WEYL STRUCTURES WITH SPECIAL HOLONOMY ON COMPACT CONFORMAL MANIFOLDS

FLORIN BELGUN, BRICE FLAMENCOURT, ANDREI MOROIANU

Abstract. We consider compact conformal manifolds $(M,[g])$ endowed with a closed Weyl structure ∇, i.e. a torsion-free connection preserving the conformal structure, which is locally but not globally the Levi-Civita connection of a metric in $[g]$. Our aim is to classify all such structures when both ∇ and ∇^g, the Levi-Civita connection of g, have special holonomy. In such a setting, $(M,[g],\nabla)$ is either flat, or irreducible, or carries a locally conformally product (LCP) structure. Since the flat case is already completely classified, we focus on the last two cases. When ∇ has irreducible holonomy we prove that (M,g) is either Vaisman, or a mapping torus of an isometry of a compact nearly Kähler or nearly parallel G_2 manifold, while in the LCP case we prove that g is neither Kähler nor Einstein, thus reducible by the Berger-Simons Theorem, and we obtain the local classification of such structures in terms of adapted metrics.

1. Introduction

A Weyl structure on a conformal manifold $(M,[g])$ is a torsion-free linear connection preserving the conformal structure $[g]$. A Weyl structure is called exact if it is the Levi-Civita connection of some metric lying in $[g]$ and closed if this property holds in the neighbourhood of each point. The analysis of exact Weyl structures thus belongs to the field of Riemannian geometry, while the closed non-exact Weyl structures fall in the area of genuine conformal geometry. In this article we will concentrate on the analysis of this latter class of connections on compact conformal manifolds.

A closed non-exact Weyl structure ∇ is better understood through the study of its lift to the universal cover $\tilde{M}$ of M. Indeed, on $\tilde{M}$ there is a metric h, unique up to multiplication by a constant, belonging to the lifted conformal class and such that $\tilde{\nabla}$ is the Levi-Civita connection of h. Moreover, the deck-transformations given by $\pi_1(M)$ act as homotheties of $(\tilde{M},h)$, and the assumption that ∇ is not exact is equivalent to saying that not all of $\pi_1(M)$ acts by isometries (i.e $\pi_1(M)$ contains one strict homothety). Conversely, the Levi-Civita connection of a metric on $\tilde{M}$ for which $\pi_1(M)$ acts by homotheties which are not all isometries descends to a closed non-exact Weyl structure on M.

When M is compact, the existence of a closed, non-exact Weyl structure ∇ imposes strong restrictions on the holonomy group of ∇. A first result in this sense was obtained in [3],

2010 *Mathematics Subject Classification.* 53A30, 53C05, 53C29.

Key words and phrases. closed Weyl structures, reducible holonomy, LCP structures, LCK structures.

where it is proved that (M, ∇) is irreducible or flat provided that there exists $r > 0$ such that if a geodesic exists for a time greater than r, then it exists for any time. This result was greatly improved successively by Matveev and Nikolayevsky [18] in the analytical case, then by Kourganoff [14] in the smooth case, who proved that only three cases can occur: either (M, ∇) is flat, or it is irreducible, or its universal cover $(\tilde{M}, h)$ is a Riemannian product between an Euclidean space and an irreducible incomplete manifold [14, Theorem 1.5]. In this last very particular case, we call $(M, [g], \nabla)$ a locally conformally product (or LCP) structure. A detailed presentation of LCP manifolds can be found in [6].

In the present work, we investigate the following:

Problem 1.1. *Describe all compact Riemannian manifolds (M, g) of dimension $n \geq 3$ whose Levi-Civita connection ∇^g has special holonomy, such that there exists a closed, non-exact Weyl structure ∇ compatible with $[g]$ which also has special holonomy.*

By special holonomy we mean here that the restricted holonomy groups of ∇ and ∇^g are both strictly contained in $\mathrm{SO}(n)$. Note that if ∇ were exact (but different from ∇^g), i.e. the Levi-Civita connection of a metric in $[g]$ non-homothetic to g, the above question amounts to characterize compact manifolds carrying two non-homothetic conformally related metrics with special holonomy. This question has been answered by the third named author [20].

We study separately the three possible cases, where the restricted holonomy of ∇ is zero, reducible but non-zero, or irreducible. The case where ∇ is flat was already classified by Fried [7]. In every dimension $n \geq 3$, the conformal structure is induced by the quotient of $\mathbb{R}^n \setminus \{0\}$ by the semi-direct product of a group of isometries of $\mathbb{R}^n$ and a homothety of ratio $\lambda > 1$, all these transformations fixing the origin. Consequently, we focus on the two remaining cases.

In the case where (M, ∇) has special *irreducible* holonomy, the Berger-Simons holonomy classification imply that the Riemannian manifold $(\tilde{M}, h)$, where h is the metric induced by ∇ on $\tilde{M}$, is either Kähler or Einstein. In the Kähler case, we can use the well-developed theory of LCK manifolds to finish the analysis, while in the Einstein case we can relate our setting to the analysis done for exact Weyl structures in [20]. We prove in Proposition 3.1 that the only possibility is that (M, g) is either Vaisman, or a mapping torus of an isometry of a compact nearly Kähler or nearly parallel G_2 manifold.

The reducible case is more technical, and is related to the notion of LCP structures (see Definition 4.1 below). In this setting, we study separately the cases where g is Kähler, Einstein or has reducible holonomy. The known examples of LCP manifolds given in [6] provide intuition concerning the results. The only examples of complex LCP manifolds constructed so far are some particular OT-manifolds, a class of manifold introduced by Oeljeklaus and Toma [23] defined by means of algebraic number fields. When the number field has exactly one real embedding, the corresponding manifold admits an LCK structure, which in turn induces an LCP structure. However, these manifolds carry no Kähler metric. Consequently, there are no examples of LCP manifolds carrying a Kähler metric in their conformal class. In Theorem 4.3, we prove that it is indeed impossible to construct such an example.

When g is an Einstein metric, the conformal relation between the metric h and the lift $\tilde{g}$ of g to the universal cover $\tilde{M}$ of M gives some link between the Ricci tensors of the two metrics. We can then exploit the particular product structure of $(\tilde{M}, h)$, which admits a non-trivial flat factor, in order to conclude in this case as well. We prove in Theorem 4.5 that no Einstein metric lies in the conformal class of an LCP manifold.

The last part of this text is devoted to the study of the case where both h and g have reducible holonomy. This particular situation occurs in the available examples only when we use the so-called adapted metrics on LCP manifolds (see [6, Section 3] or Example 4.6 below). We can thus conjecture that in this situation, the universal cover $(\tilde{M}, \tilde{g})$ is itself a Riemannian product of the universal cover of an LCP manifold endowed with an adapted metric and the universal cover of a compact manifold endowed with a lifted metric. Actually, only a slightly weaker form of this result holds. Namely, we prove in Theorem 4.7 that the metric g is adapted, and the universal cover $(\tilde{M}, \tilde{g})$ is a Riemannian product $(M_1, g_1) \times (M_2, g_2)$ such that the flat factor of $(\tilde{M}, h)$ is tangent to M_1 at each point.

At the level of the universal cover, the solutions of Problem 1.1 are summarized in Theorem 4.12 below. For a complete classification, one would need the description of discrete cocompact groups acting freely on the given manifolds and preserving the structure, but this is a very hard problem which is out of reach for the moment.

Acknowledgments. This work was supported by the GDRI ECO-Math and by the Procope Project No. 57650868 (Germany) / 48959TL (France).

2. Preliminaries

2.1. **Weyl structures.** Let M be a compact manifold of dimension $n \geq 3$ and let c be a conformal structure on M (usually c is the conformal class of a Riemannian metric g on M, denoted by $c = [g]$). A *Weyl structure* on M is a torsion-free linear connection ∇ preserving the conformal structure c, in the sense that for every Riemannian metric $g \in c$, $\nabla_X g = -2\theta_g(X)g$ for some 1-form θ_g on M called the *Lee form* of ∇ with respect to g. The Lee form of ∇ with respect to g vanishes if and only if ∇ is the Levi-Civita connection of g, denoted by ∇^g.

The Weyl structure ∇, with the Lee form θ_g with respect to the Riemannian metric $g \in c$, acts on vector fields $Y \in C^\infty(TM)$ as follows:

$$\nabla_X Y = \nabla^g_X Y + \theta_g(Y)X + \theta_g(X)Y - g(X,Y)\theta_g^{\sharp_g}, \tag{1}$$

where $\theta_g^{\sharp_g}$ is the dual vector field to the 1-form θ_g with respect to g.

Note that, if $g' := e^{2f}g$ is another metric in the conformal class, then

$$\theta_{g'} = \theta_g + df. \tag{2}$$

The Weyl structure ∇ is called *closed* if θ_g is closed for one (hence for all) metrics $g \in c$ and *exact* if θ_g is exact (for some, hence) for all $g \in c$. From (2), an exact Weyl structure on (M, c) is the Levi-Civita connection of a metric in the conformal class c.

2.2. **The metric on $\tilde{M}$ associated to a closed Weyl structure.** Assume now that g is a Riemannian metric on M and ∇ is a closed Weyl structure on $(M, [g])$ with Lee form θ_g with respect to g. We denote by $\pi : \tilde{M} \to M$ the universal cover of M and by $\tilde{g} := \pi^* g$ the induced Riemannian metric. The lift $\tilde{\nabla}$ of ∇ to $\tilde{M}$ has Lee form $\tilde{\theta} := \pi^*\theta_g$ which is exact, since $\tilde{M}$ is simply connected. Consequently, there exists a function $\varphi \in C^\infty(\tilde{M})$, defined up to a constant, such that $\tilde{\theta} = d\varphi$. By (2), $\tilde{\nabla}$ is the Levi-Civita connection of the metric $h := e^{2\varphi}\tilde{g}$. The metric h is called associated to ∇, and is uniquely determined by ∇ up to a multiplicative constant.

The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ by isometries with respect to the metric $\tilde{g}$. Moreover, it preserves $\tilde{\theta} = d\varphi$, which means that φ is a $\pi_1(M)$-*equivariant function*, i.e.

$$\varphi(\gamma(x)) = \varphi(x) + \rho(\gamma), \ \forall \gamma \in \pi_1(M),$$

for some (nontrivial) group homomorphism $\rho : \pi_1(M) \to \mathbb{R}$. We infer that

$$\gamma^* h = e^{2\rho(\gamma)} h, \ \forall \gamma \in \pi_1(M),$$

showing that $\pi_1(M)$ acts on $\tilde{M}$ by *homotheties* with respect to the metric h.

In this case, (1) applied on $\tilde{M}$ becomes

$$\nabla^h_X Y = \nabla^{\tilde{g}}_X Y + d\varphi(Y)X + d\varphi(X)Y - (d\varphi)^\sharp \langle X, Y\rangle, \tag{3}$$

where we intentionally omit referring to h or to $\tilde{g}$ in the last term, the convention being that the metric used to compute the scalar product $\langle X, Y\rangle$ is the same as the one used to "lift indices" $\sharp : T^*\tilde{M} \to T\tilde{M}$.

2.3. **Holonomy issues.** From now on we consider that the assumptions of Problem 1.1 hold. This can be equivalently stated by the fact that the restricted holonomy groups of the metrics h on $\tilde{M}$ and g on M are *special*, that is, strictly contained in $\mathrm{SO}(n)$.

According to the classical de Rham decomposition theorem [13, p. 185] and the Berger-Simons holonomy classification [5, p. 300], there are four cases when the restricted holonomy group of a Riemannian metric (or, more generally, of a closed Weyl structure) on a n-dimensional manifold is special:

(i) the metric is irreducible and locally Kähler;
(ii) the metric is irreducible and Einstein with non-zero scalar curvature (either locally symmetric or the quaternionic-Kähler case from the Berger-Simons theorem [5, p. 300]);
(iii) the metric is irreducible and the holonomy belongs to the list

$$\mathrm{SU}(n/2) \subset \mathrm{SO}(n), \ \mathrm{Sp}(n/4) \subset \mathrm{SO}(n), \ \mathrm{G}_2 \subset \mathrm{SO}(7), \ \mathrm{Spin}(7) \subset \mathrm{SO}(8) \tag{4}$$

of the Berger-Simons theorem [5, p. 300];

(iv) the metric has reducible holonomy (so is locally a product by the local de Rham theorem).

We will discuss first the case where the Weyl connection ∇ has irreducible holonomy (cases (i)–(iii) in the above list). Then, in the last section we will treat the reducible case, which is more delicate.

3. Irreducible Weyl holonomy

In this section we consider the case when (M^n, g) is a compact Riemannian manifold with special holonomy and the closed non-exact Weyl structure ∇ has special *irreducible* holonomy. As explained before, the lift $\tilde\nabla$ of ∇ to the universal cover $\tilde M$ of M is the Levi-Civita connection of a metric h on $\tilde M$ which belongs to the conformal class determined by the lift $\tilde g$ of g to $\tilde M$.

Because the restricted holonomy group $\mathrm{Hol}_0(\nabla) = \mathrm{Hol}(\tilde M, h)$ is a Riemannian holonomy, we need to consider the following cases:

Case (i) $(\tilde M, h)$ is Kähler. Then (M, g) is a *locally conformally Kähler* (LCK) compact manifold with special holonomy which is not globally conformally Kähler, since ∇ is assumed to be non-exact. By Case 1. of Thm. 1.3 from [17] it follows that (M, g) is a Vaisman manifold and ∇ is the canonical Weyl structure of the LCK structure.

Case (ii) The metric h is irreducible and locally symmetric, or $\mathrm{Hol}_0(\nabla) = \mathrm{Sp}(n/4)\mathrm{Sp}(1)$ for $n \geq 8$ and multiple of 4. It is well known that in this situation h is Einstein with non-zero constant scalar curvature (see [5, Theorem 14.39] for the case of quaternionic-Kähler manifolds). However, since h admits strict homotheties (which preserve the Ricci tensor of h, but not its trace with respect to h), this case is impossible.

Case (iii) $\mathrm{Hol}_0(\nabla)$ belongs to the Berger list (4). We will not study the first two cases, since they correspond to $(\tilde M, h)$ being Kähler, when Case (i) applies. Assume now that $\mathrm{Hol}_0(\nabla)$ is equal to G_2 for $n = 7$ or $\mathrm{Spin}(7)$ for $n = 8$. Then the manifold $\tilde M$ is spin since its frame bundle reduces to the holonomy group of h, which is simply connected. By a result of Wang [28], (M, h) carries a non-trivial parallel spinor (so in particular it is Ricci-flat).

Consequently, $(M, [g], \nabla)$ is *Einstein-Weyl*, which by a result of Tod [25, Prop. 2.2] implies that the Lee form $\theta_0 = d\varphi_0$ of ∇ with respect to the Gauduchon metric g_0 is ∇^{g_0}-parallel. The global de Rham theorem shows that $(\tilde M, \tilde g_0)$ is isometric to $(\mathbb{R} \times N^{n-1}, d\varphi_0^2 + g_N)$ for some $n-1$-dimensional complete Riemannian manifold (N^{n-1}, g_N). Using the change of coordinates $r := e^{\varphi_0}$ we deduce that the metric $h = e^{2\varphi_0}\tilde g_0$ on $\tilde M$ is a *Riemannian cone*: $(\tilde M, h) = (\mathbb{R}_+^* \times N^{n-1}, dr^2 + r^2 g_N)$. By the results in [1], (N^{n-1}, g_N) is either nearly Kähler for $n = 7$, or nearly parallel G_2 for $n = 8$, and it is easy to check that (M, g_0) is a mapping torus of an isometry of (N^{n-1}, g_N). Moreover, the restriction of the parallel spinor of $(\tilde M, h)$

to (N^{n-1}, g_N) is a real Killing spinor ψ with Killing constant $1/2$:

$$\nabla^{g_N}_X \psi = \frac{1}{2} X \cdot \psi, \qquad \forall X \in TN. \tag{5}$$

This implies in particular that (N^{n-1}, g_N) is Einstein with positive scalar curvature (thus compact by Myers' theorem).

We thus end up with two conformally related metrics with special holonomy on the compact manifold M: the original metric g, and the Gauduchon metric g_0 of ∇. We claim that g is a scalar multiple of g_0.

Indeed, if this is not the case, it follows from [20, Thm. 5.1 and Thm. 6.3] that $\mathbb{R} \times N$ admits a triple warped product metric, so N admits a metric with reducible holonomy, which is conformal to g_N. We denote this metric by g_r. As N is compact and simply connected, de Rham's decomposition theorem shows that (N, g_r) is a global Riemannian product $(N_1, g_1) \times (N_2, g_2)$. By [15, Cor. 3.4], the conformal factor between g_N and g_r only depends on N_1 or N_2. Up to permuting the indices, we can therefore assume that $g_N = e^{2f}(g_1 + g_2)$, where $f : N_1 \to \mathbb{R}$.

We will now exploit the relation between the $(4, 0)$ curvature tensors of g_N and g_r (cf. [5, Theorem 1.159]):

$$R^{g_N} = e^{2f}(R^{g_r} + g_r \owedge (\nabla^{g_r} df - df \otimes df + \frac{1}{2}|df|^2_{g_r} g_r). \tag{6}$$

Viewing the curvature applied to two vectors as a 2-form, and using the metric g_N to identify vectors and 1-forms, this relation is equivalent to

$$\begin{aligned} R^{g_N}_{X,Y} =& R^{g_r}_{X,Y} + X^{\flat_N} \wedge \left(\nabla^{g_r}_Y df - df(Y) df + \frac{1}{2} e^{-2f} |df|^2_{g_r} Y^{\flat_N} \right) \\ & - Y^{\flat_N} \wedge \left(\nabla^{g_r}_X df - df(X) df + \frac{1}{2} e^{-2f} |df|^2_{g_r} X^{\flat_N} \right), \end{aligned} \tag{7}$$

for every tangent vectors X, Y. Applying this relation to a vector $X \in TN_1$ and to a vector $Y \in TN_2$, and using the fact that $R^{g_r}(X, Y) = 0$, $\nabla^{g_r}_Y df = 0$ and $df(Y) = 0$, we obtain

$$R^{g_N}_{X,Y} = -Y^{\flat_N} \wedge \left(\nabla^{g_1}_X df - df(X) df + e^{-2f} |df|^2_{g_1} X^{\flat_N} \right).$$

On the other hand, using (5) repeatedly we obtain

$$\frac{1}{2} R^{g_N}_{X,Y} \cdot \psi = R^{g_N}_{X,Y} \psi = \frac{1}{2} Y^{\flat_N} \wedge X^{\flat_N} \cdot \psi.$$

Comparing these two equations we obtain

$$-Y^{\flat_N} \wedge \left(\nabla^{g_1}_X df - df(X) df + e^{-2f} |df|^2_{g_1} X^{\flat_N} \right) \cdot \psi = Y^{\flat_N} \wedge X^{\flat_N} \cdot \psi, \qquad \forall X \in TN_1,\ \forall Y \in TN_2,$$

whence

$$\nabla^{g_1}_X df - df(X) df + e^{-2f} |df|^2_{g_1} X^{\flat_N} = -X^{\flat_N}, \qquad \forall X \in TN_1.$$

Applying this formula to X yields $\mathrm{Hess}_{g_1} f(X,X) = df(X)^2 - |df|^2_{g_1} g_1(X,X) - e^{2f} g_r(X,X)$ for all $X \in TN_1$. On the other hand, at a point where f attains its minimum on M_1, the left hand term is non-negative, whereas the right hand term is strictly negative for $X \neq 0$, which is a contradiction. This proves that g is proportional to g_0.

Summarizing, we have proved the following:

Proposition 3.1. *The only compact manifolds (M,g) with special holonomy carrying a closed non-exact Weyl structure ∇ with special irreducible holonomy are Vaisman manifolds or mapping tori of an isometry of a compact nearly Kähler or nearly parallel G_2 manifold.*

4. Reducible Weyl holonomy

We consider now the remaining case, when the holonomy of the closed non-exact Weyl structure ∇ on $(M,[g])$ (or, equivalently, of the metric $h := e^{2\varphi}\tilde{g}$ on the universal cover $\tilde{M}$) is reducible.

Assume first that the Weyl structure ∇ is flat. This case was classified in [7], where it is shown that the universal cover $\tilde{M}$ endowed with the metric h (whose Levi-Civita connection is $\tilde{\nabla}$) is isometric to $\mathbb{R}^n \setminus \{0\}$ and $\pi_1(M)$ is a semi-direct product $K \rtimes \mathbb{Z}$ between a finite group of isometries of $\mathbb{R}^n$ fixing the origin and a group generated by an homothety of ratio $\lambda < 1$. Seeing $\mathbb{R}^n \setminus 0$ as the product $\mathbb{R}_+ \times S^{n-1}$ together with the metric $dr^2 + r^2 g_S$ where g_S is the round metric on the sphere, one deduces that $\pi_1(M)$ acts by isometries for the metric $\frac{1}{r^2}dr^2 + g_S$, which is conformal to the previous metric and descends to M. In addition, this metric is a product metric, so it is reducible, implying that all manifolds occurring in Fried's classification [7] are solutions of Problem 1.1. In fact, the same argument as in case (iii) in the previous section shows that the product metric on $S^1 \times S^{n-1}$ is the only metric with special holonomy in its conformal class, because otherwise S^{n-1} would be a product of two positive-dimensional manifolds, which is clearly impossible.

We will thus assume from now on that the Weyl structure ∇ is non-flat and has reducible holonomy. We introduce the following terminology.

Definition 4.1. A Weyl structure ∇ on a compact conformal manifold (M,c) is called a *locally conformally product* (LCP) structure if it is closed, non-exact, non-flat, and has reducible holonomy. An LCP structure (c,∇) is said to be compatible with a Riemannian metric g on M if $g \in c$.

By assumption, the metric h on the universal cover of any LCP manifold has reducible holonomy. However, as h is incomplete, we cannot apply de Rham's decomposition theorem even though $\tilde{M}$ is simply connected. Nonetheless, we have the following:

Theorem 4.2. (Kourganoff [14, Theorem 1.5]) *The universal cover $(\tilde{M},h)$ of a compact LCP manifold (M,c,∇) is globally isometric to a Riemannian product $\mathbb{R}^q \times (N,g_N)$, where $\mathbb{R}^q$ $(q \geq 1)$ is the flat Euclidean space, and (N,g_N) is an incomplete Riemannian manifold with irreducible holonomy.*

Our assumption throughout this section is thus that $([g], \nabla)$ is an LCP structure, and that g has special holonomy. We will distinguish 3 cases, according to the type of g: Kähler, irreducible Einstein, and reducible.

In view of Theorem 4.2, we fix some notations on LCP structures that we will use until the end of this section. For any LCP structure $([g], \nabla)$ on the compact manifold M we denote by $\tilde{g}$ the lift of the metric g to the universal cover $\tilde{M}$ of M and by $h := e^{2\varphi}\tilde{g}$ the reducible Riemannian metric on $\tilde{M}$ for which $\tilde{\nabla} = \nabla^h$. With respect to product decomposition $(\tilde{M}, h) \cong \mathbb{R}^q \times (N, g_N)$ where $\mathbb{R}^q$ is an Euclidean space and (N, g_N) is an irreducible incomplete manifold. the tangent bundle of $\tilde{M}$ decomposes into the orthogonal ∇^h-parallel direct sum $T\mathbb{R}^q \oplus TN$. We call $T\mathbb{R}^q$ the *flat distribution* and TN the *non-flat distribution* of the LCP structure.

4.1. **LCP structures on compact Kähler manifolds.** The aim of this subsection is to prove the following:

Theorem 4.3. *On compact Kähler manifolds, there are no LCP structures compatible with the Kähler metric.*

Proof. Assume that (M, g, J) is a compact Kähler manifold admitting an LCP structure $([g], \nabla)$. This situation is similar to the one studied in Section 6 of [17], except that loc. cit. dealt with reducible metrics which are locally conformally Kähler, while here we consider Kähler metrics which are locally conformally reducible.

On the universal cover $\tilde{M}$ of M the metric $\tilde{g}$ is Kähler with respect to the lift $\tilde{J}$. In order to use some results from [17], we introduce the notation $D_1 := T\mathbb{R}^q$ and $D_2 := TN$.

Consider first the case when the dimensions n_1 and n_2 of D_1 and D_2 are both at least 2. Most arguments in Theorem 6.2 in [17] are valid without the compactness assumption. More precisely, the second formula on page 143 of [17] shows the following:

Proposition 4.4. *Assume that $T\tilde{M} = D_1 \oplus D_2$ is a ∇^h-parallel splitting on a Riemannian manifold $(\tilde{M}^n, h)$ with $n_i := \dim(D_i) \geq 2$, and assume moreover that a conformally related metric $\tilde{g} := e^{-2\varphi}h$ on $\tilde{M}$ is Kähler. Then*

$$\frac{1}{n_1}\left(|\tilde{\theta}_1|_h^2 - \delta^h\tilde{\theta}_1\right) + \frac{1}{n_2}\left(|\tilde{\theta}_2|_h^2 - \delta^h\tilde{\theta}_2\right) - |\tilde{\theta}|_h^2 = 0, \tag{8}$$

where $\tilde{\theta}_i$ denotes the restriction of $\tilde{\theta} := d\varphi$ to D_i.

The conformal change formulas (cf. [5], Theorem 1.159) give

$$|\alpha|_h^2 = e^{-2\varphi}|\alpha|_{\tilde{g}}^2, \qquad \delta^h\alpha = e^{-2\varphi}(\delta^{\tilde{g}}\alpha - (n-2)\tilde{g}(d\varphi, \alpha))$$

for every 1-form α. Equation (8) thus becomes

$$\frac{1}{n_1}\left(|\tilde{\theta}_1|_{\tilde{g}}^2 - \delta^{\tilde{g}}\tilde{\theta}_1 + (n-2)\tilde{g}(\tilde{\theta}, \tilde{\theta}_1)\right) + \frac{1}{n_2}\left(|\tilde{\theta}_2|_{\tilde{g}}^2 - \delta^{\tilde{g}}\tilde{\theta}_2 + (n-2)\tilde{g}(\tilde{\theta}, \tilde{\theta}_1)\right) - |\tilde{\theta}|_{\tilde{g}}^2 = 0. \tag{9}$$

As $n_1+n_2=n$, $\tilde g(\tilde\theta,\tilde\theta_i)=|\tilde\theta_i|^2_{\tilde g}$ and $|\tilde\theta|^2_{\tilde g}=|\tilde\theta_1|^2_{\tilde g}+|\tilde\theta_2|^2_{\tilde g}$, we thus get

$$\frac{1}{n_1}\left((n_2-1)|\tilde\theta_1|^2_{\tilde g}-\delta^{\tilde g}\tilde\theta_1\right)+\frac{1}{n_2}\left((n_1-1)|\tilde\theta_2|^2_{\tilde g}-\delta^{\tilde g}\tilde\theta_2\right)=0. \tag{10}$$

The forms $\tilde\theta_i$ are $\pi_1(M)$-invariant, so they are pull-backs of 1- forms θ_i defined on M. The relation (10) thus projects to the compact quotient $M=\tilde M/\pi_1(M)$ into

$$\frac{1}{n_1}\left((n_2-1)|\theta_1|^2_{g}-\delta^{g}\theta_1\right)+\frac{1}{n_2}\left((n_1-1)|\theta_2|^2_{g}-\delta^{g}\theta_2\right)=0. \tag{11}$$

so after integration over M we obtain $\theta=0$, which is excluded.

It remains to study the case when one of the distributions D_1 or D_2 has dimension 1. Since $\tilde M$ is simply connected, this distribution determines a ∇^h-parallel vector field ζ on $\tilde M$ of unit length with respect to h. Then the vector field $\tilde\xi:=e^\varphi\zeta$ has unit length with respect to $\tilde g$, and by (3) (applied to $\tilde g$ instead of g) we get

$$\nabla^{\tilde g}_X\tilde\xi=\nabla^h_X\tilde\xi-d\varphi(\tilde\xi)X-d\varphi(X)\tilde\xi+\tilde g(X,\tilde\xi)(d\varphi)^\sharp=-d\varphi(\tilde\xi)X+\tilde g(X,\tilde\xi)(d\varphi)^\sharp, \tag{12}$$

where we have used that $\nabla^h_X\tilde\xi=d\varphi(X)e^\varphi\zeta=d\varphi(X)\tilde\xi$. Up to passing to a double cover of M if necessary, $\tilde\xi$ projects to a unit length vector field ξ on (M,g) which by (12) satisfies

$$\nabla^g_X\xi=-\theta(\xi)X+g(X,\xi)\theta \tag{13}$$

(we identify from now on vectors and 1-forms using the metric g). We decompose θ as

$$\theta=a\xi+bJ\xi+\theta_0,$$

where $a:=\theta(\xi)$, $b:=\theta(J\xi)$ and $\theta_0(\xi)=\theta_0(J\xi)=0$. Since J is ∇^g-parallel, (13) immediately gives

$$\nabla^g_X\xi=-aX+g(X,\xi)\theta,\qquad \nabla^g_XJ\xi=-aJX+g(X,\xi)J\theta,\qquad \forall X\in TM. \tag{14}$$

Let us denote by m the complex dimension of M, so that $n=2m$, $m>1$. Using a local orthonormal basis $\{e_i\}_{i=1,\dots,2m}$ of the tangent bundle and the relations $d=\sum e_i\wedge\nabla^g_{e_i}$, $\delta=-\sum e_i\lrcorner\nabla^g_{e_i}$, $d^c=\sum Je_i\wedge\nabla^g_{e_i}$, $\delta^c=-\sum Je_i\lrcorner\nabla^g_{e_i}$, we readily compute

$$d\xi=-\theta\wedge\xi,\qquad d^c\xi=2a\Omega-\theta\wedge J\xi,\qquad \delta\xi=(2m-1)a,\qquad \delta J\xi=b \tag{15}$$

(here $\Omega:=g(J\cdot,\cdot)$ is the Kähler form). The anti-commutation of d^c and δ (cf. [21]) yields

$$\begin{aligned}0&=d^c\delta\xi+\delta d^c\xi=(2m-1)d^ca-\sum e_i\lrcorner(2e_i(a)\Omega-\nabla^g_{e_i}\theta\wedge J\xi-\theta\wedge\nabla^g_{e_i}J\xi)\\&=(2m-1)Jda-2Jda-\delta\theta J\xi-\nabla^g_{J\xi}\theta+\nabla^g_\theta(J\xi)+(\delta(J\xi))\theta\\&=(2m-3)Jda-\delta\theta J\xi-\nabla^g_{J\xi}\theta+\nabla^g_\theta(J\xi)+b\theta.\end{aligned}$$

Since (14) implies $\nabla^g_\theta J\xi=0$, we obtain thus

$$(2m-3)Jda-\delta\theta J\xi-\nabla^g_{J\xi}\theta+b\theta=0. \tag{16}$$

From (14) together with the fact that $d\theta = 0$, we obtain for every vector field X:

$$g(X, \nabla^g_{J\xi}\theta) = g(J\xi, \nabla^g_X\theta) = X(b) - g(\nabla^g_X J\xi, \theta) = X(b) + ag(JX, \theta),$$

whence $\nabla^g_{J\xi}\theta = db - aJ\theta$. Equation (16) thus reads

$$0 = (2m-3)Jda - \delta\theta J\xi - db + aJ\theta + b\theta. \tag{17}$$

We take the scalar product with $J\xi$ in (17) and obtain

$$0 = (2m-3)\xi(a) - \delta\theta - J\xi(b) + a^2 + b^2,$$

which after an integration over M and use of the divergence theorem and (15) yields

$$\begin{aligned} 0 &= \int_M \big((2m-3)\xi(a) - J\xi(b) + a^2 + b^2\big)\, d\mu_g = \int_M \big((2m-3)a\delta\xi - b\delta(J\xi) + a^2 + b^2\big)\, d\mu_g \\ &= 4(m-1)^2 \int_M a^2 d\mu_g. \end{aligned}$$

This shows that the function a vanishes identically, and thus (17) becomes

$$0 = \delta\theta J\xi + db - b\theta. \tag{18}$$

From (14) and (18) we get

$$d(bJ\xi) = db \wedge J\xi + bdJ\xi = b\theta \wedge J\xi - bJ\theta \wedge \xi \in \Omega^{(1,1)}M. \tag{19}$$

The global $i\partial\bar{\partial}$-Lemma (cf. [21]) shows that there exists a real function ψ on M such that

$$bJ\theta \wedge \xi - b\theta \wedge J\xi = i\partial\bar{\partial}\psi.$$

Applying the Lefschetz operator Λ to this relation and using the commutation relation $[\Lambda, \partial] = i\bar{\partial}^*$ (see [21], Eq. (14.15)), we get

$$\Delta\psi = \frac{1}{2}\Delta^{\bar{\partial}}\psi = \frac{1}{2}\bar{\partial}^*\bar{\partial}\psi = -i[\Lambda, \partial]\bar{\partial}\psi = -b\Lambda(J\theta \wedge \xi - \theta \wedge J\xi) = 0,$$

since $\Lambda(J\theta \wedge \xi) = -g(\theta, \xi) = -a = 0$, and similarly $\Lambda(\theta \wedge J\xi) = 0$. It follows that ψ is constant, so

$$bJ\theta \wedge \xi = b\theta \wedge J\xi. \tag{20}$$

We now remark that the 1-form θ_0 is harmonic. Indeed, $d\theta_0 = d\theta - d(bJ\xi) = 0$ by (19) and (20) and

$$\delta\theta_0 = \delta\theta - \delta(bJ\xi) = \delta\theta - b\delta J\xi + J\xi(b) = \delta\theta - b^2 + J\xi(b) = 0$$

by (18). Since (M, g, J) is compact Kähler, $J\theta_0$ must be harmonic too, so in particular $dJ\theta_0 = 0$. We then compute

$$0 = d(dJ\xi) = -d(J\theta \wedge \xi) = -d(J\theta_0 \wedge \xi) = J\theta_0 \wedge d\xi = J\theta_0 \wedge \theta \wedge \xi.$$

Since $J\theta_0$, ξ and $\theta = \theta_0 + bJ\xi$ are mutually orthogonal, this shows that θ_0 vanishes identically, so

$$\theta = bJ\xi. \tag{21}$$

The relations (14) now read

$$\nabla^g_X \xi = bg(X,\xi)J\xi, \qquad \nabla^g_X J\xi = -bg(X,\xi)\xi, \qquad \forall X \in TM, \tag{22}$$

thus showing that the distribution spanned by ξ and $J\xi$ is ∇^g-parallel. With the equalities (22), one has

$$[e^{-\varphi}\xi, J\xi] = \nabla^g_{e^{-\varphi}\xi} J\xi - \nabla^g_{J\xi} e^{-\varphi}\xi = -be^{-\varphi}\xi + be^{-\varphi}\xi = 0, \tag{23}$$

and

$$d(e^{-\varphi}\xi) = d(J\xi) = 0. \tag{24}$$

We now consider the universal cover $\tilde{M}$ of M, endowed with the Kähler metric $\tilde{g}$. We will denote by η and $\tilde{J}\eta$ the one-forms dual to $\tilde{\xi}$ and $\tilde{J}\tilde{\xi}$ via the metric $\tilde{g}$ in order to avoid confusions. The previous analysis shows that the distribution $(\tilde{\xi}, \tilde{J}\tilde{\xi})$ is $\tilde{g}$-parallel. Since $(\tilde{M}, \tilde{g})$ is simply connected and complete, one can use the de Rham decomposition theorem to obtain

$$(\tilde{M}, \tilde{g}) = (M_0, g_0) \times (K, g_K) \tag{25}$$

where M_0 is the integral manifold of the parallel distribution $(\tilde{\xi}, \tilde{J}\tilde{\xi})$ endowed with the metric $g_0 := \eta^2 + (\tilde{J}\eta)^2$ and $(K, g_K, \tilde{J}|_K)$ is a Kähler manifold. The relations (22) give that $\tilde{J}\tilde{\xi}$ is a geodesic vector field, therefore its integral curve through any point $p \in M_0$ is the geodesic starting at p with speed $\tilde{J}\tilde{\xi}_p$. Hence, the completeness of (M_0, g_0) implies that the flow ψ^J_t of $\tilde{J}\tilde{\xi}$ is defined for all times. In addition, the flow ψ_s of the vector field $\zeta = e^{-\varphi}\tilde{\xi}$ is also defined for any time by definition of ζ. We fix $p \in M_0$ and using (24) and the simple connectedness of M_0, we know there exist two functions σ, τ such that $d\sigma = e^{-\varphi}\eta$, $d\tau = \tilde{J}\eta$, $\sigma(p) = \tau(p) = 0$.

We have defined so far all the necessary objects to construct a diffeomorphism between M_0 and $\mathbb{R}^2$. Namely, we consider the maps:

$$f_1 : \mathbb{R}^2 \to M_0, (s,t) \mapsto \psi_s \circ \psi^J_t(p), \qquad f_2 : M_0 \to \mathbb{R}^2, x \mapsto (\sigma(x), \tau(x)). \tag{26}$$

By (23), the two flow maps ψ and ψ^J commute, and then easy computations give

$$d(f_1 \circ f_2) = \mathrm{id}, \qquad f_1 \circ f_2(p) = (0,0), \qquad d(f_2 \circ f_1) = \mathrm{id}, \qquad f_2 \circ f_1(0,0) = p, \tag{27}$$

so f_1 and f_2 are inverse to each other. Consequently, f_1 is a diffeomorphism, and after computing the pull-back of g_0 one obtains in the new coordinates:

$$(M_0, g_0) \simeq (\mathbb{R}^2, e^{-2\varphi}ds^2 + dt^2). \tag{28}$$

In these coordinates one has $\eta = e^{\varphi}ds$ and $\tilde{J}\eta = dt$. Note that the function φ, viewed as a function on $\mathbb{R}^2$, only depends on t due to the fact that $\zeta(\varphi) = \tilde{\theta}(\zeta) = \tilde{\theta}(e^{-\varphi}\tilde{\xi}) = 0$ by (21).

We claim that the rank of the LCP structure on M (i.e. the rank of the subgroup of $\mathbb{R}$ consisting of the homothety factors of the action of $\pi_1(M)$ on $(\tilde{M}, h)$) is 1. Indeed, we recall that the function φ is $\pi_1(M)$-equivariant, so assume there exist two real numbers $\lambda_1, \lambda_2 > 0$ and two constants $c_1, c_2 \in \mathbb{R}$ such that

$$\varphi(t + c_i) = \varphi(t) + \ln \lambda_i, \qquad \forall i \in \{1,2\}, \forall t \in \mathbb{R}, \tag{29}$$

and $\langle \lambda_1, \lambda_2 \rangle$ is a subgroup of $(\mathbb{R}^*, \times)$ of rank 2. This implies that $\langle c_1, c_2 \rangle$ is a subgroup of $(\mathbb{R}, +)$ of rank 2, thus it is dense in $\mathbb{R}$. Consequently, there are two sequences $(a_n), (b_n) \in \mathbb{Z}^{\mathbb{N}}$ such that $a_n c_1 + b_n c_2 \to 0$, hence the property (29) gives for any $t \in \mathbb{R}$

$$\varphi'(t) = \lim_{n\to+\infty} \frac{\varphi(t + a_n c_1 + b_n c_2) - \varphi(t)}{a_n c_1 + b_n c_2} = \lim_{n\to+\infty} \frac{a_n \ln \lambda_1 + b_n \ln \lambda_2}{a_n c_1 + b_n c_2}. \tag{30}$$

Since the limit of the left-hand side exists, and the one of the right-hand side is independent of t, we conclude that φ' is constant. By (21) we get $b\tilde{J}\tilde{\xi} = \tilde{\theta} = d\varphi = \varphi'(t)dt = \varphi'(t)\tilde{J}\tilde{\xi}$, thus b is constant on M. However, by (15), one has $\delta(J\xi) = b$, so by integration on the compact manifold M, one has $b = 0$ and φ is constant, which is absurd.

From now on, we will use the coordinates (s, t, x) on $\tilde{M} \simeq M_0 \times K \simeq \mathbb{R} \times \mathbb{R} \times K$. Let $\lambda \in (0, 1)$ be a generator of the group formed by the homothety factors of $\pi_1(M)$ acting on $(\tilde{M}, h)$, which exists by the previous analysis. Let $\gamma \in \pi_1(M)$ with ratio λ. The action of γ on $\tilde{M}$ must preserve the decomposition $\mathbb{R} \times \mathbb{R} \times K$ because γ is an isometry of $(\tilde{M}, \tilde{g})$, so it preserves the decomposition $M_0 \times K$, and it is a similarity of $(\tilde{M}, h)$ so it preserves the h-parallel distribution generated by $\frac{\partial}{\partial s}$ and its orthogonal in M_0. Moreover, (28) implies that $h = ds^2 + e^{2\varphi(t)}(dt^2 + g_K)$, and φ satisfies the equivariance property (32), so γ acts as a similarity of ratio λ on the variable s, as an isometry on the variable t and as an isometry on (K, g_K). Then, by making an affine change of variable on s, we can assume that γ acts as

$$\gamma : (s, t, x) \mapsto (\epsilon_1(\gamma)\lambda s, \epsilon_2(\gamma)t + c, \alpha(x)), \tag{31}$$

where $\epsilon_i(\gamma) = \pm 1$, $c \in \mathbb{R}$ and α is an isometry of (K, g_K). First of all, if $\epsilon_2(\gamma) = -1$ one obtains by the equivariance property of φ (29) that for any $t \in \mathbb{R}$:

$$\varphi(t) + 2\ln \lambda = (\gamma^2)^*\varphi(t) = \varphi(-(-t + c) + c) = \varphi(t)$$

thus $\lambda = 1$, which contradicts $\lambda \in (0, 1)$, so $\epsilon_2(\gamma) = 1$. In addition, up to an affine change of variable on t we can assume that $c = 1$, so in particular

$$\varphi(t + 1) = \varphi(t) + \ln \lambda, \qquad \forall t \in \mathbb{R}. \tag{32}$$

Now, let $\gamma_0 \in \pi_1(M)$ which acts as an isometry on $(\tilde{M}, h)$. Then, γ_0 acts as

$$\gamma_0 : (s, t, x) \mapsto (\epsilon_1(\gamma_0)s + c_s, \epsilon_2(\gamma_0)t + c_t, \beta(x)), \tag{33}$$

where $\epsilon_i(\gamma_0) = \pm 1$, $(c_s, c_t) \in \mathbb{R}^2$ and β is an isometry of (K, g_K). Again, if $\epsilon_2(\gamma_0) = -1$, one has for any $t \in \mathbb{R}$:

$$\varphi(t + c_t/2) = (\gamma_0)^*\varphi(t + c_t/2) = \varphi(-t + c_t/2), \tag{34}$$

so $t \mapsto \varphi(t + c_t/2)$ is symmetric, but this is impossible since by (32) one has

$$\varphi(-1 + c_t/2) + 2\ln \lambda = \varphi(1 + c_t/2),$$

and $\lambda \neq 1$. Thus $\epsilon_2(\gamma_0) = 1$. Moreover, assume that $c_t \neq 0$. Then, the subgroup of $(\mathbb{R}, +)$ given by $\langle c_t, 1 \rangle$ must be of rank 2, otherwise there exists non-zero integers n_1, n_2 such that $n_1 c_t + n_2 = 0$, and using (32) we obtain

$$\varphi(t) = \varphi(t + n_1 c_t + n_2) = \varphi(t + n_1 c_t) + n_2 \ln \lambda = ((\gamma_0^{n_1})^*\varphi)(t) + n_2 \ln \lambda = \varphi(t) + n_2 \ln \lambda,$$

contradicting the fact that $\lambda \neq 1$. We then use the same argument as above (see the computations in (30) and the subsequent arguments), to prove that φ' is constant, which cannot occurs. Consequently, we have $c_t = 0$.

It is easy to see that ϵ_1, which depends on the chosen element of $\pi_1(M)$, can be extended to a group homomorphism from $\pi_1(M)$ to $\{\pm 1\}$. Its kernel is a normal subgroup of $\pi_1(M)$ of index 2, so it acts freely, properly discontinuously and co-compactly on $\tilde{M}$. The quotient $M' := \tilde{M}/\ker \epsilon_1$ is then a double cover of M which carries an LCP structure induced by the Levi-Civita of ∇^h. In other words, up to replacing M by M', we can assume that ϵ_1 is constant equal to 1.

Altogether, $\pi_1(M)$ is the semi-direct product $D \rtimes \langle \gamma \rangle$ with

$$\gamma = (s, t, x) \mapsto (\lambda s, t + 1, \alpha(x)) \tag{35}$$

and D only contains transformations of the form

$$(s, t, x) \mapsto (s + c_s, t, \beta(x)), \tag{36}$$

where β is an isometry of (K, g_K). Consequently, one has $M \simeq (\tilde{M}/D)/\langle \gamma \rangle$. In addition, it is easy to see using (36) that D acts only on $\mathbb{R}_s \times K$, where $\mathbb{R}_s$ stands for the line parametrized by the variable s, and this action is free and properly discontinuous. Denoting by D' the restriction of D to $\mathbb{R}_s \times K$, one has $\tilde{M}/D \simeq ((\mathbb{R}_s \times K)/D') \times \mathbb{R}_t$. By (35) it turns out that $M \simeq (\tilde{M}/D)/\langle \gamma \rangle$ is a fiber bundle over S^1 with fiber $(\mathbb{R}_s \times K)/D'$. Since M is compact, $C := (\mathbb{R}_s \times K)/D'$ has to be compact too. By (36), D' acts by isometries on the Riemannian manifold $(\mathbb{R}_s \times K, ds^2 + g_K)$, so this metric descends to a metric g_C on C.

The restriction γ' of γ to $\mathbb{R}_s \times K$ is a diffeomorphism, and it descends to a map f on C. Indeed, we remark that the restriction of the action of $\pi_1(M) = D \rtimes \langle \gamma \rangle$ to $\mathbb{R}_s \times K$ is the semi-direct product $D' \rtimes \langle \gamma' \rangle$. Thus if $(p, q) \in (\mathbb{R}_s \times K)^2$ are such that there exists $\gamma_0 \in D'$, $\gamma_0 p = q$, then $\gamma' \gamma_0 p = \gamma' q$, hence $(\gamma' \gamma_0 \gamma'^{-1}) \gamma' p = \gamma' q$, so $\gamma' p \sim \gamma' q$ because D' is normal. We already know that f is a local diffeomorphism by definition, and it is invertible because γ^{-1} also descends to a map on C, thus f is a diffeomorphism. Let $\bar{p}, \bar{q}$ be the equivalence classes of $(p, q) \in (\mathbb{R}_s \times K)^2$, and assume $f\bar{p} = f\bar{q}$. Then, one has $\gamma_0 \gamma' p = \gamma' q$, thus $(\gamma'^{-1} \gamma_0 \gamma') p = q$, so $\bar{p} = \bar{q}$ because D' is normal.

For any $p \in \mathbb{R}_s \times K$ one has $|\det d\gamma'| = \lambda < 1$ (with respect to the metric $ds^2 + g_K$) because α is an isometry of (K, g_K). This implies that the diffeomorphism $f : C \to C$ satisfies $|\det df| < 1$ (with respect to the metric g_C). Let v_C be the Riemannian volume element of the compact manifold (C, g_C). Then, the volume V_C of (C, g_C) is finite and satisfies:

$$V_C = \int_C v_C = \int_C f^*(v_C) = \int_C |\det df| v_C < V_C,$$

which is absurd. This concludes the proof. $\square$

4.2. **LCP structures on compact Einstein manifolds.** In this subsection we will prove the following:

Theorem 4.5. *On a compact Einstein manifold, there are no LCP structures compatible with the Einstein metric.*

Proof. Assume that $([g], \nabla)$ is an LCP structure on an Einstein manifold (M, g), i.e. satisfying

$$\mathrm{Ric}^g = \lambda g \tag{37}$$

for some real constant λ.

Since $\tilde{g} = e^{-2\varphi} h$, (3) shows that the connections $\nabla^{\tilde{g}}$ and ∇^h are related by:

$$\nabla^h_X Y - \nabla^{\tilde{g}}_X Y = d\varphi(Y)X + d\varphi(X)Y - \tilde{g}(X,Y)(d\varphi)^{\sharp_{\tilde{g}}}, \tag{38}$$

where $(d\varphi)^{\sharp_{\tilde{g}}}$ is the dual vector field to $d\varphi$ with respect to $\tilde{g}$.

Moreover, from [5, Theorem 1.159] we have the following relations between the Laplace operators on functions and the Ricci tensors of the metrics $\tilde{g}$ and h:

$$\Delta^{\tilde{g}} f = e^{2\varphi}(\Delta^h f + (n-2)h(df, d\varphi)), \tag{39}$$

and

$$\mathrm{Ric}^{\tilde{g}} - \mathrm{Ric}^h = (n-2)(\nabla^h(d\varphi) + d\varphi \otimes d\varphi) - (\Delta^h \varphi + (n-2)\|d\varphi\|_h^2)h, \tag{40}$$

Note that $\mathrm{Ric}^h(X,Y) = 0$ for every $X \in T\mathbb{R}^q$ and $Y \in TN$. Let us fix vector fields $X \in \mathcal{X}(\mathbb{R}^q)$ and $Y \in \mathcal{X}(N)$, identified with their canonical extensions to $\tilde{M}$. In particular, $\nabla^h_X Y = 0$. Plugging X, Y into (40) and using (37) yields

$$\nabla^h(d\varphi)(X,Y) + d\varphi(X)d\varphi(Y) = 0. \tag{41}$$

This implies

$$X(Y(e^\varphi)) = X(e^\varphi(d\varphi(Y))) = e^\varphi(d\varphi(X)d\varphi(Y) + \nabla^h(d\varphi)(X,Y)) = 0.$$

Since this holds for every $X \in \mathcal{X}(\mathbb{R}^q)$ and $Y \in \mathcal{X}(N)$, we see that $e^\varphi = f_1 + f_2$ for some smooth functions $f_1 \in C^\infty(\mathbb{R}^q)$ and $f_2 \in C^\infty(N)$.

Consider an element $\gamma \in \pi_1(M)$ acting on $\tilde{M}$ as a strict homothety of h. Since $\tilde{g}$ is of course $\pi_1(M)$-invariant, this means that there exists a positive real number $\mu \neq 1$ such that $\gamma^*(e^\varphi) = \mu e^\varphi$. We thus obtain

$$\gamma^* f_1 - \mu f_1 = \mu f_2 - \gamma^* f_2.$$

Since this is an equality between functions on $\mathbb{R}^q$ and N respectively, there exists a constant c such that $\gamma^* f_1 - \mu f_1 = c$. This equation can be written as

$$\gamma^*(f_1 - \frac{c}{1-\mu}) = \mu(f_1 - \frac{c}{1-\mu}).$$

On the other hand, [6, Lemma 3.4] shows that f_1 is bounded on $\mathbb{R}^q$. The above equivariance property thus shows that $f_1 - \frac{c}{1-\mu}$ vanishes, i.e. f_1 is constant.

We have thus proved that φ is the pull-back to $\tilde{M}$ of a function defined on N. We now plug in a non-zero vector X from $T\mathbb{R}^q$ in the Ricci transformation formula (40) and obtain:

$$\lambda \tilde{g}(X,X) = -(\Delta^h \varphi + (n-2)\|d\varphi\|_h^2)h(X,X),$$

whence

$$\lambda = -e^{2\varphi}(\Delta^h\varphi + (n-2)\|d\varphi\|_h^2).$$

We conclude from (39) that

$$-\lambda = \Delta^{\tilde{g}}\varphi.$$

Recall now that $d\varphi$ is the pull-back to $\tilde{M}$ of the Lee form θ on M. The previous relation thus reads $-\lambda = \delta^g\theta$ on M, which, by integration on the compact manifold M, yields that $\lambda = 0$ and thus θ is g-harmonic. The Bochner formula applied to the compact Ricci-flat manifold (M, g) then shows that θ is ∇^g-parallel. However, this is impossible by [9] (see also [3, Theorem 1.6]). $\square$

4.3. **LCP structures on reducible manifolds.** We start by recalling the construction in [6] of LCP structures on compact manifolds carrying a reducible metric in their conformal class. Let (c, ∇) be an LCP structure on M. Recall that a metric $g \in c$ is called *adapted* if the Lee form of ∇ with respect to g vanishes on the flat distribution [6, Definition 3.8]. This is equivalent to the fact that the function φ on $\tilde{M}$ defined by $h = e^{2\varphi}\tilde{g}$ is constant along $\mathbb{R}^q$, i.e. it is the pull-back of a function on N. By [6, Proposition 3.6], every LCP structure admits adapted metrics.

Example 4.6. Let g' be an adapted metric for an LCP structure on M', and let $(\tilde{M}', h') = \mathbb{R}^q \times (N', g_{N'})$ be the decomposition of the universal cover of $(\tilde{M}', h' := e^{2\varphi}\tilde{g}')$ given by Theorem 4.2. If (K, g_K) be any compact Riemannian manifold, then the Riemannian product $(M, g) := (M', g') \times (K, g_K)$ also carries an LCP structure. Indeed, the lift of the Riemannian metric g to the universal cover $\tilde{M} = \tilde{M}' \times \tilde{K}$ of M can be written

$$\tilde{g} = \tilde{g}' + \tilde{g}_K = e^{-2\varphi}h' + \tilde{g}_K = e^{-2\varphi}(g_{\mathbb{R}^q} + g_{N'} + e^{2\varphi}\tilde{g}_K)$$

so $(\tilde{M}, e^{2\varphi}\tilde{g})$ is the Riemannian product of the flat space $\mathbb{R}^q$ and $(N, g_N) := (N' \times \tilde{K}, g_{N'} + e^{2\varphi}\tilde{g}_K)$ (the latter being a warped product metric on $N' \times \tilde{K}$ since φ is a function on N').

The universal cover $\tilde{M}$ of M admits thus a Riemannian product metric $g = g_1 + g_2$ – where $g_1 := g'$ is a metric on $M_1 := \tilde{M}'$, and $g_2 := \tilde{g}_K$ on $M_2 := \tilde{K}$ – which is $\pi_1(M) = \pi_1(M') \times \pi_1(K)$-invariant. The connection ∇^h induces an LCP structure on M, for which the flat distribution $T\mathbb{R}^q$ is contained in TM_1 and the function $\varphi : \tilde{M} = \mathbb{R}^q \times N' \times \tilde{K}$ determining the conformal change from h to $\tilde{g}$ only depends on the factor N'.

Our aim is to prove that conversely, every reducible Riemannian manifold carrying an LCP structure is obtained locally by the above construction, or, equivalently, that the properties described in the paragraph above are satisfied on every compact reducible LCP manifold.

Theorem 4.7. *Assume that (M, g) is a compact reducible Riemannian manifold (thus its universal cover $(\tilde{M}, \tilde{g})$ is isometric to a Riemannian product $(M_1, g_1) \times (M_2, g_2)$ of two complete Riemannian manifolds). If $([g], \nabla)$ is an LCP structure on M, defining the $\pi_1(M)$-equivariant product metric $e^{2\varphi}\tilde{g} = h := g_{\mathbb{R}^q} + g_N$ on $\tilde{M}$, then up to interchanging the factors*

M_1 and M_2, the flat distribution $T\mathbb{R}^q$ is contained in TM_1, and the conformal factor φ is the pull-back of a function on M_1 which is constant along $\mathbb{R}^q$.

Moreover, the universal cover $\tilde{M}$ decomposes as a triple product $\mathbb{R}^q \times N' \times M_2$, where $M_1 = \mathbb{R}^q \times N'$, and $h = g_{\mathbb{R}^q} + g_{N'} + e^{-2\varphi} g_2$, with $(N', g_{N'})$ incomplete.

Proof. Let ξ be a non-zero ∇^h-parallel vector field tangent to the flat factor $\mathbb{R}^q$ of $\tilde{M}$. One can assume that $h(\xi,\xi) = 1$. Since $\nabla^h \xi = 0$, ξ is Killing with respect to h and conformal Killing with respect to the complete Riemannian product metric $\tilde{g}$, which is non-flat since we excluded the Einstein case in Theorem 4.5. Moreover, ξ is complete, therefore we can apply the following remarkable result:

Theorem (Tashiro-Miyashita) [24] *A complete conformal Killing vector field on a complete, non-flat Riemannian product is Killing.*

We infer that ξ is Killing with respect to $\tilde{g}$, which implies that $\xi(\varphi) = 0$. Thus $\tilde{\theta}(\xi) = 0$, for all h-parallel vector fields ξ tangent to the factor $\mathbb{R}^q$, showing that $\tilde{\theta}(T\mathbb{R}^q) = 0$, or equivalently that g is adapted. We will use this fact several times in the subsequent proofs without referring to it explicitly.

We consider the open set

$$W_\xi := \{x \in M \mid \xi_1 \neq 0,\ \xi_2 \neq 0\},$$

where for every tangent vector $X \in T\tilde{M}$ we will denote by X_1 and X_2 its projections to TM_1 and TM_2.

Our goal is to show that $W_\xi = \emptyset$. This will be done in two steps: first we prove that if W_ξ is non-empty, then it is dense in $\tilde{M}$. Next, assuming W_ξ is dense, we will obtain a contradiction.

Lemma 4.8. *For every ∇^h-parallel vector field ξ, the set W_ξ is either empty or dense in $\tilde{M}$.*

Proof. The statement is trivial for $\xi = 0$, so we can assume that $h(\xi,\xi) = 1$. For every $x \in \tilde{M}$ we denote by $\tilde{M}_x^\perp$ the maximal integral manifold induced by the distribution $\xi^\perp$. Since $(\tilde{M}, h)$ is isometric to $\mathbb{R}^q \times (N, g_N)$ and ξ is a constant vector on $\mathbb{R}^q$, one has $M_x^\perp \simeq \mathbb{R}^{q-1} \times N$.

Suppose that $W_\xi \neq \emptyset$ and fix $x \in W_\xi$. Applying (3) to a vector field $X \in T\tilde{M}_x^\perp$ yields

$$0 = \nabla^{\tilde{g}}_X \xi + \tilde{\theta}(X)\xi = \nabla^{\tilde{g}}_X (e^\varphi \xi) \tag{42}$$

showing that the distribution generated by ξ is parallel along $\tilde{M}_x^\perp$. Let $y \in \tilde{M}_x^\perp$. One has $\xi_y \notin TM_1$ because otherwise ξ_x would be in TM_1 since TM_1 is $\tilde{g}$-parallel. With the same argument, one has $\xi_y \notin TM_2$, and we conclude that $y \in W_\xi$ and thus $\tilde{M}_x^\perp \subset W_\xi$.

It remains to understand how the decomposition of ξ with respect to TM_1 and TM_2 varies in the direction of ξ. Let D_x be the maximal integral manifold through x of the distribution spanned by ξ. We know that $D_x \simeq \mathbb{R}$ and φ is constant along D_x since $d\varphi(\xi) = \tilde{\theta}(\xi) = 0$. This implies that $\nabla^h_\xi \tilde{\theta} = 0$. We remark that the metric duals $\tilde{\theta}^\sharp$ and $\tilde{\theta}^{\sharp_h}$ of θ with respect to $\tilde g$ and h are related by $\tilde\theta^\sharp = e^{2\varphi}\tilde\theta^{\sharp_h}$, showing that $\nabla^h_\xi\tilde\theta^\sharp = \nabla^h_\xi(e^{-2\varphi}\tilde\theta^{\sharp_h}) = e^{-2\varphi}\nabla^h_\xi\tilde\theta^{\sharp_h} = 0$. Applying (3) again, we obtain

$$0 = \nabla^h_\xi\xi = \nabla^{\tilde g}_\xi\xi - \tilde\theta^\sharp\tilde g(\xi,\xi) = \nabla^{\tilde g}_\xi\xi - e^{-2\varphi}\tilde\theta^\sharp$$
$$0 = \nabla^h_\xi\tilde\theta^\sharp = \nabla^{\tilde g}_\xi\tilde\theta^\sharp + \tilde\theta(\tilde\theta^\sharp)\xi = \nabla^{\tilde g}_\xi\tilde\theta^\sharp + |\tilde\theta|^2_g\xi,$$

which can be rewritten

$$\begin{aligned}\nabla^{\tilde g}_\xi\xi &= e^{-2\varphi}\tilde\theta^\sharp\\ \nabla^{\tilde g}_\xi\tilde\theta^\sharp &= -|\tilde\theta|^2_g\xi.\end{aligned}\tag{43}$$

Taking the $\tilde g$-scalar product with $\tilde\theta^\sharp$ in the second equation of (43) shows that $\tilde\theta^\sharp$ has constant norm along D_x, so the subspace $E := \mathrm{span}(\xi,\theta^\sharp)$ defines a $\nabla^{\tilde g}$-parallel distribution along D_x. We distinguish two cases.

Case 1. Assume that $E_x\cap T_xM_1 = \{0\} = E_x\cap T_xM_2$. Since $E\cap TM_1$ and $E\cap TM_2$ are both $\nabla^{\tilde g}$-parallel along D_x, this implies that $E_y\cap T_yM_1 = \{0\} = E_y\cap T_yM_2$ for every $y\in D_x$. In particular, ξ is contained neither in TM_1 nor in TM_2 along D_x, so $D_x\subset W_\xi$.

Case 2. Assume now that $E_x\cap T_xM_1 \neq \{0\}$. The case $E_x\cap T_xM_2\neq\{0\}$ will be treated similarly. Since $x\in W_\xi$, one has $\xi_x\notin T_xM_1$, so $\tilde\theta^\sharp_x\neq 0$ and the dimension of E is 2 along D_x. Moreover, $E_x\cap T_xM_1$ has dimension 1, so the distribution $E\cap TM_1$ has dimension 1 along D_x. This allows us to define a $\nabla^{\tilde g}$-parallel vector field $X\in E\cap TM_1$ along D_x satisfying $\tilde g(X,X)=1$. Consider the $\nabla^{\tilde g}$-parallel vector field Y along D_x (uniquely defined up to a sign) which is orthogonal to X, belongs to E, and satisfies $\tilde g(Y,Y)=1$. We take a scalar product with respect to X and Y in equation (43) and obtain

$$\xi(\tilde g(\xi,X)) = e^{-2\varphi}\tilde g(\tilde\theta^\sharp,X) \qquad \xi(\tilde g(\tilde\theta^\sharp,X)) = -|\tilde\theta|^2_{\tilde g}\tilde g(\xi,X)$$
$$\xi(\tilde g(\xi,Y)) = e^{-2\varphi}\tilde g(\tilde\theta^\sharp,Y) \qquad \xi(\tilde g(\tilde\theta^\sharp,Y)) = -|\tilde\theta|^2_{\tilde g}\tilde g(\xi,Y).$$

Defining $\xi_X := \tilde g(\xi,X)$, $\xi_Y := \tilde g(\xi,Y)$, $c := e^{-\varphi}|\tilde\theta|_{\tilde g}$ and taking a further derivative with respect to ξ in the first and third equations above leads to

$$\xi^2(\xi_X) = -c^2\xi_X, \qquad \xi^2(\xi_Y) = -c^2\xi_Y.$$

We conclude that there exist $c_1,c_2,c_3,c_4\in\mathbb{R}$ such that

$$\xi_X(t) = c_1\cos(ct)+c_2\sin(ct) \qquad \xi_Y(t) = c_3\cos(ct)+c_4\sin(ct),\tag{44}$$

where we parameterized D_x via $t\mapsto \exp_x(t\xi)$. Due to the initial conditions, $(c_1,c_2)\neq(0,0)$ and $(c_3,c_4)\neq(0,0)$, so ξ_X and ξ_Y are analytic non-zero functions of t. We now write the decomposition of X, Y and ξ according to $TM_1\oplus TM_2$ as $X =: X_1$, $Y =: Y_1+Y_2$ and $\xi =: \xi_1+\xi_2$ respectively, and we obtain along D_x:

$$\xi_1 = \xi_X X_1 + \xi_Y Y_1, \qquad \xi_2 = \xi_Y Y_2.$$

One has $\tilde g(X_1,Y_1) = \tilde g(X_1,Y) = \tilde g(X,Y) = 0$, so for any $y\in D_x$

$$\xi_y\in T_yM_2 \Leftrightarrow (\xi_X X_1)_y = 0 \text{ and } (\xi_Y Y_1)_y = 0, \qquad \xi_y\in T_yM_1 \Leftrightarrow (\xi_Y Y_2)_y = 0.$$

Note that $X = X_1$ is non-vanishing along D_x, and Y_2 is also non-vanishing since otherwise Y would belong to $E \cap TM_1$ which has dimension 1 and is spanned by X. Thus, a necessary condition for ξ_y to belong to T_yM_2 is that $\xi_X(y) = 0$ and a necessary condition for ξ_y to belong to T_yM_1 is that $\xi_Y(y) = 0$. However, since the functions ξ_X and ξ_Y are analytic and non-zero, these two conditions occur only on a discrete subset of D_x. Since in Case 1 we have $D_x \subset W_\xi$, this argument shows that $U_x := D_x \cap W_\xi$ is dense in D_x in both cases.

Now, using the fact that $\tilde{M}_y^\perp \subset W_\xi$ for every $y \in W_\xi$, we conclude that $W_\xi = U_x \times \tilde{M}_x^\perp$ which is dense in $\tilde{M}$, thus proving the lemma. □

We will now prove that W_ξ is actually empty for every ξ.

Lemma 4.9. *For every ∇^h-parallel vector field ξ, the set W_ξ is empty.*

Proof. Like before, one can assume that $h(\xi, \xi) = 1$. By (3) we then obtain $0 = \nabla^{\tilde{g}}_X \xi + \tilde{\theta}(X)\xi - \tilde{\theta}^\sharp \tilde{g}(X, \xi)$ for every $X \in T\tilde{M}$. Denoting by $\eta := e^\varphi \xi$, this equation reads

$$\nabla^{\tilde{g}}_X \eta = \tilde{\theta}^\sharp \tilde{g}(X, \eta) \qquad \forall X \in T\tilde{M}. \tag{45}$$

By taking a further covariant derivative in this relation and skew-symmetrizing, we obtain

$$R^{\tilde{g}}_{X,Y}\eta = \tilde{g}(\eta, Y)(\nabla^{\tilde{g}}_X \tilde{\theta}^\sharp - \tilde{\theta}(X)\tilde{\theta}^\sharp) - \tilde{g}(\eta, X)(\nabla^{\tilde{g}}_Y \tilde{\theta}^\sharp - \tilde{\theta}(Y)\tilde{\theta}^\sharp) \qquad \forall X, Y \in T\tilde{M}.$$

For $X := X_1 \in TM_1$ and $Y := X_2 \in TM_2$ and using the notation $\alpha := -e^{-\varphi}\tilde{\theta}$, the above relation becomes

$$0 = \tilde{g}(\eta, X_2)\nabla^{\tilde{g}}_{X_1}\alpha - \tilde{g}(\eta, X_1)\nabla^{\tilde{g}}_{X_2}\alpha \qquad \forall X_1 \in TM_1,\ \forall X_2 \in TM_2. \tag{46}$$

Let us assume W_ξ is non-empty, thus dense in $\tilde{M}$ by Lemma 4.8. The relation (46) implies that $\nabla^{\tilde{g}}_X \alpha = 0$ at each point of W_ξ, for all X orthogonal to the 2-plane P spanned by η_1 and η_2, the components of η in TM_1 and TM_2 which are both non-zero. In particular

$$d\|\alpha\|^2_{\tilde{g}}(X) = 0 \qquad \forall X \perp P, \tag{47}$$

which implies that $d\|\alpha\|^2_{\tilde{g}}$ belongs to P^* the dual 2-plane to P. Take now in (46) the scalar product with 2α. We obtain

$$(\eta^\flat \wedge d\|\alpha\|^2_{\tilde{g}})(X_1, X_2) = 0, \quad \forall X_i \in TM_i,\ i = 1, 2. \tag{48}$$

But the 2-form $\eta^\flat \wedge d\|\alpha\|^2_{\tilde{g}}$ is a decomposable form whose factors are both in the dual 2-plane P^*, and, considering the basis of P defined by $X_i := \eta_i$, $i = 1, 2$, the above relation shows that $\eta^\flat \wedge d\|\alpha\|^2_{\tilde{g}} = 0$ on W_ξ, thus everywhere on $\tilde{M}$. But the factor $\eta^\flat$ is unitary and, as α only depends on φ which is constant on $\mathbb{R}^q$, $d\|\alpha\|^2_{\tilde{g}}(\eta) = 0$, thus $d\|\alpha\|^2_{\tilde{g}} \perp \eta^\flat$.

We thus obtain $d\|\alpha\|^2_{\tilde{g}} = 0$ on $\tilde{M}$, so $\|\alpha\|^2_{\tilde{g}} = e^{-2\varphi}\|\tilde{\theta}\|$ is constant on $\tilde{M}$. But $\|\tilde{\theta}\|^2_{\tilde{g}} = \|\theta\|^2_g$ is bounded, whereas φ is unbounded on $\tilde{M}$. This contradiction shows that $W_\xi = \emptyset$. □

Lemma 4.10. *The distribution $T\mathbb{R}^q$ is either contained in TM_1 or in TM_2.*

Proof. Let x be any point in $\tilde{M}$. By Lemma 4.9, $T_x\mathbb{R}^q \subset T_xM_1 \cup T_xM_2$. This clearly implies that $T_x\mathbb{R}^q \subset T_xM_1$ or $T_x\mathbb{R}^q \subset T_xM_2$. For $i = 1, 2$, the sets

$$C_i := \{x \in \tilde{M},\ T_x\mathbb{R}^q \subset T_xM_i\}$$

are closed and disjoint. Since $\tilde{M}$ is connected, one of them is equal to $\tilde{M}$, thus proving the lemma. ☐

We can now finish the proof of Theorem 4.7.

Up to exchanging M_1 and M_2 we can assume that $T_x\mathbb{R}^q$ is contained in T_xM_1 for every $x \in \tilde{M}$. Using (3) for some non-zero ∇^h-parallel section ξ of $T\mathbb{R}^q$, we get for every $X \in T\tilde{M}$

$$0 = \nabla^{\tilde{g}}_X \xi + \tilde{\theta}(\xi)X + \tilde{\theta}(X)\xi - \tilde{\theta}^\sharp \tilde{g}(X, \xi). \tag{49}$$

The vector field ξ and is tangent to TM_1 and the same holds for $\nabla^{\tilde{g}}_X\xi$ since TM_1 is $\nabla^{\tilde{g}}$-parallel. Therefore, projecting the equation above on TM_2 and taking $X = \xi$ yields $\tilde{\theta}_2 = 0$, so

$$X_2(\varphi) = 0 \text{ for all } X_2 \in TM_2. \tag{50}$$

The action of the Lie group $\mathbb{R}^q$ on $\tilde{M} = M_1 \times M_2$ defined by the ∇^h-parallel vector fields ξ from the Riemannian factor $\mathbb{R}^q$ is free, proper, isometric with respect to h and $\tilde{g}$, and preserves all slices $M_1 \times \{x_2\}$, $x_2 \in M_2$. Fix a point $y_2 \in M_2$ and denote by N' the quotient space $(M_1 \times \{y_2\})/\mathbb{R}^q$ with respect to the above free proper action of $\mathbb{R}^q$. The projection $p : M_1 \times \{y_2\} \to N'$ is an $\mathbb{R}^q$ principal bundle with an $\mathbb{R}^q$-invariant horiontal (integrable) distribution D. As M_1 is simply connected, N' is as well, thus the flat $\mathbb{R}^q$-connection D in the above principal bundle has no monodromy and is therefore trivial, which makes every leaf of the foliation tangent to D inside $M_1 \times \{y_2\}$ diffeomorphic to the base N'. In fact they are also isometric when considered the metric induced by h, thus $(M_1 \times \{y_2\}, h|_{M_1\times\{y_2\}})$ is isometric to the Riemannian product $(\mathbb{R}^q, g_{\mathbb{R}^q}) \times (N', g_{N'})$.

In order to prove that $\tilde{M} = M_1 \times M_2$ is a triple product, we need to show that the decomposition $M_1 \times \{y_2\} = \mathbb{R}^q \times N'$ is independent of the choice of $y_2 \in M_2$. This in turn holds if and only if the distributions $\mathbb{R}^q$ and D are invariant by the infinitesimal action of any vertical vector field $\bar{X}_2 \in \mathcal{X}(\tilde{M})$ with respect to the projection $p_2 : \tilde{M} \to M_2$, i.e. for example for $\bar{X}_2$ induced by a vector field $X_2 \in \mathcal{X}(M_2)$. In fact $[X_1, \bar{X}_2]$ is a section of TN (as both X_1 and $\bar{X}_2$ are) for any vector field tangent to $D \subset TN$. That makes $D \subset TM_1$ automatically stable along $\bar{X}_2$.

Consider now ξ a ∇^h-parallel vector field tangent to $\mathbb{R}^q$ and compute

$$[\xi, \bar{X}_2] = \nabla^{\tilde{g}}_\xi \bar{X}_2 - \nabla^{\tilde{g}}_{\bar{X}_2}\xi.$$

The first term belongs to TM_2 (in fact it is zero for our choice of $\bar{X}_2$ being a lift of $X_2 \in \mathcal{X}(M_2)$) and the second vanishes from (42) since $\bar{X}_2 \perp \xi$ and $\theta(\bar{X}_2) = 0$ from (50). This implies that the quotient space M_1 of $\tilde{M}$ inherits the product structure defined by the pair of integrable distributions $\mathbb{R}^q$ and D on each slice $M_1 \times \{y_2\}$, therefore $\tilde{M}$ is diffeomorphic to the triple product $\mathbb{R}^q \times N' \times M_2$. From a metric viewpoint, we have $h = g_{\mathbb{R}^q} + g_{N'} + e^{-2\varphi} g_2$ and $\tilde{g} = e^{2\varphi}(g_{\mathbb{R}^q} + g_{N'}) + g_2$ as claimed.

In particular, (N, g_N) is isometric to $(N' \times M_2, g_{N'} + e^{-2\varphi} g_2)$, so $(N', g_{N'})$ is not complete because otherwise (N, g_N) would be complete as a warped product of two complete Riemannian manifolds. □

Although the universal cover of M turns out to be a triple product as in Example 4.6 of a Riemannian product of an LCP manifold M' with a compact manifold K, the next example shows that the fundamental group of M is not necessarily a product of two groups acting separately on the factors M_1 and M_2, so the reducible metric g on the LCP manifold M is not globally a product in general.

Example 4.11. Let (a, b) be the canonical coordinate system of $\mathbb{R}^2$. We consider the transformation of $\mathbb{R}^2$ given by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and we chose a basis of eigenvectors of A with associated eigenvalues (λ, λ^{-1}), inducing a coordinate system (x, y) in $\mathbb{R}^2$. We define $\tilde{M} = \mathbb{R}^2 \times \mathbb{R}_+^* \times \mathbb{R}$, and we endow this manifold with the metric

$$h := dx^2 + t^4 dy^2 + dt^2 + t^2 ds^2, \tag{51}$$

written in the coordinate system (x, y, t, s). We now define the group of transformations G generated by the maps

$$\begin{aligned} (a, b, t, s) &\mapsto (a + 1, b, t, s) \\ (a, b, t, s) &\mapsto (a, b + 1, t, s) \\ (a, b, t, s) &\mapsto (a, b, t, s + \sqrt{2}) \\ (a, b, t, s) &\mapsto (A(a, b)^T, \lambda t, s + 1), \end{aligned}$$

written in the coordinate system (a, b, t, s). It is easy to check that G acts freely, properly and co-compactly on $\tilde{M}$ by homotheties, and the last map is an homothety of ratio $\lambda \neq 1$, so it is a strict homothety. Thus the metric h defines an LCP structure on $M := \tilde{M}/G$ (see [6, Remark 2.6] for more details). In addition, the metric

$$\tilde{g} := t^{-2} h = t^{-2} dx^2 + t^2 dy^2 + t^{-2} dt^2 + ds^2 \tag{52}$$

descends to a reducible metric g on M. With the notations of this section, we can write

$$(M_1, g_1) = (\mathbb{R}^2 \times \mathbb{R}_+^*, t^{-2} dx^2 + t^2 dy^2 + t^{-2} dt^2), \qquad (M_2, g_2) = (\mathbb{R} ds^2), \tag{53}$$

but the group $G = \pi_1(M)$ is not a product of two groups acting separately on M_1 and M_2.

The above example shows that our results do not answer completely Problem 1.1, since we cannot describe the structure of the fundamental groups of the solutions. However, we do have a complete classification at the level of the universal covers. To make this precise, note that there is a one-to-one correspondence between the solutions (M, g, ∇) of Problem 1.1, and tuples $(\tilde{M}, \tilde{g}, \varphi, \Gamma)$, where:

(1) $(\tilde{M}, \tilde{g})$ is a complete simply connected Riemannian manifold of dimension $n \geq 3$ with special holonomy;
(2) φ is a smooth function on $\tilde{M}$ such that the metric $h := e^{2\varphi}\tilde{g}$ has special holonomy;
(3) Γ is a discrete co-compact group acting on $\tilde{M}$ by isometries of $\tilde{g}$ and homotheties of h, not all of them being isometries.

Summarizing the results in Proposition 3.1, Theorem 4.3, Theorem 4.5 and Theorem 4.7, we obtain the following classification result:

Theorem 4.12. *The triples $(\tilde{M}, \tilde{g}, \varphi)$ satisfying conditions $(1)-(2)$ above, for which there exists a group Γ satisfying condition (3), are of the following form:*

- $\tilde{M} = \mathbb{R} \times S$, $\tilde{g} = dt^2 + g_S$, *and* $\varphi = dt$, *where* (S, g_S) *is a either a complete Sasakian manifold, a round sphere, or a compact nearly Kähler or nearly parallel* G_2 *manifold.*
- $\tilde{M} = \mathbb{R}^q \times N' \times M_2$, $\tilde{g} = e^{-2\varphi}(g_{\mathbb{R}^q} + g_{N'}) + g_2$, *with* $q \geq 1$, $\varphi \in C^\infty(N')$, *where* (M_2, g_2) *is a complete Riemannian manifold and* $(N', g_{N'})$ *is incomplete.*

References

[1] C. Bär, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993), 509–521.
[2] F. Belgun, *On the structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), 1–40.
[3] F. Belgun, A. Moroianu, *On the irreducibility of locally metric connections*, J. reine angew. Math. **714** (2016), 123–150.
[4] F. Belgun, A. Moroianu, *Weyl-parallel forms, conformal products and Einstein-Weyl manifolds*, Asian J. Math. **15** (2011), 499–520.
[5] A. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 10. Springer-Verlag, Berlin, 1987.
[6] B. Flamencourt, *Locally conformally product structures*, arxiv:2205.02943.
[7] D. Fried, *Closed similarity manifolds.* Comment. Math. Helv. **55** (4) (1980), 576–582.
[8] Th. Friedrich, I. Kath, A. Moroianu, U. Semmelmann, *On nearly-parallel* G_2*-structures*, J. Geom. Phys. **23** (1997), 269–286.
[9] S. Gallot, *Équations différentielles caractéristiques de la sphère,* Ann. Sci. Ec. Norm. Sup. Paris **12** (1979), 235–267.
[10] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), 495–518.
[11] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type* $S^1 \times S^3$, J. reine angew. Math. **469** (1995), 1–50.
[12] A. Gray, *The structure of nearly Kähler manifolds*, Math. Ann. **223** (1976), 233–248.
[13] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry I*, New York, Interscience Publishers, 1963.
[14] M. Kourganoff, *Similarity structures and de Rham decomposition.* Math. Ann. **373** (2019), 1075–1101.
[15] W. Kühnel, H.-B. Rademacher, *Conformally Einstein product spaces*, Differential Geom. Appl. **49** (2016), 65–96.

[16] C. R. LeBrun, *Anti-self-dual Hermitian metrics on blown-up Hopf surfaces*, Math. Ann. **389** (1991), 383–392.

[17] F. Madani, A. Moroianu, M. Pilca, *Conformally related Kähler metrics and the holonomy of lcK manifolds*. J. Eur. Math. Soc. **22** (1) (2020), 119–149.

[18] V.S. Matveev, Y. Nikolayevsky, *Locally conformally Berwald manifolds and compact quotients of reducible manifolds by homotheties*. Annales de l'Institut Fourier **67** (2) (2017), 843–862.

[19] S. Merkulov, L. Schwachhöfer, *Classification of irreducible holonomies of torsion-free affine connections*, Ann. of Math. **150** (1999), 77–149.

[20] A. Moroianu, *Conformally related metrics with non-generic holonomy*. J. reine angew. Math. **755** (2019), 279–292.

[21] A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts, vol. 69, Cambridge University Press, Cambridge, 2007.

[22] A. Moroianu, L. Ornea, *Conformally Einstein products and nearly Kähler manifolds*, Ann. Global Anal. Geom. **33** (2008), 11–18.

[23] K. Oeljeklaus, M. Toma, *Non-Kähler compact complex manifolds associated to number fields*. Ann. Inst. Fourier **55** (2005), 161–171.

[24] Y. Tashiro, K. Miyashita, *Conformal transformations in complete product Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 328–346.

[25] K.P. Tod, *Compact 3-dimensional Einstein-Weyl structures*, J. London Math. Soc. (2) **45** (1992), 341–351.

[26] F. Tricceri, *Some examples of locally conformal Kähler manifolds*, Rend. Semin. Mat. Univ. Politecn. Torino **40** (1982), 81–92.

[27] I. Vaisman, *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Mat. (6) **12** (1979), 263–284.

[28] M. Wang, *Parallel Spinors and Parallel Forms,* Ann Global Anal. Geom. **7** (1989), 59–68.

[29] H. Weyl, *Raum. Zeit. Materie,* (German), Heidelberger Taschenbücher, **251**, Springer-Verlag, Berlin, 1988.

Florin Belgun, IMAR, Calea Griviţei 21, Bucharest, Romania,

Email address: `florin.belgun@imar.ro`

Brice Flamencourt, Institut für Geometrie und Topologie, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

Email address: `brice.flamencourt@igt.uni-stuttgart.de`

Andrei Moroianu, Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France

Email address: `andrei.moroianu@math.cnrs.fr`