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On some problems in spectral analysis, spin geometry and conformal geometry

Sur quelques problèmes d'analyse spectrale, géométrie spinorielle et géométrie conforme

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À mes parents, à mon frère.
À Laury.

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Introduction

Ces pages contiennent le travail que j'ai effectué pendant les trois années de ma thèse de doctorat. Il est cependant difficile d'en présenter le sujet et les résultats en quelques mots, pour une raison qui devient évidente à la lecture des titres des différents chapitres : les domaines mathématiques couverts ne semblent avoir que peu de relations entre eux. Aussi, l'introduction de ce texte ne saurait être autre chose qu'un éclaircissement sur les liens entre les sujets des travaux regroupés ici, liens qui relèvent de l'évolution de l'encadrement de ma thèse, et expliquent que cette progression m'a paru naturelle. C'est au cours de cette exposition que seront introduits les concepts étudiés.

Deux grands axes d'étude feront l'objet de ce document. Le premier concerne l'analyse spectrale, et plus particulièrement le comportement asymptotique du spectre de certains opérateurs à paramètres. Il s'agit des résultats que j'ai obtenus pendant la première moitié de ma thèse, sur des sujets proposés par mon directeur de thèse Konstantin Pankrashkin, avant qu'il ne quitte le laboratoire de mathématiques d'Orsay. Le second axe est celui de la géométrie, Riemannienne et conforme, où les sujets abordés m'ont été proposés par Andrei Moroianu, mon second directeur de thèse. Néanmoins, ces deux parties se confondent dans l'étude asymptotique des opérateurs de Dirac avec masses, où le cadre géométrique prend une place importante aux côtés de l'analyse spectrale.

Suivant naturellement l'ordre chronologique de mes recherches et de mon apprentissage, je présenterai tout d'abord la partie portant sur l'analyse avant d'aborder les sujets géométriques. En proposant une courte introduction sur les différents chapitres qui composent ma thèse, j'espère, sans toutefois entrer dans les détails, que le lecteur pourra se faire une idée des motivations qui m'ont poussé à l'étude de ces sujets, et de l'intérêt des résultats énoncés.

Sur les δ-interactions sur des courbes à point de rebroussement

L'étude du comportement de particules assujetties à se déplacer dans une région restreinte de l'espace est un problème récurrent de la physique quantique. Différents modèles peuvent être proposés afin de décrire de telles particules, le plus simple, mais qui ouvre déjà de nombreuses perspectives d'études, étant celui des graphes quantiques. Cependant, cette simplicité a un prix : elle ne permet pas de prendre en compte certains effets physiques importants, comme l'effet tunnel. Pour pallier ce défaut, on peut se pencher sur un modèle plus complexe qui inclut ces phénomènes : les δ-interactions. Dans le contexte des particules dont le mouvement est régi par l'équation de Schrödinger, et contraintes à se déplacer sur un graphe métrique Γ plongé dans $\mathbb{R}^n$, on perturbe l'opérateur de Schrödinger par un potentiel attractif et singulier supporté par Γ. Une explication détaillée de ce modèle et des questions qu'il implique peut être trouvée dans [21].

D'un point de vue mathématique, et dans le cas où le graphe est un sous-espace de $\mathbb{R}^2$, l'opérateur qui décrit le comportement des particules est donné formellement par l'expression

$$H_\alpha := -\Delta - \alpha\delta(x - \Gamma),$$

où δ est la distribution de Dirac et $\alpha > 0$. Afin de rendre cette définition rigoureuse, on définit H_α comme l'opérateur auto-adjoint associé à la forme quadratique

$$H^1(\mathbb{R}^2) \ni u \mapsto h_\alpha(u, u) := \iint_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \alpha \int_\Gamma u^2 \mathrm{d}s,$$

par les théorèmes de représentations usuels [41]. Le paramètre $\alpha > 0$ représente l'attractivité de la région Γ, et l'on peut se demander comment évolue cet opérateur lorsque Γ devient très attractive, c'est-à-dire lorsque $\alpha \to +\infty$. En effet, le modèle précédent n'interdit pas la présence d'une particule en dehors de la région Γ, et c'est uniquement dans le cas de cette limite que l'on confine effectivement le mouvement.

On s'intéresse donc au comportement asymptotique des valeurs propres de H_α, qui correspondent physiquement aux niveaux d'énergie des particules, quand α tend vers $+\infty$. Ce problème particulier a déjà été étudié pour différents types de graphes comme des courbes lisses ou des domaines à coins [17, 20, 23, 24, 26, 50, 69]. Dans tous ces cas, on peut proposer un développement asymptotique des valeurs propres, sachant que le spectre discret contient un nombre arbitraire d'éléments lorsque α devient grand.

Le premier chapitre de cette thèse retranscrit un article que j'ai rédigé avec Konstantin Pankrashkin, et qui étudie le comportement asymptotique des valeurs propres de H_α lorsque Γ est une courbe fermée, de classe C^4 en tout point différent de l'origine, et il existe $\varepsilon_0 > 0$ et $p > 1$ tels que

$$\Gamma \cap (-\varepsilon_0, \varepsilon_0)^2 = \big\{(x_1, x_2) : x_1 \in (0, \varepsilon_0),\, |x_2| = x_1^p\big\}.$$

Cela signifie que la courbe forme un angle de mesure nulle à l'origine. Lorsque l'on s'intéresse à une courbe lisse, on obtient un développement asymptotique en exhibant un opérateur effectif par séparation de variables. Cette manipulation est rendue possible grâce à l'existence d'un voisinage tubulaire régulier autour de Γ. Ici, la difficulté et la particularité du problème résident dans l'absence d'un tel voisinage, forçant l'utilisation d'autres méthodes.

Par un découpage bien choisi de l'espace, on remarque que les fonctions propres se localisent près de l'origine dans le régime asymptotique considéré, et la n-ième valeur propre de H_α, notée $E_n(H_\alpha)$, se comporte de la manière suivante :

$$E_n(H_\alpha) = -\alpha^2 + 2^{\frac{2}{p+2}} E_n(A)\, \alpha^{\frac{6}{p+2}} + \mathcal{O}(\alpha^{\frac{6}{p+2} - \eta})$$

où $\eta := \min\big\{\frac{p-1}{2(p+2)}, \frac{2(p-1)}{(p+1)(p+2)}\big\} > 0$.

La preuve de cette estimation repose principalement sur l'utilisation du principe du Min-Max afin de localiser le problème au voisinage de l'origine et de se ramener à des opérateurs effectifs pour lesquels les calculs sont plus faciles à mener.

Modèle MIT Bag dans des limites de grandes masses

Le deuxième chapitre s'inscrit dans la continuité du précédent, au sens où il porte sur l'étude d'opérateurs de Dirac avec une masse, et le comportement asymptotique des valeurs propres de leurs carrés lorsque cette masse devient infinie. Il s'agit donc également d'analyse asymptotique en géométrie spectrale. Par ailleurs, précisons que, le carré de l'opérateur de Dirac étant à peu de chose près le Laplacien, son étude a un lien fort avec l'opérateur de Schrödinger, que nous avions regardé dans le chapitre précédent. Ce sujet m'a été suggéré par Konstantin Pankrashkin, qui souhaitait généraliser un article qu'il avait écrit en collaboration avec Andrei Moroianu et Thomas Ourmières-Bonafos [59].

Afin d'expliquer l'apport de ce second chapitre, il est nécessaire d'exposer une fois encore le contexte physique avant de rentrer dans le cadre mathématique.

Le modèle MIT Bag a été imaginé par des chercheurs de l'université éponyme pour décrire des particules telles que des quarks, qui seraient emprisonnées dans un hadron, représenté par une région bornée $\mathcal{K}$ de l'espace ambiant [40]. Dans le cadre de la théorie quantique des champs, ces particules sont considérées comme des champs quantiques, et donc régies par l'équation de Dirac. De son côté, le confinement est cette fois imposé par une condition de bord, indiquant que le flux du champ à travers la frontière de $\mathcal{K}$ est nul. Ce modèle diffère fortement du cas des δ-interactions, où le potentiel permettait de localiser la particule dans un voisinage de la zone d'attraction avec une certaine probabilité.

Les objets analysés ici présentent de nombreuses dissemblances avec ceux du chapitre précédent, ceci venant de la complexité conceptuelle de l'équation de Dirac. Celle-ci a été introduite par Paul Dirac afin de concilier la relativité restreinte et la mécanique quantique pour la description de l'électron. Aussi, les solutions de cette équation ne sont plus des fonctions complexes de l'espace et du temps, mais des fonctions à valeurs dans l'espace $\mathbb{C}^4$ appelées spineurs, dont le module ne s'interprète plus comme une densité de probabilité. On peut écrire cette équation dans l'espace-temps à quatre dimensions sous la forme

$$H_m\psi := \left(-i\sum_{k=1}^{3}\alpha_k\partial_k + m\beta\right)\psi = i\frac{\partial}{\partial t}\psi,$$

où H_m est l'opérateur de Dirac, les α_k sont des matrices 4×4 qui satisfont aux conditions de Clifford $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_i^j \mathrm{I}_4$, m est le paramètre de masse, et ψ est le spineur solution de l'équation.

Dans ce cadre, et en notant $\mathbf{n}$ le vecteur normal unitaire extérieur à $\partial\mathcal{K}$, la condition de bord du modèle MIT Bag s'écrit $-i\beta(\alpha\cdot\mathbf{n})\psi = \psi$. L'opérateur de Dirac avec la condition de bord MIT Bag sur $\mathbb{R}^3$ est alors H_m défini sur le domaine $\{\psi \in H^1(\mathcal{K},\mathbb{C}^4) \mid -i\beta(\alpha\cdot\mathbf{n})\psi = \psi\}$. Récemment, un intérêt a été porté au comportement des valeurs propres de cet opérateur dans la limite de grandes masses [4, 5]. Ce régime s'interprète physiquement comme une limite non-relativiste. En effet, dans l'équation physique, la masse présente dans l'expression de l'opérateur H_m est remplacée par le terme mc^2, ainsi, l'assertion "m tend vers l'infini" signifie que la vitesse des particules devient très faible devant la vitesse de la lumière.

Pour permettre l'utilisation d'outils tels que le principe du Min-Max, on regardera désormais les carrés des opérateurs, ces-derniers n'étant pas bornés inférieurement.

Lorsque la masse croît, on observe une localisation des fonctions propres de H_m^2 près du bord de $\mathcal{K}$, et les valeurs propres de H_m^2 convergent vers celles d'un opérateur effectif sur $\partial\mathcal{K}$. Par ailleurs, dans [4], les auteurs se sont penchés sur un opérateur à deux paramètres de masse $m, m' \in \mathbb{R}$ donné par

$$H_{m,m'} := H_m + \mathbf{1}_{\mathcal{K}^c}(m'-m)\beta,$$

qui peut s'interpréter comme un opérateur de Dirac avec deux potentiels de masse localisés dans les régions distinctes $\mathcal{K}$ et $\mathcal{K}^c$. Ils ont démontré que lorsque m était fixé et que l'on faisait tendre m' vers l'infini, les valeurs propres de $H_{m,m'}^2$ convergeaient vers celles du carré de l'opérateur MIT Bag avec masse m.

Ces résultats de convergence ont été généralisés au cadre Euclidien en dimension quelconque dans [59]. De plus, les auteurs ont montré que l'opérateur modèle intervenant dans la limite des valeurs propres du carré de l'opérateur de Dirac avec condition MIT Bag est en fait le carré de l'opérateur de Dirac sur l'hypersurface $\partial\mathcal{K}$. Puisque seul l'opérateur de Dirac sur l'espace plat a été introduit jusqu'ici, et afin d'expliquer cette dernière phrase à un public non averti, il est nécessaire de digresser sur l'opérateur de Dirac sur les variétés.

L'opérateur de Dirac général a été construit par Atiyah et Singer dans les années 1960 à la suite de leur travail sur la théorie de l'indice. Ils ont remarqué que les variétés Riemaniennes orientées dont le SO_n-fibré principal induit admet un revêtement par un Spin_n-fibré principal permettent une construction similaire à celle de l'opérateur de Dirac des espaces Euclidiens. Les variétés disposant d'une telle structure sont désormais appelées variétés spin, et il est à noter que cette propriété est topologique, malgré l'apparente dépendance à une métrique : elle est équivalente à l'annulation de la deuxième classe de Stiefel-Whitney.

Plus précisément, une variété Riemaninenne orientée (M^n, g) est spin si le SO_n-fibré principal des repères orthonormés $P_{\mathrm{SO}_n}M$ admet un revêtement à deux feuillets par un Spin_n-fibré principal $P_{\mathrm{Spin}_n}M$, tel que, en notant $\chi : P_{\mathrm{Spin}_n}M \to P_{\mathrm{SO}_n}M$ la projection, pour tout $u \in P_{\mathrm{Spin}_n}M$ on a le diagramme commutatif suivant:

$$\begin{tikzcd}
\mathrm{Spin}_n \arrow[r, "s\mapsto us"] \arrow[dd, "\widetilde{Ad}"'] & P_{\mathrm{Spin}_n}M \arrow[dd, "\chi"] \arrow[dr] & \\
 & & M. \\
\mathrm{SO}_n \arrow[r, "p\mapsto\chi(u)p"] & P_{\mathrm{SO}_n}M \arrow[ur] &
\end{tikzcd}$$

Fixons une variété spin (M^n, g). Considérons le module de Clifford $\Sigma_n := \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$, sur lequel se représente de manière irréductible l'algèbre de Clifford complexe $\mathbb{C}\mathrm{l}_n$. Ceci induit une représentation spinorielle $\rho_n : \mathrm{Spin}_n \to \mathrm{End}(\Sigma_n)$. On peut alors construire le fibré des spineurs $\Sigma M := P_{\mathrm{Spin}_n}M \times_{\rho_n} \Sigma_n$ par la construction usuelle des fibrés associés [43], et les sections de ce fibré sont appelées des spineurs. Dans le cas de l'espace Euclidien $\mathbb{R}^n$, où la structure spin est triviale, il s'agit de fonctions de $\mathbb{R}^n$ dans Σ_n, et pour $n = 4$, on retrouve les spineurs introduits dans l'équation de Dirac.

Il existe plusieurs structures naturelles sur le fibré ΣM. Premièrement, on définit la multiplication de Clifford. Pour cela, remarquons déjà que l'on peut étendre la représentation du groupe $\mathrm{SO}_n(\mathbb{R})$ sur $\mathbb{R}^n$ à l'algèbre de Clifford complexe $\mathbb{C}\mathrm{l}_n$ en une représentation l. En utilisant cette action, on définit le fibré de Clifford complexe sur M par $\mathbb{C}\mathrm{l}(M) := P_{\mathrm{SO}_n}(M) \times_l \mathbb{C}\mathrm{l}_n$. Alors, la multiplication de Clifford, notée "$\cdot$", entre une section de $\mathbb{C}\mathrm{l}(M)$ et un spineur est donnée par :

$$[\chi(u), v] \cdot [u, \psi] := [u, \rho_n(v)\psi]$$

où $u \in P_{\mathrm{Spin}_n}M$, $v \in \mathbb{C}\mathrm{l}_n$ et $\psi \in \Sigma_n$. Deuxièmement, le produit Hermitien canonique sur Σ_n, pour lequel la multiplication de Clifford par les vecteurs de norme 1 dans $\mathbb{R}^n$ est unitaire, se transporte sur ΣM en un produit Hermitien $\langle\cdot,\cdot\rangle$. Il satisfait donc l'identité

$$\langle X \cdot \Psi_1, X \cdot \Psi_2\rangle = \|X\|^2\langle\Psi_1, \Psi_2\rangle$$

pour tous spineurs Ψ_1, Ψ_2 et $X \in TM$. Enfin, la connexion de Levi-Civita sur $P_{\mathrm{SO}_n}M$ se relève en une connexion sur $P_{\mathrm{Spin}_n}M$, qui induit une connexion métrique ∇ sur ΣM [43].

Les outils introduits dans le paragraphe précédent permettent de définir l'opérateur de Dirac sur M. Soit $(e_1, \ldots, e_n)$ un repère orthonormé en un point x de M. Alors, l'opérateur de Dirac $\not{D}$ est l'opérateur différentiel d'ordre 1 défini en x par l'expression

$$\not{D} := \sum_{k=1}^{n} e_k \cdot \nabla_{e_k}.$$

Il est facile de montrer que cette définition ne dépend pas du repère fixé. De plus, $\not{D}$ est formellement auto-adjoint [15].

Ayant désormais exposé cette construction, une remarque est toutefois nécessaire pour conclure sur l'apparition de l'opérateur de Dirac sur $\partial\mathcal{K}$ dans le problème MIT Bag. Avec les notations précédemment introduites, si N est une hypersurface orientée de M, alors la structure spin de M induit canoniquement une structure spin sur N, telle que la restrition $\Sigma M|_N$ fibré des spineurs à N s'identifie à ΣN si n est impair, et à $\Sigma N \oplus \Sigma N$ si n est pair. En utilisant cette identification, le produit Hermitien, la multiplication de Clifford et la dérivée covariante sur ΣN sont reliés à leurs équivalents sur $\Sigma M|_N$ (voir [33, Proposition 1.4.1], ou la Proposition 2.2.6 ci-après). La frontière de $\mathcal{K}$ hérite alors d'une structure spin induite par celle de l'espace ambiant, qui permet de définir l'opérateur effectif.

L'intervention de cet opérateur classique de la géométrie différentielle dans le régime asymptotique de grande masse pose une question naturelle : le résultat de convergence est-il généralisable au cadre de la géométrie spinorielle? C'est donc à cette question que l'on s'intéressera au cours du second chapitre de cette thèse.

Néanmoins, le problème n'est pas encore bien posé sous cette forme, car l'opérateur étudié dans le cadre Euclidien n'est pas l'opérateur de Dirac tel que nous l'avons défini sur les variétés. On introduit donc un opérateur de Dirac qui généralise le modèle MIT Bag sur les variétés spin, et une fois le cadre géométrique correctement déterminé, on démontre la convergence des valeurs propres du carré de cet opérateur dans la limite de grandes masses. Plus précisément, on s'intéresse au régime $m \to -\infty$, le régime $m \to +\infty$ ne donnant pas une convergence des valeurs propres. On généralise également le résultat de convergence de l'opérateur à deux masses en étudiant la limite $m' \to +\infty$ à m fixée, et le régime où les deux limites $m \to -\infty$, $m' \to +\infty$ sont considérées simultanément.

Spineurs de Cauchy sur les variétés de dimension 3

A la fin du projet décrit précédemment, l'encadrement de ma thèse a changé puisque Konstantin Pankrashkin a quitté le Laboratoire de mathématiques d'Orsay. Même si il a continué à encadrer mes travaux, j'ai, à partir de ce moment là, d'avantage travaillé sous la direction d'Andrei Moroianu, qui m'avait formé à la géométrie spinorielle et m'avait aidé à mener à bien mon entreprise précédente. Pour cette raison, je me suis orienté vers la géométrie Riemannienne, qui fait partie de son domaine d'expertise.

Afin d'utiliser les compétences que j'avais acquises en me formant sur les spineurs et l'opérateur de Dirac, il m'a été proposé un sujet comportant une forte composante d'analyse sur une classe particulière de spineurs. Le troisième chapitre de ce manuscrit contient les résultats de ce travail : il s'agit d'un article sur les spineurs de Cauchy en dimension 3, co-écrit avec Sergiu Moroianu.

Optons dans un premier temps pour une approche historique, ou du moins chronologique, de la définition des spineurs de Cauchy. En 1980, Thomas Friedrich [29] démontre une inégalité améliorant la borne inférieure des valeurs propres de l'opérateur de Dirac, donnée par l'inégalité de Lichnerowicz [48]. Sur toute variété spin compacte (M^n, g), si λ est valeur propre de l'opérateur de Dirac, alors

$$\lambda^2 \geq \frac{n}{4(n-1)} \operatorname{Scal}_0,$$

où Scal_0 est l'infimum de la courbure scalaire. Lorsque l'égalité a lieu pour la valeur propre λ_0, la variété (M, g) est Einstein [15, Théorème 5.3], et il existe donc un spineur propre Ψ sur M pour la valeur propre λ_0, i.e. $D\!\!\!/\Psi = \lambda_0 \Psi$. Un tel spineur satisfait alors une équation particulière, appelée équation de Killing [15, Remarque 5.5] :

$$\nabla_X \Psi + \frac{\lambda_0}{n} X \cdot \Psi = 0, \qquad \forall X \in TM,$$

dont les solutions sont dénommées *spineurs de Killing réels*. Cette appellation provient d'une propriété de ces spineurs, affirmant que le champ de vecteurs

$$X_\Psi := \sum_{k=1}^{n} i \langle \Psi, e_k \cdot \Psi \rangle e_k$$

qui leur est associé est un champ de vecteurs de Killing [15, Lemme 5.9], i.e. $\mathcal{L}_{X_\Psi} g = 0$. Notons par ailleurs que le cas $\lambda_0 = 0$ correspond aux spineurs *parallèles*.

On peut démontrer que remplacer la valeur propre λ_0 par une fonction réelle quelconque f dans l'équation de Killing ne définit pas une plus grande famille de spineurs sur une variété compacte [15, Proposition 5.11]. Il existe néanmoins une généralisation naturelle, qui consiste à étudier les solutions de l'équation

$$\nabla_X \Psi + A(X) \cdot \Psi = 0, \qquad \forall X \in TM,$$

où A est un endomorphisme symétrique de TM. Les spineurs solutions de cette dernière équation ont été étudiés dans différents contextes sous le nom de *spineurs de Killing généralisés* [3, 60–62].

Malgré les étapes que nous avons franchies pour arriver à la définition de ces derniers, ils apparaissent naturellement lorsque l'on restreint des spineurs parallèles à des hypersurfaces. Expliquons-nous. Supposons que (M, g) est une hypersurface d'une variété Riemannienne spin $(\mathcal{Z}, g_\mathcal{Z})$, munie de la métrique induite. Comme nous l'avons expliqué auparavant, M hérite d'une structure spin induite par celle de $\mathcal{Z}$, et les dérivées covariantes sur $\Sigma\mathcal{Z}$ et ΣM, notées respectivement $\nabla^\mathcal{Z}$ et ∇, sont reliées par l'dentité

$$\nabla_X \Psi + \frac{A(X)}{2} \cdot \Psi = \nabla^\mathcal{Z}_X \Psi, \qquad \forall X \in TM, \Psi \in \Sigma\mathcal{Z}|_M,$$

où cette fois A est le tenseur de Weingarten de M. Si Ψ est un spineur parallèle sur $\mathcal{Z}$, i.e. $\nabla^\mathcal{Z}\Psi = 0$, sa restriction $\psi := \Psi|_M$ à M est solution de l'équation

$$\nabla_X \psi = -\frac{A(X)}{2} \cdot \psi, \qquad \forall X \in TM,$$

et on reconnaît l'équation des spineurs de Killing généralisés, introduite ci-dessus, pour l'endomorphisme symétrique particulier $A/2$.

Cette construction par restriction à des hypersurfaces pose la question naturelle de la possibilité d'une réciproque. En effet, donnons-nous un spineur $\psi \in \Sigma M$ satisfaisant cette dernière équation pour un endomorphisme symétrique A de TM. On peut alors se demander si il existe une variété Riemannienne spin $\mathcal{Z}$, telle que M se plonge isométriquement dans $\mathcal{Z}$, que A soit le tenseur de Weingarten de M et que ψ soit la restriction à M d'un spineur parallèle sur $\mathcal{Z}$? De manière plus imagée, cela revient à demander si l'on peut "épaissir" la variété M en une variété $\mathcal{Z}$, et prolonger ψ en un spineur parallèle sur $\mathcal{Z}$.

La réponse à cette question se révèle positive dans le cas analytique mais négative en général lorsque les données sont seulement lisses [3]. Ce problème pose la question de l'existence des prolongements d'une métrique et d'un spineur, prolongements qui satisfont à certaines équations différentielles, avec des conditions initiales sur l'hypersurface M. Il s'agit dès lors d'un problème de Cauchy à résoudre. Pour cette raison, et afin de distinguer d'avantage ces spineurs particuliers, nous les avons nommés *spineurs de Cauchy*, terminologie qui a reçu l'approbation des différents auteurs ayant travaillé sur le sujet. Notons qu'un spineur de Cauchy est accompagné d'un endomorphisme symétrique, appelé endormorphisme de Cauchy, qui lui est naturellement associé par l'équation le définissant.

Le cas des variétés simplement connexes de dimension 3 est particulier, car les structures additionnelles conduisent à supprimer les spineurs du problème. Ainsi, en dimension 3, un endomorphisme symétrique A est associé à un spineur de Cauchy si et seulement si il vérifie l'équation

$$0 = \mathrm{R}(X,Y) + *d^{\nabla}A(X,Y) + A(X) \wedge A(Y), \qquad\qquad X, Y \in TM,$$

où R est le tenseur de courbure de la variété ambiante, "$*$" est l'opérateur de Hodge et d^{∇} est la dérivée covariante extérieure, définie par l'expression $d^{\nabla}A(X,Y) = (\nabla_X A)Y - (\nabla_Y A)X$. Dans le cas non simplement connexe, cette équation devient une condition nécessaire mais non suffisante à l'existence d'un spineur de Cauchy associé à A.

C'est en utilisant cette équation constituve que nous démontrons les résultats du troisième chapitre. On s'y intéresse à la structure de l'espace des endomorphismes de Cauchy : on démontre que l'espace des déformations infinitésimales autour d'un endomorphisme donné est de dimension finie. On résout également le problème de l'épaississement de la sphère ronde $\mathbf{S}^3$ pour la donnée de certains spineurs de Cauchy. Enfin, on démontre des résultats de classification des spineurs sur cette même variété.

Avant de clore la présentation de cette partie, il est nécessaire de signaler que même dans le cas simple de $\mathbf{S}^3$, cette classification n'est pas complète. Deux familles d'exemples de spineurs de Cauchy sont connues sur la sphère ronde [60–62], mais la question de savoir si ce sont les seuls exemples reste ouverte.

Variétés localement conformément produit

L'étude des spineurs de Cauchy m'avait été suggérée afin d'inscrire mon projet de thèse dans une continuité logique, en poursuivant un travail sur les spineurs après m'être formé sur l'opérateur de Dirac. Toutefois, le cœur de la recherche de mon second directeur de thèse, Andrei Moroianu, est désormais la géométrie conforme. Les problèmes qu'il avait à me proposer, une fois rédigé l'article faisant l'objet du troisième chapitre, s'inscrivaient donc dans cette dernière discipline. Cependant, il est à souligner que de nombreuses questions concernant les spineurs de Cauchy restent en suspens, mais l'absence de piste pour y répondre, même après de longues réflexions, semble devoir reculer leur résolution. C'est ainsi que j'achevais ma transition depuis l'analyse vers la géométrie conforme, qui sera l'objet des deux derniers chapitres de cette thèse.

Les classes conformes sur les variétés on été introduites par Weyl dans son livre fondateur *Raum, Zeit, Materie* [75], afin de construire une théorie unificatrice de l'électromagnétisme et de la relativité. La volonté de Weyl était de rompre avec l'idée que l'espace-temps est une variété Riemannienne, en considérant qu'il n'existe pas de mesure de distance absolue définie en tout point de l'univers. Au lieu de cela, il a considéré que la notion de distance en physique était régie par des principes proches de ceux de la théorie des connexions affines, et que le déplacement parallèle était l'unique moyen de comparer des longueurs entre elles. Bien que cette vision a été abandonnée par les physiciens, elle pose tout de même les bases d'une large théorie mathématique.

Une classe conforme c sur une variété M est un ensemble de métriques Riemanniennes tel que si g, g' sont dans c, il existe une fonction lisse f sur M donnant l'identité $e^{2f}g = g'$. La donnée d'une métrique Riemannienne g induit donc naturellement une classe conforme en considérant toutes les métriques multiples g par une fonction strictement positive. De cette définition, il découle que la notion de distance n'a plus de sens sur une variété conforme, alors que les angles sont les mêmes pour toutes les métriques de la classe.

Dans les mathématiques modernes, on opte pour une description différente des structures conformes, qui permet de s'affranchir d'une métrique de référence. Définissons tout d'abord pour $k \in \mathbb{R}$ le fibré des poids $\mathcal{L}^k := \mathrm{Fr}(M) \times_{|\det|^{\frac{k}{n}}} \mathbb{R}$, où $\mathrm{Fr}(M)$ est le fibré des repères sur M^n. On remarque alors qu'étant donnée une métrique Riemannienne g dans la classe conforme c sur M, son élément de volume $v_g : \mathrm{Fr}(M) \to \mathbb{R}$ est identifié à la section de $\mathcal{L}^{-n}$ donnée par $s(x) := [u, v_h(u)]$ pour tout $x \in M$, où u est un repère quelconque au-dessus de x. Via cette identification, $g \otimes v_g^{-\frac{2}{n}}$ est une section de $\mathrm{Sym}(T^*M \otimes T^*M) \otimes \mathcal{L}^2$, qui ne dépend pas de la métrique choisie. Ainsi, cette section caractérise la classe conforme et peut être confondue avec elle.

Afin de suivre l'idée originale de Weyl, il est nécessaire de définir une connexion sur le fibré tangent TM pour comparer les vecteurs en utilisant le déplacement parallèle. Lorsque l'on travaille en géométrie Riemannienne, il existe une connexion métrique sans torsion préférentielle, la connexion de Levi-Civita. Dans le cadre de la géométrie conforme, on s'intéresse plus largement aux connexions sans torsion qui préservent la classe conforme, appelées structures de Weyl.

La classe conforme c définit une réduction de $\mathrm{Fr}(M)$ au groupe conforme $\mathrm{CO}_n(\mathbb{R})$, que l'on notera $P_{\mathrm{CO}_n}M$. Ceci est mis en évidence par l'identification opérée précédemment entre c et une section de $\mathrm{Sym}(T^*M \otimes T^*M) \otimes \mathcal{L}^2$. Une structure de Weyl est donc une connexion sans torsion sur $P_{\mathrm{CO}_n}M$, ou de manière équivalente une connexion D sur TM qui préserve c.

En regardant désormais c comme un ensemble de métrique, la propriété de préservation de c par D signifie que pour toute métrique g dans c, il existe une 1-forme θ_g, appelée forme de Lee de D par rapport à g telle que $Dg = -2\theta_g \otimes g$. Dans cette dernière expression, le coefficient -2, de peu d'importance, permet simplement des simplifications dans le jeu des écritures.

Les structures de Weyl offrent un large choix de connexions sur TM, qui englobe notamment les connexions de Levi-Civita des métriques dans c. En général, une structure de Weyl ne préserve cependant aucune métrique dans la classe conforme. Lorsqu'il existe localement une métrique préservée par D, cette structure de Weyl est dite fermée, et si cette métrique est globale, elle est dite exacte. Cette terminologie provient directement de celle utilisée pour les 1-formes : D est fermée (respectivement exacte) si et seulement si sa forme de Lee par rapport à une métrique $g \in c$ est fermée (respectivement exacte), propriété alors vraie pour toutes les métriques dans c.

Le relèvement $\widetilde{D}$ d'une structure de Weyl D d'une variété conforme (M, c) à son revêtement universel $\widetilde{M}$ est une structure de Weyl exacte sur $(\widetilde{M}, \widetilde{c})$, où $\widetilde{c}$ est le relèvement de la classe conforme c. Ceci signifie qu'il existe une métrique h, définie sur $\widetilde{M}$ à un facteur multiplicatif près, telle que $\widetilde{D}$ coïncide avec la connexion de Levi-Civita de h. Les éléments de $\pi_1(M)$ agissent alors sur $\widetilde{M}$ comme des h-similitudes, qui se trouvent être uniquement des isométries si et seulement si D est exacte. De plus, il existe une dualité entre les propriétés de (M, c, D) et celles de $(\widetilde{M}, h)$.

Dans le quatrième chapitre, nous nous penchons sur l'étude des structures de Weyl fermées, non-exactes, non-plates et à holonomie réductible sur les variétés compactes. Toutes ces propriétés mises bout à bout semblent être un artifice de simplification hasardeux, mais il répond pourtant à une logique précise que nous exposons maintenant. Soulignons tout d'abord que les structures exactes n'ont qu'un intérêt limité en géométrie conforme, puisque leur étude relève de géométrie Riemannienne, ce qui explique qu'on les écarte ici.

Il y a de cela quelques années, Florin Belgun et Andrei Moroianu ont formulé une conjecture énonçant que les structures de Weyl fermées, non-exactes sur des variétés compactes étaient soit irréductibles, soit plates [9]. Cet énoncé s'est toutefois révélé faux [52], et il a été démontré par la suite, tout d'abord dans le cadre analytique par Vladimir S. Matveev et Yuri Nikolayevsky [53], puis dans dans le cadre général des variétés lisses par Mickaël Kourganoff [45], que la variété Riemannienne $(\widetilde{M}, h)$ pouvait également être le produit Riemannien $\mathbb{R}^q \times (N, g_N)$, où $\mathbb{R}^q$, $q \geq 1$, est un espace Euclidien, et (N, g_N) est une variété incomplète et irréductible. Ce dernier cas constitue néanmoins la seule alternative possible. La connexion D sur M est alors localement la connexion de Levi-Civita d'une métrique qui se relève en une métrique produit sur $\widetilde{M}$. Pour cette raison, le triplet (M, c, D) est appelé une structure *localement conformément produit* (ou *LCP*).

Le chapitre quatre se veut une ouverture à une classification des structures LCP. On y construit de nouveaux exemples et l'on y démontre que les variétés LCP admettent des métriques particulières, par rapport auxquelles la forme de Lee de D s'annule sur la distribution plate. On y établit également un lien particulier entre la théorie des nombres et les structures LCP.

Connexions sans torsion sur les G-structures

Enfin, nous introduisons brièvement le dernier chapitre, qui est une simple note portant sur les structures de Weyl. On s'y intéresse aux connexions compatibles avec une G-structure sur une variété M^n, où G est un sous-groupe fermé de $\mathrm{GL}_n(\mathbb{R})$ contenant $\mathrm{SO}_n(\mathbb{R})$. On démontre qu'il existe alors une telle connexion, et que celle-ci provient d'une structure de Weyl fermée pour une certaine classe conforme sur M.

Ce résultat a été motivé par la lecture d'un exercice dans [58]. Si l'existence d'une telle connexion sans torsion est une application classique de la théorie de la torsion intrinsèque, le résultat, plus fort, que nous proposons, se démontre grâce à une classification des sous-groupes de $\mathrm{GL}_n(\mathbb{R})$ contenant $\mathrm{SO}_n(\mathbb{R})$.

Chapter 1

δ-interactions on curves with cusps

Ce chapitre est la retranscription d'un article co-écrit avec Konstantin Pankrashkin et paru dans *Journal of Mathematical Analysis and Applications*, **491**, 124287 (2020). Il porte sur l'étude asymptotique des valeurs propres d'un opérateur de Schrödinger sur $\mathbb{R}^2$ avec un potentiel singulier, porté par une courbe présentant un point de rebroussement.

1.1 Introduction

Schrödinger operators with singular interactions supported by submanifolds represent an important class of models in mathematical physics, and they have been the subject of an intensive study during the last decades. In the present work we deal with two-dimensional operators, so we assume that Γ is a metric graph embedded in the Euclidean space $\mathbb{R}^2$, and we will be interested in the spectral study of the operators formally written as

$$H_\alpha := -\Delta - \alpha\delta(x - \Gamma)$$

with δ being the Dirac distribution and $\alpha > 0$ being the coupling constant. Such operators describe the motion of particles confined to the graph Γ but allowing for a quantum tunneling between its different parts. The above definition is made rigorous by considering first the quadratic form

$$H^1(\mathbb{R}^2) \ni u \mapsto h_\alpha(u, u) := \iint_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \alpha \int_\Gamma u^2 \mathrm{d}s,$$

where $\mathrm{d}s$ is the one-dimensional Hausdorff measure on Γ. Under suitable regularity assumptions on Γ (e.g. a finite union of bounded Lipschitz curves) the quadratic form h_α is closed and semibounded from below, and, hence, generate in a canonical way a unique self-adjoint operator H_α in $L^2(\mathbb{R}^2)$ whose domain is contained in $H^1(\mathbb{R}^2)$ and such that

$$\iint_{\mathbb{R}^2} u\, H_\alpha u \, \mathrm{d}x = h_\alpha(u, u)$$

for any function u in the domain. In informal language, the operator H_α is the distributional Laplacian in $\mathbb{R}^2 \setminus \Gamma$ with interface conditions $[\partial u] + \alpha u = 0$ on Γ, where $[\partial u]$ denotes a suitably defined jump of the normal derivative of u on Γ, see e.g. [8, 16] for a more detailed discussion.

The well-known review paper [21] provides an introduction to the topic and proposes a number of research directions. An interesting problem setting is provided by the strong coupling regime, i.e. the case $\alpha \to +\infty$. It can be easily seen that the lowest eigenfunctions of H_α concentrate exponentially near Γ, so that one might expect that an "effective operator" on Γ governing the spectral behavior could come in play. This was first proved in [26] for the case when Γ is a C^4-smooth loop: for any fixed $n \in \mathbb{N}$ the operator H_α admits at least n negative eigenvalues if α is sufficiently large, and the nth eigenvalue $E_n(H_\alpha)$ behaves as

$$E_n(H_\alpha) = -\tfrac{1}{4}\,\alpha^2 + E_n(P) + \mathcal{O}\big(\tfrac{\log \alpha}{\alpha}\big), \tag{1.1.1}$$

where P is the operator on $L^2(\Gamma)$ acting in the arc-length parametrization as $f \mapsto -f'' - \frac{1}{4}\,\gamma^2 f$ with γ being the curvature. A similar result holds for finite open arcs as well [24]. To our knowledge, no sufficiently detailed analysis for non-smooth Γ was carried out so far. Being based on the general machinery for problems with corners [13, 18, 42] one might expect that if Γ is piecewise smooth with non-zero angles, then at least several lowest eigenvalues behave as $E_n(H_\alpha) \simeq -\mu_n \alpha^2$ as $\alpha \to +\infty$, where $\mu_n \in (\frac{1}{4}, 1)$ are spectral quantities associated with some model operators (so-called star leaky graphs) whose exact values are not known: we refer to [17, 20, 23, 50, 69] for a number of estimates.

It seems that no work analyzed the case of non-Lipschitz Γ, and we make the first step in this direction in the present text by considering curves with power cusps. More precisely, we assume that Γ is a Jordan curve satisfying $0 \in \Gamma$ and the following two conditions:

$$\Gamma \text{ is } C^4\text{-smooth at all points except at the origin,}$$

there exist $\varepsilon_0 > 0$ and $p > 1$ such that

$$\Gamma \cap (-\varepsilon_0, \varepsilon_0)^2 = \big\{(x_1, x_2) : x_1 \in (0, \varepsilon_0),\ |x_2| = x_1^p\big\}. \tag{1.1.2}$$

The value p is indeed unique. It is easily seen that the essential spectrum of H_α covers the half-axis $[0, +\infty)$ (use [16, Theorem 3.1] for example) and that for any $\alpha > 0$ the discrete spectrum is non-empty and finite. Our result on the asymptotics of individual eigenvalues of H_α for large α involves an auxiliary one-dimensional operator A in $L^2(0, +\infty)$ acting as

$$(Af)(x) = -f''(x) + x^p f(x)$$

on the functions f satisfying the Dirichlet condition $f(0) = 0$. It is directly seen that A has compact resolvent and that all its eigenvalues $E_n(A)$ are strictly positive and simple.

Theorem 1.1.1. *For any fixed $n \in \mathbb{N}$ one has, as α tends to $+\infty$,*

$$E_n(H_\alpha) = -\alpha^2 + 2^{\frac{2}{p+2}} E_n(A)\, \alpha^{\frac{6}{p+2}} + \mathcal{O}(\alpha^{\frac{6}{p+2} - \eta})$$

where $\eta := \min\big\{\frac{p-1}{2(p+2)}, \frac{2(p-1)}{(p+1)(p+2)}\big\} > 0.$

Remark 1.1.2. For the quadratic cusp, $p = 2$, the eigenvalues $E_n(A)$ can be computed explicitly. The operator A in this case is unitarily equivalent to the restriction of the harmonic oscillator to the odd functions, and its eigenvalues are the usual harmonic oscillator eigenvalues with even numbers, i.e. $E_n(A) = 4n - 1$ for any $n \in \mathbb{N}$. Hence, the asymptotics of Theorem 1.1.1 takes the very explicit form

$$E_n(H_\alpha) = -\alpha^2 + (4n-1)\sqrt{2}\, \alpha^{\frac{3}{2}} + \mathcal{O}(\alpha^{\frac{11}{8}}).$$

We are not aware of other values of $p > 1$ admitting a simple expression for the eigenvalues of A.

Remark 1.1.3. Both main and secondary terms in the result of Theorem 1.1.1 are different from the asymptotics (1.1.1) for the smooth curves and from the expectations for the curves with non-zero angles. In particular, the distance between the individual eigenvalues is of order α^k, where the power $k = \frac{6}{p+2}$ can be given any value between 0 and 2 by a suitable choice of $p \in (1, +\infty)$. Such a control of the eigenvalue gap asymptotics represents a new feature of the model, which is not observed for δ-potentials supported by curves of a higher regularity. Nevertheless we recall that similar effects can be seen in other boundary eigenvalue problems by a suitable control of the boundary curvature, see e.g. [28, 70].

Remark 1.1.4. One should remark that the presence of a singularity does not involve any problem with the semiboundedness of the form h_α, and arbitrary values of p are allowed due to the fact that both sides of Γ are involved. In fact, this directly follows from the fact that Γ can be decomposed into two smooth open arcs, and the L^2-trace of a function from $H^1(\mathbb{R}^2)$ to such an arc is well-defined. This is in contrast with the one-sided Robin problems for the Laplacian in a domain surrounded by Γ, for which the cusp is not allowed to be very sharp: see e.g. [46] for the study of the eigenvalues and [54, 66] for the issues concerning the definition of the operator.

The proof of Theorem 1.1.1 is almost entirely based on the min-max tools for the study of the eigenvalues: we recall them in Section 1.2. We first apply some truncations in order to localize the problem near the cusp and then extend it to a suitable half-place and rescale it in order to have a semiclassical formulation admitting a more explicit analysis (Section 1.3). The resulting problem in the half-plane is analyzed by considering first the action of the operator in one of the variables and then by showing that only the projection onto the lowest mode contributes to the individual eigenvalues. At some points the problem shows a number of similarities to the case when Γ is a sharply broken line [20], and we were able to use a part of that analysis. The overall proof scheme is rather classical, see e.g [28], but a big number of various new technical ingredients and adapted variables are required in order to carry out the complete study. In Section 1.4 we show the upper bound for $E_n(H_\alpha)$, which is rather straightforward. The lower bound is obtained in Section 1.5, and is much more demanding, both for the dimension reduction and for the analysis of the resulting one-dimensional effective operator.

1.2 Preliminaries

We will recall some notation and basic facts on the min-max principle for the eigenvalues of self-adjoint operators.

In this paper we only deal with real-valued operators, so we prefer to work with real Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space and $u \in \mathcal{H}$, then we denote by $\|u\|_{\mathcal{H}}$ the norm of u. For a linear operator T we denote $\mathcal{D}(T)$ its domain. If the operator T is self-adjoint and semibounded from below, then $\mathcal{Q}(T)$ denotes the domain of its bilinear form, and the value of the bilinear form on $u, v \in \mathcal{Q}(T)$ will be denoted by $T[u, v]$. For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, by $E_n(T)$ we denote the nth discrete eigenvalue of T (if it exists) when enumerated in the non-decreasing order and taking the multiplicities into account.

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and T be a lower semibounded self-adjoint operator in $\mathcal{H}$. If T is with compact resolvent, we set $\Sigma := +\infty$, otherwise let Σ denote the bottom of the essential spectrum of T. The nth *Rayleigh quotient* $\Lambda_n(T)$ of T is defined by

$$\Lambda_n(T) := \inf_{\substack{\mathcal{L} \subset \mathcal{Q}(T) \\ \dim \mathcal{L} = n}} \sup_{u \in \mathcal{L} \setminus \{0\}} \frac{T[u, u]}{\|u\|^2_{\mathcal{H}}}.$$

The well-known min-max principle, see e.g. Section 4.5 of [22], states that one and only one of the following assertions is true:

(a) $\Lambda_n(T) < \Sigma$ for all n, $\lim_{m \to +\infty} \Lambda_m(T) = \Sigma$ and $E_n(T) = \Lambda_n(T)$ for all n.

(b) $\Sigma < +\infty$ and there is $N < +\infty$ such that the interval $(-\infty, \Sigma)$ contains exactly N eigenvalues of T counted with multiplicity and for all $n \leq N$, one has $\Lambda_n(T) = E_n(T)$ and $\Lambda_m(T) = \Sigma$ for all $m > N$.

In what follows we will actively work with the Rayleigh quotients of various operators instead of eigenvalues as the former are easier to deal with. The passage from the Rayleigh quotients to the eigenvalues will be done at suitable points by simply checking that the values are below the essential spectrum.

One of the most classical applications of the min-max principle is recalled in the next assertion (the proof is by a direct application of the definition). It will be used systemically through the whole text.

Proposition 1.2.1. *Let T and T' be lower semibounded self-adjoint operators in infinite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively. Assume that there exists a linear map $J : \mathcal{Q}(T) \to \mathcal{Q}(T')$ such that*

$$\|Ju\|_{\mathcal{H}'} = \|u\|_{\mathcal{H}}, \quad T'[Ju, Ju] \leq T[u, u] \text{ for all } u \in \mathcal{Q}(T).$$

Then for any $n \in \mathbb{N}$ there holds $\Lambda_n(T') \le \Lambda_n(T)$.

At the last steps of the proof of Theorem 1.1.1 we will also need the following result, which is a slight reformulation of [25, Lemma 2.1] or of [71, Lemma 2.2]. As some details are different, we prefer to give a complete proof, which is quite short.

Proposition 1.2.2. *Let $\mathcal{H}$, $\mathcal{H}'$ be two infinite-dimensional Hilbert spaces and T be a* non-negative *self-adjont operator in $\mathcal{H}$ and T' be a lower semibounded self-adjoint operator in $\mathcal{H}'$. Assume that there exist a linear map $J : \mathcal{Q}(T) \to \mathcal{Q}(T')$ and non-negative numbers δ_1 and δ_2 such that for all $u \in \mathcal{Q}(T)$ there holds*

$$\begin{aligned}\|u\|^2_{\mathcal{H}} - \|Ju\|^2_{\mathcal{H}'} &\le \delta_1\big(T[u,u] + \|u\|^2_{\mathcal{H}}\big),\\ T'[Ju,Ju] - T[u,u] &\le \delta_2\big(T[u,u] + \|u\|^2_{\mathcal{H}}\big),\end{aligned}$$

and that for some $n \in \mathbb{N}$ one has the strict inequality

$$\delta_1\big(\Lambda_n(T) + 1\big) < 1, \tag{1.2.1}$$

then

$$\Lambda_n(T') \le \Lambda_n(T) + \frac{\big(\delta_1\Lambda_n(T) + \delta_2\big)\big(\Lambda_n(T) + 1\big)}{1 - \delta_1\big(\Lambda_n(T) + 1\big)}.$$

Proof. During the proof we abbreviate $\lambda_n := \Lambda_n(T)$. By (1.2.1), for any sufficiently small $\varepsilon > 0$ one has

$$\delta_1(\lambda_n + 1 + \varepsilon) < 1. \tag{1.2.2}$$

In view of the definition of λ_n, one can find an n-dimensional subspace $F \subset \mathcal{Q}(T)$ such that $T[u,u] \le (\lambda_n + \varepsilon)\|u\|^2_{\mathcal{H}}$ for all $u \in F$. Therefore, for any $u \in F$ one has

$$\|Ju\|^2_{\mathcal{H}'} \ge (1 - \delta_1)\|u\|^2_{\mathcal{H}} - \delta_1 T[u,u] \ge \big(1 - \delta_1(\lambda_n + 1 + \varepsilon)\big)\|u\|^2_{\mathcal{H}}.$$

The first factor on the right-hand side is strictly positive by (1.2.2), and it follows that $J : F \to J(F)$ is injective. In particular, $\dim J(F) = n$. Therefore, for $u \in F \setminus \{0\}$ one has $Ju \ne 0$ and

$$\begin{aligned}\frac{T'[Ju,Ju]}{\|Ju\|^2_{\mathcal{H}'}} &\le \frac{T[u,u] + \delta_2\big(T[u,u] + \|u\|^2_{\mathcal{H}}\big)}{\|Ju\|^2_{\mathcal{H}'}}\\ &\le \frac{T[u,u] + \delta_2\big(T[u,u] + \|u\|^2_{\mathcal{H}}\big)}{\big(1 - \delta_1(\lambda_n + 1 + \varepsilon)\big)\|u\|^2_{\mathcal{H}}} \le \frac{\lambda_n + \varepsilon + \delta_2(\lambda_n + 1 + \varepsilon)}{1 - \delta_1(\lambda_n + 1 + \varepsilon)}\\ &= \lambda_n + \frac{\lambda_n + \varepsilon + \delta_2(\lambda_n + 1 + \varepsilon) - \lambda_n\big(1 - \delta_1(\lambda_n + 1 + \varepsilon)\big)}{1 - \delta_1(\lambda_n + 1 + \varepsilon)}\\ &= \lambda_n + \frac{\varepsilon + (\delta_1\lambda_n + \delta_2)(\lambda_n + 1 + \varepsilon)}{1 - \delta_1(\lambda_n + 1 + \varepsilon)}.\end{aligned}$$

Due to the definition of $\Lambda_n(T')$ one has

$$\begin{aligned}\Lambda_n(T') &\le \sup_{v \in J(F)\setminus\{0\}} \frac{T'[v,v]}{\|v\|^2_{\mathcal{H}'}} = \sup_{u \in F\setminus\{0\}} \frac{T'[Ju,Ju]}{\|Ju\|^2_{\mathcal{H}'}}\\ &\le \lambda_n + \frac{\varepsilon + (\delta_1\lambda_n + \delta_2)(\lambda_n + 1 + \varepsilon)}{1 - \delta_1(\lambda_n + 1 + \varepsilon)},\end{aligned}$$

and the claim follows by sending ε to zero. □

1.3 Reduction to a problem in a moving half-plane

We first apply some truncations in order to obtain a model problem which only takes into account the cusp and neglects the rest of Γ. For $\varepsilon > 0$ we denote

$$\Gamma_\varepsilon := \big\{(x_1, x_2) : x_1 \in (0, \varepsilon), |x_2| = x_1^p\big\}$$

and consider the half-plane

$$\Omega_\varepsilon := (-\infty, \varepsilon) \times \mathbb{R}.$$

One clearly has $\Gamma_\varepsilon \subset \Omega_\varepsilon$, and by $H_{\alpha,\varepsilon}$ we denote the self-adjoint operator in $L^2(\Omega_\varepsilon)$ given by

$$H_{\alpha,\varepsilon} = \iint_{\Omega_\varepsilon} |\nabla u|^2 \mathrm{d}x - \alpha \int_{\Gamma_\varepsilon} u^2 \mathrm{d}s, \quad \mathcal{Q}(H_{\alpha,\varepsilon}) = H_0^1(\Omega_\varepsilon)$$

where H_0^1 is the standard Sobolev space. We start with the following result, taking ε_0 from (1.1.2):

Lemma 1.3.1. *Let $\varepsilon \in (0, \varepsilon_0)$ and $n \in \mathbb{N}$. Assume that*

$$\text{for some } c > \tfrac{1}{4} \text{ there holds } \Lambda_n(H_{\alpha,\varepsilon}) \le -c\alpha^2 \text{ for large } \alpha > 0, \tag{1.3.1}$$

then $\Lambda_n(H_\alpha) = \Lambda_n(H_{\alpha,\varepsilon}) + \mathcal{O}(1)$ for $\alpha \to +\infty$.

Proof. The proof will be in two steps. We first reduce the problem to a bounded neighborhood of the origin, and then to the half-plane Ω_ε, as the latter is easier to analyze.
For $\varepsilon > 0$ denote $\square_\varepsilon := (-\varepsilon, \varepsilon)^2$, then the assumption (1.1.2) rewrites as

$$\text{there exists } \varepsilon_0 > 0 \text{ such that } \Gamma \cap \square_{\varepsilon_0} = \Gamma_{\varepsilon_0},$$

and then for any $\varepsilon \in (0, \varepsilon_0)$ one has $\Gamma \cap \square_\varepsilon = \Gamma_\varepsilon$ as well; we remark that we can take $\varepsilon_0 \le 1$ in condition (1.1.2).
From now on let us pick some $\varepsilon \in (0, \varepsilon_0)$ and let $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^2)$ such that $\chi_1^2 + \chi_2^2 = 1$ and

$$\chi_1 = 1 \text{ in } \square_{\frac{\varepsilon}{2}}, \quad \chi_1 = 0 \text{ outside } \square_\varepsilon.$$

An easy computation shows that for any $u \in \mathcal{Q}(H_\alpha) \equiv H^1(\mathbb{R}^2)$ one has

$$\begin{aligned} H_\alpha[u,u] &= H_\alpha[\chi_1 u, \chi_1 u] + H_\alpha[\chi_2 u, \chi_2 u] - \int_{\mathbb{R}^2} \left(|\nabla \chi_1|^2 + |\nabla \chi_2|^2\right) u^2 \mathrm{d}x \\ &\ge H_\alpha[\chi_1 u, \chi_1 u] + H_\alpha[\chi_2 u, \chi_2 u] - C\|u\|^2_{L^2(\mathbb{R}^2)}, \end{aligned} \tag{1.3.2}$$

where $C = \big\||\nabla \chi_1|^2 + |\nabla \chi_2|^2\big\|_\infty$.
Denote by $D_{\alpha,\varepsilon}$ the self-adjoint operator in $L^2(\square_\varepsilon)$ given by

$$D_{\alpha,\varepsilon}[u,u] = \iint_{\square_\varepsilon} |\nabla u|^2 \mathrm{d}x - \alpha \int_{\Gamma_\varepsilon} u^2 \mathrm{d}s, \quad \mathcal{Q}(D_{\alpha,\varepsilon}) = H_0^1(\square_\varepsilon).$$

Due to $\mathrm{supp}\chi_1 \subset \square_\varepsilon$ we have

$$\chi_1 u \in \mathcal{Q}(D_{\alpha,\varepsilon}), \quad H_\alpha[\chi_1 u, \chi_1 u] = D_{\alpha,\varepsilon}[\chi_1 u, \chi_1 u].$$

On the other hand, by the initial assumption of Γ (C^4-smoothness except at the origin) one can find a C^4-smooth Jordan curve Γ' which coincides with Γ outside $\square_{\frac{\varepsilon}{2}}$. Denote by H'_α the self-adjoint operator in $L^2(\mathbb{R}^2)$ given by

$$H'_\alpha[u,u] = \iint_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x - \alpha \int_{\Gamma'} u^2\,\mathrm{d}s, \quad \mathcal{Q}(H'_\alpha) = H^1(\mathbb{R}^2).$$

As $\operatorname{supp}\chi_2 \cap \square_{\frac{\varepsilon}{2}} = \emptyset$, one has $H_\alpha[\chi_2 u, \chi_2 u] = H'_\alpha[\chi_2 u, \chi_2 u]$, and the inequality (1.3.2) takes the form

$$H_\alpha[u,u] + C\|u\|^2_{L^2(\mathbb{R}^2)} \geq D_{\alpha,\varepsilon}[\chi_1 u, \chi_1 u] + H'_\alpha[\chi_2 u, \chi_2 u]. \tag{1.3.3}$$

Noting that $J : L^2(\mathbb{R}^2) \ni u \mapsto (\chi_1 u, \chi_2 u) \in L^2(\square_\varepsilon) \oplus L^2(\mathbb{R}^2)$ is isometric and that (1.3.3) can be rewritten as

$$H_\alpha[u,u] + C\|u\|^2_{L^2(\mathbb{R}^2)} \geq (D_{\alpha,\varepsilon} \oplus H'_\alpha)[Ju, Ju],$$

we conclude by the min-max principle (Proposition 1.2.1) that

$$\Lambda_n(H_\alpha) \geq \Lambda_n(D_{\alpha,\varepsilon} \oplus H'_\alpha) - C \text{ for all } n \in \mathbb{N},\ \alpha > 0.$$

As discussed in the introduction, see e.g. Eq. (1.1.1), due to the smoothness of Γ', for some $C_0 > 0$ one has $H'_\alpha \geq -\frac{1}{4}\,\alpha^2 - C_0$ for large $\alpha > 0$. Hence, if

$$\text{for some } c > \tfrac{1}{4} \text{ there holds } \Lambda_n(D_{\alpha,\varepsilon}) \leq -c\alpha^2 \text{ for large } \alpha > 0, \tag{1.3.4}$$

then $\Lambda_n(D_{\alpha,\varepsilon} \oplus H'_\alpha) = \Lambda_n(D_{\alpha,\varepsilon})$, and then $\Lambda_n(H_\alpha) \geq \Lambda_n(D_{\alpha,\varepsilon}) - C$ for large $\alpha > 0$. On the other hand, by the min-max principle one directly has $\Lambda_n(H_\alpha) \leq \Lambda_n(D_{\alpha,\varepsilon})$. Therefore, the assumption (1.3.4) implies

$$\Lambda_n(H_\alpha) = \Lambda_n(D_{\alpha,\varepsilon}) + \mathcal{O}(1) \text{ for } \alpha \to +\infty. \tag{1.3.5}$$

Now we need to pass from $D_{\alpha,\varepsilon}$ to $H_{\alpha,\varepsilon}$, which is done in a very similar way. First, by the min-max principle we have

$$\Lambda_n(H_{\alpha,\varepsilon}) \leq \Lambda_n(D_{\alpha,\varepsilon}) \tag{1.3.6}$$

for any $\alpha > 0$. Furthermore, let us pick $\xi_1, \xi_2 \in C^\infty(\mathbb{R}^2)$ such that $\xi_1^2 + \xi_2^2 = 1$ and

$$\xi_1(x) = 1 \text{ for } x \in (0,+\infty) \times (-\varepsilon^p, \varepsilon^p),$$
$$\xi_1(x) = 0 \text{ for } x \notin (-\varepsilon,+\infty) \times (-\varepsilon, \varepsilon).$$

For any $u \in \mathcal{Q}(H_{\alpha,\varepsilon})$ we have then, with $W(x) := |\nabla\xi_1|^2 + |\nabla\xi_2|^2 \leq C'$,

$$\begin{aligned} H_{\alpha,\varepsilon}[u,u] &= H_{\alpha,\varepsilon}[\xi_1 u, \xi_1 u] + H_{\alpha,\varepsilon}[\xi_2 u, \xi_2 u] - \iint_{\Omega_\varepsilon} W\,u^2\mathrm{d}x \\ &\equiv D_{\alpha,\varepsilon}[\xi_1 u, \xi_1 u] + \iint_{\Omega_\varepsilon} \big|\nabla(\xi_2 u)\big|^2\mathrm{d}x - \iint_{\Omega_\varepsilon} W\,u^2\mathrm{d}x \\ &\geq D_{\alpha,\varepsilon}[\xi_1 u, \xi_1 u] - C'\|u\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

As in the first part of the proof, this implies

$$\Lambda_n(H_{\alpha,\varepsilon}) \geq \Lambda_n(D_{\alpha,\varepsilon} \oplus \mathbb{O}) - C' \tag{1.3.7}$$

with $\mathbb{O}$ being the zero operator in $L^2(\Omega_\varepsilon)$. Let (1.3.1) hold, then by (1.3.7) we also have $\Lambda_n(D_{\alpha,\varepsilon} \oplus \mathbb{O}) \leq -c\alpha^2$ for large α. Then $\Lambda_n(D_{\alpha,\varepsilon} \oplus \mathbb{O}) = \Lambda_n(D_{\alpha,\varepsilon})$, and (1.3.4) holds, which implies the estimate (1.3.5). At the same time, Eq. (1.3.7) reads now as $\Lambda_n(H_{\alpha,\varepsilon}) \geq \Lambda_n(D_{\alpha,\varepsilon}) - C'$, and together with (1.3.6) we arrive at $\Lambda_n(D_{\alpha,\varepsilon}) = \Lambda_n(H_{\alpha,\varepsilon}) + \mathcal{O}(1)$ for large α. Substituting this estimate into (1.3.5) we prove the claim. $\square$

Let us apply an additional scaling in order to pass to the semiclassical framework. For $h > 0$ and $b > 0$ consider the self-adjoint operator $F_{h,b}$ in $L^2(\Omega_b)$ defined for $\mathcal{Q}(F_{h,b}) = H^1_0(\Omega_b)$ by

$$F_{h,b}[u,u] = \iint_{\Omega_b} \big(h^2(\partial_1 u)^2 + (\partial_2 u)^2\big)\mathrm{d}x \\ - \int_0^b \sqrt{1 + p^2 h^2 s^{2(p-1)}}\big(u(s,s^p)^2 + u(s,-s^p)^2\big)\mathrm{d}s.$$

Lemma 1.3.2. *For any $\varepsilon > 0$ and $\alpha > 0$ and $n \in \mathbb{N}$ one has*

$$\Lambda_n(H_{\alpha,\varepsilon}) = \alpha^2 \Lambda_n(F_{h,b}) \text{ for } h = \alpha^{\frac{1-p}{p}}, \quad b = \varepsilon\alpha^{\frac{1}{p}} \equiv \varepsilon h^{\frac{1}{1-p}}.$$

Proof. We prefer to give a detailed explicit computation. Consider the unitary operator $\Theta : L^2(\Omega_b) \to L^2(\Omega_\varepsilon)$ given by

$$(\Theta u)(x_1,x_2) = \alpha^{\frac{1}{2}(\frac{1}{p}+1)}\, u(\alpha^{\frac{1}{p}}x_1, \alpha x_2),$$

then $\Theta\mathcal{Q}(F_{h,b}) = \mathcal{Q}(H_{\alpha,\varepsilon})$. By writing the one-dimensional Hausdorff measure on Γ_ε in an explicit form, for any $u \in \mathcal{Q}(H_{\alpha,\varepsilon})$ we have

$$H_{\alpha,\varepsilon}[u,u] = \iint_{\Omega_\varepsilon} \big[(\partial_1 u)^2 + (\partial_2 u)^2\big]\mathrm{d}x \\ - \alpha \int_0^\varepsilon \sqrt{1 + p^2 s^{2p-2}}\big(u(s,s^p)^2 + u(s,-s^p)^2\big)\mathrm{d}s.$$

Then for any $v \in \mathcal{Q}(F_{h,b})$ one obtains

$$\begin{aligned} H_{\alpha,\varepsilon}[\Theta v, \Theta v] = \alpha^{\frac{1}{p}+1} \iint_{\Omega_\varepsilon} \Big[&\alpha^{\frac{2}{p}} \partial_1 v(\alpha^{\frac{1}{p}}x_1, \alpha x_2)^2 \\ &+ \alpha^2 \partial_2 v(\alpha^{\frac{1}{p}}x_1, \alpha x_2)^2\Big]\mathrm{d}x_1\mathrm{d}x_2 \\ - \alpha^{\frac{1}{p}+2} \int_0^\varepsilon &\sqrt{1 + p^2 s^{2(p-1)}}\big(v(\alpha^{\frac{1}{p}}s, \alpha s^p)^2 \\ &+ v(\alpha^{\frac{1}{p}}s, -\alpha s^p)^2\big)\mathrm{d}s. \end{aligned}$$

Using the new variables $y_1 = \alpha^{\frac{1}{p}}x_1$, $x_2 = \alpha y_2$, $t = \alpha^{\frac{1}{p}}s$ we rewrite it as

$$\begin{aligned} H_{\alpha,\varepsilon}[\Theta v, \Theta v] &= \iint_{\Omega_{\varepsilon\alpha^{\frac{1}{p}}}} \Big[\alpha^{\frac{2}{p}} \partial_1 v(y_1,y_2)^2 + \alpha^2 \partial_2 v(y_1,y_2)^2\Big]\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad - \alpha^2 \int_0^{\varepsilon\alpha^{\frac{1}{p}}} \sqrt{1 + p^2 \alpha^{\frac{2-2p}{p}} s^{2p-2}}\big(v(t,t^p) + v(t,-t^p)\big)\mathrm{d}t \\ &= \alpha^2 F_{h,b}[v,v], \end{aligned}$$

which shows that $H_{\alpha,\varepsilon}$ is unitarily equivalent to $\alpha^2 F_{h,b}$. □

By combining Lemma 1.3.1 with Lemma 1.3.2 we arrive at the following reformulation:

Lemma 1.3.3. *Let $\varepsilon > 0$, $h_0 > 0$, $n \in \mathbb{N}$ be such that*

$$\Lambda_n(F_{h,\varepsilon h^{\frac{1}{1-p}}}) \le -c \text{ for all } h \in (0,h_0) \text{ and some } c > \tfrac{1}{4}. \tag{1.3.8}$$

Then $\Lambda_n(H_\alpha) = \alpha^2 \Lambda_n(F_{h,\varepsilon h^{\frac{1}{1-p}}}) + \mathcal{O}(1)$ for $h := \alpha^{\frac{1-p}{p}}$ and $\alpha \to +\infty$.

1.4 Upper bound

1.4.1 Reduction to a one-dimensional effective operator

For some $k > 0$, to be chosen later, denote

$$\Omega'_h := (0, h^k) \times \mathbb{R}$$

and denote by G_h the self-adjoint operator in $L^2(\Omega'_h)$ given by

$$G_h[u, u] = \iint_{\Omega'_h} \big(h^2(\partial_1 u)^2 + (\partial_2 u)^2\big)\mathrm{d}x - \int_0^{h^k} \big(u(s, s^p)^2 + u(s, -s^p)^2\big)\mathrm{d}s$$

and $\mathcal{Q}(G_h) = H_0^1(\Omega'_h)$. For sufficiently small $h > 0$ one has the inclusion $\Omega'_h \subset \Omega_{\varepsilon h^{\frac{1}{1-p}}}$, and for $u \in H_0^1(\Omega'_h)$ we denote u_0 its extension by zero to $\Omega_{\varepsilon h^{\frac{1}{1-p}}}$, then $F_{h,b}[u_0, u_0] \le G_h[u, u]$. It follows by the min-max principle that:

Lemma 1.4.1. *For any $\varepsilon > 0$ there exists $h_0 > 0$ such that for $h \in (0, h_0)$ and $n \in \mathbb{N}$ there holds $\Lambda_n(F_{h,\varepsilon h^{\frac{1}{1-p}}}) \le \Lambda_n(G_h)$.*

In order to study G_h we will use some facts on a simple one-dimensional operator T_x, $x > 0$, which is the self-adjoint operator in $L^2(\mathbb{R})$ given by

$$T_x[f, f] = \int_{\mathbb{R}} f'(y)^2 \mathrm{d}y - \big(f(x)^2 + f(-x)^2\big), \quad \mathcal{Q}(T_x) = H^1(\mathbb{R}). \tag{1.4.1}$$

We recall some simple properties of T_x established in [20, Proposition 2.3]. The bottom of the spectrum of T_x is a simple isolated eigenvalue, which we denote by $\sigma(x)$ due to its special role in what follows,

$$\sigma(x) := \Lambda_1(T_x), \quad x > 0,$$

and we denote by Ψ_x the respective eigenfunction chosen L^2-normalized and positive. We will use the following properties of their dependence on $x > 0$:

Proposition 1.4.2. *The following holds:*

(a) $-1 < \sigma(x) < -\frac{1}{4}$ *for all* $x \in (0, +\infty)$,

(b) σ *is non-decreasing,*

(c) $\sigma(x) = -1 + 2x + \mathcal{O}(x^2)$ *for* $x \to 0^+$,

(d) the function $x \mapsto \|\partial_x \Psi_x\|_{L^2(\mathbb{R})}$ *is bounded on* $(0, +\infty)$,

(e) for $x < 1$ *one has* $\Lambda_2(T_x) = 0$.

The above properties allow one to give an upper bound for the Rayleigh quotients of G_h by those of a one-dimensional operator on $(0, h^k)$. Namely, denote by K_h the self-adjoint operator in $L^2(0, h^k)$ given by

$$K_h[f, f] = \int_0^{h^k} \big(h^2 f'(x)^2 + 2x^p f(x)^2\big)\mathrm{d}x, \quad \mathcal{Q}(K_h) = H_0^1(0, h^k). \tag{1.4.2}$$

Lemma 1.4.3. *There exists $a_0 > 0$ such that*

$$\Lambda_n(G_h) \leq -1 + \Lambda_n(K_h) + a_0(h^{2+2k(p-1)} + h^{2kp}) \text{ for all } h > 0 \text{ and } n \in \mathbb{N}.$$

Proof. If $f \in H_0^1(0, h^k)$, then for the function $u \in H_0^1(\Omega'_h)$ defined by

$$u(x_1, x_2) = f(x_1)\Psi_{x_1^p}(x_2)$$

we have $\|f\|_{L^2(0,h^k)} = \|u\|_{L^2(\Omega'_h)}$ and

$$\iint_{\Omega'_h} (\partial_2 u)^2 \mathrm{d}x - \int_0^{h^k} \big(u(s, s^p)^2 + u(s, -s^p)^2\big)\mathrm{d}s = \int_0^{h_k} \sigma(x_1^p) f(x_1)^2 \mathrm{d}x_1.$$

The L^2-normalization of $\Psi_{x_1^p}$ implies

$$\int_{\mathbb{R}} \Psi_{x_1^p}(x_2)\, \partial_{x_1} \Psi_{x_1^p}(x_2)\mathrm{d}x_2 = \frac{1}{2}\, \partial_{x_1} \|\Psi_{x_1^p}\|^2_{L^2(\mathbb{R})} = 0,$$

hence,

$$\begin{aligned}\iint_{\Omega'_h} (\partial_1 u)^2 \mathrm{d}x &= \int_0^{h^k} \int_{\mathbb{R}} \Big[f'(x_1)^2 \Psi_{x_1^p}(x_2)^2 \\ &\qquad + 2f(x_1)f'(x_1)\Psi_{x_1^p}(x_2)\partial_{x_1}\Psi_{x_1^p}(x_2) \\ &\qquad + f(x_1)^2(\partial_{x_1}\Psi_{x_1^p}(x_2))^2\Big]\mathrm{d}x_2\, \mathrm{d}x_1 \\ &= \int_0^{h^k} \big(f'(x_1)^2 + w(x_1)f(x_1)^2\big)\mathrm{d}x_1,\end{aligned}$$

where we denote $w(x_1) := \big\|\partial_{x_1}\Psi_{x_1^p}\big\|^2_{L^2(\mathbb{R})} \equiv p^2 x_1^{2(p-1)}\big\|(\partial_z \Psi_z)_{z=x_1^p}\big\|^2_{L^2(\mathbb{R})}$, and

$$G_h[u, u] = \int_0^{h^k} \Big(h^2 f'(x_1)^2 + \big[\sigma(x_1^p) + h^2 w(x_1)\big]\, f(x_1)^2\Big)\mathrm{d}x_1$$

Due to Proposition 1.4.2(c,d) for a sufficiently large $a_0 > 0$ one can estimate

$$p^2\big\|(\partial_z \Psi_z)_{z=x_1^p}\big\|^2_{L^2(\mathbb{R})} \leq a_0, \quad \sigma(x_1^p) \leq -1 + 2x_1^p + a_0 h^{2kp}, \quad x_1 \in (0, h^k),$$

and then

$$\begin{aligned}G_h[u, u] \leq -\|f\|^2_{L^2(0,h^k)} + \int_0^{h^k} \big(h^2 f'(x_1)^2 + 2x_1^p f(x_1)^2\big)\mathrm{d}x_1 \\ + a_0(h^{2+2k(p-1)} + h^{2kp})\|f\|^2_{L^2(0,h^k)}.\end{aligned}$$

Therefore, the linear operator $J : \mathcal{Q}(K_h) \ni f \mapsto u \in \mathcal{Q}(G_h)$ satisfies, for all $f \in \mathcal{Q}(K_h)$, the equality $\|Jf\|_{L^2(\Omega'_h)} = \|f\|_{L^2(0,h^k)}$ and the inequality

$$G_h[Jf, Jf] \leq -\|f\|^2_{L^2(0,h^k)} + K_h[f, f] + a_0(h^{2+2k(p-1)} + h^{2kp})\|f\|^2_{L^2(0,h^k)},$$

which implies the claim by the min-max principle. □

1.4.2 Analysis of the effective operator

Now we are reduced to the study of the eigenvalues of K_h for small $h > 0$. We will show that the principal term of their asymptotics is determined by the eigenvalues of the model operator A.

For $\mu > 0$, we introduce first two auxiliary operators $C^\mu_{N/D}$, which are the self-adjoint operators in $L^2(0,\mu)$ given by

$$C^\mu_{N/D}[f,f] = \int_0^\mu \big(f'(x)^2 + x^p f(x)^2\big)\mathrm{d}x,$$
$$\mathcal{Q}(C^\mu_N) = \big\{f \in H^1(0,\mu) : f(0) = 0\big\}, \quad \mathcal{Q}(C^\mu_D) = H^1_0(0,\mu).$$

An elementary scaling argument gives the following result:

Lemma 1.4.4. *For any $n \in \mathbb{N}$ and $h > 0$ one has*

$$\Lambda_n(K_h) = 2^{\frac{2}{2+p}} h^{\frac{2p}{2+p}} \Lambda_n(C^\mu_D), \quad \mu := 2^{\frac{1}{2+p}} h^{k-\frac{2}{2+p}}.$$

Remark that if $k < \frac{2}{2+p}$ then in the above representation one has $\mu \to +\infty$ as $h \to 0^+$. Let us now study the behavior of the eigenvalues of $C^\mu_{N/D}$ for large $\mu > 0$.

Lemma 1.4.5. *Let $n \in \mathbb{N}$ be fixed, then $\Lambda_n(C^\mu_{N/D}) = \Lambda_n(A) + \mathcal{O}(\mu^{-2})$ for $\mu \to +\infty$.*

Proof. Directly by the min-max principle, for any $\mu > 0$ one has the inequality

$$\Lambda_n(A) \le \Lambda_n(C^\mu_D). \tag{1.4.3}$$

Furthermore, consider the self-adjoint operator D_μ in $L^2(\mu,+\infty)$ given by

$$D_\mu[f,f] = \int_\mu^\infty \big(f'(x)^2 + x^p f(x)^2\big)\mathrm{d}x,$$
$$\mathcal{Q}(D_\mu) = \big\{f \in H^1(\mu,+\infty) : x^{\frac{p}{2}} f \in L^2(\mu,+\infty)\big\},$$

then one clearly has $\Lambda_n(A) \ge \Lambda_n(C^\mu_N \oplus D_\mu)$ for any $\mu > 0$. The left-hand side is independent of μ, while $D_\mu \ge \mu^p \to +\infty$ as $\mu \to +\infty$. Therefore, there exists $\mu_n > 0$ such that

$$\Lambda_n(A) \ge \Lambda_n(C^\mu_N) \text{ for } \mu \ge \mu_n. \tag{1.4.4}$$

Now let $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ such that

$$\chi_1^2 + \chi_2^2 = 1, \quad \chi_1(t) = 1 \text{ for } t \le \tfrac{1}{2}, \quad \chi_1(t) = 0 \text{ for } t \ge \tfrac{3}{4},$$

and denote $\chi_{j,\mu} := \chi_j(\cdot/\mu)$. Consider the self-adjoint operator D'_μ in $L^2(\frac{\mu}{2},\mu)$ given by

$$D'_\mu[f,f] = \int_{\frac{\mu}{2}}^\mu \big(f'(x)^2 + x^p f(x)^2\big)\,\mathrm{d}x, \quad \mathcal{Q}(D'_\mu) = H^1\big(\tfrac{\mu}{2},\mu\big).$$

Then a direct computation shows that for any $f \in \mathcal{Q}(C^\mu_N)$ one has, with $K := \big\|(\chi_1')^2 + (\chi_2')^2\big\|_\infty$,

$$C^\mu_N[f,f] = C^\mu_N[\chi_{1,\mu}f, \chi_{1,\mu}f] + C^\mu_N[\chi_{2,\mu}f, \chi_{2,\mu}f]$$

$$
\begin{aligned}
&\qquad - \int_0^\mu \big((\chi'_{1,\mu})^2 + (\chi'_{2,\mu})^2\big) f^2 \mathrm{d}x \\
&\qquad \geq C_N^\mu[\chi_{1,\mu} f, \chi_{1,\mu} f] + C_N^\mu[\chi_{2,\mu} f, \chi_{2,\mu} f] - K\mu^{-2}\|f\|^2_{L^2(0,\mu)} \\
&\qquad = C_D^\mu[\chi_{1,\mu} f, \chi_{1,\mu} f] + D'_\mu[\chi_{2,\mu} f, \chi_{2,\mu} f] - K\mu^{-2}\|f\|^2_{L^2(0,\mu)}, \\
&\qquad = (C_D^\mu \oplus D'_\mu)[Jf, Jf] - K\mu^{-2}\|f\|^2_{L^2(0,\mu)}, \\
&Jf := (\chi_{1,\mu} f, \chi_{2,\mu} f),
\end{aligned}
$$

which implies $\Lambda_n(C_N^\mu) \geq \Lambda_n(C_D^\mu \oplus D'_\mu) - K\mu^{-2}$ for any $\mu > 0$. By (1.4.4), for $\mu \to +\infty$ the left-hand side of the last inequality remains bounded, while $D'_\mu \geq \mu^p 2^{-p} \to +\infty$. Therefore, the value of μ_n in (1.4.4) can be assumed such that, in addition,

$$
\Lambda_n(C_N^\mu) \geq \Lambda_n(C_D^\mu) - K\mu^{-2} \text{ for any } \mu \geq \mu_n. \tag{1.4.5}
$$

By putting together the above estimates, for $\mu \geq \mu_n$ we obtain

$$
\Lambda_n(C_D^\mu) - K/\mu^2 \overset{(1.4.5)}{\leq} \Lambda_n(C_N^\mu) \overset{(1.4.4)}{\leq} \Lambda_n(A) \overset{(1.4.3)}{\leq} \Lambda_n(C_D^\mu),
$$

which implies first $\Lambda_n(C_D^\mu) = \Lambda_n(A) + \mathcal{O}(\mu^{-2})$ and then $\Lambda_n(C_N^\mu) = \Lambda_n(C_D^\mu) + \mathcal{O}(\mu^{-2}) = \Lambda_n(A) + \mathcal{O}(\mu^{-2})$. □

By combining Lemma 1.4.4 with Lemma 1.4.5 we arrive at

Lemma 1.4.6. *For any $n \in \mathbb{N}$ and $k \in (0, \frac{2}{2+p})$ there holds*

$$
\Lambda_n(K_h) = 2^{\frac{2}{2+p}} h^{\frac{2p}{2+p}} \Lambda_n(A) + \mathcal{O}(h^{2-2k}) \text{ as } h \to 0^+.
$$

1.4.3 Proof of the upper eigenvalue bound

The substitution of the asymptotics of Lemma 1.4.6 (passage from K_h to A) into Lemma 1.4.3 (passage from G_h to K_h) shows that for every fixed $n \in \mathbb{N}$ and $k \in (0, \frac{2}{2+p})$ there holds

$$
\Lambda_n(G_h) \leq -1 + 2^{\frac{2}{2+p}} h^{\frac{2p}{2+p}} \Lambda_n(A) + \mathcal{O}(h^{2+2k(p-1)} + h^{2kp} + h^{2-2k})
$$

as $h \to 0^+$. For $k \in (0, \frac{2}{2+p})$ one has

$$
\begin{gathered}
2 + 2k(p-1) = 2kp + 2(1-k) \geq 2kp, \\
\mathcal{O}(h^{2+2k(p-1)} + h^{2kp} + h^{2-2k}) = \mathcal{O}(h^{2kp} + h^{2-2k}).
\end{gathered}
$$

Taking $k := \frac{1}{1+p} \in (0, \frac{2}{2+p})$ and then applying Lemma 1.4.1 we see that for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there holds, as $h \to 0^+$,

$$
\Lambda_n(F_{h,\varepsilon h^{\frac{1}{1-p}}}) \leq \Lambda_n(G_h) \leq -1 + 2^{\frac{2}{2+p}} h^{\frac{2p}{2+p}} \Lambda_n(A) + \mathcal{O}(h^{\frac{2p}{1+p}}) < -\tfrac{1}{2}. \tag{1.4.6}
$$

It follows that the assumption (1.3.8) is satisfied *for any* $\varepsilon > 0$ and $n \in \mathbb{N}$, which gives a stronger version of Lemma 1.3.3:

Lemma 1.4.7. *For any $n \in \mathbb{N}$ and $\varepsilon > 0$ there holds*

$$
\Lambda_n(H_\alpha) = \alpha^2 \Lambda_n(F_{h,\varepsilon h^{\frac{1}{1-p}}}) + \mathcal{O}(1) \text{ for } h := \alpha^{\frac{1-p}{p}} \text{ and } \alpha \to +\infty. \tag{1.4.7}
$$

Applying again (1.4.6) to the right-hand side of (1.4.7) one arrives at

$$\begin{aligned}\Lambda_n(H_\alpha) &\le -\alpha^2 + 2^{\frac{2}{2+p}}\Lambda_n(A)\alpha^{\frac{6}{2+p}} + \mathcal{O}(\alpha^{\frac{4}{1+p}})\\ &\equiv -\alpha^2 + 2^{\frac{2}{2+p}}\Lambda_n(A)\alpha^{\frac{6}{2+p}} + \mathcal{O}(\alpha^{\frac{6}{2+p}-\eta}), \quad \alpha\to+\infty.\end{aligned}$$

where $\eta := \frac{6}{2+p} - \frac{4}{1+p} = \frac{2(p-1)}{(p+1)(p+2)} > 0$. As the upper bound obtained for $\Lambda_n(H_\alpha)$ is strictly negative for large α, it lies below the essential spectrum of H_α which is $[0,+\infty)$ as we emphasize it in the introduction, and it follows by the min-max principle that $\Lambda_n(H_\alpha)$ is the nth eigenvalue of H_α.

1.5 Lower bound

1.5.1 Reduction to a smaller half-plane

Now we need to obtain a lower bound for the eigenvalues of $F_{h,\varepsilon h^{\frac{1}{1-p}}}$ with a suitably chosen $\varepsilon > 0$. Recall that

$$\begin{aligned}F_{h,\varepsilon h^{\frac{1}{1-p}}}[u,u] = \iint_{\Omega_{\varepsilon h^{\frac{1}{1-p}}}} &\big(h^2(\partial_1 u)^2 + (\partial_2 u)^2\big)\mathrm{d}x\\ &- \int_0^{\varepsilon h^{\frac{1}{1-p}}} \sqrt{1+p^2h^2s^{2(p-1)}}\big(u(s,s^p)^2 + u(s,-s^p)^2\big)\mathrm{d}s.\end{aligned}$$

Let $k>0$, to be chosen later, and $h>0$ sufficiently small to have $h^k < \varepsilon h^{\frac{1}{1-p}}$. Let R_h be the self-adjoint operator in $L^2(\Omega_{h^k})$ given by

$$\begin{aligned}R_h[u,u] &= \iint_{\Omega_{h^k}} \big(h^2(\partial_1 u)^2 + (\partial_2 u)^2\big)\mathrm{d}x\\ &\quad - \int_0^{h^k} \sqrt{1+p^2h^{2+2k(p-1)}}\big(u(s,s^p)^2 + u(s,-s^p)^2\big)\mathrm{d}s,\\ \mathcal{Q}(R_h) &= H^1(\Omega_{h^k}).\end{aligned}$$

Lemma 1.5.1. *Let $k \in \big(0, \frac{2}{2+p}\big)$. There exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0,\varepsilon_1)$ and any $n\in\mathbb{N}$ there holds*

$$\Lambda_n(F_{h,\varepsilon h^{\frac{1}{1-p}}}) \ge \Lambda_n(R_h) \text{ as } h\to 0^+.$$

For the proof of Lemma 1.5.1 we need an auxiliary one-dimensional operator, which will also play a role on later steps. For $x>0$ and $\beta>0$ we denote by $T_{x,\beta}$ the self-adjoint operator in $L^2(\mathbb{R})$ given by

$$T_{x,\beta}[f,f] = \int_{\mathbb{R}} f'(y)^2\mathrm{d}y - \beta\big(f(x)^2 + f(-x)^2\big), \quad \mathcal{Q}(T_{x,\beta}) = H^1(\mathbb{R}),$$

which is closely related to the operator T_x from (1.4.1) and Proposition 1.4.2: a simple scaling argument shows that $T_{x,\beta}$ is unitarily equivalent to $\beta^2 T_{\beta x}$ and $\Lambda_n(T_{x,\beta}) = \beta^2\Lambda_n(T_{\beta x})$ for any $n\in\mathbb{N}$. In particular,

$$\Lambda_1(T_{x,\beta}) = \beta^2\sigma(\beta x).$$

Proof of Lemma 1.5.1. By considering separately the integrals for $x_1 < h^k$ and $x_1 > h^k$ we arrive at $F_{h,\varepsilon h^{\frac{1}{1-p}}}[u,u] = I_1 + I_2$ with

$$\begin{aligned}
I_1 &= \iint_{\Omega_{h^k}} \big(h^2(\partial_1 u)^2 + (\partial_2 u)^2\big)\mathrm{d}x\\
&\quad - \int_0^{h^k} \sqrt{1+p^2h^2x_1^{2(p-1)}}\,\big(u(s,s^p)^2 + u(s,-s^p)^2\big)\mathrm{d}s,\\
I_2 &= \int_{h^k}^{\varepsilon h^{\frac{1}{1-p}}} \bigg[\int_{\mathbb{R}} \big(h^2(\partial_1 u)^2 + (\partial_2 u)^2\big)\mathrm{d}x_2\\
&\quad - \sqrt{1+p^2h^2x_1^{2(p-1)}}\,\big(u(x_1,x_1^p)^2 + u(x_1,-x_1^p)^2\big)\bigg]\mathrm{d}x_1,
\end{aligned}$$

and one has obviously $I_1 \geq R_h[u_1,u_1]$ with $u_1 := u|_{\Omega_{h^k}}$.

Now one needs a lower bound for I_2. First, by dropping the non-negative term $(\partial_1 u)^2$ and using the above one-dimensional operator operator $T_{x,\beta}$ we estimate

$$I_2 \geq \int_{h^k}^{\varepsilon h^{\frac{1}{1-p}}} \lambda(x_1,h) \int_{\mathbb{R}} u(x_1,x_2)^2 \mathrm{d}x_2 \mathrm{d}x_1,$$

where we denoted

$$\begin{aligned}
\lambda(x_1,h) &:= \Lambda_1\big(T_{x_1^p,\sqrt{1+p^2h^2x_1^{2(p-1)}}}\big)\\
&\equiv \big(1+p^2h^2x_1^{2(p-1)}\big)\sigma\big(\sqrt{1+p^2h^2x_1^{2(p-1)}}\,x_1^p\big).
\end{aligned}$$

To estimate $\lambda(x_1,h)$ from below let us pick $q \in (0,\frac{1}{p-1})$, then for small h one has $h^k < h^{-q} < \varepsilon h^{\frac{1}{1-p}}$.

Consider first the values $x_1 \in (h^k, h^{-q})$. Due to

$$\sqrt{1+p^2h^2x_1^{2(p-1)}}\,x_1^p > x_1^p > h^{kp},$$

by Proposition 1.4.2(a,b) one obtains

$$\sigma(h^{kp}) \leq \sigma\big(\sqrt{1+p^2h^2x_1^{2(p-1)}}\,x_1^p\big) < 0.$$

On the other hand, $1+p^2h^2x_1^{2(p-1)} < 1+p^2h^{2-2q(p-1)}$, which together with the preceding estimate gives

$$\big(1+p^2h^2x_1^{2(p-1)}\big)\sigma\big(\sqrt{1+p^2h^2x_1^{2(p-1)}}\,x_1^p\big) \geq (1+p^2h^{2-2q(p-1)})\sigma(h^{kp}).$$

Using Proposition 1.4.2(c) to estimate $\sigma(h^{kp})$, for small $h>0$ we arrive at

$$\lambda(x_1,h) \geq (1+p^2h^{2-2q(p-1)})\Big(-1+\frac{3}{2}h^{kp}\Big) \geq -1+\tfrac{3}{2}h^{kp} - p^2h^{2-2q(p-1)}.$$

As k and q were rather arbitrary so far, we may assume that

$$kp < 2, \quad 0 < q < \tfrac{2-kp}{2(p-1)} \equiv \tfrac{1-\frac{kp}{2}}{p-1} < \tfrac{1}{p-1},$$

then $kp < 2 - 2q(p-1)$ and $h^{2-2q(p-1)} = o(h^{kp})$. Therefore,

$$\lambda(x_1, h) \geq -1 + h^{kp} \text{ for } x_1 \in (h^k, h^{-q}) \text{ and } h \to 0^+. \tag{1.5.1}$$

Keeping the above value of q consider now $x_1 \in \big(h^{-q}, \varepsilon h^{\frac{1}{1-p}}\big)$. We have first

$$\sqrt{1 + p^2 h^2 x_1^{2(p-1)}}\, x_1^p > x_1^p > h^{-pq}$$

and then, by Proposition 1.4.2(a,b),

$$\sigma(h^{-pq}) \leq \sigma\Big(\sqrt{1 + p^2 h^2 x_1^{2(p-1)}}\, x_1^p\Big) < 0.$$

In addition, $1 + p^2 h^2 x_1^{2(p-1)} \leq 1 + p^2 \varepsilon^{2(p-1)}$, and $\sigma(h^{-pq}) < 0$, therefore,

$$\lambda(x_1, h) \geq (1 + p^2 \varepsilon^{2(p-1)}) \sigma(h^{-pq}).$$

In view of Proposition 1.4.2(b,c), one can choose $\delta > 0$ sufficiently small such that $\sigma(h^{-pq}) \geq -1+2\delta$ for small $h > 0$. In addition, we may take $\varepsilon_1 > 0$ sufficiently small to have $p^2 \varepsilon_1^{2(p-1)} < \delta$, then for any $\varepsilon \in (0, \varepsilon_1)$ one $\lambda(x_1, h) \geq (1+\delta)(-1+2\delta) \geq -1+\delta$ for small h. By combining with (1.5.1) we see that $\lambda(x_1, h) \geq -1 + h^{kp}$ for all $x_1 \in (h^k, \varepsilon h^{\frac{1}{1-p}})$ if h is sufficiently small, and then

$$I_2 \geq (-1 + h^{kp}) \int_{h^k}^{\varepsilon h^{\frac{1}{1-p}}} \int_{\mathbb{R}} u(x_1, x_2)^2 \mathrm{d}x_2 \mathrm{d}x_1.$$

We summarize the above estimates as follows: there exist $\varepsilon \in (0, \varepsilon_1)$ and $h_1 > 0$ such that for all $h \in (0, h_1)$ and $u \in \mathcal{Q}(F_{h, \varepsilon h^{\frac{1}{1-p}}})$ there holds

$$F_{h, \varepsilon h^{\frac{1}{1-p}}} \geq R_n[u_1, u_1] + (-1 + h^{kp}) \|u_2\|^2_{L^2(\Omega_{\varepsilon h^{\frac{1}{1-p}}} \setminus \Omega_{h^k})},$$
$$u_1 := u|_{\Omega_{h^k}}, \quad u_2 := u|_{\Omega_{\varepsilon h^{\frac{1}{1-p}}} \setminus \Omega_{h^k}},$$

and then for any fixed $n \in \mathbb{N}$ and small h one has

$$\Lambda_n(F_{h, \varepsilon h^{\frac{1}{1-p}}}) \geq \min\big\{\Lambda_n(R_h), -1 + h^{kp}\big\}. \tag{1.5.2}$$

The min-max principle shows that $\Lambda_n(R_h) \leq \Lambda_n(G_h)$ for the operator G_h from Subsection 1.4.1, and the estimate (1.4.6) for $\Lambda_n(G_h)$ yields $\Lambda_n(R_h) \leq -1 + \mathcal{O}(h^{\frac{2p}{2+p}})$. For $k \in (0, \frac{2}{2+p})$ one has $h^{\frac{2p}{2+p}} = o(h^{kp})$ and then $\Lambda_n(R_h) < -1 + h^{kp}$. The substitution into (1.5.2) concludes the proof. □

1.5.2 Reduction to a one-dimensional problem

In the present section we will provide a lower bound for the eigenvalues of $\Lambda_n(R_h)$ in terms of a one-dimensional operator. Namely, consider the function

$$V : x \mapsto \begin{cases} 1, & x < 0, \\ 2x^p, & x > 0 \end{cases},$$

and the operator Z_h in $L^2(-\infty, h^k)$ given by $Z_h f = -h^2 f'' + Vf$ with Neumann condition at the right end, $f'(h^k) = 0$, i.e.

$$Z_h[f,f] = h^2 \int_{-\infty}^{h^k} f'(x)^2 \mathrm{d}x + \int_{-\infty}^{0} f(x)^2 \mathrm{d}x + 2\int_0^{h^k} x^p f(x)^2 \mathrm{d}x$$

with $\mathcal{Q}(Z_h) = H^1(-\infty, h^k)$.

Lemma 1.5.2. *For any $n \in \mathbb{N}$, $k \in (0, \frac{2}{2+p})$ and $s > 0$ there holds*

$$\Lambda_n(R_h) \geq -1 + \Lambda_n(Z_{h_0}) + \mathcal{O}(h^{2+2k(p-1)-s} + h^{2kp}), \quad h \to 0^+,$$

where we denote

$$h_0 := h\sqrt{1-h^s}.$$

The proof will occupy the rest of the subsection.

It will be convenient to use the one-dimensional operator

$$L_{x_1,h} := T_{x_1^p, \sqrt{1+p^2h^{2+2k(p-1)}}},$$

its first eigenvalue

$$\begin{aligned}\kappa(x_1,h) &:= \Lambda_1(L_{x_1,h}) \equiv \Lambda_1\big(T_{x_1^p,\sqrt{1+p^2h^{2+2k(p-1)}}}\big)\\ &\equiv \big(1+p^2h^{2+2k(p-1)}\big)\sigma\big(\sqrt{1+p^2h^{2+2k(p-1)}}\, x_1^p\big),\end{aligned}$$

and the associated eigenfunction $\Phi_{x_1,h}$ chosen positive and normalized by $\|\Phi_{x_1,h}\|_{L^2(\mathbb{R})} = 1$. In terms of the first eigenfunction Ψ_x of T_x one has clearly

$$\Phi_{x_1,h}(t) = \sqrt[4]{1+p^2h^{2+2k(p-1)}}\, \Psi_{\sqrt{1+p^2h^{2+2k(p-1)}}\, x_1^p}\big(\sqrt{1+p^2h^{2+2k(p-1)}}\, t\big).$$

Due to Proposition 1.4.2 for any $h > 0$ the function $x_1 \mapsto \Phi_{x_1,h}$ admits a finite limit $\Phi_{0,h}$ at $x_1 = 0^+$, so we define

$$\widehat{\Phi}_{x_1,h} = \begin{cases} \Phi_{x_1,h}, & x_1 > 0, \\ \Phi_{0,h}, & x_1 < 0. \end{cases}$$

Consider the following closed subspace $\mathcal{G}$ of $L^2(\Omega_{h^k})$,

$$\mathcal{G} := \big\{(x_1,x_2) \mapsto f(x_1)\widehat{\Phi}_{x_1,h}(x_2) : f \in L^2(-\infty, h^k)\big\},$$

and denote by Π the orthogonal projector onto $\mathcal{G}$ in $L^2(\Omega_{h^k})$, then the operator $\Pi^\perp := 1 - \Pi$ is the orthogonal projector onto $\mathcal{G}^\perp$. One easily checks that for $u \in L^2(\Omega_{h^k})$ there holds

$$\begin{aligned}(\Pi u)(x_1,x_2) = f(x_1)\widehat{\Phi}_{x_1,h}(x_2) \text{ with } f(x_1) = \int_{\mathbb{R}} \widehat{\Phi}_{x_1,h}(x_2)u(x_1,x_2)\mathrm{d}x_2,\\ \|\Pi u\|^2_{L^2(\Omega_{h^k})} = \|f\|^2_{L^2(-\infty,h^k)},\end{aligned}$$

and that for $u \in \mathcal{Q}(R_h)$ one has $f \in H^1(-\infty, h^k)$. We keep this correspondence between u and f for subsequent computations. Recall that

$$R_h[u,u] = \iint_{\Omega_{h^k}} \big(h^2(\partial_1 u)^2 + (\partial_2 u)^2\big)\mathrm{d}x$$

$$- \int_0^{h^k} \sqrt{1+p^2h^{2+2k(p-1)}}\big(u(s,s^p)^2 + u(s,-s^p)^2\big)\mathrm{d}s.$$

Using the spectral theorem for the above operator $L_{x_1,h}$ we obtain

$$\begin{aligned}
I &:= \iint_{\Omega_{h^k}} (\partial_2 u)^2 \mathrm{d}x \\
&\qquad - \int_0^{h^k} \sqrt{1+p^2h^{2+2k(p-1)}}\big(u(s,s^p)^2 + u(s,-s^p)^2\big)\mathrm{d}s \\
&\geq \iint_{\Omega_{h^k}\cap\{x_1>0\}} (\partial_2 u)^2 \mathrm{d}x \\
&\qquad - \int_0^{h^k} \sqrt{1+p^2h^{2+2k(p-1)}}\big(u(s,s^p)^2 + u(s,-s^p)^2\big)\mathrm{d}s \\
&= \int_0^{h^k} \Big[\int_{\mathbb{R}} \partial_2 u(x_1,x_2)^2 \mathrm{d}x_2 \\
&\qquad - \sqrt{1+p^2h^{2+2k(p-1)}}\,\big(u(x_1,x_1^p)^2 + u(x_1,-x_1^p)^2\big)\Big]\mathrm{d}x_1 \\
&\geq \int_0^{h^k} \Big(\Lambda_1(L_{x_1,h})\|\Pi u(x_1,\cdot)\|^2_{L^2(\mathbb{R})} + \Lambda_2(L_{x_1,h})\|\Pi^\perp u(x_1,\cdot)\|^2_{L^2(\mathbb{R})}\Big)\mathrm{d}x_1.
\end{aligned}$$

Assuming that h is small, by Proposition 1.4.2(e) one obtains, for any $x_1 \in (0,h^k)$,

$$\Lambda_2(L_{x_1,h}) = \big(1+p^2h^{2+2k(p-1)}\big)\Lambda_2\big(T_{\sqrt{1+p^2h^{2+2k(p-1)}}\,x_1^p}\big) = 0,$$

which gives

$$I \geq \int_0^{h^k} \kappa(x_1,h)\|\Pi u(x_1,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}x_1 \equiv \int_0^{h^k} \kappa(x_1,h) f(x_1)^2 \mathrm{d}x_1.$$

Hence, if h is sufficiently small, for any $u \in \mathcal{Q}(R_h)$ we have

$$R_h[u,u] \geq h^2 \iint_{\Omega_{h^k}} (\partial_1 u)^2 \mathrm{d}x + \int_0^{h^k} \kappa(x_1,h) f(x_1)^2 \mathrm{d}x_1. \tag{1.5.3}$$

To obtain a lower bound for the first summand on the right-hand side we start with

$$\begin{aligned}
\Pi\partial_1 u(x_1,x_2) &= \int_{\mathbb{R}} \widehat{\Phi}_{x_1,h}(t)\partial_1 u(x_1,t)\mathrm{d}t\,\widehat{\Phi}_{x_1,h}(x_2), \\
\partial_1 \Pi u(x_1,x_2) &= \frac{\partial}{\partial x_1}\Big(\int_{\mathbb{R}} \widehat{\Phi}_{x_1,h}(t)u(x_1,t)\mathrm{d}t\,\widehat{\Phi}_{x_1,h}(x_2)\Big) \\
&= \int_{\mathbb{R}} \widehat{\Phi}_{x_1,h}(t)\partial_1 u(x_1,t)\mathrm{d}t\,\widehat{\Phi}_{x_1,h}(x_2) \\
&\qquad + \int_{\mathbb{R}} (\partial_{x_1}\widehat{\Phi}_{x_1,h})(t)u(x_1,t)\mathrm{d}t\,\widehat{\Phi}_{x_1,h}(x_2) \\
&\qquad + \int_{\mathbb{R}} \widehat{\Phi}_{x_1,h}(t)u(x_1,t)\mathrm{d}t\,(\partial_{x_1}\widehat{\Phi}_{x_1,h})(x_2).
\end{aligned}$$

Therefore, using $(a+b)^2 \leq 2a^2 + 2b^2$ and Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\big|(\Pi\partial_1-\partial_1\Pi)u(x_1,x_2)\big|^2\\
&\qquad\qquad = \bigg|\int_{\mathbb{R}}(\partial_{x_1}\widehat{\Phi}_{x_1,h})(t)u(x_1,t)\mathrm{d}t\,\widehat{\Phi}_{x_1,h}(x_2)\\
&\qquad\qquad\qquad + \int_{\mathbb{R}}\widehat{\Phi}_{x_1,h}(t)u(x_1,t)\mathrm{d}t\,(\partial_{x_1}\widehat{\Phi}_{x_1,h})(x_2)\bigg|^2\\
&\qquad\qquad \le 2\|\partial_{x_1}\widehat{\Phi}_{x_1,h}\|^2_{L^2(\mathbb{R})}\|u(x_1,\cdot)\|^2_{L^2(\mathbb{R})}\widehat{\Phi}_{x_1,h}(x_2)^2\\
&\qquad\qquad\qquad + 2\|\widehat{\Phi}_{x_1,h}\|^2_{L^2(\mathbb{R})}\|u(x_1,\cdot)\|^2_{L^2(\mathbb{R})}(\partial_{x_1}\widehat{\Phi}_{x_1,h})(x_2)^2.
\end{aligned}
$$

We further recall that $\|\widehat{\Phi}_{x_1,h}\|^2_{L^2(\mathbb{R})}=1$ for all x_1 and that

$$
\partial_{x_1}\widehat{\Phi}_{x_1,h}=\begin{cases}\partial_{x_1}\Phi_{x_1,h}, & x_1>0,\\ 0, & x_1<0.\end{cases}
$$

This gives

$$
\begin{aligned}
&\big\|(\Pi\partial_1-\partial_1\Pi)u\big\|^2_{L^2(\Omega_{h^k})}\\
&\qquad \le 2\int_0^{h^k}\|\partial_{x_1}\Phi_{x_1,h}\|^2_{L^2(\mathbb{R})}\|u(x_1,\cdot)\|^2_{L^2(\mathbb{R})}\Big(\int_{\mathbb{R}}\Phi_{x_1,h}(x_2)^2\mathrm{d}x_2\Big)\mathrm{d}x_1\\
&\qquad\quad + 2\int_0^{h^k}\|u(x_1,\cdot)\|^2_{L^2(\mathbb{R})}\Big(\int_{\mathbb{R}}(\partial_{x_1}\Phi_{x_1,h})(x_2)^2\mathrm{d}x_2\Big)\mathrm{d}x_1\\
&\qquad \le 4\int_0^{h^k} w(x_1,h)\|u(x_1,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}x_1,
\end{aligned}
$$

where we denoted

$$
w(x_1,h):=\|\partial_{x_1}\Phi_{x_1,h}\|^2_{L^2(\mathbb{R})}.
$$

With $\lambda:=\sqrt{1+p^2h^{2+2k(p-1)}}$ we have $\Phi_{x_1,h}(t)=\sqrt{\lambda}\Psi_{\lambda\,x_1^p}(\lambda\,t)$ and

$$
\begin{aligned}
w(x_1,h)&=\lambda^3\int_{\mathbb{R}}p^2x_1^{2(p-1)}(\partial_z\Psi_z)_{z=\lambda\,x_1^p}(\lambda\,t)^2\mathrm{d}t\\
&=\lambda^2p^2x_1^{2(p-1)}\int_{\mathbb{R}}(\partial_z\Psi_z)_{z=\lambda\,x_1^p}(t)^2\mathrm{d}t\\
&\le p^2(1+p^2h^{2+2k(p-1)})x_1^{2(p-1)}\sup_{z>0}\|\partial_z\Psi_z\|^2_{L^2(\mathbb{R})}
\end{aligned}
$$

Due to Proposition 1.4.2(d) the last factor on the right-hand side is finite, and for a suitable $b_0>0$ one obtains $w(x_1,h)\le b_0x_1^{2(p-1)}$, and then

$$
\begin{aligned}
\big\|(\Pi\partial_1-\partial_1\Pi)u\big\|^2_{L^2(\Omega_{h^k})}&\le 4\int_0^{h^k}b_0x_1^{2(p-1)}\|u(x_1,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}x_1\\
&\le 4b_0h^{2k(p-1)}\|u\|^2_{L^2(\Omega_{h^k})}.
\end{aligned}
$$

In addition, the function $(\Pi^\perp\partial_1-\partial_1\Pi^\perp)u\equiv-(\Pi\partial_1-\partial_1\Pi)u$ admits the same norm estimate. Using $(a+b)^2\ge(1-\delta)a^2-\delta^{-1}b^2$ for $a,b\in\mathbb{R}$ and $\delta>0$ we estimate, with any $\delta>0$,

$$
\big\|\partial_1u\big\|^2_{L^2(\Omega_{h^k})}=\big\|\Pi\partial_1u\big\|^2_{L^2(\Omega_{h^k})}+\big\|\Pi^\perp\partial_1u\big\|^2_{L^2(\Omega_{h^k})}
$$

$$\begin{aligned}
&= \big\|\partial_1\Pi u + (\Pi\partial_1 - \partial_1\Pi)u\big\|^2_{L^2(\Omega_{h^k})}\\
&\quad + \big\|\partial_1\Pi^\perp u + (\Pi^\perp\partial_1 - \partial_1\Pi^\perp)\big\|^2_{L^2(\Omega_{h^k})}\\
&\geq (1-\delta)\big\|\partial_1\Pi u\big\|^2_{L^2(\Omega_{h^k})} - \delta^{-1}\,\big\|(\Pi\partial_1 - \partial_1\Pi)u\big\|^2_{L^2(\Omega_{h^k})}\\
&\quad + (1-\delta)\big\|\partial_1\Pi^\perp u\big\|^2_{L^2(\Omega_{h^k})}\\
&\quad - \delta^{-1}\,\big\|(\Pi^\perp\partial_1 - \partial_1\Pi^\perp)u\big\|^2_{L^2(\Omega_{h^k})}\\
&\geq (1-\delta)\big\|\partial_1\Pi u\big\|^2_{L^2(\Omega_{h^k})} - b\delta^{-1}h^{2k(p-1)}\,\|u\|^2_{L^2(\Omega_{h^k})},
\end{aligned}$$

where we took $b := 8b_0$. To estimate the term with $\partial_1\Pi u$ we compute

$$(\partial_1\Pi)u(x_1,x_2) = f'(x_1)\widehat{\Phi}_{x_1,h}(x_2) + f(x_1)\partial_{x_1}\widehat{\Phi}_{x_1,h}(x_2)$$

and remark that due to

$$\int_{\mathbb{R}} \widehat{\Phi}_{x_1,h}(x_2)\partial_{x_1}\widehat{\Phi}_{x_1,h}(x_2)\mathrm{d}x_1 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x_1}\|\widehat{\Phi}_{x_1,h}\|^2_{L^2(\mathbb{R})} = 0$$

we have

$$\begin{aligned}
\|\partial_1\Pi u\|^2_{L^2(\Omega_{h^k})} &= \int_{-\infty}^{h^k} f'(x_1)^2\int_{\mathbb{R}}\Phi_{x_1,h}(x_2)^2\mathrm{d}x_2\,\mathrm{d}x_1\\
&\quad + \int_{-\infty}^{h^k} f(x_1)^2\int_{\mathbb{R}}\big(\partial_{x_1}\widehat{\Phi}_{x_1,h}\big)(x_2)^2\mathrm{d}x_2\,\mathrm{d}x_1\\
&\geq \int_{-\infty}^{h^k} f'(x_1)^2\mathrm{d}x_1.
\end{aligned}$$

Therefore,

$$\big\|\partial_1 u\big\|^2_{L^2(\Omega_{h^k})} \geq (1-\delta)\|f'\|^2_{L^2(-\infty,h^k)} - b\delta^{-1}h^{2k(p-1)}\,\|u\|^2_{L^2(\Omega_{h^k})},$$

and the substitution into (1.5.3) gives

$$\begin{aligned}
&R_h[u,u] + b\delta^{-1}h^{2+2k(p-1)}\,\|u\|^2_{L^2(\Omega_{h^k})}\\
&\qquad\qquad \geq h^2(1-\delta)\int_{-\infty}^{h^k} f'(x_1)^2\mathrm{d}x_1 + \int_0^{h^k}\kappa(x_1,h)f(x_1)^2\mathrm{d}x_1.
\end{aligned}$$

For what follows it is convenient to set $\delta := h^s$ with $s > 0$ to be chosen later, then

$$\begin{aligned}
&R_h[u,u] + bh^{2+2k(p-1)-s}\,\|u\|^2_{L^2(\Omega_{h^k})}\\
&\qquad \geq h^2(1-h^s)\int_{-\infty}^{h^k} f'(x_1)^2\mathrm{d}x_1 + \int_0^{h^k}\kappa(x_1,h)f(x_1)^2\mathrm{d}x_1. \quad (1.5.4)
\end{aligned}$$

In view of Proposition 1.4.2(c) one can find constants $a_0, a > 0$ such that for small h and $x_1 \in (0, h^k)$ there holds

$$\kappa(x_1,h) = \big(1 + p^2h^{2+2k(p-1)}\big)\sigma\big(\sqrt{1+p^2h^{2+2k(p-1)}}\,x_1^p\big)$$

$$\begin{aligned}&\geq \big(1+p^2h^{2+2k(p-1)}\big)\big(-1+2\sqrt{1+p^2h^{2+2k(p-1)}}\,x_1^p\\&\qquad -a_0(1+p^2h^{2+2k(p-1)})\,x_1^{2p}\big)\\&\geq \big(1+p^2h^{2+2k(p-1)}\big)(-1+2x^p-2a_0h^{2kp}\big)\\&\geq -1+2x^p-a(h^{2+2k(p-1)}+h^{2kp}).\end{aligned}$$

Substituting this inequality into (1.5.4) and taking into account the inequality $\|f\|^2_{L^2(0,h^k)} \equiv \|\Pi u\|^2_{L^2(\Omega_{h^k})} \leq \|u\|^2_{L^2(\Omega_{h^k})}$ we obtain, with some constant $B>0$,

$$\begin{aligned}R_h[u,u]&+B(h^{2+2k(p-1)-s}+h^{2+2k(p-1)}+h^{2kp})\,\|u\|^2_{L^2(\Omega_{h^k})}\\&\geq h^2(1-h^s)\int_{-\infty}^{h^k} f'(x_1)^2\mathrm{d}x_1+\int_0^{h^k}(-1+2x_1^p)f(x_1)^2\big)\mathrm{d}x_1.\end{aligned}$$

For $s>0$ we clearly have $h^{2+2k(p-1)}=o(h^{2+2k(p-1)-s})$, hence, with some $B'>B$,

$$\begin{aligned}R_h[u,u]&+B'(h^{2+2k(p-1)-s}+h^{2kp})\,\|u\|^2_{L^2(\Omega_{h^k})}\\&\geq h^2(1-h^s)\int_{-\infty}^{h^k} f'(x_1)^2\mathrm{d}x_1+\int_0^{h^k}(-1+2x_1^p)f(x_1)^2\big)\mathrm{d}x_1\\&\equiv(-\mathbb{1}+Z_{h_0})[f,f].\end{aligned}\tag{1.5.5}$$

Consider now the isometric map

$$J: L^2(\Omega_{h^k})\ni u\mapsto (f,\Pi^\perp u)\in L^2(-\infty,h^k)\oplus\mathcal{G}^\perp,$$

then the estimate (1.5.5) can be rewritten as

$$\big(R_h+B'(h^{2+2k(p-1)-s}+h^{2kp})\big)[u,u]\geq\big((-\mathbb{1}+Z_{h_0})\,\oplus\,0\big)[Ju,Ju].$$

As this holds for all $u\in\mathcal{Q}(R_h)$, the min-max principle shows that for any fixed $n\in\mathbb{N}$ one has, as $h\to0^+$,

$$\begin{aligned}\Lambda(R_h)+B'(h^{2+2k(p-1)-s}+h^{2kp})&\geq\Lambda_n\big((-\mathbb{1}+Z_{h_0})\,\oplus\,0\big)\\&=\min\big\{\Lambda_n(-\mathbb{1}+Z_{h_0}),0\big\}=-1+\min\big\{\Lambda_n(Z_{h_0}),1\big\}.\end{aligned}$$

The min-max principle also shows that for any $n\in\mathbb{N}$ and $h>0$ one has $\Lambda_n(Z_h)\leq\Lambda_n(K_h)$, where the operator K_h was defined in (1.4.2), and it was shown in Lemma 1.4.4 that $\Lambda_n(K_h)=o(1)$ for small h. It follows that $\Lambda_n(Z_{h_0})=o(1)$, and then $\min\big\{\Lambda_n(Z_{h_0}),1\big\}=\Lambda_n(Z_{h_0})$. This gives finally $\Lambda(R_h)\geq-1+\Lambda_n(Z_{h_0})+\mathcal{O}(h^{2+2k(p-1)-s}+h^{2kp})$. This proves Lemma 1.5.2.

1.5.3 One-dimensional analysis

Now we need a more precise analysis of Z_h for small h. We are going to prove the following result, whose proof will occupy the rest of the subsection:

Lemma 1.5.3. *Let $0<k<\frac{2}{p+2}$, then for any $n\in\mathbb{N}$ there holds*

$$E_n(Z_h)=2^{\frac{2}{2+p}}E_n(A)\,h^{\frac{2p}{p+2}}+\mathcal{O}(h^{\frac{5p}{2p+4}}+h^{2-2k})\ \text{as}\ h\to0^+.$$

It appears more convenient to change the scale in order to work with large constants. Namely, for $\lambda > 0$ and $\mu > 0$ we introduce self-adjoint operators $B^{\mu,\lambda}$ in $L^2(-\infty, \mu)$ by

$$B^{\mu,\lambda}[f,f] = \int_{-\infty}^{\mu} f'(x)^2 \mathrm{d}x + \lambda \int_{-\infty}^{0} f(x)^2 \mathrm{d}x + \int_{0}^{\mu} x^p f(x)^2 \mathrm{d}x,$$
$$\mathcal{Q}(B^{\mu,\lambda}) = H^1(-\infty, \mu).$$

An elementary scaling argument gives the following result:

Lemma 1.5.4. *For any $n \in \mathbb{N}$ one has $\Lambda_n(Z_h) = 2^{\frac{2}{2+p}} h^{\frac{2p}{2+p}} \Lambda_n(B^{\lambda,\mu})$ with $\lambda = 2^{\frac{2}{2+p}} h^{-\frac{2p}{2+p}}$ and $\mu = 2^{\frac{1}{2+p}} h^{k-\frac{2}{2+p}}$.*

In view of Lemma 1.5.4 the behavior of the eigenvalues of Z_h for $h \to 0^+$ can be deduced from that of the eigenvalues of $B^{\lambda,\mu}$ for $\lambda \to +\infty$ and $\mu \to +\infty$. The latter will be again approached using the auxiliary operators $C^{\mu}_{N/D}$ already studied in Subsection 1.4.2.

Lemma 1.5.5. *For any $n \in \mathbb{N}$ there exists $\lambda_n > 0$ and $M_n > 0$ such that*

$$\Lambda_j(C^{\mu}_N) - K\lambda^{-\frac{1}{4}} \le \Lambda_j(B^{\lambda,\mu}) \le \Lambda_j(C^{\mu}_D). \tag{1.5.6}$$

for all $(\lambda, \mu) \in (\lambda_n, +\infty) \times (1, +\infty)$.

Proof. Remark first that all operators $B^{\lambda,\mu}$ and $C^{\mu}_{N/D}$ are non-negative. For $\mu > 1$ and $\lambda > 0$ the min-max principle gives

$$0 \le \Lambda_n(B^{\mu,\lambda}) \le \Lambda_n(C^{\mu}_D) \le \Lambda_n(C^1_D), \tag{1.5.7}$$

and it follows, in particular, that the eigenvalue $\Lambda_n(B^{\mu,\lambda})$ is uniformly bounded. It remains to show the first inequality in (1.5.6). As the participating operators act in different spaces, it will be convenient to use Proposition 1.2.2, and we remark that this proof scheme is inspired by the constructions of [71]. Consider the linear map

$$J : \mathcal{Q}(B^{\lambda,\mu}) \to \mathcal{Q}(C^{\mu}_N), \quad (Jf)(x) = f(x) - f(0)e^{-x}, \quad x \in (0, \mu).$$

For any $\varepsilon > 0$ and $a, b \in \mathbb{R}$ one has $(a+b)^2 \ge (1-\varepsilon)a^2 - \varepsilon^{-1}b^2$. Therefore, for any $f \in H^1(-\infty, \mu)$ and $\varepsilon > 0$ one has

$$\begin{aligned}
\|Jf\|^2_{L^2(0,\mu)} &= \int_0^{\mu} \big(f(x) - f(0)e^{-x}\big)^2 \mathrm{d}x \\
&\ge (1-\varepsilon) \int_0^{\mu} f(x)^2 \mathrm{d}x - \varepsilon^{-1} \int_0^{\mu} f(0)^2 e^{-2x} \mathrm{d}x \\
&\ge (1-\varepsilon)\|f\|^2_{L^2(0,\mu)} - \varepsilon^{-1} f(0)^2,
\end{aligned}$$

resulting in

$$\|f\|^2_{L^2(-\infty,\mu)} - \|Jf\|^2_{L^2(0,\mu)} \le \varepsilon\|f\|^2_{L^2(0,\mu)} + \varepsilon^{-1} f(0)^2 + \|f\|^2_{L^2(-\infty,0)}. \tag{1.5.8}$$

For any $\delta > 0$ one can estimate

$$f(0)^2 = 2\int_{-\infty}^{0} f(x)f'(x)\mathrm{d}x \le \delta\|f'\|^2_{L^2(-\infty,0)} + \delta^{-1}\|f\|^2_{L^2(-\infty,0)},$$

and the substitution into (1.5.8) yields

$$\begin{aligned}
&\|f\|^2_{L^2(-\infty,\mu)} - \|Jf\|^2_{L^2(0,\mu)} \\
&\qquad \le \varepsilon\|f\|^2_{L^2(0,\mu)} + \delta\varepsilon^{-1}\|f'\|^2_{L^2(-\infty,0)} + \varepsilon^{-1}\delta^{-1}\,\|f\|^2_{L^2(-\infty,0)} + \|f\|^2_{L^2(-\infty,0)} \\
&\qquad = \varepsilon\|f\|^2_{L^2(0,\mu)} + \delta\varepsilon^{-1}\|f'\|^2_{L^2(-\infty,0)} + (\varepsilon^{-1}\delta^{-1}\lambda^{-1} + \lambda^{-1})\,\lambda\|f\|^2_{L^2(-\infty,0)}.
\end{aligned}$$

We now set $\delta := \lambda^{-\frac{1}{2}}$ and $\varepsilon := \lambda^{-\frac{1}{4}}$, then for $\lambda > 1$ we have

$$\begin{aligned}
&\|f\|^2_{L^2(-\infty,\mu)} - \|Jf\|^2_{L^2(0,\mu)} \\
&\qquad \le \lambda^{-\frac{1}{4}}\|f\|^2_{L^2(0,\mu)} + \lambda^{-\frac{1}{4}}\|f'\|^2_{L^2(-\infty,0)} + (\lambda^{-\frac{1}{4}} + \lambda^{-1})\,\lambda\|f\|^2_{L^2(-\infty,0)} \\
&\qquad \le 2\lambda^{-\frac{1}{4}}\big(\|f\|^2_{L^2(0,\mu)} + \|f'\|^2_{L^2(-\infty,0)} + \lambda\|f\|^2_{L^2(-\infty,0)}\big),
\end{aligned}$$

and it follows that

$$\|f\|^2_{L^2(-\infty,\mu)} - \|Jf\|^2_{L^2(0,\mu)} \le 2\lambda^{-\frac{1}{4}}\big(B^{\lambda,\mu}[f,f] + \|f\|^2_{L^2(-\infty,\mu)}\big). \tag{1.5.9}$$

Now let us estimate the difference $C^\mu_N[Jf,Jf] - B^{\lambda,\mu}[f,f]$. For any $\varepsilon \in (0,1)$ and $a, b \in \mathbb{R}$ one has $(a+b)^2 \le (1+\varepsilon)a^2 + 2\varepsilon^{-1}b^2$. Therefore, for any $\delta > 0$ and $\varepsilon \in (0,1)$ we have, with some $K > 0$,

$$\begin{aligned}
C^\mu_N[Jf,Jf] &= \int_0^\mu \big(f'(x) + f(0)e^{-x}\big)^2 \mathrm{d}x + \int_0^\mu x^p\big(f(x) - f(0)e^{-x}\big)^2 \mathrm{d}x \\
&\le (1+\varepsilon)\int_0^\mu \big(f'(x)^2 + x^p f(x)^2\big)\mathrm{d}x \\
&\qquad + 2\varepsilon^{-1} f(0)^2 \int_0^\mu (1+x^p)e^{-2x}\mathrm{d}x \\
&\le (1+\varepsilon)\int_0^\mu \big(f'(x)^2 + x^p f(x)^2\big)\mathrm{d}x + K\varepsilon^{-1}f(0)^2 \\
&\le (1+\varepsilon)\int_0^\mu \big(f'(x)^2 + x^p f(x)^2\big)\mathrm{d}x \\
&\qquad + K\varepsilon^{-1}\big(\delta\|f'\|^2_{L^2(-\infty,0)} + \delta^{-1}\|f\|^2_{L^2(-\infty,0)}\big).
\end{aligned}$$

As previously, set $\delta := \lambda^{-\frac{1}{2}}$ and $\varepsilon := \lambda^{-\frac{1}{4}}$, then, with some $K' > 0$,

$$\begin{aligned}
C^\mu_N[Jf,Jf] &\le (1+\lambda^{-\frac{1}{4}})\int_0^\mu \big(f'(x)^2 + x^p f(x)^2\big)\mathrm{d}x \\
&\qquad + K\lambda^{-\frac{1}{4}}\|f'\|^2_{L^2(-\infty,0)} + K\lambda^{-\frac{1}{4}}\cdot\lambda\|f\|^2_{L^2(-\infty,0)} \\
&\le \int_0^\mu \big(f'(x)^2 + x^p f(x)^2\big)\mathrm{d}x \\
&\qquad + K'\lambda^{-\frac{1}{4}}\Big(\int_0^\mu \big(f'(x)^2 + x^p f(x)^2\big)\mathrm{d}x + \|f'\|^2_{L^2(-\infty,0)} \\
&\qquad\qquad + \lambda\|f\|^2_{L^2(-\infty,0)}\Big) \\
&\le B^{\lambda,\mu}[f,f] + K'\lambda^{-\frac{1}{4}}\Big(B^{\lambda,\mu}[f,f] + \|f\|^2_{L^2(-\infty,\mu)}\Big),
\end{aligned}$$

resulting in

$$C^\mu_N[Jf,Jf] - B^{\lambda,\mu}[f,f] \le K'\lambda^{-\frac{1}{4}}\big(B^{\lambda,\mu}[f,f] + \|f\|^2_{L^2(-\infty,\mu)}\big). \tag{1.5.10}$$

By (1.5.9) and (1.5.10) we are in the situation of Proposition 1.2.2 with

$$T := B^{\lambda,\mu}, \quad T' := C_N^{\mu}, \quad \delta_1 = 2\lambda^{-\frac{1}{4}}, \quad \delta_2 = K'\lambda^{-\frac{1}{4}}.$$

Furthermore, in view of (1.5.7) one has $\Lambda_n(B^{\lambda,\mu}) \le M := \Lambda_n(C_D^1)$ for all $(\lambda,\mu) \in (0,+\infty) \times (1,+\infty)$. Therefore, one can find $\lambda_n > 0$ such that

$$\delta_1\big(1+\Lambda_n(T)\big) \equiv 2\lambda^{-\frac{1}{4}}\big(1+\Lambda_n(B^{\lambda,\mu})\big) \le 2(M+1)\lambda^{-\frac{1}{4}}$$

for all $(\lambda,\mu) \in (\lambda_n,+\infty) \times (1,+\infty)$. Hence, Proposition 1.2.2 implies

$$\begin{aligned}\Lambda_n(B^{\lambda,\mu}) &\ge \Lambda_n(C_N^{\mu}) - \frac{\big(2\Lambda_n(B^{\lambda,\mu})+K'\big)\lambda^{-\frac{1}{4}}\big(\Lambda_n(B^{\lambda,\mu})+1\big)}{1-2\lambda^{-\frac{1}{4}}\big(\Lambda_n(B^{\lambda,\mu})+1\big)}\\ &\ge \Lambda_n(C_N^{\mu}) - \frac{(2M+K')(M+1)}{1-2\lambda_n^{-\frac{1}{4}}(M+1)}\,\lambda^{-\frac{1}{4}}\\ &=: \Lambda_n(C_N^{\mu}) - M_n\lambda^{-\frac{1}{4}}\end{aligned}$$

for all $(\lambda,\mu) \in (\lambda_n,+\infty) \times (1,+\infty)$. □

By combining Lemma 1.5.5 with the estimate of the eigenvalues of C_N^{μ} obtained in Lemma 1.4.5) one arrives at the following result:

Lemma 1.5.6. *For any $n \in \mathbb{N}$ there exist $m > 0$ and $M > 0$ such that*

$$\big|\Lambda_n(B^{\lambda,\mu}) - \Lambda_n(A)\big| \le M(\lambda^{-\frac{1}{4}} + \mu^{-2})$$

for all $(\lambda,\mu) \in (m,+\infty) \times (m,+\infty)$.

Now we can complete the proof of Lemma 1.5.3. Choosing $\lambda = 2^{\frac{2}{2+p}}\,h^{-\frac{2p}{2+p}}$ and $\mu = 2^{\frac{1}{2+p}}\,h^{k-\frac{2}{2+p}}$ and using Lemma 1.5.4, for $h \to 0^+$ we obtain

$$\Lambda_n(Z_h) = 2^{\frac{2}{2+p}}\,h^{\frac{2p}{2+p}}\,\Lambda_n(B^{\lambda,\mu}). \tag{1.5.11}$$

By Lemma 1.5.6 we have

$$\Lambda_n(B^{\lambda,\mu}) = \Lambda_n(A) + \mathcal{O}(\lambda^{-\frac{1}{4}} + \mu^{-2}) \equiv \Lambda_n(A) + \mathcal{O}(h^{\frac{p}{4+2p}} + h^{\frac{4}{2+p}-2k}),$$

and the substitution into (1.5.11) completes the proof of Lemma 1.5.3.

1.5.4 Proof of the lower eigenvalue bound

We now use all the preceding components to obtain the sought lower bound for the eigenvalues of R_h and then for those of H_α. For any $m > 0$ we have $h_0^m = h^m(1-h^s)^{\frac{m}{2}} = h^m + \mathcal{O}(h^{m+s})$, and then we conclude by Lemma 1.5.3 that

$$\begin{aligned}E_n(Z_{h_0}) &= 2^{\frac{2}{2+p}}\,E_n(A)\,h_0^{\frac{2p}{p+2}} + \mathcal{O}(h_0^{\frac{5p}{2p+4}} + h_0^{2-2k})\\ &= 2^{\frac{2}{2+p}}\,E_n(A)\,h^{\frac{2p}{p+2}} + \mathcal{O}(h^{\frac{2p}{p+2}+s} + h^{\frac{5p}{2p+4}} + h^{2-2k}).\end{aligned}$$

The substitution into Lemma 1.5.2 gives then

$$\Lambda_n(R_h) \ge -1 + 2^{\frac{2}{2+p}}\,E_n(A)\,h^{\frac{2p}{p+2}} + \rho(h),$$

$$\rho(h) = \mathcal{O}(h^{\frac{2p}{p+2}+s} + h^{\frac{5p}{2p+4}} + h^{2-2k} + h^{2+2k(p-1)-s} + h^{2kp}).$$

It is convenient to set first $k = \frac{1}{p+1}$ to have

$$\rho(h) = \mathcal{O}(h^{\frac{2p}{p+2}+s} + h^{\frac{5p}{2p+4}} + h^{2+2\frac{p-1}{p+1}-s} + h^{\frac{2p}{1+p}}).$$

Furthermore, choosing $s = 1 + \frac{p-1}{p+1} - \frac{p}{p+2} \equiv \frac{p(p+3)}{(p+1)(p+2)}$ we have

$$\frac{2p}{p+2} + s = 2 + 2\frac{p-1}{p+1} - s = \frac{p}{p+2} + 1 + \frac{p-1}{p+1} = \frac{p(3p+5)}{(p+1)(p+2)},$$
$$\rho(h) = \mathcal{O}(h^{\frac{p(3p+5)}{(p+1)(p+2)}} + h^{\frac{5p}{2p+4}} + h^{\frac{2p}{1+p}}).$$

(One can prove that this choice of s and k optimizes the order in h.) We compute then

$$\frac{p(3p+5)}{(p+1)(p+2)} - \frac{2p}{1+p} = \frac{p(3p+5)-2p(p+2)}{(p+1)(p+2)} = \frac{p^2+p}{(p+1)(p+2)} > 0,$$

which yields $h^{\frac{p(3p+5)}{(p+1)(p+2)}} = o(h^{\frac{2p}{1+p}})$ and $\rho(h) = \mathcal{O}(h^{\frac{5p}{2p+4}} + h^{\frac{2p}{1+p}})$. To summarize,

$$\Lambda_n(R_h) \geq -1 + 2^{\frac{2}{2+p}} E_n(A)\, h^{\frac{2p}{p+2}} + \mathcal{O}(h^{\frac{5p}{2p+4}} + h^{\frac{2p}{1+p}}).$$

By Lemma 1.5.1 we have then, with a suitably small $\varepsilon > 0$,

$$\Lambda_n(F_{h,\varepsilon h^{\frac{1}{1-p}}}) \geq -1 + 2^{\frac{2}{2+p}} E_n(A)\, h^{\frac{2p}{p+2}} + \mathcal{O}(h^{\frac{5p}{2p+4}} + h^{\frac{2p}{1+p}}).$$

Applying now Lemma 1.4.7, for $h := \alpha^{\frac{1-p}{p}}$ and $\alpha \to +\infty$ we obtain

$$\begin{aligned}\Lambda_n(H_\alpha) &\geq \alpha^2 \Lambda_n(F_{h,\varepsilon h^{\frac{1}{1-p}}}) + \mathcal{O}(1),\\ &\geq \alpha^2\Big(-1 + 2^{\frac{2}{2+p}} E_n(A)\, \alpha^{\frac{2(1-p)}{p+2}} + \mathcal{O}(\alpha^{\frac{5(1-p)}{2p+4}} + \alpha^{\frac{2(1-p)}{1+p}})\Big) + \mathcal{O}(1)\\ &= -\alpha^2 + 2^{\frac{2}{2+p}} E_n(A)\, \alpha^{\frac{6}{p+2}} + \mathcal{O}(\alpha^{\frac{13-p}{2p+4}} + \alpha^{\frac{4}{1+p}}).\end{aligned}$$

Noting that

$$\eta_1 := \frac{6}{p+2} - \frac{13-p}{2p+4} = \frac{p-1}{2(p+2)} > 0, \quad \eta_2 := \frac{6}{p+2} - \frac{4}{p+1} = \frac{2(p-1)}{(p+1)(p+2)} > 0$$

we obtain

$$\Lambda_n(H_\alpha) \geq -\alpha^2 + 2^{\frac{2}{2+p}} E_n(A)\, \alpha^{\frac{6}{p+2}} + \mathcal{O}(\alpha^{\frac{6}{p+2}-\eta}), \quad \eta := \min\{\eta_1, \eta_2\} > 0.$$

Recall that in Subsection 1.4.3 we already obtained a suitable upper bound and noted that $\Lambda_n(H_\alpha)$ is the nth eigenvalue of H_α if α is large. This completes the proof of Theorem 1.1.1.

Chapter 2

A MIT Bag model on spin manifolds

Le présent chapitre retranscrit un article paru dans *Journal of Geometry and Physics*, **178**, 104534 (2022), généralisant le modèle MIT Bag au cadre des variétés spin. On y étudie la convergence des valeurs propres de l'opérateur de Dirac MIT Bag dans des limites de grandes masses.

2.1 Introduction

The MIT Bag model was developed by the physicists in order to describe the behaviour of quarks fields inside hadrons. Mathematically, the hadron is seen as a compact region $\mathcal{K}$ with smooth boundary of the ambient space, where the quarks are supposed to be confined. This could be quantified by saying that the quantum flux through the border of $\mathcal{K}$ is null, a condition which is satisfied if we add the so-called MIT Bag condition on the boundary of $\mathcal{K}$ (see [40] for the details). Moreover, the quarks fields inside the hadron are Dirac fields, which means they are governed by the Dirac equation.

A Dirac field in the case of the space of dimension 3 is a $\mathbb{C}^4$-valued function ψ also depending on time, and the Dirac equation takes the form

$$H_m\psi := \left(-i\sum_{k=1}^{3}\alpha_k\partial_k + m\beta\right)\psi = i\,\frac{\partial}{\partial t}\psi \tag{2.1.1}$$

where $\alpha_1, \alpha_2, \alpha_3, \beta \in M_4(\mathbb{C})$ are four Hermitian matrices satisfying the conditions $\alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_k^l \mathrm{I}_4$, $\beta^2 = \mathrm{I}_4$ and α_k anti-commutes with β for all $k, l \in \{1, 2, 3\}$. In view of this equation, the Dirac operator H_m can be interpreted as a Hamiltonian, and the description of its spectrum is a natural question. Thus, in the context of the MIT Bag model, we are interested in the operator resulting from the combination of H_m restricted to the region $\mathcal{K}$ together with the MIT Bag boundary condition, namely

$$H_m^{\mathcal{K}}\psi := H_m\psi, \quad \mathrm{dom}(H_m^{\mathcal{K}}) = \{\psi \in H^1(\mathcal{K}, \mathbb{C}^4),\ -i\,\beta(\alpha\cdot\mathbf{n})\,\psi_{|\partial\mathcal{K}} = \psi_{|\partial\mathcal{K}}\}, \tag{2.1.2}$$

where $\mathbf{n}$ is the outer normal vector field along $\partial\mathcal{K}$. The spectrum of this operator has been investigated in [5], where the non-relativistic limit was considered, i.e. the asymptotic regime where the mass goes to infinity. From a physical point of view, this last fact means that the speed of light becomes large, since this constant is hidden in the mass term in (2.1.1). It was shown that if we denote by $(\mu_j)_{j\geq 1}$ the non-decreasing sequence of positive eigenvalues of $H_m^{\mathcal{K}}$, one has the asymptotic

$$\mu_j \underset{m\to-\infty}{=} \widetilde{\mu}_j^{\frac{1}{2}} + \mathcal{O}(m^{-\frac{1}{2}}) \tag{2.1.3}$$

where $(\widetilde{\mu}_j)$ is the non-decreasing sequence of eigenvalues of an effective operator acting on the boundary of $\mathcal{K}$.

In the same framework, the MIT Bag Dirac operator was interpreted as the limit of a Dirac-type operator with a potential corresponding to two masses m and M in the regions $\mathcal{K}$ and $\mathcal{K}^c$ respectively [4]. More precisely, if we define the operator

$$H_{m,M} := H_m + (M-m)\mathbf{1}_{\mathcal{K}^c}, \quad \mathrm{dom}(H_{m,M}) := H^1(\mathbb{R}^3, \mathbb{C}^4), \tag{2.1.4}$$

then the eigenvalues of $H_{m,M}$ converge to the corresponding ones of $H_m^{\mathcal{K}}$ when $M \to +\infty$.

In the recent article [59], the case of Euclidean spaces was studied in order to enlarge the precedent results. The expression of the operator in dimension 3 given by (2.1.2) was generalized to dimension n by considering $n+1$ Hermitian matrices $\alpha_1, \ldots, \alpha_{n+1} \in M_N(\mathbb{C})$ ($N := 2^{\lfloor\frac{n+1}{2}\rfloor}$) satisfying the Clifford conditions $\alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_k^l \mathrm{I}_N$ and by setting

$$D_m\psi := \left(-i\sum_{k=1}^{n}\alpha_k\partial_k + m\alpha_{n+1}\right)\psi, \mathrm{dom}(D_m) = H^1(\mathbb{R}^n, \mathbb{C}^N). \tag{2.1.5}$$

This last operator is not the intrinsic Dirac operator in $\mathbb{R}^n$ but it can be interpreted like in (2.1.1) as the Hamiltonian appearing in the Dirac equation of a Lorentzian space of dimension $n+1$. From these considerations, the MIT Bag Dirac operator A_m can be defined by

$$A_m := D_m, \mathrm{dom}(A_m) := \{\psi \in H^1(\mathcal{K}, \mathbb{C}^4),\ -i\,\alpha_{n+1} \sum_{k=1}^{n} \mathbf{n}_k \alpha_k\, \psi_{|\partial\mathcal{K}} = \psi_{|\partial\mathcal{K}}\}. \tag{2.1.6}$$

With this definition, the result on the convergence of the eigenvalues of A_m still holds, and the effective operator on the boundary can be explicited. Namely, the eigenvalues of A_m^2 converge to the eigenvalues of the square of the intrinsic Dirac operator on $\partial\mathcal{K}$. Moreover, if $n \notin 4\mathbb{Z}$, the spectra of the operators are symmetric with respect to the origin, and we recover the result stated in dimension 3.

As for the Minkowski space, the operator A_m can be viewed as the limit of an operator with two masses [59, Theorem 1.2]. This operator is defined in the same way as before:

$$B_{m,M} := D_m + (M-m)\mathbf{1}_{\mathcal{K}}\alpha_{n+1}, \quad \mathrm{dom}(B_{m,M}) := H^1(\mathbb{R}^n, \mathbb{C}^N), \tag{2.1.7}$$

and the eigenvalues of $B_{m,M}^2$ converge to the eigenvalues of A_m^2 when $M \to +\infty$. In addition, a combination of the two previous asymptotic behaviours is also true [59, Theorem 1.3]: in the asymptotic regime $m \to -\infty$ and $M \to +\infty$ with $\frac{m}{M} \to 0$, one has that the eigenvalues of $B_{m,M}^2$ converge to the corresponding ones of the intrinsic Dirac operator on the boundary $\partial\mathcal{K}$.

In the precedent discussion, the spaces considered where always flat, but the Dirac operator can be defined in a more general setting, for example over a manifold admitting a Spin-structure. Consequently, our aim in the present text is to extend the results of [59] to this more general framework. In order to do so, the first step is to understand the geometrical meaning of the operator considered in the MIT Bag model, because we recall that the Dirac operator considered in [59] is not the intrinsic Dirac operator of the Euclidean space. Indeed, the operator D_m is the so-called Dirac-Witten operator on $\mathbb{R}^n$ seen as an hypersurface of $\mathbb{R}^{n+1}$, plus a mass term which is actually the Clifford multiplication by the vector $i\,mx_{n+1}$ in $\mathbb{R}^{n+1}$. Note that D_m acts on spinors of $\mathbb{R}^{n+1}$ so it is not exactly the Dirac Hamiltonian on $\mathbb{R}^n$ plus the mass term.

Nevertheless, even if the expression (2.1.6) is a direct generalization of equation (2.1.2), the Dirac-Witten operator is not the operator we obtain from the physical model [40]. Indeed, in (2.1.1) we used the alpha matrices, but the Dirac equation is more often written using the gamma matrices defined by

$$\gamma^0 := \beta, \quad \gamma^k := -i\,\gamma^0\alpha_k, \qquad k = 1, 2, 3.$$

If one rewrites (2.1.1) with the γ matrices, one obtains

$$H_m\psi = \left(\sum_{k=1}^{3} \gamma^0\gamma^k\partial_k + m\gamma^0\right)\psi, \tag{2.1.8}$$

and this last operator is, up to a change of sign, the extrinsic Dirac operator on the hypersurface $\mathbb{R}^3$ plus the mass term. Moreover, the boundary condition defined in [5] by $-i\,\beta(\alpha \cdot \mathbf{n})\psi = \psi$ reads $i\,(\gamma \cdot \mathbf{n})\psi = \psi$ and this last boundary condition is the MIT Bag boundary condition as introduced in [40].

Altogether, we have two natural ways of setting the problem in the case of a complete spin manifold $\mathcal{N}$. In both cases, we have to see $\mathcal{N}$ as an hypersurface of the Riemannian product

$\mathcal{C} := \mathcal{N} \times \mathbb{R}$, and we denote by ν the outer normal vector field over $\mathcal{N}$. In addition, the region $\mathcal{K}$ is now a compact submanifold of $\mathcal{N}$ with boundary. The theory of Spin-structures restricted to hypersurfaces gives that $\mathcal{C}$ and $\partial\mathcal{K}$ are also spin manifolds. Consequently, we can define the spinor bundle $\Sigma\mathcal{C}$ over $\mathcal{C}$, and the extrinsic Dirac operator $\mathcal{D}^{\mathcal{N}}$, which acts on spinors of $\mathcal{C}$ restricted to $\mathcal{N}$.

From the previous discussion, the obvious generalization of the MIT Bag Dirac operator in the Euclidean spaces (2.1.6) is defined as the Dirac-Witten operator on $\mathcal{N}$ plus a mass term, and we add the boundary condition $i\,\nu \cdot \mathbf{n} \cdot \Psi = \Psi$ on $\partial\mathcal{K}$. This last condition is not the MIT Bag boundary condition, but the condition associated with a chirality operator, and it is consistent with the condition imposed in (2.1.6). Namely, we have

$$A_m := \nu \cdot \mathcal{D}^{\mathcal{N}} + i\,m\,\nu\cdot,\ \mathrm{dom}(A_m) = \left\{\Psi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}}), i\,\nu \cdot \mathbf{n} \cdot \Psi = \Psi \text{ on } \partial\mathcal{K}\right\}. \tag{2.1.9}$$

Furthermore, the cylinder $\mathcal{C}$ can be endowed with a Lorentzian metric such that ν is a time-like vector, and in this case, solving the Dirac equation in $\mathcal{C}$ in the same way as for dimension 3 lets us with the study of the extrinsic Dirac operator on $\mathcal{N}$ plus the mass term. The boundary condition imposed in this case is the original MIT Bag boundary condition $i\,\mathbf{n} \cdot \Psi = \Psi$.

Actually, the two operators we defined this way are unitarily equivalent since the manifold $\mathcal{N}$ is totally geodesic in $\mathcal{C}$. This last result explains how the operator studied in [59] is obtained from the physical model, and the two definitions we gave above are equivalent.

In the same way as before, the two-masses operator is obtained by adding a potential corresponding to two masses in $\mathcal{K}$ and $\mathcal{K}^c$ in the expression of the operator A_m. Since in our framework the manifold $\mathcal{N}$ is complete but not necessarily compact, $B_{m,M}$ is defined as the closure of the operator

$$\widetilde{B}_{m,M} := \nu \cdot \mathcal{D}^{\mathcal{N}} + i\,(m\mathbf{1}_{\mathcal{K}} + M\mathbf{1}_{\mathcal{K}^c})\nu\cdot, \tag{2.1.10}$$

whose domain is the set of smooth sections with compact support in $\Sigma\mathcal{C}_{|\mathcal{N}}$. This definition is consistent with (2.1.7) because it was shown in [59] that the two-masses operator is essentially self-adjoint on the smooth functions with compact support.

The operators A_m and $B_{m,M}$ are self-adjoint and we are interested in the behaviour of the spectrum of A_m when $m \to -\infty$ and the spectrum of $B_{m,M}$ in the asymptotic regime $M \to +\infty$ and $\min(-m, M) \to +\infty$. These limits are the ones studied in [59], and the three main theorems we state below are the counterparts of [59, Theorems 1.1, 1.2, 1.3].

From now on, we use for $j \in \mathbb{N}$ and a lower semibounded operator T the notation $E_j(T)$, which stands for the j-th eigenvalue of T when counted with multiplicity in the non-decreasing order.

First of all, one has the convergence of the eigenvalues of A_m^2 to the eigenvalues of the square of the Dirac operator on $\partial\mathcal{K}$:

Theorem 2.1.1. *For any $j \in \mathbb{N}$, one has $E_j(A_m^2) \underset{m\to-\infty}{\longrightarrow} E_j\left((\not{D}^{\partial\mathcal{K}})^2\right)$.*

The two operators A_m^2 and $B_{m,M}^2$ are surprisingly related in the asymptotic regime $M \to +\infty$:

Theorem 2.1.2. *For any $j \in \mathbb{N}$, there is $M_0 \in \mathbb{R}$ such that for all $M \geq M_0$, $B_{m,M}^2$ has at least j eigenvalues, and one has $E_j(B_{m,M}^2) \underset{M\to+\infty}{\longrightarrow} E_j(A_m^2)$.*

In addition, one has a combination of these two results:

Theorem 2.1.3. *For any $j \in \mathbb{N}$, there is $\tau_j \in \mathbb{R}$ such that for all $M \geq \tau_j$ and $m \leq -\tau_j$, the operator $B^2_{m,M}$ has at least j eigenvalues, and one has $E_j(B^2_{m,M}) \underset{\min(M,-m)\to+\infty}{\longrightarrow} E_j\left((\not{D}^{\partial\mathcal{K}})^2\right)$.*

Note that Theorem 2.1.3 is an improvement of [59, Theorem 1.3] since we drop the assumption $\frac{m}{M} \to 0$.

Remark 2.1.4. We can also look at the operator A^2_m when $m \to +\infty$ and the operator $B^2_{m,M}$ when $m, M \to +\infty$ (or $m, M \to -\infty$). We can prove that in these two cases, the spectrum escapes to infinity (see Remarks 2.7.2 and 2.9.1 below).

Remark 2.1.5. It is not easy to understand the implications of the three theorems above for the spectrums of the operators A_m and $B_{m,M}$. However, as in the Euclidean case [59], we can show that the spectrum of these operators is symmetric when the dimension n of $\mathcal{N}$ is not in $4\mathbb{Z}$. Indeed, in this case there is a parallel antilinear map J on $\Sigma\mathcal{C}$ which commutes with the Clifford multiplication by elements of $T\mathcal{C}$ (see [15, Theorem 1.39] for example). Then, $\theta := J \circ (\nu\cdot)$ anticommutes with the operators A_m and $B_{m,M}$ because ν is parallel. In addition, θ preserves the domains of these operators. Consequently, if Ψ is an eigenspinor for A_m, i.e. $A_m\Psi := \lambda\Psi$ for a $\lambda \in \mathbb{R}$, one has $A_m\theta\Psi = -\lambda\theta\Psi$, implying that $-\lambda$ is an eigenvalue of A_m. The case of $B_{m,M}$ is done in the same way.

Organization of the paper

The proofs of the three theorems are really close to the ones written in [59] once we have stated the correct geometrical context. The global strategy is thus to compute sesquilinear forms for the operators A^2_m and $B^2_{m,M}$ in order to find lower and upper bounds for the limits of the eigenvalues by use of the Min-Max principle.

In section 2.2 we first recall some fundamental results in spectral theory on the correspondence between self-adjoint operator and sesquilinear forms on Hilbert space. The Min-Max principle, which is the key point of our proof, is stated, and we also give a quick review on the monotone convergence theorem in the case of sesquilinear forms. This last theorem is helpful to find the lower bounds for the limits of the eigenvalues, since it gives a description of the asymptotic domain of the operators. After these preliminaries on operators theory, we introduce the basic tools needed to understand the geometrical context. Indeed, the theory of restriction of the spin structure of spin manifolds to oriented hypersurfaces plays a significant role in the understanding of the generalized MIT Bag operator.

Section 2.3 is devoted to the construction of the operators. We develop here the discussion about the two equivalent ways of defining A_m. We also define the operator $B_{m,M}$ and we show that it is self-adjoint as a direct consequence of the completeness of $\mathcal{N}$. The self-adjointness of A_m is more difficult to prove, and we need to compute the sesquilinear form for A^2_m in order to understand its graph norm and its domain. The computations for the forms of square operators are done in Section 2.4 and the main tool used to this aim is the Schrödinger-Lichnerowicz formula, which gives the expression of the square of the Dirac operator on a spin manifold. Once we get the sesquilinear forms, the graph norm of A_m is shown to be equivalent to the H^1 norm on its domain, and we can use the analysis done in [36] to conclude on self-adjointness.

An important idea to prove the main results is that we can restrict the analysis to a tubular neighbourhood of the boundary of $\mathcal{K}$. Thanks to this restriction of domain, we only have to understand the operators on a generalized cylinder $\partial\mathcal{K} \times (-\delta, \delta)$ with $\delta > 0$. However, there is an additional difficulty since we cannot compare the covariant derivatives on the different slices of the cylinder as it is done in [59]. Thus, we prove some comparison lemmas in section 2.5, where we express the operators in tubular coordinates.

The aim of this restriction is to be able to separate the variables in the generalized cylinder previously introduced. Thus, some one-dimensional operators will appear later in the analysis, and we devote section 2.6 to the spectral analysis of these operators, even if a large part of this work has already been done in [59, Section 3].

In section 2.7 we prove Theorem 2.1.1. The geometrical context is well-defined, and it remains to follow the lines of [59, Section 4]. The proof is done by restricting the analysis to the tubular neighbourhood of $\partial\mathcal{K}$ intersected with the interior of $\mathcal{K}$ thanks to the Min-Max principle. Next, an upper bound can be found for the limit by choosing good test functions which are tensorial products between eigenspinors of a model operator on $\partial\mathcal{K}$ and the first eigenfunction of a one-dimensional operator. The proof of the lower bound relies on the monotone convergence theorem after operating a transformation on the operator in tubular coordinates.

The result stated in Theorem 2.1.2 is proved in section 2.8. We find an appropriate extension operator which sends eigenspinors of A_m^2 into $\mathrm{dom}(B_{m,M})$, and this gives the upper bound. The lower bound is once again a consequence of the monotone convergence theorem together with the Min-Max principle.

Finally, we prove Theorem 2.1.3 in section 2.9 using a combination of the precedent arguments. After restricting the problem to the tubular neighbourhood of $\partial\mathcal{K}$, the upper bound is found in the same way as for Theorem 2.1.1 by choosing good test functions in the Min-Max principle, and the lower bound is a consequence of the monotone convergence theorem.

2.2 Notations and preliminaries.

2.2.1 About spectral theory.

Let $\mathbf{H}$ be an infinite-dimensional Hilbert space endowed with the inner product $(\cdot,\cdot)_{\mathbf{H}}$. For a self-adjoint and lower semibounded operator T on $\mathbf{H}$, we denote by $\mathrm{dom}\, T$ its domain, and for any $j \in \mathbb{N}$, $E_j(T)$ is the jth eigenvalue of T, counted with multiplicity in the non-decreasing order. We also note $\sigma(T)$, $\sigma_{ess}(T)$ and $\sigma_d(T)$ the spectrum, the essential spectrum and the discrete spectrum of T respectively.

We denote the adjoint of an operator T by T^* and its closure by $\overline{T}$.

For a sesquilinear form t in $\mathbf{H}$, we denote its domain by $\mathcal{Q}(t)$. There is a one-to-one correspondence between densely defined, closed, symmetric, lower semibounded forms and lower semibounded self-adjoint operators (see [41, VI, Theorem 2.1] for details). For a lower semibounded self-adjoint operator T, we will denote by $\mathcal{Q}(T)$ the domain of the associated form. If T and T' are two such operators, and t, t' are the associated forms, we write $T \leq T'$ if $\mathcal{Q}(T') \subset \mathcal{Q}(T)$ and $t(u,u) \leq t'(u,u)$ for all $u \in \mathcal{Q}(T')$.

For $j \in \mathbb{N}$, we define the jth Rayleigh quotient of the form t by

$$\Lambda_j(t) := \inf_{\substack{V \subset \mathcal{Q}(t) \\ \dim V = j}} \sup_{u \in V \setminus \{0\}} \frac{t(u,u)}{\|u\|^2_{\mathcal{H}}}. \tag{2.2.1}$$

We recall that if t and t' are two semibounded from below bilinear forms, we write $t \leq t'$ if $\mathcal{Q}(t') \subset \mathcal{Q}(t)$ and $t(u,u) \leq t'(u,u)$ for all $u \in \mathcal{Q}(t')$.

Let t be a closed symmetric lower semibounded form, and T its associated operator. The well-known Min-Max principle gives a link between the Rayleigh quotients of t and the eigenvalues of T. More precisely, we have the following theorem:

Theorem 2.2.1 (Min-Max principle)**.** *Let $\Sigma := \inf \sigma_{ess}T$. We are in one of the following cases:*

(a) $\Lambda_j(t) < \Sigma$ for all j, $\lim_{m\to+\infty} \Lambda_m(t) = \Sigma$ and $E_j(T) = \Lambda_j(t)$ for all j.

(b) $\sigma_{ess}T < +\infty$ and there is $N < +\infty$ such that the interval $(-\infty, \Sigma)$ contains exactly N eigenvalues of T counted with multiplicity and for all $j \leq N$, one has $\Lambda_j(t) = E_j(T)$ and $\Lambda_m(t) = \Sigma$ for all $m > N$.

The proofs of the spectral part of this text will use monotone convergence of operators. The result stated below is a reformulation of [7, Theorem 4.2].

Theorem 2.2.2. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of lower semibounded self-adjoint operators in closed subspaces $(\mathbf{H}_n)_{n\in\mathbb{N}}$ of $\mathbf{H}$, and let $(t_n)_{n\in\mathbb{N}}$ be the sequence of associated forms. Assume there exists $\gamma \in \mathbb{R}$ such that $t_n \geq \gamma$ for all n and suppose moreover that the sequence (t_n) (or equivalently (T_n)) is non-decreasing. Then, the form t_∞ defined by*

$$\mathcal{Q}(t_\infty) = \left\{ h \in \bigcap_{n\in\mathbb{N}} \mathcal{Q}(t_n),\ \lim_{n\to\infty} t_n(h,h) < \infty \right\} \tag{2.2.2}$$

and $t_\infty(h,h) = \lim_{n\to\infty} t_n(h,h)$ for all $h \in \mathcal{Q}(t_\infty)$ is closed, symmetric, and $t_\infty \geq \gamma$. Moreover, if $\mathbf{H}_\infty := \overline{\mathcal{Q}(t_\infty)}$, one can define the self-adjoint operator T_∞ on $\mathbf{H}_\infty$ associated with t_∞, and the sequence (T_n) strongly converges to T_∞ in the generalized resolvent sense, i.e. for all $\lambda < \gamma$, one has

$$((T_n - \lambda)^{-1} \oplus 0_{\mathbf{H}_n^\perp})h \underset{n\to\infty}{\longrightarrow} ((T_\infty - \lambda)^{-1} \oplus 0_{\mathbf{H}_\infty^\perp})h,\ \forall h \in \mathbf{H}. \tag{2.2.3}$$

Since we are interested in the behaviour of the spectrum, we claim that in the framework of Theorem 2.2.2, one has actually the convergence of the eigenvalues of T_n to the corresponding eigenvalues of T_∞. To show this, we first recall [74, Theorem 2.1]:

Theorem 2.2.3. *Let (T_n) be a sequence of self-adjoint operators which are bounded from below with $T_n \leq T_{n+1}$, strongly converging to T in the generalized resolvent sense. Assume that the essential spectrum of T_n is contained in $[0, +\infty)$ for all $n \in \mathbb{N}$. Suppose that T has j_0 negative eigenvalues (j_0 might be infinite). Then,*

$$E_j(T_n) \underset{n\to+\infty}{\longrightarrow} E_j(T) \text{ for all } j \leq j_0$$

$$\lim_{n\to+\infty} E_j(T_n) \geq 0 \text{ for all } j > j_0.$$

Moreover,

$$\|\mathbf{1}_{(-\infty,\lambda)}(T_n) - \mathbf{1}_{(-\infty,\lambda)}(T)\| \underset{n\to+\infty}{\longrightarrow} 0 \text{ for all } \lambda < 0.$$

From Theorem 2.2.2 and Theorem 2.2.3 we deduce the following corollary:

Corollary 2.2.4. *Let $(T_n)_{n\in\mathbb{N}}$ and T_∞ be like in Theorem 2.2.2. Assume moreover that $\sigma_{ess}(T_{n_0}) \subset [\eta, +\infty)$ for some $n_0 \in \mathbb{N}$ and that T_∞ has j_0 eigenvalues below η (j_0 might be infinite). Then, one has*

$$E_j(T_n) \underset{n\to+\infty}{\longrightarrow} E_j(T) \text{ for all } j \leq j_0 \tag{2.2.4}$$

and

$$\|\mathbf{1}_{(-\infty,\lambda)}(T_n) - \mathbf{1}_{(-\infty,\lambda)}(T_\infty)\| \underset{n\to+\infty}{\longrightarrow} 0, \quad \forall\lambda < \eta. \tag{2.2.5}$$

Proof. We consider for $n \geq n_0$ large enough the bounded self-adjoint operators in $\mathbf{H}$

$$B_n := \frac{1}{\eta-\gamma} - ((T_n-\gamma)^{-1} \oplus 0_{\mathbf{H}_n^\perp})$$
$$B_\infty := \frac{1}{\eta-\gamma} - ((T_\infty-\gamma)^{-1} \oplus 0_{\mathbf{H}_\infty^\perp}).$$

From [7, Proposition 2.2], it comes that for all $n \geq n_0$, one has $B_n \leq B_{n+1} \leq B_\infty$. In addition, $\sigma_{ess}(B_n) \subset [0, \frac{1}{\eta-\gamma}]$, $\sigma_{ess}(B_\infty) \subset [0, \frac{1}{\eta-\gamma}]$, and (B_n) converges strongly to B_∞. Thus, Theorem 2.2.3 gives that for all $j \in \mathbb{N}$ such that $E_j(B_\infty) < 0$ one has

$$E_j(B_n) \underset{n\to+\infty}{\longrightarrow} E_j(B_\infty) \tag{2.2.6}$$

and that for all $t < 0$, there holds

$$\|\mathbf{1}_{(-\infty,t)}(B_n) - \mathbf{1}_{(-\infty,t)}(B_\infty)\| \underset{n\to\infty}{\longrightarrow} 0. \tag{2.2.7}$$

For $\lambda > \gamma$, we define the strictly increasing function $f(\lambda) := \frac{1}{\eta-\gamma} - \frac{1}{\lambda-\gamma}$. One has $B_n = f(T_n)$ and $B_\infty = f(T_\infty)$ and we deduce that for all $j \leq j_0$

$$E_j(T_n) \underset{n\to+\infty}{\longrightarrow} E_j(T) \text{ for all } j \leq j_0$$

and from

$$\mathbf{1}_{(-\infty,f(\lambda))}(B_n) = \mathbf{1}_{(-\infty,\lambda)}(T_n), \quad \mathbf{1}_{(-\infty,f(\lambda))}(B_\infty) = \mathbf{1}_{(-\infty,\lambda)}(T_\infty),$$

we deduce that for all $\lambda < \eta$

$$\|\mathbf{1}_{(-\infty,\lambda)}(T_n) - \mathbf{1}_{(-\infty,\lambda)}(T_\infty)\| \underset{n\to\infty}{\longrightarrow} 0. \qquad \square$$

2.2.2 Clifford algebra

We recall here the basic facts about Clifford algebra, and we refer to [15] for the details. For any $d \in \mathbb{N}$, the real Clifford algebra Cl_d is the quotient of the tensorial algebra over $\mathbb{R}^d$ by the two-sided ideal generated by the elements $x \otimes x + \|x\|^2 1$. The induced product on the quotient algebra is called the Clifford product, and is denoted by "$\cdot$". The complex Clifford algebra is defined by $\mathbb{C}\mathrm{l}_d := \mathrm{Cl}_d \otimes_{\mathbb{R}} \mathbb{C}$. The spin group is the subgroup of Cl_d given by

$$\mathrm{Spin}_d := \{x_1 \cdot \ldots \cdot x_{2k} \in \mathrm{Cl}_d,\ k \in \mathbb{N} \text{ and } x_j \in \mathbb{R}^d,\ \|x_j\| = 1 \text{ for all } 1 \leq j \leq 2k\}.$$

We define the complex volume form as the element of $\mathbb{C}\mathrm{l}_d$

$$\omega_d^{\mathbb{C}} := i^{\lfloor \frac{d+1}{2} \rfloor} e_1 \cdot \ldots \cdot e_d \tag{2.2.8}$$

where $(e_1, \ldots, e_d)$ is any positively-oriented orthonormal frame of $\mathbb{R}^d$, canonically identified with a basis of $\mathbb{C}^d$.

If d is even, $\mathbb{C}\mathrm{l}_d$ admits an unique irreducible complex representation (ρ_d, Σ_d) where Σ_d is a complex vector space of dimension $2^{\frac{d}{2}}$. When restricted to the Spin group, this Clifford module decomposes into $\Sigma_d = \Sigma_d^+ \oplus \Sigma_d^-$ and the representation splits in two irreducible inequivalent representations $(\rho_d^\pm, \Sigma_d^\pm)$. These submodules are characterized by the action of the complex volume form, namely $\omega_d^{\mathbb{C}}$ acts as $\pm\mathrm{Id}$ on $\Sigma_d^\pm$.

When d is odd, $\mathbb{C}\mathrm{l}_d$ admits two irreducible inequivalent representations over complex vector spaces of dimension $2^{\frac{d-1}{2}}$. They are characterized by the action of the complex volume form which acts as $\pm\mathrm{Id}$. We denote by (ρ_d, Σ_d) the representation on which $\omega_d^{\mathbb{C}}$ acts as the identity.

2.2.3 Notations for manifolds and bundles

In all this text, the manifolds will be considered smooth and paracompact.

Let $(\mathcal{M}, g)$ be a Riemannian manifold of dimension $d+1$, with boundary $\partial\mathcal{M}$ (possibly empty). If $\mathcal{M}$ is oriented, we denote by $v_{\mathcal{M}}$ the volume form on $\mathcal{M}$ compatible with the metric. Throughout this article, integrations will be done with respect to the Riemannian measure, which coincides with the integration with respect to the volume form $v_{\mathcal{M}}$ in the oriented case.

We denote by $\nabla^{\mathcal{M}}$ the Levi-Civita connection of $(\mathcal{M}, g)$ and by $\mathrm{R}^{\mathcal{M}}$, $\mathrm{Ric}^{\mathcal{M}}$, $\mathrm{Scal}^{\mathcal{M}}$ the Riemann curvature tensor, the Ricci tensor, and the scalar curvature of $\mathcal{M}$ respectively.

If E is a vector bundle over $\mathcal{M}$, we denote respectively by $\Gamma(E)$, $\Gamma_c(E)$ and $\Gamma_{cc}(E)$ the smooth sections of E, the smooth sections of E with compact support in $\mathcal{M}$, and the smooth sections of E with compact support in $\mathcal{M} \setminus \partial\mathcal{M}$. If moreover E is a Hermitian bundle, we note $L^2(E)$ the space of square integrable sections of E. If it is necessary, we will write $L^2(E, v_{\mathcal{M}})$ to specify the measure used for the integration.

We now assume that $\mathcal{M}$ is oriented. The manifold $\mathcal{M}$ admits a spin structure if there exists a map χ and a principal bundle $P_{\mathrm{Spin}_{d+1}}\mathcal{M}$ over $\mathcal{M}$ such that for every $u \in P_{\mathrm{Spin}_{d+1}}\mathcal{M}$ we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{Spin}_{d+1} \arrow[r, "s\mapsto us"] \arrow[dd] & P_{\mathrm{Spin}_{d+1}}\mathcal{M} \arrow[dd, "\chi"] \arrow[dr] & \\
 & & \mathcal{M} \\
\mathrm{SO}_{d+1} \arrow[r, "g\mapsto \chi(u)g"] & P_{\mathrm{SO}_{d+1}}\mathcal{M} \arrow[ur] &
\end{tikzcd} \tag{2.2.9}$$

Given a spin structure on $\mathcal{M}$, we define the associated complex spinor bundle by $\Sigma\mathcal{M} := P_{\mathrm{Spin}_{d+1}}\mathcal{M} \times_{\rho_{d+1}} \Sigma_{d+1}$ where we recall that $(\rho_{d+1}, \Sigma_{d+1})$ is an irreducible representation of the Clifford algebra $\mathbb{C}\mathrm{l}_{d+1}$ as defined in section 2.2.2.

There is a natural action of the Clifford bundle $\mathbb{C}\mathcal{M} := P_{\mathrm{SO}_{d+1}} \times_r \mathbb{C}\mathrm{l}_{d+1}$ (where r is the action of SO_{d+1} on $\mathbb{R}^d$ extended to a representation on $\mathbb{C}\mathrm{l}_d$) defined by:

$$[\chi(u), v]([u, \psi]) := [u, \rho_{d+1}(v)\psi] \tag{2.2.10}$$

for all $u \in P_{\mathrm{Spin}_{d+1}}\mathcal{M}$, $v \in \mathbb{C}l_{d+1}$ and $\psi \in \Sigma_{d+1}$. This action is called the Clifford product and will be denoted by "$\cdot$".

One has a canonical Hermitian product $\langle \cdot, \cdot \rangle$ on $\Sigma\mathcal{M}$ for which the Clifford product by a unit vector is unitary. Moreover, one obtains a metric connection on $\Sigma\mathcal{M}$ by lifting the Levi-Civita connection on the orthonormal frame bundle of $\mathcal{M}$ through the map χ. The covariant derivative obtained this way will still be denoted by $\nabla^{\mathcal{M}}$.

We define the intrinsic Dirac operator $\not{D}^{\mathcal{M}}$ on $\mathcal{M}$, by its pointwise expression

$$\not{D}^{\mathcal{M}}\Psi = \sum_{k=1}^{d+1} e_k \cdot \nabla^{\mathcal{M}}_{e_k}\Psi, \ \mathrm{dom}(\not{D}^{\mathcal{M}}) = \Gamma_c(\Sigma\mathcal{M}), \tag{2.2.11}$$

where $(e_1, \ldots, e_{d+1})$ is an orthonormal frame. This definition does not depend on the choice of the frame.

Finally, we remind the Schrödinger-Lichnerowicz formula, which will be a fundamental tool to compute sesquilinear forms of operators. A proof can be found in [33, Theorem 1.3.8].

Theorem 2.2.5 (Schrödinger-Lichnerowicz formula)**.** *The Dirac operator $\not{D}^{\mathcal{M}}$ satisfies the formula*

$$(\not{D}^{\mathcal{M}})^2 = \left(\nabla^{\mathcal{M}}\right)^* \nabla^{\mathcal{M}} + \frac{\mathrm{Scal}^{\mathcal{M}}}{4}, \tag{2.2.12}$$

where $(\nabla^{\mathcal{M}})^* : \Gamma(T^*\mathcal{M} \otimes \Sigma\mathcal{M}) \to \Gamma(\Sigma\mathcal{M})$ is the formal adjoint of $\nabla^{\mathcal{M}}$ and $\mathrm{Scal}^{\mathcal{M}}$ is the scalar curvature of $\mathcal{M}$.

2.2.4 Restriction of the spinor bundle to hypersurfaces

We take $(\mathcal{M}, g)$ as in the previous section.

Let $\mathcal{H}$ be a smooth oriented hypersurface of $\mathcal{M}$. Let ν be the outer unit normal vector field on $\mathcal{H}$, that is, the only vector field such that if $(e_1, \ldots, e_d)$ is an oriented frame of $\mathcal{H}$, then $(e_1, \ldots, e_d, \nu)$ is an oriented frame of $\mathcal{M}$. We define the Weingarten operator of $\mathcal{H}$ as the endomorphism of $T\mathcal{H}$ given by

$$W_{\mathcal{H}}(X) := -\nabla^{\mathcal{M}}_X \nu, \tag{2.2.13}$$

and $H_{\mathcal{H}} : \mathcal{M} \to \mathbb{R}$ will be the pointwise trace of this operator.

The hypersurface $\mathcal{H}$ inherits a spin structure from the one of $\mathcal{M}$, and we can define the spinor bundle $\Sigma\mathcal{H}$ (for the details, see [15, Section 2.4]). This last bundle is endowed with the natural Hermitian product on spinors, still denoted by $\langle\cdot,\cdot\rangle$. The covariant derivative on $\Sigma\mathcal{H}$ induced by the Levi-Civita connection will be denoted by $\nabla^{\mathcal{H}}$. We will also write $\nabla^{\mathcal{H}}$ for the covariant derivative on $\Sigma\mathcal{H} \oplus \Sigma\mathcal{H}$ (where $\oplus$ stands for the Whitney product), and for all $X \in T\mathcal{H}$, the Clifford product by X on $\Sigma\mathcal{H} \oplus \Sigma\mathcal{H}$ is given by

$$X \cdot (\Psi_1, \Psi_2) := (X \cdot \Psi_1, -X \cdot \Psi_2), \quad \forall (\Psi_1, \Psi_2) \in \Sigma\mathcal{H} \oplus \Sigma\mathcal{H}. \tag{2.2.14}$$

There is a link between the restricted spinor bundle $\Sigma\mathcal{M}_{|\mathcal{H}}$ and $\Sigma\mathcal{H}$, given by the following proposition (see [33, Proposition 1.4.1]):

Proposition 2.2.6. *Let $\mathcal{M}$ and $\mathcal{H}$ be as above. There exists an isomorphism ζ from $\Sigma\mathcal{M}_{|\mathcal{H}}$ into $\Sigma\mathcal{H}$ if d is even and into $\Sigma\mathcal{H} \oplus \Sigma\mathcal{H}$ otherwise, which satisfies the following properties:*

1. *For all $x \in \mathcal{H}$, $X \in \Gamma(T_x\mathcal{H})$ and $\Psi \in (\Sigma\mathcal{M})_{|\{x\}}$, the Clifford product on $\mathcal{H}$ satisfies*
$$X \cdot \zeta(\Psi) = \zeta(X \cdot \nu(x) \cdot \Psi), \tag{2.2.15}$$
2. *The isomorphism ζ is unitary,*
3. *For all $\Psi \in \Gamma(\Sigma\mathcal{M}_{|\mathcal{H}})$ and $X \in T\mathcal{H}$,*
$$\zeta(\nabla^{\mathcal{M}}_X \Psi) = \nabla^{\mathcal{H}}_X \zeta(\Psi) + \frac{1}{2} W_{\mathcal{H}} X \cdot \zeta(\Psi). \tag{2.2.16}$$

4. For $\Psi \in \Sigma\mathcal{M}_{|\mathcal{H}}$,

$$\zeta(i\,\nu\cdot\Psi) = \begin{cases} \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \zeta(\Psi) & \text{if } d \text{ is odd} \\ \omega_d^{\mathbb{C}} \cdot \zeta(\Psi) & \text{if } d \text{ is even} \end{cases}, \tag{2.2.17}$$

where the complex volume form $\omega_d^{\mathbb{C}}$ was defined in section 2.2.2.

We can define a covariant derivative $\overline{\nabla}^{\mathcal{M}}$ on $\Sigma\mathcal{M}_{|\mathcal{H}}$ such that $\overline{\nabla}^{\mathcal{M}}\Psi$ is the restriction of $\nabla^{\mathcal{M}}\Psi$ to $\Gamma(T^*\mathcal{H}\otimes E)$. This notation will be useful as we will often consider the norm of the restricted covariant derivative on hypersurfaces.

The link between $\Sigma\mathcal{M}_{|\mathcal{H}}$ and $\Sigma\mathcal{H}$ gives rise to a natural operator called the extrinsic Dirac operator. This is actually the Dirac operator of $\mathcal{H}$ which acts on the spinor bundle $\Sigma\mathcal{M}_{|\mathcal{H}}$. This extrinsic Dirac operator on $\mathcal{H}$ is the operator acting on $\Gamma_c(\Sigma\mathcal{M})$ defined by

$$\mathcal{D}^{\mathcal{H}} := \zeta^* \not{D}^{\mathcal{H}} \zeta \text{ if } d \text{ is odd}, \quad \mathcal{D}^{\mathcal{H}} := \zeta^*(\not{D}^{\mathcal{H}} \oplus -\not{D}^{\mathcal{H}})\zeta \text{ if } d \text{ is even}. \tag{2.2.18}$$

where ζ is the isomorphism given by Proposition 2.2.6. It can be explicitly computed, and its expression at $x \in \mathcal{H}$ for $\Psi \in \Sigma\mathcal{M}$ is

$$\mathcal{D}^{\mathcal{H}}\Psi(x) = \frac{H_{\mathcal{H}}(x)}{2}\Psi(x) - \nu(x)\cdot\sum_{k=1}^{d} e_k \cdot \nabla_{e_k}^{\mathcal{M}}\Psi(x) \tag{2.2.19}$$

where $(e_1, \dots, e_d)$ is an orthonormal frame of $T_x\mathcal{H}$ [33, Proposition 1.4.1], [39].

2.2.5 Sobolev spaces on manifolds

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension $d+1$ with boundary $\partial\mathcal{M}$. We denote by $\nu_{\mathcal{M}}$ the normal unit vector field over $\partial\mathcal{M}$.

Let $(E, \nabla^E, \langle\cdot,\cdot\rangle_E)$ be an Hermitian bundle of dimension q over $\mathcal{M}$. The construction of the Sobolev spaces on E is done for example in [36, Definition 3.5], but we recall the idea to be self-contained.

In what follows, we will denote by $\exp^{\mathcal{M}}$ the Riemannian exponential map on $\mathcal{M}$ and by $B_x^{\mathcal{M}}(r)$ the ball of radius $r > 0$ and of center 0 in $T_x\mathcal{M}$ where $x \in \mathcal{M}$. This notation will be used for the boundary $\partial\mathcal{M}$ with an obvious modification. By the compactness of $\mathcal{M}$, there is $r_t > 0$ such that:

- the map
$$F : \partial\mathcal{M} \times [0, 2r_t) \ni (x,t) \mapsto \exp_x^{\mathcal{M}}(t\nu_{\mathcal{M}}(x)) \tag{2.2.20}$$
is a diffeomorphism on its image;
- for all $x \in \mathcal{M} \setminus F(\partial\mathcal{M} \times [0, 2r_t))$, $\exp^{\mathcal{M}}$ is injective on the open ball of radius r_t of $T_x\mathcal{M}$;
- for all $x \in \partial\mathcal{M}$, $\exp^{\partial\mathcal{M}}$ is injective on the open ball of radius r_t of $T_x\partial\mathcal{M}$.

Let $(U_j)_{j\in J}$ be a finite covering of $\mathcal{M}$ such that $U_j = \exp_x^{\mathcal{M}}(B_x^{\mathcal{M}}(r_t))$ with $x \in \mathcal{M} \setminus F(\partial\mathcal{M} \times [0, 2r_t))$ (Gaussian coordinates) or $U_j = F(B_x^{\partial\mathcal{M}}(r_t) \times [0, 2r_t))$ with $x \in \partial\mathcal{M}$ (normal coordinates). The maps given by these charts are denoted by $(f_j)_{j\in J}$. We trivialize E over U_j with Gaussian coordinates by identifying E_x with $\mathbb{C}^q$ and by making parallel transport along the radial geodesics. Over the set U_j with normal coordinates, we trivialize E by identifying E_x with $\mathbb{C}^q$ and by making parallel transport first along the radial geodesics in $\partial\mathcal{M}$ and then along the geodesics normal to $\partial\mathcal{M}$. The trivializations obtained are denoted by ξ_j.

Let $(h_j)_{j\in J}$ be a partition of unity adapted to the covering $(U_j)_{j\in J}$. For $s\in\mathbb{R}$ we define the H^s norm by

$$\|\Psi\|^2_{H^s(E)} := \sum_{j\in J} \|(\xi_j)_*(h_j\Psi)\circ f_j^{-1}\|^2_{H^s(\mathbf{R}_j^{d+1},\mathbb{C}^q)}, \tag{2.2.21}$$

where $\mathbf{R}_j^{d+1} := \mathbb{R}^{d+1}$ when $U_j\cap\partial\mathcal{M}=\emptyset$ and $\mathbf{R}_j^{d+1} := \mathbb{R}^d\times\mathbb{R}^+$ otherwise.

Definition 2.2.7. *Let $s\in\mathbb{R}$. The Sobolev space $H^s(E)$ is the completion of the space $\Gamma_c(E)$ for the H^s norm.*

Remark 2.2.8. The Sobolev spaces defined in this way are a generalization of the H^s spaces in $\mathbb{R}^{d+1}$, and for $k\in\mathbb{N}$, the H^s norm is equivalent to the norm defined by the square root of $\sum_{j=0}^{k}\|(\nabla^E)^j\cdot\|^2$ (see [37, Theorem 5.7], or [36, Remark 3.6]).

A direct consequence of Definition 2.2.7 is that the intrinsic Dirac operator on a compact manifold without boundary is essentially self-adjoint and the domain of its closure is the Sobolev space H^1:

Proposition 2.2.9. *If $(\mathcal{M},g)$ is a compact Riemannian spin manifold without boundary, $\not{D}^{\mathcal{M}}$ is essentially self-adjoint, and the domain of its closure is $H^1(\Sigma\mathcal{M})$.*

Proof. The Dirac operator is symmetric, and then it is closable. By compactness, there exists $C>0$ such that $|\mathrm{Scal}^{\mathcal{M}}|\le C$. Moreover, by the Schrödinger-Lichnerowicz formula (Theorem 2.2.5), the graph norm of $\not{D}^{\mathcal{M}}$ is equivalent to

$$(1+C)\|\cdot\|^2_{L^2(\mathcal{M})} + \|\not{D}^{\mathcal{M}}\cdot\|^2_{L^2(\mathcal{M})} = \left(1+C+\frac{\mathrm{Scal}^{\mathcal{M}}}{4}\right)\|\cdot\|^2_{L^2(\mathcal{M})} + \|\nabla^{\mathcal{M}}\cdot\|^2_{L^2(\mathcal{M})}$$

and this last norm is equivalent to the $H^1(\Sigma\mathcal{M})$-norm because of the boundedness of $\mathrm{Scal}^{\mathcal{M}}$. Then, the domain of the closure of $\not{D}^{\mathcal{M}}$ is the completion of $\Gamma_c(\Sigma\mathcal{M})$ for the graph norm, which is exactly $H^1(\Sigma\mathcal{M})$.

The manifold $(\mathcal{M},g)$ is compact, and then the Dirac operator is essentially self-adjoint in $L^2(\Sigma\mathcal{M})$ [33, Proposition 1.3.5], which concludes the proof. □

By the definition of the Sobolev spaces, one can observe that it is possible to extend the results valid for Euclidean spaces. We state a trace theorem which is a modification of [36, Theorem 3.7], where we add a bound for the L^2-norm of the trace.

Theorem 2.2.10. *Let $(\mathcal{M},g)$ be a compact Riemannian manifold with boundary $\partial\mathcal{M}$. Let $(E,\nabla^E,\langle\cdot,\cdot\rangle_E)$ be an Hermitian vector bundle with base $\mathcal{M}$.*

Then, the pointwise restriction operator $\gamma_{\mathcal{M}}:\Gamma_c(E)\to\Gamma_c(E_{|\partial\mathcal{M}})$ extends to a bounded operator from $H^1(E)$ onto $H^{\frac12}(E_{|\partial\mathcal{M}})$, and there is a bounded right inverse to $\gamma_{\mathcal{M}}:H^1(E)\to H^{\frac12}(E_{|\partial\mathcal{M}})$ denoted by $\epsilon_{\mathcal{M}}$, which maps $\Gamma_c(E_{|\partial\mathcal{M}})$ into $\Gamma_c(E)$. Moreover, there exists $K>0$ such that for any $\varepsilon\in(0,1)$,

$$\|\gamma_{\mathcal{M}}\Psi\|^2_{L^2(\partial\mathcal{M})} \le K\left(\varepsilon^{\frac12}\|\nabla^E\Psi\|^2_{L^2(\mathcal{M})} + \varepsilon^{-\frac12}\|\Psi\|^2_{L^2(\mathcal{M})}\right),\ \Psi\in H^1(E).$$

Proof. The proof of the first part of the theorem is done in [36, Theorem 3.7]. We prove the last estimate.

With the notations of (2.2.21), we denote by J_N the set of all $j \in J$ such that $U_j \cap \partial\mathcal{M} \neq \emptyset$, and there is a constant $C > 0$ and a constant $\widetilde{K} > 0$ given by [35, Theorem 1.5.1.10] such that for any $\varepsilon \in (0,1)$ and for all $\Psi \in H^1(E)$

$$\begin{aligned}
\|\gamma_{\mathcal{M}}\Psi\|^2_{L^2(\partial\mathcal{M})} \leq& C\sum_{j\in J_N}\|(\xi_j)_*(h_j\Psi)\circ f_j^{-1}\|^2_{L^2(\mathbb{R}^d\times\{0\},\mathbb{C}^q)}\\
\leq& C\widetilde{K}\sum_{j\in J}[\varepsilon^{\frac{1}{2}}\|(\xi_j)_*(h_j\Psi)\circ f_j^{-1}\|^2_{H^1(\mathbf{R}_j^{d+1},\mathbb{C}^q)}\\
&+\varepsilon^{-\frac{1}{2}}\|(\xi_j)_*(h_j\Psi)\circ f_j^{-1}\|^2_{L^2(\mathbf{R}_j^{d+1},\mathbb{C}^q)}]\\
=& C\widetilde{K}\left(\varepsilon^{\frac{1}{2}}\|\nabla^E\Psi\|^2_{L^2(\mathcal{M})}+\varepsilon^{-\frac{1}{2}}\|\Psi\|^2_{L^2(\mathcal{M})}\right). \qquad \square
\end{aligned}$$

The Rellich-Kondrachov theorem still holds for the Sobolev spaces on compact manifolds. Consequently, the operators with domain included in the first Sobolev space on a vector bundle with compact base have compact resolvent. We refer to [72, Proposition 3.13] for the following theorem.

Theorem 2.2.11 (Rellich-Kondrachov-type theorem)**.** *Let E be an Hermitian vector bundle over a compact manifold $\mathcal{M}$. Then, the inclusion $H^1(E) \subset L^2(E)$ is compact.*

We end this section with a direct consequence of Proposition 2.2.6. We assume that $(\mathcal{M}, g)$ is a compact Riemannian spin manifold of dimension $d+1$ and we take an oriented hypersurface $\mathcal{H}$ of $\mathcal{M}$. We use the notation of section 2.2.3.

Corollary 2.2.12. *The isomorphism ζ given by Proposition 2.2.6 is an isomorphism between $H^1(\Sigma\mathcal{M}_{|\mathcal{H}})$ and $H^1(\Sigma\mathcal{H})$ if d is even or $H^1(\Sigma\mathcal{H}\oplus\Sigma\mathcal{H})$ if d is odd.*

Proof. We define $\|W_{\mathcal{H}}\|_\infty := \sup_{x\in\mathcal{H}}\sup_{X\in T_x\mathcal{H}\setminus\{0\}}\frac{|g(WX,X)|}{g(X,X)} < \infty$. Let $\Psi \in \Gamma_c(\Sigma\mathcal{M}_{|\mathcal{H}})$ and $(e_1,\dots,e_d)$ a local orthonormal frame of $\mathcal{H}$ at a point $x \in \mathcal{H}$. At this point, one has, using Proposition 2.2.6, (3),

$$\begin{aligned}
|\nabla^{\mathcal{H}}\zeta\Psi|^2 &= \sum_{k=1}^d|\zeta\left(\nabla^{\mathcal{M}}_{e_k}\Psi\right)-\frac{1}{2}W_{\mathcal{H}}e_k\cdot\zeta(\Psi)|^2_{L^2(\mathcal{H})}\\
&\leq 2|\zeta(\overline{\nabla}^{\mathcal{M}}\Psi)|^2_{L^2(\mathcal{H})}+\frac{1}{2}\sum_{k=1}^d|W_{\mathcal{H}}e_k\cdot\nu\cdot\Psi|^2_{L^2(\mathcal{H})}\\
&\leq 2|\overline{\nabla}^{\mathcal{M}}\Psi|^2_{L^2(\mathcal{H})}+\frac{d}{2}\|W_{\mathcal{H}}\|^2_\infty\|\Psi|^2_{L^2(\mathcal{H})}
\end{aligned}$$

and then, by integration we obtain

$$\begin{aligned}
\|\zeta\Psi\|^2_{H^1(\mathcal{H})} &= \|\zeta\Psi\|^2_{L^2(\mathcal{H})}+\|\nabla^{\mathcal{H}}\zeta\Psi\|^2_{L^2(\mathcal{H})}\\
&\leq \|\Psi\|^2_{L^2(\mathcal{H})}+2\|\overline{\nabla}^{\mathcal{M}}\Psi\|^2_{L^2(\mathcal{H})}+\frac{d}{2}\|W_{\mathcal{H}}\|^2_\infty\|\Psi\|^2_{L^2(\mathcal{H})}\\
&\leq C_1\|\Psi\|^2_{H^1(\mathcal{H})},
\end{aligned}$$

where $C_1 > 0$. The same argument shows that there exists $C_2 > 0$ such that for all $\Psi \in \zeta(\Gamma_c(\Sigma\mathcal{M}_{|\mathcal{H}}))$, one has $\|\zeta^{-1}\Psi\|^2_{H^1(\mathcal{H})} \leq C_2\|\Psi\|^2_{H^1(\mathcal{H})}$. $\square$

2.3 Definition of the operators

2.3.1 The generalized MIT Bag Dirac operator

In this section, we would like to give a generalization of the MIT Bag Dirac operator in the context of spin manifolds. Our construction will be done by considering the Riemannian product of a manifold $\mathcal{N}$ with $\mathbb{R}$ and interpreting the operator as the extrinsic Dirac operator on the hypersuface $\mathcal{N} \times \{0\}$, modified by a Clifford multiplication with the normal vector field. Since the hypersurface $\mathcal{N}$ is totally geodesic, this operator is the so-called Dirac-Witten operator (see the remark in the proof of [33, Theorem 5.2.3] for example).

We first introduce the context of our study. Let $n \in \mathbb{N}$ and let $(\mathcal{N}, g)$ be a n-dimensional smooth Riemannian manifold which is spin and complete.

Let $(\mathcal{C}, g_{\mathcal{C}}) := (\mathcal{N}, g) \times (\mathbb{R}, \mathrm{d}t^2)$ be the Riemannian product of $\mathcal{N}$ and $\mathbb{R}$. We identify $\mathcal{N}$ with $\mathcal{N} \times \{0\}$. Let p_1 be the projection on $\mathcal{N}$ in $\mathcal{C}$. We endow $\mathcal{C}$ with a spin structure as follows: we denote by P the pull-back to $\mathcal{C}$ of the bundle $P_{\mathrm{Spin}_n}\mathcal{N}$ by the projection p_1, and then the extension of P to Spin_{n+1} is a spin structure on $\mathcal{C}$ (see [6, Section 5] for example).

We denote by ν the outer unit normal vector field on $\mathcal{N} \times \{0\}$ in $\mathcal{C}$, i.e. the vector field $(0, \frac{\partial}{\partial t})$. By construction, the Weingarten tensor of $\mathcal{N}$ vanishes, so the mean curvature $H_{\mathcal{N}}$ is zero.

We denote by ι be the isomorphism given by in Proposition 2.2.6, in the particular case where $\mathcal{M} := \mathcal{C}$ and $\mathcal{H} := \mathcal{N}$. It is important to remark that the spin structure originally defined on $\mathcal{N}$ and the spin structure inherited by $\mathcal{N}$ from the one of $\mathcal{C}$ according to Proposition 2.2.6 are the same.

Let $\mathcal{K}$ be a submanifold of $\mathcal{N}$ of dimension n, and assume that $\mathcal{K}$ is compact with non-empty boundary $\partial\mathcal{K}$. From these assumptions, we know that $\partial\mathcal{K}$ is oriented. Thus, we denote by

$$\mu : \Sigma\mathcal{N} \to \begin{cases} \Sigma(\partial\mathcal{K}) & \text{if } n \text{ is odd} \\ \Sigma(\partial\mathcal{K}) \oplus \Sigma(\partial\mathcal{K}) & \text{if } n \text{ is even} \end{cases}$$

the isomorphism given by Proposition 2.2.6 and by $\mathbf{n}$ the unit outer normal vector field over $\partial\mathcal{K}$ viewed as a submanifold of $\mathcal{N}$.

The operators $\mathcal{D}^{\mathcal{N}}$, $\not{D}^{\mathcal{N}}$, $\mathcal{D}^{\partial\mathcal{K}}$ and $\not{D}^{\partial\mathcal{K}}$ defined in (2.2.11) and (2.2.18) are essentially self-adjoint [33, Proposition 1.3.5]. We keep the same notation for their closures.

In what follows, we will simply write W for $W_{\partial\mathcal{K}}$ and H for $H_{\partial\mathcal{K}}$.

Let $m \in \mathbb{R}$. To any $\Psi \in \Gamma(\Sigma\mathcal{C}_{|\mathcal{N}})$, we associate an element $\hat{\Psi}_m$ of $\Gamma(\Sigma\mathcal{C})$ defined for $(x,t) \in \mathcal{C}$ by $\hat{\Psi}_m(x,t) = e^{imt}\widetilde{\Psi}(x,t)$ where $\widetilde{\Psi}(x,t)$ is obtained by parallel transport of $\Psi(x)$ along the curves $s \mapsto (x,s)$.

Let $(e_1, \dots, e_n)$ be a local orthonormal frame at $x \in \mathcal{N}$. Then, we compute

$$\begin{aligned}(\not{D}^{\mathcal{C}}\hat{\Psi}_m)(x) &= \left(\sum_{j=1}^{n} e_j \cdot \nabla^{\mathcal{C}}_{e_j}\hat{\Psi}_m + i\,m\nu \cdot \hat{\Psi}_m\right)(x,0) \\ &= \left(-\sum_{j=1}^{n} \nu \cdot \nu \cdot e_j \cdot \nabla^{\mathcal{C}}_{e_j}\Psi\right)(x) + i\,m\nu \cdot \Psi(x) \\ &= \nu \cdot \left(\mathcal{D}^{\mathcal{N}} + im\right)\Psi(x),\end{aligned}$$

where the extrinsic Dirac operator $\mathcal{D}^{\mathcal{N}}$ is the operator given by the expression (2.2.19). The operator obtained in the last line is precisely the operator that we want to study, as it can be interpreted as a Dirac operator with a mass.

We remark that the above construction can be done by restricting the domain of the operator to $\mathcal{K}$. We thus introduce the generalized MIT Bag operator

$$\widetilde{A}_m := \nu \cdot \left(\mathcal{D}^{\mathcal{N}} + im\right), \ \mathrm{dom}(\widetilde{A}_m) := \left\{\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}}), i\,\nu \cdot \mathbf{n} \cdot \Psi = \Psi \text{ on } \partial\mathcal{K}\right\}. \tag{2.3.1}$$

Remark 2.3.1. One can observe that in the case of Euclidean spaces, the expression (2.3.1) coincides with [59, Equation (1)], which is already a generalization of the MIT Bag Dirac operator in dimension 3 (see [4, Equation 1.1]). Indeed, the only difference comes from the convention on the Clifford multiplication, because in the present text we have the identity $X \cdot X = -|X|^2$.

Remark 2.3.2. It is easily seen that the operator $\widetilde{A}_m$ is symmetric since ν anti-commutes with $\mathcal{D}^{\mathcal{N}}$ (see [39, Proposition 1] for the general case, or simply remark that ν is parallel in our framework). Since symmetric operators are closable, we denote by A_m its closure.

Actually, the boundary condition imposed in the domain of the operator is not the Lorentzian MIT Bag boundary condition as stated by the physicists [40] because of the Clifford multiplication by ν. However, this is consistent with the boundary conditions imposed in [5], [4] and [59]. To understand this, we can give another interpretation of the operator $\widetilde{A}_m$ which seems more physical, and appears to give a unitarily equivalent operator.

Until the end of this section, we will deal with Clifford algebra and spin structures in the Lorentzian case. We refer to [6, section 2] for a detailed presentation.

One can endow $\mathcal{C}$ with the Lorentzian metric $g - \mathrm{d}t^2$. There is a Spin_0-structure over $\mathcal{C}$ given by the pull-back of the Spin-structure on $\mathcal{N}$ and extending the fiber. One can construct the associated spinor bundle $\Sigma_L\mathcal{C}$, whose Clifford multiplication will be denoted by " $\cdot_L$". Moreover, we write ∇^L for the covariant derivative on $\Sigma_L\mathcal{C}$, and we denote by $\langle\cdot,\cdot\rangle_L$ the Hermitian product on this spinor bundle. We recall that this inner product is not necessarily definite. In this framework, the Dirac operator with a mass on $\Sigma_L\mathcal{C}$ admits the pointwise expression

$$\not{D}_L^{\mathcal{C}}\Psi := i\left(-\nu \cdot_L \nabla^L_\nu \Psi + \sum_{j=1}^n e_j \cdot_L \nabla^L_{e_j}\Psi\right) - m\Psi \tag{2.3.2}$$

where $(e_1, \ldots, e_n)$ is any orthonormal frame on $\mathcal{N}$ (see [6, section 2]). Consequently, the Dirac equation $\not{D}_L^{\mathcal{C}}\Psi = 0$ is equivalent to

$$i\,\nabla^L_\nu\Psi = i\sum_{j=1}^n \nu \cdot_L e_j \cdot_L \nabla^L_{e_j}\Psi - m\,\nu \cdot_L \Psi. \tag{2.3.3}$$

Now, if we take $\Psi(x,t) = e^{i\omega t}\phi(x)$ for all $(x,t) \in \mathcal{C}$, where ϕ is parallel along the time lines, we arrive at

$$\omega\phi = -i\sum_{j=1}^n \nu \cdot_L e_j \cdot_L \nabla^L_{e_j}\phi + m\,\nu \cdot_L \phi. \tag{2.3.4}$$

We have the counterpart of Proposition 2.2.6 for the Lorentzian case. Namely, the spinor bundle $\Sigma_L\mathcal{C}$ can be identified to one or two copies of $\Sigma\mathcal{N}$ as in the Riemannian case.

Proposition 2.3.3. *There is an isomorphism ι_L from $\Sigma_L\mathcal{C}_{|\mathcal{N}}$ into $\Sigma\mathcal{N}$ if n is even and into $\Sigma\mathcal{N} \oplus \Sigma\mathcal{N}$ if n is odd such that:*

- $\iota_L(-i\, X \cdot_L \nu \cdot_L \Psi) = X \cdot \iota_L \Psi$ *for all* $X \in T\mathcal{N}$ *and* $\Psi \in \Sigma_L \mathcal{C}$,
- $\iota_L \nu \cdot_L = \omega_n^{\mathbb{C}} \cdot \iota_L$ *when* n *is even, and* $\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$ *when* n *is odd.*
- $\langle \iota_L \Psi, \iota_L \Phi \rangle = \langle \Psi, \nu \cdot_L \Phi \rangle_L$ *for all* $\Phi, \Psi \in \Sigma_L \mathcal{C}_{|\mathcal{N}}$,
- $\iota_L \nabla^L_X \Psi = \nabla^{\mathcal{N}}_X \iota_L \Psi$ *for* $X \in T\mathcal{N}$ *and* $\Psi \in \Sigma_L \mathcal{C}_{|\mathcal{N}}$.

Proof. We recall that the notations for Clifford algebras were introduced in Section 2.2.2. Consider the space $\mathbb{R}^{n,1}$ endowed with the Lorentzian quadratic form of signature $(n, 1)$ and let $(e_1, \dots, e_{n+1})$ be the canonical basis of $\mathbb{R}^{n,1}$, so that e_{n+1} is timelike. The Clifford algebra over this Lorentzian space is denoted by $\mathbb{C}\mathrm{l}_{n,1}$. We turn the representation $(\rho_{n+1}, \Sigma_{n+1})$ into a complex representation of $\mathbb{C}\mathrm{l}_{n,1}$ $(\rho_{n,1}, \Sigma_{n+1})$ by setting

$$\rho_{n,1}(e_i) := \rho_{n+1}(e_i) \text{ for } 1 \le i \le n, \text{ and } \rho_{n,1}(e_{n+1}) := i\, \rho_{n+1}(e_{n+1}).$$

We remark that when n is even, $i^{\frac{n}{2}} \rho_{n,1}(e_1 \cdot \dots \cdot e_{n+1})$ acts as the identity.

Following [6, section 2], the Hermitian product $\langle \cdot, \cdot \rangle_L$ on Σ_{n+1} for the Lorentzian structure is defined for all $\psi, \phi \in \Sigma_{n+1}$ by

$$\langle \psi, \phi \rangle_L := \langle \psi, \rho_{n,1}(e_{n+1}) \phi \rangle$$

where $\langle \cdot, \cdot \rangle$ is the natural Spin_{n+1}-invariant Hermitian product on Σ_{n+1}.

One can define a representation ρ of $\mathbb{C}\mathrm{l}_n$ over the space Σ_{n+1} by

$$\rho(x) = -i\, \rho_{n,1}(x \cdot e_{n+1}) \quad \text{for all } x \in \mathbb{R}^n.$$

For n even, this representation is equivalent to (ρ_n, Σ_n), so we have an isomorphism $U : \Sigma_{n+1} \to \Sigma_n$ such that $\rho_n U = U \rho$. Moreover, since $i^{\frac{n}{2}} \rho_{n,1}(e_1 \cdot \dots \cdot e_{n+1})$ acts as the identity on Σ_{n+1}, an easy computation gives $U \rho_{n,1}(e_{n+1}) U^{-1} = \rho_n(\omega_n^{\mathbb{C}})$.

We still denote by $\langle \cdot, \cdot \rangle$ the Hermitian product on Σ_n and we remark that U can be chosen unitary for this inner product. Thus, for all $\psi, \phi \in \Sigma_{n+1}$ one has

$$\langle U\psi, U\phi \rangle = \langle \psi, \phi \rangle = \langle \psi, \rho_{n,1}(e_{n+1})^2 \phi \rangle = \langle \psi, \rho_{n,1}(e_{n+1}) \phi \rangle_L.$$

For n odd, the restriction of ρ to Σ^+_{n+1} is equivalent to (ρ_n, Σ_n), so we have an isomorphism $U_0 : \Sigma^+_{n+1} \to \Sigma_n$ such that $\rho_n U_0 = U_0 \rho$. In addition, $\rho_{n,1}(e_{n+1})$ is an isomorphism from $\Sigma^\pm_{n+1}$ into $\Sigma^\mp_{n+1}$, so we set

$$U : \Sigma_{n+1} = \Sigma^+_{n+1} \oplus \Sigma^-_{n+1} \to \Sigma_n \oplus \Sigma_n, \; U := (U_0 \oplus U_0)(\mathrm{Id} \oplus \rho_{n,1}(e_{n+1})).$$

Easy computations give $U \rho(x) U^{-1} = \rho_n(x) \oplus -\rho_n(x)$ for all $x \in \mathbb{R}^n \subset \mathbb{R}^{n+1}$ and $U \rho_{n,1}(x) U^{-1}(\psi_1, \psi_2) = (\psi_2, \psi_1)$ for all $(\psi_1, \psi_2) \in \Sigma_n \oplus \Sigma_n$.

The Hermitian product on Σ_n extends to $\Sigma_n \oplus \Sigma_n$ and this extension is still denoted by $\langle \cdot, \cdot \rangle$. The isomorphism U can be chosen unitary for this inner product, and one has for all $\psi, \phi \in \Sigma_{n+1}$

$$\langle U\psi, U\phi \rangle = \langle \psi, \phi \rangle = \langle \psi, \rho^2_{n,1} \phi \rangle = \langle \psi, \rho_{n,1} \phi \rangle_L.$$

Now, all these properties transport to manifolds by identifying e_{n+1} with ν since the Spin_0 structure over $\mathcal{C}$ is defined by pull-back of the Spin structure over $\mathcal{N}$.

The last point follows from the explicit formula of the covariant derivative on spinor [6, formula 2.5] and the fact that $\mathcal{N}$ is totally geodesic in $\mathcal{C}$. □

We infer that $\Sigma\mathcal{C}_{|\mathcal{N}}$ and $\Sigma_L\mathcal{C}_{|\mathcal{N}}$ are both isomorphic to $\Sigma\mathcal{N}$ if n is even and to $\Sigma\mathcal{N}\oplus\Sigma\mathcal{N}$ if n is odd, so we can identify them via the isomorphism $\iota^{-1}\iota_L$.

Corollary 2.3.4. *The isomorphism $\iota^{-1}\iota_L : \Sigma_L\mathcal{C} \to \Sigma\mathcal{C}$ satisfies:*

- $\langle(\iota^{-1}\iota_L)\Psi, i\,\nu\cdot(\iota^{-1}\iota_L)\Phi\rangle = \langle\Psi,\Phi\rangle_L$ *for all* $\Psi,\Phi\in\Sigma_L\mathcal{C}$.
- $\nabla^{\mathcal{C}}_X(\iota^{-1}\iota_L)\Psi = (\iota^{-1}\iota_L)\nabla^L_X\Psi$ *for all* $X\in T\mathcal{N}$ *and* $\Psi\in\Gamma(\Sigma_L\mathcal{C})$.
- $X\cdot(\iota^{-1}\iota_L)\Psi = (\iota^{-1}\iota_L)(X\cdot_L\Psi)$ *for all* $X\in T\mathcal{N}$
- $i\,\nu\cdot(\iota^{-1}\iota_L) = (\iota^{-1}\iota_L)\nu\cdot_L$.

Under the identification of Corollary 2.3.4, Equation (2.3.4) reads

$$\omega\phi = \sum_{j=1}^{n}\nu\cdot e_j\cdot\nabla^{\mathcal{C}}_{e_j}\phi + i\,m\,\nu\cdot\phi = (-\mathcal{D}^{\mathcal{N}} + i\,m\,\nu\cdot)\phi. \tag{2.3.5}$$

This is an eigenvalue equation, and it is now natural to look at the spectrum of the operator defined by the right-hand side. We just need to add a boundary condition to define a generalized MIT Bag operator. Since the physical condition imposed in [40] is that the flux $\langle\phi,\mathbf{n}\cdot_L\phi\rangle_L$ of the field vanishes at the boundary, we consider the MIT Bag boundary condition $i\,\mathbf{n}\cdot\phi=\phi$. One has

$$-\langle\phi,\phi\rangle_L = \langle\phi,-i\,\mathbf{n}\cdot_L\phi\rangle_L = \langle i\,\mathbf{n}\cdot_L\phi,\phi\rangle_L = \langle\phi,\phi\rangle_L,$$

and we conclude that $\langle\phi,-i\,\mathbf{n}\cdot_L\phi\rangle_L = 0$, so the condition of the physical model is verified. We can now define another generalization of the MIT Bag Dirac operator by

$$\widehat{A}_m := \mathcal{D}^{\mathcal{N}} + i\,m\,\nu\cdot,\quad \mathrm{dom}(\widehat{A}_m) = \left\{\Psi\in\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}}), i\,\mathbf{n}\cdot\Psi=\Psi\right\}. \tag{2.3.6}$$

The change of sign for the mass in (2.3.6) compared to (2.3.5) comes from the fact that we consider a model where $m\to-\infty$ (see [5, section 1.3.3] for more explanations).

We have now two candidates for the generalization of the MIT Bag Dirac operator. However, one can remark that the difference between $\widetilde{A}_m$ and $\widehat{A}_m$ is only a matter of how the Clifford product is defined, and the two operators are unitarily equivalent.

Proposition 2.3.5. *The operators $\widetilde{A}_m$ and $\widehat{A}_m$ are unitarily equivalent via a $\nabla^{\mathcal{C}}$-parallel operator.*

Proof. We define a new Clifford representation on the vector bundle $\Sigma\mathcal{C}$ by setting $X*\Psi := \nu\cdot X\cdot\Psi$ and $\nu*\Psi := \nu\cdot\Psi$ for $X\in T\mathcal{N}$ and $\Psi\in\Sigma\mathcal{C}$. This new product still satisfies the Clifford conditions in each fiber, and when n is even the complex volume form $\omega^{\mathbb{C}}_{n+1}$ acts as

$$\begin{aligned}\omega^{\mathbb{C}}_{n+1}*\Psi &= i^{\lfloor\frac{n+2}{2}\rfloor}e_1*\ldots*e_n*\nu*\Psi\\ &= i^{\lfloor\frac{n+2}{2}\rfloor}(\nu\cdot e_1)\cdot\ldots(\nu\cdot e_n)\cdot\nu\cdot\Psi = \omega^{\mathbb{C}}_{n+1}\cdot\Psi,\end{aligned}$$

where $(e_1,\ldots,e_n)$ is a direct orthonormal basis of $T\mathcal{N}$. It follows by the general theory of Clifford representations that there is a unitary isomorphism $U:\Sigma\mathcal{C}\to\Sigma\mathcal{C}$ such that $X\cdot U\Psi = U(X*\Psi)$ for all $X\in T\mathcal{C}$ and $\Psi\in\Sigma\mathcal{C}$.

Actually, one can give such an isomorphism explicitly. If n is even, we use the decomposition $\Sigma\mathcal{N} = \Sigma^+\mathcal{N} \oplus \Sigma^-\mathcal{N}$ (see [15, Proposition 1.32]) and the pointwise identification $\Sigma\mathcal{C}_{|(x,t)} \cong \Sigma\mathcal{N}_{|x}$ for all $(x,t) \in \mathcal{C}$ given by Proposition 2.2.6. Under this identification, one has

$$\nu \cdot (\Psi^+, \Psi^-) = (-i\,\Psi^+, i\,\Psi^-),\ X \cdot (\Psi^+, \Psi^-) = i\,(-X \cdot \Psi^-, X \cdot \Psi^+) \text{ for all } X \in T\mathcal{N},$$

and we deduce that U can be defined by

$$U(\Psi^+, \Psi^-) := (\Psi^+, -i\,\Psi^-).$$

Indeed, one has for any $X \in T\mathcal{N}$

$$\begin{aligned} U(X * (\Psi^+, \Psi^-)) &= U(\nu \cdot X \cdot (\Psi^+, \Psi^-)) = U(i\,\nu \cdot (-X \cdot \Psi^-, X \cdot \Psi^+)) \\ &= -U(X \cdot \Psi^-, X \cdot \Psi^+) = (-X \cdot \Psi^-, i\,X \cdot \Psi^+) \end{aligned}$$

and

$$X \cdot U(\Psi^+, \Psi^-) = X \cdot (\Psi^+, -i\,\Psi^-) = (-X \cdot \Psi^-, i\,X \cdot \Psi^+),$$

thus $U(X * (\Psi^+, \Psi^-)) = X \cdot U(\Psi^+, \Psi^-)$. In addition, U obviously commutes with ν.
In the case where n is odd, one has the pointwise identification $\Sigma\mathcal{C}_{|(x,t)} \cong \Sigma\mathcal{N}_{|x} \oplus \Sigma\mathcal{N}_{|x}$ for all $(x,t) \in \mathcal{C}$ and under this identification,

$$\nu \cdot (\Psi_1, \Psi_2) = (-i\,\Psi_2, -i\,\Psi_1),\ X \cdot (\Psi_1, \Psi_2) = i\,(X \cdot \Psi_2, -X \cdot \Psi_1) \text{ for all } X \in T\mathcal{N},$$

It follows that U can be defined by

$$U(\Psi_1, \Psi_2) := \frac{1}{\sqrt{2}}(\Psi_1 + i\,\Psi_2, i\,\Psi_1 + \Psi_2).$$

Indeed, for all $X \in T\mathcal{N}$ one has

$$\begin{aligned} U(X * (\Psi_1, \Psi_2)) &= i\,U(\nu \cdot (X \cdot \Psi_2, -X \cdot \Psi_1)) = U(-X \cdot \Psi_1, X \cdot \Psi_2) \\ &= \frac{1}{\sqrt{2}}(X \cdot (-\Psi_1 + i\,\Psi_2), X \cdot (-i\,\Psi_1 + \Psi_2)) \end{aligned}$$

and

$$\begin{aligned} X \cdot U(\Psi_1, \Psi_2) &= \frac{1}{\sqrt{2}} X \cdot (\Psi_1 + i\,\Psi_2, i\,\Psi_1 + \Psi_2) \\ &= \frac{1}{\sqrt{2}}(X \cdot (-\Psi_1 + i\,\Psi_2), X \cdot (-i\,\Psi_1 + \Psi_2)), \end{aligned}$$

thus $X \cdot U(\Psi_1, \Psi_2) = U(X * (\Psi_1, \Psi_2))$. Again, ν commutes with U.
In both cases, U is parallel with respect to $\nabla^{\mathcal{C}}$ and we remark that $U(\mathrm{dom}(\widetilde{A}_m)) = \mathrm{dom}(\widehat{A}_m)$.
We deduce from these considerations that

$$U^* \widehat{A}_m U \Psi = \widetilde{A}_m \Psi \qquad \text{for all } \Psi \in \mathrm{dom}(\widetilde{A}_m), \tag{2.3.7}$$

which is the statement we wanted to prove. □

Remark 2.3.6. The key point in Proposition 2.3.5 is of course that $H_{\mathcal{N}} = 0$. It is only under this condition that the isomorphism U is parallel with respect to $\nabla^{\mathcal{C}}$. Thus, it is equivalent to study any of the two operators, but we wanted to insist on the physical meaning of $\widehat{A}_m$.

2.3.2 The two-masses Dirac operator

We introduce now an operator that can be interpreted as a Dirac operator on $\mathcal{N}$ with two masses in the two separated regions $\mathcal{K}$ and $\mathcal{K}^c$. The interest of this operator, as we will show later, is that when the mass in $\mathcal{K}^c$ goes to infinity, its spectrum converges to the spectrum of the MIT Bag Dirac operator.

Let $m, M \in \mathbb{R}$. We define the operator $\widetilde{B}_{m,M}$ by

$$\widetilde{B}_{m,M} := \nu \cdot \mathcal{D}^{\mathcal{N}} + i\,(m\mathbf{1}_{\mathcal{K}} + M\mathbf{1}_{\mathcal{K}^c})\nu\cdot,\ \mathrm{dom}(\widetilde{B}_{m,M}) := \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}}). \tag{2.3.8}$$

Since the Clifford multiplication by ν is an endomorphism of $\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$, the range of this operator is included in $\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$.

Until the end of this subsection, we make a differentiation between the Dirac operators on complete manifolds and their closures.

The operator $\widetilde{B}_{m,M}$ is symmetric because ν anti-commutes with $\mathcal{D}^{\mathcal{N}}$ [39, Proposition 1] and by Corollary 2.4.2 below. Since the manifold $\mathcal{N}$ is complete by assumption, the intrinsic Dirac operator on $\mathcal{N}$ is essentially self-adjoint in $L^2(\Sigma\mathcal{C}_{|\mathcal{N}})$ [33, Proposition 1.3.5]. Moreover, (2.2.18) gives that $\mathcal{D}^{\mathcal{N}}$ is unitarily equivalent to $\not{D}^{\mathcal{N}}$ if n is even and $\not{D}^{\mathcal{N}} \oplus -\not{D}^{\mathcal{N}}$ if n is odd, and the isomorphism ι sends $\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$ into $\Gamma_c(\Sigma\mathcal{N})$. Thus, $\mathcal{D}^{\mathcal{N}}$ is essentially self-adjoint, and it is easy to see that its closure still anti-commutes with ν. Using the fact that the Clifford multiplication by ν is a unitary isomorphism in $L^2(\Sigma\mathcal{C}_{|\mathcal{N}})$ we have

$$(\nu \cdot \overline{\mathcal{D}^{\mathcal{N}}})^* = -\overline{\mathcal{D}^{\mathcal{N}}}\nu\cdot = \nu \cdot \overline{\mathcal{D}^{\mathcal{N}}}, \quad \text{and} \quad \overline{\nu \cdot \mathcal{D}^{\mathcal{N}}} = \nu \cdot \overline{\mathcal{D}^{\mathcal{N}}},$$

so $\overline{\nu \cdot \mathcal{D}^{\mathcal{N}}}$ is self-adjoint.

We conclude that $\widetilde{B}_{m,M}$ is essentially self-adjoint because the potential is a bounded self-adjoint operator. We define the self-adjoint operator $B_{m,M}$ as the closure of $\widetilde{B}_{m,M}$.

2.4 Sesquilinear forms for the operators with mass

An important tool for the asymptotic analysis will be the sesquilinear forms associated with the square of the operators. We begin this section by recalling some useful formulas involving the Dirac operator. After that, we compute the sesquilinear forms for the operators A_m^2 and $B_{m,M}^2$ and we show that A_m is self-adjoint. We end this section with the study of a model operator which appears naturally in the asymptotic analysis, and we prove that it is unitarily equivalent to the square of the Dirac operator on $\partial\mathcal{K}$.

2.4.1 Integration by parts with the Dirac operator

We first recall the well-known result:

Lemma 2.4.1. *Let $\Psi, \Phi \in \Gamma_c(\Sigma\mathcal{N})$. Then, one has the pointwise equality*

$$\langle \not{D}^{\mathcal{N}}\Psi, \Phi\rangle = -\mathrm{div}\, V + \langle \Psi, \not{D}^{\mathcal{N}}\Phi\rangle$$

where V is the complex vector field on $\mathcal{N}$ defined by

$$g(V, X) := \langle \Psi, X \cdot \Phi\rangle\,,\ \forall X \in T\mathcal{N}.$$

Proof. Let $\Psi, \Phi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$, $x \in \mathcal{N}$ and let $(e_1, \ldots, e_n)$ be a normal coordinate system at x for $\nabla^{\mathcal{N}}$, i.e. $\nabla^{\mathcal{N}}_{e_i} e_j(x) = 0$ for all $i, j \in \{1, \ldots, n\}$. One has at x,

$$\left\langle \not{D}^{\mathcal{N}}\Psi, \Phi \right\rangle = \langle \sum_{j=1}^{n} e_j \cdot \nabla^{\mathcal{N}}_{e_j}\Psi, \Phi\rangle.$$

On the other hand, for all $j \in \{1, \ldots, n\}$,

$$\begin{aligned}\left\langle e_j \cdot \nabla^{\mathcal{N}}_{e_j}\Psi, \Phi \right\rangle &= -\left\langle \nabla^{\mathcal{N}}_{e_j}\Psi, e_j \cdot \Phi \right\rangle \\ &= -e_j \left\langle \Psi, e_j \cdot \Phi\right\rangle + \left\langle \Psi, \nabla^{\mathcal{N}}_{e_j}(e_j \cdot \Phi) \right\rangle.\end{aligned}$$

Thus, $\langle \not{D}^{\mathcal{N}}\Psi, \Phi\rangle = -\sum_{j=1}^{n} e_j \left\langle \Psi, e_j \cdot \Phi\right\rangle + \langle \Psi, \not{D}^{\mathcal{N}}\Psi\rangle$. We recognize in the first term of this last sum the divergence of a complex vector field. To see this, we introduce $V \in \Gamma(T\mathcal{N})$ as in the statement of the lemma. Then, we have at the point x

$$\begin{aligned}\operatorname{div} V &= \sum_{j=1}^{n} g(\nabla^{\mathcal{N}}_{e_j} V, e_j) = \sum_{j=1}^{n} e_j\, g(V, e_j) - g(V, \nabla^{\mathcal{N}}_{e_j} e_j) \\ &= \sum_{j=1}^{n} e_j\, g(V, e_j) = \sum_{j=1}^{n} e_j \left\langle \Psi, e_j \cdot \Psi\right\rangle.\end{aligned}$$ □

A direct corollary is an integral version of Lemma 2.4.1.

Corollary 2.4.2. *One has*

$$\langle \not{D}^{\mathcal{N}}\Psi, \Phi\rangle_{L^2(\mathcal{K})} = \langle \Psi, \not{D}^{\mathcal{N}}\Phi\rangle_{L^2(\mathcal{K})} - \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n} \cdot \Phi\rangle v_{\partial\mathcal{K}}$$

for all $\Psi, \Phi \in H^1(\Sigma\mathcal{K})$, *and*

$$\langle \mathcal{D}^{\mathcal{N}}\Psi, \Phi\rangle_{L^2(\mathcal{K})} = \langle \Psi, \mathcal{D}^{\mathcal{N}}\Phi\rangle_{L^2(\mathcal{K})} - \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n} \cdot \nu \cdot \Phi\rangle v_{\partial\mathcal{K}}$$

for all $\Psi, \Phi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$.

Proof. The first identity is proved by integrating the formula obtained in Lemma 2.4.1 for $\Psi, \Phi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}})$ and using the divergence theorem. We conclude by density. For the second one, we use the definition of the extrinsic Dirac operator given by (2.2.18) together with the first equation. □

Finally, we obtain an integration by parts formula for the Dirac operator with a mass defined in the previous section.

Corollary 2.4.3. *For any* $\Psi, \Phi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$, *one has*

$$\left\langle \nu \cdot (\mathcal{D}^{\mathcal{N}} + i\,m)\Psi, \Phi \right\rangle_{L^2(\mathcal{K})} = \left\langle \Psi, \nu \cdot (\mathcal{D}^{\mathcal{N}} + i\,m)\Phi \right\rangle_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \left\langle \Psi, \mathbf{n} \cdot \Phi\right\rangle v_{\partial\mathcal{K}}.$$

Proof. Let $\Psi, \Phi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$, using Corollary 2.4.2 one has

$$\begin{aligned}
\left\langle \nu \cdot (\mathcal{D}^{\mathcal{N}} + i\, m)\Psi, \Phi \right\rangle_{L^2(\mathcal{K})} &= - \left\langle (\mathcal{D}^{\mathcal{N}} + i\, m)\Psi, \nu \cdot \Phi \right\rangle_{L^2(\mathcal{K})} \\
&= - \left\langle \Psi, (\mathcal{D}^{\mathcal{N}} - i\, m)(\nu \cdot \Phi) \right\rangle_{L^2(\mathcal{K})} \\
&\quad - \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n} \cdot \nu \cdot \nu \cdot \Phi \rangle \, v_{\partial\mathcal{K}} \\
&= \left\langle \Psi, \nu \cdot (\mathcal{D}^{\mathcal{N}} + i\, m)\Phi \right\rangle_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \langle \Psi, \mathbf{n} \cdot \Phi \rangle v_{\partial\mathcal{K}}. \qquad \square
\end{aligned}$$

2.4.2 Sesquilinear form for $\widetilde{A}_m^2$ and essential self-adjointness

In this section we show that the operator $\widetilde{A}_m$ is essentially self-adjoint, and the domain of its closure is an extension of $\mathrm{dom}(\widetilde{A}_m)$ to the space $H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$. The proof of this fact is done in two steps. First, we compute the sesquilinear form of $\widetilde{A}_m^2$ to get the domain of the closure and secondly, we show the essential self-adjointness following the analysis of [36].

From Corollary 2.4.3, we see that $\widetilde{A}_m$ is symmetric since for any $\Psi, \Phi \in \mathrm{dom}(\widetilde{A}_m)$ one has

$$\langle \Psi, \mathbf{n} \cdot \Phi \rangle = \langle \Psi, i\, \nu \cdot \Phi \rangle = \langle i\, \nu \cdot \Psi, \Phi \rangle = \langle \mathbf{n} \cdot \Psi, \Phi \rangle = -\langle \Psi, \mathbf{n} \cdot \Phi \rangle = 0.$$

Proposition 2.4.4. *For all* $\Psi \in \mathrm{dom}(\widetilde{A}_m)$,

$$\begin{aligned}
\|\widetilde{A}_m \Psi\|^2_{L^2(\mathcal{K})} = &\int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4} |\Psi|^2 \right) v_{\mathcal{N}} \\
&+ m^2 \|\Psi\|^2_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \left(m - \frac{H}{2} \right) |\Psi|^2 v_{\partial\mathcal{K}}.
\end{aligned}$$

Moreover, the graph norm of $\widetilde{A}_m$ *and the* H^1*-norm are equivalent on* $\mathrm{dom}(\widetilde{A}_m)$.

Proof. We recall that $\mathrm{dom}(\widetilde{A}_m)$ was defined in (2.3.1). Let $\Psi \in \mathrm{dom}(\widetilde{A}_m)$. With Corollary 2.4.2 one has

$$\begin{aligned}
\|\widetilde{A}_m \Psi\|^2_{L^2(\mathcal{K})} &= \left\langle (\mathcal{D}^{\mathcal{N}} + i\, m)\Psi, (\mathcal{D}^{\mathcal{N}} + i\, m)\Psi \right\rangle_{L^2(\mathcal{K})} \\
&= \|\mathcal{D}^{\mathcal{N}}\Psi\|^2_{L^2(\mathcal{K})} + m^2 \|\Psi\|^2_{L^2(\mathcal{K})} + m \left\langle \mathcal{D}^{\mathcal{N}}\Psi, i\, \Psi \right\rangle_{L^2(\mathcal{K})} \\
&\quad + m \left\langle i\, \Psi, \mathcal{D}^{\mathcal{N}}\Psi \right\rangle_{L^2(\mathcal{K})} \\
&= \|\mathcal{D}^{\mathcal{N}}\Psi\|^2_{L^2(\mathcal{K})} + m^2 \|\Psi\|^2_{L^2(\mathcal{K})} - m \int_{\partial\mathcal{K}} \langle \Psi, i\, \mathbf{n} \cdot \nu \cdot \Psi \rangle \, v_{\partial\mathcal{K}} \\
&= \|\mathcal{D}^{\mathcal{N}}\Psi\|^2_{L^2(\mathcal{K})} + m^2 \|\Psi\|^2_{L^2(\mathcal{K})} + m \int_{\partial\mathcal{K}} |\Psi|^2 v_{\partial\mathcal{K}},
\end{aligned}$$

where we used the property $\Psi = i\, \nu \cdot \mathbf{n} \cdot \Psi$ on $\partial\mathcal{K}$.

We consider the operator $\widetilde{\mathcal{D}}^{\partial\mathcal{K}} := \mathcal{D}^{\partial\mathcal{K}}$ if n is even and $\widetilde{\mathcal{D}}^{\partial\mathcal{K}} := \mathcal{D}^{\partial\mathcal{K}} \oplus \mathcal{D}^{\partial\mathcal{K}}$ if n is odd. From [39, Formula (13)] we have for all $\Phi \in \Gamma(\Sigma\mathcal{K})$

$$\begin{aligned}
\int_{\mathcal{K}} |\not{D}^{\mathcal{N}}\Phi|^2 v_{\mathcal{N}} = &\int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}\Phi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4} |\Phi|^2 \right) v_{\mathcal{N}} \\
&+ \int_{\partial\mathcal{K}} \left(-\frac{H}{2} |\Phi|^2 - \langle \mathcal{D}^{\partial\mathcal{K}}\Phi, \Phi \rangle \right) v_{\partial\mathcal{K}}.
\end{aligned}$$

Using this equation together with the definition of the extrinsic Dirac operator (2.2.18), one has

$$\begin{aligned}\int_{\mathcal{K}} |\mathcal{D}^{\mathcal{N}}\Psi|^2 v_{\mathcal{N}} = &\int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 \right) v_{\mathcal{N}} \\ &+ \int_{\partial\mathcal{K}} \left(-\frac{H}{2}|\Psi|^2 + \left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle \right) v_{\partial\mathcal{K}}.\end{aligned} \tag{2.4.1}$$

On the other hand, as $\widetilde{\mathcal{D}}^{\partial\mathcal{K}}$ anti-commutes with the Clifford multiplication by $\mathbf{n}$ [39, Proposition 1],

$$\begin{aligned}\left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle &= \left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota(-i\,\mathbf{n}\cdot\nu\cdot\Psi)), \iota\Psi \right\rangle = \left\langle -i\,\widetilde{\mathcal{D}}^{\partial\mathcal{K}}\mathbf{n}\cdot(\iota\Psi), \iota\Psi \right\rangle \\ &= \left\langle i\,\mathbf{n}\cdot\widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle = \left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), i\,\mathbf{n}\cdot(\iota\Psi) \right\rangle \\ &= \left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), -\iota(i\,\nu\cdot\mathbf{n}\cdot\Psi) \right\rangle = -\left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle\end{aligned}$$

and we deduce that $\left\langle \widetilde{\mathcal{D}}^{\partial\mathcal{K}}(\iota\Psi), \iota\Psi \right\rangle = 0$.

Finally, using this equation together with (2.4.1), we get

$$\begin{aligned}\|\widetilde{A}_m\Psi\|^2_{L^2(\mathcal{K})} = &\int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 \right) v_{\mathcal{N}} \\ &+ m^2\|\Psi\|^2_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \left(m - \frac{H}{2} \right) |\Psi|^2 v_{\partial\mathcal{K}}.\end{aligned}$$

It remains to prove the equivalence of the norms. As $\mathcal{K}$ is a compact manifold with boundary, Theorem 2.2.10 applies and there is $C_1 > 0$ such that for all $\Psi \in \mathrm{dom}(\widetilde{A}_m)$,

$$\begin{aligned}\|\Psi\|^2_{L^2(\mathcal{K})} + \|\widetilde{A}_m\Psi\|^2_{L^2(\mathcal{K})} =& \|\iota\Psi\|^2_{L^2(\mathcal{K})} + \int_{\mathcal{K}} \left(|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\iota\Psi|^2 \right) v_{\mathcal{N}} \\ &+ m^2\|\iota\Psi\|^2_{L^2(\mathcal{K})} + \int_{\partial\mathcal{K}} \left(m - \frac{H}{2} \right) |\iota\Psi|^2 v_{\partial\mathcal{K}} \\ \leq& C_1\|\iota\Psi\|^2_{L^2(\mathcal{K})} + \|\nabla^{\mathcal{N}}(\iota\Psi)\|^2_{L^2(\mathcal{K})} + C_1\|\iota\Psi\|^2_{H^1(\mathcal{K})} \\ \leq& 2(C_1+1)\|\iota\Psi\|^2_{H^1(\mathcal{K})}.\end{aligned}$$

Moreover, using Theorem 2.2.10 with ε small enough, there exists a constant $C_2 > 0$ such that

$$\|\Psi\|^2_{L^2(\mathcal{K})} + \|\widetilde{A}_m\Psi\|^2_{L^2(\mathcal{K})} \geq C_2\|\iota\Psi\|^2_{H^1(\mathcal{K})}.$$

Thus, the graph norm is equivalent to the $H^1(\iota(\Sigma\mathcal{C}_{|\mathcal{K}}))$ norm, which is equivalent to the $H^1(\Sigma\mathcal{C}_{|\mathcal{K}})$ norm thanks to Corollary 2.2.12. □

We now show that A_m is self-adjoint. For this purpose, it is sufficient to prove that $\nu\cdot\mathcal{D}^{\mathcal{N}}$ is essentially self-adjoint on $\mathrm{dom}(\widetilde{A}_m)$ because the potential is a bounded operator. From Proposition 2.2.6 and (2.2.18), one has

$$\iota^{-1}(\nu\cdot\mathcal{D}^{\mathcal{N}})\iota = -i\,\omega_n^{\mathbb{C}}\cdot \not{D}^{\mathcal{N}} \quad \text{if } n \text{ is even,} \tag{2.4.2}$$

and

$$\iota^{-1}(\nu\cdot\mathcal{D}^{\mathcal{N}})\iota = -i \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} (\not{D}^{\mathcal{N}} \oplus -\not{D}^{\mathcal{N}}) \quad \text{if } n \text{ is odd.} \tag{2.4.3}$$

Having these considerations in mind, we define

$$A := \not{D}^{\mathcal{N}} \text{ if } n \text{ is even, } A := \not{D}^{\mathcal{N}} \oplus -\not{D}^{\mathcal{N}} \text{ is } n \text{ is odd,} \tag{2.4.4}$$

and

$$T := -i\,\omega_n^{\mathbb{C}} \cdot \text{ if } n \text{ is even, } T := -i \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \text{ if } n \text{ is odd.} \tag{2.4.5}$$

We remark that T is a unitary skew-Hermitian operator which anti-commutes with A. Consider the operators

$$P_\pm := \frac{1 \pm i\,\mathbf{n}\cdot}{2} \text{ on } \iota(\Sigma\mathcal{C}_{|\mathcal{K}}), \text{ and } \mathcal{P}_\pm := \frac{1 \pm i\,\nu \cdot \mathbf{n}\cdot}{2} \text{ on } \Sigma\mathcal{C}_{|\mathcal{K}}. \tag{2.4.6}$$

Let $A_\pm$ be the restriction of A to the domain $\{\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}}),\, P_\pm\Psi = 0\}$. Then, the operator $\nu \cdot \mathcal{D}^{\mathcal{N}}$ with domain $\mathrm{dom}(\widetilde{A}_m)$ is unitarily equivalent to TA_+ for any parity of n.

Lemma 2.4.5. *For any $s \in \mathbb{R}$, $P_\pm$ and $\mathcal{P}_\pm$ define bounded operators from H^s to itself.*

Proof. The proof is straightforward, see [36, Lemma 5.1 (ii)]. □

Theorem 2.4.6. *The operator A_m is self-adjoint, and the equality in Proposition 2.4.4 holds for any $\Psi \in \mathrm{dom}(A_m) = \big\{\Psi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}}), \mathcal{P}_-\Psi = 0\big\}$.*

Proof. We first prove that $E := \big\{\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}}), \mathcal{P}_-\Psi = 0\big\}$ is dense in $F := \big\{\Psi \in H^1(\Sigma\mathcal{C}_{|\mathcal{K}}), \mathcal{P}_-\Psi = 0\big\}$ for the H^1 norm. Let $\Psi \in F$. There exists a sequence $(\Psi_j)_{j\in\mathbb{N}}$ in $\Gamma_c(\Sigma\mathcal{C}_{|\mathcal{K}})$ converging to Ψ in the H^1 norm. Let $\Phi_j := \Psi_j - \epsilon_{\mathcal{K}}\mathcal{P}_-\gamma_{\mathcal{K}}\Psi_j$, where we recall that $\epsilon_{\mathcal{K}}$ is the extension operator defined in Theorem 2.2.10. One has $\mathcal{P}_-\gamma_{\mathcal{K}}\Phi_j = 0$ and from Theorem 2.2.10 and Lemma 2.4.5 we obtain

$$\begin{aligned}
\|\Phi_j - \Psi\|_{H^1(\mathcal{K})} &= \|\Psi_j - \epsilon_{\mathcal{K}}\mathcal{P}_-\gamma_{\mathcal{K}}\Psi_j - \Psi\|_{H^1(\mathcal{K})} \\
&\le \|\Psi_j - \Psi\|_{H^1(\mathcal{K})} + \|\epsilon_{\mathcal{K}}\mathcal{P}_-\gamma_{\mathcal{K}}\Psi_j\|_{H^1(\mathcal{K})} \\
&\le \|\Psi_j - \Psi\|_{H^1(\mathcal{K})} + C_1\|\mathcal{P}_-\gamma_{\mathcal{K}}\Psi_j - \mathcal{P}_-\gamma_{\mathcal{K}}\Psi\|_{H^{\frac{1}{2}}(\mathcal{K})} \\
&\le C_2\|\Psi_j - \Psi\|_{H^1(\mathcal{K})} \underset{j\to+\infty}{\longrightarrow} 0
\end{aligned}$$

with $C_1, C_2 > 0$.

Thus, E is dense in F, and as the graph norm of $\widetilde{A}_m$ and the H^1 norm are equivalent on E by Proposition 2.4.4. We conclude that $F \subset \mathrm{dom}(A_m)$. By density, the expression of Proposition 2.4.4 holds for any $\Psi \in F$, and the graph norm and the H^1 norm are still equivalent on F. But F is closed for the H^1 norm, so we deduce that $F = \mathrm{dom}(A_m)$, and using Corollary 2.2.12, we have $\mathrm{dom}(\overline{A_+}) = \big\{\Psi \in H^1(\iota\Sigma\mathcal{C}_{|\mathcal{K}}), P_+\Psi = 0\big\}$. This means that $\overline{A_+}$ is exactly one or two copies of the operator D_+ (up to a sign) studied in [36, Lemma 5.1]. By the same method, we can show that $\mathrm{dom}(\overline{A_-}) = \big\{\Psi \in H^1(\iota\Sigma\mathcal{C}_{|\mathcal{K}}), P_-\Psi = 0\big\}$ and $\overline{A_-}$ is one or two copies of the operator D_- (up to a sign) studied in [36, Lemma 5.1].

Finally, [36, Lemma 5.1 (v)] gives us $(\overline{A_\pm})^* = \overline{A_\mp}$, and we deduce that

$$(T\overline{A_+})^* = -(\overline{A_+})^*T = -\overline{A_-}T = T\overline{A_+}.$$

Consequently, TA_- is self-adjoint, and so is A_m by unitary equivalence. □

2.4.3 Sesquilinear form for $B^2_{m,M}$

As for the operator A_m, we compute the sesquilinear form of the operator $B^2_{m,M}$ defined in section 2.3.2. As a consequence of the Schrödinger-Lichnerowicz formula, we can first compute the square of the extrinsic Dirac operator acting on smooth sections with compact support in $\mathcal{N}$.

Lemma 2.4.7. *Let $\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$. Then*

$$\|\nu \cdot \left(\mathcal{D}^{\mathcal{N}} + i\, m\right) \Psi\|^2_{L^2(\mathcal{N})} = \int_{\mathcal{N}} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 + m^2|\Psi|^2 \right] v_{\mathcal{N}}.$$

Proof. Let $\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$. One has

$$\begin{aligned}
\|\nu \cdot \left(\mathcal{D}^{\mathcal{N}} + i\, m\right) \Psi\|^2_{L^2(\mathcal{N})} &= \left\langle \nu \cdot \left(\mathcal{D}^{\mathcal{N}} + i\, m\right) \Psi, \nu \cdot \left(\mathcal{D}^{\mathcal{N}} + im\right) \Psi \right\rangle_{L^2(\mathcal{N})} \\
&= \left\langle \left(\mathcal{D}^{\mathcal{N}} + im\right) \Psi, \left(\mathcal{D}^{\mathcal{N}} + im\right) \Psi \right\rangle_{L^2(\mathcal{N})} \\
&= \left\langle \mathcal{D}^{\mathcal{N}}\Psi, \mathcal{D}^{\mathcal{N}}\Psi \right\rangle_{L^2(\mathcal{N})} + m^2 \left\langle \Psi, \Psi \right\rangle_{L^2(\mathcal{N})} \\
&\quad + m \left[\left\langle \mathcal{D}^{\mathcal{N}}\Psi, i\,\Psi \right\rangle_{L^2(\mathcal{N})} + \left\langle i\,\Psi, \mathcal{D}^{\mathcal{N}}\Psi \right\rangle_{L^2(\mathcal{N})} \right].
\end{aligned}$$

Using Lemma 2.4.1, one has at any point $x \in \mathcal{N}$,

$$\left\langle \mathcal{D}^{\mathcal{N}}\Psi, i\,\Psi \right\rangle + \left\langle i\,\Psi, \mathcal{D}^{\mathcal{N}}\Psi \right\rangle = -\mathrm{div}\ V.$$

By the divergence theorem, the Schrödinger-Lichnerowicz formula (Proposition 2.2.5) and Equation 2.2.18, one can integrate over $\mathcal{N}$ to obtain

$$\begin{aligned}
\|\nu \cdot \left(\mathcal{D}^{\mathcal{N}} + i\, m\right) \Psi\|^2_{L^2(\mathcal{N})} &= \left\langle \mathcal{D}^{\mathcal{N}}\Psi, \mathcal{D}^{\mathcal{N}}\Psi \right\rangle_{L^2(\mathcal{N})} + m^2 \left\langle \Psi, \Psi \right\rangle_{L^2(\mathcal{N})} \\
&= \int_{\mathcal{N}} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 + m^2|\Psi|^2 \right] v_{\mathcal{N}}. \qquad \square
\end{aligned}$$

We can now compute the quadratic form for the operator $B_{m,M}$ by integration over $\mathcal{N}$, and it comes out that its domain is a subspace of the Sobolev space H^1.

Proposition 2.4.8. *One has* $\mathrm{dom}(B_{m,M}) \subset H^1(\Sigma\mathcal{C}_{|\mathcal{N}})$ *and for* $\Psi \in \mathrm{dom}(B_{m,M})$*,*

$$\begin{aligned}
\|B_{m,M}\Psi\|^2_{L^2(\mathcal{N})} = &\int_{\mathcal{N}} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 \right] v_{\mathcal{N}} + m^2\|\Psi\|^2_{L^2(\mathcal{K})} \\
&+ M^2\|\Psi\|^2_{L^2(\mathcal{K}^c)} + (M - m)\int_{\partial\mathcal{K}} \left(|\mathcal{P}_-\Psi|^2 - |\mathcal{P}_+\Psi|^2 \right) v_{\partial\mathcal{K}}
\end{aligned}$$

where we recall that $\mathcal{P}_\pm$ *were defined in* (2.4.6)*.*

Proof. Let $\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$. One has

$$\begin{aligned}
\|B_{m,M}\Psi\|^2_{L^2(\mathcal{N})} =& \|\nu \cdot (\mathcal{D}^{\mathcal{N}} + i\, M)\Psi + i\,(m - M)\mathbf{1}_{\mathcal{K}}\nu \cdot \Psi\|^2_{L^2(\mathcal{N})} \\
=& \|(\mathcal{D}^{\mathcal{N}} + i\, M)\Psi\|^2_{L^2(\mathcal{N})} + (m - M)^2\|\Psi\|^2_{L^2(\mathcal{K})}
\end{aligned}$$

$$+ (m - M)2\Re\left\langle(\mathcal{D}^{\mathcal{N}} + i\,M)\Psi, i\,\mathbf{1}_{\mathcal{K}}\Psi\right\rangle_{L^2(\mathcal{N})}$$

With Lemma 2.4.1

$$2\Re\langle(\mathcal{D}^{\mathcal{N}} + i\,M)\Psi, i\,\Psi\rangle_{L^2(\mathcal{K})} = -\int_{\partial\mathcal{K}} \langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}} + 2M\,\langle\Psi,\Psi\rangle_{L^2(\mathcal{K})}\,.$$

Thus, we have

$$\begin{aligned}
\|B_{m,M}\|^2_{L^2(\mathcal{N})} &= \|(\mathcal{D}^{\mathcal{N}} + i\,M)\Psi\|^2_{L^2(\mathcal{N})} + (m-M)^2\|\Psi\|^2_{L^2(\mathcal{K})} \\
&\quad + (M-m)\int_{\partial\mathcal{K}} \langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}} + 2M(m-M)\|\Psi\|^2_{L^2(\mathcal{K})} \\
&= \|(\mathcal{D}^{\mathcal{N}} + i\,M)\Psi\|^2_{L^2(\mathcal{N})} + (m^2-M^2)\|\Psi\|^2_{L^2(\mathcal{K})} \\
&\quad + (M-m)\int_{\partial\mathcal{K}} \langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}} \\
&= \int_{\mathcal{N}}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2 + M^2|\Psi|^2\right] v_{\mathcal{N}} + (m^2-M^2)\|\Psi\|^2_{L^2(\mathcal{K})} \\
&\quad + (M-m)\int_{\partial\mathcal{K}} \langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}} \\
&= \int_{\mathcal{N}}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2|\Psi|^2\right] v_{\mathcal{N}} + m^2\|\Psi\|^2_{L^2(\mathcal{K})} + M^2\|\Psi\|^2_{L^2(\mathcal{K}^c)} \\
&\quad + (M-m)\int_{\partial\mathcal{K}} \langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle\, v_{\partial\mathcal{K}}
\end{aligned} \tag{2.4.7}$$

and

$$\langle\Psi, i\,\mathbf{n}\cdot\nu\cdot\Psi\rangle = \langle\Psi, -i\,\nu\cdot\mathbf{n}\cdot\Psi\rangle = \langle\Psi, \mathcal{P}_-\Psi\rangle - \langle\Psi, \mathcal{P}_+\Psi\rangle = |\mathcal{P}_-\Psi|^2 - |\mathcal{P}_+\Psi|^2.$$

It follows from Theorem 2.2.10 that there is a constant $C > 0$ such that for all $\Psi \in \Gamma_c(\Sigma\mathcal{C}_{|\mathcal{N}})$,

$$\|B_{m,M}\Psi\|^2_{L^2(\mathcal{N})} \geq C\left(\|\nabla^{\mathcal{N}}(\iota\Psi)\|^2_{L^2(\mathcal{N})} - \|\Psi\|^2_{L^2(\mathcal{N})}\right).$$

This shows that the graph norm of $\widetilde{B}_{m,M}$ is larger than the $H^1(\Sigma\mathcal{C}_{|\mathcal{N}})$-norm up to a constant. Thus $\mathrm{dom}(B_{m,M}) \subset H^1(\Sigma\mathcal{C}_{|\mathcal{N}})$, and one can conclude by density. □

2.4.4 The limit operator

In this section, we introduce the effective operator L which will appear naturally as the limit operator for A_m when $m \to -\infty$. We define it as the operator acting on the Hilbert space

$$\mathbf{H} := \left\{\Psi \in L^2(\Sigma\mathcal{C}_{|\partial\mathcal{K}}), \Psi = i\,\nu\cdot\mathbf{n}\cdot\Psi\right\} \tag{2.4.8}$$

associated with the quadratic form

$$\begin{aligned}
\ell[\Psi,\Psi] &= \int_{\partial\mathcal{K}}\left[|\overline{\nabla}^{\mathcal{N}}\iota\Psi|^2 + \frac{1}{4}\left(\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)\right)|\Psi|^2\right] v_{\partial\mathcal{K}}, \\
\mathcal{Q}(\ell) &:= \left\{\Psi \in H^1(\Sigma\mathcal{C}_{|\partial\mathcal{K}}), \Psi = i\,\nu\cdot\mathbf{n}\cdot\Psi\right\}.
\end{aligned} \tag{2.4.9}$$

By the compactness of $\mathcal{K}$, it follows that the form (2.4.9) is closed and semibounded from below, so the operator L is well-defined.

The operator L is actually unitarily equivalent to the square of the Dirac operator on $\partial\mathcal{K}$. This fact can be established using the link between the spinor bundles of the spaces $\partial\mathcal{K} \subset \mathcal{N} \subset \mathcal{C}$.

Remark 2.4.9. Using Gauss-Codazzi equations (see [6, Proposition 4.1], for example), one has

$$\mathrm{Tr}(W^2) = H^2 + \mathrm{Scal}^{\mathcal{N}} - \mathrm{Scal}^{\partial\mathcal{K}} - 2\mathrm{Ric}^{\mathcal{N}}(\mathbf{n}, \mathbf{n}).$$

Thus, the operator we are considering here is a generalization of the operator L defined in [59, section 2.2] and we generalize the result of [59, Lemma 2.4].

Lemma 2.4.10. *The operator L is unitarily equivalent to $(\not{D}^{\partial\mathcal{K}})^2$.*

Proof. We consider separately the case of n even and n odd.

Case n odd: One can represent any $\Psi \in \mathbf{H}$ as $\Psi =: (\Psi^+, \Psi^-) \in L^2(\Sigma^+\mathcal{C}_{|\partial\mathcal{K}}) \times L^2(\Sigma^-\mathcal{C}_{|\partial\mathcal{K}})$, and then

$$\Psi = i\,\nu \cdot \mathbf{n} \cdot \Psi \Leftrightarrow \iota\Psi = i\,\iota(\nu \cdot \mathbf{n} \cdot \Psi) \Leftrightarrow \iota\Psi = -i\,\mathbf{n} \cdot \iota\Psi.$$

Thus, the isomorphism ι induces the isomorphisms $\iota^{\pm} : \Sigma^{\pm}\mathcal{C} \to \Sigma\mathcal{N}$, and one has

$$\begin{pmatrix} \iota^+\Psi^+ \\ \iota^-\Psi^- \end{pmatrix} = \begin{pmatrix} -i\,\mathbf{n} \cdot \iota^+\Psi^+ \\ i\,\mathbf{n} \cdot \iota^-\Psi^- \end{pmatrix}.$$

We introduce the (pointwise) unitary operator $U : L^2(\Sigma\mathcal{N}_{|\partial\mathcal{K}}) \to \mathbf{H}$, which sends $H^1(\Sigma\mathcal{N}_{|\partial\mathcal{K}})$ into $\mathcal{Q}(\ell)$, and is defined by

$$U\Psi = \frac{1}{2}\iota^{-1} \begin{pmatrix} (1 - i\,\mathbf{n}) \cdot \Psi \\ (1 + i\,\mathbf{n}) \cdot \Psi \end{pmatrix}.$$

We compute now $|\overline{\nabla}^{\mathcal{N}}\iota(U\Psi)|^2$ for $\Psi \in H^1(\Sigma\mathcal{N}_{|\partial\mathcal{K}})$. Let $(e_1, \dots, e_{n-1})$ be a pointwise local orthonormal frame of $T(\partial\mathcal{K})$. The vector fields $(e_j)_{1\leq j\leq n-1}$ are naturally identified with elements of $T\mathcal{N}$. Using the Schrödinger-Lichnerowicz formula and Proposition 2.2.6, (3) one has

$$\begin{aligned}
|\overline{\nabla}^{N}\iota(U\Psi)|^2 &= \frac{1}{4}\left(|\overline{\nabla}^{N}((1 + i\,\mathbf{n}\cdot)\Psi)|^2 + |\overline{\nabla}^{N}((1 - i\,\mathbf{n}\cdot)\Psi)|^2\right) \\
&= \frac{1}{2}\sum_{k=1}^{n-1}\left(|\nabla^{\mathcal{N}}_{e_k}\Psi|^2 + |(\nabla^{\mathcal{N}}_{e_k}\mathbf{n}) \cdot \Psi + \mathbf{n} \cdot \nabla^{\mathcal{N}}_{e_k}\Psi|^2\right) \\
&= \sum_{k=1}^{n-1}|\nabla^{\mathcal{N}}_{e_k}\Psi + \frac{1}{2}\mathbf{n} \cdot We_k \cdot \Psi|^2 + \frac{1}{4}\sum_{k=1}^{n-1}|We_k \cdot \Psi|^2 \\
&= |\mu^{-1}\nabla^{\partial\mathcal{K}}\mu\Psi|^2 + \frac{1}{4}\mathrm{Tr}(W^2)|\Psi|^2 \\
&= |\mathcal{D}^{\partial\mathcal{K}}\Psi|^2 + \frac{1}{4}\left(-\mathrm{Scal}^{\partial\mathcal{K}} + \mathrm{Tr}(W^2)\right)|\Psi|^2.
\end{aligned}$$

Thus,

$$\ell[U\Psi, U\Psi] = \int_{\partial\mathcal{K}} |\mathcal{D}^{\partial\mathcal{K}}\Psi|^2 v_{\partial\mathcal{K}} = \int_{\partial\mathcal{K}} |\not{D}^{\partial\mathcal{K}}\mu\Psi|^2 v_{\partial\mathcal{K}}.$$

Case n even : The isomorphism μ induces the isomorphisms $\mu^{\pm} : \Sigma^{\pm}\mathcal{N} \to \Sigma\mathcal{K}$. According to Proposition 2.2.6, as $n-1$ is odd, for all $f \in \Gamma(\Sigma\mathcal{N}_{|\partial\mathcal{K}})$ one has

$$\mu(i\,\mathbf{n}\cdot f) = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} \mu^+ f^+ \\ \mu^- f^- \end{pmatrix}.$$

Then, for $\Psi \in \mathbf{H}$ one has

$$\begin{aligned} &i\,\nu\cdot\mathbf{n}\cdot\Psi = \Psi \Leftrightarrow -\iota(i\,\mathbf{n}\cdot\nu\cdot\Psi) = \iota\Psi \Leftrightarrow -\mu(i\,\mathbf{n}\cdot\iota\Psi) = \mu\iota\Psi \\ &\Leftrightarrow -\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} \mu^+(\iota\Psi)^+ \\ \mu^-(\iota\Psi)^- \end{pmatrix} = \mu\iota\Psi \Leftrightarrow (\iota\Psi)^- = -(\mu^-)^{-1}\mu^+(\iota\Psi)^+. \end{aligned}$$

Thus, the unitary operator

$$\begin{array}{rcl} U : L^2(\Sigma(\partial\mathcal{K})) & \longrightarrow & \mathbf{H} \\ \Psi & \longmapsto & \frac{1}{\sqrt{2}}\iota^{-1}\mu^{-1}\begin{pmatrix} -\Psi \\ \Psi \end{pmatrix} \end{array}$$

sends $H^1(\Sigma(\partial\mathcal{K}))$ into $\mathcal{Q}(\ell)$. Now we compute $|\overline{\nabla}^N\iota(U\Psi)|^2$ for $\Psi \in H^1(\Sigma(\partial\mathcal{K}))$. Let $(e_1,\dots,e_{n-1})$ be a pointwise local orthonormal frame of $T(\partial\mathcal{K})$. One has, using Proposition 2.2.6, (3)

$$\begin{aligned} |\overline{\nabla}^N\iota(U\Psi)|^2 &= |\mu\overline{\nabla}^N\iota(U\Psi)|^2 \\ &= \frac{1}{2}\left|\mu\overline{\nabla}^N\mu^{-1}\begin{pmatrix} -\Psi \\ \Psi \end{pmatrix}\right|^2 \\ &= \sum_{k=1}^{n-1}\frac{1}{2}\left|\left(\nabla^{\partial\mathcal{K}}_{e_k} + \frac{1}{2}We_k\right)\begin{pmatrix} -\Psi \\ \Psi \end{pmatrix}\right|^2 \\ &= \frac{1}{2}\sum_{k=1}^{n-1}\left(\left|\left(\nabla^{\partial\mathcal{K}}_{e_k} + \frac{1}{2}We_k\right)\Psi\right|^2 + \left|\left(\nabla^{\partial\mathcal{K}}_{e_k} - \frac{1}{2}We_k\right)\Psi\right|^2\right) \\ &= \sum_{k=1}^{n-1}\left(|\nabla^{\partial\mathcal{K}}_{e_k}\Psi|^2 + \frac{1}{4}|We_k|^2|\Psi|^2\right) \\ &= |\not{D}^{\partial\mathcal{K}}\Psi|^2 + \frac{1}{4}\left(-\mathrm{Scal}^{\partial\mathcal{K}} + \mathrm{Tr}(W^2)\right)|\Psi|^2 \end{aligned}$$

Thus

$$\ell[U\Psi, U\Psi] = \int_{\partial\mathcal{K}} |\not{D}^{\partial\mathcal{K}}\Psi|^2 v_{\partial\mathcal{K}}$$

which concludes the proof. □

2.5 Operators in tubular coordinates

When the masses m and M become large, one can localize the eigenvalue problem in a neighbourhood of $\partial\mathcal{K}$ since the potential in the square of the operators is large outside of this region. For this reason, it is useful to express the operators in tubular coordinates around $\partial\mathcal{K}$. Thus, we identify a collar near the boundary of $\mathcal{K}$ with the cylinder $\partial\mathcal{K}\times(-\delta,\delta)$ and we look at the operator obtained via this identification. However, the aim of this procedure is to simplify the expression, so we would like to change the induced metric on the cylinder into the product metric. This last step cannot be done without a way to compare the spinor bundles involved, and in particular the way we modify the covariant derivative.

2.5.1 Tubular coordinates

For $\delta > 0$ we define the tubular neighbourhood of $\partial\mathcal{K}$ by

$$\mathbf{n}_\delta(\partial\mathcal{K}) := \{x \in \mathcal{N}, \mathrm{dist}(x, \partial\mathcal{K}) < \delta\}. \tag{2.5.1}$$

Since $\partial\mathcal{K}$ is compact, $\mathbf{n}_\delta(\partial\mathcal{K})$ can be identified with the product $\partial\mathcal{K} \times (-\delta, \delta)$ through the Riemannian exponential map when δ is small. To make this precise, we define

$$\Pi_\delta := \partial\mathcal{K} \times (-\delta, \delta), \Pi_\delta^+ := \partial\mathcal{K} \times (0, \delta),\ \Pi_\delta^- := \partial\mathcal{K} \times (-\delta, 0),\ \Pi^t := \partial\mathcal{K} \times \{t\}, \tag{2.5.2}$$

and it is standard that there exists $\delta_0 > 0$ such that the map

$$\begin{array}{ccc} \Pi_{\delta_0} & \longrightarrow & \mathbf{n}_{\delta_0}(\partial\mathcal{K}) \\ (x,t) & \longmapsto & \exp_x^{\mathcal{N}}(t\mathbf{n}(x)) \end{array} \tag{2.5.3}$$

is a diffeomorphism on its image.

For every $\delta < \delta_0$, Π_δ inherits an orientation via the previous identification. Moreover, one has $T(\Pi_\delta) \cong T(\partial\mathcal{K}) \times T\mathbb{R}$ and we denote by $\frac{\partial}{\partial t}$ the vector field $(0,1) \in T(\partial\mathcal{K}) \times T\mathbb{R}$.

Recall now the definition of a generalized cylinder introduced in [6]:

Definition 2.5.1. *A generalized cylinder is a Riemannian manifold of the form $\mathcal{Z} := \mathcal{M} \times I$ where $I \subset \mathbb{R}$ is an interval, $\mathcal{M}$ is a differentiable manifold and the Riemannian metric on $\mathcal{Z}$ has the form $g_{\mathcal{Z}} = g_t + dt^2$ where $(g_t)_{t\in I}$ is a smooth 1-parameter family of Riemannian metrics of $\mathcal{M}$.*

We identify any vector field X on the hypersurface $\partial\mathcal{K}$ with the vector field on $T\Pi_{\delta_0}$ also denoted by X and defined by $X_{(y,t)} := X_y$ for all $(y,t) \in \Pi_{\delta_0}$. Note that in this case $[\frac{\partial}{\partial t}, X] = 0$.

We have two natural metrics on Π_{δ_0}. First, the metric g of $\mathcal{N}$ via the previous identification, and secondly, the Riemannian product metric $h := g_{|\partial\mathcal{K}} + \mathrm{d}t^2$. Furthermore, $\Sigma\Pi_{\delta_0}$ is the spinor bundle of $\mathcal{N}$ restricted to Π_{δ_0}.

With these notations, we have the useful property:

Lemma 2.5.2. *The Riemannian manifold (Π_{δ_0}, g) is a generalized cylinder.*

Proof. It is sufficient to prove that $g = g_t + dt^2$ with $(g_t)_t$ a family of metrics on $\partial\mathcal{K}$. This is equivalent to show that the vector field $\frac{\partial}{\partial t}$ is normal to Π^t for all $t \in (-\delta_0, \delta_0)$. Let $(x,t) \in \Pi_{\delta_0}$ and $X \in T(\partial\mathcal{K})$, identified with a vector field on Π_{δ_0} as before. One has

$$\begin{aligned} \frac{d}{dt} g\left(X, \frac{\partial}{\partial t}\right) &= g\left(\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} X, \frac{\partial}{\partial t}\right) + g\left(X, \nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\right) \\ &= \overbrace{g\left(\nabla^{\mathcal{N}}_X \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)}^{=0} + g\left(\left[\frac{\partial}{\partial t}, X\right], \frac{\partial}{\partial t}\right) \\ &= g\left(\left[\frac{\partial}{\partial t}, X\right], \frac{\partial}{\partial t}\right) = 0. \end{aligned}$$

This shows that $g\left(X, \frac{\partial}{\partial t}\right)$ is constant along the curves $s \mapsto (\cdot, s)$ since $g\left(X, \frac{\partial}{\partial t}\right)_{(x,0)} = 0$. We get $g\left(X, \frac{\partial}{\partial t}\right)_{(x,t)} = 0$, which concludes the proof. $\square$

From Proposition 2.5.2, we deduce that there exists a family of metrics $(g_t)_t$ on $\partial\mathcal{K}$ such that $g = g_t + \mathrm{d}t^2$. One can observe that $h = g_0 + \mathrm{d}t^2$ in these notations.

We define for any $(s,t) \in (-\delta_0, \delta_0)$ the map Γ_s^t which acts as the parallel transport from s to t along the curves $r \mapsto (\cdot, r)$ with respect to the connection $\nabla^{\mathcal{N}}$.

We recall that $v_{\mathcal{N}}$ is the volume form on Π_{δ_0} compatible with the metric g. Let $v_h := v_{\partial\mathcal{K}} \wedge \mathrm{d}t$ be the volume form compatible with h on Π_{δ_0}.

The bilinear form g is identified with an endomorphism of $T\Pi_{\delta_0}$ via the metric h. Let $(x,t) \in \Pi_{\delta_0}$. For any direct orthonormal frame f of $T_{(x,t)}\Pi_{\delta_0}$ endowed with the metric h we define

$$\phi(x,t) := \sqrt{\det_f g}. \tag{2.5.4}$$

One can show that this does not depend on the choice of the basis, and the volume forms with respect to the different metrics are related by

$$v_{\mathcal{N}} = \phi v_h. \tag{2.5.5}$$

Our aim in this section is relates all the objects on (Π_{δ_0}, g) in terms of those over (Π_{δ_0}, h). The function ϕ defined above relates the integration over these two Riemannian manifolds, and in particular the corresponding L^2 spaces. More precisely, the map

$$\begin{array}{rccc} \Theta: & L^2(\Sigma\Pi_{\delta_0}, v_{\mathcal{N}}) & \longrightarrow & L^2(\Sigma\Pi_{\delta_0}, v_h) \\ & \Psi & \longmapsto & \sqrt{\phi}\Psi \end{array} \tag{2.5.6}$$

is a unitary isomorphism from $L^2(\Sigma\Pi_{\delta_0}, v_{\mathcal{N}})$ onto $L^2(\Sigma\Pi_{\delta_0}, v_h)$.

2.5.2 Estimates in the generalized cylinder

We now fix $\delta < \frac{\delta_0}{2}$. In order to compare the structures over the hypersurfaces Π^t for $t \in (-\delta, \delta)$, we first show that the norm of a vector field defined on Π^t and extended by parallel transport with respect to $\nabla^{\mathcal{N}}$ does not vary too much when δ is small.

Lemma 2.5.3. *We endow Π_δ with the metric g. There exists $C > 0$ depending only on δ_0 such that for all $t, t' \in (-\delta, \delta)$ and $X \in \Gamma(T\Pi^t)$, for all $x \in \partial\mathcal{K}$, one has the estimate*

$$|X_{(x,t')} - \Gamma_t^{t'}(X_{(x,t)})|_g \leq C|t - t'||X_{(x,t)}|_g,$$

where X is extended to $T\Pi_\delta$ as before.

Proof. First, we remark that $C_1 := \sup\limits_{(y,s)\in\Pi_{\delta_0/2}} \sup\limits_{Z\in T_{(y,s)}\setminus\{0\}} \frac{|g(W_{\Pi^s}Z,Z)|}{g(Z,Z)}$ is finite by compactness. Let $t \in (-\delta, \delta)$ and $X \in \Gamma(T\Pi^t)$. We define the vector field $Y \in \Gamma(T\Pi_\delta)$ by $Y_{(y,s)} := \Gamma_t^s(X_{(y,t)})$ for any $(y,s) \in \Pi_\delta$.

One has for all $t' \in (-\delta, \delta)$,

$$\left|\frac{\partial}{\partial t} g(X,X)\right|_{|(\cdot,t')} = \left|2g\left(\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} X, X\right)\right|_{|(\cdot,t')} \leq 2C_1 g(X,X)_{(\cdot,t')}.$$

By integration, we obtain the inequality $g(X,X)_{(\cdot,t')} \leq g(X,X)_{(\cdot,t)} \exp(2C_1|t'-t|)$, and for $C_2 := \exp(2\delta_0 C_1)$ one has $g(X,X)_{(\cdot,t')} \leq C_2 g(X,X)_{(\cdot,t)}$.

Now, one has

$$\begin{aligned}\left|\frac{\partial}{\partial t}g(X-Y,X-Y)\right|_{(\cdot,t')} &= \left|2g(\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}X,X-Y)\right|_{(\cdot,t')} \\ &= |2g(W_{\Pi^{t'}}X,X-Y)|_{(\cdot,t')} \\ &\leq 2C_1|X_{(\cdot,t')}|_g|(X-Y)_{(\cdot,t')}|_g \\ &\leq 2C_1C_2|X_{(\cdot,t)}|_g|(X-Y)_{(\cdot,t')}|_g.\end{aligned}$$

We need the following technical lemma to conclude.

Lemma 2.5.4. *Let I be an interval of $\mathbb{R}$ containing 0 and let $f : I \to \mathbb{R}$ be a differentiable non-negative function. Assume there is $C > 0$ such that $|f'| \leq C\sqrt{f}$. Then, one has $|\sqrt{f}(x) - \sqrt{f}(0)| \leq \frac{C}{2}|x|$ for all $x \in I$.*

Using Lemma 2.5.4 we arrive at

$$g(X-Y,X-Y)_{(\cdot,t')} \leq C_1C_2|X_{(\cdot)}|^2_g(t'-t)^2$$

and the claim follows by taking the square root in this inequality. □

Proof of Lemma 2.5.4. Let $\varepsilon > 0$. One has $|f'| \leq C\sqrt{f+\varepsilon}$, which gives $\left|\frac{\mathrm{d}\sqrt{f+\varepsilon}}{\mathrm{d}x}\right| \leq \frac{C}{2}$. By integration, we obtain that for all $x \in I$, $|\sqrt{f(x)+\varepsilon} - \sqrt{f(0)+\varepsilon}| \leq \frac{C}{2}|x|$. Letting ε tend to zero, one gets the result. □

We are now able to compare the norms of the covariant derivatives on the different hypersurfaces of Π_δ. For this purpose, we recall that $\overline{\nabla}^{\mathcal{N}}\Psi$ is defined as the restriction of $\nabla^{\mathcal{N}}\Psi$ to $T^*\partial\mathcal{K} \otimes \Sigma\Pi_\delta$.

Lemma 2.5.5. *There exists $C > 0$ only depending on δ_0 such that for any $t \in (-\delta,\delta)$ and $\Psi \in \Gamma(\Sigma\Pi_\delta)$,*

$$\begin{aligned}(1-C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma^0_t\Psi(\cdot,t)\right|^2 - C\delta|\Psi(\cdot,t)|^2 &\leq \left|\overline{\nabla}^{\mathcal{N}}\Psi(\cdot,t)\right|^2 \\ &\leq (1+C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma^0_t\Psi(\cdot,t)\right|^2 + C\delta|\Psi|^2(\cdot,t).\end{aligned}$$

Proof. Let $\Psi \in \Gamma(\Sigma\Pi_\delta)$. Let $(x,t) \in \Pi_\delta$ and $X \in T(\partial\mathcal{K})$ such that $|X_{(x,t)}|_{g_t} = 1$, extended constantly to Π_δ. The Riemannian curvature of (Π_δ, g) is bounded, so for any $s \in (-\delta,\delta)$ one can find $C_1 > 0$ such that

$$\begin{aligned}\left|\frac{\partial}{\partial s}|(\nabla^{\mathcal{N}}_X\Gamma^s_t\Psi)(x,s)|^2\right| =&2\left|\Re\left\langle(\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\nabla^{\mathcal{N}}_X\Gamma^s_t\Psi)(x,s),(\nabla^{\mathcal{N}}_X\Gamma^s_t\Psi)(x,s)\right\rangle\right| \\ =&\left|\Re\left\langle R^{\mathcal{N}}\left(\frac{\partial}{\partial t},X\right)\cdot(\Gamma^s_t\Psi)(x,s),(\nabla^{\mathcal{N}}_X\Gamma^t_s\Psi)(x,s)\right\rangle\right| \\ \leq&C_1|X_{(x,s)}|_g|\Psi(x,t)||(\nabla^{\mathcal{N}}_X\Gamma^t_s\Psi)(x,s)|.\end{aligned}$$

By Lemma 2.5.3, one can find $C > 0$ independent of X such that

$$|X_{(x,s)}|_g \leq 1 + C|t-s| \leq 1 + C\delta_0.$$

Thus,

$$\left|\frac{\partial}{\partial s}|(\nabla^{\mathcal{N}}_X \Gamma^s_t \Psi)(x,s)|^2\right| \leq C_1(1+C\delta_0)|\Psi(x,t)||(\nabla^{\mathcal{N}}_X \Gamma^s_t \Psi)(x,s)|.$$

Using Lemma 2.5.4, we obtain

$$\big||(\nabla^{\mathcal{N}}_X \Gamma^0_t \Psi)(x,0)| - |\nabla^{\mathcal{N}}_X \Psi(x,t)|\big| \leq C_1(1+C\delta_0)|t||\Psi(x,t)|.$$

On the other hand,

$$\begin{aligned}|(\nabla^{\mathcal{N}}_X \Gamma^0_t \Psi)(x,0) - (\nabla^{\mathcal{N}}_{\Gamma^0_t X} \Gamma^0_t \Psi)(x,0)| \leq& |X_{(x,0)} - \Gamma^0_t(X_{(x,t)})|_g |(\overline{\nabla}^{\mathcal{N}} \Gamma^0_t \Psi)(x,0)| \\ \leq& C|t||(\overline{\nabla}^{\mathcal{N}} \Gamma^0_t \Psi)(x,0)|.\end{aligned}$$

Thus, combining the previous estimates, one can find $C_2 > 0$ such that

$$\left||(\nabla^{\mathcal{N}}_{\Gamma^0_t X} \Gamma^0_t \Psi)(x,0)| - |\nabla^{\mathcal{N}}_X \Psi(x,t)|\right| \leq C_2|t|\left(|\Psi(x,t)| + |(\overline{\nabla}^{\mathcal{N}} \Gamma^0_t \Psi)(x,0)|\right).$$

Now, let $(e_1,\dots,e_n)$ be an orthonormal frame at the point (x,t). One obtains

$$\begin{aligned}\left||(\overline{\nabla}^{\mathcal{N}} \Gamma^0_t \Psi)(x,0)| - |\overline{\nabla}^{\mathcal{N}} \Psi(x,t)|\right| \leq& \sum_{k=1}^n \left||(\nabla^{\mathcal{N}}_{\Gamma^0_t e_k} \Gamma^0_t \Psi)(x,0)| - |\nabla^{\mathcal{N}}_{e_k} \Psi(x,t)|\right| \\ \leq& \sum_{k=1}^n C_2|t|\left(|\Psi(x,t)| + |(\overline{\nabla}^{\mathcal{N}} \Gamma^0_t \Psi)(x,0)|\right) \\ \leq& nC_2\delta\left(|\Psi(x,t)| + |(\overline{\nabla}^{\mathcal{N}} \Gamma^0_t \Psi)(x,0)|\right).\end{aligned}$$

The result is then a consequence of the following lemma:

Lemma 2.5.6. *For all $C > 0$ and $\delta < \delta_0/2$, there is $C' > 0$ depending only on δ_0 and C such that for all $a,b,d > 0$ verifying $|a-b| \leq C\delta(b+d)$, one has $|a^2-b^2| \leq C'\delta(b^2+d^2)$.* □

Proof of Lemma 2.5.6. One has

$$\begin{aligned}|a^2-b^2| =& |(a-b+b)^2 - b^2| = |(a-b)^2 + 2(a-b)b| \leq |a-b|^2 + |2(a-b)b| \\ \leq& C^2\delta^2(b+d)^2 + 2C\delta(b+d)b \leq C^2\delta^2(b+d)^2 + C\delta(b+d)^2 + C\delta b^2 \\ \leq& (2C^2\delta^2 + C\delta)(b^2+d^2) + C\delta b^2 \leq (2C^2\delta_0 + 2C)\delta(b^2+d^2),\end{aligned}$$

which is equivalent to the statement of the lemma. □

2.5.3 Bracketing for the quadratic form of A_m^2

We end this section by finding a lower and an upper bound for the quadratic form of A_m^2 expressed in the tubular coordinates.

Lemma 2.5.7. *There exists $c > 0$ depending only on δ_0 such that the following estimates hold:*

$$\|\phi - 1\|_{L^\infty(\Pi_\delta)} \leq c\delta \qquad (2.5.7) \qquad\qquad \|\overline{\nabla}^{\mathcal{N}}\phi\|^2_{L^\infty(\Pi_\delta)} \leq c\delta^2 \qquad (2.5.8)$$

$$\left\|\frac{(\partial_t\phi)(\cdot,\delta)}{2\phi(\cdot,\delta)}\right\|_{L^\infty(\partial\mathcal{K})} \leq c \qquad (2.5.9) \qquad\qquad \partial_t\phi(\cdot,0) = -\frac{H}{2} \qquad (2.5.10)$$

$$\left|\frac{\partial_t^2\phi}{2\phi}(x,t) - \frac{(\partial_t\phi)^2}{4\phi^2}(x,t) - \frac{1}{4}(\mathrm{Scal}^{\partial\mathcal{K}}(x) - \mathrm{Tr}(W^2)(x) - \mathrm{Scal}^{\mathcal{N}}(x,t))\right| \leq c\delta, \tag{2.5.11}$$

for all $(x,t) \in \Pi_\delta$.

Proof. To show (2.5.7), (2.5.8) and (2.5.9), we just remark that ϕ is a smooth function on the closure of Π_δ which is compact, so it is bounded on Π_δ as well as all its derivatives.

Thanks to Lemma 2.5.2 we can use the explicit expression of the Weingarten tensor [6, formula (4.1)], so (2.5.10) follows from:

$$\frac{\partial_t\phi(\cdot,0)}{2\phi(\cdot,0)} = \frac{\partial_t\sqrt{\det_f g}(\cdot,0)}{2} = \frac{\mathrm{Tr}(\partial_t g)(\cdot,0)}{4\sqrt{\det_f g}(\cdot,0)} = -\frac{2\mathrm{Tr}(W)}{4} = -\frac{H}{2}.$$

Finally, we prove (2.5.11). Let $(x,t) \in \Pi_\delta$ and let f be a direct orthonormal frame of (Π_δ, h) at (x,t). One has, using lemma 2.5.2 and the formula for the scalar curvature of generalized cylinders [6, equation (4.8)],

$$\begin{aligned}\frac{\partial_t^2\phi}{2\phi}(x,t) - \frac{(\partial_t\phi)^2}{4\phi^2}(x,t) =& \frac{\partial_t^2 \det_f g}{4\det_f g}(x,t) - \frac{3(\partial_t \det_f g)^2}{16(\det_f g)^2}(x,t)\\ =& \left(\frac{\partial_t^2 \det_f g}{4} - \frac{3(\partial_t \det_f g)^2}{16}\right)(x,0) + \mathcal{O}(t)\\ =& \left(\frac{H^2}{4} - \mathrm{Tr}(W^2) + \frac{\mathrm{Tr}(\ddot{g}_t|_{t=0})}{4}\right)(x) + \mathcal{O}(t)\\ =& \frac{1}{4}(\mathrm{Scal}^{\partial\mathcal{K}}(x) - \mathrm{Tr}(W^2)(x) - \mathrm{Scal}^{\mathcal{N}}(x,t)) + \mathcal{O}(t),\end{aligned}$$

which gives the result. $\square$

For $\alpha \in \mathbb{R}$, $\delta \in (0, \delta_0/2)$ and $\Psi \in H^1\left(\Sigma\overline{\Pi^\pm_\delta}\right))$ we define

$$J_\pm(\Psi) := \int_{\Pi^\pm_\delta}\left[|\nabla^{\mathcal{N}}\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right]v_{\mathcal{N}} + \int_{\partial\mathcal{K}}\left(\alpha \pm \frac{H}{2}\right)|\Psi|^2 v_{\partial\mathcal{K}}. \tag{2.5.12}$$

Proposition 2.5.8. *There is a constant $c > 0$ depending only on δ_0 such that for all $\alpha \in \mathbb{R}$ and $\delta \in (0, \delta_0/2)$, the following inequalities hold:*

1. *For every $\Psi \in H^1\left(\Sigma\overline{\Pi^\pm_\delta}\right)$, one has*

$$\begin{aligned}J_\pm(\Psi) \geq& \int_{\Pi^\pm_\delta}\left[(1-c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi)(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi|^2\right]v_h(x,t)\\ &+\int_{\Pi^\pm_\delta}\left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} - c\delta\right)|\Theta\Psi|^2\right]v_h\\ &+\int_{\partial\mathcal{K}}\left[\alpha|(\Theta\Psi)(\cdot,0)|^2 - c|(\Theta\Psi)(\cdot,\delta)|^2\right]v_{\partial\mathcal{K}}.\end{aligned} \tag{2.5.13}$$

2. *If moreover $\Psi = 0$ on the outer boundary $\Pi^{\pm\delta}$, one has*

$$J_\pm(\Psi) \leq \int_{\Pi_\delta^\pm} \left[(1+c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi)(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi|^2\right] v_h(x,t)$$
$$+ \int_{\Pi_\delta^\pm}\left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + c\delta\right)|\Theta\Psi|^2\right] v_h + \alpha\int_{\partial\mathcal{K}} |(\Theta\Psi)(\cdot,0)|^2 v_{\partial\mathcal{K}} \quad (2.5.14)$$

Proof. It is sufficient to prove the result for $\Psi \in \Gamma_c\left(\Sigma\overline{\Pi_\delta^\pm}\right)$ and to conclude by density. One has

$$J_\pm(\Psi) = \int_{\Pi_\delta^\pm}\left[|\nabla^{\mathcal{N}}\phi^{-\frac{1}{2}}\Theta\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\phi^{-\frac{1}{2}}\Theta\Psi|^2\right]\phi v_h + \int_{\partial\mathcal{K}}\left(\alpha \pm \frac{H}{2}\right)|\Psi|^2 v_{\partial\mathcal{K}}.$$

We remark that $\phi = 1$ on $\partial\mathcal{K}$ and Lemma 2.5.5 gives a constant $C > 0$ such that

$$\int_{\Pi_\delta^\pm}\left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\phi^{-\frac{1}{2}}\Theta\Psi\right|^2 + (1-C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\phi^{-\frac{1}{2}}\Theta\Psi\right|^2(\cdot,0) - C\delta|\phi^{-\frac{1}{2}}\Theta\Psi|^2\right]\phi v_h$$
$$+ \int_{\Pi_\delta^\pm}\frac{\mathrm{Scal}^{\Pi}}{4}|\Theta\Psi|^2\phi v_h + \int_{\partial\mathcal{K}}\left(\alpha\pm\frac{H}{2}\right)|\Theta\Psi|^2 v_{\partial\mathcal{K}} \leq J_\pm(\Psi)$$
$$\leq \int_{\Pi_\delta^\pm}\left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\phi^{-\frac{1}{2}}\Theta\Psi\right|^2 + (1+C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\phi^{-\frac{1}{2}}\Theta\Psi\right|^2(\cdot,0) + C\delta|\phi^{-\frac{1}{2}}\Theta\Psi|^2\right]\phi v_h$$
$$+ \int_{\Pi_\delta^\pm}\frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\phi^{-\frac{1}{2}}\Theta\Psi|^2\phi v_h + \int_{\partial\mathcal{K}}\left(\alpha\pm\frac{H}{2}\right)|\Theta\Psi|^2 v_{\partial\mathcal{K}}.$$

Moreover, for all $(x,t) \in \Pi_\delta$ and $X \in T_x\partial\mathcal{K}$,

$$\left|\overline{\nabla}^{\mathcal{N}}_X\Gamma_t^0(\phi^{-\frac{1}{2}}\Theta\Psi)\right|^2(x,0)\phi(x,t)$$
$$= \left|\overline{\nabla}^{\mathcal{N}}_X\Gamma_t^0\Theta\Psi - \frac{1}{2\phi(x,t)}X(\phi)(x,t)\Gamma_t^0\Theta\Psi\right|^2(x,0)$$
$$= \left|\overline{\nabla}^{\mathcal{N}}_X\Gamma_t^0\Theta\Psi\right|^2(x,0) + \left|\frac{1}{2\phi(x,t)}X(\phi)(x,t)\Gamma_t^0\Theta\Psi\right|^2(x,0)$$
$$-\frac{1}{\phi(x,t)}\Re\left\langle\overline{\nabla}^{\mathcal{N}}_X\Gamma_t^0\Theta\Psi, X(\phi)(x,t)\Gamma_t^0\Theta\Psi\right\rangle(x,0)$$

and

$$\left|\Re\left\langle\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi, X(\phi)(x,t)\Gamma_t^0\Theta\Psi\right\rangle(x,0)\right| \leq \delta|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi|^2(x,0) + |\Theta\Psi|^2|X(\phi)|^2(x,t)/\delta.$$

Using this together with the inequality (2.5.8) shows the existence of $C' > 0$ such that

$$(1-C'\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi\right|^2(x,0) - C'\delta\,|\Theta\Psi|^2(x,t)$$
$$\leq (1\pm C\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\phi^{-\frac{1}{2}}\Theta\Psi\right|^2(x,0)\phi(x,t)$$
$$\leq (1+C'\delta)\left|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Theta\Psi\right|^2(x,0) + C'\delta\,|\Theta\Psi|^2(x,t).$$

It remains to compute

$$\phi|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\phi^{-\frac{1}{2}}\Theta\Psi|^2 = \left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi - \frac{1}{2\phi}\partial_t\phi(\Theta\Psi)\right|^2$$

$$
\begin{aligned}
&= \left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2 + \frac{(\partial_t\phi)^2}{4\phi^2}\left|\Theta\Psi\right|^2 - \frac{\partial_t\phi}{\phi}\Re\left\langle \nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi, \Theta\Psi\right\rangle \\
&= \left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2 + \frac{(\partial_t\phi)^2}{4\phi^2}\left|\Theta\Psi\right|^2 - \frac{\partial_t\phi}{2\phi}\partial_t\left|\Theta\Psi\right|^2.
\end{aligned}
$$

Integrating by parts yields

$$
\begin{aligned}
\int_{\Pi^{\pm}_{\delta}} |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\phi^{-\frac{1}{2}}\Theta\Psi|^2\phi v_h &= \int_{\Pi^{\pm}_{\delta}}\left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2 + \frac{(\partial_t\phi)^2}{4\phi^2}\left|\Theta\Psi\right|^2 - \frac{\partial_t\phi}{2\phi}\partial_t\left|\Theta\Psi\right|^2\right] v_h \\
&= \int_{\Pi^{\pm}_{\delta}}\left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2 + \frac{(\partial_t\phi)^2}{4\phi^2}\left|\Theta\Psi\right|^2 + \left(\frac{\partial_t^2\phi}{2\phi} - \frac{(\partial_t\phi)^2}{2\phi^2}\right)\left|\Theta\Psi\right|^2\right] v_h \\
&\qquad \mp \int_{\Pi^{\pm\delta}} \frac{\partial_t\phi}{2\phi}|\Theta\Psi|^2 v_h \pm \int_{\Pi^0}\frac{\partial_t\phi}{2\phi}|\Theta\Psi|^2 v_{\partial\mathcal{K}} \\
&= \int_{\Pi^{\pm}_{\delta}}\left[\left|\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi\right|^2 + \left(\frac{\partial_t^2\phi}{2\phi} - \frac{(\partial_t\phi)^2}{4\phi^2}\right)\left|\Theta\Psi\right|^2\right] v_h \\
&\qquad \mp \int_{\Pi^{\pm\delta}} \frac{\partial_t\phi}{2\phi}|\Theta\Psi|^2 v_{\partial\mathcal{K}} \mp \int_{\Pi^0}\frac{H}{2}|\Theta\Psi|^2 v_{\partial\mathcal{K}}
\end{aligned}
$$

where we used (2.5.10). Thus, we have

$$
\begin{aligned}
J_{\pm}(\Psi) \leq \int_{\Pi^{\pm}_{\delta}}\Bigg[&(1+C\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma^0_t\Theta\Psi)(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi|^2 \\
&+\left(\frac{\partial_t^2\phi}{2\phi} - \frac{(\partial_t\phi)^2}{4\phi^2} + \frac{\mathrm{Scal}^{\mathcal{N}}}{4} + C\delta\right)|\Theta\Psi|^2(x,t)\Bigg] v_h(x,t) \\
&+\alpha\int_{\Pi^0}|\Theta\Psi|^2 v_{\partial\mathcal{K}} \quad \text{if } \Psi = 0 \text{ on } \Pi^{\pm\delta}
\end{aligned}
$$

$$
\begin{aligned}
J_{\pm}(\Psi) \geq \int_{\Pi^{\pm}_{\delta}}\Bigg[&(1-C\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma^0_t\Theta\Psi)(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Theta\Psi|^2 \\
&+\left(\frac{\partial_t^2\phi}{2\phi} - \frac{(\partial_t\phi)^2}{4\phi^2} + \frac{\mathrm{Scal}^{\mathcal{N}}}{4} - C\delta\right)|\Theta\Psi|^2(x,t)\Bigg] v_h \\
&+\alpha\int_{\Pi^0}|\Theta\Psi|^2 v_{\partial\mathcal{K}} \mp \int_{\Pi^{\pm\delta}}\frac{\partial_t\phi}{2\phi}|\Theta\Psi|^2 v_{\partial\mathcal{K}}.
\end{aligned}
$$

These estimates, together with (2.5.9) and (2.5.11) give the result. □

2.6 Analysis of the one-dimensional operators

The proofs of the main results will use some separation of variables in the generalized cylinder Π_δ. For this reason, we will need to analyse various one-dimensional operators. We define them in this section and we state the properties that we need on the behaviour of their eigenvalues in some asymptotic regimes.

We recall the following results from [59, Section 3]:

Lemma 2.6.1. *Let $\varepsilon > 0$. Let $\alpha > 0$ and let S be the self-adjoint operator on $L^2(0,\delta)$ associated with the quadratic form*

$$s[f,f] = \int_0^\varepsilon |f'|^2 dt - \alpha|f(0)|^2,\ \mathcal{Q}(s) = \left\{f \in H^1(0,\varepsilon),\ f(\varepsilon) = 0\right\}.$$

Then, when $\alpha \to +\infty$, one has $E_1(S) = -\alpha^2 + \mathcal{O}(e^{-\varepsilon\alpha})$, and the associated L^2-normalized eigenfunction f satisfies $|f(0)|^2 = 2\alpha + \mathcal{O}(1)$.

Lemma 2.6.2. *Let $\varepsilon > 0$. Let $\alpha, \beta > 0$ and let S' be the self-adjoint operator on $L^2(0,\varepsilon)$ associated with the quadratic form*

$$s'[f,f] = \int_0^\varepsilon |f'|^2 dt + m|f(0)|^2 - \beta|f(\varepsilon)|^2,\ \mathcal{Q}(S') = H^1(0,\varepsilon).$$

Then, when $\alpha \to +\infty$, one has $E_1(S') = -\alpha^2 + \mathcal{O}(e^{-\varepsilon\alpha})$, and there exist $b^\pm > 0$ and $b > 0$ such that

$$b^- j^2 - b \le E_j(S') \le b^+ j^2 \text{ for all } j \ge 2 \text{ and } \alpha > 0.$$

A third one-dimensional operator will be of interest for the proof of Theorem 2.1.3. It can be interpreted as the Laplacian on an interval $(-\delta,\delta)$ with a potential consisting of two masses on the two sides of the origin and a δ-interaction at 0. For this last operator, we state the result in the very specific case of our framework, for $m, M \in \mathbb{R}$ and $\delta \in (0, \delta_0/2)$.

For $\beta > 0$, let X be the operator associated with the quadratic form

$$\begin{aligned} x[f,f] = \int_{-\delta}^{\delta} |f'|^2 \mathrm{d}t - \beta(|f(\delta)|^2 + |f(-\delta)|^2) \\ + \int_{-\delta}^{0} M^2 |f|^2 \mathrm{d}t + \int_0^\delta m^2|f|^2 \mathrm{d}t - (M-m)|f(0)|^2, \\ \mathcal{Q}(x) = H^1(-\delta,\delta). \end{aligned} \tag{2.6.1}$$

Lemma 2.6.3. *For $\delta > 0$ and $\beta > 0$ fixed, one has $E_1(X) = \mathcal{O}(e^{-\frac{\min(|m|,M)}{2}\delta})$ when $\min(-m, M) \to +\infty$. Moreover, for all $j \ge 2$, one can find $C_1, C_2 > 0$ such that*

$$\min(m^2, M^2) + C_1 j^2 - C_2 \le E_j(X).$$

Proof. One can see that the operator X acts as $f \mapsto -f'' + (M^2\mathbf{1}_{(-\delta,0)} + m^2\mathbf{1}_{(0,\delta)})f$ on the functions $f \in H^1(-\delta,\delta) \cap (H^2(-\delta,0) \cup H^2(0,\delta))$ satisfying $f'(\delta) - \beta f(\delta) = f'(-\delta) + \beta f(-\delta) = 0$ and $f'(0^+) - f(0^-) + (|m| + M)f(0) = 0$. We search for a negative eigenvalue for X of the form $-k^2$ with $k > 0$. The associated eigenfunction must be of the form

$$f(t) = \left\{ \begin{array}{ll} a_1 e^{-k_1 t} + b_1 e^{k_1 t} & \text{if } t \in (-\delta, 0) \\ a_2 e^{k_2 t} + b_2 e^{-k_2 t} & \text{if } t \in (0,\delta) \end{array} \right. \tag{2.6.2}$$

where $k_1 := \sqrt{M^2 + k^2}$ and $k_2 := \sqrt{m^2 + k^2}$.

We can rewrite the equations satisfied by f as

$$\begin{aligned} 0 &= a_2(k_2 - \beta)e^{k_2\delta} - b_2(k_2 + \beta)e^{-k_2\delta} \\ 0 &= a_1(k_1 - \beta)e^{k_1\delta} - b_1(k_2 + \beta)e^{-k_2\delta} \\ a_1 + b_1 &= a_2 + b_2 \end{aligned}$$

$$0 = a_2k_2 - b_2k_2 + a_1k_1 - b_1k_1 + (|m| + M)(a_1 + b_1).$$

The first two equations give $b_2 = \frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}a_2$ and $b_1 = \frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}a_1$. Thus, with the equation of continuity we have

$$a_1\left(1 + \frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}\right) = a_2\left(1 + \frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}\right).$$

We conclude that

$$a_2 = a_1\left(1 + \frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}\right)^{-1}\left(1 + \frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}\right)$$

because for $\min(|m|, M)$ large enough, one has that the different terms are not zero.

We arrive at

$$|m| + M = k_2\left(\frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta} - 1\right)\left(1 + \frac{k_2-\beta}{k_2+\beta}e^{2k_2\delta}\right)^{-1} + k_1\left(\frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta} - 1\right)\left(1 + \frac{k_1-\beta}{k_1+\beta}e^{2k_1\delta}\right)^{-1}.$$

Let $F(x) := x\left(\frac{x-\beta}{x+\beta}e^{2x\delta} - 1\right)\left(1 + \frac{x-\beta}{x+\beta}e^{2x\delta}\right)^{-1}$ defined on $(\min(|m|, M), +\infty)$. The previous equation reads $|m| + M = F(k_1) + F(k_2)$, and when $k = 0$ the right-hand side is $F(|m|) + F(M) < |m| + M$. Since $F(k_1) + F(k_2) \to +\infty$ when $k \to +\infty$ and F is strictly increasing there exists an unique $k \in (0, +\infty)$ such that $|m| + M = F(k_1) + F(k_2)$.

Now, one has

$$F(x) = x(1 + \mathcal{O}(e^{-2x\delta})) = x + \mathcal{O}(e^{-3x\delta/2}).$$

Thus, for $\zeta := \min(|m|, M)$ large enough one has

$$k_2 + k_1 - 2e^{-\zeta\delta} \leq |m| + M \leq k_2 + k_1 + 2e^{-\zeta\delta}$$

and

$$0 \leq \sqrt{m^2 + k^2} - |m| + \sqrt{M^2 + k^2} - M \leq 2e^{-\zeta\delta}.$$

Then, $\sqrt{\zeta^2 + k^2} - \zeta \leq 2e^{-\zeta\delta}$ and we arrive at

$$k^2 = \mathcal{O}(e^{-\zeta\delta/2}).$$

To conclude, we consider the operator X' defined by the same quadratic form as X but with the form domain $\{f \in H^1(-\delta, \delta), f(0) = 0\}$. From the Min-Max principle, one has $E_{j-1}(X') \leq E_j(X_\alpha) \leq E_j(X')$ for all $j \geq 2$ because X is a rank-one perturbation of X'. But $X' \cong (S_D + m^2) \oplus (S_D + M^2)$ where S_D is the operator acting in $L^2(0, \delta)$ as $f \mapsto -f''$ for $f \in H^2(0, \delta)$ with $f(0) = f'(\delta) - \beta f(\delta) = 0$. We conclude by remarking that $E_j(S_D) \sim \pi^2 j^2/\delta^2$ when $j \to +\infty$, so $E_j(X') \geq \min(m^2, M^2) - C_2 + C_1 j^2$ for suitable $C_1, C_2 > 0$. □

2.7 Asymptotics analysis for the operator A_m

In this section, we prove Theorem 2.1.1 following the analysis of [59, Section 4]. The proof is made by localizing the problem near the boundary of $\mathcal{K}$ and using the analysis done in the previous section to find a lower and an upper bound for the limits of the eigenvalues. These bounds coincide and are equal to the eigenvalues of the model operator L introduced in (2.4.9). We begin by showing a Dirichlet-Neumann bracketing for the operator A_m.

Let $\delta \in (0, \delta_0/2)$. We introduce several new operators. Let Z_m^+, Z_m^-, Z_m' be the operators defined by their quadratic forms z_m^+, z_m^-, z_m' which admit the same expression as the quadratic form of A_m^2 given in Proposition 2.4.4 with

$$\mathrm{dom}(z_m^+) = \left\{\Psi \in H^1(\Sigma\mathcal{C}_{|\overline{\Pi_\delta^-}}), \Psi = i\,\nu\cdot\mathbf{n}\cdot\Psi \text{ on } \partial\mathcal{K} \text{ and } \Psi = 0 \text{ on } \Pi^{-\delta}\right\}, \tag{2.7.1}$$

$$\mathrm{dom}(z_m^-) = \left\{\Psi \in H^1(\Sigma\mathcal{C}_{|\overline{\Pi_\delta^-}}), \Psi = i\,\nu\cdot\mathbf{n}\cdot\Psi \text{ on } \Pi^0\right\}, \tag{2.7.2}$$

$$\mathrm{dom}(z_m') = H^1\left(\Sigma\mathcal{C}_{|\mathcal{K}\setminus(\Pi_\delta^-\cup\Pi^0)}\right). \tag{2.7.3}$$

We define the maps $J_1 : \mathrm{dom}(A_m) \to \mathrm{dom}(z_m^-) \oplus \mathrm{dom}(z_m')$, $\Psi \mapsto (\Psi_{|\overline{\Pi_\delta^-}}, \Psi_{|\mathcal{K}\setminus(\Pi_\delta^-\cup\Pi^0)})$ and $J_2 : \mathrm{dom}(z_m^+) \to \mathrm{dom}(A_m)$ which is the extension by zero. For $\Psi_1 \in \mathrm{dom}(A_m)$ one has

$$(z_m^- \oplus z_m')\,[J_1(\Psi_1), J_1(\Psi_1)] \le \langle A_m\Psi_1, A_m\Psi_1\rangle_{L^2(\mathcal{K})},$$

and for $\Psi_2 \in \mathrm{dom}(z_m^+)$,

$$\langle A_m J_2(\Psi_2), A_m J_2(\Psi_2)\rangle_{L^2(\mathcal{K})} \le z_m^+\,[\Psi_2, \Psi_2].$$

Then, the Min-Max principle gives

$$E_j\left(Z_m^- \oplus Z_m'\right) \le E_j\left(A_m^2\right) \le E_j\left(Z_m^+\right). \tag{2.7.4}$$

We remark that $Z_m' \ge m^2$ and then, for any $j \in N$ such that $E_j\left(Z_m^+\right) < m^2$, one has

$$E_j\left(Z_m^-\right) \le E_j\left(A_m^2\right) \le E_j\left(Z_m^+\right). \tag{2.7.5}$$

We introduce the notation $\mathbf{S}_\delta^- := \iota(\Sigma\mathcal{C}_{|\overline{\Pi_\delta^-}})$. Let $c > 0$ be the constant given by Proposition 2.5.8. We consider the two quadratic forms in $L^2(\mathbf{S}_\delta^-, v_h)$ given by

$$\begin{aligned} y_m^+[\Psi,\Psi] :=& \int_{\Pi_\delta^-}\left[(1+c\delta)|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi|^2\right]v_h \\ &+ \int_{\Pi_\delta^-}\left[\left(m^2 + \frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + c\delta\right)|\Psi|^2\right]v_h + m\int_{\partial\mathcal{K}}|\Psi(\cdot,0)|^2 v_{\partial\mathcal{K}} \\ &\mathcal{Q}(y_m^+) := \left\{\Psi \in H^1\left(\mathbf{S}_\delta^-\right), \mathcal{P}_-\iota^{-1}(\Psi(\cdot,0)) = 0 \text{ and } \Psi(\cdot,\delta) = 0\right\}, \end{aligned} \tag{2.7.6}$$

and

$$\begin{aligned} y_m^-[\Psi,\Psi] :=& \int_{\Pi_\delta^-}\left[(1-c\delta)|\overline{\nabla}^{\mathcal{N}}\Gamma_t^0\Psi|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi|^2\right]v_h \\ &+ \int_{\Pi_\delta^-}\left[\left(m^2 + \frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} - c\delta\right)|\Psi|^2\right]v_h \\ &+ \int_{\partial\mathcal{K}}\left[m|\Psi(\cdot,0)|^2 - c|\Psi(\cdot,\delta)|^2\right]v_{\partial\mathcal{K}} \end{aligned}$$

$$\mathcal{Q}(y_m^-) := \left\{\Psi \in H^1\left(\mathbf{S}_\delta^-\right), \mathcal{P}_-\iota^{-1}\Psi(\cdot, 0) = 0\right\}. \quad (2.7.7)$$

Remarking that $\mathcal{Q}(y_m^\pm) = \Theta\iota(\mathrm{dom}(z_m^\pm))$, and that $\Theta\iota$ is unitary from $L^2(\Sigma\mathcal{C}_{|\Pi_\delta^-}, \upsilon_\mathcal{N})$ onto $L^2\left(\mathbf{S}_\delta^-, \upsilon_h\right)$, Proposition 2.5.8 and the Min-Max principle give

$$\Lambda_j\left(y_m^-\right) \le E_j\left(A_m^2\right) \le \Lambda_j\left(y_m^+\right) \text{ for any } j \in N \text{ such that } \Lambda_j(y_m^+) < m^2. \quad (2.7.8)$$

2.7.1 Upper bound

The upper bound is found by taking good test functions in the Min-Max principle. The first observation is that the quadratic form y_m^+ admits a separation of variables. Indeed, it can be seen as the tensor product of a sesquilinear form on $\partial\mathcal{K}$ and a one-dimensional sesquilinear form S. The behaviour of its first eigenvalue allows us to find the bound we are searching for.

Let S be the self-adjoint operator on $L^2(0, \delta)$ associated with the quadratic form

$$s[f, f] = \int_0^\delta |f'|^2 dt + m|f(0)|^2, \, \mathcal{Q}(s) = \left\{f \in H^1(0, \delta), \, f(\delta) = 0\right\}, \quad (2.7.9)$$

and let f be a normalized eigenfunction for the first eigenvalue of S. According to Lemma 2.6.1, when $-m$ is large, there is $b > 0$ such that $S[f, f] + m^2 \le b\exp(-\delta|m|)$.

For $a > 0$, we introduce the quadratic form

$$\ell_a[\Psi, \Psi] = \int_{\partial\mathcal{K}} \left[(1 + ca)|\overline{\nabla}^N \iota\Psi|^2 + \left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + ca\right)|\Psi|^2\right] \upsilon_{\partial\mathcal{K}},$$
$$\mathcal{Q}(\ell_a) = \mathcal{Q}(\ell), \quad (2.7.10)$$

where ℓ was defined in (2.4.9). The sesquilinear form ℓ_a is lower semibounded and closed. We denote by L_a the associated self-adjoint operator.

Let $\xi_1, \ldots, \xi_j$ be linearly independant eigenspinors for the first j eigenvalues of L_δ. We define the set

$$V := \left\{\Psi \in L^2\left(\mathbf{S}_\delta^-\right), \Psi(x, t) = f(t)\Gamma_0^t(\iota\xi(x)), \, \xi \in \mathrm{Span}(\xi_1, \ldots, \xi_j)\right\}. \quad (2.7.11)$$

With all these notations, for $\Psi(x, t) := f(t)\Gamma_0^t(\iota\xi(x)) \in V$ and $-m$ large enough, one has, using Leibniz's rule

$$\begin{aligned}
y_m^+[\Psi, \Psi] = &\int_{\Pi_\delta^-} \left[|\nabla^\mathcal{N}_{\frac{\partial}{\partial t}}\Psi|^2 + (1 + c\delta)|\overline{\nabla}^\mathcal{N}\Gamma_0^t\Psi|^2\right] \upsilon_h \\
&+ \int_{\Pi_\delta^-} \left[\left(m^2 + \frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + c\delta\right)|\Psi|^2\right] \upsilon_h + m\int_{\partial\mathcal{K}} |\Psi(., 0)|^2 \upsilon_{\partial\mathcal{K}} \\
= &\int_{\Pi_\delta^-} \left[|\frac{\partial}{\partial t} f|^2|\xi|^2 + (1 + c\delta)|\overline{\nabla}^N\iota\xi|^2|f|^2\right] \upsilon_h \\
&+ \int_{\Pi_\delta^-} \left[\left(m^2 + \frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + c\delta\right)|\Psi|^2\right] \upsilon_h + m\int_{\partial\mathcal{K}} |\Psi(\cdot, 0)|^2 \upsilon_{\partial\mathcal{K}} \\
= &\,\ell_\delta[\xi, \xi]\|f\|^2_{L^2(0,\delta)} + \left(S[f, f] + m^2\right)\|\xi\|^2_{L^2(\partial\mathcal{K})}
\end{aligned}$$

$$\leq \ell_\delta[\xi,\xi] + b\exp(-\delta|m|)\|\xi\|^2_{L^2(\partial\mathcal{K})}$$
$$\leq \left(E_j(L_\delta) + b\exp(-\delta|m|)\right)\|\xi\|^2_{L^2(\partial\mathcal{K})}.$$

Thus, $\Lambda_j(y_m^+) \leq E_j(L_\delta) + b\exp(-\delta|m|)$. We remark that $\lim_{\delta\to 0} E_j(L_\delta) = E_j(L)$ so we get the bound

$$\limsup_{m\to-\infty} E_j(A_m^2) \leq E_j(L). \tag{2.7.12}$$

2.7.2 Lower bound

The strategy to obtain the lower bound is to relax the constraint in the domain of y_m^- in order to obtain a separation of variable. In this way, we arrive are in the good setting to apply the monotone convergence theorem. This analysis will be done in the remaining part of this section.

Let S' be the self-adjoint operator on $L^2(0,\delta)$ associated with the quadratic form

$$S'[f,f] = \int_0^\delta |f'|^2 dt + m|f(0)|^2 - c|f(\delta)|^2,\ \mathcal{Q}(S') = H^1(0,\delta), \tag{2.7.13}$$

and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of normalized eigenfunctions for the eigenvalues $E_k(S')$. According to Lemma 2.6.2, there exist $b^\pm > 0$, $b > 0$ and $b_0 > 0$ such that $E_1(S') \geq -m^2 - be^{-\delta|m|}$ when $m\to-\infty$ and $b^-k^2 - b_0 \leq E_k(S') \leq b^+k^2$ for all $k\geq 2$.

If $c>0$ is the constant given by Proposition 2.5.8, we define the quadratic form y_m by the same formula as y_m^-, but with the domain $Q(y_m) = H^1\left(\mathbf{S}_\delta^-\right)$.

We also define for $a\in\mathbb{R}$ the sesquilinear form

$$\ell'_a[\Psi,\Psi] = \int_{\partial\mathcal{K}}\left[(1+ca)|\overline{\nabla}^N \iota\Psi|^2\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + ca\right)|\Psi|^2\right] v_{\partial\mathcal{K}},$$
$$\mathcal{Q}(\ell'_a) = H^1(\Sigma\mathcal{C}_{|\partial\mathcal{K}}). \tag{2.7.14}$$

This form is closed and lower semibounded. We denote by L'_a the associated self-adjoint operator.

We state the following density result, which allows us to express Y_m as the sum of tensor products of operators.

Lemma 2.7.1. *Let*

$$F := \left\{\Psi,\, \exists(f,\Psi_0)\in L^2(0,\delta)\times L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right),\, \Psi(x,t) = f(-t)\Gamma_0^t\left(\iota\Psi_0(x)\right)\right\}.$$

Then, $\mathrm{Span}(F)$ *is dense in* $L^2\left(\Sigma\Pi_\delta^-\right)$*, so one has a natural isomorphism* $L^2\left(\mathbf{S}_\delta^-, v_h\right) \cong L^2(0,\delta)\otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$.

Proof. Let $E := (-\delta,0)\times\mathbb{R}$ viewed as a vector bundle over $(-\delta,0)$, and $P := E\otimes\Sigma\mathcal{C}_{|\partial\mathcal{K}}$. The statement of the lemma is then equivalent to the density of $\mathrm{Span}(F')$ in $L^2(P,v_h)$ where

$$F' := \left\{\Psi,\, \exists(f,\Psi_0)\in L^2(-\delta,0)\times L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right),\, \Psi(x,t) = f(t)\Psi_0(x)\right\},$$

and this fact is standard. $\square$

We denote by Y_m the self-adjoint operator associated with y_m, and using the identification of Lemma 2.7.1, one can write

$$Y_m = (S' + m^2) \otimes 1 + 1 \otimes L'_{-\delta}.$$

Now, we define the unitary transformation

$$\mathcal{U} : L^2\left(\mathbf{S}_\delta^-\right) \longrightarrow \ell^2(\mathbb{N}) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$$
$$\mathcal{U}\Psi = (\Psi_k),\ \Psi_k = \int_0^\delta f_k(t)\, \iota^{-1}\Gamma_t^0(\Psi(\cdot,t))\mathrm{d}t.$$

By the spectral theorem, $\widehat{Y}_m := \mathcal{U}Y_m\mathcal{U}^*$ is given by its quadratic form denoted by $\widehat{y}_m$:

$$\widehat{y}_m[(\Psi_k),(\Psi_k)] = \sum_{k\in\mathbb{N}} \left(\ell'_{-\delta}[\Psi_k,\Psi_k] + (E_k(S') + m^2)\|\Psi_k\|^2_{L^2(\partial\mathcal{K})}\right),$$

and the form domain is the subset of $\ell^2(\mathbb{N}) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$ for which the right-hand side converges. Thus,

$$\begin{aligned}\mathcal{Q}(\widehat{y}_m) = \Big\{(\Psi_k) \in \ell^2(\mathbb{N}) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right),\ \Psi_k \in H^1\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)& \\ \text{and } \sum\left(\|\Psi_k\|^2_{H^1(\partial\mathcal{K})} + k^2\|\Psi_k\|^2_{L^2(\partial\mathcal{K})}\right) < \infty\Big\}.&\end{aligned} \tag{2.7.15}$$

Setting $\widehat{Y}_m^- := \mathcal{U}Y_m^-\mathcal{U}^*$, the sesquilinear form for $\widehat{Y}_m^-$ is the same as for $\widehat{Y}_m$ with the domain

$$\mathcal{Q}(\widehat{y_m^-}) = \left\{\widehat{\Psi} = (\Psi_k) \in \mathcal{Q}(\widehat{y}_m) : \mathcal{P}_-\mathcal{U}^*\widehat{\Psi}(\cdot,0) = 0\right\}. \tag{2.7.16}$$

Then, if we define

$$\begin{aligned}w_m[\widehat{\Psi},\widehat{\Psi}] := \ell'_{-\delta}[\Psi_1,\Psi_1] - b\exp(-\delta|m|)\|\Psi_1\|^2_{L^2(\partial\mathcal{K})}& \\ + \sum_{k\geq 2}\ell'_{-\delta}[\Psi_k,\Psi_k] + (b^-k^2 - b_0 + m^2)\|\Psi_k\|^2_{L^2(\partial\mathcal{K})},& \\ \mathcal{Q}(w_m) := \mathcal{Q}(\widehat{y_m^-}),&\end{aligned} \tag{2.7.17}$$

we have $\widehat{y_m^-} \geq w_m$. The form w_m is semibounded form below and closed. Let W_m be the associated self-adjoint operator. By Theorem 2.2.11, this operator has compact resolvent. For all $j \in \mathbb{N}$, one has

$$E_j(A_m^2) \geq \Lambda_j(y_m^-) = \Lambda_j(\widehat{y_m^-}) \geq E_j(W_m).$$

We can now apply the monotone convergence theorem to the non-decreasing family of self-adjoint operators (W_m). The form domain of the limit operator will be:

$$\mathcal{Q}_\infty := \left\{\widehat{\Psi} = (\Psi_k) \in \bigcap_{m<0}\mathcal{Q}(W_m),\ \sup_{m<0} W_m[\widehat{\Psi},\widehat{\Psi}] < \infty\right\}. \tag{2.7.18}$$

One has $\widehat{\Psi} := (\Psi_k) \in \mathcal{Q}_\infty$ iff $\Psi_k = 0$ for all $k \geq 2$ and $0 = \mathcal{P}_-\mathcal{U}^*\widehat{\Psi}(\cdot,0) = f_1(0)\mathcal{P}_-\Psi_1$. If we denote by $e_1 := (1,0,0,\ldots) \in \ell^2(\mathbb{N})$ this gives

$$\mathcal{Q}_\infty = \left\{\widehat{\Psi} = e_1 \otimes \Psi_1 : \Psi_1 \in \mathcal{Q}(\ell)\right\}.$$

Thus, for any $\widehat{\Psi} \in \mathcal{Q}_\infty$ one has

$$\lim_{m\to-\infty} W_m[\widehat{\Psi}, \widehat{\Psi}] = L_{-\delta}[\Psi_1, \Psi_1].$$

We define the Hilbert space $\mathbf{H}_\infty := e_1 \otimes \mathbf{H}$ and the sesquilinear form

$$w_\infty[e_1 \otimes \Psi_1, e_1 \otimes \Psi_1] = L_{-\delta}[\Psi_1, \Psi_1], \ \mathcal{Q}(w_\infty) = \mathbf{H}_\infty. \tag{2.7.19}$$

Let W_∞ be the associated self-adjoint operator. By Corollary 2.2.4 (monotone convergence), one has $\lim_{m\to-\infty} E_j(W_m) = E_j(W_\infty) = E_j(L_{-\delta})$ for all $j \in \mathbb{N}$. Letting δ go to 0 we obtain

$$\liminf_{m\to\infty} E_j(A_m^2) \geq E_j(L). \tag{2.7.20}$$

The estimates (2.7.12) and (2.7.20) together with Lemma 2.4.10 give

$$\lim_{m\to\infty} E_j\left(A_m^2\right) = E_j\left((\not{D}^{\partial\mathcal{K}})^2\right). \tag{2.7.21}$$

Remark 2.7.2. With the help of the sesquilinear form, we can investigate another asymptotic regime. Let $\Psi \in \mathrm{dom}(A_m)$ and assume $m > 0$. Proposition 2.4.6 gives that for m large enough, $\|A_m\Psi\|^2_{L^2(\mathcal{N})} \geq m^2\|\Psi\|^2_{L^2(\mathcal{N})}$. Hence, when $m \to +\infty$, one has $E_j(A_m) \to +\infty$ for all $j \in \mathbb{N}$ by the Min-Max principle.

2.8 The operator $B^2_{m,M}$ in the limit of large M

We now prove Theorem 2.1.2 following the lines of [59, Section 5]. Again, this is done by finding a lower and an upper bound for the limit of the eigenvalues of $B^2_{m,M}$. The proof relies on the localization of the problem in a neighbourhood of $\mathcal{K}$ and the construction of an appropriated extension for the spinors in $\mathcal{K}$. For the lower bound, we make another use of the monotone convergence theorem to observe that the projection $\mathcal{P}_+$ on the boundary of $\mathcal{K}$ must vanish in the asymptotic regime.

We begin with some preliminary estimates and the definition of the extension operator.

Lemma 2.8.1. *Let r'_α be the sesquilinear form given by*

$$r'_\alpha[\Psi, \Psi] := \int_{\mathcal{K}^c\setminus\Pi^+_\delta} \left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right) v_{\mathcal{N}}$$

with $\mathcal{Q}(r'_\alpha) = \{\Psi_{|\mathcal{K}^c\setminus\Pi^+_\delta},\ \Psi \in \mathrm{dom}(B_{m,M})\}$. Then, r'_α is semibounded from below.

Proof. Let $\Psi \in \mathcal{Q}(r'_\alpha)$. Let χ_1, χ_2 be two non-negative real smooth functions on $\mathcal{N}$ such that $\chi_1^2 + \chi_2^2 = 1$, χ_1 is supported in $\mathcal{K} \cup \Pi^+_{\frac{3\delta}{2}}$ and χ_2 is supported in $\mathcal{N} \setminus (\mathcal{K} \cup \Pi^+_{\frac{5\delta}{4}})$.

An easy computation gives

$$r'_\alpha[\Psi,\Psi] = r'_\alpha[\chi_1\Psi, \chi_1\Psi] + r'_\alpha[\chi_2\Psi, \chi_2\Psi] - \int_{\mathcal{K}^c\setminus\Pi^+_\delta} (|(\mathrm{d}\chi_1)\iota\Psi|^2 + |(\mathrm{d}\chi_2)\iota\Psi|^2)v_{\mathcal{N}},$$

and then there exists a constant $C_1 > 0$ such that

$$r'_\alpha[\Psi,\Psi] \geq r'_\alpha[\chi_1\Psi, \chi_1\Psi] + r'_\alpha[\chi_2\Psi, \chi_2\Psi] - C_1\|\Psi\|^2_{L^2(\mathcal{N})}.$$

Now, the Schrödinger-Lichnerowicz formula gives

$$r'_\alpha[\chi_2\Psi, \chi_2\Psi] = \|\mathcal{D}^{\mathcal{N}}\chi_k\Psi\|^2_{L^2(\mathcal{N})} \geq 0.$$

Moreover, there exists $C_2 > 0$ such that

$$r'_\alpha[\chi_1\Psi, \chi_1\Psi] \geq -C_2\|\chi_1\Psi\|^2_{L^2(\mathcal{N})}$$

because χ_1 has compact support.
Altogether, we have $r'_\alpha[\Psi, \Psi] \geq -C\|\Psi\|^2_{L^2(\mathcal{N})}$ for a constant $C > 0$. □

We define $\mathbf{S}^+_\delta := \iota(\Sigma\mathcal{C}_{|\overline{\Pi^+_\delta}})$.

Lemma 2.8.2. *For $\Psi \in \{\Phi_{|\mathcal{K}^c},\ \Phi \in \mathrm{dom}(B_{m,M})\}$ and $\alpha > 0$ we define the sesquilinear form*

$$r_\alpha[\Psi, \Psi] = \int_{\mathcal{K}^c} \left(\left|\nabla^{\mathcal{N}}\iota\Psi\right|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right) v_{\mathcal{N}} + \int_{\partial\mathcal{K}} \left(\frac{H}{2} - \alpha\right) |\Psi|^2 v_{\partial\mathcal{K}}.$$

Then, there exists $C > 0$ such that for $\alpha > 0$ large enough, one has a map F_α : $H^1(\iota(\Sigma\mathcal{C}_{|\partial\mathcal{K}})) \to \mathrm{dom}(r_\alpha)$ with $F_\alpha\Psi = \Psi$ on $\partial\mathcal{K}$ and

$$r_\alpha[F_\alpha\Psi, F_\alpha\Psi] + \alpha^2\|F_\alpha\Psi\|^2_{L^2(\mathcal{K}^c)} \leq \frac{c}{\alpha}\|\Psi\|^2_{H^1(\partial\mathcal{K})}.$$

Moreover there exists a constant $C_0 > 0$, such that $\Lambda_1(r_\alpha) \geq -\alpha^2 - C_0$.

Proof. We recall that for $\alpha > 0$ we defined in (2.7.9) the operator S associated with the sesquilinear form

$$s[f, f] = \int_0^\delta |f'|^2 dt - \alpha|f(0)|^2,\ \mathcal{Q}(s) = \left\{f \in H^1(0, \delta),\ f(\delta) = 0\right\}.$$

Let f be the first eigenfunction of the operator S normalized by $f(0) = 1$.
We define the map F_α by

$$F_\alpha\Psi(x) := \begin{cases} (\Theta\iota)^{-1}v(x) & \text{if } x \in \Pi^+_\delta \\ 0 & \text{if } x \in \mathcal{K}^c \setminus \Pi^+_\delta \end{cases}$$

where $v := f \otimes \Psi$. From Lemma 2.6.1 there exists $C > 0$ such that $\|f\|^2_{L^2(0,\delta)} \leq \frac{C}{\alpha}$ and $\alpha^2 + E_1(S) \leq Ce^{-\delta\alpha}$. Then, using Proposition 2.5.8, one can find $a > 0$ such that

$$\begin{aligned} r_\alpha[F_\alpha\Psi, F_\alpha\Psi] + \alpha^2\|F_\alpha\Psi\|^2_{L^2(\mathcal{K}^c)} &= J_\alpha(F_\alpha\Psi) + \alpha^2\|f_\alpha\|^2_{L^2(\mathcal{K}^c)} \\ &\leq \int_{\Pi^+_\delta} \left(a|\overline{\nabla}^{\mathcal{N}}\Gamma^0_t v|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}} v|^2 + (\alpha^2 + a)|v|^2\right) v_h - \alpha\int_{\partial\mathcal{K}} |\Psi|^2 v_{\partial\mathcal{K}} \\ &= \int_{\partial\mathcal{K}} \left[a|\overline{\nabla}^{\mathcal{N}}\Psi|^2 + (E_1(S) + \alpha^2 + a)|\Psi|^2\right] v_h\|f\|^2_{L^2(0,\delta)} \\ &\leq \frac{C(C+a)}{\alpha}\|\Psi\|^2_{H^1(\partial\mathcal{K})}. \end{aligned}$$

For the second assertion, we introduce the sesquilinear forms

$$r^0_\alpha[\Psi, \Psi] := \int_{\Pi^+_\delta} \left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}|\Psi|^2\right) v_{\mathcal{N}} + \int_{\partial\mathcal{K}} \left(\frac{H}{2} - \alpha\right) |\Psi|^2 v_{\partial\mathcal{K}}$$

with $\mathcal{Q}(r_\alpha^0) = \{\Psi_{|\Pi_\delta^+}, \Psi \in \mathrm{dom}(B_{m,M})\}$ and

$$r'_\alpha[\Psi,\Psi] := \int_{\mathcal{K}^c \setminus \Pi_\delta^+} \left(|\nabla^{\mathcal{N}} \iota \Psi|^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4} |\Psi|^2 \right) v_{\mathcal{N}}$$

with $\mathcal{Q}(r'_\alpha) = \{\Psi_{|\mathcal{K}^c \setminus \Pi_\delta^+}, \Psi \in \mathrm{dom}(B_{m,M})\}$. One has the inequality $\Lambda_1(r_\alpha) \geq \min(\Lambda_1(r'_\alpha), \Lambda_1(r_\alpha^0))$. Since r'_α is lower semibounded by Lemma 2.8.1, another use of Proposition 2.5.8 gives that when α is large $\Lambda_1(r_\alpha^0) \geq \Lambda_1(q_\alpha)$ with

$$q_\alpha[\Psi,\Psi] = \int_{\Pi_\delta^+} \left[\frac{1}{a} |\overline{\nabla}^{\mathcal{N}} \Gamma_t^0 \Psi|^2 + |\nabla^h_{\frac{\partial}{\partial t}} \Psi|^2 - a|\Psi|^2 \right] v_h \\ - \alpha \int_{\partial\mathcal{K}} |\Psi(\cdot,0)|^2 v_{\partial\mathcal{K}} - a \int_{\partial\mathcal{K}} |\Psi(\cdot,\delta)|^2 v_{\partial\mathcal{K}}$$

where $a > 0$ and $\mathcal{Q}(q_\alpha) = H^1\left(\mathbf{S}_\delta^+\right)$. We recall that the operator S' with parameter α was defined in Lemma 2.6.2.

For any $x \in \partial\mathcal{K}$, we define an orthonormal basis $(\psi_1, \dots, \psi_l)$ of the fiber $\mathbf{S}^+_{\delta\,|(x,0)}$, and we extend it by parallel transport along $\{x\} \times (0,\delta)$ in Π_δ^+. Let $\Psi \in \mathcal{Q}(q_\alpha)$, then for any $x \in \partial\mathcal{K}$ there are functions $(f_1^x, \dots, f_l^x)$ from $(0,\delta)$ to $\mathbb{C}$ such that $\Psi(x,t) = \sum_1^l f_k^x \psi_k$. With these notations and using Fubini's theorem, we obtain

$$\begin{aligned} q_\alpha[\Psi,\Psi] &\geq \int_{\Pi_\delta^+} \left[|\nabla^h_{\frac{\partial}{\partial t}} \Psi|^2 - a|\Psi|^2 \right] v_h - \alpha \int_{\partial\mathcal{K}} |\Psi(\cdot,0)|^2 v_{\partial\mathcal{K}} - a \int_{\partial\mathcal{K}} |\Psi(\cdot,\delta)|^2 v_{\partial\mathcal{K}} \\ &= \int_{\partial\mathcal{K}} \left[\sum_{k=1}^l (s' - a)[f_k^x, f_k^x] \right] v_{\partial\mathcal{K}} \geq \int_{\partial\mathcal{K}} \left[(\Lambda_1(S') - a) \sum_{k=1}^l \|f_k^x\|^2_{L^2(0,\delta)} \right] v_{\partial\mathcal{K}} \\ &= (\Lambda_1(S') - a) \|\Psi\|^2_{L^2(\Pi_\delta^+)}. \end{aligned}$$

We deduce that $\Lambda_1(r_\alpha) \geq \Lambda_1(q_\alpha) \geq \Lambda_1(S') - a \geq -\alpha^2 - C$ with $C > 0$ when $\alpha \to +\infty$. □

Using Proposition 2.4.8, the sesquilinear form for $B_{m,M}^2$ can be written for any spinor $\Psi \in \mathrm{dom}(B_{m,M})$ and any $\varepsilon > 0$ as

$$\begin{aligned} \|B_{m,M}\Psi\|^2_{L^2(\mathcal{N})} = &\int_{\mathcal{K}} \left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + m^2 \right) |\Psi|^2 \right] v_{\mathcal{N}} \\ &+ \int_{\partial\mathcal{K}} \left(m - \varepsilon - \frac{H}{2} \right) |\Psi|^2 v_{\partial\mathcal{K}} + 2(M - m) \int_{\partial\mathcal{K}} |\mathcal{P}_-\Psi|^2 v_{\partial\mathcal{K}} \\ + \int_{\mathcal{K}^c} &\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + M^2 \right) |\Psi|^2 \right] v_{\mathcal{N}} - \int_{\partial\mathcal{K}} \left(M - \varepsilon - \frac{H}{2} \right) |\Psi|^2 v_{\partial\mathcal{K}} \end{aligned} \tag{2.8.1}$$

where we recall that $\mathcal{P}_- = \frac{1 - i\nu\cdot\mathbf{n}\cdot}{2}$.

2.8.1 Upper bound

We are now able to find an upper bound for the limit of $E_j(B_{m,M}^2)$ when $M \to +\infty$ for $j \in \mathbb{N}$. Let $\eta > 0$ and pick $(\Psi_1, \dots, \Psi_j)$ in $\Gamma(\Sigma\mathcal{C}_{|\mathcal{K}})$, smooth spinors such that

$$\inf_{\Psi \in \mathrm{Span}(\Psi_1,\dots,\Psi_j)} \frac{\langle A_m^2 \Psi, \Psi \rangle_{L^2(\mathcal{K})}}{\|\Psi\|^2_{L^2(\mathcal{K})}} \leq E_j(A_m^2) + \eta.$$

We define $a := \sup\left\{\|\Psi\|^2_{H^1(\partial\mathcal{K})}, \Psi \in \mathrm{Span}(\Psi_1,\dots,\Psi_j), \|\Psi\|_{L^2(\mathcal{K})} = 1\right\}$. Let $\Psi \in V := \mathrm{Span}(\Psi_1,\dots,\Psi_j)$ and

$$\widetilde{\Psi} := \begin{cases} \Psi & \text{in } \mathcal{K} \\ F_M(\Psi_{|\partial\mathcal{K}}) & \text{in } \mathcal{K}^c. \end{cases}$$

By Lemma 2.8.2 there is a constant $C > 0$ such that

$$\int_{\mathcal{K}^c}\left[|\nabla^{\mathcal{N}}(\iota\widetilde{\Psi})|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + M^2\right)|\widetilde{\Psi}|^2\right]v_{\mathcal{N}} - \int_{\partial\mathcal{K}}\left(M - \frac{H}{2}\right)|\widetilde{\Psi}|^2 v_{\partial\mathcal{K}}$$
$$= r_M[\widetilde{\Psi},\widetilde{\Psi}] + M^2\|\widetilde{\Psi}\|^2_{L^2(\mathcal{K}^c)} \le \frac{C}{M}\|\widetilde{\Psi}\|^2_{H^1(\partial\mathcal{K})} \le \frac{Ca}{M}\|\Psi\|^2_{L^2(\mathcal{K})}.$$

Then, using the expression (2.8.1) with $\varepsilon = 0$,

$$\|B_{m,M}\widetilde{\Psi}\|^2_{L^2(\mathcal{N})} \le A_m^2[\Psi,\Psi] + \frac{Ca}{M}\|\Psi\|^2_{L^2(\mathcal{K})} \le \left(E_j(A_m^2) + \eta + \frac{Ca}{M}\right)\|\Psi\|^2_{L^2(\mathcal{K})}$$
$$\le \left(E_j(A_m^2) + \eta + \frac{Ca}{M}\right)\|\widetilde{\Psi}\|^2_{L^2(\mathcal{K})}$$

and letting η go to zero one gets $\limsup_{M\to+\infty} E_j(B^2_{m,M}) \le E_j(A_m^2)$.

2.8.2 Lower bound

It remains to find a lower bound for the eigenvalues. In order to do so, we separate the representation (2.8.1) in the two parts corresponding to $\mathcal{K}$ and $\mathcal{K}^c$ and we remark that the outer part becomes very large when M goes to $+\infty$ so the eigenvalues must converge to the eigenvalues of an operator in $\mathcal{K}$.

Let $j \in \mathbb{N}$. One has

$$E_j(B^2_{m,M}) \ge \min\left\{\Lambda_j(k^c_{M,\varepsilon}), E_j(K_{m,M,\varepsilon})\right\}$$

where $K_{m,M,\varepsilon}$ is the operator associated with the sesquilinear form

$$k_{m,M,\varepsilon}[\Psi,\Psi] := \int_{\mathcal{K}}\left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \left(m^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}\right)|\Psi|^2\right)v_{\mathcal{N}}$$
$$+ \int_{\partial\mathcal{K}}(m - \varepsilon - \frac{H}{2})|\Psi|^2 v_{\partial\mathcal{K}} + 2(M-m)\int_{\partial\mathcal{K}}|P_-\Psi|^2 v_{\partial\mathcal{K}} \quad (2.8.2)$$

and $k^c_{M,\varepsilon}$ is the sesquilinear form

$$k^c_{M,\varepsilon}[\Psi,\Psi] := \int_{\mathcal{K}^c}\left(|\nabla^{\mathcal{N}}\iota\Psi|^2 + \left(M^2 + \frac{\mathrm{Scal}^{\mathcal{N}}}{4}\right)|\Psi|^2\right)v_N$$
$$- \int_{\partial\mathcal{K}}(M - \varepsilon - \frac{H}{2})|\Psi|^2 v_{\partial\mathcal{K}} \quad (2.8.3)$$

where the respective domains are the restrictions of $\mathrm{dom}(B_{m,M})$ to $\mathcal{K}$ and $\mathcal{K}^c$.

One has $k^c_{M,\varepsilon} = r_{M-\varepsilon} + M^2$, where $r_{M-\varepsilon}$ was defined in Lemma 2.8.2). The same lemma gives

$$\Lambda_1(k^c_{M,\varepsilon}) = \Lambda_1(r_{M-\varepsilon} + M^2) \ge -(M-\varepsilon)^2 - C_0 + M^2 = 2\varepsilon M - \varepsilon^2 - C_0$$

$$= \varepsilon M + (\varepsilon M - \varepsilon^2 - C_0) \geq \varepsilon M \text{ when } M \to +\infty.$$

It follows that $E_j(B^2_{m,M}) = E_j(K_{m,M,\varepsilon})$ when $M \to +\infty$. But $k_{M,m,\varepsilon}$ is increasing in$M$, and

$$k_{M,m,\varepsilon}[\Psi,\Psi] \underset{M\to+\infty}{\longrightarrow} \langle A_m\Psi, A_m\Psi\rangle_{L^2(\mathcal{K})} - \varepsilon\|\Psi\|_{L^2(\partial\mathcal{K})}.$$

Furthermore,

$$\left\{\Psi \in \bigcap_{M>0} \mathrm{dom}(k_{m,M,\varepsilon}),\ \lim_{M\to+\infty} k_{m,M,\varepsilon}[\Psi,\Psi] < \infty\right\} = \mathrm{dom}(A_m),$$

thus, by monotone convergence (Corollary 2.2.4) and letting ε go to 0, we obtain $\liminf_{M\to+\infty} E_j(B^2_{m,M}) = E_j(A^2_m)$. Taking into account the upper bound obtained above, one gets $\lim_{M\to+\infty} E_j(B^2_{m,M}) = E_j(A^2_m)$.

2.9 The operator $B_{m,M}$ for large masses

In this section, we investigate the asymptotic regime $m \to -\infty$ and $M \to +\infty$ and we give a proof of Theorem 2.1.3. The method we use is very similar to the one of section 2.8. The difference lies in the proof of the lower bound, where we do not make the analysis on the operator outside and inside $\mathcal{K}$, but we rather divide the ambient space into three pieces: the tubular neighbourhood of $\partial\mathcal{K}$, and the remaining regions lying inside and outside the compact $\mathcal{K}$. By Dirichlet-Neumann bracketing, it is then sufficient to study the operator restricted to the tubular neighbourhood to conclude.

2.9.1 Upper bound

In this section, we write $\mathbf{S}_\delta := \iota\left(\Sigma\mathcal{C}_{|\overline{\Pi_\delta}}\right)$. We recall that for $\alpha \in \mathbb{R}$ we defined the self-adjoint operator S_α associated with the quadratic form

$$s_\alpha[f,f] = \int_0^\delta |f'|^2 \mathrm{d}t - \alpha|f(0)|^2,\ \mathcal{Q}(s_\alpha) = \left\{f \in H^1(0,\delta), f(\delta) = 0\right\}, \tag{2.9.1}$$

and denoting by f_α the L^2-normalized eigenfunction associated with $E_1(S_\alpha)$, one has $|f_\alpha(0)|^2 = 2\alpha + \mathcal{O}(1)$ and $E_1(S_\alpha) = \alpha^2 + \mathcal{O}(e^{-\alpha\delta})$ when $\alpha \to +\infty$ (see Lemma 2.6.1).

The operator L_a was defined by the quadratic form (2.7.10).

Let $j \in \mathbb{N}$ and $\Psi_1, \ldots, \Psi_j$ be j eigenspinors for the first j eigenvalues of L_δ. For $\Psi \in V := \mathrm{Span}(\Psi_1, \ldots, \Psi_j)$, we define the extension operator $\mathcal{E} : H^1(\mathbf{S}_\delta) \to H^1(\Sigma\mathcal{C}_{|\mathcal{N}})$ by

$$\mathcal{E}\Psi := \begin{cases} \frac{|f_{-m}(0)|}{|f_M(0)|}(\Theta\iota)^{-1}(\Psi\otimes f_M) & \text{in } \Pi^+_\delta \\ (\Theta\iota)^{-1}(\Psi\otimes f_{-m}) & \text{in } \Pi^-_\delta \\ 0 & \text{in } \mathcal{N}\setminus\Pi_\delta \end{cases}. \tag{2.9.2}$$

One easily sees that $\|\mathcal{E}\Psi\|^2_{L^2(\mathcal{N})} = \left(1 + \left(\frac{f_{-m}(0)}{f_M(0)}\right)^2\right)\|\Psi\|^2_{L^2(\partial\mathcal{K})}$, so the operator $\mathcal{E}$ is injective.

We use the expression (2.8.1) and Proposition 2.5.8 to compute:

$$
\begin{aligned}
\|B^2_{m,M}\mathcal{E}\Psi\|^2_{L^2(\mathcal{N})} &= \int_{\mathcal{K}} \left[|\nabla^{\mathcal{N}}(\iota\mathcal{E}\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + m^2\right)|\mathcal{E}\Psi|^2\right] v_{\mathcal{N}} \\
&+ \int_{\partial\mathcal{K}} \left(m - \varepsilon - \frac{H}{2}\right)|\mathcal{E}\Psi|^2 v_{\partial\mathcal{K}} + \int_{\mathcal{K}^c}\left[|\nabla^{\mathcal{N}}(\iota\mathcal{E}\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + M^2\right)|\mathcal{E}\Psi|^2\right] v_{\mathcal{N}} \\
&- \int_{\partial\mathcal{K}}\left(M - \varepsilon - \frac{H}{2}\right)|\mathcal{E}\Psi|^2 v_{\partial\mathcal{K}} \\
&\leq \int_{\Pi^-_\delta}\left[(1+c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma^0_t\Psi\otimes f_{-m})(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi\otimes f_{-m}|^2\right] v_h \\
&+ \int_{\Pi^-_\delta}\left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + m^2 + c\delta\right)|\Psi\otimes f_{-m}|^2\right] v_{\partial\mathcal{K}}\mathrm{d}t \\
&+ \int_{\Pi^+_\delta}\left[(1+c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma^0_t\Psi\otimes f_{M})(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi\otimes f_{M}|^2\right] v_h \\
&+ \int_{\Pi^+_\delta}\left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + M^2 + c\delta\right)|\Psi\otimes f_{M}|^2\right] v_{\partial\mathcal{K}}\mathrm{d}t \\
&+ \int_{\partial\mathcal{K}}(-m|\Psi\otimes f_{-m}(\cdot,0)|^2 + M|(\Psi\otimes f_M)(\cdot,0)|^2) v_{\partial\mathcal{K}} \\
&\leq \left(1 + \left(\frac{f_{-m}(0)}{f_M(0)}\right)^2\right)\left[\ell_\delta[\Psi,\Psi] + C\|\Psi\|^2_{L^2(\partial\mathcal{K})}\left(e^{-M\delta} + e^{-|m|\delta}\right)\right]
\end{aligned}
$$

where $C > 0$.
The Min-Max principle gives

$$
\begin{aligned}
E_j(B^2_{m,M}) &\leq \sup_{\Psi\in V}\frac{B^2_{m,M}[\mathcal{E}\Psi,\mathcal{E}\Psi]}{\|\mathcal{E}\Psi\|^2_{L^2(\mathcal{N})}} \\
&\leq \sup_{v\in V}\left[L_\delta[\Psi,\Psi] + C\|\Psi\|^2_{L^2(\partial\mathcal{K})}\left(e^{-M\delta} + e^{-|m|\delta}\right)\right]\|\Psi\|^{-2}_{L^2(\partial\mathcal{K})} \\
&\leq E_j(L_\delta) + C\left(e^{-M\delta} + e^{-|m|\delta}\right).
\end{aligned}
$$

We now let $\min(-m,M)\to+\infty$, so we obtain

$$
\limsup_{\min(-m,M)\to+\infty} E_j(B^2_{m,M}) \leq E_j(L_\delta).
$$

On the other hand, δ can be taken arbitrary small, and one has the obvious limit $E_j(L) \underset{\delta\to 0}{\longrightarrow} E_j(L)$, so we arrive at

$$
\limsup_{\min(-m,M)\to+\infty} E_j(B^2_{m,M}) \leq E_j(L). \tag{2.9.3}
$$

2.9.2 Lower bound

We consider the lower semibounded sesquilinear forms

$$
k_{m,M}[\Psi,\Psi] = \int_{\mathcal{N}\setminus\Pi_\delta}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + m^2\mathbf{1}_{\mathcal{K}} + M^2\mathbf{1}_{\mathcal{K}^c}\right)|\Psi|^2\right] v_{\mathcal{N}}
$$

$$\mathcal{Q}(K_{m,M}) = \{\Psi_{\mathcal{N}\setminus\Pi_\delta},\ \Psi \in \mathrm{dom}(B_{m,M})\} \quad (2.9.4)$$

and

$$\begin{aligned} k'_{m,M}[u,u] = &\int_{\Pi_\delta^-}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + m^2\right)|\Psi|^2\right] v_{\mathcal{N}} \\ &+ \int_{\partial\mathcal{K}}\left(m - \varepsilon - \frac{H}{2}\right)|\Psi|^2 v_{\partial\mathcal{K}} + 2(M-m)\int_{\partial\mathcal{K}} |\mathcal{P}_-\Psi|^2 v_{\partial\mathcal{K}} \\ &+ \int_{\Pi_\delta^+}\left[|\nabla^{\mathcal{N}}(\iota\Psi)|^2 + \left(\frac{\mathrm{Scal}^{\mathcal{N}}}{4} + M^2\right)|\Psi|^2\right] v_{\mathcal{N}} - \int_{\partial\mathcal{K}}\left(M - \varepsilon - \frac{H}{2}\right)|\Psi|^2 v_{\partial\mathcal{K}}, \\ &\mathcal{Q}(K'_{m,M}) = H^1(\Sigma\mathcal{C}_{\overline{\Pi_\delta}}). \quad (2.9.5) \end{aligned}$$

We denote by $K'_{m,M}$ the operator associated with $k'_{m,M}$.

Let $j \in \mathbb{N}$. The Min-Max principle gives the lower estimate $E_j(B^2_{m,M}) \geq \min(E_j(K'_{m,M}), \Lambda_1(k_{m,M}))$, and by Lemma 2.8.1 there is a constant $C > 0$ such that $\Lambda_1(k_{m,M}) \geq \min(m^2, M^2) - C$. This last quantity goes to $+\infty$ in the asymptotic regime under consideration, and we know thanks to the upper bound that $E_j(B^2_{m,M}) = \mathcal{O}(1)$. Thus, in the given asymptotic regime one has $E_j(B^2_{m,M}) \geq E_j(K'_{m,M})$.

We now apply a transformation to the operator $K'_{m,M}$ written in tubular coordinates, and we consider the operator $P_{m,M}$ associated with the quadratic form

$$\begin{aligned} p_{m,M}[\Psi,\Psi] = &\int_{\Pi_\delta}\left[(1-c\delta)\left|(\overline{\nabla}^{\mathcal{N}}\Gamma^0_t\Psi)(x,0)\right|^2 + |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi|^2\right] v_h \\ &+ \int_{\Pi_\delta}\left[\left(\frac{\mathrm{Scal}^{\partial\mathcal{K}} - \mathrm{Tr}(W^2)}{4} + m^2\mathbf{1}_{\mathcal{K}} + M^2\mathbf{1}_{\mathcal{K}^c} - c\delta\right)|\Psi|^2\right] v_{\partial\mathcal{K}}\mathrm{d}t \\ &+ (m-M)\int_{\partial\mathcal{K}}|\Psi(\cdot,0)|^2 v_{\partial\mathcal{K}} - c\int_{\partial\mathcal{K}}|\Psi(\cdot,\delta)| v_{\partial\mathcal{K}} + 2(M-m)\int_{\partial\mathcal{K}}|\mathcal{P}_-\Psi|^2 v_{\mathcal{K}}, \\ &\mathcal{Q}(p_{m,M}) = H^1(S_\delta), \quad (2.9.6) \end{aligned}$$

where $c > 0$ is chosen so that Proposition 2.5.8 is valid, implying that $E_j(K'_{m,M}) \geq E_j(P_{m,M})$.

For $a \in \mathbb{R}$, let L''_a be the operator given by the sesquilinear form ℓ''_a, having the same expression as (2.7.10) but with the domain $H^1(\Sigma\mathcal{C}_{|\partial\mathcal{K}})$.

Let $P'_{m,M}$ be the operator defined by the same quadratic form as in (2.9.6) but without the term involving the operator $\mathcal{P}_-$. We recall that the one-dimensional operator X was defined by (2.6.1), so one has

$$P'_{m,M} = \ell''_{-\delta} \otimes 1 + 1 \otimes X.$$

Let (f_k) be a sequence of L^2-normalized eigenfunctions for the eigenvalues $E_k(X)$. We define the unitary transformation

$$\mathcal{U} : L^2(\mathbf{S}_\delta) \longrightarrow \ell^2(\mathbb{N}) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$$

$$\mathcal{U}\Psi = (\Psi_k),\ \Psi_k = \int_{-\delta}^{\delta} f_k(t)\,\iota^{-1}\Gamma^0_t(\Psi(t,\cdot))\mathrm{d}t.$$

Let $\widehat{P}'_{m,M} := \mathcal{U}P'_{m,M}\mathcal{U}^*$. This is a self-adjoint operator acting on $\ell^2(\mathbb{N}) \otimes L^2\left(\Sigma\mathcal{C}_{|\partial\mathcal{K}}\right)$. One can write

$$\widehat{P}'_{m,M}[\widehat{v},\widehat{v}] = \sum_{k\in\mathbb{N}} \left(\ell''_{-\delta}[\Psi_k,\Psi_k] + E_k(X)\|\Psi_k\|^2_{L^2(\Sigma)}\right),$$

$$\mathcal{Q}(\widehat{P}'_{m,M}) = \Big\{\widehat{\Psi} \in \ell^2(\mathbb{N}) \otimes L^2(\Sigma\mathcal{C}_{|\partial\mathcal{K}}), \Psi_k \in H^1(\Sigma\mathcal{C}_{|\partial\mathcal{K}}),$$

$$\sum_{k\in\mathbb{N}} \left(\|\Psi_k\|^2_{H^1(\partial\mathcal{K})} + k^2\|\Psi_k\|^2_{L^2(\partial\mathcal{K})}\right)\Big\}. \quad (2.9.7)$$

The operator $\widehat{P}_{m,M} = \mathcal{U}^* P_{m,M}\mathcal{U}$ has the same form domain as $\widehat{P}'_{m,M}$ and

$$\widehat{P}_{m,M}[\widehat{\Psi},\widehat{\Psi}] = \sum_{k\in\mathbb{N}} \left(\ell''_{-\delta}[\Psi_k,\Psi_k] + E_k(X)\|\Psi_k\|^2_{L^2(\Sigma)}\right) + 2(M+|m|)\int_\Sigma |\mathcal{P}_-\mathcal{U}^*\widehat{\Psi}|^2 \mathrm{d}s.$$

where the operator X was defined in (2.6.1). We set

$$\zeta := \min(M,-m). \quad (2.9.8)$$

Using Lemma 2.6.3, we consider the quadratic form w_ζ defined by

$$w_\zeta[\widehat{\Psi},\widehat{\Psi}] = \ell''_{-\delta}[\Psi_1,\Psi_1] - Ce^{-\zeta\delta/2} + 4\zeta\int_\Sigma |\mathcal{P}_-\mathcal{U}^*\widehat{\Psi}|^2\mathrm{d}s$$
$$+ \sum_{k\geq 2}\left(\ell''_{-\delta}[\Psi_k,\Psi_k] + (C_1k^2 - C_2)\|\Psi_k\|^2_{L^2(\Sigma,\mathbb{C}^N)} + \zeta^2\|\Psi_k\|^2_{L^2(\Sigma)}\right),$$

$$\mathcal{Q}(w_\zeta) = \mathcal{Q}(\widehat{P}_{m,M}), \quad (2.9.9)$$

and we claim that $\widehat{P}_{m,M} \geq w_\zeta$ for a suitable $C > 0$. The form w_ζ is semibounded from below and closed, and we define the associated self-adjoint operator W_ζ with compact resolvent. The previous discussion gives the lower estimate $E_j(B^2_{m,M}) \geq E_j(W_\zeta)$ in the asymptotic regime.

In order to apply the monotone convergence theorem, we define

$$\mathcal{Q}_\infty = \left\{\widehat{\Psi} \in \bigcap_{\zeta>0} \mathcal{Q}(W_\zeta) = \mathcal{Q}(w_\zeta),\ \sup_{\zeta>0} w_\zeta[\widehat{\Psi},\widehat{\Psi}] < +\infty\right\}. \quad (2.9.10)$$

We easily see that $\widehat{\Psi}$ is in $\mathcal{Q}_\infty$ if and only if $\Psi_k = 0$ for all $k \geq 2$ and $\mathcal{P}_-\mathcal{U}^*\widehat{\Psi} = 0$, which is equivalent to $\widehat{\Psi} = e_1 \otimes \Psi_1$ with $e_1 := (1,0,0,\ldots)$ and $\mathcal{P}_-\Psi_1 = 0$. It follows that $\mathcal{Q}_\infty = \{e_1 \otimes \Psi_1 : \Psi_1 \in H^1(\Sigma,\mathbb{C}^N) \cap \mathcal{H}\}$. Moreover, we have

$$\lim_{\zeta\to\infty} W_\zeta[e_1\otimes\Psi_1, e_1\otimes\Psi_1] = L_{-\delta}[\Psi_1,\Psi_1]. \quad (2.9.11)$$

Thus, if we define the operator $W_\infty[e_1\otimes\Psi_1, e_1\otimes\Psi_1] := L_{-\delta}[\Psi_1,\Psi_1]$ on $e_1\otimes\mathcal{H}$, the monotone convergence theorem gives $\lim_{\zeta\to\infty} E_j(W_\zeta) = E_j(L_{-\delta})$. Altogether, we arrive at $\liminf_{\min(-m,M)\to+\infty} E_j(B^2_{m,M}) \geq E_j(L_{-\delta})$. We now let δ go to zero and we obtain $\liminf_{\min(-m,M)\to+\infty} E_j(B^2_{m,M}) \geq E_j(L)$. The upper and the lower bounds together give

$$\lim_{\min(-m,M)\to+\infty} E_j(B^2_{m,M}) = E_j(L) = E_j\left((\not{D}^{\partial\mathcal{K}})^2\right). \quad (2.9.12)$$

Remark 2.9.1. We can look at the asymptotic regime $M \to +\infty$ and $m \to +\infty$. Let $(m_k, M_k)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^2$ such that $m_k, M_k \underset{k\to+\infty}{\longrightarrow} +\infty$. In this case, we can use the inequality $E_1(B^2_{m,M}) \geq E_1(P_{m,M})$, and for any $\Psi \in \mathcal{Q}(p_{m,M})$ there exists a constant $C > 0$ such that

$$p_{m,M}[\Psi,\Psi] \geq \int_{\Pi_\delta} |\nabla^{\mathcal{N}}_{\frac{\partial}{\partial t}}\Psi|^2 v_h + \int_{\Pi_\delta} \left[m^2\mathbf{1}_{(0,\delta)} + M^2\mathbf{1}_{(-\delta,0)} - C\right] |\Psi|^2 v_h \\ - C\int_{\partial\mathcal{K}} |\Psi(\cdot,\delta)| v_{\partial\mathcal{K}} - |M-m| \int_{\partial\mathcal{K}} |\Psi|^2 v_{\mathcal{K}}.$$

Without loss of generality, we can assume that there is a subsequence of (M_k, m_k) still denoted by (M_k, m_k) such that $M_k \geq m_k$ for all k. We have

$$p_{m_k,M_k}[\Psi,\Psi] \geq m_k^2\|\Psi\|^2_{L^2(\Pi^-_\delta)} + \|\Psi\|^2_{L^2(\Pi^+_\delta)}(M_k^2 + E_1(S_{M_k-m_k})) - C\|\Psi\|^2_{L^2(\Pi_\delta)},$$

but when k is large there is a constant C_1 such that

$$\begin{aligned} M_k^2 + E_1(S_{M_k-m_k}) &\geq M_k^2 - M_k^2 - m_k^2 + 2M_km_k - C_1 \\ &\geq 2M_km_k - m_k^2 - C_1 \geq m_k^2 - C_1. \end{aligned}$$

Thus, $E_1(B^2_{m_k,M_k}) \geq E_1(P_{m_k,M_k}) \geq m_k^2 - C - C_1 \underset{k\to+\infty}{\longrightarrow} +\infty$. This means that every sequence $E_1(B^2_{m_k,M_k})$ admits a divergent subsequence, an we conclude that $E_1(B^2_{m,M}) \to +\infty$ in this regime.

By similar constructions, the same result holds for $m, M \to -\infty$ as well.

Chapter 3

Cauchy spinors on 3-manifolds

Les lignes de ce chapitre proviennent d'un article co-écrit avec Sergiu Moroianu, et paru dans *Journal of Geometric Analysis*, **32**, 186 (2022). On y démontre certaines propriétés des spineurs de Cauchy sur les 3-variétés, et on établit des résultats de classification de ces mêmes spineurs sur $\mathbf{S}^3$.

3.1 Introduction

The Cauchy problem for parallel spinors

If (M, g) is an oriented hypersurface in a spin manifold $(\mathcal{Z}, g_{\mathcal{Z}})$ and Ψ is a parallel spinor on $\mathcal{Z}$, then the restriction $\psi := \Psi|_M$ is a spinor on M satisfying the initial condition

$$\nabla_X \psi = -\frac{A(X)}{2} \cdot \psi, \qquad (\forall) X \in TM, \tag{3.1.1}$$

where A is the second fundamental form of M (see [33, Proposition 1.4.1]).

Conversely, given a spinor ψ on M satisfying the constraint (3.1.1) for some symmetric endomorphism A, is it always possible to embed M as a hypersurface in an ambient manifold $\mathcal{Z}$ with second fundamental form A, and such that ψ is the restriction to M of a parallel spinor Ψ on $\mathcal{Z}$? In other words, is (3.1.1) the only constraint for the existence of $\mathcal{Z}$ and Ψ? This is the so-called Cauchy problem for parallel spinors studied in [3]. The answer is positive when all the objects involved are real analytic, and negative in general in the smooth setting for $\dim(M) \geq 3$ [3, Theorems 1.1 and 4.27].

A spinor and symmetric 2-tensor on M satisfying Equation (3.1.1) will be called below a *Cauchy spinor*, respectively a *Cauchy endomorphism*. Since (3.1.1) includes as a particular case the Killing spinor equation (i.e., when A is a constant multiple of the identity), Cauchy spinors were sometimes called *generalized Killing spinors* e.g. in [60–62]. Different generalizations of the notion of Killing spinors appear however in the literature, e.g. in the papers [38] or [30, 31]. We believe that the current name should be more appropriate, as it describes more accurately the property of being the restriction of a parallel spinor to a hypersurface.

The classification problem for Cauchy spinors and endomorphisms on a given manifold M requires us to describe all pairs (ψ, A) verifying (3.1.1). In dimension 3, partial results in this direction were found in [60], where Cauchy spinors are characterized in terms of an orthonormal frame of divergence-free vector fields on M. The same authors investigated in [61, 62] the case of the sphere $\mathbf{S}^3$, classifying all Cauchy endomorphisms having at most 2 distinct eigenvalues. This example illustrates the little understanding we have of Cauchy spinors in dimension 3, as we are unable to classify them even on the round sphere. Note that a complete description can be given in several other dimensions [62].

Cauchy spinors on 3-manifolds and flat connections

Spin geometry in dimension 3 is special because the Hodge $*$ operator allows an exceptional identification between 1- and 2-forms, and moreover the real spinor bundle carries a quaternionic structure. Using these algebraic structures, on simply connected 3-manifolds (which by the Poincaré conjecture must be diffeomorphic to the sphere) we can restate the classification problem for Cauchy spinors without mentioning spinors at all! Indeed, Equation (3.1.1) implies a constraint for the symmetric endomorphism field A:

$$0 = \mathrm{R}(X, Y) + *d^{\nabla} A(X, Y) + A(X) \wedge A(Y), \qquad (\forall) X, Y \in TM. \tag{3.1.2}$$

Here R is the Riemann curvature tensor and d^{∇} is the exterior covariant derivative on M mapping sections of $\Lambda^1 M \otimes TM$ to sections of $\Lambda^2 M \otimes TM$. The set of symmetric endomorphisms satisfying (3.1.2) is denoted $\mathcal{C}_M^{\mathrm{loc}}$. If M is simply connected, every solution of Equation (3.1.2) also satisfies (3.1.1) for some Cauchy spinor, unique up to right multiplication by a quaternion. Our strategy below is to exploit Equation (3.1.2) in order to obtain new results on the Cauchy problem for parallel spinors, and also on the classification problem for Cauchy spinors.

Equation (3.1.2) amounts to the flatness of the modified metric connection $\nabla^A = \nabla + *A$ on TM. Even in the compact and simply connected case, the structure of the set of flat connections $\mathcal{C}_M^{\mathrm{loc}}$ remains elusive, in part because Equation (3.1.2) is non-linear and not elliptic. For this reason, we first study the linearization of (3.1.2). We show that if the scalar curvature of M is positive, the space of infinitesimal deformations, defined as the space of symmetric endomorphism fields solution to the linearization of (3.1.2), is finite-dimensional (Theorem 3.3.2). This can be interpreted as a finiteness result for the dimension of the "tangent space" of $\mathcal{C}_M^{\mathrm{loc}}$, with the caveat that this set of flat connections is a priori not a smooth manifold. The hypothesis on the sign of the scalar curvature is necessary: we exhibit flat compact 3-manifolds for which the dimension of the space of deformations is infinite.

Cauchy endomorphisms on the round three-sphere

We view $\mathbf{S}^3$ as the Lie group of unit-length quaternions. Four examples of symmetric endomorphisms A fulfilling the flatness condition (3.1.2) on $\mathbf{S}^3$ are known from [60, Example 3.2]: $\pm\mathrm{Id}$, and the endomorphism fields constant in a left (resp. right)-invariant orthonormal frame, with eigenvalues $1, -3, -3$ (respectively $-1, 3, 3$). It was already shown in [62] that there are no deformations around $A = \pm\mathrm{Id}$. We prove that the space of infinitesimal deformations around the other two examples has dimension 2, and corresponds to the Lie derivative of A in the direction of a left (or right-) invariant vector field from $\ker(A \pm 3\mathrm{Id})$. In particular, there are no other deformations of the above solutions. If $\mathcal{C}_{\mathbf{S}^3}^{\mathrm{loc}}$ were a smooth manifold, it would therefore necessarily have at least 4 connected components.

The examples of endomorphisms on $\mathbf{S}^3$ given above are analytic, so by [3] they can be realized as second fundamental forms of the three-sphere embedded as a hypersurface in a generalized cylinder $\mathcal{Z} := (-\epsilon, \epsilon) \times \mathbf{S}^3$ carrying a parallel spinor. The cases $A = \pm\mathrm{Id}$ both induce the standard embedding of $\mathbf{S}^3$ into $\mathbb{R}^4$. We calculate in Section 3.4 an explicit expression of this metric in the other two cases, finding an extension of the family of Euclidean Taub-NUT metrics with a negative parameter. This computation solves the long-time Cauchy problem on the three-sphere for the four known examples of Cauchy spinors on $\mathbf{S}^3$.

Classification results

In the final section we prove three classification results for symmetric endomorphisms solving Equation (3.1.2) on $\mathbf{S}^3$. The known examples of Cauchy endomorphisms have constant matrices in a left or right-invariant orthonormal basis of the tangent bundle of the Lie group $\mathbf{S}^3$. It is natural to ask if these are the only symmetric endomorphisms solutions to (3.1.2) with this property. We prove in Proposition 3.6.1 that this is indeed the case. Moreover, since all the solutions of (3.1.2) with at most two eigenvalues are known [62, Theorem 4.10], we investigate the case where A has three distinct constant eigenvalues. We show that there is no solution in this case, using a characterization of the Hopf fields on $\mathbf{S}^3$ (Proposition 3.6.2). Finally, we show that the solutions which are constant only in the direction of a left or right-invariant eigenvector ξ of A, i.e., such that $\mathcal{L}_\xi A = 0$, must also be constant in a left (respectively right) invariant orthonormal frame, so they fall in the class of previously known examples (Proposition 3.6.4). This result turns out to imply (a slight extension of) the Liebmann rigidity theorem [49].

Related results

Cauchy spinors in $3+1$ Lorentzian signature were recently classified by Murcia and Shahbazi [64], [65], thanks to the fact that one of the elements in the global coframe defined by the Cauchy spinor on the hypersurface is privileged, and determines an integrable foliation. On the other hand, unlike in the Riemannian case, the Lorentzian initial value problem for parallel spinors is of hyperbolic nature and has solutions in the smooth category. Thus the two problems are in fact rather different, despite some formal similarities.

3.2 Preliminaries

3.2.1 Spinors in dimension 3

The real Clifford algebra Cl_2 is canonically isomorphic to the quaternion algebra $\mathbb{H}$ by the map sending e_1 to i and e_2 to j, where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. It follows that the even Clifford algebra $\mathrm{Cl}_3^{\mathrm{even}}$ is also isomorphic to $\mathbb{H}$ by the unique algebra map sending e_1e_3 to i, e_2e_3 to j, and hence e_2e_3 to k.

Multiplication by the central element $P := \frac{1-e_1e_2e_3}{2}$ is a projector in Cl_3. Let Σ_3 be the image of this projector. Then P maps $\mathrm{Cl}_3^{\mathrm{even}}$ isomorphically onto Σ_3, thus identifying Σ_3 to $\mathbb{H}$. Since P is central, $\mathrm{Cl}_3^{\mathrm{even}}$ acts on Σ_3 by left multiplication, and this representation commutes with the right action of $\mathbb{H} \simeq \Sigma^3$ on itself. The restriction of this quaternionic representation to the spin group Spin_3 is the spinor representation. By construction, the spinor representation is thus the restriction of a Cl_3 representation under which the volume element $e_1e_2e_3$ acts as minus the identity.

The spinor representation is orthogonal with respect to the natural scalar product on Σ_3. The right multiplication by quaternions is also compatible with the scalar product, in the sense that $\langle \psi a, \phi a\rangle = |a|^2\langle \psi, \phi\rangle$ for all $a \in \mathbb{H}$ and $\psi, \phi \in \Sigma_3$.

Recall now that every oriented 3-manifold is parallelizable, hence spin. Let (M, g) be an oriented Riemannian 3-manifold with a fixed spin structure. The real spinor bundle ΣM over M is the vector bundle associated to the spin bundle and the Spin_3 spinor representation Σ_3. It is endowed with a natural inner product $\langle \cdot, \cdot\rangle$ and a $\mathrm{Cl}(TM)$-action such that the Clifford product by vectors on ΣM is skew-symmetric. The right action of $\mathbb{H}$ on Σ_3 induces a right $\mathbb{H}$ action on ΣM commuting with Clifford multiplication by tangent vectors, and satisfying $\langle \psi a, \phi a\rangle = |a|^2\langle \psi, \phi\rangle$ for all $a \in \mathbb{H}$ and $\psi, \phi \in \Sigma M$.

By construction, Clifford multiplication with the volume form acts as $-\mathrm{Id}$ on ΣM. This choice implies that for a 2-form $\omega \in \Lambda^2(M)$ and a spinor $\psi \in \Sigma M$ we have

$$\omega \cdot \psi = *\omega \cdot \psi, \tag{3.2.1}$$

where $*$ denotes the Hodge star-operator.

3.2.2 Cauchy spinors and endomorphisms

Let $E \to M$ be a vector bundle endowed with a connection ∇. The exterior differential twisted by ∇ is defined on E-valued p-forms using Einstein's summation convention:

$$d^\nabla(\omega \otimes V) := d\omega \otimes V + (-1)^p \omega \wedge X^j \otimes \nabla_{X_j} V,$$

where $(X_j)_{1\leq j\leq n}$ is any local orthonormal frame on M. For an endomorphism field $A \in \mathrm{End}(TM)$, the above formula becomes

$$d^\nabla A(X,Y) := (\nabla_X A)Y - (\nabla_Y A)X, \qquad (\forall) X, Y \in TM. \tag{3.2.2}$$

The divergence operator δ^∇ is the formal adjoint of d^∇ with respect to the L^2-inner product on vector-valued forms.

Let now ∇ be the Levi-Civita covariant derivative on an oriented 3-manifold M, and $\mathrm{R} = d^\nabla \circ \nabla$ its Riemann curvature tensor. The Levi-Civita covariant derivative on the spinor bundle ΣM is still denoted by ∇, and its curvature tensor is written $\mathcal{R}$.

Definition 3.2.1. *Let (M,g) be a spin 3-manifold. A non-zero section $\psi \in \Gamma(\Sigma M)$ is a* Cauchy spinor *if there exists a symmetric endomorphism field $A \in \Gamma(T^*M \times TM)$ such that the pair (ψ, A) satisfies Equation (3.1.1). In this situation, A is called a* Cauchy endomorphism.

We denote by $\mathcal{C}_M$ the set of all Cauchy endomorphisms on M, and by $\mathcal{C}_M^{\mathrm{loc}}$ the set of all symmetric endomorphisms on M satisfying Equation (3.1.2).

In other words, an endomorphism field A on M is a Cauchy endomorphism if it is symmetric and there exists some non-zero spinor $\psi \in \Gamma(\Sigma M)$ satisfying $\nabla_X\psi = -\frac{1}{2}A(X)\cdot\psi$ for all vectors $X \in TM$. We stress that the symmetry assumption on A is crucial here, since in dimension 3 every spinor of constant length determines uniquely some general endomorphism field A so that Equation (3.1.1) holds.

The sets $\mathcal{C}_M$ and $\mathcal{C}_M^{\mathrm{loc}}$ will form our main object of study in this paper.

3.2.3 Parallel spinors in dimension 4

Let us review below the main results of [3] about parallel spinors in dimension 4, Ricci-flat metrics and Cauchy spinors.

Let (M,g) be a hypersurface in a 4-dimensional manifold $(\mathcal{Z}, g^{\mathcal{Z}})$. If $\mathcal{Z}$ is Ricci-flat, the second fundamental form W of the embedding $M \hookrightarrow \mathcal{Z}$ satisfies the contracted Codazzi and Gauss equations:

$$\mathrm{Scal}^M = \mathrm{tr}(A)^2 - \mathrm{tr}(A^2), \qquad \delta^\nabla A + d\mathrm{tr}(A) = 0. \tag{3.2.3}$$

Conversely, if (M, g^M) is real analytic and the constraints (3.2.3) hold on M for some analytic symmetric endomorphism A, Koiso [44] proved that there exists a Ricci-flat real-analytic ambient manifold $(\mathcal{Z}, g^{\mathcal{Z}})$ in which M imbeds isometrically with second fundamental form A (see also [3, Theorem 2.1]).

Upon replacing $\mathcal{Z}$ by a collar neighborhood of M, we can assume that $\mathcal{Z}$ is also parallelizable. Fix a spin structure on $\mathcal{Z}$. The restriction of each of the spinor bundles $\Sigma^\pm\mathcal{Z}$ to M is isomorphic to ΣM as right $\mathbb{H}$-modules. If $\mathcal{Z}$ admits a nonzero parallel spinor Ψ, up to reversing the orientation on $\mathcal{Z}$ we can assume that the chiral component Ψ^+ is nonzero. The parallel spinor induces an algebra map $\phi : \mathbb{H} \to \Gamma(\mathrm{End}(TM))$ via the quaternionic structure of the spinor bundle:

$$\mathbb{H} \ni q \mapsto \phi(q) := Q, \qquad Q(V)\cdot\Psi^+ = V\cdot\Psi^+ q, \qquad (\forall) V \in \Gamma(TM).$$

The endomorphisms in the image of ϕ are parallel, implying that the metric $g^{\mathcal{Z}}$ is hyperkähler, hence self-dual and therefore Ricci flat (if Ψ^+ and Ψ^- are both nonzero then $g^{\mathcal{Z}}$ is flat).

Moreover, the restriction of Ψ^+ to M is a Cauchy spinor, and the second fundamental form of $M \hookrightarrow \mathcal{Z}$ is a Cauchy endomorphism.

Conversely, if there exists a Cauchy spinor ψ and a Cauchy endomorphism A on M satisfying (3.1.1), then the identities (3.2.3) are satisfied (an alternate derivation of these identities can be found in [61, Lemma 3.1, Equation (12)]). If moreover M, g^M, A and ψ are real-analytic, then by the above cited result of Koiso there exists a Ricci-flat 4-manifold $(\mathcal{Z}, g^{\mathcal{Z}})$ into which (M, g^M) embeds isometrically with second fundamental form A, and locally there exists a parallel spinor of positive chirality on $\mathcal{Z}$ extending ψ, see [3, Theorem 1.1]. In particular, $g^{\mathcal{Z}}$ is self-dual and hence hyperkähler.

3.2.4 The modified metric connection ∇^A

Throughout the paper we identify 1-forms and vectors on M using the metric g. Given any $A \in \Gamma(\Lambda^1 M \otimes TM)$, we define $\overline{A} \in \Gamma(\Lambda^1 M \otimes \Lambda^2 M)$ by $\overline{A}(X) := *(A(X))$ for all $X \in TM$. We can view the 2-form $\overline{A}(X)$ as a skew-symmetric endomorphism of TM in the usual way: for $Y \in TM$, $\overline{A}(X)(Y)$ is the unique vector Z satisfying $g(Z, W) = \overline{A}(X)(Y, W)$ for all $W \in TM$.

We introduce the connection on TM

$$\nabla^A_X Y := \nabla_X Y + \overline{A}(X)(Y), \qquad X, Y \in TM, \tag{3.2.4}$$

and we denote by R^A its curvature tensor. Since by construction $\overline{A}(X)$ is skew symmetric, ∇^A is compatible with the Riemannian metric and hence induces a connection, still denoted ∇^A, on the spinor bundle ΣM by pull-back from the orthonormal frame bundle:

$$\nabla^A_X \psi = \nabla_X \psi + \frac{A(X)}{2} \cdot \psi, \qquad (\forall) X \in TM, \psi \in \Sigma M.$$

We denote by $\mathcal{R}^A$ the curvature of ∇^A on spinors.

The problem of finding solutions to (3.1.1) can be reduced to solving an equation involving the endomorphism field A alone, at least when M is simply-connected.

Proposition 3.2.2. *Let (M, g) be a Riemannian 3-manifold and $A \in \Gamma(\Lambda^1 M \otimes TM)$ a symmetric endomorphism field. The following conditions are equivalent:*

1. *Locally on M there exist nonzero Cauchy spinors with respect to A;*
2. $\mathcal{R}^A = 0$*;*
3. $\mathrm{R}^A = 0$*;*
4. *The symmetric endomorphism A satisfies the equation (3.1.2), i.e., $A \in \mathcal{C}^{\mathrm{loc}}_M$.*

When M is simply connected, the first condition is equivalent to the global existence of Cauchy spinors, hence $\mathcal{C}^{\mathrm{loc}}_M = \mathcal{C}_M$.

Proof. If there exists a locally defined, nonzero spinor ψ satisfying (3.1.1), then the four mutually orthogonal spinors $(\psi, \psi i, \psi j, \psi k)$, defined with the help of the quaternionic structure on ΣM, are parallel for the covariant derivative ∇^A. This implies that the curvature endomorphism $\mathcal{R}^A$ of $(\Sigma M, \nabla^A)$ vanishes. Conversely, if $\mathcal{R}^A = 0$, locally there exist non-zero spinors parallel with respect to the connection ∇^A, i.e., Cauchy spinors. Moreover, if M is simply connected then such spinors exist globally on M.

Since the connection ∇^A on ΣM is induced from the connection with the same name on TM and the spinor representation ΣM, it is well known that $\mathcal{R}^A(X,Y) = \frac{1}{2}\mathrm{R}^A(X,Y)\cdot$, where $\cdot$ denotes Clifford multiplication (see e.g. [15, Theorem 2.7]). In dimension 3 the map associating to a 2-form its action by Clifford multiplication on spinors is injective, thus $\mathcal{R}^A(X,Y) = 0$ if and only if $\mathrm{R}^A(X,Y) = 0$.

For all $X, Y \in TM$, we compute from the definition of ∇^A its curvature:

$$\mathrm{R}^A(X,Y) = \mathrm{R}(X,Y) + d^\nabla \overline{A}(X,Y) + [\overline{A}(X), \overline{A}(Y)]$$

Since the Hodge $*$ operator is parallel, it commutes with d^∇, so $d^\nabla \overline{A} = *d^\nabla A$. Also, we check directly that

$$\overline{A}(X) \circ \overline{A}(Y) - \overline{A}(Y) \circ \overline{A}(X) = A(X) \wedge A(Y).$$

Hence $\mathrm{R}^A = \mathrm{R} + *d^\nabla A + A \wedge A$ as claimed. □

Proposition 3.2.2 shows that $\mathcal{C}_M \subset \mathcal{C}_M^{\mathrm{loc}}$, with equality when M is simply connected.

3.3 Deformation of endomorphism fields

There is not much one can say about the structure of the sets $\mathcal{C}_M$ or $\mathcal{C}_M^{\mathrm{loc}}$, even in the simply connected case when they coincide. Let us introduce the following definition:

Definition 3.3.1. *The space of infinitesimal deformations of $\mathcal{C}_M^{\mathrm{loc}}$ at $A \in \mathcal{C}_M^{\mathrm{loc}}$ is*

$$\{\dot{C}(0) |\, (\exists)\varepsilon > 0, (\exists)C \in C^\infty((-\varepsilon,\varepsilon), \Gamma(\mathrm{Sym}^2(TM)), C(0) = A, \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{R}^{C(t)}|_{t=0} = 0\}.$$

In the case where $\mathcal{C}_M^{\mathrm{loc}}$ is a differentiable manifold, its tangent space $T_A\mathcal{C}_M^{\mathrm{loc}}$ is the space of tangent vectors at $t = 0$ to smooth curves $C : (-\varepsilon,\varepsilon) \to \mathcal{C}_M^{\mathrm{loc}}$ with $C(0) = A$. In general, since we do not know any differentiable structure on $\mathcal{C}_M^{\mathrm{loc}}$, the above cone might even not be a vector space. The space of infinitesimal deformations always contains this formal tangent cone.

Theorem 3.3.2. *Let (M, g) be a compact oriented Riemannian 3-manifold with strictly positive scalar curvature and let $A \in \mathcal{C}_M^{\mathrm{loc}}$. Then the space of infinitesimal deformations of A is finite-dimensional.*

Proof. Let $C \in C^\infty((-\varepsilon,\varepsilon), \Gamma(\mathrm{Sym}^2(TM))$ be a smooth curve as in Definition 3.3.1, with $C(0) = A$. We define $\dot{A} := \dot{C}(0)$. By differentiating with respect to t in (3.1.2) the condition $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{R}^{C(t)}|_{t=0} = 0$ rewrites

$$0 = *(\mathrm{d}^\nabla \dot{A})(X,Y) + \dot{A}(X) \wedge A(Y) + A(X) \wedge \dot{A}(Y). \tag{3.3.1}$$

We recall some elementary identities about the Hodge star-operator in dimension 3. Let $X, Y \in TM$ and $\alpha \in \Lambda^2 M$. Then we easily check that

$$X \lrcorner * Y = - * (X \wedge Y), \qquad\qquad X \wedge *\alpha = - * (X \lrcorner \alpha). \tag{3.3.2}$$

Using these identities, one has for all $X, Y \in TM$

$$\dot{A}(X) \wedge A(Y) + A(X) \wedge \dot{A}(Y) = - * \overline{A}(Y)(\dot{A}(X)) + *\overline{A}(X)(\dot{A}(Y)).$$

Consequently, Equation (3.3.1) rewrites

$$0 = (\mathrm{d}^{\nabla}\dot{A})(X,Y) - \overline{A}(Y)(\dot{A}(X)) + \overline{A}(X)(\dot{A}(Y)) = (d^{\nabla^A}\dot{A})(X,Y).$$

We are hence led to the equation for the infinitesimal deformation of a flat connection:

$$d^{\nabla^A}\dot{A} = 0. \tag{3.3.3}$$

According to Proposition 3.2.2, the connection ∇^A is flat, meaning that $\mathrm{R}^A = d^{\nabla^A} \circ \nabla^A = 0$ on $\Gamma(TM)$. But $d^{\nabla^A} \circ d^{\nabla^A}$ is given by the action of R^A also on $\Gamma(\Lambda^k M \otimes TM)$ for all $k \geq 0$. This implies that $d^{\nabla^A} \circ d^{\nabla^A} = 0$ on $\Gamma(\Lambda^* M \otimes TM)$. The differential operators

$$d^{\nabla^A} : \Gamma(\Lambda^* M \otimes TM) \to \Gamma(\Lambda^{*+1} M \otimes TM)$$

form therefore an elliptic complex. By Hodge theory, Equation (3.3.3) implies that there exists a vector field $X_{\dot{A}} \in \Gamma(TM)$ and a d^{∇^A}-harmonic vector-valued 1-form $B_{\dot{A}} \in \Gamma(\Lambda^1 M \otimes TM)$ (i.e., $\delta^{\nabla^A} B_{\dot{A}} = 0$ and $d^{\nabla^A} B_{\dot{A}} = 0$) such that $\dot{A} = \nabla^A X_{\dot{A}} + B_{\dot{A}}$. Notice that this equation still involves the symmetric endomorphism A which defines the flat connection ∇^A. The symmetry condition on $\dot{A}$ can be rewritten as an equation on $X_{\dot{A}}$, $B_{\dot{A}}$ and A:

Lemma 3.3.3. *Let $X \in \Gamma(TM)$ and $B \in \Gamma(\Lambda^1 M \otimes TM)$. The endomorphism $d^{\nabla^A} X + B$ is symmetric if and only if X, viewed as a 1-form, satisfies $dX - *(X\mathrm{tr}(A) - AX) + X_k \wedge B(X_k) = 0$, where (X_1, X_2, X_3) is any orthonormal basis.*

Proof. The endomorphism $d^{\nabla^A} X + B$ is symmetric if and only if its skew-symmetric part is zero. Using the identities (3.3.2) we compute

$$\begin{aligned}
0 &= X_k \wedge \nabla^A_{X_k} X + X_k \wedge B(X_k) \\
&= X_k \wedge (\nabla_{X_k} X + \overline{A}(X_k)(X)) + X_k \wedge B(X_k) \\
&= X_k \wedge (\nabla_{X_k} X + X \lrcorner \overline{A}(X_k)) + X_k \wedge B(X_k) \\
&= X_k \wedge (\nabla_{X_k} X - *(X \wedge A(X_k))) + X_k \wedge B(X_k) \\
&= X_k \wedge \nabla_{X_k} X - X_k \wedge *(X \wedge A(X_k)) + X_k \wedge B(X_k) \\
&= X_k \wedge \nabla_{X_k} X + *(X_k \lrcorner (X \wedge A(X_k))) + X_k \wedge B(X_k) \\
&= dX + *(X(X_k)A(X_k) - A(X_k)(X_k)X) + X_k \wedge B(X_k) \\
&= dX + *(A(X) - X\mathrm{tr}A) + X_k \wedge B(X_k).
\end{aligned}$$

□

The space of d^{∇^A}-harmonic vector-valued 1-forms B is finite dimensional by ellipticity. Lemma 3.3.3 implies the identity $dX_{\dot{A}} - *(X_{\dot{A}}\mathrm{tr}(A) - AX_{\dot{A}}) = -X_k \wedge B_{\dot{A}}(X_k)$, and for a given B, the solutions $X \in TM$ of

$$dX - *(X\mathrm{tr}(A) - AX) + X_k \wedge B(X_k) = 0$$

form an affine space of direction $\ker(X \mapsto dX - *(X\mathrm{tr}(A) - AX))$. Thus, to show that the space of deformations is finite dimensional, it is sufficient to prove that the solution space of

$$dX - *(X\mathrm{tr}(A) - AX) = 0 \tag{3.3.4}$$

is finite dimensional.

By applying the exterior derivative to equation (3.3.4) we have $0 = d * (X\mathrm{tr}(A) - AX) = *\delta(X\mathrm{tr}(A) - AX)$, and then

$$\delta(X\mathrm{tr}(A) - AX) = 0.$$

We define the differential operator

$$\Xi : \Lambda^1 M \longrightarrow \Lambda^2 M \oplus \Lambda^0 M, \qquad \Xi(X) = (dX - *(X\mathrm{tr}(A) - AX), \delta(X\mathrm{tr}A - AX)).$$

To compute the principal symbol of Ξ, recall that for any $x \in M$, any element of $\Lambda^1_x M$ is the differential at x of some smooth function. Let thus f be a smooth function on M. Again by using (3.3.2) we obtain

$$\begin{aligned}\sigma_\Xi(df)(X) &= \Xi(fX) - f\Xi X \\ &= (d(fX) - fdX, \delta[f(X\mathrm{tr}A - AX)] - f\delta[X\mathrm{tr}A - AX]) \\ &= (df \wedge X, -\langle df, (X\mathrm{tr}A - AX)\rangle).\end{aligned}$$

Hence, the principal symbol of Ξ at $\alpha \in \Lambda^1_x M$ is given by

$$\sigma_\Xi(\alpha)(X) = (\alpha \wedge X, \langle \alpha, (X\mathrm{tr}A - AX)\rangle).$$

We want to show that this principal symbol is injective in order to use the theory of elliptic operators. To do so, we first remark that equation (3.2.3) holds for any manifold carrying a Cauchy spinor even if it is not embedded into a Ricci-flat manifold [61, Lemma 3.1]. Then, the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of A satisfy $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{1}{2}\mathrm{Scal}^M$. We now use the hypothesis $\mathrm{Scal}^M > 0$.

Lemma 3.3.4. *Let $B \in M_3(\mathbb{R})$ be a symmetric matrix with eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ such that $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$. Then $B - \mathrm{tr}(B)\mathrm{Id}$ is definite.*

Proof. Since B is symmetric and real, there is $P \in \mathrm{O}_3(\mathbb{R})$ such that

$$P^T BP = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3).$$

Thus, $P^T(B - \mathrm{tr}(B)\mathrm{Id})P = -\mathrm{diag}(\lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2)$. Moreover, one has

$$(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) = \lambda_2\lambda_1 + \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_3^2 > 0$$

and similarly $(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_2) > 0$, $(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2) > 0$. We conclude that $\lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2$ have the same sign and are different from 0, so $B - \mathrm{tr}(B)\mathrm{Id}$ is definite. $\square$

As a consequence of Lemma 3.3.4, $A - \mathrm{tr}(A)\mathrm{Id}$ is definite under the assumption that $\mathrm{Scal}^M > 0$. Now, let $\alpha \in \Lambda^1 M$ such that there is a non-zero vector X with $\sigma_\Xi(\alpha)(X) = 0$. In particular, we have $g(\alpha, [A - \mathrm{tr}(A)\mathrm{Id}]X) = 0$ and $\alpha \wedge X = 0$. We deduce that $X = f\alpha$ and $g(\alpha, [A - \mathrm{tr}(A)\mathrm{Id}]\alpha) = 0$, so $\alpha = 0$ because the endomorphism $A - \mathrm{tr}(A)\mathrm{Id}$ is invertible. Consequently, the principal symbol of Ξ is injective.

The operator $\Xi^*\Xi$ has the same kernel as Ξ and is elliptic, so its kernel is finite dimensional (see e.g. [67, Theorem 5.2]). Thus the space of infinitesimal deformations of A is finite dimensional, ending the proof. $\square$

The assumption $\mathrm{Scal}^M > 0$ is necessary, as shown by the following example:

Remark 3.3.5. We look at a flat Riemannian product $M = \mathbf{S}^1 \times E$, where $E = \mathbb{R}^2/\Gamma$ is an elliptic curve. Let $p : M \to \mathbf{S}^1$ be the projection on the first factor, and $P : TM \to TM$ the orthogonal projection on the first factor in the tangent bundle. The endomorphism P is parallel and symmetric, and it clearly satisfies Equation (3.1.2). Moreover, for any smooth function $f : \mathbf{S}^1 \to \mathbb{R}$, the symmetric endomorphism field $A_f := (p^* f)P$ also satisfies Equation (3.1.2) since all the terms in this equation vanish when at least one of the vectors X, Y are tangent to the second factor E. Thus the space of infinitesimal deformations of A_f contains the infinite-dimensional space $C^\infty(\mathbf{S}^1)$.

Not every such $A_f \in \mathcal{C}_M^{\mathrm{loc}}$ is necessarily associated to a Cauchy spinor, because the torus is not simply connected. Take a non-zero parallel spinor Ψ on the flat torus $F \times E$, where F is also an elliptic curve, and consider any closed simple curve γ in F of length 2π, hence isometric to $\mathbf{S}^1$. Then the manifold $\gamma \times E$ is isometric to M, so the restriction of Ψ to M is a Cauchy spinor on M with respect to $A_{\mathfrak{k}}$, where $\mathfrak{k}$ is the geodesic curvature function of γ. Since the set of curvature functions of curves of length 2π in F parametrized by arc-length is not finite-dimensional, it is evident that the deformation space of flat connections near such a $A_{\mathfrak{k}}$ cannot be finite dimensional either.

3.4 Deformations of Cauchy spinors on the three-sphere

Let us illustrate the above result in the case of the round 3-sphere, noting that even in this simplest possible case Equation (3.1.1) is not yet fully understood.

We identify $\mathbf{S}^3$ with the unit sphere in the quaternions $\mathbb{H} \simeq \mathbb{R}^4$. In this way, $\mathbf{S}^3$ becomes a Lie group with Lie algebra the space of imaginary quaternions $\mathrm{Im}\mathbb{H}$. Let (e_1, e_2, e_3) be the three left-invariant vector fields corresponding to the quaternions i, j, k. They form an orthonormal frame at any point of $\mathbf{S}^3$. Recall that the Levi-Civita covariant derivative of left-invariant vector fields on compact Lie groups is given by $\nabla_X Y = \frac{1}{2}[X, Y]$. Recall also that for an even permutation (a, b, c) of the indices $(1, 2, 3)$, the Lie brackets are given by

$$[e_a, e_b] = 2e_c. \tag{3.4.1}$$

We compute from here the covariant derivatives of these orthonormal vector fields:

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0 & \nabla_{e_2} e_2 = 0 & \nabla_{e_3} e_3 = 0 \\ \nabla_{e_1} e_2 = e_3 & \nabla_{e_2} e_3 = e_1 & \nabla_{e_3} e_1 = e_2 \\ \nabla_{e_2} e_1 = -e_3 & \nabla_{e_3} e_2 = -e_1 & \nabla_{e_1} e_3 = -e_2. \end{array} \tag{3.4.2}$$

On the round sphere, the curvature tensor satisfies $\mathrm{R}(X, Y) = -X \wedge Y$ so Equation (3.1.2) rewrites

$$*d^\nabla A(X, Y) = X \wedge Y - A(X) \wedge A(Y). \tag{3.4.3}$$

Remark 3.4.1. From [60, Example 3.2], we know four families of Cauchy endomorphisms in $\mathcal{C}_{\mathbf{S}^3}$:

- plus or minus the identity
- the symmetric endomorphism fields constant in a left-invariant orthonormal frame, with eigenvalues 1, -3, -3
- the symmetric endomorphism fields constant in a right-invariant orthonormal frame, with eigenvalues -1, 3, 3.

It was already shown in [62, Theorem 5.1] that $\mathcal{C}^{\mathrm{loc}}_{\mathbf{S}^3}$ does not admit infinitesimal deformations at the endomorphisms $\pm\mathrm{Id}$. Let us thus study the infinitesimal deformations of $\mathcal{C}^{\mathrm{loc}}_{\mathbf{S}^3}$ at the symmetric endomorphism field

$$A_0 := e_1 \otimes e_1 - 3(e_2 \otimes e_2 + e_3 \otimes e_3). \tag{3.4.4}$$

Lemma 3.4.2. *Let (M, g) be a simply connected 3-manifold and $A \in \Gamma(\mathrm{End}(TM))$ such that the connection ∇^A defined by (3.2.4) is flat. Then the cohomology space $H^1(M, d^{\nabla^A})$ vanishes, hence there are no nonzero d^{∇^A}-harmonic sections in $\Gamma(\Lambda^1 M \otimes TM)$.*

Proof. Fix a global frame $s_1, s_2, s_3 \in \Gamma(TM)$ consisting of ∇^A-parallel vector fields, possible since M is simply-connected. In this basis, the elliptic complex $(\Gamma(\Lambda^* M \otimes TM), d^{\nabla^A})$ is isomorphic to the tensor product of the standard de Rham complex with $\mathbb{R}^3$. It follows that $H^1(M, d^{\nabla^A}) \simeq H^1(M) \otimes \mathbb{R}^3$, and this space vanishes since the first Betti number of a simply-connected manifold is zero. □

By the analysis from the proof of Theorem 3.3.2 and Lemma 3.4.2, any infinitesimal deformation $\dot{A}$ can be written as the covariant derivative ∇^{A_0} of a vector field $X_{\dot{A}}$. By Lemma 3.3.3, the symmetry of $\dot{A}$ leads to the equation

$$dX + *(A_0 X + 5X) = 0. \tag{3.4.5}$$

If we write $X =: x^k e_k$, we have $dX = d(x^k e_k) = dx^k \wedge e_k + x^k de_k$ and the exterior derivatives are given by $de_k = -2 * e_k$ for all $k \in \{1, 2, 3\}$. Finally, equation (3.4.5) rewrites

$$dx^k \wedge e_k + 4x^1 e_2 \wedge e_3 = 0.$$

This means that we have the differential system of equations in the unknown functions $x^1, x^2, x^3 \in C^\infty(\mathbf{S}^3)$:

$$e_2(x^1) = e_1(x^2), \qquad e_3(x^1) = e_1(x^3), \qquad e_3(x^2) = e_2(x^3) + 4x^1.$$

By taking further partial derivatives one has

$$\begin{aligned} e_3 e_2(x^1) &= e_3 e_1(x^2) = e_1 e_3(x^2) + 2e_2(x^2) \\ e_2 e_3(x^1) &= e_2 e_1(x^3) = e_1 e_2(x^3) - 2e_3(x^3) \\ 4e_1(x^1) &= e_1 e_3(x^2) - e_1 e_2(x^3). \end{aligned}$$

We subtract the first equation from the sum of the last two, and we obtain

$$3e_1(x^1) + e_2(x^2) + e_3(x^3) = 0.$$

Hence,

$$3e_1 e_1(x^1) = -e_1 e_2(x^2) - e_1 e_3(x^3), \quad e_2 e_2(x^1) = e_2 e_1(x^2), \quad e_3 e_3(x^1) = e_3 e_1(x^3),$$

and summing these three equations one has

$$3e_1 e_1(x^1) + e_2 e_2(x^1) + e_3 e_3(x^1) = 2e_2(x^3) - 2e_3(x^2) = -8x^1.$$

Consequently, we have to solve

$$\Delta_B x^1 = 8x^1$$

where $\Delta_B := -(3e_1e_1 + e_2e_2 + e_3e_3)$. Note that this operator is the Laplacian on the Berger sphere with metric $\frac{1}{3}e_1^2+e_2^2+e_3^2$. Let $\Delta = -(e_1e_1+e_2e_2+e_3e_3)$ be the Laplacian for the round metric. Since e_1 is a Killing vector field, it commutes with Δ, and since it is divergence-free it follows that $e_1^* = -e_1$. Thus the operator $-e_1^2$ is symmetric and positive, so Δ_B commutes with Δ and $\Delta_B \geq \Delta$. It follows that an eigenfunction of Δ_B for the eigenvalue 8 is a sum of eigenfunctions of Δ for the eigenvalue 3 or 8. From the analysis done in [10, Section 5 and 6.2] or by direct computation, we know that all eigenfunctions of Δ for the eigenvalue 3 (the spherical harmonics of degree 1) are also eigenfunctions of Δ_B of eigenvalue 5. Moreover, an eigenfunction of Δ for the eigenvalue 8 (a spherical harmonic of degree 2) is a sum of eigenfunctions of Δ_B for the eigenvalues 8 and 16, and the multiplicity of the eigenvalue 8 for Δ_B turns out to be 3. More precisely, the associated eigenspace V_8 is spanned by the functions π^*y_k for $k \in \{1,2,3\}$ where $\pi : \mathbf{S}^3 \to \mathbf{S}^2(\frac{1}{2})$ is the Hopf fibration for which e_1 is tangent to the fibers and y_k is the k^{th} coordinate in $\mathbb{R}^3 \supset \mathbf{S}^2$. In particular, the action of e_1 is trivial on V_8, so $e_1(x^1) = 0$.

Lemma 3.4.3. *For any $k \in \{1,2,3\}$ one has $e_2e_3(\pi^*y_k) = e_3e_2(\pi^*y_k) = 0$ and $e_2e_2(\pi^*y_k) = e_3e_3(\pi^*y_k) = -4\pi^*y_k$.*

Proof. The space $\mathrm{Span}(\pi^*y_1, \pi^*y_2, \pi^*y_3)$ is generated by the three harmonic quadratic polynomials $\{a_1^2 + a_2^2 - a_3^2 - a_4^2, a_1a_4 + a_2a_3, a_1a_3 - a_2a_4\}$ restricted to $\mathbf{S}^3$, where a_k stands for the k^{th} coordinate in $\mathbb{R}^4$. The lemma follows by a direct computation. □

Now we have

$$\begin{aligned}
e_1e_1(x^2) &= e_1e_2(x^1) = e_2e_1(x^1) + 2e_3(x^1) = 2e_3(x^1)\\
e_2e_2(x^2) &= -e_2e_3(x^3)\\
e_3e_3(x^2) &= e_3e_2(x^3) + 4e_3(x^1)
\end{aligned}$$

and by adding these equations we obtain for the Laplacian $\Delta = -(e_1e_1 + e_2e_2 + e_3e_3)$ of the round metric:

$$\Delta x^2 = -2e_3(x^1) + 2e_1(x^3) - 4e_3(x^1) = -4e_3(x^1).$$

In the same way we have

$$\begin{aligned}
e_1e_1(x^3) &= e_1e_3(x^1) = -2e_2(x^1)\\
e_2e_2(x^3) &= e_2e_3(x^2) - 4e_2(x^1)\\
e_3e_3(x^3) &= -e_3e_2(x^2),
\end{aligned}$$

thus,

$$\Delta x^3 = 2e_2(x^1) - 2e_1(x^2) + 4e_2(x^1) = 4e_2(x^1).$$

We are left with the system

$$\Delta x^2 = -4e_3(x^1), \qquad \Delta x^3 = 4e_2(x^1).$$

Since e_1, e_2, e_3 are Killing vector fields, Δ commutes with e_1, e_2 and e_3. This implies that

$$\Delta(x^2 + \tfrac{1}{2}e_3(x^1)) = 0, \qquad \Delta(x^3 - \tfrac{1}{2}e_2(x^1)) = 0.$$

Since harmonic functions on a compact manifold must be constant, we deduce that $x_2 = -\frac{1}{2}e_3(x^1) + c_2$ and $x_3 = \frac{1}{2}e_2(x^1) + c_3$ for some constants $c_2, c_3 \in \mathbb{R}$. Finally, the space of solutions

$$\mathcal{S} := \{x^1e_1 + (-\tfrac{1}{2}e_3(x^1) + c_2)e_2 + (\tfrac{1}{2}e_2(x^1) + c_3)e_3; x^1 \in V_8,\ c_2, c_3 \in \mathbb{R}\} \tag{3.4.6}$$

is 5-dimensional, since the eigenspace V_8 of Δ_B for the eigenvalue 8 has dimension 3. From (3.4.1), the Lie derivatives of A_0 in the direction of e_2, e_3 are given by

$$\begin{aligned}
\mathcal{L}_{e_2} A_0 &= \mathcal{L}_{e_2}(e_1 \otimes e_1 - 3e_2 \otimes e_2 - 3e_3 \otimes e_3) \\
&= -2(e_3 \otimes e_1 + e_1 \otimes e_3) - 6(e_1 \otimes e_3 + e_3 \otimes e_1) \\
&= -8(e_3 \otimes e_1 + e_1 \otimes e_3); \\
\mathcal{L}_{e_3} A_0 &= \mathcal{L}_{e_3}(e_1 \otimes e_1 - 3e_2 \otimes e_2 - 3e_3 \otimes e_3) \\
&= 2(e_2 \otimes e_1 + e_1 \otimes e_2) + 6(e_1 \otimes e_2 + e_2 \otimes e_1) \\
&= 8(e_2 \otimes e_1 + e_1 \otimes e_2).
\end{aligned}$$

Let $X := x^k e_k \in \mathcal{S}$. One has

$$\begin{aligned}
\nabla^{A_0} X &= dx^k \otimes e_k + x^k \nabla^{A_0} e_k \\
&= dx^k \otimes e_k + 2x^1(e_2 \otimes e_3 - e_3 \otimes e_2) \\
&\quad + 2x^2(e_1 \otimes e_3 + e_3 \otimes e_1) - 2x^3(e_1 \otimes e_2 + e_2 \otimes e_1),
\end{aligned}$$

and we can compute the coefficients of $\nabla^{A_0} X$ using Lemma 3.4.3:

$$\begin{aligned}
&2\langle \nabla^{A_0} X, e_1 \otimes e_1 \rangle = 2e_1(x^1) = 0 \\
&2\langle \nabla^{A_0} X, e_2 \otimes e_2 \rangle = -e_2 e_3(x^1) = 0 \\
&2\langle \nabla^{A_0} X, e_3 \otimes e_3 \rangle = e_3 e_2(x^1) = 0 \\
&2\langle \nabla^{A_0} X, e_1 \otimes e_2 \rangle = -e_1 e_3(x^1) - 2e_2(x^1) - 4c_3 = 2e_2(x^1) - 2e_2(x^1) - 4c_3 = -4c_3 \\
&2\langle \nabla^{A_0} X, e_1 \otimes e_3 \rangle = e_1 e_2(x^1) - 2e_3(x^1) + 4c_2 = 2e_3(x^1) - 2e_3(x^1) + 4c_2 = 4c_2 \\
&2\langle \nabla^{A_0} X, e_2 \otimes e_3 \rangle = e_2 e_2 x^1 + 4x^1 = 0.
\end{aligned}$$

From the symmetry of $\nabla^{A_0} X$ we conclude that

$$\begin{aligned}
\nabla^{A_0} X &= 2c_3(e_1 \otimes e_3 + e_3 \otimes e_1) - 2c_2(e_1 \otimes e_2 + e_2 \otimes e_1) \\
&= -\frac{1}{4}(c_3 \mathcal{L}_{e_3} A_0 + c_2 \mathcal{L}_{e_2} A_0).
\end{aligned}$$

Consequently, we proved the following proposition:

Proposition 3.4.4. *The space of infinitesimal deformations of $\mathcal{C}^{\mathrm{loc}}_{\mathbf{S}^3}$ at A_0 is of dimension 2, and consists exactly of the Lie derivatives of A_0 in the directions spanned by (e_2, e_3).*

The same analysis holds for endomorphism fields constant in a right-invariant orthonormal frame.

Remark 3.4.5. The infinitesimal deformations described in Proposition 3.4.4 can be obtained as the tangent vectors to actual smooth curves in $\mathcal{C}^{\mathrm{loc}}_{\mathbf{S}^3}$, as described after Definition 3.3.1. This comes from the fact that the solutions we know form a differentiable manifold with four connected components.

3.5 Endomorphisms fields on the 3-sphere and the second fundamental form

3.5.1 Thickening of the three-sphere

In the real-analytic case, it was shown in [44] and [3, Theorem 1.1] that the existence of a Cauchy spinor ψ over (M, g) is equivalent to the existence of a Ricci-flat metric of the form $g^{\mathcal{Z}} = \mathrm{d}t^2 + g_t$ on a cylinder $\mathcal{Z} := (-\epsilon, \epsilon) \times M$ with $g_0 = g$, which carries a parallel spinor Ψ. In this case, the Cauchy endomorphism A from (3.1.1) is the second fundamental form of the hypersurface $\{0\} \times M$, and ψ is the restriction of Ψ to this hypersurface. Moreover, the germ near $t = 0$ of the Ricci-flat metric $g^{\mathcal{Z}}$ is unique by analyticity.

When A is plus or minus the identity on the sphere $\mathbf{S}^3$, the resulting metric on $\mathcal{Z}$ is flat and isometric to the induced metric on a tubular neighborhood of $\mathbf{S}^3$ through the canonical embedding in $\mathbb{R}^4$.

In this section we shall describe the ambient metric obtained by this thickening procedure in the case where the Cauchy endomorphism field is A_0 defined in (3.4.4) in terms of the left-invariant orthonormal frame (e_1, e_2, e_3) on the standard sphere $\mathbf{S}^3$. Once again, a similar analysis can be carried out in the case of an endomorphism field constant in a *right*-invariant orthonormal frame, as it amounts to reversing the orientation on $\mathcal{Z}$.

Since both the standard metric on $\mathbf{S}^3$ and A_0 are real-analytic and S^1-invariant, by uniqueness it follows that the Ricci-flat metric $g^{\mathcal{Z}}$ is also S^1-invariant. For an interval $I \ni 0$ (to be defined later), we thus make the following Ansatz: let η_1, η_2, η_3 be the 1-forms dual to the Hopf vector fields e_1, e_2, e_3, we look for a metric on $\mathcal{Z} := \mathbf{S}^3 \times I$ of the form

$$g^{\mathcal{Z}} := \mathrm{d}t^2 + g_t, \qquad g_t =: a(t)^2\eta_1^2 + b(t)^2(\eta_2^2 + \eta_3^2) \tag{3.5.1}$$

such that $\Sigma\mathcal{Z}$ carries a parallel spinor Ψ and A_0 is the Weingarten map of $\mathbf{S}^3 \times \{0\}$. Moreover, g_0 has to coincide with the metric of the round sphere, so $a(0) = b(0) = 1$, and we assume that the functions a and b are non-negative.

We introduce the notation $M_t := \mathbf{S}^3 \times \{t\}$. Because of the form of the metric g_t defined by (3.5.1), the hypersurfaces M_t are Berger spheres.

The covariant derivative on (M_t, g_t) is denoted ∇^t. We also denote by R^t the Riemann curvature tensor of (M_t, g_t), and by A_t the Weingarten map of the hypersurface M_t in $\mathcal{Z}$.

The fact that Ψ is parallel in $\Sigma\mathcal{Z}$ implies that each hypersurface M_t carries a Cauchy spinor (the restriction of Ψ to M_t) with associated endomorphism field A_t. In particular, Equation (3.1.2) gives for all t

$$0 = \mathrm{R}^t(X, Y) + *d^{\nabla^t} A_t(X, Y) + A_t(X) \wedge A_t(Y) \tag{3.5.2}$$

for all $X, Y \in TM_t \simeq TM$. In turn, the identity (3.5.2) for all t will be shown to determine uniquely the metric $g^{\mathcal{Z}}$, hence, as explained above, it implies the vanishing of the Ricci tensor of $g^{\mathcal{Z}}$ by [44] and the existence of a parallel spinor on $\mathcal{Z}$ by the main result of [3].

Using Koszul's formula and the expression (3.4.1) for the Lie brackets of e_1, e_2, e_3, we obtain by a straightforward computation that

$$\nabla^t_{e_1} e_2 = \left(2 - \frac{a^2}{b^2}(t)\right) e_3, \qquad \nabla^t_{e_2} e_1 = -\frac{a^2}{b^2}(t) e_3, \qquad \nabla^t_{e_1} e_3 = \left(\frac{a^2}{b^2}(t) - 2\right) e_2,$$
$$\nabla^t_{e_3} e_1 = \frac{a^2}{b^2}(t) e_2, \qquad \nabla^t_{e_2} e_3 = e_1, \qquad \nabla^t_{e_3} e_2 = -e_1$$

$$\nabla^t_{e_1} e_1 = 0, \qquad \nabla^t_{e_2} e_2 = 0, \qquad \nabla^t_{e_3} e_3 = 0.$$

From this we immediately get

$$\begin{aligned}
&\mathrm{R}^t_{e_1,e_2} e_1 = -\frac{a^4}{b^4}(t) e_2, && \mathrm{R}^t_{e_1,e_2} e_2 = \frac{a^2}{b^2}(t) e_1, && \mathrm{R}^t_{e_1,e_2} e_3 = 0 \\
&\mathrm{R}^t_{e_1,e_3} e_1 = -\frac{a^4}{b^4}(t) e_3, && \mathrm{R}^t_{e_1,e_3} e_2 = 0, && \mathrm{R}^t_{e_1,e_3} e_3 = \frac{a^2}{b^2}(t) e_1 \\
&\mathrm{R}^t_{e_2,e_3} e_1 = 0, && \mathrm{R}^t_{e_2,e_3} e_2 = \left(3\frac{a^2}{b^2}(t) - 4\right) e_3 && \mathrm{R}^t_{e_2,e_3} e_3 = \left(4 - 3\frac{a^2}{b^2}(t)\right) e_3.
\end{aligned}$$

Thus, in the basis (e_1, e_2, e_3) one has

$$\mathrm{R}^t_{e_1,e_2} = \begin{bmatrix} 0 & \frac{a^2}{b^2} & 0 \\ -\frac{a^4}{b^4} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{R}^t_{e_1,e_3} = \begin{bmatrix} 0 & 0 & \frac{a^2}{b^2} \\ 0 & 0 & 0 \\ -\frac{a^4}{b^4} & 0 & 0 \end{bmatrix}$$

$$\mathrm{R}^t_{e_2,e_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 - 3\frac{a^2}{b^2} \\ 0 & 3\frac{a^2}{b^2} - 4 & 0 \end{bmatrix}.$$

We shall identify vectors and 1-forms on M_t using the metric g_t. Notice that the Hopf frame (e_1, e_2, e_3) is orthogonal, but not orthonormal on M_t. In terms of 2-forms, the t-dependent curvature matrices become

$$\mathrm{R}^t(e_1, e_2) = -\frac{a^2}{b^4} e_1 \wedge e_2 \qquad \mathrm{R}^t(e_1, e_3) = -\frac{a^2}{b^4} e_1 \wedge e_3$$

$$\mathrm{R}^t(e_2, e_3) = \frac{3a^2 - 4b^2}{b^4} e_2 \wedge e_3.$$

Let us now analyze the Weingarten maps A_t. By [6, Proposition 4.1], A_t is computed in the frame $\{e_1, e_2, e_3\}$ by the formula $g_t(A_t(X), Y) = -\frac{1}{2}\frac{\partial}{\partial t}(g_t(X, Y))$. We obtain

$$A_t = -\frac{\dot{a}}{a^3}(t) e_1 \otimes e_1 - \frac{\dot{b}}{b^3}(t) e_2 \otimes e_2 - \frac{\dot{b}}{b^3}(t) e_3 \otimes e_3.$$

The twisted exterior differential of A_t is

$$\begin{aligned}
(\mathrm{d}^{\nabla^t} A_t)(e_1, e_2) &= \nabla^t_{e_1}(A_t e_2) - A_t \nabla^t_{e_1} e_2 - \nabla^t_{e_2}(A_t e_1) + A_t \nabla^t_{e_2} e_1 \\
&= -\frac{\dot{b}}{b}(t) \nabla^t_{e_1} e_2 - \left(2 - \frac{a^2}{b^2}(t)\right) A_t e_3 + \frac{\dot{a}}{a}(t) \nabla^t_{e_2} e_1 - \frac{a^2}{b^2}(t) A_t e_3 \\
&= \frac{a^2}{b^2}(t) \left(\frac{\dot{b}}{b}(t) - \frac{\dot{a}}{a}(t)\right) e_3 \\
(\mathrm{d}^{\nabla^t} A_t)(e_1, e_3) &= \nabla^t_{e_1}(A_t e_3) - A_t \nabla^t_{e_1} e_3 - \nabla^t_{e_3}(A_t e_1) + A_t \nabla^t_{e_3} e_1 \\
&= -\frac{\dot{b}}{b}(t) \nabla^t_{e_1} e_3 - \left(\frac{a^2}{b^2}(t) - 2\right) A_t e_2 + \frac{\dot{a}}{a}(t) \nabla^t_{e_3} e_1 + \frac{a^2}{b^2}(t) A_t e_2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^2}{b^2}(t)\left(\frac{\dot a}{a}(t) - \frac{\dot b}{b}(t)\right) e_2 \\
(\mathrm{d}^{\nabla^t} A_t)(e_2, e_3) &= \nabla^t_{e_2}(A_t e_3) - A_t \nabla^t_{e_2} e_3 - \nabla^t_{e_3}(A_t e_2) + A_t \nabla^t_{e_3} e_2 \\
&= -\frac{\dot b}{b}(t)\nabla^t_{e_2} e_3 - A_t e_1 + \frac{\dot b}{b}(t)\nabla^t_{e_3} e_2 - A_t e_1 \\
&= 2\left(\frac{\dot a}{a}(t) - \frac{\dot b}{b}(t)\right) e_1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
*(\mathrm{d}^{\nabla^t} A_t)(e_1, e_2) &= \frac{a}{b^2}(t)\left(\frac{\dot b}{b}(t) - \frac{\dot a}{a}(t)\right) e_1 \wedge e_2, \\
*(\mathrm{d}^{\nabla^t} A_t)(e_1, e_3) &= \frac{a}{b^2}(t)\left(\frac{\dot b}{b}(t) - \frac{\dot a}{a}(t)\right) e_1 \wedge e_3, \\
*(\mathrm{d}^{\nabla^t} A_t)(e_2, e_3) &= \frac{2a}{b^2}(t)\left(\frac{\dot a}{a}(t) - \frac{\dot b}{b}(t)\right) e_2 \wedge e_3.
\end{aligned}
$$

Finally, Equation (3.5.2) rewrites as the system

$$
\begin{cases}
0 = \mathrm{R}^t_{e_1,e_2} + *(\mathrm{d}^{\nabla^t} A_t)(e_1, e_2) + A_t(e_1) \wedge A_t(e_2) \\
0 = \mathrm{R}^t_{e_1,e_3} + *(\mathrm{d}^{\nabla^t} A_t)(e_1, e_3) + A_t(e_1) \wedge A_t(e_3) \\
0 = \mathrm{R}^t_{e_2,e_3} + *(\mathrm{d}^{\nabla^t} A_t)(e_2, e_3) + A_t(e_2) \wedge A_t(e_3)
\end{cases}
$$

and taking into account the previous computations, this system reads

$$
\begin{cases}
0 = -\dfrac{a^2}{b^4} e_1 \wedge e_2 + \dfrac{a}{b^2}\left(\dfrac{\dot b}{b} - \dfrac{\dot a}{a}\right) e_1 \wedge e_2 + \dfrac{\dot a \dot b}{ab} e_1 \wedge e_2 \\
0 = -\dfrac{a^2}{b^4} e_1 \wedge e_3 + \dfrac{a}{b^2}\left(\dfrac{\dot b}{b} - \dfrac{\dot a}{a}\right) e_1 \wedge e_3 + \dfrac{\dot a \dot b}{ab} e_1 \wedge e_3 \\
0 = \dfrac{3a^2 - 4b^2}{b^4} e_2 \wedge e_3 + \dfrac{2a}{b^2}\left(\dfrac{\dot a}{a} - \dfrac{\dot b}{b}\right) e_2 \wedge e_3 + \left(\dfrac{\dot b}{b}\right)^2 e_2 \wedge e_3
\end{cases}
$$

so we are left with the two independent equations

$$
\begin{cases}
0 = -\dfrac{a^2}{b^4} - \dfrac{\dot a}{b^2} + \dfrac{a\dot b}{b^3} + \dfrac{\dot a \dot b}{ab} \\
0 = \dfrac{3a^2 - 4b^2}{b^4} + \dfrac{2\dot a}{b^2} - \dfrac{2a\dot b}{b^3} + \left(\dfrac{\dot b}{b}\right)^2.
\end{cases}
\tag{3.5.3}
$$

Moreover, from the identity $g_0(A_0(X), Y) = -\frac{1}{2}\dot g_0(X, Y)$ for all $X, Y \in TM_0$, we have the initial conditions

$$
a(0) = b(0) = 1 \qquad \dot a(0) = -1 \qquad \dot b(0) = 3.
$$

The first equation of (3.5.3) can be rewritten as

$$0 = \frac{1}{ab}\left(\dot{b} - \frac{a}{b}\right)\left(\frac{a^2}{b^2} + \dot{a}\right),$$

thus either $\dot{b} - \frac{a}{b} = 0$ or $\frac{a^2}{b^2} + \dot{a} = 0$. However, from the initial conditions one has $\dot{b}(0) - \frac{a}{b}(0) = 2 \neq 0$ so the first case never occurs. Consequently, we get

$$\dot{a} = -\frac{a^2}{b^2}.$$

Substituting this $\dot{a}$ in the second equation of the system (3.5.3), one obtains by factorization

$$0 = \frac{1}{b^4}(a - b\dot{b} - 2b)(a - b\dot{b} + 2b).$$

Thus, either $a - b\dot{b} - 2b = 0$ or $a - b\dot{b} + 2b = 0$. The initial conditions give $a(0) - b\dot{b}(0) = 1 - 3 = -2$ and we conclude that the second case occurs. We have reduced (3.5.3) to the simpler system

$$\begin{cases} \dot{a} = -\dfrac{a^2}{b^2} \\ \dot{b} = \dfrac{a}{b} + 2 \\ a(0) = b(0) = 1,\ \dot{a}(0) = -1,\ \dot{b}(0) = 3. \end{cases} \tag{3.5.4}$$

In order to solve this system, we will find a conserved quantity and make a well-chosen change of variable. We begin by computing the derivative of $\frac{b}{a}$:

$$\left(\frac{b}{a}\right)' = \frac{\dot{b}a - \dot{a}b}{a^2} = \frac{\left(\frac{a}{b} + 2\right)a + \frac{a^2}{b^2}b}{a^2} = \frac{2}{b} + \frac{2}{a},$$

and then,

$$0 = \left(\frac{b}{a}\right)' - 2\left(\frac{1}{b} + \frac{1}{a}\right) \Leftrightarrow 0 = ab\left(\frac{b}{a}\right)' - 2(a+b).$$

In addition, one has

$$(ab)' = \dot{a}b + \dot{b}a = -\frac{a^2}{b^2}b + a\left(\frac{a}{b} + 2\right) = 2a,$$

thus

$$2(a+b) = 2a + 2\frac{b}{a}a = 2a\left(\frac{b}{a} + 1\right) = (ab)'\left(\frac{b}{a} + 1\right),$$

and finally we have

$$0 = ab\left(\frac{b}{a}\right)' - (ab)'\left(\frac{b}{a} + 1\right) \Leftrightarrow \left(\frac{1}{ab}\left(\frac{b}{a} + 1\right)\right)' = 0.$$

We conclude that the quantity $\frac{1}{ab}\left(\frac{b}{a} + 1\right)$ is constant, so

$$\frac{1}{ab}\left(\frac{b}{a} + 1\right) = 2.$$

A natural change of variable is to set $s = \phi(t) := a(t)b(t)$, so $\frac{b(t)}{a(t)} = 2\phi(t) - 1 = 2s - 1$. Composing by ϕ^{-1} on the right, we obtain for s in a neighborhood of 1

$$\begin{aligned}
&\dot{a}(\phi^{-1}(s)) = -\frac{a^2}{b^2}(\phi^{-1}(s)) \\
&\dot{b}(\phi^{-1}(s)) = \frac{a}{b}(\phi^{-1}(s)) + 2 \\
&a(\phi^{-1}(s))b(\phi^{-1}(s)) = \phi(\phi^{-1}(s)) \\
&\frac{b}{a}(\phi^{-1}(s)) = 2\phi(\phi^{-1}(s)) - 1
\end{aligned}$$

and setting $\alpha := a \circ \phi^{-1}$ and $\beta := b \circ \phi^{-1}$ one arrives at

$$\begin{aligned}
&\dot{\alpha} = -(\phi^{-1})'\frac{\alpha^2}{\beta^2} = -(\phi^{-1})'\frac{1}{(2s-1)^2} \\
&\dot{\beta} = (\phi^{-1})'\left(\frac{\alpha}{\beta} + 2\right) = (\phi^{-1})'\frac{4s-1}{2s-1} \\
&\alpha\beta = s \\
&\frac{\beta}{\alpha} = 2s - 1.
\end{aligned}$$

Differentiating the last two equations one gets

$$\dot{\alpha}\beta + \alpha\dot{\beta} = 1 \tag{3.5.5}$$

and

$$\begin{aligned}
\alpha\dot{\beta} - \dot{\alpha}\beta = 2\alpha^2 \Leftrightarrow 2\alpha = \dot{\beta} - \dot{\alpha}(2s-1) &= (\phi^{-1})'\frac{4s-1}{2s-1} + (\phi^{-1})'\frac{1}{2s-1} \\
\Leftrightarrow \alpha &= (\phi^{-1})'\frac{2s}{2s-1}.
\end{aligned}$$

Re-injecting this last equation in (3.5.5) we have

$$\begin{aligned}
&\alpha\left(\dot{\alpha}\frac{\beta}{\alpha} + \dot{\beta}\right) = 1 \Leftrightarrow (\phi^{-1})'\frac{2s}{2s-1}\left(-(\phi^{-1})'\frac{1}{(2s-1)^2}(2s-1) + (\phi^{-1})'\frac{4s-1}{2s-1}\right) = 1 \\
&\Leftrightarrow \left((\phi^{-1})'\right)^2\frac{4s}{2s-1} = 1 \Leftrightarrow (\phi^{-1})' = \sqrt{\frac{2s-1}{4s}},
\end{aligned}$$

where we used that the derivative of ϕ is positive. Thus, one has

$$\begin{aligned}
&\dot{\alpha} = -(\phi^{-1})'\frac{1}{(2s-1)^2} = -\frac{1}{2\sqrt{s}(2s-1)^{\frac{3}{2}}} \\
&\dot{\beta} = (\phi^{-1})'\frac{4s-1}{2s-1} = \frac{4s-1}{2\sqrt{2s^2-s}}
\end{aligned}$$

and by integration we finally obtain

$$\alpha = \sqrt{\frac{s}{2s-1}}, \qquad \beta = \sqrt{s(2s-1)}.$$

We conclude that in terms of the new variable s, the metric on $S^3 \times I$ introduced in (3.5.1) is

$$g^{\mathcal{Z}} = \frac{2s-1}{4s}\mathrm{d}s^2 + \frac{s}{2s-1}\eta_1^2 + s(2s-1)(\eta_2^2+\eta_3^2), \qquad s \in (\tfrac{1}{2}, \infty).$$

We can compute the interval to which the variable t belongs by calculating

$$\begin{aligned}\int_{\frac{1}{2}}^{1} (\phi^{-1})'(s)\mathrm{d}s &= \int_{\frac{1}{2}}^{1} \sqrt{\frac{2s-1}{4s}}\mathrm{d}s \\ &= \frac{1}{\sqrt{2}}\left[s\sqrt{1-\frac{1}{2s}} - \frac{1}{4}\ln\left(4s-1+\sqrt{16s^2-8s}\right)\right]_{\frac{1}{2}}^{1} \\ &= \frac{1}{2} - \frac{1}{4\sqrt{2}}\ln(3+2\sqrt{2}) = \frac{\sqrt{2}-\ln(1+\sqrt{2})}{2\sqrt{2}} \approx 0{,}1884,\end{aligned}$$

and so the family of metrics g_t from (3.5.1) admitting Cauchy spinors exists for $t \in \left(\frac{1}{2}\left(\frac{1}{\sqrt{2}}\ln(1+\sqrt{2})-1\right), \infty\right)$.

We now show the metric obtained on $\mathcal{Z}$ is indeed the Ricci-flat metric we were searching for. Being able to extend a Cauchy spinor on $\{0\} \times S^3$ to a parallel spinor on $\mathcal{Z}$ is equivalent to $\Sigma^+\mathcal{Z}$ being flat because of the quaternionic structure. This is equivalent to

$$*\mathrm{R}^{\mathcal{Z}}_{X,Y} = -\mathrm{R}^{\mathcal{Z}}_{X,Y}, \qquad \forall X, Y \in T\mathcal{Z}, \tag{3.5.6}$$

where $\mathrm{R}^{\mathcal{Z}}$ is the curvature tensor of $\mathcal{Z}$, thus we just check this last fact. Equation (3.5.2) already gives that (3.5.6) is true for any $X, Y \in TM_t$, $t \in I$. We will use the notation $\nu := \frac{\partial}{\partial t}$. By [6, Proposition 4.1], one has

$$g^{\mathcal{Z}}(\mathrm{R}_{X,\nu}Y, Z) = g_t(\mathrm{d}^{\nabla^t}A_t(Y,Z), X), \qquad \forall t \in I, \forall X, Y, Z \in TM_t.$$

This together with system (3.5.4) gives

$$\begin{aligned} g^{\mathcal{Z}}(\mathrm{R}^{\mathcal{Z}}_{e_1,\nu}e_2, e_3) &= 2a^2\left(\frac{\dot a}{a} - \frac{\dot b}{b}\right) = -4\frac{a^3+a^2b}{b^2} \\ g^{\mathcal{Z}}(\mathrm{R}^{\mathcal{Z}}_{e_2,\nu}e_1, e_3) &= a^2\left(\frac{\dot a}{a} - \frac{\dot b}{b}\right) = -2\frac{a^3+a^2b}{b^2} \\ g^{\mathcal{Z}}(\mathrm{R}^{\mathcal{Z}}_{x_3,\nu}x_1, x_2) &= -a^2\left(\frac{\dot a}{a} - \frac{\dot b}{b}\right) = 2\frac{a^3+a^2b}{b^2}. \end{aligned}$$

Moreover, by [6, Proposition 4.1], one has

$$g^{\mathcal{Z}}(\mathrm{R}_{X,\nu}Y, \nu) = \frac{1}{2}(\ddot g_t(X,Y) + \dot g_t(A_t(X), Y)), \qquad \forall t \in I, \forall X, Y \in TM_t,$$

thus, another use of system (3.5.4) gives

$$\begin{aligned} g^{\mathcal{Z}}(\mathrm{R}^{\mathcal{Z}}_{e_1,\nu}e_1, \nu) &= \frac{1}{2}\left(2a\ddot a + 2\dot a^2 - 2\dot a^2\right) = a\ddot a = 4\frac{a^4+a^3b}{b^4} \\ g^{\mathcal{Z}}(\mathrm{R}^{\mathcal{Z}}_{e_2,\nu}e_2, \nu) &= \frac{1}{2}\left(2b\ddot b + 2\dot b^2 - 2\dot b^2\right) = b\ddot b = -2\frac{a^2+ab}{b^2} \end{aligned}$$

$$g^{\mathcal{Z}}(\mathrm{R}^{\mathcal{Z}}_{e_3,\nu}e_3,\nu) = \frac{1}{2}\left(2b\ddot{b} + 2\dot{b}^2 - 2\dot{b}^2\right) = b\ddot{b} = -2\frac{a^2+ab}{b^2},$$

and $g^{\mathcal{Z}}(\mathrm{R}^{\mathcal{Z}}_{e_k,\nu}e_l,\nu)$ for all $k \neq l$. We now define the oriented orthonormal base $(x_1, x_2, x_3, \nu) := (e_1/a, e_2/b, e_3/b, \nu)$. All together, one has

$$\begin{aligned}\mathrm{R}^{\mathcal{Z}}_{\cdot,\nu} =& \frac{a^2+ab}{b^4}(4x_1 \otimes (x_1 \wedge \nu - x_2 \wedge x_3) - 2x_2 \otimes (x_1 \wedge x_3 + x_2 \wedge \nu) \\ &+ 2x_3 \otimes (x_1 \wedge x_2 - x_3 \wedge \nu)),\end{aligned} \tag{3.5.7}$$

and we easily check that property (3.5.6) is true. This means that $\mathcal{Z}$ carries a parallel spinor so it is hyperkähler, and then Ricci-flat.

3.5.2 Link with the family of Euclidean Taub-NUT metrics

Let us first extend the previous study to the case where the initial sphere has radius $r > 0$. It is easy to see that this does not change the form of the endomorphism A_0 defined in (3.4.4). This modification results in the rescaling of the metric by a factor r^2. Subsequently, if we keep the same notations as in the case $r = 1$, the metric $g^{\mathcal{Z}}$ on $\mathcal{Z}$ is given by

$$g^{\mathcal{Z}} = \frac{2s-1}{4s}r^2\mathrm{d}s^2 + \frac{r^2 s}{2s-1}\eta_1^2 + r^2 s(2s-1)(\eta_2^2+\eta_3^2), \qquad s \in (\tfrac{1}{2}, \infty).$$

With the change of variable $u := rs$ we can express this metric by

$$g^{\mathcal{Z}} = \frac{2u-r}{4u}\mathrm{d}u^2 + \frac{r^2 u}{2u-r}\eta_1^2 + u(2u-r)(\eta_2^2+\eta_3^2), \qquad u \in (\tfrac{r}{2}, \infty). \tag{3.5.8}$$

This family of metrics is strikingly similar to the well-known family of Euclidean Taub-NUT metrics on $\mathbb{R}^4$ (see e.g. [63] and the references therein). In polar coordinates, the Euclidean Taub-NUT metrics are given (up to a constant) by the expression

$$g_{TN} = \frac{as+b}{4s}\left(\mathrm{d}s^2 + \frac{4b^2s^2}{(as+b)^2}\eta_1^2 + 4s^2(\eta_2^2+\eta_3^2)\right)$$

where a and b are *positive* parameters and we recall that η_1, η_2, η_3 are the 1-forms dual to the Hopf vector fields e_1, e_2, e_3. Through a change of variable $u = \sqrt{\frac{a}{2}}s$ in the radial variable s, we can always normalize the parameter a to be equal to 2:

$$g_{TN} = \frac{2u+r}{4u}\left(\mathrm{d}u^2 + \frac{4r^2u^2}{(2u+r)^2}\eta_1^2 + 4u^2(\eta_2^2+\eta_3^2)\right). \tag{3.5.9}$$

where $r = \sqrt{\frac{2}{a}}b$. The metric from Equation (3.5.8) belongs formally to the extension for *negative* values of the parameter r of the family of Taub-NUT metrics normalized with $a = 2$. Note that the parameter r cannot vanish, else the metric degenerates. Alternately, we recover (3.5.8) formally as the Taub-NUT metric (3.5.9) with $r > 0$, but on the interval $(-\infty, -\frac{r}{2}) \times \mathbf{S}^3$. The metrics in this family admit a nonzero parallel spinor. This implies that they are hyperkähler, hence Ricci-flat, and also (anti-) self-dual, according to the chirality of the nonzero parallel spinor.

Miyake [56, Prop. 2.2 (2)] found the above family of metrics for non-zero r in his study of self-dual metrics of Iwai-Katayama type.

In the presentation (3.5.9) of the Taub-NUT metric, $s = 0$ is an apparent singularity, but in fact the metric extends smoothly in the origin of $\mathbb{R}^4$. In contrast, the metric (3.5.8) has a true singularity at $s = \frac{1}{2}$. The horizontal directions e_2 and e_3 collapse, while the vertical direction of e_1 *explodes* in finite time as $s \searrow \frac{1}{2}$. As a result, the curvature operator is unbounded near $s = \frac{1}{2}$.

Note that on $(-\frac{r}{2}, 0) \times \mathbf{S}^3$, (3.5.9) describes the original (Lorentzian) Taub-NUT metric.

3.6 Classification results on $\mathbf{S}^3$

In this section we analyze the set of symmetric solutions of (3.1.2) in the case $M = \mathbf{S}^3$. Recall that $\mathcal{C}_{\mathbf{S}^3} = \mathcal{C}^{\mathrm{loc}}_{\mathbf{S}^3}$ because the sphere is simply-connected. Since the known examples of solutions can be expressed as constant matrices in a frame of left (or right-) invariant vector fields, we will investigate some classes of endomorphisms related to these vector fields.

3.6.1 Endomorphisms constant in a left or right invariant orthonormal frame

The four examples we recalled in Remark 3.4.1 can be interpreted as constant matrices either in a left- or a right-invariant orthonormal frame. For this reason, it is legitimate to search for all symmetric endomorphisms in $\mathcal{C}_{\mathbf{S}^3}$ that verify this property. We shall prove that there exist no other examples besides the ones already known from Remark 3.4.1.

Proposition 3.6.1. *Let $A \in \mathcal{C}_{\mathbf{S}^3}$. Assume that A is constant in a left (resp. right)-invariant orthonormal frame. Then, either $A = \pm\mathrm{Id}$ or A has eigenvalues $1, -3, -3$ (resp. $-1, 3, 3$). In particular, A is one of the endomorphism fields described in Remark 3.4.1.*

Proof. We recall that (e_1, e_2, e_3) are the three left-invariant vector fields corresponding to the quaternions i, j, k on $\mathbf{S}^3$.

Let $A \in \mathcal{C}_{\mathbf{S}^3}$ (i.e., A is symmetric and satisfies Equation (3.4.3) on $\mathbf{S}^3$), and assume that A is constant in a left-invariant orthonormal frame. Hence, A can be viewed as a real symmetric 3×3 matrix, in particular it is diagonalizable. From these considerations, up to an isometry of the sphere we can assume without loss of generality that $A = ae_1 \otimes e_1 + be_2 \otimes e_2 + ce_3 \otimes e_3$ for $a, b, c \in \mathbb{R}$.

Equation (3.4.3) applied to $X, Y \in \{e_1, e_2, e_3\}$ together with (3.2.2) and (3.4.2) give the system

$$\begin{cases} a + b - 2c = 1 - ab \\ b + c - 2a = 1 - bc \\ c + a - 2b = 1 - ca \end{cases} \Leftrightarrow \begin{cases} (a+1)(b+1) = 2(c+1) \\ (b+1)(c+1) = 2(a+1) \\ (a+1)(c+1) = 2(b+1). \end{cases}$$

We easily see that $a + 1 = 0 \Leftrightarrow b + 1 = 0 \Leftrightarrow c + 1 = 0$, and in this case $A = -\mathrm{Id}$. We assume now that $a + 1 \neq 0$. The product of all the equations give

$$(a+1)(b+1)(c+1) = 8,$$

and we conclude that $(a+1)^2 = (b+1)^2 = (c+1)^2 = 4$. Therefore $a+1, b+1, c+1 \in \{-2, 2\}$ and moreover an even number among them are negative, concluding the proof when A is constant in a left-invariant frame.

The case of a right-invariant orthonormal frame is treated similarly and produces the additional solution $-1, 3, 3$. □

3.6.2 Endomorphism fields with three distinct constant eigenvalues

The case of an endomorphism solution of (3.4.3) with at most two distinct eigenvalues was already studied in [62], where it was shown that the only possibilities are the ones given in Remark 3.4.1.

Proposition 3.6.2. *There is no element of $\mathcal{C}_{\mathbf{S}^3}$ with three distinct constant eigenvalues.*

Proof. Let $A \in \mathcal{C}_{\mathbf{S}^3}$ with three constant eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$. The associated unitary eigenvectors are global vector fields on $\mathbf{S}^3$, which form an orthonormal frame, and are denoted by X_1, X_2, X_3. Equation (3.4.3) means that for every cyclic permutation (a, b, c) of the index set $(1, 2, 3)$ one has

$$\lambda_b \nabla_{X_a} X_b - \lambda_a \nabla_{X_b} X_a - A[X_a, X_b] = (1 - \lambda_a \lambda_b) X_c.$$

Projecting on X_a, we see that

$$\begin{aligned}
&\lambda_b g(\nabla_{X_a} X_b, X_a) - \lambda_a g(\nabla_{X_b} X_a, X_a) - \lambda_a g([X_a, X_b], X_a) = 0 \\
\Leftrightarrow &(\lambda_b - \lambda_a) g(\nabla_{X_a} X_b, X_a) = 0 \\
\Leftrightarrow &(\lambda_b - \lambda_a) g([X_a, X_b], X_a) = 0.
\end{aligned}$$

This last equation is true for any $a, b \in \{1, 2, 3\}$, and this means $[X_a, X_b] \in \mathrm{Span}(X_c)$.

As a direct consequence of Koszul formula, $\nabla_{X_a} X_a = 0$ for $a \in \{1, 2, 3\}$. We can compute for any a

$$\delta(X_a) = -X_k \lrcorner \nabla_{X_k} X_a = g(X_a, \nabla_{X_k} X_k) = 0.$$

This shows that the vector fields X_k are geodesic and divergence free. By a result of Gluck and Gu [34, Theorem A], every geodesic and divergence free vector field on $\mathbf{S}^3$ is a Hopf vector field (i.e. a unit vector field tangent to the fiber of a Hopf fibration). Moreover, since they form an orthonormal basis at any point, they are all either left or right-invariant. However we can give a simpler argument in our case:

Lemma 3.6.3. *Let (X_1, X_2, X_3) be a global orthonormal frame of geodesic vector fields on $\mathbf{S}^3$ (i.e. $\nabla_{X_k} X_k = 0$). Then (X_1, X_2, X_3) is either a left- or a right-invariant frame.*

Proof. Using Koszul's formula, one sees that the assumption $\nabla_{X_k} X_k = 0$ is actually equivalent to the existence of three real functions $\alpha_1, \alpha_2, \alpha_3$ on $\mathbf{S}^3$ such that $[X_a, X_b] = \alpha_c X_c$ for any cyclic permutation (a, b, c) of the indices $1, 2, 3$.

We define the real-valued functions $2\beta_k = (-1)^{\delta_{1,k}} \alpha_1 + (-1)^{\delta_{2,k}} \alpha_2 + (-1)^{\delta_{3,k}} \alpha_3$. By the Koszul formula,

$$\nabla_{X_a} X_b = \beta_a X_c, \qquad \nabla_{X_a} X_c = -\beta_a X_b,$$

Moreover, the curvature tensor on the sphere satisfies $\mathrm{R}(X_a, X_b) X_b = X_a$. Since the vector fields X_1, X_2, X_3 are geodesic, we also have

$$\begin{aligned}
\mathrm{R}(X_a, X_b) X_b &= -\nabla_{X_b} \nabla_{X_a} X_b - \nabla_{[X_a, X_b]} X_b \\
&= -X_b(\beta_a) X_b - \beta_a \beta_b X_a + 2\alpha_c \beta_c X_a \\
&= -X_b(\beta_a) X_b + (-\beta_a \beta_b + \beta_a \beta_c + \beta_b \beta_c) X_a,
\end{aligned}$$

so the projection of this equation on X_a yields

$$-\beta_a \beta_b + \beta_a \beta_c + \beta_b \beta_c = 1.$$

Since this is true for any value of (a,b,c) in $\{(1,2,3),(2,3,1),(3,1,2)\}$, one has $\beta_a\beta_b = 1$ and we conclude that $\beta_k = \pm 1$ for any $k \in \{1,2,3\}$.

Assume first that $\beta_k = 1$ for any $k \in \{1,2,3\}$. We define for any $X, Y \in T\mathbf{S}^3$ the covariant derivative

$$\overline{\nabla}_X Y := \nabla_X Y - *(X \wedge Y),$$

which was already considered in (3.2.4). The vector fields X_k are parallel for $\overline{\nabla}$ and so are the left-invariant vector fields with value $(X_k)_e$ at e. Thus, these vector fields coincide.

In the case $\beta_1 = \beta_2 = \beta_3 = -1$, the same proof shows that X_1, X_2, X_3 are right-invariant. □

Hence, A is constant in a left or right-orthonormal frame and according to Proposition (3.6.1) it must have at most 2 different eigenvalues, which contradicts the hypothesis. □

3.6.3 Endomorphisms constant in the direction of a left-invariant vector field

We will now weaken the condition from Section 3.6.1, and search for solutions A of (3.4.3) on $\mathbf{S}^3$ that are constant in the direction of a left-invariant vector field ξ, i.e. $\mathcal{L}_\xi A = 0$. Assuming this invariance, all the objects can be expressed on the basis of the Hopf fibration with fibers tangent to ξ. We decompose A under the form:

$$A = f\xi \otimes \xi + v \otimes \xi + \xi \otimes v + B \tag{3.6.1}$$

where f is a function on $\mathbf{S}^3$, $v \in \xi^\perp$ and B is the restriction of A to $\xi^\perp$. The condition $\mathcal{L}_\xi A = 0$ gives

$$0 = \mathcal{L}_\xi A = (\xi f)\xi \otimes \xi + \mathcal{L}_\xi v \otimes \xi + \xi \otimes \mathcal{L}_\xi v + \mathcal{L}_\xi B,$$

and we know that for all $X \in \xi^\perp$, $\mathcal{L}_\xi X \in \xi^\perp$ since ξ is a Killing field, so we deduce

$$\xi f = 0, \qquad \mathcal{L}_\xi v = 0, \qquad \mathcal{L}_\xi B = 0. \tag{3.6.2}$$

As a direct consequence of equations (3.6.2) we can interpret f, v and B respectively as a function, a vector and an endomorphism on the basis $\mathbf{S}^2(\frac{1}{2})$ of the Hopf fibration.

We define the endomorphism J of $\xi^\perp$ by $JX := -\nabla_X \xi$. This endomorphism is skew-symmetric and satisfies $J^2 = -1$; it is actually the lift of the standard almost complex structure from $\mathbf{S}^2(\frac{1}{2})$ through the Hopf fibration, so we will see it as an endomorphism of the base.

The invariance equations (3.6.2) give

$$\begin{aligned}
\nabla_\xi v &= \nabla_v \xi = -Jv \\
(\nabla_\xi B)X &= \nabla_\xi(BX) - B\nabla_\xi X = \nabla_{BX}\xi + [\xi, BX] + BJX - B[\xi, X] \\
&= [B,J]X + \mathcal{L}_\xi(BX) - B\mathcal{L}_\xi X = [B,J]X + (\mathcal{L}_\xi B)X = [B,J]X.
\end{aligned}$$

Now, we express Equation (3.4.3) in terms of f, v and B by considering horizontal and vertical vectors for X and Y.

Let X, Y be two orthogonal vector fields in $\xi^\perp$. One has,

$$\begin{aligned}
d^\nabla A(\xi, X) &= (\nabla_\xi A)X - (\nabla_X A)\xi \\
&= -(Jv \otimes \xi + \xi \otimes Jv - [B,J])X - \nabla_X(A\xi) - AJX
\end{aligned}$$

$$= -g(Jv, X)\xi - JBX + BJX - \nabla_X(f\xi + v) - g(v, JX)\xi - BJX$$
$$= -JBX - \nabla_X(f\xi + v).$$

We now use the fact that for any X, $g(JX, X) = 0$ to infer that $*\xi \wedge X = JX$ and $*v \wedge X = -g(v, JX)\xi$, so Equation (3.4.3) implies

$$\begin{aligned} -JBX - \nabla_X(f\xi + v) &= *(\xi \wedge X) - *(f\xi + v) \wedge (g(X, v)\xi + BX) \\ &= JX - fJBX + g(v, JBX)\xi + g(X, v)Jv. \end{aligned} \tag{3.6.3}$$

Projecting equation (3.6.3) on ξ, one has

$$-g(\nabla_X(f\xi + v), \xi) = g(v, JBX)$$

thus

$$g(BJv + Jv - df, X) = 0.$$

Since this last equation is true for any $X \in \xi^\perp$ we conclude

$$BJv = -Jv + df. \tag{3.6.4}$$

We define the orthogonal projector P on $\xi^\perp$. We now project equation (3.6.3) on the orthogonal of ξ, and we obtain

$$-JBX + fJX - P\nabla_X v = JX - fJBX + g(X, v)Jv$$

thus

$$(f - 1)(JBX + JX) = P\nabla_X v + g(X, v)Jv.$$

Thus, we have the system

$$\begin{cases} BJv = -Jv + df \\ (f - 1)(JBX + JX) = P\nabla_X v + g(X, v)Jv \end{cases}$$

We now compute:

$$\begin{aligned} (\nabla_X A)Y =&(X(f)\xi \otimes \xi - fJX \otimes \xi - f\xi \otimes JX + \nabla_X v \otimes \xi \\ &+ \xi \otimes \nabla_X v - JX \otimes v - v \otimes JX + (\nabla_X B))Y \\ =& - fg(JX, Y)\xi + g(\nabla_X v, Y)\xi - g(JX, Y)v \\ &- g(v, Y)JX + (\nabla_X B)Y. \end{aligned}$$

We deduce that

$$\begin{aligned} d^\nabla A(X, Y) =& - fg(JX, Y)\xi + g(\nabla_X v, Y)\xi - g(JX, Y)v - g(v, Y)JX + (\nabla_X B)Y \\ &+ fg(JY, X)\xi - g(\nabla_Y v, X)\xi + g(JY, X)v + g(v, X)JY - (\nabla_Y B)X \\ =& 2fg(X, JY)\xi + dv(X, Y)\xi + 2g(X, JY)v \\ &- g(v, Y)JX + g(v, X)JY + d^\nabla B(X, Y). \end{aligned}$$

Equation (3.4.3) leads to

$$\begin{aligned} &2fg(X, JY)\xi + dv(X, Y)\xi + 2g(X, JY)v - g(v, Y)JX + g(v, X)JY + d^\nabla B(X, Y) \\ =& * X \wedge Y - *(g(X, v)\xi + BX) \wedge (g(Y, v)\xi + BY) \end{aligned}$$

$$= -\,[g(X,JY) - g(BX,JBY)]\xi - g(X,v)JBY + g(Y,v)JBX$$

which leads to

$$\begin{aligned}[-g(X,JY) + g(BX,JBY) - 2fg(X,JY) - dv(X,Y)]\xi - 2g(X,JY)v \\ = d^{\nabla}B(X,Y) + g(X,v)J(B+1)Y - g(Y,v)J(B+1)X.\end{aligned}$$

Since

$$\begin{aligned}g(d^{\nabla}B(X,Y),\xi) &= g(\nabla_X(BY) - \nabla_Y(BX) - B[X,Y],\xi) \\ &= g(\nabla_X(BY),\xi) - g(\nabla_Y(BX),\xi) \\ &= -g(BX,JY) + g(BY,JX),\end{aligned}$$

the projection on ξ gives

$$-g(X,JY) + g(BX,JBY) - 2fg(X,JY) - dv(X,Y) = -g(BX,JY) + g(BY,JX)$$

and

$$2(1+f)g(X,JY) + dv(X,Y) = g((B+1)X, J(B+1)Y).$$

We remark that for any symmetric endomorphism S, one has $SJS = \det(S)J$ and the last equation is rewritten

$$[2(1+f) - \det(B+1)]g(X,JY) = -dv(X,Y).$$

Moreover, the projection on $\xi^{\perp}$ provides the equation

$$-Pd^{\nabla}B(X,Y) = g(X,v)J(B+1)Y - g(Y,v)J(B+1)X + 2g(X,JY)v.$$

For the remainder of this section, we will denote by $\overline{\nabla}$ the covariant derivative on $\mathbf{S}^2(\frac{1}{2})$ (the basis of the Hopf fibration). We recall that if U,V are basic vector fields on $\mathbf{S}^3$ we have the equation

$$\nabla_U V = \overline{\nabla}_U V + g(V,JU)\xi.$$

Now, we study the objects on $\mathbf{S}^2(\frac{1}{2})$. For any $X,Y \in T\mathbf{S}^2(\frac{1}{2})$, one has $dv(X,Y) = d^*(Jv)g(JX,Y)$ and, using a unit vector field X at a point of $\mathbf{S}^2(\frac{1}{2})$ and the fact that J is parallel, one has

$$\begin{aligned}*d^{\overline{\nabla}}B = d^{\overline{\nabla}}B(X,JX) &= (\overline{\nabla}_X B)JX - (\overline{\nabla}_{JX}B)X \\ &= (\overline{\nabla}_X BJ)X + (\overline{\nabla}_{JX}BJ)JX = -\delta^{\overline{\nabla}}(BJ).\end{aligned}$$

Consequently, we obtain four equations on the sphere $\mathbf{S}^2(\frac{1}{2})$

$$\begin{cases}(B+1)Jv = df \\ (f-1)J(B+1) = \overline{\nabla}v + v\otimes Jv \\ 2(1+f) - \det(B+1) = d^*(Jv) \\ \delta^{\overline{\nabla}}(BJ) = J(B+3)Jv.\end{cases} \tag{3.6.5}$$

The system (3.6.5) seems too difficult to solve in full generality for the time being, and is left as an open problem.

3.6.4 A particular case: $v = 0$

We proceed by treating only the special case in the system (3.6.5) where ξ is an eigenvector of A, i.e. $v = 0$ in (3.6.1). In this situation, the system reduces to

$$\begin{cases} df = 0 \\ (f-1)J(B+1)X = 0 \\ 2(1+f) - \det(B+1) = 0 \\ \delta^{\overline{\nabla}}(BJ) = 0. \end{cases} \tag{3.6.6}$$

Proposition 3.6.4. *Let $A \in \mathcal{C}_{\mathbf{S}^3}$ be the symmetric endomorphism corresponding to a Cauchy spinor on $\mathbf{S}^3$. Assume that there exists a left (resp. right)-invariant vector field ξ such that $\mathcal{L}_\xi A = 0$ and such that ξ is an eigenvector of A. Then $A = \pm\mathrm{Id}$ or $A = \xi \otimes \xi - 3P$ (resp. $A = -\xi \otimes \xi + 3P$), where P is the orthogonal projector on $\xi^\perp$.*

Proof. We start with the case where ξ is left-invariant. We have seen that A must equal $f\xi\otimes\xi + B$ where the Killing vector field ξ is an eigenvector of A, and f and B are ξ-invariant, hence they are pull-back of objects from the base $\mathbf{S}^2(\frac{1}{2})$ of the Hopf fibration. Define the open set $\mathcal{O} = \{x \in \mathbf{S}^2,\, B(x) \neq -\mathrm{Id}\} \subset \mathbf{S}^2$. On $\mathcal{O}$, the second equation gives $f = 1$, and the third one gives $\det(B + 1) = 4$. Since f is constant by the first equation, $\mathcal{O}$ is either empty or equal to $\mathbf{S}^2$.

If $\mathcal{O}$ is empty, $B = -\mathrm{Id}$ on $\mathbf{S}^2$, so $f = -1$ and the only solution of the system is $A = -\mathrm{Id}$.

If $\mathcal{O} = \mathbf{S}^2$, we have seen that $f = 1$ so we are left with the two equations $\det(B+1) = 4$ and $\delta^{\overline{\nabla}}(BJ) = 0$. Equivalently, we search for a symmetric endomorphism field $C = \frac{B+1}{2}$ on the unit sphere $\mathbf{S}^2$ with $\det C = 1$ and $\delta^{\overline{\nabla}}(CJ) = 0$. Notice that $\delta^{\overline{\nabla}}(JCJ) = -J\delta^{\overline{\nabla}}(CJ) = 0$, hence the symmetric endomorphism $U := JCJ$ satisfies $\det U = 1$ and $\delta^{\overline{\nabla}}U = 0$. The following proposition contains the result we need:

Proposition 3.6.5. *Let U be a symmetric endomorphism on $\mathbf{S}^2$ which satisfies $\det U = 1$ and $\delta^{\overline{\nabla}}U = 0$. Then $U = \pm\mathrm{Id}$.*

Proof. Let $t := \frac{1}{2}\mathrm{tr}U$, and $S := J(U - t\mathrm{Id})$. Since $U - t\mathrm{Id}$ is symmetric and traceless, so is S. We will use several times below that traceless symmetric endomorphisms anticommute with J. One has $\delta^{\overline{\nabla}}S = \delta^{\overline{\nabla}}(JU) - \delta^{\overline{\nabla}}(Jt\mathrm{Id}) = Jdt$. Since $U^2 - 2tU + \det U = 0$ by Cayley-Hamilton's theorem, one has

$$S^2 = (JS)^2 = (U - tId)^2 = U^2 - 2tU + t^2\mathrm{Id} = (t^2 - 1)\mathrm{Id}. \tag{3.6.7}$$

Thus, $|t| \geq 1$ because $S^2 = SS^*$ is non-negative. The function t is continuous on $\mathbf{S}^2$, so it cannot change sign. It follows that either $t \geq 1$ on $\mathbf{S}^2$ or $t \leq -1$ on $\mathbf{S}^2$. We shall show below that in the first case $U = \mathrm{Id}$. In the second case, $-U$ also satisfies the hypotheses of the proposition and moreover $\mathrm{tr}(-U) > 0$, so by the first case we get $-U = \mathrm{Id}$. It suffices therefore to solve the first case, i.e., we can assume in the rest of the proof that $t \geq 1$ on $\mathbf{S}^2$.

We want to show that the open set $E := \{x \in \mathbf{S}^2; t(x) > 1\}$ is empty. One has $S \neq 0$ on E, thus we can define $T := \frac{S}{\|S\|}$. Since S is symmetric and traceless, so are T and JT, hence (T, JT) is an orthonormal frame over E in the bundle of symmetric traceless endomorphisms of $T\mathbf{S}^2$. Therefore,

$$\overline{\nabla}T =: \alpha \otimes JT \tag{3.6.8}$$

for some 1-form $\alpha \in \Lambda^1(E)$. By taking a further covariant derivative in this equation and skew-symmetrizing one has for any $X, Y \in T\mathbf{S}^2$

$$d\alpha(X,Y)JT = \bar{\mathrm{R}}_{X,Y}T$$

where $\bar{\mathrm{R}}$ is the curvature of $\mathbf{S}^2$, which acts on any endomorphism field W as $\bar{\mathrm{R}}_{X,Y}W = [\bar{\mathrm{R}}_{X,Y}, W]$. Then, using the identity $\bar{\mathrm{R}}_{X,Y} = -X \wedge Y$ on $\mathbf{S}^2$, we obtain $d\alpha = -2vol$, where vol is the Riemannian volume form.

We identify α with a vector field $\alpha^\sharp$ via the metric, and one has $\delta^{\overline{\nabla}}T = -JT\alpha^\sharp$ by Equation (3.6.8). Since $T^2 = (JT)^2 = \frac{1}{2}\mathrm{Id}$ we obtain $\alpha^\sharp = -2JT(\delta^{\overline{\nabla}}T)$. The condition $\delta^{\overline{\nabla}}S = Jdt$ gives $\delta^{\overline{\nabla}}T = \frac{\delta^{\overline{\nabla}}S}{\|S\|} + T\mathrm{grad}\ln(\|S\|) = \frac{Jdt}{\|S\|} + T\mathrm{grad}\ln(\|S\|)$, hence

$$\alpha^\sharp = -\frac{2S(\mathrm{grad}(t))}{\|S\|^2} - J\mathrm{grad}\ln(\|S\|).$$

We now differentiate this equation to obtain

$$d^*J\alpha^\sharp = -d^*\left(\frac{2JS(\mathrm{grad}(t))}{\|S\|^2}\right) + \Delta\ln(\|S\|)$$

where $\Delta = d^*d$ is the positive Laplacian on $\mathbf{S}^2$. Using the fact that $d\alpha = -2V$, which is equivalent to $d^*J\alpha^\sharp = -2$, one has

$$-2 = -d^*\left(\frac{2JS(\mathrm{grad}(t))}{\|S\|^2}\right) + \Delta\ln(\|S\|).$$

We know from (3.6.7) that $\|S\|^2 = 2(t^2-1)$, so $t = \sqrt{\frac{1}{2}\|S\|^2 + 1}$. This leads to $dt = \frac{\|S\|d(\|S\|)}{2\sqrt{\frac{1}{2}\|S\|^2+1}}$, and finally

$$-2 = -d^*\left(\frac{JSd\ln(\|S\|)}{\sqrt{\frac{1}{2}\|S\|^2+1}}\right) + \Delta\ln(\|S\|).$$

We define $\zeta := \frac{1}{2}\|S\|^2$ and we rewrite the above equation as

$$-1 = -d^*(\frac{JSd\ln\zeta}{\sqrt{\zeta+1}}) + \Delta\ln\zeta.$$

Let $x \in \mathbf{S}^2$ be a point where t, and thus $\|S\|$, reaches its maximum. Clearly, $x \in E$. Let (e_1, e_2) be an orthonormal frame which is parallel at x. At this point, one has

$$\begin{aligned}
d^*(\frac{JSd\ln\zeta}{\sqrt{\zeta+1}}) &= -\langle J\overline{\nabla}_{e_1}(\frac{Sd\ln\zeta}{\sqrt{\zeta+1}}), e_1\rangle - \langle J\overline{\nabla}_{e_2}(\frac{Sd\ln\zeta}{\sqrt{\zeta+1}}), e_2\rangle \\
&= \frac{\left[\langle\overline{\nabla}_{e_1}d\ln\zeta, SJe_1\rangle + \langle\overline{\nabla}_{e_2}d\ln\zeta, SJe_2\rangle\right]}{\sqrt{\zeta+1}} \\
&= \frac{\langle\mathrm{Hess}\ \ln\zeta, SJ\rangle}{\sqrt{\zeta+1}},
\end{aligned}$$

where the Hessian is defined for any function β by $\mathrm{Hess}\,\beta := \overline{\nabla}d\beta$. Thus,

$$-1 = -\frac{\langle\mathrm{Hess}\ \ln\zeta, SJ\rangle}{\sqrt{\zeta+1}} - \langle\mathrm{Hess}\ \ln\zeta, \mathrm{Id}\rangle$$

$$= -\langle \text{Hess} \ln \zeta, \text{Id} + M \rangle,$$

where $M := \frac{SJ}{\sqrt{\zeta+1}}$. Since $\zeta = \frac{1}{2}\|S\|^2$ one has

$$\|M\| = \sqrt{\frac{2\zeta}{\zeta+1}} < \sqrt{2}.$$

Since the trace of M vanishes, $\|M\| = \sqrt{2}\rho(M)$, where $\rho(M)$ is the spectral radius of M. Thus one has $\rho(M) < 1$, so $\text{Id} + M$ is positive definite. We now use the following elementary result:

Lemma 3.6.6. *Let $N_1, N_2 \in \mathcal{S}_n(\mathbb{R})$ be two symmetric matrices such that N_1 is positive and N_2 is non-positive. Then, $\langle N_1 N_2 x, x\rangle \leq 0$ for all $x \in \mathbb{R}^n$.*

Proof of the lemma. By working in a eigenbasis $(f_1, \dots, f_n)$ of N_1, we can suppose that N_1 is diagonal. Thus, for any $j \in \{1, \dots, n\}$ we have $N_1 f_j =: \lambda_j f_j$ and

$$\langle N_1 N_2 f_j, f_j\rangle = \langle N_2 f_j, N_1 f_j\rangle = \lambda_j \langle N_2 f_j, f_j\rangle \leq 0. \qquad \square$$

The matrix Hess $\ln(\|S\|)$ is non-positive because we are at a maximum point, and $\text{Id} + M$ is positive definite, so the previous lemma yields:

$$-1 = -\langle \text{Hess} \ln \zeta, \text{Id} + M\rangle \geq 0$$

which is absurd. Thus $E = \emptyset$, so $t = 1$ on $\mathbf{S}^2$, and hence $S = 0$ by (3.6.7). Therefore, in the case $t > 0$ we finally get $U = \text{Id}$.

The solution $U = -\text{Id}$ is obtained in the case $t < 0$, as explained in the beginning of the proof. $\square$

Recall that in the case $B \neq -\text{Id}$ we defined $U = C^{-1}$ where $C = \frac{B+\text{Id}}{2}$. As a consequence of Proposition 3.6.5, we get the additional solutions $B = -3\text{Id}$ or $B = \text{Id}$. We obtain therefore three solutions to equations (3.6.6), which lead to the endomorphism fields $A = \pm\text{Id}$ and $A = \xi \otimes \xi - 3P$, where we recall that P is the orthogonal projector on $\xi^\perp$.

The previous analysis adapts as usual in the case where ξ is right-invariant, yielding the fourth solution $A = -\xi \otimes \xi + 3P$. $\square$

3.6.5 Link with the sphere rigidity theorem

A classical result due to Liebmann [49] states that the only isometric immersions of the round sphere $\mathbf{S}^2$ in $\mathbb{R}^3$ are the totally umbilical embeddings (hence they differ from the standard embedding by an isometry of $\mathbb{R}^3$). Let us recall a property of Codazzi tensors in dimension 2:

Lemma 3.6.7. *Let Σ be a surface endowed with a Riemannian metric h and S a field of endomorphisms on Σ. Then S is a Codazzi tensor (i.e., $d^\nabla S = 0$) if and only if JSJ is divergence-free, where J is the Hodge star on 1-forms.*

Proof. Let X be a locally-defined unit vector field on Σ. Using $\nabla J = 0$ we compute

$$\begin{aligned} -\delta^\nabla(JSJ) =& \nabla_X(JSJ)(X) + \nabla_{JX}(JSJ)(JX) \\ =& J\nabla_X(S)(JX) + J\nabla_{JX}(S)(JJX) \\ =& Jd^\nabla(S)(X, JX). \end{aligned} \qquad \square$$

Proposition 3.6.5 implies the following slight extension of the sphere rigidity theorem:

Proposition 3.6.8. *Every isometric immersion of the round* 2*-sphere in a flat* 3*-manifold is totally umbilical.*

Indeed, the Gauss and Codazzi equations of the embedding $\mathbf{S}^2 \hookrightarrow M$ with second fundamental form S tell us that $\det(S) = 1$ and $d^\nabla S = 0$. By the above lemma, this is equivalent to $\det(JSJ) = 1$ and $\delta^\nabla(JSJ) = 0$, so by Proposition 3.6.5 we deduce that $JSJ = \pm\mathrm{Id}$, which means that S itself is $\pm\mathrm{Id}$.

We refer to [2] for a modern proof of Liebmann's theorem using the curvature of the metric defined by the second fundamental form S.

The interplay between solving the system (3.6.6) and a nontrivial classical result might explain why the more general system (3.6.5) is not so easy to solve.

Chapter 4

Locally conformally product structures

On étudie dans ce chapitre les structures de Weyl fermées, non-exactes, non plates et à holonomie réductibles sur les variétés compactes. On démontre l'existence de métriques pour lesquelles la forme de Lee de la structure de Weyl est tangente à la distribution non-plate, et on élargit le champ des examples des variétés admettant de telles structures en utilisant la théorie des nombres.

4.1 Introduction

On any Riemannian manifold, there exists a unique torsion-free metric connection, called the Levi-Civita connection, which is the basic tool of Riemannian geometry. However, if one consider the slightly more general context of conformal geometry, the uniqueness of compatible connection does not hold anymore.

Conformal structures were introduced in 1919 by Weyl in the third edition of the book *Raum, Zeit, Materie* [75], in an attempt to unify electromagnetism and gravity. He defined conformal classes of Riemannian metrics, and considered the set of torsion-free compatible connections, nowadays called Weyl structures. The fundamental theorem of conformal geometry states that they form an affine space modelled on the space of one-forms.

In general, a Weyl structure does not preserve any metric in the conformal class, even locally. Those which satisfy this property in a neighbourhood of each point are called closed, and those which preserve a global metric are called exact Weyl structures. In this article we are mostly interested in the closed, non-exact Weyl structures on compact conformal manifolds.

The study of closed Weyl structures on a conformal manifold M can be better understood in terms of the universal cover $\widetilde{M}$. Indeed, the lift of a closed Weyl structure D to $\widetilde{M}$ is exact, meaning that it is the Levi-Civita connection of a Riemannian metric h_D on $\widetilde{M}$, uniquely defined up to a constant factor. The fundamental group of M acts by h_D-similarities on $\widetilde{M}$, all of them being isometries if and only if D is exact.

Every geometrical property of the closed Weyl connection D can be interpreted on the Riemannian manifold $(\widetilde{M}, h_D)$, and conversely. One natural question to study is the reducibility of the holonomy group of D, or equivalently of the Riemannian metric h_D.

A first step in this direction was done by Belgun and Moroianu in [9], where the authors, motivated by a result of Gallot [32], conjectured that a closed non-exact Weyl structure on a compact conformal manifold has reducible holonomy if and only if it is flat. They showed that the conjecture holds under an additional assumption about the lifetime of half-geodesics on the universal cover. However, soon after the formulation of the conjecture, a counter-example was proposed by Matveev and Nikolayevsky [52] who constructed a cocompact action by a group of similarities on the Riemannian product of an Euclidean space and an incomplete irreducible Riemannian manifold. Additionally, the same authors proved that this is the only possible type of counter-example in the analytic setting [53].

More recently, Kourganoff extended this result to the smooth setting [45, Theorem 1.5]. More precisely, he proved that if a closed, non-exact Weyl structure D on a compact conformal manifold (M, c) is non-flat and has reducible holonomy, then the Riemannian manifold $(\widetilde{M}, h_D)$ is isometric to the Riemannian product $\mathbb{R}^q \times (N, g_N)$ where $\mathbb{R}^q$ (the *flat part*) is an Euclidean space and (N, g_N) (the *non-flat part*) is an irreducible, non-complete manifold. In this case, (M, c, D) is called a *locally conformally product* structure, or *LCP* structure for short. This article is devoted to the study of these particular structures on compact manifolds.

There are up to now only few examples of LCP manifolds. As mentioned before, the first one was given in [52], and generalized in [45, Example 1.6] (we outline the construction in Example 4.2.8 below). This example is very restrictive because it only provides LCP manifolds of dimension 3 or 4, with a flat part of dimension 1 or 2 [51]. Nevertheless, they are the only examples when the non-flat part is of dimension 2 [45, Theorem 1.8].

The other class of example comes from the theory of *locally conformally Kähler* (or *LCK*) manifolds. A conformal complex manifold is LCK if for any point there exists a metric in the conformal class which is Kähler in a neighbourhood of this point. This is equivalent to the existence of a Kähler metric on the universal cover, which belongs to the lift of the conformal class. In [68], Oeljeklaus and Toma constructed a class of complex manifolds called OT-manifolds, some of which admit LCP structures (we recall the construction in Example 4.2.18 below). These LCP manifolds have flat parts of dimension 2, so they are still restrictive examples.

One can define several invariants on LCP manifolds. On the one hand, the dimensions of the flat and the non-flat parts, and on the other hand, the rank of the subgroup of $\mathbb{R}^*_+$ generated by the similarity ratios of $\pi_1(M)$ acting on $(\widetilde{M}, h_D)$, which we call the *rank* of the LCP manifold. As noticed before, in the known examples the possibilities for these numbers are limited: the flat part is always of dimension 1 or 2, and it is not clear whether or not the rank can be higher than 1. Our first goal in the present text is to extend the examples of LCP manifolds, and to show, in particular, that the three invariants previously introduced can be chosen arbitrarily large.

Let us now describe the organization of the paper. In Section 4.2, we recall the background of Weyl structures and we define LCP manifolds. We also remind some basics about algebraic number fields, which will be needed in the sequel. Indeed, it turns out that the study of LCP manifolds is closely related to number theory, a fact that we can already notice from the previous examples, which involve matrices in $\mathrm{GL}_n(\mathbb{Z})$ [45] and algebraic number fields [68]. The structure theorem for LCP manifold proved by Kourganoff [45, Theorem 1.9], is also restated. This last article will actually be our main tool, so we will often refer to it in the subsequent lines.

Section 4.3 is devoted to the proof of several properties of LCP manifolds. First, we prove in Proposition 4.3.6 that there exists a metric in the conformal class c on M with respect to which the Lee form of the Weyl structure D vanishes on the flat distribution of D. This property is equivalent to the existence of a smooth function defined on the non-flat factor N, having the same equivariance as the metric h_D on $\widetilde{M}$ with respect to the action of $\pi_1(M)$. In turn, the existence of such functions allows us to construct, starting from a given compact LCP manifold (M, c, D), infinitely many new examples, by taking the product of $\widetilde{M}$ with the universal cover of a compact manifold, endowed with a warped product metric admitting a free cocompact action by similarities. This leads to the concept of reducible LCP manifolds. Moreover, in Proposition 4.3.12 we prove that the similarity ratios of $\pi_1(M)$ acting on $(\widetilde{M}, h_D)$ are always units in some algebraic number field.

In Section 4.4, we construct new examples of LCP manifolds. We give a general construction which contains all previous examples of LCP manifolds, and using some Galois theory and Dirichlet's unit theorem, we construct LCP manifolds with arbitrary rank in Proposition 4.4.9. We also find LCP manifolds with flat and non-flat part of arbitrarily large dimension.

4.2 Preliminaries

4.2.1 Locally conformally product manifolds

Let M be a smooth manifold of dimension n and denote by $\mathrm{Fr}(M)$ its frame bundle. For every $k \in \mathbb{R}$ we define the weight bundle $\mathcal{L}^k := \mathrm{Fr}(M) \times_{|\det|^{\frac{k}{n}}} \mathbb{R}$, which is an oriented bundle.

A *conformal class* on M is a positive definite section of the fibre bundle $\mathrm{Sym}(T^*M \otimes T^*M) \otimes \mathcal{L}^2$. The manifold M together with this section is called a conformal manifold. Equivalently, a conformal manifold is given by M and a class of metrics c which are related in the following manner: for any $g, g' \in c$, there is $f : M \to \mathbb{R}$ such that $g' = e^{2f} g$.

On a conformal manifold, there is no preferential connection as in the Riemannian case with the Levi-Civita connection, because the metric is defined up to multiplication by a positive function. However, a new class of connections is relevant:

Definition 4.2.1. *A Weyl structure on a conformal manifold (M, c) is a torsion-free connection D on TM which preserves c i.e. such that for any $g \in c$, there is a 1-form θ_g on M, called the Lee form of D with respect to g, satisfying $Dg = -2\theta_g \otimes g$.*

It comes from the definition that if θ_g is the Lee form of D with respect to $g \in c$, then for any $g' := e^{2f} g \in c$, the Lee form of D with respect to g' is $\theta_g - df$. Then, the Lee form of D with respect to g is closed (resp. exact) if and only if the Lee form of any metric in c is closed (resp. exact). For this reason, we introduce the following terminology:

Definition 4.2.2. *A Weyl structure D on a conformal manifold (M, c) is closed (resp. exact) if the Lee form of at least one metric (and then of all metrics) in c is closed (resp. exact).*

An easy consequence of the definition is that a closed Weyl structure is locally the Levi-Civita connection of a metric in c, and an exact Weyl structure is the Levi-Civita connection of a metric in c.

We recall that a similarity between two Riemannian manifolds (M_1, g_1) and (M_2, g_2) is a diffeomorphism $s : M_1 \to M_2$ such that $s^* g_2 = \lambda^2 g_1$ for some positive real number $\lambda > 0$ called the similarity ratio. In order to define the main object of this text, we need the following definition:

Definition 4.2.3. *A similarity structure on a compact manifold M is a metric h on its universal cover $\widetilde{M}$ such that $\pi_1(M)$ acts by similarities on $(\widetilde{M}, h)$. A similarity structure is said to be Riemannian if in addition $\pi_1(M)$ acts only by isometries.*

It turns out that this notion is closely related to closed Weyl structures. More precisely, we have the following result:

Proposition 4.2.4. *On a conformal manifold (M, c) there is a one-to-one correspondence between closed Weyl structures and similarity structures h in the lifted conformal structure $\widetilde{c}$ on the universal cover, defined up to multiplication by a positive real number. This correspondence takes exact Weyl structures to Riemannian similarity structures.*

Proof. Let D be a closed Weyl structure on (M, c). Let $\widetilde{M}$ be the universal cover of M and $\widetilde{c}$ the induced conformal structure on $\widetilde{M}$. The connection D induces a Weyl structure $\widetilde{D}$ on $\widetilde{M}$ which is exact since $\widetilde{M}$ is simply connected. Thus, there is a metric $h_D \in \widetilde{c}$, unique up to multiplication by a positive number, such that $\nabla^{h_D} = \widetilde{D}$, where ∇^{h_D} is the Levi-Civita connection of h_D. If $g \in c$ is a metric on M, the induced metric $\widetilde{g}$ on $\widetilde{M}$ can be written $\widetilde{g} = e^{-2f} h_D$ for some real-valued function f of $\widetilde{M}$, and a simple calculation shows that the Lee form of $\widetilde{D}$ with respect to $\widetilde{g}$ is $\mathrm{d}f$, which means that the pull-back $\widetilde{\theta}_g$ of the Lee form θ_g is equal to df. Now, let $\gamma \in \pi_1(M)$. One has $df = \widetilde{\theta}_g = \gamma^* \widetilde{\theta}_g = \gamma^* df$, thus there is $\lambda > 0$ such that $\gamma^* f = f + \ln \lambda$ and $\gamma^* h_D = \lambda^2 h_D$. We conclude that the elements of $\pi_1(M)$ act on $(\widetilde{M}, h_D)$ as similarities. Moreover, if these similarities are all isometries, the Weyl structure D is exact.

Conversely, assume one has a compact manifold M and a metric h on its universal cover $\widetilde{M}$ such that $\pi_1(M)$ acts by similarities on $(\widetilde{M}, h)$. Then, the metric h does not define a metric on M, but it induces a conformal class c, and the Levi-Civita connection ∇^h descends to a closed Weyl structure on (M, c). If the elements of $\pi_1(M)$ are all isometries, this Weyl structure is exact. □

As we mentioned in the introduction, it was conjectured by Belgun an Moroianu [9] that given a conformal manifold together with a closed, non-exact Weyl structure, the induced connection on the universal cover must be flat or irreducible. A counter-example to this conjecture was found by Matveev and Nikolayevsky [52], who showed that in the non-flat, analytic case, the universal cover is a Riemannian product $\mathbb{R}^q \times N$ where $q \geq 0$ and N is a non-complete, irreducible manifold of dimension at least 2. This result was extended by Kourganoff to the smooth setting. More precisely, he proved the following theorem [45, Theorem 1.5]:

Theorem 4.2.5. *Consider a compact manifold M endowed with a non-Riemannian similarity structure, and its universal cover $\widetilde{M}$ is equipped with the corresponding Riemannian metric h_D (D being the closed non-exact Weyl structure associated via Proposition 4.2.4). Then we are in exactly one of the following situations:*

1. *$(\widetilde{M}, h_D)$ is flat.*
2. *$(\widetilde{M}, h_D)$ has irreducible holonomy and $\dim(\widetilde{M}) \geq 2$.*
3. *$(\widetilde{M}, h_D) = \mathbb{R}^q \times (N, g_N)$, where $q \geq 1$, $\mathbb{R}^q$ is the Euclidean space, and (N, g_N) is a non-flat, non-complete Riemannian manifold which has irreducible holonomy.*

In the third case of Theorem 4.2.5, we say that M is a *locally conformally product manifold*, or *LCP manifold* for short. Then, a LCP manifold (M, c, D) is the data of a compact manifold, a conformal class, and a closed, non-exact Weyl structure, with reducible, non-flat holonomy.

Remark 4.2.6. We recall that the Cauchy border of a Riemannian manifold $\mathcal{Z}$ is $\partial\mathcal{Z} := C\mathcal{Z} \setminus \mathcal{Z}$, where $C\mathcal{Z}$ is the metric completion of $\mathcal{Z}$. The classification of flat similarity structures was done in [27]. From this result, it comes that in the first case of Theorem 4.2.5, the Cauchy border of $\widetilde{M}$ must be a single point. But this cannot happen in the case of an LCP manifold, because the flat part is a Riemannian factor of $\widetilde{M}$ and the non-flat part is incomplete, so $\partial\widetilde{M}$ must have infinite cardinal. A direct consequence of this observation is that on a compact conformal manifold (M, c), a closed, non-exact Weyl structure D defines an LCP structure if and only if $(\widetilde{M}, h_D)$ (where $\widetilde{M}$ is the universal cover of M, and h_D is the similarity structure induced by D) has reducible holonomy and infinite Cauchy border, or equivalently if $(\widetilde{M}, h_D)$ has a flat Riemannian factor $\mathbb{R}$.

We will often write the universal cover of an LCP manifold (M, c, D) as $(\widetilde{M}, h_D) = \mathbb{R}^q \times (N, g_N)$. In this case, $\mathbb{R}^q$ will always stand for the flat part of the de Rham decomposition of $\widetilde{M}$, (N, g_N) is the non-flat, incomplete, irreducible part, and h_D is the similarity structure induced by D, defined up to a constant factor.

We define the following invariant on LCP manifolds:

Definition 4.2.7. *The rank of an LCP manifold (M, c, D) is the rank of the subgroup of $\mathbb{R}^*_+$ generated by the ratios of the elements of $\pi_1(M)$ viewed as similarities acting on $(\widetilde{M}, h_D)$.*

Equivalently, the rank of an LCP manifold (M, c, D) is the minimal rank of a subgroup of $H^1(M, \mathbb{Z})$ whose span in $H^1(M, \mathbb{R})$ contains the cohomology class $[\theta]$ of the Lee form of D.

To prove this last fact, we recall that there is a canonical isomorphism

$$\Xi : H^1(M,\mathbb{R}) \to \mathrm{Hom}(\pi_1(M),\mathbb{R}), \qquad [w] \mapsto \left([\gamma] \mapsto \int_\gamma w\right). \tag{4.2.1}$$

This map induces an isomorphism from $H^1(M,\mathbb{Z})$ to $\mathrm{Hom}(\pi_1(M),\mathbb{Z})$. In addition, an easy computation shows that $\Xi([\theta])$ is exactly the composition of the logarithm and the morphism associating to an element of $\pi_1(M)$ its similarity ratio, so the rank of the image of $\Xi([\theta])$ is the rank of the LCP manifold, denoted by r. Since Ξ is an isomorphism, it is sufficient to prove that the rank s of the smallest subgroup of $\mathrm{Hom}(\pi_1(M),\mathbb{Z})$ whose span contains $\Xi(\theta)$ is r. Since the image of $\Xi(\theta)$ is of rank r, there exist r morphisms $p_1,\dots,p_r \in \mathrm{Hom}(\pi_1(M),\mathbb{R})$, whose images are of rank 1, such that $\Xi(\theta) = \sum_{k=1}^r p_k$. For all $1 \le k \le r$, there is $a_k \in \mathbb{R}$ such that $p_k = a_k p'_k$ where $p'_k \in \mathrm{Hom}(\pi_1(M),\mathbb{Z})$. Consequently, $\Xi(\theta) = \sum_{k=1}^r a_k p'_k$, thus $s \le r$. In addition, $r \le s$ because if $\Xi(\theta) = \sum_{k=1}^s p_k$ with the p_k's being morphisms with images of rank 1, then the rank of the image of $\Xi(\theta)$ is smaller than s.

A first example of LCP manifold was given by Matveev and Nikolayevsky [52] and generalized by Kourganoff [45, Example 1.6]. We outline it here:

Example 4.2.8. Let $\widetilde{M} := \mathbb{R}^{q+1} \times \mathbb{R}^*_+$ with $q \ge 1$. Let b be a symmetric positive definite bilinear form on $\mathbb{R}^{q+1}$ and $A \in \mathrm{SL}_{q+1}(\mathbb{Z})$ such that there exist $\lambda \in (0,1)$ and a decomposition $\mathbb{R}^{q+1} = E^u \perp E^s$ (where the orthogonal symbol refers to the metric induced by b) stable by A with $A|_{E^s} = \lambda O$ where $O \in O(E^s, b|_{E^s})$, and E^u is one-dimensional.

Let G be the group of transformation of $\widetilde{M}$ generated by the translations $\mathbb{R}^{q+1} \times \mathbb{R}^*_+ \ni (x,t) \mapsto (x+e_k,t)$, $k \in \{1,\dots,q+1\}$ where e_k is the k-th vector of the canonical basis of $\mathbb{R}^{q+1}$, and the transformation $\mathbb{R}^{q+1} \times \mathbb{R}^*_+ \ni (x,t) \mapsto (Ax, \lambda t)$.

Let $\varphi : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ be a function satisfying $\varphi(\lambda t) = \lambda^{2q+2}\varphi(t)$. We define a metric h on $\widetilde{M}$ by

$$h_{x,t} := b|_{E^s} + \varphi(t) b|_{E^u} + dt^2$$

for any $(x,t) \in \widetilde{M}$. Then, the metric h defines a similarity structure on the manifold $\widetilde{M}/G$.

However, as it was pointed out in [51, Proposition 1], the only admissible values of q in Example 4.2.8 are $q = 1, 2$, so this construction only provides examples of LCP manifolds of dimension 3 or 4.

In the remaining part of this section, (M,c,D) is an LCP manifold, and $(\widetilde{M}, h_D) = \mathbb{R}^q \times (N, g_N)$ is its universal cover.

Let $\gamma \in \pi_1(M)$. Since γ acts as a similarity on $(\widetilde{M}, h_D)$, it must preserve the de Rham decomposition, meaning that there is a similarity γ_E (for Euclidean) of $\mathbb{R}^q$ and a similarity γ_N of N such that $\gamma = (\gamma_E, \gamma_N)$.

Thus, we introduce the following definitions:

Definition 4.2.9. *We define $P = \{p \in \mathrm{Sim}(N), \exists \gamma \in \pi_1(M), \gamma_N = p\}$, the restriction of $\pi_1(M)$ to the non-flat part N. We also introduce $\overline{P}$, $\overline{P}^0$ which are respectively the closure of P in $\mathrm{Sim}(N)$, and the identity connected component of this closure.*

The groups considered in Definition 4.2.9 were introduced by Kourganoff in [45], and their analysis provides several useful results on LCP manifolds. We will keep these notations throughout this text. From [45, Lemma 4.1] we know that $\overline{P}^0$ is abelian and by [45, Lemma 4.13] that $\overline{P}^0$ acts on N by isometries.

There is actually a correspondence between P and $\pi_1(M)$:

Lemma 4.2.10. *The group P is isomorphic to $\pi_1(M)$.*

Proof. The second projection $\pi_1(M) \to P, \gamma \mapsto \gamma_N$ is a group morphism. We will show that it is an isomorphism. Assume there is $\gamma \in \pi_1(M) \setminus \mathrm{id}$ such that $\gamma_N = \mathrm{id}$. Let $v \in \pi_1(M)$ whose similarity ratio is $\lambda \in (0,1)$. By the Banach fixed point theorem, v_E has a fixed point, and we can assume without loss of generality that it is 0. Then, we can find $R_v, R_\gamma \in O_q(\mathbb{R})$ and $t_\gamma \in \mathbb{R}^q$ such that $v_E(a) = R_v a$ and $\gamma_E(a) = R_\gamma a + t_\gamma$ for any $a \in \mathbb{R}^q$. Since γ cannot have a fixed point, because $\pi_1(M)$ acts freely on $\widetilde{M}$, one has $t_\gamma \neq 0$.

One has, for any $k \in \mathbb{N}$ and $(a,x) \in \mathbb{R}^q \times N$:

$$v^k \gamma v^{-k}(a,x) = (R_v^k R_\gamma R_v^{-k} a + R_v^k t_\gamma, x). \tag{4.2.2}$$

Since $v^k \gamma v^{-k}(0,x) = (R_v^k t_\gamma, x) \underset{k\to+\infty}{\longrightarrow} 0$, the orbit of $(0,x)$ by $\pi_1(M)$ admits an accumulation point, which contradicts the fact that $\pi_1(M)$ acts properly on $\widetilde{M}$. □

4.2.2 Number theory

We will need a few notions coming from number theory in order to give examples of locally conformally product manifolds having arbitrary high rank.

First, we recall that an algebraic number field K, or number field for short, is an extension of $\mathbb{Q}$ of finite dimension. The degree $[K : \mathbb{Q}]$ of such an extension is its dimension as $\mathbb{Q}$-vector space. If α is an algebraic number, we will denote by $\mathbb{Q}[\alpha]$ the smallest extension of $\mathbb{Q}$ containing α. In this case, the degree of α is the degree of its (monic) minimal polynomial. The conjugates of an algebraic number α are the roots of its minimal polynomial.

Definition 4.2.11. *An algebraic number field is called totally real if all its embeddings in $\mathbb{C}$ lie in $\mathbb{R}$.*

Equivalently, a number field $K := \mathbb{Q}[\alpha]$ is totally real if and only if the minimal polynomial of α has only real roots, i.e. all the conjugates of α are real.

We recall that an extension K/L is a Galois extension if it is normal, meaning that all the conjugates of an element $\alpha \in K$ lie in K, and separable, i.e. the minimal polynomial of any $\alpha \in K$ has simple roots in an algebraic closure of K. In this case, the Galois group of K/L is the set of automorphisms of K which fixes L. When $L = \mathbb{Q}$, all the algebraic extensions are separable, so for an extension $\mathbb{Q}[\alpha]$, to be a Galois extension means that all the conjugates of α lie in $\mathbb{Q}[\alpha]$. These considerations lead us to introduce the following definition:

Definition 4.2.12. *An extension K/L is called cyclic if it is a Galois extension and its Galois group is cyclic.*

One object of interest for our analysis will be the ring of integer of an extension K, and more specifically its group of units.

Definition 4.2.13. *An element β of an algebraic number field K is an algebraic integer if its monic minimal polynomial is in $\mathbb{Z}[X]$.*

One basic result is that the set $\mathcal{O}_K$ of the algebraic integers in K is indeed a ring.

Definition 4.2.14. *The group $\mathcal{O}_K^\times$ of invertible algebraic integers in K is called the group of units of K and its elements are called units.*

Remark 4.2.15. A useful characterization of units is the following: an algebraic integer of K is a unit if and only if the constant coefficient of its minimal polynomial in $\mathbb{Z}[X]$ is equal to ± 1.

A fundamental result on the structure of the group of unit is the Dirichlet's units theorem (for a proof, see [55, Theorem 5.1]):

Theorem 4.2.16 (Dirichlet's units theorem)**.** *The group of units in a number field K is finitely generated with rank equal to $s+t-1$, where s is the number of real embeddings of K, and $2t$ is the number of nonreal complex embeddings of K (so $s+2t=[K:\mathbb{Q}]$).*

In particular, $\mathcal{O}_K^\times \simeq T \oplus \mathbb{Z}^{s+t-1}$, where T is the subgroup of torsion elements in $\mathcal{O}_K^\times$. When K is totally real, there is no torsion element different from ± 1, and then $\mathcal{O}_K^\times \simeq \{\pm 1\} \oplus \mathbb{Z}^{s-1}$.

The last notion that we need concerns the bases of an algebraic number field K. More precisely, we are interested in the case where K admits a basis which is adapted to the ring of integers.

Definition 4.2.17. *An algebraic number field K is called monogenic if there exists a power integral basis in K, i.e. there is an element $\alpha \in K$ such that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.*

We also recall that the n-th cyclotomic extension is the extension of $\mathbb{Q}$ generated by a primitive n-th root of unity. The degree of this extension is the value of the Euler's totient function at n.

We now have the tools to construct the so-called OT-manifolds, which where introduced by Oeljeklaus and Toma in [68]. Before giving the construction, we emphasize that throughout this article, a *lattice* in an Abelian Lie group G will be a discrete subgroup H of G. If the quotient G/H is compact, then H will be called a *full lattice.*

Example 4.2.18 (OT-manifolds)**.** Let K by a number field with s real embeddings $\sigma_1, \dots, \sigma_s$ and $2t$ complex embeddings $\sigma_{s+1}, \dots, \sigma_{s+2t}$ such that σ_{s+i} and σ_{s+t+i} are conjugated for any $1 \le i \le t$ (such a field always exists, see [68, Remark 1.1]). We define the geometric representation of K

$$\sigma : K \to \mathbb{C}^{s+t}, a \mapsto (\sigma_1(a), \dots, \sigma_{s+t}(a)).$$

The image of the ring of integers $\mathcal{O}_K$ of K by σ is a lattice of rank $s+2t$ in $\mathbb{C}^{s+t}$. Moreover, we consider

$$\mathcal{O}_K^{\times,+} := \{a \in \mathcal{O}_K^\times, \sigma_i(a) > 0, 1 \le i \le s\},$$

and we define an action of this set on $\mathbb{C}^{s+t}$ by $az := (\sigma_1(a)z_1, \dots, \sigma_{s+t}(a)z_{s+t})$ for any $a \in \mathcal{O}_K^{\times,+}$. Let U be a subgroup of $\mathcal{O}_K^{\times,+}$ such that the image of U by the composition $p_{\mathbb{R}^s} \circ l$ of the logarithmic representation

$$\begin{aligned} &\ell : \mathcal{O}_K^{\times,+} \to \mathbb{R}^{s+t}, \\ &\ell(u) := (\ln|\sigma_1(u)|, \dots, \ln|\sigma_s(u)|, 2\ln|\sigma_{s+1}(u)|, \dots, 2\ln|\sigma_{s+t}(u)|) \end{aligned} \tag{4.2.3}$$

and the projection $p_{\mathbb{R}^s} : \mathbb{R}^{s+t} \to \mathbb{R}^s$ on the first s coordinates is a full lattice.

Let $H := \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$. Combining the additive action of $\mathcal{O}_K$ and the multiplicative action of U, the group $U \ltimes \mathcal{O}_K$ acts freely, cocompactly and properly on $H^s \times \mathbb{C}^t$. Thus, the quotient $X(K,U) := (H^s \times \mathbb{C}^t)/(U \ltimes \mathcal{O}_K)$ is a compact manifold.

When $t=1$, the manifold $X(K,U)$ admits an LCK structure, which is determined by a Kähler potential

$$F(z) := \prod_{k=1}^{s} \frac{i}{z_k - \bar{z}_k} + |z_{s+1}|^2$$

on its universal cover [68]. This induces in turn a similarity structure on $X(K,U)$. If this structure was Riemannian, the Kähler metric $i\partial\bar{\partial}F$ would descend to $X(K,U)$. This is impossible because an OT-manifold admits no Kähler metric [68, Proposition 2.5]. In addition, from the form of the Kähler potential, the second factor $\mathbb{C}^t(=\mathbb{C})$ of the universal cover of $X(K,U)$ is a Riemannian factor. Thus, by Remark 4.2.6, $X(K,U)$ admits an LCP structure when $t=1$.

Remark 4.2.19. In Example 4.2.18, when $s=t=1$, the Kähler potential of the lift of the LCK metric to the universal cover is [68]

$$F : H \times \mathbb{C} \to \mathbb{R}, \quad F(z) := \frac{i}{z_1 - \bar{z}_1} + |z_2|^2. \tag{4.2.4}$$

Writing the Kähler form as $\sum\limits_{k\neq l} \omega_{kl} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_l$, one has:

$$\omega_{11} = \partial_{z_1}\bar{\partial}_{z_1}\left(\frac{i}{z_1-\bar{z}_1} + |z_2|^2\right) = \frac{1}{4}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}\right)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}\right)\frac{1}{2y_1} = \frac{1}{4}\frac{1}{y_1^3}$$

$$\omega_{22} = 1 \qquad \omega_{12} = 0.$$

Then, the metric can be rewritten as $g := \frac{1}{4y_1^3}(\mathrm{d}x_1^2 + \mathrm{d}y_1^2) + (\mathrm{d}x_2^2 + \mathrm{d}y_2^2)$. We make the change of variable $v_1 := x_1/2$, $w_1 := \frac{1}{\sqrt{y_1}}$ and the metric becomes

$$g = (w_1^6 \mathrm{d}v_1^2 + \mathrm{d}w_1^2) + (\mathrm{d}x_2^2 + \mathrm{d}y_2^2). \tag{4.2.5}$$

Moreover, the group U is generated by a single unit $u \in \mathcal{O}_K^{\times,+}$ which satisfies $\sigma_1(u) = |\sigma_2(u)|^{-2}$. After the change of variable, the multiplicative action of u is given, for any $(v_1, w_1, x_2 + iy_2) \in \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{C}$, by

$$\begin{aligned} u \cdot (v_1, w_1, x_2 + iy_2) &= (\sigma_1(u)v_1, \sigma_1(u)^{-\frac{1}{2}}w_1, \sigma_2(u)(x_2 + iy_2)) \\ &= (\sigma_1(u)v_1, |\sigma_2(u)|w_1, \sigma_2(u)(x_2 + iy_2)). \end{aligned}$$

If we look at the restriction of this action to $\mathbb{R} \times \mathbb{C}$ by dropping the variable w_1, we remark that the matrix of the transformation in a basis of the lattice $\sigma(\mathcal{O}_K)$ belongs to $\mathrm{SL}_3(\mathbb{Z})$ (see the proof of Corollary 4.4.6 below for more details). Then, we recognize the example 4.2.8 in the case $q=2$.

4.2.3 Foliations and LCP manifolds

A foliation of dimension p of an n-dimensional manifold M is a maximal atlas $(U_i, \phi_i)_{i\in I}$ on M such that for each $i, j \in I$ the transition map $\Phi_{i,j} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ satisfies

$$\frac{\partial \Phi_{i,j}^l}{\partial x_k} = 0 \quad \text{for all } p+1 \leq l \leq n, \text{ and } 1 \leq k \leq p \tag{4.2.6}$$

where x_k is the k-th coordinate of $\mathbb{R}^n$.

A foliation induces a p-dimensional distribution on M, taking at each point $x \in U_i$ the subspace of T_xM given by $d\phi_i^{-1}(\phi_i(x))(\mathbb{R}^p \times \{0\})$. From this, one can define the leaves of the foliation as follows: if $x \in M$, the leaf passing through x is the set of all the points that can be reached from x by continuous, piecewise differentiable paths whose tangent vector at each smooth point is in the distribution previously defined. For more details, see [57].

When the manifold M is compact, one can extract a finite covering $(U_i)_{i\in J}$, $J \subset I$ such that for any $i \in J$ the open set U_i is diffeomorphic to a product $V_i \times T_i$ where V_i and T_i are open cubes of $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$ respectively. This induces maps $f_i : U_i \to T_i$ in a natural way, and we define the transition maps $\gamma_{ij} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ by $f_j = \gamma_{ij} \circ f_i$. The disjoint union $T := \bigsqcup_{i\in J} T_i$ is called the *transversal* of the foliation. The foliation is said to be Riemannian if there exists a metric on the transversal such that the transition maps are isometries.

In [45, Theorem 1.9], it was shown that an LCP manifold carries a Riemannian foliation. More precisely, one has the following theorem:

Theorem 4.2.20. *Let (M, c, D) be a LCP manifold, and let $(\widetilde{M}, h_D) = \mathbb{R}^q \times (N, g_N)$ be its universal cover endowed with the metric h_D induced by D. Here, (N, g_N) is the non-flat, irreducible factor of the de Rham decomposition. The foliation $\widetilde{\mathcal{F}}$ tangent to $\mathbb{R}^q$ induces by projection a foliation $\mathcal{F}$ on M. Then $\mathcal{F}$ is a Riemannian foliation on M, and the closures of the leaves form a singular Riemannian foliation $\overline{\mathcal{F}}$ on M, such that each leaf of $\overline{\mathcal{F}}$ is a smooth manifold of dimension d, depending of the leaf, with $q < d < q + n$, where $n = \dim(N)$.*

Moreover, on each leaf of $\overline{\mathcal{F}}$, there is a flat Riemannian metric which is compatible with the similarity structure of M.

Definition 4.2.21. *In Theorem 4.2.20, we call the distribution tangent to the leaves of $\mathcal{F}$ the* flat distribution *on M, and the orthogonal distribution is called the* non-flat distribution*.*

Again, we recall several results and observations from [45]. In the setting of Theorem 4.2.20, we can describe the leaves of $\mathcal{F}$ using the canonical surjection $\pi : \widetilde{M} \to M$, and the group P previously defined. The leaf of $\mathcal{F}$ passing through $\pi(a, x)$ for $(a, x) \in \mathbb{R}^q \times N$ is equal to $\pi(\mathbb{R}^q \times Px)$, and its closure is $\overline{\mathcal{F}}_x := \pi(\mathbb{R}^q \times \overline{P}^0 x)$ [45, Lemma 4.11]. By Theorem 4.2.20, the metric h_D restricted to $\mathbb{R}^q \times \overline{P}^0 x$ descends to a metric g_x on $\overline{\mathcal{F}}_x$. Thus, the metric h_D induces a Riemannian metric, up to a multiplicative factor, on the closure of the leaves of $\mathcal{F}$. Since $\overline{P}^0$ is abelian and acts by isometries, for any $x \in N$, the closed leaf $\overline{\mathcal{F}}_x$ is the product of an Euclidean space and a flat torus. In particular, it is a complete space, which implies that an element of $\pi_1(M)$ with ratio $\neq 1$ acts freely on $N/\overline{P}^0$.

We consider the subgroup of $\pi_1(M)$ defined by $\Gamma_0 := \pi_1(M) \cap (\mathrm{Sim}(\mathbb{R}^q) \times \overline{P}^0)$. From [45, Lemma 4.18], we know that this group is a full lattice in $\mathbb{R}^q \times \overline{P}^0$ where $\mathbb{R}^q$ is identified with its translations. In Example 4.2.8 for instance, Γ_0 is the group of translations $\mathbb{Z}^{q+1}$ acting on $\mathbb{R}^{q+1}$. This observation explains why we will always consider such lattices in order to construct examples.

4.3 Properties of LCP manifolds

Let (M, c, D) be an LCP manifold and $(\widetilde{M}, h_D) = \mathbb{R}^q \times (N, g_N)$ be its universal cover, endowed with the similarity structure h_D induced by D. We denote by $\pi : \widetilde{M} \to M$ the canonical surjection.

4.3.1 Adapted metrics

In this subsection, we prove that there exists a metric $g \in c$ such that the Lee form θ_g of D with respect to g vanishes on the flat distribution (Definition 4.2.21) of D on M. This is equivalent to the existence of a function of N having the same equivariance (the term *same automorphy* is also often used in the litterature) as h_D with respect to $\pi_1(M)$. For this reason, we introduce the following definition:

Definition 4.3.1. *Let G be a group acting on a Riemannian manifold $(\mathcal{Z}, g_{\mathcal{Z}})$ by similarities. A smooth function $f : \mathcal{Z} \to \mathbb{R}$ is said to be G-equivariant if for every $\gamma \in G$, one has $\gamma^* e^{2f} = \lambda_\gamma^2 e^{2f}$ where λ_γ is the similarity ratio of γ. Equivalently, a function f is G-equivariant if G consists of isometries of $e^{-2f} g_{\mathcal{Z}}$.*

We now give an important property of the equivariant functions on the universal cover $\mathbb{R}^q \times N$ of LCP manifolds: they are bounded on sets of the form $\mathbb{R}^q \times K$ where K is a compact subset of N. In order to prove this result, we recall that the Cauchy boundary $\partial\mathcal{Z}$ of a Riemannian manifold $\mathcal{Z}$ is the set $C\mathcal{Z} \setminus \mathcal{Z}$ where $C\mathcal{Z}$ is the metric completion of $\mathcal{Z}$. The Riemannian distance $d^{\mathcal{Z}}$ on $\mathcal{Z}$, is extended to $C\mathcal{Z}$ in the following natural way: if $(x_n), (y_n)$ are representatives of elements $x, y \in C\mathcal{Z}$ (which consists of equivalence classes of Cauchy sequences in $\mathcal{Z}$), $(d^{\mathcal{Z}}(x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, and $d^{\mathcal{Z}}(x, y)$ is defined as the limit of this sequence. We first state the following easy lemma:

Lemma 4.3.2. *Let A be a subset of a Riemannian manifold $\mathcal{Z}$. Assume that ∂Z is non-empty. We define $\alpha := \inf_{x\in A} d^{\mathcal{Z}}(x, \partial\mathcal{Z})$ and $\beta := \sup_{x\in A} d^{\mathcal{Z}}(x, \partial\mathcal{Z})$. Then, if γ is a similarity of $\mathcal{Z}$ of ratio $\lambda \in \mathbb{R}_+^*$, it extends uniquely to $C\mathcal{Z}$ as a uniformly continuous function on a dense subset of $C\mathcal{Z}$ and one has the property*

$$\forall x \in A, d^{\mathcal{Z}}(\gamma x, \partial\mathcal{Z}) \in [\lambda\alpha, \lambda\beta].$$

Proof. Let $x \in A$ and γ a similarity of $\mathcal{Z}$ of ratio $\lambda \in \mathbb{R}_+^*$. One has, $\alpha \leq d^{\mathcal{Z}}(x, \partial\mathcal{Z}) \leq \beta$. It is easy to see from the definition that $\gamma(\partial\mathcal{Z}) = \partial\mathcal{Z}$, thus $\lambda\alpha \leq d^{\mathcal{Z}}(\gamma x, \partial\mathcal{Z}) \leq \lambda\beta$. □

Corollary 4.3.3. *In the setting of Lemma 4.3.2, for any compact subsets $K_1, K_2 \subset \mathcal{Z}$, the similarity ratios of the elements of $\Gamma = \{\gamma \in \mathrm{Sim}(\mathcal{Z}), (\gamma K_1) \cap K_2 \neq \emptyset\}$ are included in a compact subset of $\mathbb{R}_+^*$.*

Proof. Let $\rho : \mathrm{Sim}(\mathcal{Z}) \to \mathbb{R}_+^*$ be the group morphism which associates to an element of $\mathrm{Sim}(\mathcal{Z})$ its similarity ratio. We also introduce

$$\alpha_1 := \inf_{x\in K_1} d^{\mathcal{Z}}(x, \partial\mathcal{Z}) \qquad \beta_1 := \sup_{x\in K_1} d^{\mathcal{Z}}(x, \partial\mathcal{Z})$$

and

$$\alpha_2 := \inf_{x\in K_2} d^{\mathcal{Z}}(x, \partial\mathcal{Z}) \qquad \beta_2 := \sup_{x\in K_2} d^{\mathcal{Z}}(x, \partial\mathcal{Z}).$$

Let $\gamma \in \Gamma$. By definition, there exists $x \in K_1$ such that $\gamma x \in K_2$ so in particular we have

$$\alpha_2 \leq d^{\mathcal{Z}}(\gamma x, \partial\mathcal{Z}) \leq \beta_2.$$

Moreover by Lemma 4.3.2 one has

$$\rho(\gamma)\alpha_1 \leq d^{\mathcal{Z}}(\gamma x, \partial\mathcal{Z}) \leq \rho(\gamma)\beta_1,$$

which implies

$$\rho(\gamma)\alpha_1 \leq \beta_2 \qquad\qquad \alpha_2 \leq \rho(\gamma)\beta_1,$$

so we conclude

$$\alpha_2/\beta_1 \leq \rho(\gamma) \leq \beta_2/\alpha_1.$$

Thus, $\rho(\Gamma)$ is included in the compact set $[\alpha_2/\beta_1, \beta_2/\alpha_1]$. □

We have now all the tools to prove the boundedness property for equivariant functions:

Lemma 4.3.4. *Let $f : \widetilde{M} \to \mathbb{R}$ be a smooth $\pi_1(M)$-equivariant function. Then, for any compact subset K of N, f is bounded on $\mathbb{R}^q \times K$.*

Proof. Let $K \subset N$ be a compact set. Since $\pi_1(M)$ acts cocompactly on $\widetilde{M}$, there is a compact set $C \subset \widetilde{M}$ such that $\pi_1(M)C = \widetilde{M}$. Moreover C can be assumed to be equal to $C_E \times C_N$ where C_E is a compact of $\mathbb{R}^q$ and C_N is a compact of N. Let

$$\Gamma := \{\gamma \in \pi_1(M), (\gamma C) \cap (\mathbb{R}^q \times K) \neq \emptyset\} = \{\gamma \in \pi_1(M), (\gamma_N C_N) \cap K \neq \emptyset\}.$$

Let $\rho : \pi_1(M) \to \mathbb{R}_+^*$ be the group morphism which associates to an element of $\pi_1(M)$ its similarity ratio. By Corollary 4.3.3, $\rho(\Gamma)$ is included in a compact set $[\alpha, \beta]$, with $\alpha, \beta > 0$. We know that f is bounded on C, meaning there are $\alpha', \beta' \in \mathbb{R}$ such that $\alpha' \leq f \leq \beta'$ on C. In addition, for any $x \in \mathbb{R}^q \times K$, there is $\gamma \in \Gamma$ and $y \in C$ such that $\gamma y = x$. Thus, the equivariance property of e^{2f} yields $\alpha' + \ln \alpha \leq f(x) \leq \beta' + \ln \beta$, which gives the desired result. □

For $x \in N$, let $S_x := \{\gamma \in \pi_1(M) | \gamma_N \cdot x \in \overline{P}^0 x\}$ ($\overline{P}^0$ was defined in Definition 4.2.9). We recall that in Section 4.2.3 we defined the closed leaf $\overline{\mathcal{F}}_x \subset M$, and showed that the metric h_D descends to a metric g_x on it. We give here a short proof of a result partially stated in the proof of [45, Lemma 4.18].

Lemma 4.3.5. *Let $x \in N$. Then, $\overline{\mathcal{F}}_x$ is isomorphic to $(\mathbb{R}^q \times \overline{P}^0 x)/S_x$ and S_x acts on $(\widetilde{M}, h_D)$ by isometries. Moreover, if $\gamma \in \pi_1(M)$ with similarity ratio $\lambda > 0$, there is a similarity $\overline{\gamma} : (\overline{\mathcal{F}}_x, g_x) \to (\overline{\mathcal{F}}_{\gamma x}, g_{\gamma x})$ of ratio λ for which the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{R}^q \times \overline{P}^0 x \arrow[r, "\gamma"] \arrow[d, "\pi"] & \mathbb{R}^q \times \overline{P}^0 \gamma_N x \arrow[d, "\pi"] \\ \overline{\mathcal{F}}_x \arrow[r, "\overline{\gamma}"] & \overline{\mathcal{F}}_{\gamma_N x} \end{tikzcd} \tag{4.3.1}$$

Proof. The proof of [45, Proposition 4.16] shows that the elements of $\pi_1(M)$ with ratio different from 1 act freely on $N/\overline{P}^0$. Since the set S_x stabilizes $\overline{P}^0 x$ in $N/\overline{P}^0$, it contains only isometries.

Let (a, y) and (a', y') in $\mathbb{R}^q \times \overline{P}^0 x$. Assume there is $\gamma \in S_x$ such that $\gamma(a, y) = (a', y')$. By definition, $\pi(a, x) = \pi(a', y')$, thus, the application π induces a surjective map $\phi : (\mathbb{R}^q \times \overline{P}^0 x)/S_x \to \overline{\mathcal{F}}_x$. We will show that ϕ is injective. Assume $\phi(S_x(a, y)) = \phi(S_x(a', y'))$. Thus, one has $\pi(a, y) = \pi(a', y')$, meaning there is $\gamma \in \pi_1(M)$ such that $\gamma(a, y) = (a', y')$, implying $\gamma_N y = y'$. By definition, there are $p, p' \in \overline{P}^0$ such that $y = p \cdot x$ and $y' = p' \cdot x$, so we obtain $\gamma_N p \cdot x = p' \cdot x$, whence $\gamma_N p \gamma_N^{-1} \gamma_N \cdot x \in \overline{P}^0 x$. Using that $\overline{P}^0$ is normal in $\overline{P}$, because it is the connected component of the identity, one gets $\gamma_N \cdot x \in \gamma_N p^{-1} \gamma_N^{-1} \overline{P}^0 x = \overline{P}^0 x$. We conclude that $\gamma \in S_x$ and $S_x(a, y) = S_x(a', y')$, providing that ϕ is injective.

Now, let $\gamma \in \pi_1(M)$ with similarity ratio λ. One has

$$\gamma(\mathbb{R}^q \times \overline{P}^0 x) = \mathbb{R}^q \times \gamma_N \overline{P}^0 x = \mathbb{R}^q \times \gamma_N \overline{P}^0 \gamma_N^{-1} \gamma_N x = \mathbb{R}^q \times \overline{P}^0 \gamma_N x,$$

justifying the first line of the diagram. On the other hand, if $(a, y), (a', y')$ are elements of $\mathbb{R}^q \times \overline{P}^0 x$ such that there is $\gamma' \in S_x$ with $\gamma'(a, y) = (a', y')$, one has

$$\pi \circ \gamma(a, y) = \pi(a, y) = \pi(\gamma'(a, y)) = \pi(a', y').$$

Thus, $\pi \circ \gamma$ induces a surjective map from $(\mathbb{R}^q \times \overline{P}^0 x)/S_x$ to $(\mathbb{R}^q \times \overline{P}^0 \gamma_N x)/S_{\gamma_N x}$. To prove that γ descends to an isomorphism $\overline{\gamma}$, it is then sufficient to prove that this map is injective, or equivalently that $S_x(a,y) = S_x(a',y')$ implies $S_{\gamma_N x}\gamma(a,y) = S_{\gamma_N x}\gamma(a',y')$. It is sufficient to show that $\gamma^{-1}S_{\gamma x}\gamma = S_x$, which follows from

$$\begin{aligned}
\gamma^{-1}S_{\gamma_N x}\gamma &= \{\gamma^{-1}\gamma'\gamma, \gamma' \in S_{\gamma_N x}\} \\
&= \{\gamma^{-1}\gamma'\gamma, \gamma'_N\gamma_N \cdot x \in \overline{P}^0\gamma_N x\} \\
&= \{\gamma^{-1}\gamma'\gamma, \gamma_N^{-1}\gamma'_N\gamma_N \cdot x \in \overline{P}^0 x\} \\
&\subset S_x,
\end{aligned}$$

using again that $\overline{P}^0$ is a normal subgroup of $\overline{P}$. The same proof shows that $\gamma S_x \gamma^{-1} \subset S_{\gamma_N x}$, so we conclude that $S_x = \gamma^{-1}S_{\gamma_N x}\gamma$, which shows the existence of $\overline{\gamma}$.

We easily see that $\overline{\gamma}$ is a similarity of ratio λ using the commutative diagram and the fact that h_D descends to the closure of the leaves. □

Proposition 4.3.6. *Let (M,c,D) be an LCP manifold and $(\widetilde{M}, h_D) = \mathbb{R}^q \times (N, g_N)$ be its universal cover, endowed with the similarity structure h_D induced by D. Then, there exists a smooth P-equivariant function $\varphi : N \to \mathbb{R}$ (P was defined in Definition 4.2.9). In particular, if we denote by $\pi_N : \widetilde{M} \to N$ the second projection, $\pi_N^*\varphi$ is a $\pi_1(M)$-equivariant function on $\widetilde{M}$ depends only on the non-flat factor N.*

Proof. We first prove that there always exists a $\pi_1(M)$-equivariant function on $\widetilde{M}$. Let g be any Riemannian metric on M in the conformal class c. The pull-back $\widetilde{g}$ of g to $\widetilde{M}$ satisfies $e^{2f}\widetilde{g} = h_D$ for a function $f : \widetilde{M} \to \mathbb{R}$, which is clearly $\pi_1(M)$-equivariant.

By Lemma 4.3.5, $\overline{\mathcal{F}}_x \simeq (\mathbb{R}^q \times \overline{P}^0 x)/S_x$ and S_x acts by isometries, so the function $f|_{\mathbb{R}^q \times \overline{P}^0 x}$ descends to a function $\bar{f}_x$ on $\overline{\mathcal{F}}_x$. The manifold $\overline{\mathcal{F}}_x$ being compact, we can define

$$e^{2w(x)} := \left(\int_{\overline{\mathcal{F}}_x} \mathrm{d}\mu_x\right)^{-1}\left(\int_{\overline{\mathcal{F}}_x} e^{2\bar{f}_x}\mathrm{d}\mu_x\right), \tag{4.3.2}$$

where $\mathrm{d}\mu_x$ is the measure induced by the metric g_x. Doing this for any $x \in N$ gives a function $w : N \to \mathbb{R}$. We claim that this function is bounded on any compact subset of N. Indeed, if $K \subset N$ is compact, by Lemma 4.3.4 there is a constant $\beta_K > 0$ such that $f(a,x) \leq \beta_K$ for any $(a,x) \in \mathbb{R}^q \times K$. Since $\overline{P}^0$ acts by isometries, $f(a,x) \leq \beta_K$ for any $(a,x) \in \mathbb{R}^q \times (P \cap \overline{P}^0)K$, and by density this still holds for $(a,x) \in \mathbb{R}^q \times \overline{P}^0 K$. Thus, for any $x \in K$ one has $\bar{f}_x \leq \beta_K$ and consequently $w(x) \leq \beta_K$.

We now check that the function w still has the desired equivariance. Let $p \in P$, and let $\lambda > 0$ be its similarity ratio with respect to the metric g_N. By Lemma 4.2.10, there is a unique $\gamma \in \pi_1(M)$ such that $p = \gamma_N$. Denoting $y := p \cdot x$, Lemma 4.3.5 allows us to define $\overline{\gamma} : \overline{\mathcal{F}}_x \to \overline{\mathcal{F}}_y$, which is a similarity of ratio λ. Thus, one has

$$\begin{aligned}
e^{2w(y)} &= \left(\int_{\overline{\mathcal{F}}_y} \mathrm{d}\mu_y\right)^{-1}\left(\int_{\overline{\mathcal{F}}_y} e^{2\bar{f}_y}\mathrm{d}\mu_y\right) \\
&= \left(\int_{\overline{\mathcal{F}}_x} \overline{\gamma}^*(\mathrm{d}\mu_y)\right)^{-1}\left(\int_{\overline{\mathcal{F}}_x} \overline{\gamma}^*(e^{2\bar{f}_y})\overline{\gamma}^*(\mathrm{d}\mu_y)\right) \\
&= \left(\int_{\overline{\mathcal{F}}_x} \lambda^n\mathrm{d}\mu_x)\right)^{-1}\left(\int_{\overline{\mathcal{F}}_x} \lambda^2 e^{2\bar{f}_x}\lambda^n\mathrm{d}\mu_x\right)
\end{aligned}$$

$$= \lambda^2 e^{2w(x)}.$$

However, the function w is not necessarily smooth. We will use a convolution process to obtain the desired smooth equivariant function. Since the foliation $\mathcal{F}$ is Riemannian, one can define a complete Riemannian metric $\widetilde{g}_N$ on N with respect to which P acts by isometries (see [45, Lemma 4.9] for further details).

As P acts cocompactly by isometries on $(N, \widetilde{g}_N)$, the injectivity radius r_0 of $(N, \widetilde{g}_N)$ is positive i.e. for any $x \in N$ the Riemannian exponential $\exp_x$ defined by $\widetilde{g}_N$ is a diffeomorphism on $B_x(r_0)$, the open ball of radius r_0 and center 0 in T_xN. Let $0 < 3r < r_0$ and let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth plateau function in a neighbourhood of 0, compactly supported in $[0, r]$. For every $x \in N$, let $\mathrm{d}V_x$ be the measure induced on T_xN by the metric $\widetilde{g}_N$. Consider the function $\varphi : N \to \mathbb{R}$ given by

$$e^{2\varphi(x)} := \int_{T_xN} e^{2w} \circ \exp_x(v)\chi(\|v\|)\mathrm{d}V_x, \tag{4.3.3}$$

which is well-defined because the function e^{2w} is bounded on any compact subset of N.

We claim that the function φ is smooth. To prove this fact, we first remark that for any x, if one denotes by $B_N(y, a) := \exp_y(B_y(a))$ the ball of radius $0 < a < r_0$ and center y in $(N, \widetilde{g}_N)$, for any $y \in B_N(x, r)$ the Riemannian exponential is a diffeomorphism from $B_y(2r)$ to $B_N(y, 2r)$ because $r < r_0/3$. In particular, $B_N(x, 2r)$ does not meet the cut-locus of y, and the square of the distance function $d^{\widetilde{g}_N}$ induced by $\widetilde{g}_N$ is smooth on $B_N(x, 2r)$. Consequently, we can apply a differentiation under integral argument if we remark that for $y := \exp_x(v_0) \in B_N(x, r)$ (with $v_0 \in T_xN$), one has

$$\begin{aligned}
e^{2\varphi(y)} &= \int_{T_yN} e^{2w} \circ \exp_y(v)\chi(\|v\|)\mathrm{d}V_y \\
&= \int_{T_yN} e^{2w} \circ \exp_x \circ \exp_x^{-1} \circ \exp_y(v)\chi(d^{\widetilde{g}_N}(\exp_x \circ \exp_x^{-1} \circ \exp_y(v), 0))\mathrm{d}V_y \\
&= \int_{T_xN} e^{2w} \circ \exp_x(v)\chi(d^{\widetilde{g}_N}(\exp_x(v), \exp_x(v_0)))(\exp_y^{-1} \circ \exp_x)^*(\mathrm{d}V_y) \\
&= \int_{T_xN} e^{2w} \circ \exp_x(v)\chi(d^{\widetilde{g}_N}(\exp_x(v), y))vol(y, v)\mathrm{d}V_x
\end{aligned}$$

where vol is a smooth function giving the change of volume element.

It remains to check the equivariance property. Let $p \in P$, and let $\lambda > 0$ be its similarity ratio for the metric h_D. One has, denoting $y := p \cdot x$, and using the fact that p is an isometry of $(N, \widetilde{g}_N)$:

$$\begin{aligned}
e^{2\varphi(y)} &= \int_{T_yN} e^{2w} \circ \exp_y(v)\chi(\|v\|)\mathrm{d}V_y \\
&= \int_{T_xN} (p^* e^{2w}) \circ \exp_x(v)\chi(\|v\|)p^*(\mathrm{d}V_y) \\
&= \int_{T_xN} \lambda^2 e^{2w} \circ \exp_x(v)\chi(\|v\|)\mathrm{d}V_x \\
&= \lambda^2 e^{2\varphi}.
\end{aligned}$$

Then, φ is a P-equivariant function. $\square$

Remark 4.3.7. It is easy to show that the P-equivariant function φ given by Proposition 4.3.6 is in fact $\overline{P}$-equivariant. Indeed, for any $p \in P \cap \overline{P}^0$ one has $p^*\varphi = \varphi$ since $\overline{P}^0$ acts by isometries. As $P \cap \overline{P}^0$ is dense in $\overline{P}^0$, we actually have $\varphi = p^*\varphi$ for all $p \in \overline{P}^0$. Our claim thus follows from [45, Lemma 4.10], which states that $\overline{P} = P\overline{P}^0$.

We define a particular class of metric on M:

Definition 4.3.8. *A metric g on M with lift $\widetilde{g}$ on $\widetilde{M}$ is said to be adapted if there exists a smooth function $f : N \to \mathbb{R}$ such that $e^{2f}\widetilde{g} = h_D$.*

With this definition, Proposition 4.3.6 just states that there exist adapted metric.

As a direct application of Proposition 4.3.6, we show that given a compact manifold K with universal cover $\widetilde{K}$, it is possible to construct an LCP manifold with universal cover $\widetilde{M} \times \widetilde{K}$. Indeed, let $\varphi : N \to \mathbb{R}$ be the smooth equivariant function given by Proposition 4.3.6. Let g_K be a metric on K and $\widetilde{g}_K$ its pull-back to $\widetilde{K}$. The metric

$$h_{M,K} := h_D + e^{2\varphi}\widetilde{g}_K \tag{4.3.4}$$

on $\widetilde{M} \times \widetilde{K}$ defines a similarity structure on $M \times K$, and thus an LCP structure $(M \times K, c_K, D_K)$, which proves our claim.

We give a name to the previous construction

Definition 4.3.9. *The LCP structure $(M \times K, c_K, D_K)$ is called an* extension *of (M, c, D) (by K).*

Proposition 4.3.10. *Let $(M \times K, c_K, D_K)$ be an extension by K of (M, c, D). Then, the non-flat part of $(\widetilde{M} \times \widetilde{K}, h_{M,K})$ ($h_{M,K}$ is defined in Equation (4.3.4)) is $N \times \widetilde{K}$.*

Proof. It is easy to see that the non-flat distribution (Definition 4.2.21) of $(\widetilde{M} \times \widetilde{K}, h_{M,K})$ is a subdistribution of $T(N \times \widetilde{K})$ since it has to be orthogonal to the flat distribution, and then orthogonal to $\mathbb{R}^q$. From the definition of LCP manifold (see Theorem 4.2.5 and the definition below), $(N \times \widetilde{K}, g_N + e^{2\varphi}\widetilde{g}_K)$ has a de Rham decomposition of the form $\mathbb{R}^{q'} \times (N', g_{N'})$, where q' migth be 0 and $(N', g_{N'})$ is an incomplete non-flat manifold.

We introduce the notations $g := g_N + e^{2\varphi}\widetilde{g}_K$ and $g' := e^{-2\varphi}g_N + \widetilde{g}_K$, so that $g = e^{2\varphi}g'$, and let ∇' be the Levi-Civita covariant derivative of g'. We recall that the restriction of $\widetilde{D}_K$ to $N \times \widetilde{K}$ is the Levi-Civita of the metric g. Let $k \in \widetilde{K}$, and $X, Y \in T(N \times \{k\})$. Now, we use the formula for the Levi-Civita connection under conformal change [11, Theorem 1.159, a)] and we obtain:

$$\nabla'_X Y = (\widetilde{D}_K)_X Y - d\varphi(X)Y - d\varphi(Y)X + g(X, Y)\widetilde{D}_K\varphi.$$

We identify $N \times \{k\}$ with N in the canonical way, and using again the formula of conformal change for the metric $g'|_N = e^{-2\varphi}g_N$, one obtains:

$$\nabla'_X Y = \widetilde{D}_X Y - d\varphi(X)Y - d\varphi(Y)X + g_N(X, Y)\widetilde{D}\varphi.$$

Combining these two equations and remarking that $g(X, Y)\widetilde{D}_K\varphi = g_N(X, Y)\widetilde{D}\varphi$ we obtain $(\widetilde{D}_K)_X Y = \widetilde{D}_X Y$, which means that $N \times \{k\}$ is totally geodesic in $N \times \widetilde{K}$.

Suppose now that $q' \neq 0$. Let $X \in T\mathbb{R}^{q'}$ be a parallel vector field of norm 1. It induces canonically a parallel vector field of norm 1, still denoted by X, on the Riemannian manifold $(N \times \widetilde{K}, g_N + e^{2\varphi}\widetilde{g}_K)$. We claim that X is tangent to $\widetilde{K}$. Indeed, for any $k \in \widetilde{K}$, the projection of X onto $T(N \times \{k\})$ is parallel because $N \times \{k\}$ is totally geodesic. However, (N, g_N) is irreducible and of dimension greater than 2, so it does not admit a non-zero parallel vector field, thus this projection is equal to zero. Now we remark that g' is a product metric, so $\nabla'_X X \in T\widetilde{K}$ and another use of the formula for the Levi-Civita connection under conformal change gives:

$$0 = (\widetilde{D}_K)_X X = \nabla'_X X + 2d\varphi(X)X - g'(X,X)\nabla\varphi = \nabla'_X X - g'(X,X)\nabla\varphi$$

because φ is a function of N. Thus $T\widetilde{K} \ni \nabla'_X X = g'(X,X)\nabla\varphi$, and $g'(X,X) \neq 0$ so $\nabla\varphi \in T\widetilde{K}$ and $\nabla\varphi \in TN$, again because φ is a function of N, which implies $\nabla\varphi = 0$ and φ is constant. This is absurd because of the $\pi_1(M)$-equivariance of φ, so $q' = 0$ and we conclude that $(N \times \widetilde{K}, g_N + e^{2\varphi}\widetilde{g}_K)$ is irreducible, thus it is the non-flat part of the LCP manifold. $\square$

In particular, the dimension of the non-flat part of the universal cover of an LCP manifold can be of any integer higher or equal to 2.

These observations lead to the definition of reducible LCP manifolds:

Definition 4.3.11. *A LCP manifold is called reducible if it arises from the previous construction, up to a finite covering. A non-reducible LCP manifold is called irreducible.*

4.3.2 Similarity ratios of $\pi_1(M)$

In the known examples of LCP manifolds, the similarity ratios are always algebraic numbers because they are roots of characteristic polynomials of matrices with coefficients in $\mathbb{Z}$. We will prove that this property is always true.

Proposition 4.3.12. *Let (M, c, D) be an LCP manifold. For any $\gamma \in \pi_1(M)$, the ratio of γ viewed as a similarity of $(\widetilde{M}, h_D)$ is a unit of an algebraic number field.*

Proof. Let $\gamma \in \pi_1(M)$ and let λ be its similarity ratio. For any $a \in \mathbb{R}^q$ we will denote by τ_a the translation by a in $\mathbb{R}^q$, so $\mathbb{R}^q$ is naturally identified with the space of translations. The restriction of γ to $\mathbb{R}^q$ can be written as $\gamma_E =: \tau_\alpha \circ \lambda\iota$ where ι is an isometry of $\mathbb{R}^q$ endowed with the metric induced by h_D, and $\alpha \in \mathbb{R}^q$.

Since $\overline{P}^0$ is an abelian Lie group, the group $\mathbb{R}^q \times \overline{P}^0$ is abelian too. We define the group automorphism $\phi : \mathbb{R}^q \times \overline{P}^0 \to \mathbb{R}^q \times \overline{P}^0$ by

$$\phi(\tau_a, p) := \gamma(\tau_a, p)\gamma^{-1} = (\tau_{\lambda\iota a}, \gamma_N p \gamma_N^{-1}). \tag{4.3.5}$$

Our proof relies on the crucial fact that the group $\Gamma_0 := \pi_1(M) \cap (\mathrm{Sim}(\mathbb{R}^q) \times \overline{P}^0)$ defined in Section 4.2.3 is a full lattice in $\mathbb{R}^q \times \overline{P}^0$ by [45, Lemma 4.18].

The preimage of Γ_0 by the Lie group exponential map is a full lattice Γ'_0 of the Lie algebra of $\mathbb{R}^q \times \overline{P}^0$, which is canonically identified with $\mathbb{R}^{q+t}$, for some $t \geq 1$. The differential of ϕ at e is a linear map satisfying $d_e\phi(\Gamma'_0) \subset \Gamma'_0$ because $\phi(\Gamma_0) \subset \Gamma_0$. Moreover, ϕ is invertible and the symmetry between γ and γ^{-1} in the previous discussion gives that $d_e\phi^{-1}(\Gamma'_0) \subset \Gamma'_0$. Thus, if we take a basis $\mathcal{B}$ of the lattice Γ'_0, the matrix $A := \mathrm{Mat}_{\mathcal{B}}(d_e\phi)$ is in $\mathrm{GL}_{q+t}(\mathbb{Z})$. But ϕ stabilizes $\mathbb{R}^q$ and its restriction to this space coincides with $\lambda\iota$. It means that there exists a complex number z of modulus 1 such that λz and $\lambda\bar{z}$ are roots of the characteristic polynomial χ_A of A. Since $A \in \mathrm{GL}_{q+t}(\mathbb{Z})$, λz and $\lambda\bar{z}$ are units of the algebraic field K generated by the roots of χ_A. Thus, $\lambda^2 = (\lambda z)(\lambda\bar{z})$ is a unit of K, and therefore so is λ.

Remark that in the case where $\overline{P}^0$ is a simply-connected space (thus isomorphic to $\mathbb{R}^t$) we don't need to use the exponential map, because ϕ is then linear already. $\square$

4.4 Examples of LCP manifolds

We begin this section by stating a well-known result which will be useful for constructing LCP manifolds:

Proposition 4.4.1. *Let G be a discrete topological group acting on a manifold M. Let $D \trianglelefteq G$ be a normal subgroup. Then, G/D acts on M/D, and $(M/D)/(G/D)$ is in bijection with M/G.*

If moreover D and G/D act freely and properly discontinuously on M and M/D respectively, then so does G on M. In particular, $(M/D)/(G/D)$ and M/G are diffeomorphic manifolds.

Proof. The action of G/D on M/D is given by $gD \cdot Dx := Dgx$ for any $(g, x) \in G \times M$.

We define $\phi : (M/D)/(G/D) \to M/G$ by $\phi(G/D \cdot Dx) := Gx$ for any $x \in M$. This map is clearly surjective. In addition, if there are $(x, y) \in M^2$ such that $Gx = Gy$, there exists $g \in G$ such that $gx = y$, implying $gD \cdot Dx = Dy$ and then $G/D \cdot Dx = G/D \cdot Dy$, so ϕ is one-to-one.

Now, assume that D and G/D act freely and properly discontinuously on M and M/D respectively. Let $g \in G$ and $x \in M$ such that $gx = x$. Then, $gD \cdot Dx = Dx$, so $gD = 1_{G/D}$ because G/D acts freely on M/D, implying $g \in D$, and $g = 1_G$ because D acts freely on M. Thus G acts freely on M.

To see that G acts properly discontinuously on M, we pick a compact $K \subset M$. Let $g \in G$ satisfying $(gK) \cap K \neq \emptyset$. Since DK is a compact subset of the manifold M/D, the set $\{g'D \in G/D \mid g'D \cdot (DK) \cap (DK) \neq \emptyset\}$ is finite: let $(g_jD)_{j \in J}$ be the family of its elements, where J is a finite set. Now, since $(gK) \cap K \neq \emptyset$, we also have $gD \cdot DK \cap DK \neq \emptyset$, so there is $j \in J$ such that $g_jD = gD$. This show that we can find $d \in D$ with $dg_j = g$ because D is normal. Then, $(dg_jK) \cap K \neq \emptyset$. But there are only finitely many elements $d \in D$ satisfying this property because D acts properly discontinuously on M. Let $(d_{j,i})_{i \in I_j}$ be the family of these elements, where I_j is a finite set for every $j \in J$. Consequently, there exist $j \in J$ and $i \in I_j$ such that $g = d_{j,i}g_j$, and conversely any element of this form satisfy $(gK) \cap K \neq \emptyset$. Thus,

$$|\{g \in G \mid (gK) \cap K \neq \emptyset\}| = \sum_{j \in J} |I_j| < +\infty,$$

so G acts properly discontinuously on M.

Finally, denote by $\pi_D : M \to M/D$, $\pi_{G/D} : M/D \to (M/D)/(G/D)$ and by $\pi_G : M \to M/G$ the canonical projections. One has the following commutative diagram:

$$\begin{tikzcd} M \arrow[d, "\pi_{G/D} \circ \pi_D"'] \arrow[dr, "\phi \circ \pi_{G/D} \circ \pi_D = \pi_G"] & \\ (M/D)/(G/D) \arrow[r, "\phi"] & M/G \end{tikzcd} \tag{4.4.1}$$

and [47, Proposition 7.17] implies that ϕ is smooth. □

4.4.1 General construction

Inspired by the known examples, we will now make a more general construction which includes all the models of LCP manifolds previously described.

Let $\overline{\mathcal{N}}$ be a compact manifold. We will denote by $\mathcal{N}$ its universal cover and by Γ its fundamental group, so $\overline{\mathcal{N}} \simeq \mathcal{N}/\Gamma$. Let $p \in \mathbb{N}$ and let $\phi : \Gamma \to \mathrm{Aff}_p(\mathbb{Z})$ be a group morphism, where $\mathrm{Aff}_p(\mathbb{Z}) := \mathbb{R}^p \rtimes \mathrm{GL}_p(\mathbb{Z})$ is the set of affine transformations of $\mathbb{R}^p$ with linear part in $\mathrm{GL}_p(\mathbb{Z})$. We denote by $\phi_L : \Gamma \to \mathrm{GL}_p(\mathbb{Z})$ the group morphism associating to $\gamma \in \Gamma$ the linear part of $\phi(\gamma)$.

We consider the simply connected manifold $\widetilde{M} := \mathbb{R}^p \times \mathcal{N}$. Let $D \simeq \mathbb{Z}^p$ be the group of translations $\widetilde{M} \ni (a, x) \mapsto (a + z, x)$ for $z \in \mathbb{Z}^p$. Let H be the group defined by $H := (\phi, \mathrm{id})(\Gamma) = \{(\phi(\gamma), \gamma) | \gamma \in \Gamma\} \subset \mathrm{Aff}_p(\mathbb{Z}) \times \Gamma \subset \mathrm{Diff}(\widetilde{M})$. Let G be the subgroup of $\mathrm{Diff}(\widetilde{M})$ generated by D and H. It is clear that D is a normal subgroup of G and $G := D \rtimes H$. We claim that G acts freely, properly discontinuously and cocompactly on $\widetilde{M}$. Indeed, one has $\widetilde{M}/D \simeq (S^1)^p \times \mathcal{N}$ and H acts freely on this quotient because Γ acts freely on $\mathcal{N}$. Moreover, H also acts properly discontinuously because the map $(S^1)^p \times \mathcal{N} \to \mathcal{N}$ being proper and H acting separately on $\mathbb{R}^p$ and $\mathcal{N}$, it is sufficient to observe that Γ acts properly on $\mathcal{N}$. In addition, this action is cocompact because Γ acts cocompactly on $\mathcal{N}$ and $(S^1)^p$ is compact. Altogether, by Proposition 4.4.1 the quotient $\widetilde{M}/G$ is a compact manifold which we denote by $Q(\overline{\mathcal{N}}, \phi)$, and whose fundamental group is G.

We now wish to construct an LCP structure on $Q(\overline{\mathcal{N}}, \phi)$. To do so, we assume that the following conditions hold:

(J_1) there exist $\delta \in \mathbb{N}$, a decomposition $\mathbb{R}^p =: E_1 \oplus \ldots \oplus E_\delta$ stabilized by the action of $\phi_L(\Gamma)$, and a positive definite bilinear form b on $\mathbb{R}^p$ such that the previous decomposition is orthogonal with respect to b and for any $1 \leq k \leq \delta$, the restriction of $\phi_L(\Gamma)$ to $(E_k, b|_{E_k})$ consists of similarities;

(J_2) $O(E_1, b|_{E_1})$ does not contain $\phi_L(\Gamma)|_{E_1}$.

Remark 4.4.2. In particular, condition (J_1) allows us to define a group morphism $\Lambda : \Gamma \to (\mathrm{R}_+^*)^\delta$ which associates to any $\gamma \in \Gamma$ the δ-tuple given by the similarity ratios of $\phi_L(\gamma)|_{E_1}, \ldots, \phi_L(\gamma)|_{E_\delta}$. For any $1 \leq k \leq \delta$, Λ_k will denote the k-th coordinate of Λ. Condition (J_2) implies that $2 \leq p$. Indeed, if $p = 1$, $\phi_L(\Gamma) \subset \{\pm 1\} = O(\mathbb{R}^p, b)$. In addition, from (J_2) we also deduce that $2 \leq \delta$, because otherwise $\mathbb{R}^p = E_1$ and there would exist an element $\gamma \in \Gamma$ such that $\pm 1 \neq \Lambda_1(\gamma)^p = \det \phi_L(\gamma) = \pm 1$, which is absurd. In particular, $Q(\overline{\mathcal{N}}, \phi)$ has dimension at least 3.

We will need the following standard lemma:

Lemma 4.4.3. *Let $\mathcal{Z}$ be a smooth manifold on which a group Γ' acts freely and properly discontinuously, so in particular $\mathcal{Z}/\Gamma'$ is a smooth manifold. Let $\rho : \Gamma' \to \mathbb{R}_+^*$ be a group morphism. Then, there exists a function $f \in C^\infty(\mathcal{Z}, \mathbb{R})$ such that for any $\gamma \in \Gamma'$, $\gamma^* e^{2f} = \rho(\gamma)^2 e^{2f}$.*

Proof. Let $\pi_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}/\Gamma'$ be the canonical submersion. We define the oriented line bundle $L := \mathcal{Z} \times_{\rho^{-1}} \mathbb{R}$. Since any orientable line bundle is trivial, there exists $s : \mathcal{Z}/\Gamma' \to L$ a nowhere vanishing smooth section of L. Then, after replacing s by $-s$ if necessary, there is a function $f : \mathcal{Z} \to \mathbb{R}$ such that for all $x \in \mathcal{Z}$ one has $s(\pi_{\mathcal{Z}}(x)) = [x, e^f(x)]$. Moreover, for any $\gamma \in \Gamma'$, we have

$$[x, e^f(x)] = s(\pi_{\mathcal{Z}}(x)) = s(\pi_{\mathcal{Z}}(\gamma x)) = [\gamma x, e^f(\gamma x)] = [x, \rho(\gamma)^{-1} e^f(\gamma x)],$$

which implies $\rho(\gamma) e^f(x) = e^f(\gamma x)$, so the function f has the desired equivariance property. $\square$

We are now in position to construct an LCP structure on $Q(\overline{\mathcal{N}}, \phi)$.

Proposition 4.4.4. *Under the assumptions* (J_1), (J_2) *there exists an LCP structure on* $Q(\overline{\mathcal{N}}, \phi)$. *The LCP manifold obtained in this way has rank equal to* $\mathrm{rk}(\Lambda_1(\Gamma))$ *and the flat part of its universal cover contains* E_1.

Proof. Let $\bar{g}$ be any Riemannian metric on $\overline{\mathcal{N}}$ and let $\widetilde{g}$ be its lift to $\mathcal{N}$. Let $f \in C^\infty(\mathcal{N}, \mathbb{R})$ be the function given by Lemma 4.4.3 applied to the morphism $\rho := \Lambda_1$. By definition, an element $\gamma \in \Gamma$ acts as a similarity of ratio $\Lambda_1(\gamma)$ on $(\mathcal{N}, g := e^{2f}\widetilde{g})$.
For any $2 \leq k \leq \delta$, we define the morphism

$$\rho_k : \Gamma \to \mathbb{R}_+^*, \qquad \gamma \mapsto \Lambda_1(\gamma)/\Lambda_k(\gamma). \tag{4.4.2}$$

By Lemma 4.4.3, we know that the set

$$\mathcal{F}_{eq}(k) := \{f \in C^\infty(\mathcal{N}, \mathbb{R}) \mid \forall \gamma \in \Gamma, \gamma^* e^{2f} = \rho_k(\gamma)^2 e^{2f}\}. \tag{4.4.3}$$

is non-empty.
We identify the tangent bundle $T\mathbb{R}^p$ with $\mathbb{R}^p \times \mathbb{R}^p$ in the canonical way, and the bilinear form b thus defines a Riemannian metric on $\mathbb{R}^p$. Then, we define a metric h on $\widetilde{M} = \mathbb{R}^p \times \mathcal{N}$ by

$$h := b|_{E_1} + \sum_{k=2}^{\delta} e^{2f_k} b|_{E_k} + g, \tag{4.4.4}$$

where for all $2 \leq k \leq \delta$, $f_k \in \mathcal{F}_{eq}(k)$.
One clearly has for any $T \in D$ that $T^*h = h$. For any $\gamma \in \Gamma$, one has

$$\begin{aligned}
(\phi(\gamma), \gamma)^* h =& \Lambda_1(\gamma)^2 b|_{E_1} + \sum_{k=2}^{\delta} \gamma^* e^{2f_k} \Lambda_k(\gamma)^2 b|_{E_k} + \gamma^* g \\
=& \Lambda_1(\gamma)^2 b|_{E_1} + \sum_{k=2}^{\delta} \left(\frac{\Lambda_1(\gamma)}{\Lambda_k(\gamma)}\right)^2 e^{2f_k} \Lambda_k(\gamma)^2 b|_{E_k} + \Lambda_1(\gamma)^2 g \\
=& \Lambda_1(\gamma)^2 h.
\end{aligned}$$

Since $G = D \rtimes H$, the elements of G act as similarities, and $\widetilde{g}$ is a similarity structure on $Q(\overline{\mathcal{N}}, \phi)$ which is not Riemannian because of condition (J_2).

It remains to prove that $(\widetilde{M}, h)$ is non-flat with reducible holonomy. But E_1 is a Riemannian factor of $\widetilde{M}$, so the claim follows from Remark 4.2.6. ☐

Example 4.4.5. We consider the matrix

$$B := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{4.4.5}$$

Let $q \geq 1$. Let $A \in \mathrm{SL}_{2q}(\mathbb{Z})$ which is the matrix diagonal by blocks with q times the block B. We consider a bilinear symmetric form b_0 on $\mathbb{R}^2$ for which the two eigenspaces of B are orthogonal, and we define the symmetric bilinear form $b := \bigoplus_{k=1}^{q} b_0$ on $\mathbb{R}^{2q}$. We consider $\overline{\mathcal{N}} := S^1$, whose fundamental group is $\Gamma := \mathbb{Z}$, and the group morphism $\phi : \Gamma \to \mathrm{SL}_{2q}(\mathbb{Z})$, $n \mapsto A^n$. By Proposition 4.4.4, $Q(S^1, \phi)$ admits an LCP structure whose universal cover has a flat part of dimension q. Thus the dimension of the flat part can be any integer.

As an application of Proposition 4.4.4, we will show that on any OT-manifold (recall that they were defined in Example 4.2.18) carries an LCP structure. The proof of this fact just relies on the remark that an OT-manifold is a particular case of the construction above.

Corollary 4.4.6. *Any OT-manifold $X(K,U)$ can be endowed with an LCP structure.*

Proof. We use the notations of Example 4.2.18. By definition one has

$$X(K,U) = (H^s \times \mathbb{C}^t)/(\mathcal{O}_K \rtimes U)$$

so its universal cover is naturally isomorphic to $(\mathbb{R}^*_+)^s \times \mathbb{R}^s \times \mathbb{R}^{2t} \simeq \mathbb{R}^s \times \mathbb{R}^{s+2t}$ using the logarithm map. By construction, the group $\Gamma := p_{\mathbb{R}^s} \circ \ell(U)$ acts freely, properly discontinuously and cocompactly on $\mathcal{N} := \mathbb{R}^s$ because it is a full lattice. Moreover, U is of rank s, so $\psi := (p_{\mathbb{R}^s} \circ \ell)^{-1}$ is a group isomorphism between Γ and U.

Let $\mathcal{B} = (e_1, \dots, e_{s+2t})$ be the canonical basis of $\mathbb{R}^{s+2t}$. Let $\mathcal{B}'$ be a basis of the lattice $\sigma(\mathcal{O}_K)$, so in particular another basis of $\mathbb{R}^{s+2t}$. With respect to the basis $\mathcal{B}'$, the action of U restricted to $\mathbb{R}^{s+2t}$ consists of multiplication by matrices of $\mathrm{GL}_{s+2t}(\mathbb{Z})$ because U preserves $\sigma(\mathcal{O}_K)$. This induces a group morphism $U \to \mathrm{GL}_{s+2t}(\mathbb{Z})$ and then a group morphism $\phi : \Gamma \to \mathrm{GL}_{s+2t}(\mathbb{Z})$ using the isomorphism ψ between Γ and U. Consequently, $X(K,U) \simeq Q(\mathcal{N}/\Gamma, \phi)$.

It is now sufficient to check that conditions (J_1), (J_2) hold, so we can apply Proposition 4.4.4 to conclude. Let b be the Euclidean metric on $\mathbb{R}^{s+2t}$ for which $\mathcal{B}$ is orthonormal. By construction, for any $\gamma \in \Gamma$, the matrix of $\phi(\gamma)$ in the basis $\mathcal{B}$ is of the form

$$\begin{pmatrix} \sigma_1(u) & & & & & \\ & \ddots & & & & \\ & & \sigma_s(u) & & & \\ & & & |\sigma_{s+1}(u)|O_1 & & \\ & & & & \ddots & \\ & & & & & |\sigma_{s+t}(u)|O_t \end{pmatrix} \tag{4.4.6}$$

where $u \in \mathcal{O}_K^{\times,+}$ and $O_1, \dots, O_t \in \mathrm{SO}_2(\mathbb{R})$. Then, the spaces

$$E_j := \mathrm{Span}(e_j)$$

for $1 \leq j \leq s$ and

$$E_{s+j} := \mathrm{Span}(e_{s+2j-1}, e_{s+2j})$$

for $1 \leq j \leq t$ give a decomposition of $\mathbb{R}^{s+2t}$ in orthogonal subspaces stable by the action of $\phi(\Gamma)$, so (J_1) is verified because of the form of the matrix (4.4.6). Finally, σ_1 is injective so for any $u \in U$, $\sigma_1(u) = 1$ implies $u = 1$. Thus there exists $u \in U$ such that $\sigma_1(u) \in (0,1)$ (because we recall that $\sigma_1(u) > 0$ by construction) so (J_2) holds. □

It is important to notice that the LCP metrics constructed by using the proof of Proposition 4.4.4 on OT-manifolds with the approach of Corollary 4.4.6 do not contain the LCK structures introduced in [68] when $t = 1$. However, we can extend the family of LCP metrics defined in the proof of Proposition 4.4.4. For any $2 \leq k, k' \leq \delta$ with $k \neq k'$, consider the morphism

$$\rho_{k,k'} : \Gamma \to \mathbb{R}^*_+, \qquad \gamma \mapsto \Lambda_1(\gamma)/\sqrt{\Lambda_k \Lambda_{k'}}(\gamma), \tag{4.4.7}$$

and let $b_{k,k'} : E_k \times E_{k'} \to \mathbb{R}$ be a bilinear form satisfying $\phi(\gamma)^* b_{k,k'} = b_{k,k'}$ for any $\gamma \in \Gamma$ (such forms always exists, since we can take $b_{k,k'} = 0$), and let $f_{k,k'}$ be an element of the set

$$\mathcal{F}_{eq}(k,k') := \{f \in C^\infty(\mathcal{N}, \mathbb{R}) \mid \forall \gamma \in \Gamma, \gamma^* e^{2f} = \rho_{k,k'}(\gamma)^2 e^{2f}\}. \tag{4.4.8}$$

Then, we consider the metric h on $\mathbb{R}^p \times \mathcal{N}$ defined by

$$h := b|_{E_1} + \sum_{k=2}^{\delta} e^{2f_k} b|_{E_k} + \sum_{k=2}^{\delta} \sum_{k'=2}^{\delta} e^{2f_{k,k'}} b_{k,k'} + g. \tag{4.4.9}$$

If the functions $f_{k,k'}$ are taken small enough on a relatively compact fundamental domain of $\mathcal{N}$, h is positive definite, and an argument similar to the one used in Proposition 4.4.4 shows that the elements of the group G act as h-similarities.

On an OT-manifold with $t = 1$, the LCK metric on its universal cover $H^s \times \mathbb{C}$ defined in [68] is of the form

$$h := \left(\prod_{j=1}^{s} \frac{1}{y_j}\right) \left(\sum_{k,k'=1}^{s} \frac{1}{y_k y_{k'}} dx_k \otimes dx_{k'} + dy_k \otimes dy_{k'}\right) + dx_{s+1}^2 + dy_{s+1}^2, \tag{4.4.10}$$

where $z_k := x_k + iy_k$, $1 \leq k \leq s+1$ are the canonical complex coordinates. This falls on the construction above, with the functions $f_{k,k'} := \left(\prod_{j=1}^{s} \frac{1}{y_j}\right) \frac{1}{y_k y_{k'}}$ and the bilinear forms $b_{k,k'} := dx_k \otimes dx_{k'}$.

4.4.2 Rank of an LCP manifold

Our next goal is to construct LCP manifolds of arbitrary rank using again Proposition 4.4.4 again. For this purpose, we need a special family of commuting matrices, which will be constructed by means of number theory. This makes the object of the two following two lemmas:

Lemma 4.4.7. *For any $n \in \mathbb{N}$ there exists a cyclic, totally real and monogenic algebraic number field of degree $p \geq n+1$.*

Proof. Let $n \in \mathbb{N}$, and let $m \geq 2n+3$ be a prime number. Let K be the maximal real subfield of the m-th cyclotomic extension. Then K is an extension of $\mathbb{Q}$ of degree $p := (m-1)/2 \geq n+1$, which is totally real, monogenic by [73, Proposition 2.16], and cyclic. □

Lemma 4.4.8. *Let $n \geq 2$. There exists an integer $p \geq n+1$ and diagonalizable matrices $A_1, \dots, A_n \in \mathrm{GL}_p(\mathbb{Z})$ with the following properties:*

- *The matrices $A_1, \dots, A_n$ commute, so their are codiagonalizable.*
- *Let $(e_1, \dots, e_p)$ be a common basis of diagonalization for $A_1, \dots, A_n$. For any $1 \leq k \leq p$, let $E_k = \mathrm{Span}(e_k)$, and denote by $\lambda_k(A_l)$ the eigenvalue of A_l associated to the eigenspace E_k. Then, the subgroup $\langle |\lambda_1(A_1)|, \dots, |\lambda_1(A_n)| \rangle$ of $\mathbb{R}_+^*$ has rank n.*

Proof. Let K be a cyclic, totally real and monogenic algebraic number field of degree $p \geq n+1$, which exists by Lemma 4.4.7. There is an algebraic integer α such that α generates a power basis of K, in particular $K = \mathbb{Q}[\alpha]$. By Dirichlet's units theorem, the group of units of

$\mathbb{Q}[\alpha]$ has rank $p-1$. Since $p-1 \geq n$, we can take n independent fundamental units $u_1, \dots, u_n$ in $\mathbb{Q}[\alpha]$. By monogeneity, there are polynomials $P_1, \dots, P_n \in \mathbb{Z}_{p-1}[X]$ such that $P_l(\alpha) = u_l$ for any $1 \leq l \leq n$.

Now, let $A \in \mathrm{GL}_p(\mathbb{Z})$ be the companion matrix of the minimal polynomial of α and let $A_l := P_l(A) \in \mathrm{GL}_p(\mathbb{Z})$ for $1 \leq l \leq n$. Since the minimal polynomial of α is irreducible over $\mathbb{Q}$, it is separable and A is diagonalizable in $\mathbb{R}$, with eigenvalues equal to the conjugates of α, namely $\alpha, \sigma(\alpha), \dots, \sigma^{p-1}(\alpha)$, where σ is a generator of the (cyclic) Galois group of $\mathbb{Q}[\alpha]$. Then, the matrices A_l are diagonalizable with eigenvalues $u_l, \sigma(u_l), \dots, \sigma^{p-1}(u_l)$. Moreover, their determinants are $\Pi_{k=0}^{p-1}\sigma^k(u_l) = \pm 1$ because u_l is a unit.

Finally, let e_1 be an eigenvector of A for the eigenvalue α. Then, $E_1 := \mathrm{Span}(e_1)$ is a one-dimensional eigenspace of any A_l for the eigenvalue u_l, and $\langle u_1, \dots, u_n\rangle$ is of rank n. We can complete (e_1) in a basis of diagonalization of A to obtain the last property of the lemma. $\square$

The matrices defined in Lemma 4.4.8 will be used to define the morphism ϕ needed for the construction of Proposition 4.4.4, so we prove the following:

Proposition 4.4.9. *Let $n \geq 1$. Let $p \geq n+1$ and $A_1, \dots, A_n \in \mathrm{GL}_p(\mathbb{Z})$ be the matrices given by Lemma 4.4.8. The group $H := \langle A_1, \dots, A_n\rangle$ is canonically isomorphic to $\mathbb{Z}^n$, defining a group isomorphism $\phi : \mathbb{Z}^n \to H$. Then, there exists a LCP structure on $Q((S^1)^n, \phi)$ of rank n.*

In particular, the rank of an LCP manifold can be any positive integer.

Proof. We keep the notations of Lemma 4.4.8 in this proof. Let $\mathcal{B}$ be a basis adapted to the decomposition $E_1 \oplus \dots \oplus E_p$ and let b be the symmetric, positive definite bilinear form for which $\mathcal{B}$ is orthonormal. Then, the conditions (J_1) and (J_2) are satisfied, so by Proposition 4.4.4 $Q((S^1)^n, \phi)$ carries an LCP structure of rank n. $\square$

Example 4.4.10. We can make an explicit computation of the matrices given by Lemma 4.4.8 in the case $n = 2$ by following the constructive approach of the proof. Taking $m = 7$ in the proof of Lemma 4.4.7 shows that $K := \mathbb{Q}[2\cos(\frac{2\pi}{7})]$ is a totally real, monogenic, cyclic extension of $\mathbb{Q}$ of degree $p = 3$. From now on, we denote by $\alpha := 2\cos(\frac{2\pi}{7})$. From [73, Proposition 2.16], one has $\mathcal{O}_K = \mathbb{Z}[\alpha]$. The minimal polynomial of α is $X^3 + X^2 - 2X - 1$, and its conjugates are $2\cos(\frac{4\pi}{7})$ and $2\cos(\frac{6\pi}{7})$. Let σ be the automorphism of K such that $\sigma(\alpha) = 2\cos(\frac{4\pi}{7})$. Then $\sigma^2(\alpha) = 2\cos(\frac{6\pi}{7})$ and $\sigma^3 = \mathrm{id}_K$.

We claim that the (multiplicative) group $\langle \alpha, \sigma(\alpha)\rangle$ has rank 2. Indeed, if there were $a, b \in \mathbb{Z}$ such that $\alpha^a = \sigma(\alpha)^b$,then the two vectors of $\mathbb{R}^3$ given by

$$X_1 := (\ln|\alpha|, \ln|\sigma(\alpha)|, \ln|\sigma^2(\alpha)|), \qquad X_2 := (\ln|\sigma(\alpha)|, \ln|\sigma^2(\alpha)|, \ln|\alpha|)$$

would be collinear. But X_1 and X_2 have the same norm for the standard Euclidean metric in $\mathbb{R}^3$ because the coefficients of X_2 are a permutation of the ones of X_1, so they are collinear if and only if $X_1 = \pm X_2$. But this is false because $\cos(\frac{2\pi}{7}) \neq \cos(\frac{4\pi}{7})^{\pm 1}$.

Now, we have the equality $\sigma(\alpha) = \alpha^2 - 2$. Thus, we consider the companion matrix of the minimal polynomial of α:

$$A_1 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}, \tag{4.4.11}$$

and the matrix

$$A_2 := A_1^2 - 2\mathrm{I}_3 = \begin{pmatrix} -2 & 1 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \tag{4.4.12}$$

One easily checks that eigenvectors corresponding to the eigenvalues $\alpha, \sigma(\alpha), \sigma^2(\alpha)$ of A_1 can be taken respectively as

$$x_1 = \begin{pmatrix} 1 \\ \alpha + \alpha^2 \\ \alpha \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ \sigma(\alpha) + \sigma(\alpha)^2 \\ \sigma(\alpha) \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ \sigma^2(\alpha) + \sigma^2(\alpha)^2 \\ \sigma^2(\alpha) \end{pmatrix}, \tag{4.4.13}$$

and they are eigenvectors of A_2 for the eigenvalues $\sigma(\alpha), \sigma^2(\alpha), \alpha$ respectively.

Using these matrices, we can now give the explicit construction of an LCP manifold of rank 2 following Proposition 4.4.9 and Proposition 4.4.4. On the manifold $\widetilde{M} := \mathbb{R}^3 \times (\mathbb{R}_+^*)^2$, the group

$$G := D \rtimes \langle (A_1, (|\alpha|, 1)), (A_2, (1, |\sigma(\alpha)|)) \rangle \tag{4.4.14}$$

acts freely, properly discontinuously and cocompactly (here the group D is defined as in Section 4.4.1, as the group of translations $\mathbb{Z}^3$ acting on $\mathbb{R}^3$). Let (t_1, t_2) be the canonical coordinates of $(\mathbb{R}_+^*)^2$. We define the metric

$$h_{x,t} := dx_1^2 + \phi_2(t_1,t_2)^2 dx_2^2 + \phi_3(t_1,t_2)^2 dx_3^2 + t_2^2 dt_1^2 + t_1^2 dt_2^2. \tag{4.4.15}$$

where

$$\phi_2(t_1,t_2) := \left| \frac{\alpha}{\sigma(\alpha)} \right|^{\ln(t_1)/\ln(|\alpha|)} \left| \frac{\sigma(\alpha)}{\sigma^2(\alpha)} \right|^{\ln(t_2)/\ln(|\sigma(\alpha)|)} \tag{4.4.16}$$

$$\phi_3(t_1,t_2) := \left| \frac{\alpha}{\sigma^2(\alpha)} \right|^{\ln(t_1)/\ln(|\alpha|)} \left| \frac{\sigma(\alpha)}{\alpha} \right|^{\ln(t_2)/\ln(|\sigma(\alpha)|)}. \tag{4.4.17}$$

The manifold $M := \widetilde{M}/G$ admits a non-Riemannian similarity structure given by h, which in turn defines an LCP structure of rank 2 on M.

4.5 Some open questions

Some questions arise naturally from the analysis and the discussions done in the previous sections. We make here a non-exhaustive list of such ones, whose answers would lead to a better understanding of LCP manifolds. Throughout this section, we will use the notations of Section 4.2.3.

First of all, it was noticed by Kourganoff [45, Theorem 1.9] that the dimension of the closures of the leaves, which are the elements of $\overline{\mathcal{F}}$ in the setting of Theorem 4.2.20, may vary. However, in all the examples given in this article, this dimension is constant, so we ask the following:

- In the setting of Theorem 4.2.20, do all the elements of $\overline{\mathcal{F}}$ have the same dimension?

We can propose a strategy to answer this first question. Indeed, assume that $\overline{P}^0$ is simply connected, i.e. it is isomorphic to the group $\mathbb{R}^t$ for some $t \in \mathbb{N}$. Then, since the group Γ_0 is a full lattice in $\mathbb{R}^q \times \overline{P}^0 \simeq \mathbb{R}^{q+t}$, the group Γ_0 is of rank $q+t$. In addition, for any $x \in N$ (the non-flat part), the closed leaf $\overline{\mathcal{F}}_x = \pi(\mathbb{R}^q \times \overline{P}^0 x)$ has the same dimension as $\mathbb{R}^q \times \overline{P}^0 x$. As we already saw, this last manifold is isomorphic to the product of an Euclidean space with a flat torus so it is a Lie group, and Γ_0 acts freely and properly discontinuously on it. Consequently, $\Gamma_0(\{(0,x)\})$ is a lattice of $\mathbb{R}^q \times \overline{P}^0 x$ with rank equal to $q+t$. Thus

$$q + t = \operatorname{rank}(\Gamma_0) \leq \dim(\mathbb{R}^q \times \overline{P}^0 x) \leq q + t, \tag{4.5.1}$$

and these inequalities turn out to be equalities, so $\overline{\mathcal{F}}_x$ has dimension $q+t$. This leads to the following question, whose answer is positive in all the examples:

- Is the group $\overline{P}^0$ simply connected, or equivalently is it isomorphic to $\mathbb{R}^t$ for some $t \in \mathbb{N}$?

In order to have a better understanding of the group P, we should specify how it acts on N. In [45, Lemma 4.17], it was shown that P acts freely on N, but the proof proposed seems incorrect, even if it does not modify the correctness of the rest of the article. The only result we can obtain is the one of Lemma 4.2.10, stated previously. We thus ask:

- Does P acts freely on N? If this is true, does $\overline{P}$ acts freely on N?

In Section 4.4.1, we have given a general construction to obtain LCP manifolds. Nevertheless, some points remain imprecise:

- What are the acceptable choices for the morphism ϕ, given a compact manifold $\mathcal{N}$ (even without asking for conditions (J_1) and (J_2))?
- Can we weaken conditions (J_1) and (J_2)?

Finally, we remark that the only known LCK manifolds which are also LCP are the OT-manifolds for $t=1$. A natural way to construct new examples would be to take extensions of OT-manifolds (see Definition 4.3.9).

- Can an extension of an LCP manifold which is also LCK be an LCK manifold?
- Are the OT-manifolds with $t=1$ the only LCP manifolds which are also LCK?

Chapter 5

Torsion-free connections on G-structures

Ce chapitre est une note concernant des G-structures particulières, à savoir celles où le groupe G contient $\mathrm{SO}_n(\mathbb{R})$. On démontre qu'elles admettent des connexions sans torsion provenant de structures de Weyl fermées.

5.1 Introduction

Let M be a smooth manifold of dimension n and G a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. A G-structure on M is a reduction of the frame bundle of M to G i.e. a principal subbundle of $\mathrm{Fr}(M)$ with structure group G.

We recall the following well-known result:

Proposition 5.1.1. *Let G be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$ and let P be a G-structure on M. Then, there exists a torsion-free connection on P.*

We quickly outline the proof of Proposition 5.1.1, using the analysis of [19, Chapter 4]. Denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathrm{ad}P$ the adjoint bundle of P (which is a vector subbundle of the bundle of endomorphisms of TM). The set of connections on TM compatible with P is an affine space of direction $\Omega^1(M, \mathrm{ad}P)$. For any $\xi \in \Omega^1(M, \mathrm{ad}P)$, we define $(\partial\xi)(X, Y) := \xi(X)(Y) - \xi(Y)(X)$ where $X, Y \in TM$ and we consider the set

$$\mathcal{T}_P := \frac{\Omega^2(M, TM)}{\partial(\Omega^1(M, \mathrm{ad}P))}. \tag{5.1.1}$$

The *intrinsic torsion* T_P^{int} of P is the equivalence class $[T_\nabla] \in \mathcal{T}_P$ where T_∇ is the torsion of any connection ∇ compatible with P. This is well-defined because if ∇' is another connection, there is $\xi \in \Omega^1(M, \mathrm{ad}P)$ such that $\nabla' = \nabla + \xi$, and an easy computation leads to $T_{\nabla'} = T_\nabla + \partial(\xi)$. Then, there exists a torsion-free connection on P if and only if $T_P^{\mathrm{int}} = 0$.

For any $x \in M$, fix a frame $u \in P_x$ (which identifies $\mathbb{R}^n$ with T_xM). For any $\phi \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, let $\xi \in (\mathbb{R}^n)^* \otimes \mathrm{End}(\mathbb{R}^n)$ be given by

$$2\xi(X)(Y) := \phi(X, Y) - \phi(X, \cdot)^*(Y) - \phi(Y, \cdot)^*(X) \qquad X, Y \in \mathbb{R}^n, \tag{5.1.2}$$

where "*" denotes the adjoint with respect to the standard metric on $\mathbb{R}^n$. By construction, one has $\partial\xi = \phi$ and $\xi(X)$ is skew-symmetric for every $X \in \mathbb{R}^n$. Since $\mathfrak{o}_n(\mathbb{R}) \subset \mathfrak{g}$, we have $\xi \in (\mathbb{R}^n)^* \otimes \mathfrak{g}$. We deduce that $\partial(\Omega^1(M, \mathrm{ad}P)) = \Omega^2(M, TM)$, implying $\mathcal{T}_P = 0$, thus $T_P^{\mathrm{int}} = 0$, which gives the result.

Proposition 5.1.1 was originally stated as an exercise in [58, Section 17.4, exercise (1)]. The author's strategy of proof was to consider a reduction of P to $\mathrm{O}_n(\mathbb{R})$ in order to take the Levi-Civita connection of the associated Riemannian metric as the desired torsion-free connection, implying the stronger result that the connection on P is induced by the Levi-Civita connection of a metric on M. However, such a reduction fails to exist in general, as shown by the following example:

Example 5.1.2. We consider the circle $S^1 \subset \mathbb{C}$, parametrized by the map $\psi : [0, 2\pi) \ni \theta \to e^{i\theta}$. Its tangent bundle is given by $TS^1 \simeq S^1 \times \mathbb{R}$, and its frame bundle is $\mathrm{Fr}(S^1) \simeq S^1 \times \mathbb{R}^*$. Let G be the closed subgroup of $\mathbb{R}^*$ generated by 2, and let P be the G-structure of S^1 given by $P_{\psi(\theta)} = \{\psi(\theta)\} \times 2^{\frac{\theta}{2\pi}} G$ for any $\theta \in [0, 2\pi)$. There is no reduction of P to $G \cap \mathrm{O}_1(\mathbb{R}) = \{1\}$ because P is a non-trivial principal bundle.

Nevertheless, we can prove that the torsion-free connection in the setting of Proposition 5.1.1 is locally induced by a Riemannian metric. More precisely, the aim of this note is to prove the following fact:

Theorem 5.1.3. *Let G be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$ and let P be a G-structure on M. Then, there is a reduction Q of P to $G \cap \mathrm{CO}_n(\mathbb{R})$ and a torsion-free connection on Q such that the connection induced on the extension of Q to $\mathrm{CO}_n(\mathbb{R})$ is a closed Weyl structure.*

5.2 Proof of Theorem 5.1.3

We recall that the conformal group $\mathrm{CO}_n(\mathbb{R})$ is the group of all matrices λS for $(\lambda, S) \in \mathbb{R}^* \times \mathrm{O}_n(\mathbb{R})$. The proof of Theorem 5.1.3 relies on the classification of the subgroups of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$.

In all this text, we will denote by $\mathrm{Diag}(a_1, \dots, a_n)$ the diagonal matrix with diagonal coefficients $a_1, \dots, a_n$. We first show the maximality of $\mathrm{SO}_n(\mathbb{R})$ in $\mathrm{SL}_n(\mathbb{R})$.

Lemma 5.2.1. *Let G be a subgroup of $\mathrm{SL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$. Then, $G = \mathrm{SL}_n(\mathbb{R})$ or $G = \mathrm{SO}_n(\mathbb{R})$.*

Proof. For $n = 1$ there is nothing to prove. For $n = 2$, suppose that there exists $A \in G \setminus \mathrm{SO}_2(\mathbb{R})$. Using the polar decomposition and the spectral theorem, one can assume that $A = \mathrm{Diag}(a, \frac{1}{a})$ with $a > 1$. For $\theta \in \mathbb{R}$ let R_θ be the rotation of angle θ. Let ψ be the map which associates to an element of $\mathrm{SL}_n(\mathbb{R})$ the largest eigenvalue of the symmetric part of its polar decomposition. This map is continuous and one has $\psi(AR_0A) = a^2$ and $\psi(AR_{\pi/2}A) = 1$. Thus, by the intermediate value theorem, for any $x \in [1, a^2]$, the matrix $\mathrm{Diag}(x, \frac{1}{x})$ is in G, and this is true for any $x > 1$ by induction, which gives the result.

Now, let $n \geq 3$. Using the polar decomposition and the spectral theorem again, it is enough to show that the group $\mathcal{D}$ of diagonal matrices with positive coefficients and determinant 1 is contained in G if $G \neq \mathrm{SO}_n(\mathbb{R})$. Suppose that there is $A \in G \setminus \mathrm{SO}_n(\mathbb{R})$. We can assume that A is diagonal using the polar decomposition, thus $A = \mathrm{Diag}(a_1, a_2, \dots, a_n)$. It is easy to see that $G \cap \mathcal{D}$ is stable by permutation of the diagonal coefficients. Thus, $G \cap \mathcal{D}$ contains the matrix $\mathrm{Diag}(a_1/a_2, a_2/a_1, 1, \dots, 1)$, and by the case $n = 2$, we know that all the matrices $\mathrm{Diag}(u, u^{-1}, 1, \dots, 1)$ with $u > 0$ are in $G \cap D$, and so are the matrices of the form $\mathrm{Diag}(1, \dots, 1, u, u^{-1}, 1, \dots, 1)$. Since all the elements of $\mathcal{D}$ are products of such elements, this concludes the proof. □

Lemma 5.2.2. *Let G be a subgroup of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$, and let $x \in \det(G)$. Then, $|x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \dots, 1) \in G$.*

Proof. Let $x \in \det(G)$. There is a matrix $A \in G$ such that $\det(A) = x$, and using the polar decomposition of A, there is a diagonal matrix $D \in G$ with $\det(D) = x$. If D is of the form $|x|^{\frac{1}{n}}\mathrm{Diag}(\pm 1, \dots, \pm 1)$ we have the conclusion of the lemma after multiplying by an element of $\mathrm{SO}_n(\mathbb{R})$ of the form $\mathrm{Diag}(\pm 1, \dots, \pm 1)$, so we assume that $D^2 \notin \mathrm{Span}(I_n)$. There is a matrix $S \in \mathrm{SO}_n(\mathbb{R})$ with $SD^2 \neq D^2 S$. Let $B := D^{-1}S^T DS \in \mathrm{SL}_n(\mathbb{R})$. One has

$$BB^T = D^{-1}S^T DSS^T DSD^{-1} = D^{-1}S^T D^2 SD^{-1} = (D^{-1}SD)^{-1}(DSD^{-1}),$$

then

$$BB^T = I_n \Leftrightarrow D^{-1}SD = DSD^{-1} \Leftrightarrow D^2 S = SD^2,$$

and this last assertion is false, thus $BB^T \neq I_n$ and $B \notin \mathrm{SO}_n(\mathbb{R})$. By Lemma 5.2.1, we conclude that $G \cap \mathrm{SL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})$, and in particular $|x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \dots, 1)D^{-1} \in G$, so $|x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \dots, 1) \in G$ after multiplication by D on the right. □

One write $\mathrm{GL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R}) \rtimes \mathbb{R}^*$ with the identification $\{\mathrm{Id}\} \rtimes \mathbb{R}^* \to \mathrm{GL}_n(\mathbb{R}), x \mapsto |x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \dots, 1)$. We finally give the classification result:

Proposition 5.2.3. *Let G be a subgroup of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$. There exists a subgroup H of $(\mathbb{R}^*, \times)$ such that G is equal to either $\mathrm{SO}_n(\mathbb{R}) \rtimes H$ or $\mathrm{SL}_n(\mathbb{R}) \rtimes H$. Moreover, if G is closed, so is H.*

Proof. One has the following short exact sequence:

$$0 \to \mathrm{SL}_n(\mathbb{R}) \cap G \to G \overset{\det}{\to} \det(G) \to 1.$$

Now, let $\phi : H := \det(G) \to G$ given by $\phi(x) = |x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \dots, 1)$, which is well-defined by Proposition 5.2.2. It is clear that ϕ is a morphism and $\det \circ \phi = id_H$, thus one has $G = (\mathrm{SL}_n(\mathbb{R}) \cap G) \rtimes H$. Moreover, by Lemma 5.2.1 one has $\mathrm{SL}_n(\mathbb{R}) \cap G = \mathrm{SL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{R}) \cap G = \mathrm{SO}_n(\mathbb{R})$ because G contains $\mathrm{SO}_n(\mathbb{R})$.

It remains to show that H is closed when G is closed. But if H is non-discrete, $H \cap \mathbb{R}^*_+$ has to be dense in $\mathbb{R}^*_+$, so, G being closed, it contains all the matrices of the form $|x|^{\frac{1}{n}} I_n$, $x \in \mathbb{R}$, and then $H = \det G = \mathbb{R}^*_+$ or $\mathbb{R}^*$. $\square$

Remark 5.2.4. Note that in Proposition 5.2.3, the semi-direct product is actually direct when $H \subset \mathbb{R}^*_+$.

Proof of Theorem 5.1.3. According to Lemma 5.2.3, there is a closed subgroup H of $\mathbb{R}^*$ such that $G \simeq \mathrm{SO}_n(\mathbb{R}) \rtimes H$ or $\mathrm{SL}_n(\mathbb{R}) \rtimes H$. From the classification of the subgroups of $\mathbb{R}^*$, H is either $\mathbb{R}^*$, $\mathbb{R}^*_+$ or discrete.

First case: $H = \mathbb{R}^*$ or $H = \mathbb{R}^*_+$. In this case, G is either $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{CO}_n(\mathbb{R})$ or $\mathrm{GL}^+_n(\mathbb{R})$ or $\mathrm{CO}^+_n(\mathbb{R})$. In all these cases, there is a metric g compatible with the G-structure, i.e. a reduction P' of P to $G \cap \mathrm{O}_n(\mathbb{R})$. Then, the Levi-Civita connection of g is torsion-free, so it induces a torsion-free connection on P', and thus a torsion-free connection on the extension Q of P' to $G \cap \mathrm{CO}_n(\mathbb{R})$. The resulting connection on the extension of Q to $\mathrm{CO}_n(\mathbb{R})$ is a closed (actually exact) Weyl structure because it is induced by the Levi-Civita connection of a metric on M.

Second case: H is discrete. Let $\widetilde{M}$ be the universal cover of M and let $\widetilde{P}$ be the pull-back of P to $\widetilde{M}$.

We first study the case $G = \mathrm{SO}_n(\mathbb{R}) \rtimes H$. Then, the H-principal bundle $\widetilde{P}/\mathrm{SO}_n(\mathbb{R})$ is a covering of $\widetilde{M}$ so it is trivial. Every element $a \in H$ thus defines an $\mathrm{SO}_n(\mathbb{R})$-structure on $\widetilde{M}$ i.e. a metric $\widetilde{g}$. Since $\pi_1(M)$ acts on $P/\mathrm{SO}_n(\mathbb{R})$ by multiplication by an element of H, we deduce that $\pi_1(M)$ acts by similarities on $(\widetilde{M}, \widetilde{g})$. Consequently, the Levi-Civita connection of $\widetilde{g}$ induces a torsion-free connection on $\widetilde{P}$ which descends to a torsion-free connection on P. We can take $Q := P$ in the statement of the theorem since $G \subset \mathrm{CO}_n(\mathbb{R})$. Finally, the resulting connection on the extension of P to $\mathrm{CO}_n(\mathbb{R})$ is a closed Weyl structure because it is locally given by the Levi-Civita covariant derivative of a Riemannian metric defined by a local reduction of P to $G \cap \mathrm{O}_n(\mathbb{R})$.

We consider now the case $G = \mathrm{SL}_n(\mathbb{R}) \rtimes H$. Just as before, the H-principal bundle $\widetilde{P}/\mathrm{SL}_n(\mathbb{R})$ is trivial. Choosing an element $a \in H$ defines a $\mathrm{SL}_n(\mathbb{R})$-structure $\widetilde{Q}$ on $\widetilde{M}$ i.e. a volume form $\widetilde{v}$, and in particular an orientation on $\widetilde{M}$. Let h be a Riemannian metric on M, and let $\widetilde{h}$ be its pull-back to $\widetilde{M}$. Let v_h be the volume with respect to $\widetilde{v}$ of a $\widetilde{h}$-orthonormal frame of $T\widetilde{M}$ (note that v_h^2 does not depend on the choice of the frame). We define $\widetilde{g} := (v_h^2)^{\frac{1}{n}} \widetilde{h}$. Then, any oriented $\widetilde{g}$-orthonormal frame has volume 1 with respect to $\widetilde{v}$. This implies that $\widetilde{g}$ defines a reduction of $\widetilde{Q}$ to $\mathrm{SO}_n(\mathbb{R})$. As in the previous case, $\pi_1(M)$ acts on $P/\mathrm{SL}_n(\mathbb{R})$ by multiplication by an element of H, so for $\gamma \in \pi_1(M)$, $\gamma^*\widetilde{v}$ is a multiple of $\widetilde{v}$. Since, $\pi_1(M)$ acts by isometries on $(\widetilde{M}, \widetilde{h})$, it acts by similarities on $(\widetilde{M}, \widetilde{g})$. We finally conclude in the same way as for the case $G = \mathrm{SO}_n(\mathbb{R}) \rtimes H$. $\square$

From the proof we see that the principal bundle Q defined in Theorem 5.1.3 has $\mathrm{SO}_n(\mathbb{R}) \rtimes H'$

as structure group, where H' is a discrete subgroup of $\mathbb{R}_+^*$ (just take $H' := \{1\}$ when $H = \mathbb{R}^*$ or $\mathbb{R}_+^*$, and $H' := H$ otherwise).

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ÉCOLE DOCTORALE
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Titre: Sur quelques problèmes d'analyse spectrale, géométrie spinorielle et géométrie conforme

Mots clés: opérateur de Dirac, modèle MIT Bag, géométrie spinorielle, spineurs de Cauchy, structures de Weyl

Résumé: Cette thèse se divise en deux grandes parties. Dans la première, on s'intéresse à deux problèmes d'analyse spectrale portant sur la convergence des valeurs propres d'opérateurs à paramètres. D'une part, on considère l'opérateur de Schrödinger dans le plan, avec un potentiel singulier supporté par une courbe fermée Γ admettant un point de rebroussement. Ce potentiel s'écrit formellement $-\alpha\delta(x-\Gamma)$, et l'on décrit le comportement du spectre de l'opérateur dans la limite $\alpha \to +\infty$. D'autre part, on étudie l'opérateur de Dirac qui apparaît dans le modèle MIT Bag, en le généralisant aux variétés spin. Lorsque le paramètre de masse de cet opérateur tend vers l'infini, on observe une convergence des valeurs propres. Dans la seconde partie, on discute différents problèmes de géométrie. On démontre tout d'abord des résultats de structure et de classification en dimension 3 pour une classe particulière de spineurs, appelés spineurs de Cauchy, qui apparaissent naturellement comme restrictions de spineurs parallèles à des hypersurfaces orientées de variétés spin. Enfin, on s'intéresse aux connexions de Weyl sur les variétés conformes. On définit les structures localement conformément produits (LCP) par la donnée d'une structure de Weyl fermée, non-exacte, non-plate et à holonomie réductible sur une variété conforme compacte. On analyse les variétés LCP afin d'initier une classification.

Title: On some problems in spectral analysis, spin geometry and conformal geometry

Keywords: Dirac operator, MIT Bag model, spin geometry, Cauchy spinors, Weyl structures

Abstract: This thesis is divided into two main parts. In the first one, we focus on two problems of spectral analysis concerning the convergence of eigenvalues of operators with parameters. On the one hand, we consider the Schrödinger operator in the plane, with a singular potential supported by a closed curve Γ admitting a cusp. This potential is formally written $-\alpha\delta(x-\Gamma)$, and we describe the behaviour of the spectrum of the operator as $\alpha \to +\infty$. On the other hand, we study the Dirac operator which appears in the MIT Bag model, by generalizing it from Euclidean spaces to spin manifolds. We observe a convergence of the eigenvalues of this operator when the mass parameter tends to infinity. In the second part, we discuss two different geometric problems. First, we prove structure and classification results in dimension 3 for a particular class of spinors, called Cauchy spinors, arising as restrictions of parallel spinors to oriented hypersurfaces of spin manifolds. Finally, we focus on Weyl connections on conformal manifolds. We define a locally conformally product (LCP) structure as a closed, non-exact, non-flat Weyl structure with reducible holonomy on a compact conformal manifold. We analyse the LCP manifolds in order to initiate a classification.