{\mathcal{M}}}{4}, \tag{2.12}$$

where $(\nabla^{\mathcal{M}})^* : \Gamma(T^*\mathcal{M} \otimes \Sigma\mathcal{M}) \to \Gamma(\Sigma\mathcal{M})$ is the formal adjoint of $\nabla^{\mathcal{M}}$.

2.4. **Restriction of the spinor bundle to hypersurfaces.** We take $(\mathcal{M}, g)$ as in the previous section.

Let $\mathcal{H}$ be a smooth oriented hypersurface of $\mathcal{M}$. Let ν be the outer unit normal vector field on $\mathcal{H}$, that is, the only vector field such that if $(e_1, \ldots, e_d)$ is an oriented frame of $\mathcal{H}$, then $(e_1, \ldots, e_d, \nu)$ is an oriented frame of $\mathcal{M}$. We define the Weingarten operator of $\mathcal{H}$ as the endomorphism of $T\mathcal{H}$ given by

$$W_{\mathcal{H}}(X) := -\nabla^{\mathcal{M}}_X \nu, \tag{2.13}$$

and $H_{\mathcal{H}} : \mathcal{M} \to \mathbb{R}$ will be the pointwise trace of this operator.

The hypersurface $\mathcal{H}$ inherits a spin structure from the one of $\mathcal{M}$, and we can define the spinor bundle $\Sigma\mathcal{H}$ (for the details, see [5, Section 2.4]). This last bundle is endowed with the natural Hermitian product on spinors, still denoted by $\langle \cdot, \cdot \rangle$. The covariant derivative on $\Sigma\mathcal{H}$ induced by the Levi-Civita connection will be denoted by $\nabla^{\mathcal{H}}$. We will also write $\nabla^{\mathcal{H}}$ for the covariant derivative on $\Sigma\mathcal{H} \oplus \Sigma\mathcal{H}$ (where $\oplus$ stands for the Whitney product), and for all $X \in T\mathcal{H}$, the Clifford product by X on $\Sigma\mathcal{H} \oplus \Sigma\mathcal{H}$ is given by

$$X \cdot (\Psi_1, \Psi_2) := (X \cdot \Psi_1, -X \cdot \Psi_2), \quad \forall (\Psi_1, \Psi_2) \in \Sigma\mathcal{H} \oplus \Sigma\mathcal{H}. \tag{2.14}$$

There is a link between the restricted spinor bundle $\Sigma\mathcal{M}_{|\mathcal{H}}$ and $\Sigma\mathcal{H}$, given by the following proposition (see [6, Proposition 1.4.1]):

Proposition 2.6. *Let $\mathcal{M}$ and $\mathcal{H}$ be as above. There exists an isomorphism ζ from $\Sigma\mathcal{M}_{|\mathcal{H}}$ into $\Sigma\mathcal{H}$ if d is even and into $\Sigma\mathcal{H} \oplus \Sigma\mathcal{H}$ otherwise, which satisfies the following properties:*

(1) For all $x \in \mathcal{H}$, $X \in \Gamma(T_x\mathcal{H})$ and $\Psi \in (\Sigma\mathcal{M})_{|\{x\}}$, the Clifford product on $\mathcal{H}$ satisfies
$$X \cdot \zeta(\Psi) = \zeta(X \cdot \nu(x) \cdot \Psi), \tag{2.15}$$
(2) The isomorphism ζ is unitary,
(3) For all $\Psi \in \Gamma(\Sigma\mathcal{M}_{|\mathcal{H}})$ and $X \in T\mathcal{H}$,
$$\zeta(\nabla^{\mathcal{M}}_X \Psi) = \nabla^{\mathcal{H}}_X \zeta(\Psi) + \frac{1}{2} W_{\mathcal{H}} X \cdot \zeta(\Psi). \tag{2.16}$$
(4) For $\Psi \in \Sigma\mathcal{M}_{|\mathcal{H}}$,
$$\zeta(i\,\nu \cdot \Psi) = \left\{ \begin{array}{ll} \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \zeta(\Psi) & \text{if } d \text{ is odd} \\ \omega_d^{\mathbb{C}} \cdot \zeta(\Psi) & \text{if } d \text{ is even} \end{array} \right. , \tag{2.17}$$
where the complex volume form $\omega_d^{\mathbb{C}}$ was defined in section 2.2.

We can define a covariant derivative $\overline{\nabla}^{\mathcal{M}}$ on $\Sigma\mathcal{M}_{|\mathcal{H}}$ such that $\overline{\nabla}^{\mathcal{M}}\Psi$ is the restriction of $\nabla^{\mathcal{M}}\Psi$ to $\Gamma(T^*\mathcal{H} \otimes E)$. This notation will be useful as we will often consider the norm of the restricted covariant derivative on hypersurfaces.

The link between $\Sigma\mathcal{M}_{|\mathcal{H}}$ and $\Sigma\mathcal{H}$ gives rise to a natural operator called the extrinsic Dirac operator. This is actually the Dirac operator of $\mathcal{H}$ which acts on the spinor bundle $\Sigma\mathcal{M}_{|\mathcal{H}}$. This extrinsic Dirac operator on $\mathcal{H}$ is the operator acting on $\Gamma_c(\Sigma\mathcal{M})$ defined by
$$\mathcal{D}^{\mathcal{H}} := \zeta^* \not{D}^{\mathcal{H}} \zeta \text{ if } d \text{ is odd}, \quad \mathcal{D}^{\mathcal{H}} := \zeta^*(\not{D}^{\mathcal{H}} \oplus -\not{D}^{\mathcal{H}})\zeta \text{ if } d \text{ is even}. \tag{2.18}$$
where ζ is the isomorphism given by Proposition 2.6. It can be explicitly computed, and its expression at $x \in \mathcal{H}$ for $\Psi \in \Sigma\mathcal{M}$ is
$$\mathcal{D}^{\mathcal{H}}\Psi(x) = \frac{H_{\mathcal{H}}(x)}{2}\Psi(x) - \nu(x) \cdot \sum_{k=1}^{d} e_k \cdot \nabla^{\mathcal{M}}_{e_k}\Psi(x) \tag{2.19}$$
where $(e_1, \ldots, e_d)$ is an orthonormal frame of $T_x\mathcal{H}$ [6, Proposition 1.4.1], [10].

2.5. **Sobolev spaces on manifolds.** Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension $d+1$ with boundary $\partial\mathcal{M}$. We denote by $\nu_{\mathcal{M}}$ the normal unit vector field over $\partial\mathcal{M}$.

Let $(E, \nabla^E, \langle\cdot,\cdot\rangle_E)$ be an Hermitian bundle of dimension q over $\mathcal{M}$. The construction of the Sobolev spaces on E is done for example in [8, Definition 3.5], but we recall the idea to be self-contained.

In what follows, we will denote by $\exp^{\mathcal{M}}$ the Riemannian exponential map on $\mathcal{M}$ and by $B^{\mathcal{M}}_x(r)$ the ball of radius $r > 0$ and of center 0 in $T_x\mathcal{M}$ where $x \in \mathcal{M}$. This notation will be used for the boundary $\partial\mathcal{M}$ with an obvious modification. By the compactness of $\mathcal{M}$, there is $r_t > 0$ such that:

- the map
$$F : \partial\mathcal{M} \times [0, 2r_t) \ni (x,t) \mapsto \exp^{\mathcal{M}}_x(t\nu_{\mathcal{M}}(x)) \tag{2.20}$$
is a diffeomorphism on its image;
- for all $x \in \mathcal{M} \setminus F(\partial\mathcal{M} \times [0, 2r_t))$, $\exp^{\mathcal{M}}$ is injective on the open ball of radius r_t of $T_x\mathcal{M}$;
- for all $x \in \partial\mathcal{M}$, $\exp^{\partial\mathcal{M}}$ is injective on the open ball of radius r_t of $T_x\partial\mathcal{M}$.

Let $(U_j)_{j\in J}$ be a finite covering of $\mathcal{M}$ such that $U_j = \exp_x^{\mathcal{M}}(B_x^{\mathcal{M}}(r_t))$ with $x \in \mathcal{M} \setminus F(\partial\mathcal{M} \times [0, 2r_t))$ (Gaussian coordinates) or $U_j = F(B_x^{\partial\mathcal{M}}(r_t) \times [0, 2r_t))$ with $x \in \partial\mathcal{M}$ (normal coordinates). The maps given by these charts are denoted by $(f_j)_{j\in J}$. We trivialize E over U_j with Gaussian coordinates by identifying E_x with $\mathbb{C}^q$ and by making parallel transport along the radial geodesics. Over the set U_j with normal coordinates, we trivialize E by identifying E_x with $\mathbb{C}^q$ and by making parallel transport first along the radial geodesics in $\partial\mathcal{M}$ and then along the geodesics normal to $\partial\mathcal{M}$. The trivializations obtained are denoted by ξ_j.

Let $(h_j)_{j\in J}$ be a partition of unity adapted to the covering $(U_j)_{j\in J}$. For $s \in \mathbb{R}$ we define the H^s norm by

$$\|\Psi\|^2_{H^s(E)} := \sum_{j\in J} \|(\xi_j)_*(h_j\Psi) \circ f_j^{-1}\|^2_{H^s(\mathbf{R}_j^{d+1},\mathbb{C}^q)}, \tag{2.21}$$

where $\mathbf{R}_j^{d+1} := \mathbb{R}^{d+1}$ when $U_j \cap \partial\mathcal{M} = \emptyset$ and $\mathbf{R}_j^{d+1} := \mathbb{R}^d \times \mathbb{R}^+$ otherwise.

Definition 2.7. *Let $s \in \mathbb{R}$. The Sobolev space $H^s(E)$ is the completion of the space $\Gamma_c(E)$ for the H^s norm.*

Remark 2.8. The Sobolev spaces defined in this way are a generalization of the H^s spaces in $\mathbb{R}^{d+1}$, and for $k \in \mathbb{N}$, the H^s norm is equivalent to the norm defined by the square root of $\sum_{j=0}^{k} \|(\nabla^E)^j \cdot \|^2$ (see [9, Theorem 5.7], or [8, Remark 3.6]).

A direct consequence of Definition 2.7 is that the intrinsic Dirac operator on a compact manifold without boundary is essentially self-adjoint and the domain of its closure is the Sobolev space H^1:

Proposition 2.9. *If $(\mathcal{M}, g)$ is a compact Riemannian spin manifold without boundary, $\not{D}^{\mathcal{M}}$ is essentially self-adjoint, and the domain of its closure is $H^1(\Sigma\mathcal{M})$.*

Proof. The Dirac operator is symmetric, and then it is closable. By compactness, there exists $C > 0$ such that $|\mathrm{Scal}^{\mathcal{M}}| \le C$. Moreover, by the Schrödinger-Lichnerowicz formula (Theorem 2.5), the graph norm of $\not{D}^{\mathcal{M}}$ is equivalent to

$$(1 + C)\| \cdot \|^2_{L^2(\mathcal{M})} + \|\not{D}^{\mathcal{M}} \cdot \|^2_{L^2(\mathcal{M})} = \left(1 + C + \frac{\mathrm{Scal}^{\mathcal{M}}}{4}\right) \| \cdot \|^2_{L^2(\mathcal{M})} + \|\nabla^{\mathcal{M}} \cdot \|^2_{L^2(\mathcal{M})}$$

and this last norm is equivalent to the $H^1(\Sigma\mathcal{M})$-norm because of the boundedness of $\mathrm{Scal}^{\mathcal{M}}$. Then, the domain of the closure of $\not{D}^{\mathcal{M}}$ is the completion of $\Gamma_c(\Sigma\mathcal{M})$ for the graph norm, which is exactly $H^1(\Sigma\mathcal{M})$.

The manifold $(\mathcal{M}, g)$ is compact, and then the Dirac operator is essentially self-adjoint in $L^2(\Sigma\mathcal{M})$ [6, Proposition 1.3.5], which concludes the proof. □

By the definition of the Sobolev spaces, one can observe that it is possible to extend the results valid for Euclidean spaces. We state a trace theorem which is a modification of [8, Theorem 3.7], where we add a bound for the L^2-norm of the trace.

Theorem 2.10. *Let $(\mathcal{M}, g)$ be a compact Riemannian manifold with boundary $\partial\mathcal{M}$. Let $(E, \nabla^E, \langle\cdot,\cdot\rangle_E)$ be an Hermitian vector bundle with base $\mathcal{M}$.*

Then, the pointwise restriction operator $\gamma_{\mathcal{M}} : \Gamma_c(E) \to \Gamma_c(E_{|\partial\mathcal{M}})$ extends to a bounded operator from $H^1(E)$ onto $H^{\frac{1}{2}}(E_{|\partial\mathcal{M}})$, and there is a bounded right inverse *to* $\gamma_{\mathcal{M}} : H^1(E) \to H^{\frac{1}{2}}(E_{|\partial\mathcal{M}})$ *denoted by* $\epsilon_{\mathcal{M}}$*, which maps* $\Gamma_c(E_{|\partial\mathcal{M}})$ *into* $\Gamma_c(E)$*. Moreover, there exists* $K > 0$ *such that for any* $\varepsilon \in (0,1)$*,*

$$\|\gamma_{\mathcal{M}}\Psi\|^2_{L^2(\partial\mathcal{M})} \leq K\left(\varepsilon^{\frac{1}{2}}\|\nabla^E\Psi\|^2_{L^2(\mathcal{M})} + \varepsilon^{-\frac{1}{2}}\|\Psi\|^2_{L^2(\mathcal{M})}\right),\ \Psi \in H^1(E).$$

Proof. The proof of the first part of the theorem is done in [8, Theorem 3.7]. We prove the last estimate.

With the notations of (2.21), we denote by J_N the set of all $j \in J$ such that $U_j \cap \partial\mathcal{M} \neq \emptyset$, and there is a constant $C > 0$ and a constant $\widetilde{K} > 0$ given by [7, Theorem 1.5.1.10] such that for any $\varepsilon \in (0,1)$ and for all $\Psi \in H^1(E)$

$$\begin{aligned}\|\gamma_{\mathcal{M}}\Psi\|^2_{L^2(\partial\mathcal{M})} \leq& C\sum_{j\in J_N}\|(\xi_j)_*(h_j\Psi)\circ f_j^{-1}\|^2_{L^2(\mathbb{R}^d\times\{0\},\mathbb{C}^q)}\\ \leq& C\widetilde{K}\sum_{j\in J}[\varepsilon^{\frac{1}{2}}\|(\xi_j)_*(h_j\Psi)\circ f_j^{-1}\|^2_{H^1(\mathbf{R}_j^{d+1},\mathbb{C}^q)}\\ &+\varepsilon^{-\frac{1}{2}}\|(\xi_j)_*(h_j\Psi)\circ f_j^{-1}\|^2_{L^2(\mathbf{R}_j^{d+1},\mathbb{C}^q)}]\\ =& C\widetilde{K}\left(\varepsilon^{\frac{1}{2}}\|\nabla^E\Psi\|^2_{L^2(\mathcal{M})} + \varepsilon^{-\frac{1}{2}}\|\Psi\|^2_{L^2(\mathcal{M})}\right).\end{aligned}$$

□

The Rellich-Kondrachov theorem still holds for the Sobolev spaces on compact manifolds. Consequently, the operators with domain included in the first Sobolev space on a vector bundle with compact base have compact resolvent. We refer to [14, Proposition 3.13] for the following theorem.

Theorem 2.11 (Rellich-Kondrachov-type theorem)**.** *Let* E *be an Hermitian vector bundle over a compact manifold* $\mathcal{M}$*. Then, the inclusion* $H^1(E) \subset L^2(E)$ *is compact.*

We end this section with a direct consequence of Proposition 2.6. We assume that $(\mathcal{M}, g)$ is a compact Riemannian spin manifold of dimension $d+1$ and we take an oriented hypersurface $\mathcal{H}$ of $\mathcal{M}$. We use the notation of section 2.3.

Corollary 2.12. *The isomorphism* ζ *given by Proposition 2.6 is an isomorphism between* $H^1(\Sigma\mathcal{M}_{|\mathcal{H}})$ *and* $H^1(\Sigma\mathcal{H})$ *if* d *is even or* $H^1(\Sigma\mathcal{H}\oplus\Sigma\mathcal{H})$ *if* d *is odd.*

Proof. We define $\|W_{\mathcal{H}}\|_\infty := \sup_{x\in\mathcal{H}}\sup_{X\in T_x\mathcal{H}\setminus\{0\}}\frac{|g(WX,X)|}{g(X,X)} < \infty$. Let $\Psi \in \Gamma_c(\Sigma\mathcal{M}_{|\mathcal{H}})$ and $(e_1,\ldots,e_d)$ a local orthonormal frame of $\mathcal{H}$ at a point $x \in \mathcal{H}$. At this point, one has, using Proposition 2.6, (3),

$$\begin{aligned}|\nabla^{\mathcal{H}}\zeta\Psi|^2 &= \sum_{k=1}^d|\zeta\left(\nabla^{\mathcal{M}}_{e_k}\Psi\right) - \frac{1}{2}W_{\mathcal{H}}e_k\cdot\zeta(\Psi)|^2_{L^2(\mathcal{H})}\\ &\leq 2|\zeta(\overline{\nabla}^{\mathcal{M}}\Psi)|^2_{L^2(\mathcal{H})} + \frac{1}{2}\sum_{k=1}^d|W_{\mathcal{H}}e_k\cdot\nu\cdot\Psi|^2_{L^2(\mathcal{H})}\\ &\leq 2|\overline{\nabla}^{\mathcal{M}}\Psi|^2_{L^2(\mathcal{H})} + \frac{d}{2}\|W_{\mathcal{H}}\|^2_\infty\|\Psi|^2_{L^2(\mathcal{H})}\end{aligned}$$

and then, by integration we obtain

$$\begin{aligned}\|\zeta\Psi\|^2_{H^1(\mathcal{H})} &= \|\zeta\Psi\|^2_{L^2(\mathcal{H})} + \|\nabla^{\mathcal{H}}\zeta\Psi\|^2_{L^2(\mathcal{H})}\\ &\leq \|\Psi\|^2_{L^2(\mathcal{H})} + 2\|\overline{\nabla}^{\mathcal{M}}\Psi\|^2_{L^2(\mathcal{H})} + \frac{d}{2}\|W_{\mathcal{H}}\|^2_\infty\|\Psi\|^2_{L^2(\mathcal{H})}\\ &\leq C_1\|\Psi\|^2_{H^1(\mathcal{H})},\end{aligned}$$

where $C_1 > 0$. The same argument shows that there exists $C_2 > 0$ such that for all $\Psi \in \zeta(\Gamma_c(\Sigma\mathcal{M}_{|\mathcal{H}}))$, one has $\|\zeta^{-1}\Psi\|^2_{H^1(\mathcal{H})} \leq C_2\|\Psi\|^2_{H^1(\mathcal{H})}$. □

3. Definition of the operators

3.1. **The generalized MIT Bag Dirac operator.** In this section, we would like to give a generalization of the MIT Bag Dirac operator in the context of spin manifolds. Our construction will be done by considering the Riemannian product of a manifold $\mathcal{N}$ with $\mathbb{R}$ and interpreting the operator as the extrinsic Dirac operator on the hypersuface $\mathcal{N} \times \{0\}$, modified by a Clifford multiplication with the normal vector field. Since the hypersurface $\mathcal{N}$ is totally geodesic, this operator is the so-called Dirac-Witten operator (see the remark in the proof of [6, Theorem 5.2.3] for example).

We first introduce the context of our study. Let $n \in \mathbb{N}$ and let $(\mathcal{N}, g)$ be a n-dimensional smooth Riemannian manifold which is spin and complete.

Let $(\mathcal{C}, g_{\mathcal{C}}) := (\mathcal{N}, g) \times (\mathbb{R}, \mathrm{d}t^2)$ be the Riemannian product of $\mathcal{N}$ and $\mathbb{R}$. We identify $\mathcal{N}$ with $\mathcal{N} \times \{0\}$. Let p_1 be the projection on $\mathcal{N}$ in $\mathcal{C}$. We endow $\mathcal{C}$ with a spin structure as follows: we denote by P the pull-back to $\mathcal{C}$ of the bundle $P_{\mathrm{Spin}_n}\mathcal{N}$ by the projection p_1, and then the extension of P to Spin_{n+1} is a spin structure on $\mathcal{C}$ (see [3, Section 5] for example).

We denote by ν the outer unit normal vector field on $\mathcal{N} \times \{0\}$ in $\mathcal{C}$, i.e. the vector field $(0, \frac{\partial}{\partial t})$. By construction, the Weingarten tensor of $\mathcal{N}$ vanishes, so the mean curvature $H_{\mathcal{N}}$ is zero.

We denote by ι be the isomorphism given by in Proposition 2.6, in the particular case where $\mathcal{M} := \mathcal{C}$ and $\mathcal{H} := \mathcal{N}$. It is important to remark that the spin structure originally defined on $\mathcal{N}$ and the spin structure inherited by $\mathcal{N}$ from the one of $\mathcal{C}$ according to Proposition 2.6 are the same.

Let $\mathcal{K}$ be a submanifold of $\mathcal{N}$ of dimension n, and assume that $\mathcal{K}$ is compact with non-empty boundary $\partial\mathcal{K}$. From these assumptions, we know that $\partial\mathcal{K}$ is oriented. Thus, we denote by

$$\mu : \Sigma\mathcal{N} \to \begin{cases} \Sigma(\partial\mathcal{K}) & \text{if } n \text{ is odd} \\ \Sigma(\partial\mathcal{K}) \oplus \Sigma(\partial\mathcal{K}) & \text{if } n \text{ is even} \end{cases}$$

the isomorphism given by Proposition 2.6 and by $\mathbf{n}$ the unit outer normal vector field over $\partial\mathcal{K}$ viewed as a submanifold of $\mathcal{N}$.

The operators $\mathcal{D}^{\mathcal{N}}$, $\not{D}^{\mathcal{N}}$, $\mathcal{D}^{\partial\mathcal{K}}$ and $\not{D}^{\partial\mathcal{K}}$ defined in (2.11) and (2.18) are essentially self-adjoint [6, Proposition 1.3.5]. We keep the same notation for their closures.

In what follows, we will simply write W for $W_{\partial\mathcal{K}}$ and H for $H_{\partial\mathcal{K}}$.

Let $m \in \mathbb{R}$. To any $\Psi \in \Gamma(\Sigma\mathcal{C}_{|\mathcal{N}})$, we associate an element $\hat{\Psi}_m$ of $\Gamma(\Sigma\mathcal{C})$ defined for $(x,t) \in \mathcal{C}$ by $\hat{\Psi}_m(x,t) = e^{imt}\widetilde{\Psi}(x,t)$ where $\widetilde{\Psi}(x,t)$ is obtained by parallel transport of $\Psi(x)$ along the curves $s \mapsto (x,s)$.

Let $(e_1, \dots, e_n)$ be a local orthonormal frame at $x \in \mathcal{N}$. Then, we compute

$$\begin{aligned}
(\not{D}^{\mathcal{C}}\hat{\Psi}_m)(x) &= \left(\sum_{j=1}^n e_j \cdot \nabla^{\mathcal{C}}_{e_j}\hat{\Psi}_m + i\, m\nu \cdot \hat{\Psi}_m\right)(x,0) \\
&= \left(-\sum_{j=1}^n \nu \cdot \nu \cdot e_j \cdot \nabla^{\mathcal{C}}_{e_j}\Psi\right)(x) + i\, m\nu \cdot \Psi(x) \\
&= \nu \cdot \left(\mathcal{D}^{\mathcal{N}} + im\right)\Psi(x),
\end{aligned}$$

where the extrinsic Dirac operator $\mathcal{D}^{\mathcal{N}}$ is the operator given by the expression (2.19). The operator obtained in the last line is precisely the operator that we want to study, as it can be interpreted as a Dirac operator with a mass.

We remark that the above construction can be done by restricting the domain of the operator to $\mathcal{K}$. We thus introduce the generalized MIT Bag operator

$$\widetilde{A}_m := \nu\cdot\big(\mathcal{D}^{\mathcal{N}} + im\big)\,,\ \mathrm{dom}(\widetilde{A}_m) := \big\{\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}}), i\,\nu\cdot\mathbf{n}\cdot\Psi = \Psi \text{ on } \partial\mathcal{K}\big\}\,. \quad (3.1)$$

Remark 3.1. One can observe that in the case of Euclidean spaces, the expression (3.1) coincides with [13, Equation (1)], which is already a generalization of the MIT Bag Dirac operator in dimension 3 (see [1, Equation 1.1]). Indeed, the only difference comes from the convention on the Clifford multiplication, because in the present text we have the identity $X\cdot X = -|X|^2$.

Remark 3.2. It is easily seen that the operator $\widetilde{A}_m$ is symmetric since ν anti-commutes with $\mathcal{D}^{\mathcal{N}}$ (see [10, Proposition 1] for the general case, or simply remark that ν is parallel in our framework). Since symmetric operators are closable, we denote by A_m its closure.

Actually, the boundary condition imposed in the domain of the operator is not the Lorentzian MIT Bag boundary condition as stated by the physicists [11] because of the Clifford multiplication by ν. However, this is consistent with the boundary conditions imposed in [2], [1] and [13]. To understand this, we can give another interpretation of the operator $\widetilde{A}_m$ which seems more physical, and appears to give an unitarily equivalent operator.

Until the end of this section, we will deal with Clifford algebra and spin structures in the Lorentzian case. We refer to [3, section 2] for a detailed presentation.

One can endow $\mathcal{C}$ with the Lorentzian metric $g - \mathrm{d}t^2$. There is a Spin_0-structure over $\mathcal{C}$ given by the pull-back of the Spin-structure on $\mathcal{N}$ and extending the fiber. One can construct the associated spinor bundle $\Sigma_L\mathcal{C}$, whose Clifford multiplication will be denoted by " $\cdot_L$". Moreover, we write ∇^L for the covariant derivative on $\Sigma_L\mathcal{C}$, and we denote by $\langle\cdot,\cdot\rangle_L$ the Hermitian product on this spinor bundle. We recall that this inner product is not necessarily definite. In this framework, the Dirac operator with a mass on $\Sigma_L\mathcal{C}$ admits the pointwise expression

$$\not{D}^{\mathcal{C}}_L\Psi := i\left(-\nu\cdot_L\nabla^L_\nu\Psi + \sum_{j=1}^n e_j\cdot_L\nabla^L_{e_j}\Psi\right) - m\Psi \quad (3.2)$$

where $(e_1,\ldots,e_n)$ is any orthonormal frame on $\mathcal{N}$ (see [3, section 2]). Consequently, the Dirac equation $\not{D}^{\mathcal{C}}_L\Psi = 0$ is equivalent to

$$i\,\nabla^L_\nu\Psi = i\sum_{j=1}^n \nu\cdot_L e_j\cdot_L\nabla^L_{e_j}\Psi - m\,\nu\cdot_L\Psi. \quad (3.3)$$

Now, if we take $\Psi(x,t) = e^{i\omega t}\phi(x)$ for all $(x,t)\in\mathcal{C}$, where ϕ is parallel along the time lines, we arrive at

$$\omega\phi = -i\sum_{j=1}^n \nu\cdot_L e_j\cdot_L\nabla^L_{e_j}\phi + m\,\nu\cdot_L\phi. \quad (3.4)$$

We have the counterpart of Proposition 2.6 for the Lorentzian case. Namely, the spinor bundle $\Sigma_L\mathcal{C}$ can be identified to one or two copies of $\Sigma\mathcal{N}$ as in the Riemannian case.

Proposition 3.3. *There is an isomorphism ι_L from $\Sigma_L\mathcal{C}_{|\mathcal{N}}$ into $\Sigma\mathcal{N}$ if n is even and into $\Sigma\mathcal{N}\oplus\Sigma\mathcal{N}$ if n is odd such that:*

- $\iota_L(-i\, X \cdot_L \nu \cdot_L \Psi) = X \cdot \iota_L \Psi$ *for all* $X \in T\mathcal{N}$ *and* $\Psi \in \Sigma_L \mathcal{C}$,
- $\iota_L \nu \cdot_L = \omega_n^{\mathbb{C}} \cdot \iota_L$ *when* n *is even, and* $\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$ *when* n *is odd.*
- $\langle \iota_L \Psi, \iota_L \Phi \rangle = \langle \Psi, \nu \cdot_L \Phi \rangle_L$ *for all* $\Phi, \Psi \in \Sigma_L \mathcal{C}_{|\mathcal{N}}$,
- $\iota_L \nabla^L_X \Psi = \nabla^{\mathcal{N}}_X \iota_L \Psi$ *for* $X \in T\mathcal{N}$ *and* $\Psi \in \Sigma_L \mathcal{C}_{|\mathcal{N}}$.

Proof. We recall that the notations for Clifford algebras were introduced in Section 2.2.

Consider the space $\mathbb{R}^{n,1}$ endowed with the Lorentzian quadratic form of signature $(n, 1)$ and let $(e_1, \dots, e_{n+1})$ be the canonical basis of $\mathbb{R}^{n,1}$, so that e_{n+1} is timelike. The Clifford algebra over this Lorentzian space is denoted by $\mathbb{C}\mathrm{l}_{n,1}$. We turn the representation $(\rho_{n+1}, \Sigma_{n+1})$ into a complex representation of $\mathbb{C}\mathrm{l}_{n,1}$ $(\rho_{n,1}, \Sigma_{n+1})$ by setting

$$\rho_{n,1}(e_i) := \rho_{n+1}(e_i) \text{ for } 1 \le i \le n, \text{ and } \rho_{n,1}(e_{n+1}) := i\, \rho_{n+1}(e_{n+1}).$$

We remark that when n is even, $i^{\frac{n}{2}} \rho_{n,1}(e_1 \cdot \ldots \cdot e_{n+1})$ acts as the identity.

Following [3, section 2], the Hermitian product $\langle \cdot, \cdot \rangle_L$ on Σ_{n+1} for the Lorentzian structure is defined for all $\psi, \phi \in \Sigma_{n+1}$ by

$$\langle \psi, \phi \rangle_L := \langle \psi, \rho_{n,1}(e_{n+1}) \phi \rangle$$

where $\langle \cdot, \cdot \rangle$ is the natural Spin_{n+1}-invariant Hermitian product on Σ_{n+1}.

One can define a representation ρ of $\mathbb{C}\mathrm{l}_n$ over the space Σ_{n+1} by

$$\rho(x) = -i\, \rho_{n,1}(x \cdot e_{n+1}) \quad \text{for all } x \in \mathbb{R}^n.$$

For n even, this representation is equivalent to (ρ_n, Σ_n), so we have an isomorphism $U : \Sigma_{n+1} \to \Sigma_n$ such that $\rho_n U = U \rho$. Moreover, since $i^{\frac{n}{2}} \rho_{n,1}(e_1 \cdot \ldots \cdot e_{n+1})$ acts as the identity on Σ_{n+1}, an easy computation gives $U \rho_{n,1}(e_{n+1}) U^{-1} = \rho_n(\omega_n^{\mathbb{C}})$.

We still denote by $\langle \cdot, \cdot \rangle$ the Hermitian product on Σ_n and we remark that U can be chosen unitary for this inner product. Thus, for all $\psi, \phi \in \Sigma_{n+1}$ one has

$$\langle U\psi, U\phi \rangle = \langle \psi, \phi \rangle = \langle \psi, \rho_{n,1}(e_{n+1})^2 \phi \rangle = \langle \psi, \rho_{n,1}(e_{n+1}) \phi \rangle_L.$$

For n odd, the restriction of ρ to Σ^+_{n+1} is equivalent to (ρ_n, Σ_n), so we have an isomorphism $U_0 : \Sigma^+_{n+1} \to \Sigma_n$ such that $\rho_n U_0 = U_0 \rho$. In addition, $\rho_{n,1}(e_{n+1})$ is an isomorphism from $\Sigma^{\pm}_{n+1}$ into $\Sigma^{\mp}_{n+1}$, so we set

$$U : \Sigma_{n+1} = \Sigma^+_{n+1} \oplus \Sigma^-_{n+1} \to \Sigma_n \oplus \Sigma_n,\ U := (U_0 \oplus U_0)(\mathrm{Id} \oplus \rho_{n,1}(e_{n+1})).$$

Easy computations give $U\rho(x)U^{-1} = \rho_n(x) \oplus -\rho_n(x)$ for all $x \in \mathbb{R}^n \subset \mathbb{R}^{n+1}$ and $U\rho_{n,1}(x)U^{-1}(\psi_1, \psi_2) = (\psi_2, \psi_1)$ for all $(\psi_1, \psi_2) \in \Sigma_n \oplus \Sigma_n$.

The Hermitian product on Σ_n extends to $\Sigma_n \oplus \Sigma_n$ and this extension is still denoted by $\langle \cdot, \cdot \rangle$. The isomorphism U can be chosen unitary for this inner product, and one has for all $\psi, \phi \in \Sigma_{n+1}$

$$\langle U\psi, U\phi \rangle = \langle \psi, \phi \rangle = \langle \psi, \rho^2_{n,1} \phi \rangle = \langle \psi, \rho_{n,1} \phi \rangle_L.$$

Now, all these properties transport to manifolds by identifying e_{n+1} with ν since the Spin_0 structure over $\mathcal{C}$ is defined by pull-back of the Spin structure over $\mathcal{N}$.

The last point follows from the explicit formula of the covariant derivative on spinor [3, formula 2.5] and the fact that $\mathcal{N}$ is totally geodesic in $\mathcal{C}$. □

We infer that $\Sigma\mathcal{C}_{|\mathcal{N}}$ and $\Sigma_L\mathcal{C}_{|\mathcal{N}}$ are both isomorphic to $\Sigma\mathcal{N}$ if n is even and to $\Sigma\mathcal{N} \oplus \Sigma\mathcal{N}$ if n is odd, so we can identify them via the isomorphism $\iota^{-1}\iota_L$.

Corollary 3.4. *The isomorphism* $\iota^{-1}\iota_L : \Sigma_L\mathcal{C} \to \Sigma\mathcal{C}$ *satisfies:*

- $\langle(\iota^{-1}\iota_L)\Psi, i\,\nu\cdot(\iota^{-1}\iota_L)\Phi\rangle = \langle\Psi,\Phi\rangle_L$ *for all* $\Psi,\Phi\in\Sigma_L\mathcal{C}$.
- $\nabla^{\mathcal{C}}_X(\iota^{-1}\iota_L)\Psi = (\iota^{-1}\iota_L)\nabla^L_X\Psi$ *for all* $X\in T\mathcal{N}$ *and* $\Psi\in\Gamma(\Sigma_L\mathcal{C})$.
- $X\cdot(\iota^{-1}\iota_L)\Psi = (\iota^{-1}\iota_L)(X\cdot_L\Psi)$ *for all* $X\in T\mathcal{N}$
- $i\,\nu\cdot(\iota^{-1}\iota_L) = (\iota^{-1}\iota_L)\nu\cdot_L$.

Under the identification of Corollary 3.4, Equation (3.4) reads

$$\omega\phi = \sum_{j=1}^{n}\nu\cdot e_j\cdot\nabla^{\mathcal{C}}_{e_j}\phi + i\,m\,\nu\cdot\phi = (-\mathcal{D}^{\mathcal{N}} + i\,m\,\nu\cdot)\phi. \tag{3.5}$$

This is an eigenvalue equation, and it is now natural to look at the spectrum of the operator defined by the right-hand side. We just need to add a boundary condition to define a generalized MIT Bag operator. Since the physical condition imposed in [11] is that the flux $\langle\phi,\mathbf{n}\cdot_L\phi\rangle_L$ of the quantum field vanishes at the boundary, we consider the MIT Bag boundary condition $i\,\mathbf{n}\cdot\phi=\phi$. One has

$$-\langle\phi,\phi\rangle_L = \langle\phi,-i\,\mathbf{n}\cdot_L\phi\rangle_L = \langle i\,\mathbf{n}\cdot_L\phi,\phi\rangle_L = \langle\phi,\phi\rangle_L,$$

and we conclude that $\langle\phi,-i\,\mathbf{n}\cdot_L\phi\rangle_L=0$, so the condition of the physical model is verified. We can now define another generalization of the MIT Bag Dirac operator by

$$\widehat{A}_m := \mathcal{D}^{\mathcal{N}} + i\,m\,\nu\cdot,\quad \mathrm{dom}(\widehat{A}_m) = \left\{\Psi\in\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}}), i\,\mathbf{n}\cdot\Psi=\Psi\right\}. \tag{3.6}$$

The change of sign for the mass in (3.6) compared to (3.5) comes from the fact that we consider a model where $m\to-\infty$ (see [2, section 1.3.3] for more explanations).

We have now two candidates for the generalization of the MIT Bag Dirac operator. However, one can remark that the difference between $\widetilde{A}_m$ and $\widehat{A}_m$ is only a matter of how the Clifford product is defined, and the two operators are unitarily equivalent.

Proposition 3.5. *The operators* $\widetilde{A}_m$ *and* $\widehat{A}_m$ *are unitarily equivalent via a* $\nabla^{\mathcal{C}}$*-parallel operator.*

Proof. We define a new Clifford representation on the vector bundle $\Sigma\mathcal{C}$ by setting $X*\Psi := \nu\cdot X\cdot\Psi$ and $\nu*\Psi := \nu\cdot\Psi$ for $X\in T\mathcal{N}$ and $\Psi\in\Sigma\mathcal{C}$. This new product still satisfies the Clifford conditions in each fiber, and when n is even the complex volume form $\omega^{\mathbb{C}}_{n+1}$ acts as

$$\begin{aligned}\omega^{\mathbb{C}}_{n+1}*\Psi &= i^{\lfloor\frac{n+2}{2}\rfloor}e_1*\ldots*e_n*\nu*\Psi\\ &= i^{\lfloor\frac{n+2}{2}\rfloor}(\nu\cdot e_1)\cdot\ldots(\nu\cdot e_n)\cdot\nu\cdot\Psi = \omega^{\mathbb{C}}_{n+1}\cdot\Psi,\end{aligned}$$

where $(e_1,\ldots,e_n)$ is a direct orthonormal basis of $T\mathcal{N}$. It follows by the general theory of Clifford representations that there is an unitary isomorphism $U:\Sigma\mathcal{C}\to\Sigma\mathcal{C}$ such that $X\cdot U\Psi = U(X*\Psi)$ for all $X\in T\mathcal{C}$ and $\Psi\in\Sigma\mathcal{C}$.

Actually, one can give such an isomorphism explicitly. If n is even, we use the decomposition $\Sigma\mathcal{N} = \Sigma^+\mathcal{N}\oplus\Sigma^-\mathcal{N}$ (see [5, Proposition 1.32]) and the pointwise identification $\Sigma\mathcal{C}_{|(x,t)}\cong\Sigma\mathcal{N}_{|x}$ for all $(x,t)\in\mathcal{C}$ given by Proposition 2.6. Under this identification, one has

$$\nu\cdot(\Psi^+,\Psi^-) = (-i\,\Psi^+, i\,\Psi^-),\ X\cdot(\Psi^+,\Psi^-) = i\,(-X\cdot\Psi^-, X\cdot\Psi^+)\text{ for all } X\in T\mathcal{N},$$

and we deduce that U can be defined by

$$U(\Psi^+,\Psi^-) := (\Psi^+,-i\,\Psi^-).$$

Indeed, one has for any $X\in T\mathcal{N}$

$$\begin{aligned}U(X*(\Psi^+,\Psi^-)) &= U(\nu\cdot X\cdot(\Psi^+,\Psi^-)) = U(i\,\nu\cdot(-X\cdot\Psi^-, X\cdot\Psi^+))\\ &= -U(X\cdot\Psi^-, X\cdot\Psi^+) = (-X\cdot\Psi^-, i\,X\cdot\Psi^+)\end{aligned}$$

and

$$X \cdot U(\Psi^+, \Psi^-) = X \cdot (\Psi^+, -i\,\Psi^-) = (-X \cdot \Psi^-, i\,X \cdot \Psi^+),$$

thus $U(X * (\Psi^+, \Psi^-)) = X \cdot U(\Psi^+, \Psi^-)$. In addition, U obviously commutes with ν.

In the case where n is odd, one has the pointwise identification $\Sigma\mathcal{C}_{|(x,t)} \cong \Sigma\mathcal{N}_{|x} \oplus \Sigma\mathcal{N}_{|x}$ for all $(x,t) \in \mathcal{C}$ and under this identification,

$$\nu \cdot (\Psi_1, \Psi_2) = (-i\,\Psi_2, -i\,\Psi_1),\ X \cdot (\Psi_1, \Psi_2) = i\,(X \cdot \Psi_2, -X \cdot \Psi_1) \text{ for all } X \in T\mathcal{N},$$

It follows that U can be defined by

$$U(\Psi_1, \Psi_2) := \frac{1}{\sqrt{2}}(\Psi_1 + i\,\Psi_2, i\,\Psi_1 + \Psi_2).$$

Indeed, for all $X \in T\mathcal{N}$ one has

$$\begin{aligned} U(X * (\Psi_1, \Psi_2)) &= i\,U(\nu \cdot (X \cdot \Psi_2, -X \cdot \Psi_1)) = U(-X \cdot \Psi_1, X \cdot \Psi_2) \\ &= \frac{1}{\sqrt{2}}(X \cdot (-\Psi_1 + i\,\Psi_2), X \cdot (-i\,\Psi_1 + \Psi_2)) \end{aligned}$$

and

$$\begin{aligned} X \cdot U(\Psi_1, \Psi_2) &= \frac{1}{\sqrt{2}} X \cdot (\Psi_1 + i\,\Psi_2, i\,\Psi_1 + \Psi_2) \\ &= \frac{1}{\sqrt{2}}(X \cdot (-\Psi_1 + i\,\Psi_2), X \cdot (-i\,\Psi_1 + \Psi_2)), \end{aligned}$$

thus $X \cdot U(\Psi_1, \Psi_2) = U(X * (\Psi_1, \Psi_2))$. Again, ν commutes with U.

In both cases, U is parallel with respect to $\nabla^{\mathcal{C}}$ and we remark that $U(\mathrm{dom}(\widetilde{A}_m)) = \mathrm{dom}(\widehat{A}_m)$. We deduce from these considerations that

$$U^* \widehat{A}_m U \Psi = \widetilde{A}_m \Psi \qquad \text{for all } \Psi \in \mathrm{dom}(\widetilde{A}_m), \tag{3.7}$$

which is the statement we wanted to prove. □

Remark 3.6. The key point in Proposition 3.5 is of course that $H_{\mathcal{N}} = 0$. It is only under this condition that the isomorphism U is parallel with respect to $\nabla^{\mathcal{C}}$. Thus, it is equivalent to study any of the two operators, but we wanted to insist on the physical meaning of $\widehat{A}_m$.

3.2. **The two-masses Dirac operator.** We introduce now an operator that can be interpreted as a Dirac operator on $\mathcal{N}$ with two masses in the two separated regions $\mathcal{K}$ and $\mathcal{K}^c$. The interest of this operator, as we will show later, is that when the mass in $\mathcal{K}^c$ goes to infinity, its spectrum converges to the spectrum of the MIT Bag Dirac operator.

Let $m, M \in \mathbb{R}$. We define the operator $\widetilde{B}_{m,M}$ by

$$\widetilde{B}_{m,M} := \nu \cdot \mathcal{D}^{\mathcal{N}} + i\,(m\mathbf{1}_{\mathcal{K}} + M\mathbf{1}_{\mathcal{K}^c})\nu\cdot,\ \mathrm{dom}(\widetilde{B}_{m,M}) := \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}}). \tag{3.8}$$

Since the Clifford multiplication by ν is an endomorphism of $\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$, the range of this operator is included in $\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$.

Until the end of this subsection, we make a differentiation between the Dirac operators on complete manifolds and their closures.

The operator $\widetilde{B}_{m,M}$ is symmetric because ν anti-commutes with $\mathcal{D}^{\mathcal{N}}$ [10, Proposition 1] and by Corollary 4.2 below. Since the manifold $\mathcal{N}$ is complete by assumption, the intrinsic Dirac operator on $\mathcal{N}$ is essentially self-adjoint in $L^2(\Sigma\mathcal{C}_{|\mathcal{N}})$ [6, Proposition 1.3.5]. Moreover, (2.18) gives that $\mathcal{D}^{\mathcal{N}}$ is unitarily equivalent to $\not{D}^{\mathcal{N}}$ if n is even and $\not{D}^{\mathcal{N}} \oplus -\not{D}^{\mathcal{N}}$ if n is odd, and the isomorphism ι sends $\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$ into

$\Gamma_c(\Sigma\mathcal{N})$. Thus, $\mathcal{D}^{\mathcal{N}}$ is essentially self-adjoint, and it is easy to see that its closure still anti-commutes with ν. Using the fact that the Clifford multiplication by ν is an unitary isomorphism in $L^2(\Sigma\mathcal{C}_{|\mathcal{N}})$ we have

$$(\nu \cdot \overline{\mathcal{D}^{\mathcal{N}}})^* = -\overline{\mathcal{D}^{\mathcal{N}}}\nu\cdot = \nu \cdot \overline{\mathcal{D}^{\mathcal{N}}}, \quad \text{and} \quad \overline{\nu \cdot \mathcal{D}^{\mathcal{N}}} = \nu \cdot \overline{\mathcal{D}^{\mathcal{N}}},$$

so $\overline{\nu \cdot \mathcal{D}^{\mathcal{N}}}$ is self-adjoint.

We conclude that $\widetilde{B}_{m,M}$ is essentially self-adjoint because the potential is a bounded self-adjoint operator. We define the self-adjoint operator $B_{m,M}$ as the closure of $\widetilde{B}_{m,M}$.

4. Sesquilinear forms for the operators with mass

An important tool for the asymptotic analysis will be the sesquilinear forms associated with the square of the operators. We begin this section by recalling some useful formulas involving the Dirac operator. After that, we compute the sesquilinear forms for the operators A_m^2 and $B_{m,M}^2$ and we show that A_m is self-adjoint. We end this section with the study of a model operator which appears naturally in the asymptotic analysis, and we prove that it is unitarily equivalent to the square of the Dirac operator on $\partial\mathcal{K}$.

4.1. Integration by parts with the Dirac operator.

We first recall the well-known result:

Lemma 4.1. *Let $\Psi, \Phi \in \Gamma_c(\Sigma\mathcal{N})$. Then, one has the pointwise equality*

$$\langle \not{D}^{\mathcal{N}}\Psi, \Phi\rangle = -\operatorname{div} V + \langle \Psi, \not{D}^{\mathcal{N}}\Phi\rangle$$

where V is the complex vector field on $\mathcal{N}$ defined by

$$g(V, X) := \langle \Psi, X \cdot \Phi\rangle\,, \ \forall X \in T\mathcal{N}.$$

Proof. Let $\Psi, \Phi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$, $x \in \mathcal{N}$ and let $(e_1, \dots, e_n)$ be a normal coordinate system at x for $\nabla^{\mathcal{N}}$, i.e. $\nabla^{\mathcal{N}}_{e_i} e_j(x) = 0$ for all $i, j \in \{1, \dots, n\}$. One has at x,

$$\left\langle \not{D}^{\mathcal{N}}\Psi, \Phi\right\rangle = \langle \sum_{j=1}^{n} e_j \cdot \nabla^{\mathcal{N}}_{e_j}\Psi, \Phi\rangle.$$

On the other hand, for all $j \in \{1, \dots, n\}$,

$$\begin{aligned}\left\langle e_j \cdot \nabla^{\mathcal{N}}_{e_j}\Psi, \Phi\right\rangle &= -\left\langle \nabla^{\mathcal{N}}_{e_j}\Psi, e_j \cdot \Phi\right\rangle \\ &= -e_j \left\langle \Psi, e_j \cdot \Phi\right\rangle + \left\langle \Psi, \nabla^{\mathcal{N}}_{e_j}(e_j \cdot \Phi)\right\rangle.\end{aligned}$$

Thus, $\langle \not{D}^{\mathcal{N}}\Psi, \Phi\rangle = -\sum_{j=1}^{n} e_j \langle \Psi, e_j \cdot \Phi\rangle + \langle \Psi, \not{D}^{\mathcal{N}}\Psi\rangle$. We recognize in the first term of this last sum the divergence of a complex vector field. To see this, we introduce $V \in \Gamma(T\mathcal{N})$ as in the statement of the lemma. Then, we have at the point x

$$\begin{aligned}\operatorname{div} V &= \sum_{j=1}^{n} g(\nabla^{\mathcal{N}}_{e_j} V, e_j) = \sum_{j=1}^{n} e_j\, g(V, e_j) - g(V, \nabla^{\mathcal{N}}_{e_j} e_j) \\ &= \sum_{j=1}^{n} e_j\, g(V, e_j) = \sum_{j=1}^{n} e_j \left\langle \Psi, e_j \cdot \Psi\right\rangle. \qquad \square\end{aligned}$$

A direct corollary is an integral version of Lemma 4.1.

Corollary 4.2. *One has*

$$\langle \not{D}^{\mathcal{N}}\Psi, \Phi\rangle_{L^2(\mathcal{K})} = \langle \Psi, \not{D}^{\mathcal{N}}\Phi\rangle_{L^2(\mathcal{K})} - \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n}\cdot\Phi\rangle v_{\partial\mathcal{K}}$$

for all $\Psi, \Phi \in H^1(\Sigma\mathcal{K})$*, and*

$$\langle \mathcal{D}^{\mathcal{N}}\Psi, \Phi\rangle_{L^2(\mathcal{K})} = \langle \Psi, \mathcal{D}^{\mathcal{N}}\Phi\rangle_{L^2(\mathcal{K})} - \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n}\cdot\nu\cdot\Phi\rangle v_{\partial\mathcal{K}}$$

for all $\Psi, \Phi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$.

Proof. The first identity is proved by integrating the formula obtained in Lemma 4.1 for $\Psi, \Phi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}})$ and using the divergence theorem. We conclude by density. For the second one, we use the definition of the extrinsic Dirac operator given by (2.18) together with the first equation. □

Finally, we obtain an integration by parts formula for the Dirac operator with a mass defined in the previous section.

Corollary 4.3. *For any* $\Psi, \Phi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$*, one has*

$$\big\langle \nu\cdot(\mathcal{D}^{\mathcal{N}} + i\,m)\Psi, \Phi\big\rangle_{L^2(\mathcal{K})} = \big\langle \Psi, \nu\cdot(\mathcal{D}^{\mathcal{N}} + i\,m)\Phi\big\rangle_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n}\cdot\Phi\rangle\, v_{\partial\mathcal{K}}.$$

Proof. Let $\Psi, \Phi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$, using Corollary 4.2 one has

$$\begin{aligned}
\big\langle \nu\cdot(\mathcal{D}^{\mathcal{N}} + i\,m)\Psi, \Phi\big\rangle_{L^2(\mathcal{K})} &= -\big\langle (\mathcal{D}^{\mathcal{N}} + i\,m)\Psi, \nu\cdot\Phi\big\rangle_{L^2(\mathcal{K})} \\
&= -\big\langle \Psi, (\mathcal{D}^{\mathcal{N}} - i\,m)(\nu\cdot\Phi)\big\rangle_{L^2(\mathcal{K})} \\
&\quad - \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n}\cdot\nu\cdot\nu\cdot\Phi\rangle\, v_{\partial\mathcal{K}} \\
&= \big\langle \Psi, \nu\cdot(\mathcal{D}^{\mathcal{N}} + i\,m)\Phi\big\rangle_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n}\cdot\Phi\rangle v_{\partial\mathcal{K}}. \ \square
\end{aligned}$$

4.2. **Sesquilinear form for $\widetilde{A}_m^2$ and essential self-adjointness.** In this section we show that the operator $\widetilde{A}_m$ is essentially self-adjoint, and the domain of its closure is an extension of $\mathrm{dom}(\widetilde{A}_m)$ to the space $H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$. The proof of this fact is done in two steps. First, we compute the sesquilinear form of $\widetilde{A}_m^2$ to get the domain of the closure and secondly, we show the essential self-adjointness following the analysis of [8].

From Corollary 4.3, we see that $\widetilde{A}_m$ is symmetric since for any $\Psi, \Phi \in \mathrm{dom}(\widetilde{A}_m)$ one has

$$\langle \Psi, \mathbf{n}\cdot\Phi\rangle = \langle \Psi, i\,\nu\cdot\Phi\rangle = \langle i\,\nu\cdot\Psi, \Phi\rangle = \langle \mathbf{n}\cdot\Psi, \Phi\rangle = -\langle \Psi, \mathbf{n}\cdot\Phi\rangle = 0.$$

Proposition 4.4. *For all* $\Psi \in \mathrm{dom}(\widetilde{A}_m)$,

$$\begin{aligned}
\|\widetilde{A}_m\Psi\|^2_{L^2(\mathcal{K})} = &\int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 \right) v_{\mathcal{N}} \\
&+ m^2\|\Psi\|^2_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \left(m - \frac{H}{2} \right) |\Psi|^2 v_{\partial\mathcal{K}}.
\end{aligned}$$

Moreover, the graph norm of $\widetilde{A}_m$ *and the* H^1*-norm are equivalent on* $\mathrm{dom}(\widetilde{A}_m)$.

Proof. We recall that $\mathrm{dom}(\widetilde{A}_m)$ was defined in (3.1). Let $\Psi \in \mathrm{dom}(\widetilde{A}_m)$. With Corollary 4.2 one has

$$\|\widetilde{A}_m\Psi\|^2_{L^2(\mathcal{K})} = \big\langle (\mathcal{D}^{\mathcal{N}} + i\,m)\Psi, (\mathcal{D}^{\mathcal{N}} + i\,m)\Psi\big\rangle_{L^2(\mathcal{K})}$$

$$
\begin{aligned}
&= \|\mathcal{D}^{\mathcal{N}}\Psi\|^2_{L^2(\mathcal{K})} + m^2\|\Psi\|^2_{L^2(\mathcal{K})} + m\left\langle \mathcal{D}^{\mathcal{N}}\Psi, i\,\Psi\right\rangle_{L^2(\mathcal{K})} \\
&\quad + m\left\langle i\,\Psi, \mathcal{D}^{\mathcal{N}}\Psi\right\rangle_{L^2(\mathcal{K})} \\
&= \|\mathcal{D}^{\mathcal{N}}\Psi\|^2_{L^2(\mathcal{K})} + m^2\|\Psi\|^2_{L^2(\mathcal{K})} - m\int_{\partial\mathcal{K}} \langle \Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}} \\
&= \|\mathcal{D}^{\mathcal{N}}\Psi\|^2_{L^2(\mathcal{K})} + m^2\|\Psi\|^2_{L^2(\mathcal{K})} + m\int_{\partial\mathcal{K}} |\Psi|^2 v_{\partial\mathcal{K}},
\end{aligned}
$$

where we used the property $\Psi = i\,\nu\cdot\mathbf{n}\cdot\Psi$ on $\partial\mathcal{K}$.
We consider the operator $\widetilde{\mathcal{D}}^{\partial\mathcal{K}} := \mathcal{D}^{\partial\mathcal{K}}$ if n is even and $\widetilde{\mathcal{D}}^{\partial\mathcal{K}} := \mathcal{D}^{\partial\mathcal{K}} \oplus \mathcal{D}^{\partial\mathcal{K}}$ if n is odd. From [10, Formula (13)] we have for all $\Phi \in \Gamma(\Sigma\mathcal{K})$

$$
\begin{aligned}
\int_{\mathcal{K}} |\not{D}^{\mathcal{N}}\Phi|^2 v_{\mathcal{N}} &= \int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}\Phi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Phi|^2 \right) v_{\mathcal{N}} \\
&\quad + \int_{\partial\mathcal{K}} \left(-\frac{H}{2}|\Phi|^2 - \left\langle \mathcal{D}^{\partial\mathcal{K}}\Phi, \Phi \right\rangle \right) v_{\partial\mathcal{K}}.
\end{aligned}
$$

Using this equation together with the definition of the extrinsic Dirac operator (2.18), one has

$$
\begin{aligned}
\int_{\mathcal{K}} |\mathcal{D}^{\mathcal{N}}\Psi|^2 v_{\mathcal{N}} &= \int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 \right) v_{\mathcal{N}} \\
&\quad + \int_{\partial\mathcal{K}} \left(-\frac{H}{2}|\Psi|^2 + \left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle \right) v_{\partial\mathcal{K}}.
\end{aligned} \tag{4.1}
$$

On the other hand, as $\widetilde{\mathcal{D}}^{\partial\mathcal{K}}$ anti-commutes with the Clifford multiplication by $\mathbf{n}$ [10, Proposition 1],

$$
\begin{aligned}
\left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle &= \left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota(-i\,\mathbf{n}\cdot\nu\cdot\Psi)), \iota\Psi \right\rangle = \left\langle -i\,\widetilde{\mathcal{D}}^{\partial\mathcal{K}}\mathbf{n}\cdot(\iota\Psi), \iota\Psi \right\rangle \\
&= \left\langle i\,\mathbf{n}\cdot\widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle = \left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), i\,\mathbf{n}\cdot(\iota\Psi) \right\rangle \\
&= \left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), -\iota(i\,\nu\cdot\mathbf{n}\cdot\Psi) \right\rangle = -\left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle
\end{aligned}
$$

and we deduce that $\left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle = 0$.
Finally, using this equation together with (4.1), we get

$$
\begin{aligned}
\|\widetilde{A}_m\Psi\|^2_{L^2(\mathcal{K})} &= \int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 \right) v_{\mathcal{N}} \\
&\quad + m^2\|\Psi\|^2_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \left(m - \frac{H}{2} \right) |\Psi|^2 v_{\partial\mathcal{K}}.
\end{aligned}
$$

It remains to prove the equivalence of the norms. As $\mathcal{K}$ is a compact manifold with boundary, Theorem 2.10 applies and there is $C_1 > 0$ such that for all $\Psi \in \mathrm{dom}(\widetilde{A}_m)$,

$$
\begin{aligned}
\|\Psi\|^2_{L^2(\mathcal{K})} + \|\widetilde{A}_m\Psi\|^2_{L^2(\mathcal{K})} &= \|\iota\Psi\|^2_{L^2(\mathcal{K})} + \int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\iota\Psi|^2 \right) v_{\mathcal{N}} \\
&\quad + m^2\|\iota\Psi\|^2_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \left(m - \frac{H}{2} \right) |\iota\Psi|^2 v_{\partial\mathcal{K}} \\
&\leq C_1\|\iota\Psi\|^2_{L^2(\mathcal{K})} + \|\nabla^{\mathcal{N}}(\iota\Psi)\|^2_{L^2(\mathcal{K})} + C_1\|\iota\Psi\|^2_{H^1(\mathcal{K})} \\
&\leq 2(C_1+1)\|\iota\Psi\|^2_{H^1(\mathcal{K})}.
\end{aligned}
$$

Moreover, using Theorem 2.10 with ε small enough, there exists a constant $C_2 > 0$ such that

$$\|\Psi\|^2_{L^2(\mathcal{K})} + \|\widetilde{A}_m \Psi\|^2_{L^2(\mathcal{K})} \geq C_2 \|\iota\Psi\|^2_{H^1(\mathcal{K})}.$$

Thus, the graph norm is equivalent to the $H^1(\iota(\Sigma\mathcal{C}_{|\mathcal{K}}))$ norm, which is equivalent to the $H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$ norm thanks to Corollary 2.12. □

We now show that A_m is self-adjoint. For this purpose, it is sufficient to prove that $\nu \cdot \mathcal{D}^{\mathcal{N}}$ is essentially self-adjoint on $\mathrm{dom}(\widetilde{A}_m)$ because the potential is a bounded operator. From Proposition 2.6 and (2.18), one has

$$\iota^{-1}(\nu \cdot \mathcal{D}^{\mathcal{N}})\iota = -i\,\omega_n^{\mathbb{C}} \cdot \not{D}^{\mathcal{N}} \quad \text{if } n \text{ is even}, \tag{4.2}$$

and

$$\iota^{-1}(\nu \cdot \mathcal{D}^{\mathcal{N}})\iota = -i \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} (\not{D}^{\mathcal{N}} \oplus -\not{D}^{\mathcal{N}}) \quad \text{if } n \text{ is odd}. \tag{4.3}$$

Having these considerations in mind, we define

$$A := \not{D}^{\mathcal{N}} \text{ if } n \text{ is even, } A := \not{D}^{\mathcal{N}} \oplus -\not{D}^{\mathcal{N}} \text{ is } n \text{ is odd}, \tag{4.4}$$

and

$$T := -i\,\omega_n^{\mathbb{C}} \cdot \text{ if } n \text{ is even, } T := -i \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \text{ if } n \text{ is odd}. \tag{4.5}$$

We remark that T is an unitary skew-Hermitian operator which anti-commutes with A.

Consider the operators

$$P_\pm := \frac{1 \pm i\,\mathbf{n}\cdot}{2} \text{ on } \iota(\Sigma\mathcal{C}_{|\mathcal{K}}), \text{ and } \mathcal{P}_\pm := \frac{1 \pm i\,\nu \cdot \mathbf{n}\cdot}{2} \text{ on } \Sigma\mathcal{C}_{|\mathcal{K}}. \tag{4.6}$$

Let $A_\pm$ be the restriction of A to the domain $\{\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}}),\, P_\pm\Psi = 0\}$. Then, the operator $\nu \cdot \mathcal{D}^{\mathcal{N}}$ with domain $\mathrm{dom}(\widetilde{A}_m)$ is unitarily equivalent to TA_+ for any parity of n.

Lemma 4.5. *For any $s \in \mathbb{R}$, $P_\pm$ and $\mathcal{P}_\pm$ define bounded operators from H^s to itself.*

Proof. The proof is straightforward, see [8, Lemma 5.1 (ii)]. □

Theorem 4.6. *The operator A_m is self-adjoint, and the equality in Proposition 4.4 holds for any $\Psi \in \mathrm{dom}(A_m) = \{\Psi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}}), \mathcal{P}_-\Psi = 0\}$.*

Proof. We first prove that $E := \{\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}}), \mathcal{P}_-\Psi = 0\}$ is dense in $F := \{\Psi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}}), \mathcal{P}_-\Psi = 0\}$ for the H^1 norm. Let $\Psi \in F$. There exists a sequence $(\Psi_j)_{j\in\mathbb{N}}$ in $\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}})$ converging to Ψ in the H^1 norm. Let $\Phi_j := \Psi_j - \epsilon_{\mathcal{K}}\mathcal{P}_-\gamma_{\mathcal{K}}\Psi_j$, where we recall that $\epsilon_{\mathcal{K}}$ is the extension operator defined in Theorem 2.10. One has $\mathcal{P}_-\gamma_{\mathcal{K}}\Phi_j = 0$ and from Theorem 2.10 and Lemma 4.5 we obtain

$$\begin{aligned}
\|\Phi_j - \Psi\|_{H^1(\mathcal{K})} &= \|\Psi_j - \epsilon_{\mathcal{K}}\mathcal{P}_-\gamma_{\mathcal{K}}\Psi_j - \Psi\|_{H^1(\mathcal{K})} \\
&\leq \|\Psi_j - \Psi\|_{H^1(\mathcal{K})} + \|\epsilon_{\mathcal{K}}\mathcal{P}_-\gamma_{\mathcal{K}}\Psi_j\|_{H^1(\mathcal{K})} \\
&\leq \|\Psi_j - \Psi\|_{H^1(\mathcal{K})} + C_1\|\mathcal{P}_-\gamma_{\mathcal{K}}\Psi_j - \mathcal{P}_-\gamma_{\mathcal{K}}\Psi\|_{H^{\frac{1}{2}}(\mathcal{K})} \\
&\leq C_2\|\Psi_j - \Psi\|_{H^1(\mathcal{K})} \underset{j\to+\infty}{\longrightarrow} 0
\end{aligned}$$

with $C_1, C_2 > 0$.

Thus, E is dense in F, and as the graph norm of $\widetilde{A}_m$ and the H^1 norm are equivalent on E by Proposition 4.4. We conclude that $F \subset \mathrm{dom}(A_m)$. By density, the expression of Proposition 4.4 holds for any $\Psi \in F$, and the graph norm and the H^1 norm are still equivalent on F. But F is closed for the H^1 norm, so we deduce that $F = \mathrm{dom}(A_m)$, and using Corollary 2.12, we have $\mathrm{dom}(\overline{A_+}) = \{\Psi \in H^1(\iota\Sigma\mathcal{C}_{|\mathcal{K}}), P_+\Psi = 0\}$. This means that $\overline{A_+}$ is exactly one or two copies of the operator D_+ (up to a sign) studied in [8, Lemma 5.1].

By the same method, we can show that $\mathrm{dom}(\overline{A_-}) = \{\Psi \in H^1(\iota\Sigma\mathcal{C}_{|\mathcal{K}}), P_-\Psi = 0\}$ and $\overline{A_-}$ is one or two copies of the operator D_- (up to a sign) studied in [8, Lemma 5.1].

Finally, [8, Lemma 5.1 (v)] gives us $(\overline{A_\pm})^* = \overline{A_\mp}$, and we deduce that

$$(T\overline{A_+})^* = -(\overline{A_+})^* T = -\overline{A_-}T = T\overline{A_+}.$$

Consequently, TA_- is self-adjoint, and so is A_m by unitary equivalence. □

4.3. **Sesquilinear form for $B^2_{m,M}$.** As for the operator A_m, we compute the sesquilinear form of the operator $B^2_{m,M}$ defined in section 3.2. As a consequence of the Schrödinger-Lichnerowicz formula, we can first compute the square of the extrinsic Dirac operator acting on smooth sections with compact support in $\mathcal{N}$.

Lemma 4.7. *Let $\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$. Then*

$$\|\nu \cdot \left(\mathcal{D}^{\mathcal{N}} + i\,m\right)\Psi\|^2_{L^2(\mathcal{N})} = \int_{\mathcal{N}} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 + m^2|\Psi|^2\right] v_{\mathcal{N}}.$$

Proof. Let $\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$. One has

$$\begin{aligned}
\|\nu \cdot \left(\mathcal{D}^{\mathcal{N}} + i\,m\right)\Psi\|^2_{L^2(\mathcal{N})} &= \left\langle \nu \cdot \left(\mathcal{D}^{\mathcal{N}} + i\,m\right)\Psi, \nu \cdot \left(\mathcal{D}^{\mathcal{N}} + im\right)\Psi\right\rangle_{L^2(\mathcal{N})} \\
&= \left\langle \left(\mathcal{D}^{\mathcal{N}} + im\right)\Psi, \left(\mathcal{D}^{\mathcal{N}} + im\right)\Psi\right\rangle_{L^2(\mathcal{N})} \\
&= \left\langle \mathcal{D}^{\mathcal{N}}\Psi, \mathcal{D}^{\mathcal{N}}\Psi\right\rangle_{L^2(\mathcal{N})} + m^2\left\langle \Psi, \Psi\right\rangle_{L^2(\mathcal{N})} \\
&\quad + m\left[\left\langle \mathcal{D}^{\mathcal{N}}\Psi, i\,\Psi\right\rangle_{L^2(\mathcal{N})} + \left\langle i\,\Psi, \mathcal{D}^{\mathcal{N}}\Psi\right\rangle_{L^2(\mathcal{N})}\right].
\end{aligned}$$

Using Lemma 4.1, one has at any point $x \in \mathcal{N}$,

$$\left\langle \mathcal{D}^{\mathcal{N}}\Psi, i\,\Psi\right\rangle + \left\langle i\,\Psi, \mathcal{D}^{\mathcal{N}}\Psi\right\rangle = -\mathrm{div}\ V.$$

By the divergence theorem, the Schrödinger-Lichnerowicz formula (Proposition 2.5) and Equation 2.18, one can integrate over $\mathcal{N}$ to obtain

$$\begin{aligned}
\|\nu \cdot \left(\mathcal{D}^{\mathcal{N}} + i\,m\right)\Psi\|^2_{L^2(\mathcal{N})} &= \left\langle \mathcal{D}^{\mathcal{N}}\Psi, \mathcal{D}^{\mathcal{N}}\Psi\right\rangle_{L^2(\mathcal{N})} + m^2\left\langle \Psi, \Psi\right\rangle_{L^2(\mathcal{N})} \\
&= \int_{\mathcal{N}} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 + m^2|\Psi|^2\right] v_{\mathcal{N}}. \qquad \square
\end{aligned}$$

We can now compute the quadratic form for the operator $B_{m,M}$ by integration over $\mathcal{N}$, and it comes out that its domain is a subspace of the Sobolev space H^1.

Proposition 4.8. *One has $\mathrm{dom}(B_{m,M}) \subset H^1(\Sigma\mathcal{C}_{|\mathcal{N}})$ and for $\Psi \in \mathrm{dom}(B_{m,M})$,*

$$\begin{aligned}
\|B_{m,M}\Psi\|^2_{L^2(\mathcal{N})} &= \int_{\mathcal{N}} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right] v_{\mathcal{N}} + m^2\|\Psi\|^2_{L^2(\mathcal{K})} \\
&\quad + M^2\|\Psi\|^2_{L^2(\mathcal{K}^c)} + (M - m)\int_{\partial\mathcal{K}} \left(|\mathcal{P}_-\Psi|^2 - |\mathcal{P}_+\Psi|^2\right) v_{\partial\mathcal{K}}
\end{aligned}$$

where we recall that $\mathcal{P}_\pm$ were defined in (4.6).

Proof. Let $\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$. One has

$$\begin{aligned}\|B_{m,M}\Psi\|^2_{L^2(\mathcal{N})} =& \|\nu\cdot(\mathcal{D}^{\mathcal{N}}+i\,M)\Psi + i\,(m-M)\mathbf{1}_{\mathcal{K}}\nu\cdot\Psi\|^2_{L^2(\mathcal{N})}\\ =& \|(\mathcal{D}^{\mathcal{N}}+i\,M)\Psi\|^2_{L^2(\mathcal{N})} + (m-M)^2\|\Psi\|^2_{L^2(\mathcal{K})}\\ &+(m-M)2\Re\left\langle(\mathcal{D}^{\mathcal{N}}+i\,M)\Psi, i\,\mathbf{1}_{\mathcal{K}}\Psi\right\rangle_{L^2(\mathcal{N})}\end{aligned}$$

With Lemma 4.1

$$2\Re\langle(\mathcal{D}^{\mathcal{N}}+i\,M)\Psi, i\,\Psi\rangle_{L^2(\mathcal{K})} = -\int_{\partial\mathcal{K}}\langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}} + 2M\,\langle\Psi,\Psi\rangle_{L^2(\mathcal{K})}\,.$$

Thus, we have

$$\begin{aligned}\|B_{m,M}\|^2_{L^2(\mathcal{N})} &= \|(\mathcal{D}^{\mathcal{N}}+i\,M)\Psi\|^2_{L^2(\mathcal{N})} + (m-M)^2\|\Psi\|^2_{L^2(\mathcal{K})}\\ &\quad+(M-m)\int_{\partial\mathcal{K}}\langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}} + 2M(m-M)\|\Psi\|^2_{L^2(\mathcal{K})}\\ &= \|(\mathcal{D}^{\mathcal{N}}+i\,M)\Psi\|^2_{L^2(\mathcal{N})} + (m^2-M^2)\|\Psi\|^2_{L^2(\mathcal{K})}\\ &\quad+(M-m)\int_{\partial\mathcal{K}}\langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}}\\ &= \int_{\mathcal{N}}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 + M^2|\Psi|^2\right]v_{\mathcal{N}} + (m^2-M^2)\|\Psi\|^2_{L^2(\mathcal{K})}\\ &\quad+(M-m)\int_{\partial\mathcal{K}}\langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}}\\ &= \int_{\mathcal{N}}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2|\Psi|^2\right]v_{\mathcal{N}} + m^2\|\Psi\|^2_{L^2(\mathcal{K})} + M^2\|\Psi\|^2_{L^2(\mathcal{K}^c)}\\ &\quad+(M-m)\int_{\partial\mathcal{K}}\langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}} \qquad (4.7)\end{aligned}$$

and

$$\langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle = \langle\Psi, -i\,\nu\cdot\mathbf{n}\cdot\Psi\rangle = \langle\Psi,\mathcal{P}_-\Psi\rangle - \langle\Psi,\mathcal{P}_+\Psi\rangle = |\mathcal{P}_-\Psi|^2 - |\mathcal{P}_+\Psi|^2.$$

It follows from Theorem 2.10 that there is a constant $C>0$ such that for all $\Psi\in\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$,

$$\|B_{m,M}\Psi\|^2_{L^2(\mathcal{N})} \geq C\left(\|\nabla^{\mathcal{N}}(\iota\Psi)\|^2_{L^2(\mathcal{N})} - \|\Psi\|^2_{L^2(\mathcal{N})}\right).$$

This shows that the graph norm of $\widetilde{B}_{m,M}$ is larger than the $H^1(\Sigma\mathcal{C}_{|\mathcal{N}})$-norm up to a constant. Thus $\mathrm{dom}(B_{m,M})\subset H^1(\Sigma\mathcal{C}_{|\mathcal{N}})$, and one can conclude by density. □

4.4. **The limit operator.** In this section, we introduce the effective operator L which will appear naturally as the limit operator for A_m when $m\to-\infty$. We define it as the operator acting on the Hilbert space

$$\mathbf{H} := \left\{\Psi\in L^2(\Sigma\mathcal{C}_{|\partial\mathcal{K}}), \Psi = i\,\nu\cdot\mathbf{n}\cdot\Psi\right\} \qquad (4.8)$$

associated with the quadratic form

$$\begin{aligned}\ell[\Psi,\Psi] &= \int_{\partial\mathcal{K}}\left[|\overline{\nabla}^{\mathcal{N}}\iota\Psi|^2 + \frac{1}{4}\left(\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)\right)|\Psi|^2\right]v_{\partial\mathcal{K}}, \qquad (4.9)\\ \mathcal{Q}(\ell) &:= \left\{\Psi\in H^1(\Sigma\mathcal{C}_{|\partial\mathcal{K}}), \Psi = i\,\nu\cdot\mathbf{n}\cdot\Psi\right\}.\end{aligned}$$

By the compactness of $\mathcal{K}$, it follows that the form (4.9) is closed and semibounded from below, so the operator L is well-defined.

The operator L is actually unitarily equivalent to the square of the Dirac operator on $\partial\mathcal{K}$. This fact can be established using the link between the spinor bundles of the spaces $\partial\mathcal{K} \subset \mathcal{N} \subset \mathcal{C}$.

Remark 4.9. Using Gauss-Codazzi equations (see [3, Proposition 4.1], for example), one has

$$\mathrm{Tr}(W^2) = H^2 + \mathrm{Scal}^{\mathcal{N}} - \mathrm{Scal}^{\partial\mathcal{K}} - 2\mathrm{Ric}^{\mathcal{N}}(\mathbf{n}, \mathbf{n}).$$

Thus, the operator we are considering here is a generalization of the operator L defined in [13, section 2.2] and we generalize the result of [13, Lemma 2.4].

Lemma 4.10. *The operator L is unitarily equivalent to $(\not{D}^{\partial\mathcal{K}})^2$.*

Proof. We consider separately the case of n even and n odd.

Case n odd: One can represent any $\Psi \in \mathbf{H}$ as $\Psi =: (\Psi^+, \Psi^-) \in L^2(\Sigma^+\mathcal{C}_{|\partial\mathcal{K}}) \times L^2(\Sigma^-\mathcal{C}_{|\partial\mathcal{K}})$, and then

$$\Psi = i\,\nu \cdot \mathbf{n} \cdot \Psi \Leftrightarrow \iota\Psi = i\,\iota(\nu \cdot \mathbf{n} \cdot \Psi) \Leftrightarrow \iota\Psi = -i\,\mathbf{n} \cdot \iota\Psi.$$

Thus, the isomorphism ι induces the isomorphisms $\iota^\pm : \Sigma^\pm\mathcal{C} \to \Sigma\mathcal{N}$, and one has

$$\begin{pmatrix} \iota^+\Psi^+ \\ \iota^-\Psi^- \end{pmatrix} = \begin{pmatrix} -i\,\mathbf{n} \cdot \iota^+\Psi^+ \\ i\,\mathbf{n} \cdot \iota^-\Psi^- \end{pmatrix}.$$

We introduce the (pointwise) unitary operator $U : L^2(\Sigma\mathcal{N}_{|\partial\mathcal{K}}) \to \mathbf{H}$, which sends $H^1(\Sigma\mathcal{N}_{|\partial\mathcal{K}})$ into $\mathcal{Q}(\ell)$, and is defined by

$$U\Psi = \frac{1}{2}\iota^{-1} \begin{pmatrix} (1 - i\,\mathbf{n}) \cdot \Psi \\ (1 + i\,\mathbf{n}) \cdot \Psi \end{pmatrix}.$$

We compute now $|\overline{\nabla}^{\mathcal{N}}\iota(U\Psi)|^2$ for $\Psi \in H^1(\Sigma\mathcal{N}_{|\partial\mathcal{K}})$. Let $(e_1, \dots, e_{n-1})$ be a pointwise local orthonormal frame of $T(\partial\mathcal{K})$. The vector fields $(e_j)_{1\le j\le n-1}$ are naturally identified with elements of $T\mathcal{N}$. Using the Schrödinger-Lichnerowicz formula and Proposition 2.6, (3) one has

$$\begin{aligned}
|\overline{\nabla}^{N}\iota(U\Psi)|^2 &= \frac{1}{4}\Big(|\overline{\nabla}^{N}((1 + i\,\mathbf{n}\cdot)\Psi)|^2 + |\overline{\nabla}^{N}((1 - i\,\mathbf{n}\cdot)\Psi)|^2\Big) \\
&= \frac{1}{2}\sum_{k=1}^{n-1}\big(|\nabla^{\mathcal{N}}_{e_k}\Psi|^2 + |(\nabla^{\mathcal{N}}_{e_k}\mathbf{n}) \cdot \Psi + \mathbf{n} \cdot \nabla^{\mathcal{N}}_{e_k}\Psi|^2\big) \\
&= \sum_{k=1}^{n-1}|\nabla^{\mathcal{N}}_{e_k}\Psi + \frac{1}{2}\mathbf{n} \cdot We_k \cdot \Psi|^2 + \frac{1}{4}\sum_{k=1}^{n-1}|We_k \cdot \Psi|^2 \\
&= |\mu^{-1}\nabla^{\partial\mathcal{K}}\mu\Psi|^2 + \frac{1}{4}\mathrm{Tr}(W^2)|\Psi|^2 \\
&= |\mathcal{D}^{\partial\mathcal{K}}\Psi|^2 + \frac{1}{4}\Big(-\mathrm{Scal}^{\partial\mathcal{K}} + \mathrm{Tr}(W^2)\Big)|\Psi|^2.
\end{aligned}$$

Thus,

$$\ell[U\Psi, U\Psi] = \int_{\partial\mathcal{K}} |\mathcal{D}^{\partial\mathcal{K}}\Psi|^2 v_{\partial\mathcal{K}} = \int_{\partial\mathcal{K}} |\not{D}^{\partial\mathcal{K}}\mu\Psi|^2 v_{\partial\mathcal{K}}.$$

Case n even : The isomorphism μ induces the isomorphisms $\mu^\pm : \Sigma^\pm\mathcal{N} \to \Sigma\mathcal{K}$. According to Proposition 2.6, as $n-1$ is odd, for all $f \in \Gamma(\Sigma\mathcal{N}_{|\partial\mathcal{K}})$ one has

$$\mu(i\,\mathbf{n} \cdot f) = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}\begin{pmatrix} \mu^+f^+ \\ \mu^-f^- \end{pmatrix}.$$

Then, for $\Psi \in \mathbf{H}$ one has

$$i\,\nu \cdot \mathbf{n} \cdot \Psi = \Psi \Leftrightarrow -\iota(i\,\mathbf{n} \cdot \nu \cdot \Psi) = \iota\Psi \Leftrightarrow -\mu(i\,\mathbf{n} \cdot \iota\Psi) = \mu\iota\Psi$$

$$\Leftrightarrow -\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}\begin{pmatrix} \mu^+(\iota\Psi)^+ \\ \mu^-(\iota\Psi)^- \end{pmatrix} = \mu\iota\Psi \Leftrightarrow (\iota\Psi)^- = -(\mu^-)^{-1}\mu^+(\iota\Psi)^+.$$

Thus, the unitary operator

$$\begin{array}{cccc} U : L^2(\Sigma(\partial\mathcal{K})) & \longrightarrow & \mathbf{H} \\ \Psi & \longmapsto & \frac{1}{\sqrt{2}}\iota^{-1}\mu^{-1}\begin{pmatrix} -\Psi \\ \Psi \end{pmatrix} \end{array}$$

sends $H^1(\Sigma(\partial\mathcal{K}))$ into $\mathcal{Q}(\ell)$. Now we compute $|\overline{\nabla}^N \iota(U\Psi)|^2$ for $\Psi \in H^1(\Sigma(\partial\mathcal{K}))$. Let $(e_1, \ldots, e_{n-1})$ be a pointwise local orthonormal frame of $T(\partial\mathcal{K})$. One has, using Proposition 2.6, (3)

$$\begin{aligned} |\overline{\nabla}^N \iota(U\Psi)|^2 &= |\mu\overline{\nabla}^N \iota(U\Psi)|^2 \\ &= \frac{1}{2}\left|\mu\overline{\nabla}^N \mu^{-1}\begin{pmatrix} -\Psi \\ \Psi \end{pmatrix}\right|^2 \\ &= \sum_{k=1}^{n-1}\frac{1}{2}\left|\left(\nabla^{\partial\mathcal{K}}_{e_k} + \frac{1}{2}We_k\right)\begin{pmatrix} -\Psi \\ \Psi \end{pmatrix}\right|^2 \\ &= \frac{1}{2}\sum_{k=1}^{n-1}\left(\left|\left(\nabla^{\partial\mathcal{K}}_{e_k} + \frac{1}{2}We_k\right)\Psi\right|^2 + \left|\left(\nabla^{\partial\mathcal{K}}_{e_k} - \frac{1}{2}We_k\right)\Psi\right|^2\right) \\ &= \sum_{k=1}^{n-1}\left(|\nabla^{\partial\mathcal{K}}_{e_k}\Psi|^2 + \frac{1}{4}|We_k|^2|\Psi|^2\right) \\ &= |\not{D}^{\partial\mathcal{K}}\Psi|^2 + \frac{1}{4}\left(-\mathrm{Scal}^{\partial\mathcal{K}} + \mathrm{Tr}(W^2)\right)|\Psi|^2 \end{aligned}$$

Thus

$$\ell[U\Psi, U\Psi] = \int_{\partial\mathcal{K}} |\not{D}^{\partial\mathcal{K}}\Psi|^2 v_{\partial\mathcal{K}}$$

which concludes the proof. □

5. Operators in tubular coordinates

When the masses m and M become large, one can localize the eigenvalue problem in a neighbourhood of $\partial\mathcal{K}$ since the potential in the square of the operators is large outside of this region. For this reason, it is useful to express the operators in tubular coordinates around $\partial\mathcal{K}$. Thus, we identify a collar near the boundary of $\mathcal{K}$ with the cylinder $\partial\mathcal{K} \times (-\delta, \delta)$ and we look at the operator obtained via this identification. However, the aim of this procedure is to simplify the expression, so we would like to change the induced metric on the cylinder into the product metric. This last step cannot be done without a way to compare the spinor bundles involved, and in particular the way we modify the covariant derivative.

5.1. **Tubular coordinates.** For $\delta > 0$ we define the tubular neighbourhood of $\partial\mathcal{K}$ by

$$\mathbf{n}_\delta(\partial\mathcal{K}) := \{x \in \mathcal{N}, \mathrm{dist}(x, \partial\mathcal{K}) < \delta\}. \tag{5.1}$$

Since $\partial\mathcal{K}$ is compact, $\mathbf{n}_\delta(\partial\mathcal{K})$ can be identified with the product $\partial\mathcal{K} \times (-\delta, \delta)$ through the Riemannian exponential map when δ is small. To make this precise, we define

$$\Pi_\delta := \partial\mathcal{K} \times (-\delta, \delta), \Pi_\delta^+ := \partial\mathcal{K} \times (0, \delta),\ \Pi_\delta^- := \partial\mathcal{K} \times (-\delta, 0),\ \Pi^t := \partial\mathcal{K} \times \{t\}, \tag{5.2}$$

and it is standard that there exists $\delta_0 > 0$ such that the map

$$\begin{array}{ccc} \Pi_{\delta_0} & \longrightarrow & \mathbf{n}_{\delta_0}(\partial\mathcal{K}) \\ (x,t) & \longmapsto & \exp_x^{\mathcal{N}}(t\mathbf{n}(x)) \end{array} \tag{5.3}$$

is a diffeomorphism on its image.

For every $\delta < \delta_0$, Π_δ inherits an orientation via the previous identification. Moreover, one has $T(\Pi_\delta) \cong T(\partial\mathcal{K}) \times T\mathbb{R}$ and we denote by $\frac{\partial}{\partial t}$ the vector field $(0,1) \in T(\partial\mathcal{K}) \times T\mathbb{R}$.

Recall now the definition of a generalized cylinder introduced in [3]:

Definition 5.1. *A generalized cylinder is a Riemannian manifold of the form $\mathcal{Z} := \mathcal{M}\times I$ where $I \subset \mathbb{R}$ is an interval, $\mathcal{M}$ is a differentiable manifold and the Riemannian metric on $\mathcal{Z}$ has the form $g_{\mathcal{Z}} = g_t + dt^2$ where $(g_t)_{t\in I}$ is a smooth 1-parameter family of Riemannian metrics of $\mathcal{M}$.*

We identify any vector field X on the hypersurface $\partial\mathcal{K}$ with the vector field on $T\Pi_{\delta_0}$ also denoted by X and defined by $X_{(y,t)} := X_y$ for all $(y,t) \in \Pi_{\delta_0}$. Note that in this case $[\frac{\partial}{\partial t}, X] = 0$.

We have two natural metrics on Π_{δ_0}. First, the metric g of $\mathcal{N}$ via the previous identification, and secondly, the Riemannian product metric $h := g_{|\partial\mathcal{K}} + \mathrm{d}t^2$. Furthermore, $\Sigma\Pi_{\delta_0}$ is the spinor bundle of $\mathcal{N}$ restricted to Π_{δ_0}.

With these notations, we have the useful property:

Lemma 5.2. *The Riemannian manifold (Π_{δ_0}, g) is a generalized cylinder.*

Proof. It is sufficient to prove that $g = g_t + dt^2$ with $(g_t)_t$ a family of metrics on $\partial\mathcal{K}$. This is equivalent to show that the vector field $\frac{\partial}{\partial t}$ is normal to Π^t for all $t \in (-\delta_0, \delta_0)$. Let $(x,t) \in \Pi_{\delta_0}$ and $X \in T(\partial\mathcal{K})$, identified with a vector field on Π_{δ_0} as before. One has

$$\begin{aligned} \frac{d}{dt} g\left(X, \frac{\partial}{\partial t}\right) &= g\left(\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} X, \frac{\partial}{\partial t}\right) + g\left(X, \nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\right) \\ &= \overbrace{g\left(\nabla^{\mathcal{N}}_X \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)}^{=0} + g\left(\left[\frac{\partial}{\partial t}, X\right], \frac{\partial}{\partial t}\right) \\ &= g\left(\left[\frac{\partial}{\partial t}, X\right], \frac{\partial}{\partial t}\right) = 0. \end{aligned}$$

This shows that $g\left(X, \frac{\partial}{\partial t}\right)$ is constant along the curves $s \mapsto (\cdot, s)$ since $g\left(X, \frac{\partial}{\partial t}\right)_{(x,0)} = 0$. We get $g\left(X, \frac{\partial}{\partial t}\right)_{(x,t)} = 0$, which concludes the proof. □

From Proposition 5.2, we deduce that there exists a family of metrics $(g_t)_t$ on $\partial\mathcal{K}$ such that $g = g_t + \mathrm{d}t^2$. One can observe that $h = g_0 + \mathrm{d}t^2$ in these notations.

We define for any $(s,t) \in (-\delta_0, \delta_0)$ the map Γ_s^t which acts as the parallel transport from s to t along the curves $r \mapsto (\cdot, r)$ with respect to the connection $\nabla^{\mathcal{N}}$.

We recall that $v_{\mathcal{N}}$ is the volume form on Π_{δ_0} compatible with the metric g. Let $v_h := v_{\partial\mathcal{K}} \wedge \mathrm{d}t$ be the volume form compatible with h on Π_{δ_0}.

The bilinear form g is identified with an endomorphism of $T\Pi_{\delta_0}$ via the metric h. Let $(x,t) \in \Pi_{\delta_0}$. For any direct orthonormal frame f of $T_{(x,t)}\Pi_{\delta_0}$ endowed with the metric h we define

$$\phi(x,t) := \sqrt{\det_f g}. \tag{5.4}$$

One can show that this does not depend on the choice of the basis, and the volume forms with respect to the different metrics are related by

$$v_{\mathcal{N}} = \phi v_h. \tag{5.5}$$

Our aim in this section is relates all the objects on (Π_{δ_0}, g) in terms of those over (Π_{δ_0}, h). The function ϕ defined above relates the integration over these two Riemannian manifolds, and in particular the corresponding L^2 spaces. More precisely, the map

$$\begin{array}{cccc} \Theta: & L^2(\Sigma\Pi_{\delta_0}, v_{\mathcal{N}}) & \longrightarrow & L^2(\Sigma\Pi_{\delta_0}, v_h) \\ & \Psi & \longmapsto & \sqrt{\phi}\Psi \end{array} \tag{5.6}$$

is a unitary isomorphism from $L^2(\Sigma\Pi_{\delta_0}, v_{\mathcal{N}})$ onto $L^2(\Sigma\Pi_{\delta_0}, v_h)$.

5.2. **Estimates in the generalized cylinder.** We now fix $\delta < \frac{\delta_0}{2}$. In order to compare the structures over the hypersurfaces Π^t for $t \in (-\delta, \delta)$, we first show that the norm of a vector field defined on Π^t and extended by parallel transport with respect to $\nabla^{\mathcal{N}}$ does not vary too much when δ is small.

Lemma 5.3. *We endow Π_δ with the metric g. There exists $C > 0$ depending only on δ_0 such that for all $t, t' \in (-\delta, \delta)$ and $X \in \Gamma(T\Pi^t)$, for all $x \in \partial\mathcal{K}$, one has the estimate*

$$|X_{(x,t')} - \Gamma_t^{t'}(X_{(x,t)})|_g \leq C|t - t'||X_{(x,t)}|_g,$$

where X is extended to $T\Pi_\delta$ as before.

Proof. First, we remark that $C_1 := \sup_{(y,s)\in\Pi_{\delta_0/2}} \sup_{Z\in T_{(y,s)}\setminus\{0\}} \frac{|g(W_{\Pi^s}Z,Z)|}{g(Z,Z)}$ is finite by compactness. Let $t \in (-\delta, \delta)$ and $X \in \Gamma(T\Pi^t)$. We define the vector field $Y \in \Gamma(T\Pi_\delta)$ by $Y_{(y,s)} := \Gamma_t^s(X_{(y,t)})$ for any $(y, s) \in \Pi_\delta$.
One has for all $t' \in (-\delta, \delta)$,

$$\left|\frac{\partial}{\partial t} g(X, X)\right|_{|(\cdot,t')} = \left|2g\left(\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} X, X\right)\right|_{|(\cdot,t')} \leq 2C_1 g(X, X)_{(\cdot,t')}.$$

By integration, we obtain the inequality $g(X, X)_{(\cdot,t')} \leq g(X, X)_{(\cdot,t)} \exp(2C_1|t'-t|)$, and for $C_2 := \exp(2\delta_0 C_1)$ one has $g(X, X)_{(\cdot,t')} \leq C_2 g(X, X)_{(\cdot,t)}$.
Now, one has

$$\begin{aligned} \left|\frac{\partial}{\partial t} g(X - Y, X - Y)\right|_{(\cdot,t')} &= \left|2g(\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} X, X - Y)\right|_{(\cdot,t')} \\ &= |2g(W_{\Pi^{t'}} X, X - Y)|_{(\cdot,t')} \\ &\leq 2C_1 |X_{(\cdot,t')}|_g |(X - Y)_{(\cdot,t')}|_g \\ &\leq 2C_1 C_2 |X_{(\cdot,t)}|_g |(X - Y)_{(\cdot,t')}|_g. \end{aligned}$$

We need the following technical lemma to conclude.

Lemma 5.4. *Let I be an interval of $\mathbb{R}$ containing 0 and let $f : I \to \mathbb{R}$ be a differentiable non-negative function. Assume there is $C > 0$ such that $|f'| \leq C\sqrt{f}$. Then, one has $|\sqrt{f}(x) - \sqrt{f}(0)| \leq \frac{C}{2}|x|$ for all $x \in I$.*

Using Lemma 5.4 we arrive at

$$g(X - Y, X - Y)_{(\cdot,t')} \leq C_1 C_2 |X_{(\cdot)}|_g^2 (t' - t)^2$$

and the claim follows by taking the square root in this inequality. □

Proof of Lemma 5.4. Let $\varepsilon > 0$. One has $|f'| \leq C\sqrt{f+\varepsilon}$, which gives $\left|\frac{\mathrm{d}\sqrt{f+\varepsilon}}{\mathrm{d}x}\right| \leq \frac{C}{2}$. By integration, we obtain that for all $x \in I$, $|\sqrt{f(x)+\varepsilon} - \sqrt{f(0)+\varepsilon}| \leq \frac{C}{2}|x|$. Letting ε tend to zero, one gets the result. $\square$

We are now able to compare the norms of the covariant derivatives on the different hypersurfaces of Π_δ. For this purpose, we recall that $\overline{\nabla}^{\mathcal{N}}\Psi$ is defined as the restriction of $\nabla^{\mathcal{N}}\Psi$ to $T^*\partial\mathcal{K} \otimes \Sigma\Pi_\delta$.

Lemma 5.5. *There exists $C > 0$ only depending on δ_0 such that for any $t \in (-\delta, \delta)$ and $\Psi \in \Gamma(\Sigma\Pi_\delta)$,*

$$\begin{aligned}(1 - C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi(\cdot,t)\right|^2 - C\delta|\Psi(\cdot,t)|^2 \leq \left|\overline{\nabla}^{\mathcal{N}}\Psi(\cdot,t)\right|^2\\ \leq (1 + C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi(\cdot,t)\right|^2 + C\delta|\Psi|^2(\cdot,t).\end{aligned}$$

Proof. Let $\Psi \in \Gamma(\Sigma\Pi_\delta)$. Let $(x,t) \in \Pi_\delta$ and $X \in T(\partial\mathcal{K})$ such that $|X_{(x,t)}|_{g_t} = 1$, extended constantly to Π_δ. The Riemannian curvature of (Π_δ, g) is bounded, so for any $s \in (-\delta,\delta)$ one can find $C_1 > 0$ such that

$$\begin{aligned}\left|\frac{\partial}{\partial s}|(\nabla_X^{\mathcal{N}}\Gamma_t^s\Psi)(x,s)|^2\right| =&2\left|\Re\left\langle(\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\nabla_X^{\mathcal{N}}\Gamma_t^s\Psi)(x,s), (\nabla_X^{\mathcal{N}}\Gamma_t^s\Psi)(x,s)\right\rangle\right|\\ =&\left|\Re\left\langle R^{\mathcal{N}}\left(\frac{\partial}{\partial t}, X\right)\cdot(\Gamma_t^s\Psi)(x,s), (\nabla_X^{\mathcal{N}}\Gamma_s^t\Psi)(x,s)\right\rangle\right|\\ \leq& C_1|X_{(x,s)}|_g|\Psi(x,t)||(\nabla_X^{\mathcal{N}}\Gamma_s^t\Psi)(x,s)|.\end{aligned}$$

By Lemma 5.3, one can find $C > 0$ independent of X such that

$$|X_{(x,s)}|_g \leq 1 + C|t-s| \leq 1 + C\delta_0.$$

Thus,

$$\left|\frac{\partial}{\partial s}|(\nabla_X^{\mathcal{N}}\Gamma_t^s\Psi)(x,s)|^2\right| \leq C_1(1+C\delta_0)|\Psi(x,t)||(\nabla_X^{\mathcal{N}}\Gamma_t^s\Psi)(x,s)|.$$

Using Lemma 5.4, we obtain

$$\left||(\nabla_X^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)| - |\nabla_X^{\mathcal{N}}\Psi(x,t)|\right| \leq C_1(1+C\delta_0)|t||\Psi(x,t)|.$$

On the other hand,

$$\begin{aligned}|(\nabla_X^{\mathcal{N}}\Gamma_t^0\Psi)(x,0) - (\nabla_{\Gamma_t^0 X}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)| \leq& |X_{(x,0)} - \Gamma_t^0(X_{(x,t)})|_g|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)|\\ \leq& C|t||(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)|.\end{aligned}$$

Thus, combining the previous estimates, one can find $C_2 > 0$ such that

$$\left||(\nabla_{\Gamma_t^0 X}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)| - |\nabla_X^{\mathcal{N}}\Psi(x,t)|\right| \leq C_2|t|\left(|\Psi(x,t)| + |(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)|\right).$$

Now, let $(e_1,\dots,e_n)$ be an orthonormal frame at the point (x,t). One obtains

$$\begin{aligned}\left||(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)| - |\overline{\nabla}^{\mathcal{N}}\Psi(x,t)|\right| \leq& \sum_{k=1}^n\left||(\nabla_{\Gamma_t^0 e_k}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)| - |\nabla_{e_k}^{\mathcal{N}}\Psi(x,t)|\right|\\ \leq& \sum_{k=1}^n C_2|t|\left(|\Psi(x,t)| + |(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)|\right)\\ \leq& nC_2\delta\left(|\Psi(x,t)| + |(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi)(x,0)|\right).\end{aligned}$$

The result is then a consequence of the following lemma:

Lemma 5.6. *For all $C > 0$ and $\delta < \delta_0/2$, there is $C' > 0$ depending only on δ_0 and C such that for all $a, b, d > 0$ verifying $|a - b| \leq C\delta(b + d)$, one has $|a^2 - b^2| \leq C'\delta(b^2 + d^2)$.*

□

Proof of Lemma 5.6. One has

$$\begin{aligned}|a^2 - b^2| =&|(a - b + b)^2 - b^2| = |(a - b)^2 + 2(a - b)b| \leq |a - b|^2 + |2(a - b)b| \\ \leq& C^2\delta^2(b + d)^2 + 2C\delta(b + d)b \leq C^2\delta^2(b + d)^2 + C\delta(b + d)^2 + C\delta b^2 \\ \leq&(2C^2\delta^2 + C\delta)(b^2 + d^2) + C\delta b^2 \leq (2C^2\delta_0 + 2C)\delta(b^2 + d^2),\end{aligned}$$

which is equivalent to the statement of the lemma. □

5.3. **Bracketing for the quadratic form of A_m^2.** We end this section by finding a lower and an upper bound for the quadratic form of A_m^2 expressed in the tubular coordinates.

Lemma 5.7. *There exists $c > 0$ depending only on δ_0 such that the following estimates hold:*

$$\|\phi - 1\|_{L^\infty(\Pi_\delta)} \leq c\delta \qquad (5.7) \qquad \|\overline{\nabla}^{\mathcal{N}}\phi\|^2_{L^\infty(\Pi_\delta)} \leq c\delta^2 \qquad (5.8)$$

$$\left\|\frac{(\partial_t\phi)(\cdot, \delta)}{2\phi(\cdot, \delta)}\right\|_{L^\infty(\partial\mathcal{K})} \leq c \quad (5.9) \qquad \partial_t\phi(\cdot, 0) = -\frac{H}{2} \qquad (5.10)$$

$$\left|\frac{\partial_t^2\phi}{2\phi}(x, t) - \frac{(\partial_t\phi)^2}{4\phi^2}(x, t) - \frac{1}{4}(\mathrm{Scal}^{\partial\mathcal{K}}(x) - \mathrm{Tr}(W^2)(x) - \mathrm{Scal}^{\mathcal{N}}(x, t))\right| \leq c\delta, \tag{5.11}$$

for all $(x, t) \in \Pi_\delta$.

Proof. To show (5.7), (5.8) and (5.9), we just remark that ϕ is a smooth function on the closure of Π_δ which is compact, so it is bounded on Π_δ as well as all its derivatives.

Thanks to Lemma 5.2 we can use [3, formula (4.1)], so (5.10) follows from:

$$\frac{\partial_t\phi(\cdot, 0)}{2\phi(\cdot, 0)} = \frac{\partial_t\sqrt{\det_f g}(\cdot, 0)}{2} = \frac{\mathrm{Tr}(\partial_t g)(\cdot, 0)}{4\sqrt{\det_f g}(\cdot, 0)} = -\frac{2\mathrm{Tr}(W)}{4} = -\frac{H}{2}.$$

Finally, we prove (5.11). Let $(x, t) \in \Pi_\delta$ and let f be a direct orthonormal frame of (Π_δ, h) at (x, t). One has, using lemma 5.2 and [3, equation (4.8)],

$$\begin{aligned}\frac{\partial_t^2\phi}{2\phi}(x, t) - \frac{(\partial_t\phi)^2}{4\phi^2}(x, t) =& \frac{\partial_t^2 \det_f g}{4\det_f g}(x, t) - \frac{3(\partial_t \det_f g)^2}{16(\det_f g)^2}(x, t) \\ =& \left(\frac{\partial_t^2 \det_f g}{4} - \frac{3(\partial_t \det_f g)^2}{16}\right)(x, 0) + \mathcal{O}(t) \\ =& \left(\frac{H^2}{4} - \mathrm{Tr}(W^2) + \frac{\mathrm{Tr}(\ddot{g}_t|_{t=0})}{4}\right)(x) + \mathcal{O}(t) \\ =& \frac{1}{4}(\mathrm{Scal}^{\partial\mathcal{K}}(x) - \mathrm{Tr}(W^2)(x) - \mathrm{Scal}^{\mathcal{N}}(x, t)) + \mathcal{O}(t),\end{aligned}$$

which gives the result. □

For $\alpha \in \mathbb{R}$, $\delta \in (0, \delta_0/2)$ and $\Psi \in H^1\left(\Sigma\overline{\Pi_\delta^\pm}\right)$ we define

$$J_\pm(\Psi) := \int_{\Pi_\delta^\pm} \left[|\nabla^{\mathcal{N}}\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 \right] v_{\mathcal{N}} + \int_{\partial\mathcal{K}} \left(\alpha \pm \frac{H}{2}\right)|\Psi|^2 v_{\partial\mathcal{K}}. \tag{5.12}$$

Proposition 5.8. *There is a constant $c > 0$ depending only on δ_0 such that for all $\alpha \in \mathbb{R}$ and $\delta \in (0, \delta_0/2)$, the following inequalities hold:*

(1) For every $\Psi \in H^1\left(\Sigma\overline{\Pi_\delta^\pm}\right)$, one has

$$\begin{aligned} J_\pm(\Psi) \geq \int_{\Pi_\delta^\pm} &\left[(1 - c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi)(x, 0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi|^2\right] v_h(x, t) \\ &+ \int_{\Pi_\delta^\pm} \left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} - c\delta\right)|\Theta\Psi|^2\right] v_h \\ &+ \int_{\partial\mathcal{K}} \left[\alpha|(\Theta\Psi)(\cdot, 0)|^2 - c|(\Theta\Psi)(\cdot, \delta)|^2\right] v_{\partial\mathcal{K}}. \end{aligned} \tag{5.13}$$

(2) If moreover $\Psi = 0$ on the outer boundary $\Pi^{\pm\delta}$, one has

$$\begin{aligned} J_\pm(\Psi) \leq \int_{\Pi_\delta^\pm} &\left[(1 + c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi)(x, 0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi|^2\right] v_h(x, t) \\ &+ \int_{\Pi_\delta^\pm} \left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + c\delta\right)|\Theta\Psi|^2\right] v_h + \alpha\int_{\partial\mathcal{K}} |(\Theta\Psi)(\cdot, 0)|^2 v_{\partial\mathcal{K}} \end{aligned} \tag{5.14}$$

Proof. It is sufficient to prove the result for $\Psi \in \Gamma_c\left(\Sigma\overline{\Pi_\delta^\pm}\right)$ and to conclude by density. One has

$$J_\pm(\Psi) = \int_{\Pi_\delta^\pm} \left[|\nabla^{\mathcal{N}}\phi^{-\frac{1}{2}}\Theta\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\phi^{-\frac{1}{2}}\Theta\Psi|^2\right]\phi v_h + \int_{\partial\mathcal{K}} \left(\alpha \pm \frac{H}{2}\right)|\Psi|^2 v_{\partial\mathcal{K}}.$$

We remark that $\phi = 1$ on $\partial\mathcal{K}$ and Lemma 5.5 gives a constant $C > 0$ such that

$$\begin{aligned} &\int_{\Pi_\delta^\pm} \left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\phi^{-\frac{1}{2}}\Theta\Psi\right|^2 + (1 - C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\phi^{-\frac{1}{2}}\Theta\Psi\right|^2(\cdot, 0) - C\delta|\phi^{-\frac{1}{2}}\Theta\Psi|^2\right]\phi v_h \\ &\qquad + \int_{\Pi_\delta^\pm} \frac{\mathrm{Scal}^{\Pi}}{4}|\Theta\Psi|^2\phi v_h + \int_{\partial\mathcal{K}} \left(\alpha \pm \frac{H}{2}\right)|\Theta\Psi|^2 v_{\partial\mathcal{K}} \leq J_\pm(\Psi) \\ &\leq \int_{\Pi_\delta^\pm} \left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\phi^{-\frac{1}{2}}\Theta\Psi\right|^2 + (1 + C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\phi^{-\frac{1}{2}}\Theta\Psi\right|^2(\cdot, 0) + C\delta|\phi^{-\frac{1}{2}}\Theta\Psi|^2\right]\phi v_h \\ &\qquad + \int_{\Pi_\delta^\pm} \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\phi^{-\frac{1}{2}}\Theta\Psi|^2\phi v_h + \int_{\partial\mathcal{K}} \left(\alpha \pm \frac{H}{2}\right)|\Theta\Psi|^2 v_{\partial\mathcal{K}}. \end{aligned}$$

Moreover, for all $(x, t) \in \Pi_\delta$ and $X \in T_x\partial\mathcal{K}$,

$$\begin{aligned} &\left|\overline{\nabla}^{\mathcal{N}}_X\Gamma_t^0(\phi^{-\frac{1}{2}}\Theta\Psi)\right|^2(x, 0)\phi(x, t) \\ &\qquad = \left|\overline{\nabla}^{\mathcal{N}}_X\Gamma_t^0\Theta\Psi - \frac{1}{2\phi(x, t)}X(\phi)(x, t)\Gamma_t^0\Theta\Psi\right|^2(x, 0) \\ &\qquad = \left|\overline{\nabla}^{\mathcal{N}}_X\Gamma_t^0\Theta\Psi\right|^2(x, 0) + \left|\frac{1}{2\phi(x, t)}X(\phi)(x, t)\Gamma_t^0\Theta\Psi\right|^2(x, 0) \\ &\qquad\qquad - \frac{1}{\phi(x, t)}\Re\left\langle\overline{\nabla}^{\mathcal{N}}_X\Gamma_t^0\Theta\Psi, X(\phi)(x, t)\Gamma_t^0\Theta\Psi\right\rangle(x, 0) \end{aligned}$$

and

$$\left|\Re\left\langle\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi, X(\phi)(x,t)\Gamma_t^0\Theta\Psi\right\rangle(x,0)\right| \leq \delta|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi|^2(x,0)+|\Theta\Psi|^2|X(\phi)|^2(x,t)/\delta.$$

Using this together with the inequality (5.8) shows the existence of $C'>0$ such that

$$\begin{aligned}(1-C'\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi\right|^2(x,0)-C'\delta\left|\Theta\Psi\right|^2(x,t)\\ \leq (1\pm C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\phi^{-\frac12}\Theta\Psi\right|^2(x,0)\phi(x,t)\\ \leq (1+C'\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi\right|^2(x,0)+C'\delta\left|\Theta\Psi\right|^2(x,t).\end{aligned}$$

It remains to compute

$$\begin{aligned}\phi|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\phi^{-\frac12}\Theta\Psi|^2 &= \left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi-\frac{1}{2\phi}\partial_t\phi(\Theta\Psi)\right|^2\\ &= \left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2+\frac{(\partial_t\phi)^2}{4\phi^2}\left|\Theta\Psi\right|^2-\frac{\partial_t\phi}{\phi}\Re\left\langle\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi,\Theta\Psi\right\rangle\\ &= \left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2+\frac{(\partial_t\phi)^2}{4\phi^2}\left|\Theta\Psi\right|^2-\frac{\partial_t\phi}{2\phi}\partial_t\left|\Theta\Psi\right|^2.\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}\int_{\Pi^\pm_\delta}|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\phi^{-\frac12}\Theta\Psi|^2\phi v_h &= \int_{\Pi^\pm_\delta}\left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2+\frac{(\partial_t\phi)^2}{4\phi^2}\left|\Theta\Psi\right|^2-\frac{\partial_t\phi}{2\phi}\partial_t\left|\Theta\Psi\right|^2\right]v_h\\ &= \int_{\Pi^\pm_\delta}\left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2+\frac{(\partial_t\phi)^2}{4\phi^2}\left|\Theta\Psi\right|^2+\left(\frac{\partial_t^2\phi}{2\phi}-\frac{(\partial_t\phi)^2}{2\phi^2}\right)\left|\Theta\Psi\right|^2\right]v_h\\ &\quad\mp\int_{\Pi^{\pm\delta}}\frac{\partial_t\phi}{2\phi}|\Theta\Psi|^2 v_h\pm\int_{\Pi^0}\frac{\partial_t\phi}{2\phi}|\Theta\Psi|^2 v_{\partial\mathcal{K}}\\ &= \int_{\Pi^\pm_\delta}\left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2+\left(\frac{\partial_t^2\phi}{2\phi}-\frac{(\partial_t\phi)^2}{4\phi^2}\right)\left|\Theta\Psi\right|^2\right]v_h\\ &\quad\mp\int_{\Pi^{\pm\delta}}\frac{\partial_t\phi}{2\phi}|\Theta\Psi|^2 v_{\partial\mathcal{K}}\mp\int_{\Pi^0}\frac{H}{2}|\Theta\Psi|^2 v_{\partial\mathcal{K}}\end{aligned}$$

where we used (5.10). Thus, we have

$$\begin{aligned}J_\pm(\Psi)\leq\int_{\Pi^\pm_\delta}&\left[(1+C\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi)(x,0)\right|^2+|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi|^2\right.\\ &\left.+\left(\frac{\partial_t^2\phi}{2\phi}-\frac{(\partial_t\phi)^2}{4\phi^2}+\frac{\mathrm{Scal}^{\mathcal{N}}}{4}+C\delta\right)|\Theta\Psi|^2(x,t)\right]v_h(x,t)\\ &+\alpha\int_{\Pi^0}|\Theta\Psi|^2 v_{\partial\mathcal{K}}\ \text{ if }\Psi=0\text{ on }\Pi^{\pm\delta}\end{aligned}$$

$$\begin{aligned}J_\pm(\Psi)\geq\int_{\Pi^\pm_\delta}&\left[(1-C\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi)(x,0)\right|^2+|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi|^2\right.\\ &\left.+\left(\frac{\partial_t^2\phi}{2\phi}-\frac{(\partial_t\phi)^2}{4\phi^2}+\frac{\mathrm{Scal}^{\mathcal{N}}}{4}-C\delta\right)|\Theta\Psi|^2(x,t)\right]v_h\\ &+\alpha\int_{\Pi^0}|\Theta\Psi|^2 v_{\partial\mathcal{K}}\mp\int_{\Pi^{\pm\delta}}\frac{\partial_t\phi}{2\phi}|\Theta\Psi|^2 v_{\partial\mathcal{K}}.\end{aligned}$$

These estimates, together with (5.9) and (5.11) give the result. □

6. Analysis of the one-dimensional operators

The proofs of the main results will use some separation of variables in the generalized cylinder Π_δ. For this reason, we will need to analyse various one-dimensional operators. We define them in this section and we state the properties that we need on the behaviour of their eigenvalues in some asymptotic regimes.

We recall the following results from [13, Section 3]:

Lemma 6.1. *Let $\varepsilon > 0$. Let $\alpha > 0$ and let S be the self-adjoint operator on $L^2(0,\delta)$ associated with the quadratic form*

$$s[f,f] = \int_0^\varepsilon |f'|^2 dt - \alpha|f(0)|^2,\ \mathcal{Q}(s) = \left\{f \in H^1(0,\varepsilon),\ f(\varepsilon) = 0\right\}.$$

Then, when $\alpha \to +\infty$, one has $E_1(S) = -\alpha^2 + \mathcal{O}(e^{-\varepsilon\alpha})$, and the associated L^2-normalized eigenfunction f satisfies $|f(0)|^2 = 2\alpha + \mathcal{O}(1)$.

Lemma 6.2. *Let $\varepsilon > 0$. Let $\alpha, \beta > 0$ and let S' be the self-adjoint operator on $L^2(0,\varepsilon)$ associated with the quadratic form*

$$s'[f,f] = \int_0^\varepsilon |f'|^2 dt + m|f(0)|^2 - \beta|f(\varepsilon)|^2,\ \mathcal{Q}(S') = H^1(0,\varepsilon).$$

Then, when $\alpha \to +\infty$, one has $E_1(S') = -\alpha^2 + \mathcal{O}(e^{-\varepsilon\alpha})$, and there exist $b^\pm > 0$ and $b > 0$ such that

$$b^- j^2 - b \le E_j(S') \le b^+ j^2 \text{ for all } j \ge 2 \text{ and } \alpha > 0.$$

A third one-dimensional operator will be of interest for the proof of Theorem 1.3. It can be interpreted as the Laplacian on an interval $(-\delta,\delta)$ with a potential consisting of two masses on the two sides of the origin and a δ-interaction at 0. For this last operator, we state the result in the very specific case of our framework, for $m, M \in \mathbb{R}$ and $\delta \in (0, \delta_0/2)$.

For $\beta > 0$, let X be the operator associated with the quadratic form

$$\begin{aligned} x[f,f] = \int_{-\delta}^{\delta} |f'|^2 \mathrm{d}t - \beta(|f(\delta)|^2 + |f(-\delta)|^2) \\ + \int_{-\delta}^{0} M^2|f|^2 \mathrm{d}t + \int_0^\delta m^2|f|^2\mathrm{d}t - (M-m)|f(0)|^2, \\ \mathcal{Q}(x) = H^1(-\delta,\delta). \quad (6.1) \end{aligned}$$

Lemma 6.3. *For $\delta > 0$ and $\beta > 0$ fixed, one has $E_1(X) = \mathcal{O}(e^{-\frac{\min(|m|,M)}{2}\delta})$ when $\min(-m, M) \to +\infty$. Moreover, for all $j \ge 2$, one can find $C_1, C_2 > 0$ such that*

$$\min(m^2, M^2) + C_1 j^2 - C_2 \le E_j(X).$$

Proof. One can see that the operator X acts as $f \mapsto -f'' + (M^2\mathbf{1}_{(-\delta,0)} + m^2\mathbf{1}_{(0,\delta)})f$ on the functions $f \in H^1(-\delta,\delta) \cap (H^2(-\delta,0) \cup H^2(0,\delta))$ satisfying $f'(\delta) - \beta f(\delta) = f'(-\delta) + \beta f(-\delta) = 0$ and $f'(0^+) - f(0^-) + (|m| + M)f(0) = 0$. We search for a negative eigenvalue for X of the form $-k^2$ with $k > 0$. The associated eigenfunction must be of the form

$$f(t) = \begin{cases} a_1 e^{-k_1 t} + b_1 e^{k_1 t} & \text{if } t \in (-\delta, 0) \\ a_2 e^{k_2 t} + b_2 e^{-k_2 t} & \text{if } t \in (0,\delta) \end{cases} \quad (6.2)$$

where $k_1 := \sqrt{M^2 + k^2}$ and $k_2 := \sqrt{m^2 + k^2}$.

We can rewrite the equations satisfied by f as

$$0 = a_2(k_2 - \beta)e^{k_2\delta} - b_2(k_2 + \beta)e^{-k_2\delta}$$

$$\begin{aligned} 0 &= a_1(k_1-\beta)e^{k_1\delta} - b_1(k_2+\beta)e^{-k_2\delta} \\ a_1+b_1 &= a_2+b_2 \\ 0 &= a_2k_2 - b_2k_2 + a_1k_1 - b_1k_1 + (|m|+M)(a_1+b_1). \end{aligned}$$

The first two equations give $b_2 = \frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}a_2$ and $b_1 = \frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}a_1$. Thus, with the equation of continuity we have

$$a_1\left(1+\frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}\right) = a_2\left(1+\frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}\right).$$

We conclude that

$$a_2 = a_1\left(1+\frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}\right)^{-1}\left(1+\frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}\right)$$

because for $\min(|m|, M)$ large enough, one has that the different terms are not zero. We arrive at

$$\begin{aligned} |m|+M = k_2&\left(\frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}-1\right)\left(1+\frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}\right)^{-1} \\ &+ k_1\left(\frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}-1\right)\left(1+\frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}\right)^{-1}. \end{aligned}$$

Let $F(x) := x\left(\frac{x-\beta}{x+\beta}e^{2x\delta}-1\right)\left(1+\frac{x-\beta}{x+\beta}e^{2x\delta}\right)^{-1}$ defined on $(\min(|m|, M), +\infty)$. The previous equation reads $|m|+M = F(k_1)+F(k_2)$, and when $k=0$ the right-hand side is $F(|m|)+F(M) < |m|+M$. Since $F(k_1)+F(k_2)\to+\infty$ when $k\to+\infty$ and F is strictly increasing there exists an unique $k\in(0,+\infty)$ such that $|m|+M = F(k_1)+F(k_2)$.

Now, one has

$$F(x) = x(1+\mathcal{O}(e^{-2x\delta})) = x+\mathcal{O}(e^{-3x\delta/2}).$$

Thus, for $\zeta := \min(|m|, M)$ large enough one has

$$k_2+k_1-2e^{-\zeta\delta} \le |m|+M \le k_2+k_1+2e^{-\zeta\delta}$$

and

$$0 \le \sqrt{m^2+k^2}-|m|+\sqrt{M^2+k^2}-M \le 2e^{-\zeta\delta}.$$

Then, $\sqrt{\zeta^2+k^2}-\zeta \le 2e^{-\zeta\delta}$ and we arrive at

$$k^2 = \mathcal{O}(e^{-\zeta\delta/2}).$$

To conclude, we consider the operator X' defined by the same quadratic form as X but with the form domain $\{f\in H^1(-\delta,\delta), f(0)=0\}$. From the Min-Max principle, one has $E_{j-1}(X') \le E_j(X_\alpha) \le E_j(X')$ for all $j\ge 2$ because X is a rank-one perturbation of X'. But $X' \cong (S_D+m^2)\oplus(S_D+M^2)$ where S_D is the operator acting in $L^2(0,\delta)$ as $f\mapsto -f''$ for $f\in H^2(0,\delta)$ with $f(0) = f'(\delta)-\beta f(\delta) = 0$. We conclude by remarking that $E_j(S_D)\sim \pi^2 j^2/\delta^2$ when $j\to+\infty$, so $E_j(X') \ge \min(m^2, M^2) - C_2 + C_1 j^2$ for suitable $C_1, C_2 > 0$. □

7. Asymptotics analysis for the operator A_m

In this section, we prove Theorem 1.1 following the analysis of [13, Section 4]. The proof is made by localizing the problem near the boundary of $\mathcal{K}$ and using the analysis done in the previous section to find a lower and an upper bound for the limits of the eigenvalues. These bounds coincide and are equal to the eigenvalues of the model operator L introduced in (4.9). We begin by showing a Dirichlet-Neumann bracketing for the operator A_m.

Let $\delta \in (0, \delta_0/2)$. We introduce several new operators. Let Z_m^+, Z_m^-, Z_m' be the operators defined by their quadratic forms z_m^+, z_m^-, z_m' which admit the same expression as the quadratic form of A_m^2 given in Proposition 4.4 with

$$\mathrm{dom}(z_m^+) = \left\{ \Psi \in H^1(\Sigma\mathcal{C}_{|\overline{\Pi_\delta^-}}), \Psi = i\,\nu \cdot \mathbf{n} \cdot \Psi \text{ on } \partial\mathcal{K} \text{ and } \Psi = 0 \text{ on } \Pi^{-\delta} \right\}, \quad (7.1)$$

$$\mathrm{dom}(z_m^-) = \left\{ \Psi \in H^1(\Sigma\mathcal{C}_{|\overline{\Pi_\delta^-}}), \Psi = i\,\nu \cdot \mathbf{n} \cdot \Psi \text{ on } \Pi^0 \right\}, \quad (7.2)$$

$$\mathrm{dom}(z_m') = H^1\left(\Sigma\mathcal{C}_{|\mathcal{K}\setminus(\Pi_\delta^- \cup \Pi^0)} \right). \quad (7.3)$$

We define the maps $J_1 : \mathrm{dom}(A_m) \to \mathrm{dom}(z_m^-) \oplus \mathrm{dom}(z_m')$, $\Psi \mapsto (\Psi_{|\overline{\Pi_\delta^-}}, \Psi_{|\mathcal{K}\setminus(\Pi_\delta^- \cup \Pi^0)})$ and $J_2 : \mathrm{dom}(z_m^+) \to \mathrm{dom}(A_m)$ which is the extension by zero. For $\Psi_1 \in \mathrm{dom}(A_m)$ one has

$$(z_m^- \oplus z_m')\,[J_1(\Psi_1), J_1(\Psi_1)] \le \langle A_m \Psi_1, A_m \Psi_1 \rangle_{L^2(\mathcal{K})},$$

and for $\Psi_2 \in \mathrm{dom}(z_m^+)$,

$$\langle A_m J_2(\Psi_2), A_m J_2(\Psi_2) \rangle_{L^2(\mathcal{K})} \le z_m^+\,[\Psi_2, \Psi_2].$$

Then, the Min-Max principle gives

$$E_j\left(Z_m^- \oplus Z_m'\right) \le E_j\left(A_m^2\right) \le E_j\left(Z_m^+\right). \quad (7.4)$$

We remark that $Z_m' \ge m^2$ and then, for any $j \in N$ such that $E_j\left(Z_m^+\right) < m^2$, one has

$$E_j\left(Z_m^-\right) \le E_j\left(A_m^2\right) \le E_j\left(Z_m^+\right). \quad (7.5)$$

We introduce the notation $\mathbf{S}_\delta^- := \iota(\Sigma\mathcal{C}_{|\overline{\Pi_\delta^-}})$. Let $c > 0$ be the constant given by Proposition 5.8. We consider the two quadratic forms in $L^2(\mathbf{S}_\delta^-, v_h)$ given by

$$\begin{aligned} y_m^+[\Psi,\Psi] := &\int_{\Pi_\delta^-} \left[(1+c\delta)|\overline{\nabla}^{\mathcal{N}} \Gamma_t^0 \Psi|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} \Psi|^2 \right] v_h \\ &+ \int_{\Pi_\delta^-} \left[\left(m^2 + \frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + c\delta \right) |\Psi|^2 \right] v_h + m \int_{\partial\mathcal{K}} |\Psi(\cdot,0)|^2 v_{\partial\mathcal{K}} \\ \mathcal{Q}(y_m^+) := &\left\{ \Psi \in H^1\left(\mathbf{S}_\delta^-\right), \mathcal{P}_- \iota^{-1}(\Psi(\cdot,0)) = 0 \text{ and } \Psi(\cdot,\delta) = 0 \right\}, \end{aligned} \quad (7.6)$$

and

$$\begin{aligned} y_m^-[\Psi,\Psi] := &\int_{\Pi_\delta^-} \left[(1-c\delta)|\overline{\nabla}^{\mathcal{N}} \Gamma_t^0 \Psi|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} \Psi|^2 \right] v_h \\ &+ \int_{\Pi_\delta^-} \left[\left(m^2 + \frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} - c\delta \right) |\Psi|^2 \right] v_h \\ &+ \int_{\partial\mathcal{K}} \left[m|\Psi(\cdot,0)|^2 - c|\Psi(\cdot,\delta)|^2 \right] v_{\partial\mathcal{K}} \\ \mathcal{Q}(y_m^-) := &\left\{ \Psi \in H^1\left(\mathbf{S}_\delta^-\right), \mathcal{P}_- \iota^{-1}\Psi(\cdot,0) = 0 \right\}. \end{aligned} \quad (7.7)$$

Remarking that $\mathcal{Q}(y_m^\pm) = \Theta\iota(\mathrm{dom}(z_m^\pm))$, and that $\Theta\iota$ is unitary from $L^2(\Sigma\mathcal{C}_{|\Pi_\delta^-}, v_{\mathcal{N}})$ onto $L^2\left(\mathbf{S}_\delta^-, v_h\right)$, Proposition 5.8 and the Min-Max principle give

$$\Lambda_j\left(y_m^-\right) \le E_j\left(A_m^2\right) \le \Lambda_j\left(y_m^+\right) \text{ for any } j \in N \text{ such that } \Lambda_j(y_m^+) < m^2. \quad (7.8)$$

7.1. **Upper bound.** The upper bound is found by taking good test functions in the Min-Max principle. The first observation is that the quadratic form y_m^+ admits a separation of variables. Indeed, it can be seen as the tensor product of a sesquilinear form on $\partial\mathcal{K}$ and a one-dimensional sesquilinear form S. The behaviour of its first eigenvalue allows us to find the bound we are searching for.

Let S be the self-adjoint operator on $L^2(0,\delta)$ associated with the quadratic form

$$s[f,f]=\int_0^\delta |f'|^2dt+m|f(0)|^2,\ \mathcal{Q}(s)=\left\{f\in H^1(0,\delta),\ f(\delta)=0\right\}, \tag{7.9}$$

and let f be a normalized eigenfunction for the first eigenvalue of S. According to Lemma 6.1, when $-m$ is large, there is $b>0$ such that $S[f,f]+m^2\le b\exp(-\delta|m|)$.

For $a>0$, we introduce the quadratic form

$$\ell_a[\Psi,\Psi]=\int_{\partial\mathcal{K}}\left[(1+ca)|\overline{\nabla}^N\iota\Psi|^2+\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}}-\mathrm{Tr}(W^2)}{4}+ca\right)|\Psi|^2\right]v_{\partial\mathcal{K}},$$
$$\mathcal{Q}(\ell_a)=\mathcal{Q}(\ell), \tag{7.10}$$

where ℓ was defined in (4.9). The sesquilinear form ℓ_a is lower semibounded and closed. We denote by L_a the associated self-adjoint operator.

Let $\xi_1,\dots,\xi_j$ be linearly independant eigenspinors for the first j eigenvalues of L_δ. We define the set

$$V:=\left\{\Psi\in L^2\left(\mathbf{S}_\delta^-\right),\Psi(x,t)=f(t)\Gamma_0^t(\iota\xi(x)),\ \xi\in\mathrm{Span}(\xi_1,\dots,\xi_j)\right\}. \tag{7.11}$$

With all these notations, for $\Psi(x,t):=f(t)\Gamma_0^t(\iota\xi(x))\in V$ and $-m$ large enough, one has, using Leibniz's rule

$$\begin{aligned}
y_m^+[\Psi,\Psi]&=\int_{\Pi_\delta^-}\left[|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi|^2+(1+c\delta)|\overline{\nabla}^{\mathcal{N}}\Gamma_0^t\Psi|^2\right]v_h\\
&+\int_{\Pi_\delta^-}\left[\left(m^2+\frac{\mathrm{Scal}^{\partial\mathcal{K}}-\mathrm{Tr}(W^2)}{4}+c\delta\right)|\Psi|^2\right]v_h+m\int_{\partial\mathcal{K}}|\Psi(.,0)|^2v_{\partial\mathcal{K}}\\
&=\int_{\Pi_\delta^-}\left[|\frac{\partial}{\partial t}f|^2|\xi|^2+(1+c\delta)|\overline{\nabla}^N\iota\xi|^2|f|^2\right]v_h\\
&+\int_{\Pi_\delta^-}\left[\left(m^2+\frac{\mathrm{Scal}^{\partial\mathcal{K}}-\mathrm{Tr}(W^2)}{4}+c\delta\right)|\Psi|^2\right]v_h+m\int_{\partial\mathcal{K}}|\Psi(\cdot,0)|^2v_{\partial\mathcal{K}}\\
&=\ell_\delta[\xi,\xi]\|f\|^2_{L^2(0,\delta)}+\left(S[f,f]+m^2\right)\|\xi\|^2_{L^2(\partial\mathcal{K})}\\
&\le\ell_\delta[\xi,\xi]+b\exp(-\delta|m|)\|\xi\|^2_{L^2(\partial\mathcal{K})}\\
&\le\left(E_j(L_\delta)+b\exp(-\delta|m|)\right)\|\xi\|^2_{L^2(\partial\mathcal{K})}.
\end{aligned}$$

Thus, $\Lambda_j(y_m^+)\le E_j(L_\delta)+b\exp(-\delta|m|)$. We remark that $\lim_{\delta\to0}E_j(L_\delta)=E_j(L)$ so we get the bound

$$\limsup_{m\to-\infty}E_j(A_m^2)\le E_j(L). \tag{7.12}$$

7.2. **Lower bound.** The strategy to obtain the lower bound is to relax the constraint in the domain of y_m^- in order to obtain a separation of variable. In this way, we arrive are in the good setting to apply the monotone convergence theorem. This analysis will be done in the remaining part of this section.

Let S' be the self-adjoint operator on $L^2(0,\delta)$ associated with the quadratic form

$$S'[f,f]=\int_0^\delta|f'|^2dt+m|f(0)|^2-c|f(\delta)|^2,\ \mathcal{Q}(S')=H^1(0,\delta), \tag{7.13}$$

and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of normalized eigenfunctions for the eigenvalues $E_k(S')$. According to Lemma 6.2, there exist $b^\pm > 0$, $b > 0$ and $b_0 > 0$ such that $E_1(S') \geq -m^2 - be^{-\delta|m|}$ when $m \to -\infty$ and $b^-k^2 - b_0 \leq E_k(S') \leq b^+k^2$ for all $k \geq 2$.

If $c > 0$ is the constant given by Proposition 5.8, we define the quadratic form y_m by the same formula as y_m^-, but with the domain $Q(y_m) = H^1\left(\mathbf{S}_\delta^-\right)$.

We also define for $a \in \mathbb{R}$ the sesquilinear form

$$\ell'_a[\Psi,\Psi] = \int_{\partial\mathcal{K}} \left[(1+ca)|\overline{\nabla}^N \iota\Psi|^2 \left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + ca\right)|\Psi|^2\right] v_{\partial\mathcal{K}},$$
$$\mathcal{Q}(\ell'_a) = H^1(\Sigma\mathcal{C}_{|\partial\mathcal{K}}). \quad (7.14)$$

This form is closed and lower semibounded. We denote by L'_a the associated self-adjoint operator.

We state the following density result, which allows us to express Y_m as the sum of tensor products of operators.

Lemma 7.1. *Let*

$$F := \left\{\Psi,\, \exists(f,\Psi_0) \in L^2(0,\delta) \times L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right),\, \Psi(x,t) = f(-t)\Gamma_0^t\left(\iota\Psi_0(x)\right)\right\}.$$

Then, $\mathrm{Span}(F)$ *is dense in* $L^2\left(\Sigma\Pi_\delta^-\right)$*, so one has a natural isomorphism* $L^2\left(\mathbf{S}_\delta^-, v_h\right) \cong L^2(0,\delta) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$.

Proof. Let $E := (-\delta,0)\times\mathbb{R}$ viewed as a vector bundle over $(-\delta,0)$, and $P := E \otimes \Sigma\mathcal{C}_{|\partial\mathcal{K}}$. The statement of the lemma is then equivalent to the density of $\mathrm{Span}(F')$ in $L^2(P,v_h)$ where

$$F' := \left\{\Psi,\, \exists(f,\Psi_0) \in L^2(-\delta,0) \times L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right),\, \Psi(x,t) = f(t)\Psi_0(x)\right\},$$

and this fact is standard. □

We denote by Y_m the self-adjoint operator associated with y_m, and using the identification of Lemma 7.1, one can write

$$Y_m = (S' + m^2)\otimes 1 + 1 \otimes L'_{-\delta}.$$

Now, we define the unitary transformation

$$\mathcal{U} : L^2\left(\mathbf{S}_\delta^-\right) \longrightarrow \ell^2(\mathbb{N}) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$$
$$\mathcal{U}\Psi = (\Psi_k),\ \Psi_k = \int_0^\delta f_k(t)\,\iota^{-1}\Gamma_t^0(\Psi(\cdot,t))\mathrm{d}t.$$

By the spectral theorem, $\widehat{Y}_m := \mathcal{U}Y_m\mathcal{U}^*$ is given by its quadratic form denoted by $\widehat{y}_m$:

$$\widehat{y}_m[(\Psi_k),(\Psi_k)] = \sum_{k\in\mathbb{N}} \left(\ell'_{-\delta}[\Psi_k,\Psi_k] + (E_k(S') + m^2)\|\Psi_k\|^2_{L^2(\partial\mathcal{K})}\right),$$

and the form domain is the subset of $\ell^2(\mathbb{N}) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$ for which the right-hand side converges. Thus,

$$\mathcal{Q}(\widehat{y}_m) = \Big\{(\Psi_k) \in \ell^2(\mathbb{N}) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right),\, \Psi_k \in H^1\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$$
$$\text{and } \sum\left(\|\Psi_k\|^2_{H^1(\partial\mathcal{K})} + k^2\|\Psi_k\|^2_{L^2(\partial\mathcal{K})}\right) < \infty\Big\}. \quad (7.15)$$

Setting $\widehat{Y}_m^- := \mathcal{U}Y_m^-\mathcal{U}^*$, the sesquilinear form for $\widehat{Y}_m^-$ is the same as for $\widehat{Y}_m$ with the domain

$$\mathcal{Q}(\widehat{y}_m^-) = \left\{\widehat{\Psi} = (\Psi_k) \in \mathcal{Q}(\widehat{y}_m) : \mathcal{P}_-\mathcal{U}^*\widehat{\Psi}(\cdot,0) = 0\right\}. \quad (7.16)$$

Then, if we define

$$w_m[\widehat{\Psi},\widehat{\Psi}] := \ell'_{-\delta}[\Psi_1,\Psi_1] - b\exp(-\delta|m|)\|\Psi_1\|^2_{L^2(\partial\mathcal{K})} + \sum_{k\geq 2} \ell'_{-\delta}[\Psi_k,\Psi_k] + (b^- k^2 - b_0 + m^2)\|\Psi_k\|^2_{L^2(\partial\mathcal{K})},$$

$$\mathcal{Q}(w_m) := \mathcal{Q}(\widehat{y_m^-}), \quad (7.17)$$

we have $\widehat{y_m^-} \geq w_m$. The form w_m is semibounded form below and closed. Let W_m be the associated self-adjoint operator. By Theorem 2.11, this operator has compact resolvent. For all $j \in \mathbb{N}$, one has

$$E_j(A_m^2) \geq \Lambda_j(y_m^-) = \Lambda_j(\widehat{y_m^-}) \geq E_j(W_m).$$

We can now apply the monotone convergence theorem to the non-decreasing family of self-adjoint operators (W_m). The form domain of the limit operator will be:

$$\mathcal{Q}_\infty := \left\{ \widehat{\Psi} = (\Psi_k) \in \bigcap_{m<0} \mathcal{Q}(W_m),\ \sup_{m<0} W_m[\widehat{\Psi},\widehat{\Psi}] < \infty \right\}. \quad (7.18)$$

One has $\widehat{\Psi} := (\Psi_k) \in \mathcal{Q}_\infty$ iff $\Psi_k = 0$ for all $k \geq 2$ and $0 = \mathcal{P}_-\mathcal{U}^*\widehat{\Psi}(\cdot,0) = f_1(0)\mathcal{P}_-\Psi_1$. If we denote by $e_1 := (1,0,0,\ldots) \in \ell^2(\mathbb{N})$ this gives

$$\mathcal{Q}_\infty = \left\{ \widehat{\Psi} = e_1 \otimes \Psi_1 : \Psi_1 \in \mathcal{Q}(\ell) \right\}.$$

Thus, for any $\widehat{\Psi} \in \mathcal{Q}_\infty$ one has

$$\lim_{m\to-\infty} W_m[\widehat{\Psi},\widehat{\Psi}] = L_{-\delta}[\Psi_1,\Psi_1].$$

We define the Hilbert space $\mathbf{H}_\infty := e_1 \otimes \mathbf{H}$ and the sesquilinear form

$$w_\infty[e_1\otimes\Psi_1, e_1\otimes\Psi_1] = L_{-\delta}[\Psi_1,\Psi_1],\ \mathcal{Q}(w_\infty) = \mathbf{H}_\infty. \quad (7.19)$$

Let W_∞ be the associated self-adjoint operator. By Corollary 2.4 (monotone convergence), one has $\lim_{m\to-\infty} E_j(W_m) = E_j(W_\infty) = E_j(L_{-\delta})$ for all $j \in \mathbb{N}$. Letting δ go to 0 we obtain

$$\liminf_{m\to\infty} E_j(A_m^2) \geq E_j(L). \quad (7.20)$$

The estimates (7.12) and (7.20) together with Lemma 4.10 give

$$\lim_{m\to\infty} E_j\left(A_m^2\right) = E_j\left((\not{D}^{\partial\mathcal{K}})^2\right). \quad (7.21)$$

Remark 7.2. With the help of the sesquilinear form, we can investigate another asymptotic regime. Let $\Psi \in \mathrm{dom}(A_m)$ and assume $m > 0$. Proposition 4.6 gives that for m large enough, $\|A_m\Psi\|^2_{L^2(\mathcal{N})} \geq m^2\|\Psi\|^2_{L^2(\mathcal{N})}$. Hence, when $m \to +\infty$, one has $E_j(A_m) \to +\infty$ for all $j \in \mathbb{N}$ by the Min-Max principle.

8. The operator $B^2_{m,M}$ in the limit of large M

We now prove Theorem 1.2 following the lines of [13, Section 5]. Again, this is done by finding a lower and an upper bound for the limit of the eigenvalues of $B^2_{m,M}$. The proof relies on the localization of the problem in a neighbourhood of $\mathcal{K}$ and the construction of an appropriated extension for the spinors in $\mathcal{K}$. For the lower bound, we make another use of the monotone convergence theorem to observe that the projection $\mathcal{P}_+$ on the boundary of $\mathcal{K}$ must vanish in the asymptotic regime.

We begin with some preliminary estimates and the definition of the extension operator.

Lemma 8.1. *Let r'_α be the sesquilinear form given by*

$$r'_\alpha[\Psi,\Psi] := \int_{\mathcal{K}^c\setminus\Pi^+_\delta}\left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right)v_{\mathcal{N}}$$

with $\mathcal{Q}(r'_\alpha) = \{\Psi_{|\mathcal{K}^c\setminus\Pi^+_\delta},\ \Psi\in\mathrm{dom}(B_{m,M})\}$. Then, r'_α is semibounded from below.

Proof. Let $\Psi\in\mathcal{Q}(r'_\alpha)$. Let χ_1,χ_2 be two non-negative real smooth functions on $\mathcal{N}$ such that $\chi_1^2+\chi_2^2 = 1$, χ_1 is supported in $\mathcal{K}\cup\Pi^+_{\frac{3\delta}{2}}$ and χ_2 is supported in $\mathcal{N}\setminus(\mathcal{K}\cup\Pi^+_{\frac{5\delta}{4}})$.

An easy computation gives

$$r'_\alpha[\Psi,\Psi] = r'_\alpha[\chi_1\Psi,\chi_1\Psi] + r'_\alpha[\chi_2\Psi,\chi_2\Psi] - \int_{\mathcal{K}^c\setminus\Pi^+_\delta}(|(\mathrm{d}\chi_1)\iota\Psi|^2 + |(\mathrm{d}\chi_2)\iota\Psi|^2)v_{\mathcal{N}},$$

and then there exists a constant $C_1>0$ such that

$$r'_\alpha[\Psi,\Psi] \geq r'_\alpha[\chi_1\Psi,\chi_1\Psi] + r'_\alpha[\chi_2\Psi,\chi_2\Psi] - C_1\|\Psi\|^2_{L^2(\mathcal{N})}.$$

Now, the Schrödinger-Lichnerowicz formula gives

$$r'_\alpha[\chi_2\Psi,\chi_2\Psi] = \|\mathcal{D}^{\mathcal{N}}\chi_k\Psi\|^2_{L^2(\mathcal{N})}\geq 0.$$

Moreover, there exists $C_2>0$ such that

$$r'_\alpha[\chi_1\Psi,\chi_1\Psi] \geq -C_2\|\chi_1\Psi\|^2_{L^2(\mathcal{N})}$$

because χ_1 has compact support.

Altogether, we have $r'_\alpha[\Psi,\Psi]\geq -C\|\Psi\|^2_{L^2(\mathcal{N})}$ for a constant $C>0$. □

We define $\mathbf{S}^+_\delta := \iota(\Sigma\mathcal{C}_{|\overline{\Pi^+_\delta}})$.

Lemma 8.2. *For $\Psi\in\{\Phi_{|\mathcal{K}^c},\ \Phi\in\mathrm{dom}(B_{m,M})\}$ and $\alpha>0$ we define the sesquilinear form*

$$r_\alpha[\Psi,\Psi] = \int_{\mathcal{K}^c}\left(\left|\nabla^{\mathcal{N}}\iota\Psi\right|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right)v_{\mathcal{N}} + \int_{\partial\mathcal{K}}\left(\frac{H}{2}-\alpha\right)|\Psi|^2 v_{\partial\mathcal{K}}.$$

Then, there exists $C>0$ such that for $\alpha>0$ large enough, one has a map F_α : $H^1(\iota(\Sigma\mathcal{C}_{|\partial\mathcal{K}}))\to\mathrm{dom}(r_\alpha)$ with $F_\alpha\Psi=\Psi$ on $\partial\mathcal{K}$ and

$$r_\alpha[F_\alpha\Psi,F_\alpha\Psi] + \alpha^2\|F_\alpha\Psi\|^2_{L^2(\mathcal{K}^c)} \leq \frac{c}{\alpha}\|\Psi\|^2_{H^1(\partial\mathcal{K})}.$$

Moreover there exists a constant $C_0>0$, such that $\Lambda_1(r_\alpha)\geq -\alpha^2 - C_0$.

Proof. We recall that for $\alpha>0$ we defined in (7.9) the operator S associated with the sesquilinear form

$$s[f,f] = \int_0^\delta |f'|^2dt - \alpha|f(0)|^2,\ \mathcal{Q}(s) = \left\{f\in H^1(0,\delta),\ f(\delta)=0\right\}.$$

Let f be the first eigenfunction of the operator S normalized by $f(0)=1$.

We define the map F_α by

$$F_\alpha\Psi(x) := \begin{cases} (\Theta\iota)^{-1}v(x) & \text{if } x\in\Pi^+_\delta \\ 0 & \text{if } x\in\mathcal{K}^c\setminus\Pi^+_\delta\end{cases}$$

where $v:=f\otimes\Psi$. From Lemma 6.1 there exists $C>0$ such that $\|f\|^2_{L^2(0,\delta)}\leq\frac{C}{\alpha}$ and $\alpha^2+E_1(S)\leq Ce^{-\delta\alpha}$. Then, using Proposition 5.8, one can find $a>0$ such that

$$r_\alpha[F_\alpha\Psi,F_\alpha\Psi] + \alpha^2\|F_\alpha\Psi\|^2_{L^2(\mathcal{K}^c)} = J_\alpha(F_\alpha\Psi) + \alpha^2\|f_\alpha\|^2_{L^2(\mathcal{K}^c)}$$

$$\leq \int_{\Pi_\delta^+} \left(a|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0 v|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} v|^2 + (\alpha^2+a)|v|^2\right) v_h - \alpha \int_{\partial\mathcal{K}} |\Psi|^2 v_{\partial\mathcal{K}}$$
$$= \int_{\partial\mathcal{K}} \left[a|\overline{\nabla}^{\mathcal{N}}\Psi|^2 + (E_1(S)+\alpha^2+a)|\Psi|^2\right] v_h \|f\|^2_{L^2(0,\delta)}$$
$$\leq \frac{C(C+a)}{\alpha}\|\Psi\|^2_{H^1(\partial\mathcal{K})}.$$

For the second assertion, we introduce the sesquilinear forms

$$r_\alpha^0[\Psi,\Psi] := \int_{\Pi_\delta^+} \left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right) v_{\mathcal{N}} + \int_{\partial\mathcal{K}} \left(\frac{H}{2}-\alpha\right)|\Psi|^2 v_{\partial\mathcal{K}}$$

with $\mathcal{Q}(r_\alpha^0) = \{\Psi_{|\Pi_\delta^+},\ \Psi \in \mathrm{dom}(B_{m,M})\}$ and

$$r'_\alpha[\Psi,\Psi] := \int_{\mathcal{K}^c\setminus\Pi_\delta^+} \left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right) v_{\mathcal{N}}$$

with $\mathcal{Q}(r'_\alpha) = \{\Psi_{|\mathcal{K}^c\setminus\Pi_\delta^+},\ \Psi \in \mathrm{dom}(B_{m,M})\}$. One has the inequality $\Lambda_1(r_\alpha) \geq \min(\Lambda_1(r'_\alpha), \Lambda_1(r_\alpha^0))$. Since r'_α is lower semibounded by Lemma 8.1, another use of Proposition 5.8 gives that when α is large $\Lambda_1(r_\alpha^0) \geq \Lambda_1(q_\alpha)$ with

$$q_\alpha[\Psi,\Psi] = \int_{\Pi_\delta^+} \left[\frac{1}{a}|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi|^2 + |\nabla^h_{\frac{\partial}{\partial t}}\Psi|^2 - a|\Psi|^2\right] v_h$$
$$- \alpha\int_{\partial\mathcal{K}} |\Psi(\cdot,0)|^2 v_{\partial\mathcal{K}} - a\int_{\partial\mathcal{K}} |\Psi(\cdot,\delta)|^2 v_{\partial\mathcal{K}}$$

where $a > 0$ and $\mathcal{Q}(q_\alpha) = H^1\left(\mathbf{S}_\delta^+\right)$. Trivializing locally the vector bundle via parallel sections along the normal geodesics and using Fubini's theorem we deduce that $\Lambda_1(r_\alpha) \geq \Lambda_1(q_\alpha) \geq \Lambda_1(S') - a \geq -\alpha^2 - C$ with $C > 0$ when $\alpha \to +\infty$. □

Using Proposition 4.8, the sesquilinear form for $B_{m,M}^2$ can be written for any spinor $\Psi \in \mathrm{dom}(B_{m,M})$ and any $\varepsilon > 0$ as

$$\begin{aligned}\|B_{m,M}\Psi\|^2_{L^2(\mathcal{N})} &= \int_{\mathcal{K}} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4}+m^2\right)|\Psi|^2\right] v_{\mathcal{N}} \\ &+ \int_{\partial\mathcal{K}} \left(m-\varepsilon-\frac{H}{2}\right)|\Psi|^2 v_{\partial\mathcal{K}} + 2(M-m)\int_{\partial\mathcal{K}} |\mathcal{P}_-\Psi|^2 v_{\partial\mathcal{K}} \\ &+ \int_{\mathcal{K}^c} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4}+M^2\right)|\Psi|^2\right] v_{\mathcal{N}} - \int_{\partial\mathcal{K}} \left(M-\varepsilon-\frac{H}{2}\right)|\Psi|^2 v_{\partial\mathcal{K}}\end{aligned} \tag{8.1}$$

where we recall that $\mathcal{P}_- = \frac{1-i\nu\cdot\mathbf{n}\cdot}{2}$.

8.1. **Upper bound.** We are now able to find an upper bound for the limit of $E_j(B_{m,M}^2)$ when $M \to +\infty$ for $j \in \mathbb{N}$. Let $\eta > 0$ and pick $(\Psi_1,\ldots,\Psi_j)$ in $\Gamma(\Sigma\mathcal{C}_{|\mathcal{K}})$, smooth spinors such that

$$\inf_{\Psi\in\mathrm{Span}(\Psi_1,\ldots,\Psi_j)} \frac{\langle A_m^2\Psi,\Psi\rangle_{L^2(\mathcal{K})}}{\|\Psi\|^2_{L^2(\mathcal{K})}} \leq E_j(A_m^2) + \eta.$$

We define $a := \sup\left\{\|\Psi\|^2_{H^1(\partial\mathcal{K})}, \Psi \in \mathrm{Span}(\Psi_1,\ldots,\Psi_j), \|\Psi\|_{L^2(\mathcal{K})} = 1\right\}$. Let $\Psi \in V := \mathrm{Span}(\Psi_1,\ldots,\Psi_j)$ and

$$\widetilde{\Psi} := \begin{cases} \Psi & \text{in } \mathcal{K} \\ F_M(\Psi_{|\partial\mathcal{K}}) & \text{in } \mathcal{K}^c. \end{cases}$$

By Lemma 8.2 there is a constant $C > 0$ such that

$$\int_{\mathcal{K}^c} \left[|\nabla^{\mathcal{N}}(\iota\widetilde{\Psi})|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + M^2 \right) |\widetilde{\Psi}|^2 \right] v_{\mathcal{N}} - \int_{\partial\mathcal{K}} \left(M - \frac{H}{2} \right) |\widetilde{\Psi}|^2 v_{\partial\mathcal{K}}$$
$$= r_M[\widetilde{\Psi}, \widetilde{\Psi}] + M^2 \|\widetilde{\Psi}\|^2_{L^2(\mathcal{K}^c)} \leq \frac{C}{M} \|\widetilde{\Psi}\|^2_{H^1(\partial\mathcal{K})} \leq \frac{Ca}{M} \|\Psi\|^2_{L^2(\mathcal{K})}.$$

Then, using the expression (8.1) with $\varepsilon = 0$,

$$\|B_{m,M}\widetilde{\Psi}\|^2_{L^2(\mathcal{N})} \leq A^2_m[\Psi, \Psi] + \frac{Ca}{M} \|\Psi\|^2_{L^2(\mathcal{K})} \leq \left(E_j(A^2_m) + \eta + \frac{Ca}{M} \right) \|\Psi\|^2_{L^2(\mathcal{K})}$$
$$\leq \left(E_j(A^2_m) + \eta + \frac{Ca}{M} \right) \|\widetilde{\Psi}\|^2_{L^2(\mathcal{K})}$$

and letting η go to zero one gets $\limsup_{M\to+\infty} E_j(B^2_{m,M}) \leq E_j(A^2_m)$.

8.2. **Lower bound.** It remains to find a lower bound for the eigenvalues. In order to do so, we separate the representation (8.1) in the two parts corresponding to $\mathcal{K}$ and $\mathcal{K}^c$ and we remark that the outer part becomes very large when M goes to $+\infty$ so the eigenvalues must converge to the eigenvalues of an operator in $\mathcal{K}$.

Let $j \in \mathbb{N}$. One has

$$E_j(B^2_{m,M}) \geq \min\left\{ \Lambda_j(k^c_{M,\varepsilon}), E_j(K_{m,M,\varepsilon}) \right\}$$

where $K_{m,M,\varepsilon}$ is the operator associated with the sesquilinear form

$$k_{m,M,\varepsilon}[\Psi, \Psi] := \int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \left(m^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4} \right) |\Psi|^2 \right) v_{\mathcal{N}}$$
$$+ \int_{\partial\mathcal{K}} (m - \varepsilon - \frac{H}{2}) |\Psi|^2 v_{\partial\mathcal{K}} + 2(M - m) \int_{\partial\mathcal{K}} |P_-\Psi|^2 v_{\partial\mathcal{K}} \quad (8.2)$$

and $k^c_{M,\varepsilon}$ is the sesquilinear form

$$k^c_{M,\varepsilon}[\Psi, \Psi] := \int_{\mathcal{K}^c} \left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \left(M^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4} \right) |\Psi|^2 \right) v_N$$
$$- \int_{\partial\mathcal{K}} (M - \varepsilon - \frac{H}{2}) |\Psi|^2 v_{\partial\mathcal{K}} \quad (8.3)$$

where the respective domains are the restrictions of $\mathrm{dom}(B_{m,M})$ to $\mathcal{K}$ and $\mathcal{K}^c$.

One has $k^c_{M,\varepsilon} = r_{M-\varepsilon} + M^2$, where $r_{M-\varepsilon}$ was defined in Lemma 8.2). The same lemma gives

$$\Lambda_1(k^c_{M,\varepsilon}) = \Lambda_1(r_{M-\varepsilon} + M^2) \geq -(M - \varepsilon)^2 - C_0 + M^2 = 2\varepsilon M - \varepsilon^2 - C_0$$
$$= \varepsilon M + (\varepsilon M - \varepsilon^2 - C_0) \geq \varepsilon M \text{ when } M \to +\infty.$$

It follows that $E_j(B^2_{m,M}) = E_j(K_{m,M,\varepsilon})$ when $M \to +\infty$. But $k_{M,m,\varepsilon}$ is increasing in$M$, and

$$k_{M,m,\varepsilon}[\Psi, \Psi] \underset{M\to+\infty}{\longrightarrow} \langle A_m\Psi, A_m\Psi \rangle_{L^2(\mathcal{K})} - \varepsilon \|\Psi\|_{L^2(\partial\mathcal{K})}.$$

Furthermore,

$$\left\{ \Psi \in \bigcap_{M>0} \mathrm{dom}(k_{m,M,\varepsilon}), \lim_{M\to+\infty} k_{m,M,\varepsilon}[\Psi, \Psi] < \infty \right\} = \mathrm{dom}(A_m),$$

thus, by monotone convergence (Corollary 2.4) and letting ε go to 0, we obtain $\liminf_{M\to+\infty} E_j(B^2_{m,M}) = E_j(A^2_m)$. Taking into account the upper bound obtained above, one gets $\lim_{M\to+\infty} E_j(B^2_{m,M}) = E_j(A^2_m)$.

9. The operator $B_{m,M}$ for large masses

In this section, we investigate the asymptotic regime $m \to -\infty$ and $M \to +\infty$ and we give a proof of Theorem 1.3. The method we use is very similar to the one of section 8. The difference lies in the proof of the lower bound, where we do not make the analysis on the operator outside and inside $\mathcal{K}$, but we rather divide the ambient space into three pieces: the tubular neighbourhood of $\partial\mathcal{K}$, and the remaining regions lying inside and outside the compact $\mathcal{K}$. By Dirichlet-Neumann bracketing, it is then sufficient to study the operator restricted to the tubular neighbourhood to conclude.

9.1. **Upper bound.** In this section, we write $\mathbf{S}_\delta := \iota\left(\Sigma\mathcal{C}_{|\overline{\Pi_\delta}}\right)$. We recall that for $\alpha \in \mathbb{R}$ we defined the self-adjoint operator S_α associated with the quadratic form

$$s_\alpha[f,f] = \int_0^\delta |f'|^2 \mathrm{d}t - \alpha|f(0)|^2,\ \mathcal{Q}(s_\alpha) = \left\{f \in H^1(0,\delta), f(\delta) = 0\right\}, \tag{9.1}$$

and denoting by f_α the L^2-normalized eigenfunction associated with $E_1(S_\alpha)$, one has $|f_\alpha(0)|^2 = 2\alpha + \mathcal{O}(1)$ and $E_1(S_\alpha) = \alpha^2 + \mathcal{O}(e^{-\alpha\delta})$ when $\alpha \to +\infty$ (see Lemma 6.1).

The operator L_a was defined by the quadratic form (7.10).

Let $j \in \mathbb{N}$ and $\Psi_1, \dots, \Psi_j$ be j eigenspinors for the first j eigenvalues of L_δ. For $\Psi \in V := \mathrm{Span}(\Psi_1, \dots, \Psi_j)$, we define the extension operator $\mathcal{E} : H^1(\mathbf{S}_\delta) \to H^1(\Sigma\mathcal{C}_{|\mathcal{N}})$ by

$$\mathcal{E}\Psi := \begin{cases} \frac{|f_{-m}(0)|}{|f_M(0)|}(\Theta\iota)^{-1}(\Psi \otimes f_M) & \text{in } \Pi_\delta^+ \\ (\Theta\iota)^{-1}(\Psi \otimes f_{-m}) & \text{in } \Pi_\delta^- \\ 0 & \text{in } \mathcal{N} \setminus \Pi_\delta \end{cases}. \tag{9.2}$$

One easily sees that $\|\mathcal{E}\Psi\|^2_{L^2(\mathcal{N})} = \left(1 + \left(\frac{f_{-m}(0)}{f_M(0)}\right)^2\right)\|\Psi\|^2_{L^2(\partial\mathcal{K})}$, so the operator $\mathcal{E}$ is injective. We use the expression (8.1) and Proposition 5.8 to compute:

$$\begin{aligned}
\|B^2_{m,M}\mathcal{E}\Psi\|^2_{L^2(\mathcal{N})} &= \int_{\mathcal{K}} \left[|\nabla^{\mathcal{N}}(\iota\mathcal{E}\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + m^2\right)|\mathcal{E}\Psi|^2\right] v_{\mathcal{N}} \\
&+ \int_{\partial\mathcal{K}} \left(m - \varepsilon - \frac{H}{2}\right)|\mathcal{E}\Psi|^2 v_{\partial\mathcal{K}} + \int_{\mathcal{K}^c} \left[|\nabla^{\mathcal{N}}(\iota\mathcal{E}\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + M^2\right)|\mathcal{E}\Psi|^2\right] v_{\mathcal{N}} \\
&- \int_{\partial\mathcal{K}} \left(M - \varepsilon - \frac{H}{2}\right)|\mathcal{E}\Psi|^2 v_{\partial\mathcal{K}} \\
&\leq \int_{\Pi_\delta^-} \left[(1 + c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi \otimes f_{-m})(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi \otimes f_{-m}|^2\right] v_h \\
&+ \int_{\Pi_\delta^-} \left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + m^2 + c\delta\right)|\Psi \otimes f_{-m}|^2\right] v_{\partial\mathcal{K}}\mathrm{d}t \\
&+ \int_{\Pi_\delta^+} \left[(1 + c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi \otimes f_M)(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi \otimes f_M|^2\right] v_h \\
&+ \int_{\Pi_\delta^+} \left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + M^2 + c\delta\right)|\Psi \otimes f_M|^2\right] v_{\partial\mathcal{K}}\mathrm{d}t \\
&+ \int_{\partial\mathcal{K}} (-m|\Psi \otimes f_{-m}(\cdot,0)|^2 + M|(\Psi \otimes f_M)(\cdot,0)|^2) v_{\partial\mathcal{K}}
\end{aligned}$$

$$\leq \left(1+\left(\frac{f_{-m}(0)}{f_M(0)}\right)^2\right)\left[\ell_\delta[\Psi,\Psi]+C\|\Psi\|^2_{L^2(\partial\mathcal{K})}\left(e^{-M\delta}+e^{-|m|\delta}\right)\right]$$

where $C>0$.
The Min-Max principle gives

$$\begin{aligned}E_j(B^2_{m,M}) &\leq \sup_{\Psi\in V}\frac{B^2_{m,M}[\mathcal{E}\Psi,\mathcal{E}\Psi]}{\|\mathcal{E}\Psi\|^2_{L^2(\mathcal{N})}}\\ &\leq \sup_{v\in V}\left[L_\delta[\Psi,\Psi]+C\|\Psi\|^2_{L^2(\partial\mathcal{K})}\left(e^{-M\delta}+e^{-|m|\delta}\right)\right]\|\Psi\|^{-2}_{L^2(\partial\mathcal{K})}\\ &\leq E_j(L_\delta)+C\left(e^{-M\delta}+e^{-|m|\delta}\right).\end{aligned}$$

We now let $\min(-m,M)\to+\infty$, so we obtain

$$\limsup_{\min(-m,M)\to+\infty} E_j(B^2_{m,M})\leq E_j(L_\delta).$$

On the other hand, δ can be taken arbitrary small, and one has the obvious limit $E_j(L)\underset{\delta\to 0}{\longrightarrow}E_j(L)$, so we arrive at

$$\limsup_{\min(-m,M)\to+\infty} E_j(B^2_{m,M})\leq E_j(L). \tag{9.3}$$

9.2. **Lower bound.** We consider the lower semibounded sesquilinear forms

$$k_{m,M}[\Psi,\Psi]=\int_{\mathcal{N}\setminus\Pi_\delta}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2+\left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4}+m^2\mathbf{1}_{\mathcal{K}}+M^2\mathbf{1}_{\mathcal{K}^c}\right)|\Psi|^2\right]v_{\mathcal{N}}$$
$$\mathcal{Q}(K_{m,M})=\{\Psi_{\mathcal{N}\setminus\Pi_\delta},\ \Psi\in\mathrm{dom}(B_{m,M})\} \tag{9.4}$$

and

$$\begin{aligned}k'_{m,M}[u,u]&=\int_{\Pi_\delta^-}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2+\left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4}+m^2\right)|\Psi|^2\right]v_{\mathcal{N}}\\ &\quad+\int_{\partial\mathcal{K}}\left(m-\varepsilon-\frac{H}{2}\right)|\Psi|^2v_{\partial\mathcal{K}}+2(M-m)\int_{\partial\mathcal{K}}|\mathcal{P}_-\Psi|^2v_{\partial\mathcal{K}}\\ &+\int_{\Pi_\delta^+}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2+\left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4}+M^2\right)|\Psi|^2\right]v_{\mathcal{N}}-\int_{\partial\mathcal{K}}\left(M-\varepsilon-\frac{H}{2}\right)|\Psi|^2v_{\partial\mathcal{K}},\end{aligned}$$
$$\mathcal{Q}(K'_{m,M})=H^1(\Sigma\mathcal{C}_{\overline{\Pi_\delta}}). \tag{9.5}$$

We denote by $K'_{m,M}$ the operator associated with $k'_{m,M}$.
Let $j\in\mathbb{N}$. The Min-Max principle gives the lower estimate $E_j(B^2_{m,M})\geq\min(E_j(K'_{m,M}),\Lambda_1(k_{m,M}))$, and by Lemma 8.1 there is a constant $C>0$ such that $\Lambda_1(k_{m,M})\geq\min(m^2,M^2)-C$. This last quantity goes to $+\infty$ in the asymptotic regime under consideration, and we know thanks to the upper bound that $E_j(B^2_{m,M})=\mathcal{O}(1)$. Thus, in the given asymptotic regime one has $E_j(B^2_{m,M})\geq E_j(K'_{m,M})$.
We now apply a transformation to the operator $K'_{m,M}$ written in tubular coordinates, and we consider the operator $P_{m,M}$ associated with the quadratic form

$$\begin{aligned}p_{m,M}[\Psi,\Psi]&=\int_{\Pi_\delta}\left[(1-c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma^0_t\Psi)(x,0)\right|^2+|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi|^2\right]v_h\\ &\quad+\int_{\Pi_\delta}\left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}}-\mathrm{Tr}(W^2)}{4}+m^2\mathbf{1}_{\mathcal{K}}+M^2\mathbf{1}_{\mathcal{K}^c}-c\delta\right)|\Psi|^2\right]v_{\partial\mathcal{K}}\mathrm{d}t\end{aligned}$$

$$+ (m-M)\int_{\partial\mathcal{K}} |\Psi(\cdot,0)|^2 v_{\partial\mathcal{K}} - c\int_{\partial\mathcal{K}} |\Psi(\cdot,\delta)| v_{\partial\mathcal{K}} + 2(M-m)\int_{\partial\mathcal{K}} |\mathcal{P}_-\Psi|^2 v_{\mathcal{K}},$$
$$\mathcal{Q}(p_{m,M}) = H^1(S_\delta), \quad (9.6)$$

where $c>0$ is chosen so that Proposition 5.8 is valid, implying that $E_j(K'_{m,M}) \geq E_j(P_{m,M})$.

For $a\in\mathbb{R}$, let L''_a be the operator given by the sesquilinear form ℓ''_a, having the same expression as (7.10) but with the domain $H^1(\Sigma\mathcal{C}_{|\partial\mathcal{K}})$.

Let $P'_{m,M}$ be the operator defined by the same quadratic form as in (9.6) but without the term involving the operator $\mathcal{P}_-$. We recall that the one-dimensional operator X was defined by (6.1), so one has

$$P'_{m,M} = \ell''_{-\delta}\otimes 1 + 1\otimes X.$$

Let (f_k) be a sequence of L^2-normalized eigenfunctions for the eigenvalues $E_k(X)$. We define the unitary transformation

$$\mathcal{U} : L^2(\mathbf{S}_\delta) \longrightarrow \ell^2(\mathbb{N})\otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$$
$$\mathcal{U}\Psi = (\Psi_k),\ \Psi_k = \int_{-\delta}^{\delta} f_k(t)\,\iota^{-1}\Gamma^0_t(\Psi(t,\cdot))\mathrm{d}t.$$

Let $\widehat{P}'_{m,M} := \mathcal{U}P'_{m,M}\mathcal{U}^*$. This is a self-adjoint operator acting on $\ell^2(\mathbb{N})\otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$. One can write

$$\widehat{P}'_{m,M}[\widehat{v},\widehat{v}] = \sum_{k\in\mathbb{N}}\left(\ell''_{-\delta}[\Psi_k,\Psi_k] + E_k(X)\|\Psi_k\|^2_{L^2(\Sigma)}\right),$$
$$\mathcal{Q}(\widehat{P}'_{m,M}) = \Big\{\widehat{\Psi}\in\ell^2(\mathbb{N})\otimes L^2(\Sigma\mathcal{C}_{|\partial\mathcal{K}}), \Psi_k\in H^1(\Sigma\mathcal{C}_{|\partial\mathcal{K}}),$$
$$\sum_{k\in\mathbb{N}}\left(\|\Psi_k\|^2_{H^1(\partial\mathcal{K})} + k^2\|\Psi_k\|^2_{L^2(\partial\mathcal{K})}\right)\Big\}. \quad (9.7)$$

The operator $\widehat{P}_{m,M} = \mathcal{U}^*P_{m,M}\mathcal{U}$ has the same form domain as $\widehat{P}'_{m,M}$ and

$$\widehat{P}_{m,M}[\widehat{\Psi},\widehat{\Psi}] = \sum_{k\in\mathbb{N}}\left(\ell''_{-\delta}[\Psi_k,\Psi_k] + E_k(X)\|\Psi_k\|^2_{L^2(\Sigma)}\right) + 2(M+|m|)\int_\Sigma |\mathcal{P}_-\mathcal{U}^*\widehat{\Psi}|^2\mathrm{d}s.$$

where the operator X was defined in (6.1). We set

$$\zeta := \min(M,-m). \quad (9.8)$$

Using Lemma 6.3, we consider the quadratic form w_ζ defined by

$$w_\zeta[\widehat{\Psi},\widehat{\Psi}] = \ell''_{-\delta}[\Psi_1,\Psi_1] - Ce^{-\zeta\delta/2} + 4\zeta\int_\Sigma |\mathcal{P}_-\mathcal{U}^*\widehat{\Psi}|^2\mathrm{d}s$$
$$+\sum_{k\geq 2}\left(\ell''_{-\delta}[\Psi_k,\Psi_k] + (C_1k^2 - C_2)\|\Psi_k\|^2_{L^2(\Sigma,\mathbb{C}^N)} + \zeta^2\|\Psi_k\|^2_{L^2(\Sigma)}\right),$$
$$\mathcal{Q}(w_\zeta) = \mathcal{Q}(\widehat{P}_{m,M}), \quad (9.9)$$

and we claim that $\widehat{P}_{m,M}\geq w_\zeta$ for a suitable $C>0$. The form w_ζ is semibounded from below and closed, and we define the associated self-adjoint operator W_ζ with compact resolvent. The previous discussion gives the lower estimate $E_j(B^2_{m,M}) \geq E_j(W_\zeta)$ in the asymptotic regime.

In order to apply the monotone convergence theorem, we define

$$\mathcal{Q}_\infty = \left\{ \widehat{\Psi} \in \bigcap_{\zeta>0} \mathcal{Q}(W_\zeta) = \mathcal{Q}(w_\zeta),\ \sup_{\zeta>0} w_\zeta[\widehat{\Psi}, \widehat{\Psi}] < +\infty \right\}. \tag{9.10}$$

We easily see that $\widehat{\Psi}$ is in $\mathcal{Q}_\infty$ if and only if $\Psi_k = 0$ for all $k \geq 2$ and $\mathcal{P}_- \mathcal{U}^* \widehat{\Psi} = 0$, which is equivalent to $\widehat{\Psi} = e_1 \otimes \Psi_1$ with $e_1 := (1, 0, 0, \ldots)$ and $\mathcal{P}_- \Psi_1 = 0$. It follows that $\mathcal{Q}_\infty = \left\{ e_1 \otimes \Psi_1 : \Psi_1 \in H^1(\Sigma, \mathbb{C}^N) \cap \mathcal{H} \right\}$. Moreover, we have

$$\lim_{\zeta \to \infty} W_\zeta[e_1 \otimes \Psi_1, e_1 \otimes \Psi_1] = L_{-\delta}[\Psi_1, \Psi_1]. \tag{9.11}$$

Thus, if we define the operator $W_\infty[e_1 \otimes \Psi_1, e_1 \otimes \Psi_1] := L_{-\delta}[\Psi_1, \Psi_1]$ on $e_1 \otimes \mathcal{H}$, the monotone convergence theorem gives $\lim_{\zeta \to \infty} E_j(W_\zeta) = E_j(L_{-\delta})$. Altogether, we arrive at $\liminf_{\min(-m,M) \to +\infty} E_j(B^2_{m,M}) \geq E_j(L_{-\delta})$. We now let δ go to zero and we obtain $\liminf_{\min(-m,M) \to +\infty} E_j(B^2_{m,M}) \geq E_j(L)$. The upper and the lower bounds together give

$$\lim_{\min(-m,M) \to +\infty} E_j(B^2_{m,M}) = E_j(L) = E_j\left((\not{D}^{\partial \mathcal{K}})^2 \right). \tag{9.12}$$

Remark 9.1. We can look at the asymptotic regime $M \to +\infty$ and $m \to +\infty$. Let $(m_k, M_k)_{k \in \mathbb{N}}$ be a sequence of $\mathbb{R}^2$ such that $m_k, M_k \underset{k \to +\infty}{\longrightarrow} +\infty$. In this case, we can use the inequality $E_1(B^2_{m,M}) \geq E_1(P_{m,M})$, and for any $\Psi \in \mathcal{Q}(p_{m,M})$ there exists a constant $C > 0$ such that

$$\begin{aligned} p_{m,M}[\Psi, \Psi] \geq \int_{\Pi_\delta} |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} \Psi|^2 v_h + \int_{\Pi_\delta} \left[m^2 \mathbf{1}_{(0,\delta)} + M^2 \mathbf{1}_{(-\delta,0)} - C \right] |\Psi|^2 v_h \\ - C \int_{\partial \mathcal{K}} |\Psi(\cdot, \delta)| v_{\partial \mathcal{K}} - |M - m| \int_{\partial \mathcal{K}} |\Psi|^2 v_{\mathcal{K}}. \end{aligned}$$

Without loss of generality, we can assume that there is a subsequence of (M_k, m_k) still denoted by (M_k, m_k) such that $M_k \geq m_k$ for all k. We have

$$p_{m_k, M_k}[\Psi, \Psi] \geq m_k^2 \|\Psi\|^2_{L^2(\Pi_\delta^-)} + \|\Psi\|^2_{L^2(\Pi_\delta^+)} (M_k^2 + E_1(S_{M_k - m_k})) - C \|\Psi\|^2_{L^2(\Pi_\delta)},$$

but when k is large there is a constant C_1 such that

$$\begin{aligned} M_k^2 + E_1(S_{M_k - m_k}) &\geq M_k^2 - M_k^2 - m_k^2 + 2 M_k m_k - C_1 \\ &\geq 2 M_k m_k - m_k^2 - C_1 \geq m_k^2 - C_1. \end{aligned}$$

Thus, $E_1(B^2_{m_k, M_k}) \geq E_1(P_{m_k, M_k}) \geq m_k^2 - C - C_1 \underset{k \to +\infty}{\longrightarrow} +\infty$. This means that every sequence $E_1(B^2_{m_k, M_k})$ admits a divergent subsequence, an we conclude that $E_1(B^2_{m,M}) \to +\infty$ in this regime.

By similar constructions, the same result holds for $m, M \to -\infty$ as well.

References

[1] N. Arrizabalaga, L. Le Treust, A. Mas, N. Raymond, *The MIT Bag Model as an infinite mass limit*. Journal de l'École polytechnique — Mathématiques **6**, 329–365 (2019).

[2] N. Arrizabalaga, L. Le Treust, N. Raymond, *On the MIT Bag model in the non-relativistic limit*. Comm. Math. Phys **354**, 641–669 (2017).

[3] C. Bär, P. Gauduchon, A. Moroianu, *Generalized cylinders in semi-Riemannian and spin geometry*. Mathematische Zeitschrift **249**, 545–580 (2005).

[4] J. Behrndt, S. Hassi, H. de Snoo, R. Wietsma: *Monotone convergence theorems for semibounded operators and forms with applications*. Proc. Roy. Soc. Edinburgh Sect. A **140**, 927–951 (2010).

[5] J-P. Bourguignon, O. Hijazi, J-L. Milhorat, A. Moroianu, S. Moroianu, *A Spinorial Approach to Riemannian and Conformal Geometry*. European Mathematical Society (2015).

[6] N. Ginoux, *The Dirac spectrum*, Lecture notes in Mathematics, vol. 1976, Springer-Verlag, Berlin (2009).

[7] P. Grisvard, *Elliptic problems in nonsmooth domains*. volume 24 of Mono-graphs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA (1985).

[8] N. Grosse, R. Nakad , *Boundary value problems for noncompact boundaries of* Spin^c *manifolds and spectral estimates*, Proceedings of the London Mathematical Society **109**, 946–974 (2018).

[9] N. Grosse, C. Schneider, *Sobolev spaces on Riemannian manifolds with bounded geometry: general coordinates and traces*, Math. Nachr. **286**, 1586–1613 (2013).

[10] O. Hijazi, S. Montiel, et X. Zhang, *Dirac Operator on Embedded Hypersurfaces*. Mathematical Research Letters **8**, 195–208 (2001).

[11] K. Johnson, *The MIT Bag model*. Acta Phys. Pol. **B** (6), 865–892 (1975).

[12] T. Kato, *Perturbation theory for linear operators*. 2nd edition. Springer (1980).

[13] A. Moroianu, T. Ourmières-Bonafos, K. Pankrashkin, *Dirac operators on hypersurfaces as large mass limits*. Commun. Math. Phys. **374**, 1963–2013 (2020).

[14] T. Schick, *Analysis and Geometry of Boundary-Manifolds of Bounded Geometry*, arXiv:math/9810107 (1998).

[15] J. Weidmann, *Monotone continuity of the spectral resolution and the eigenvalues*. Proceedings of the Royal Society of Edinburgh: Section A Mathematics **85** (1-2), 131–136 (1980).

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