THE CHARACTERISTIC GROUP OF LOCALLY CONFORMALLY PRODUCT STRUCTURES

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ABSTRACT. A compact manifold M together with a Riemannian metric h on its universal cover $\tilde{M}$ for which $\pi_1(M)$ acts by similarities is called a similarity structure. In the case where $\pi_1(M) \not\subset \mathrm{Isom}(\tilde{M}, h)$ and $(\tilde{M}, h)$ is reducible but not flat, this is a Locally Conformally Product (LCP) structure. The so-called characteristic group of these manifolds, which is a connected abelian Lie group, is the key to understand how they are built. We focus in this paper on the case where this group is simply connected, and give a description of the corresponding LCP structures. It appears that they are quotients of trivial $\mathbb{R}^p$-principal bundles over simply-connected manifolds by certain discrete subgroups of automorphisms. We prove that, conversely, it is always possible to endow such quotients with an LCP structure.

1. INTRODUCTION

A *similarity structure* on a compact manifold M is the data of a Riemannian metric h on its universal cover $\tilde{M}$ for which the deck-transformations are similarities (also called homotheties). Similarity structures can be divided in two families: those coming from lifts of Riemannian metrics on M to $\tilde{M}$, for which the deck-transformations are isometries, and those where at least one of these transformations is a strict similarity with ratio less than 1. In the first case, the structure is said to be Riemannian and its study belongs to Riemannian geometry. The second case, in which we will be interested in this article, is a pure subject of conformal geometry. We then restrict ourselves to the non-Riemannian case.

Similarity structures are quite rigid from the point of view of the holonomy group of $(\tilde{M}, h)$, and until very recently only flat and irreducible examples were known. This observation together with a previous result from Gallot [6] on Riemannian cones led Belgun and Moroianu [3] to formulate the conjecture that those where the only possibilities, a statement that they proved under an additional assumption on the existing-time of geodesics. However, it appeared that a third case can occur, closing the list of possible holonomies as shown by Matveev and Nikolayevsky [10, 11] in the analytic case and by Kourganoff [8] in the smooth case. The last possible family actually contains manifolds for whose $(\tilde{M}, h)$ is reducible and admits a de Rham decomposition with two factors. More precisely, $(\tilde{M}, h) = \mathbb{R}^q \times (N, g_N)$ where $\mathbb{R}^q$ is an Euclidean space and (N, g_N) is irreducible and incomplete. The manifolds belonging to this family are called *Locally Conformally Product* manifolds (which will be shorten to LCP in the sequel). LCP manifolds are also characterized as the ones whose similarity structure has reducible holonomy and is a Riemannian product with one of the factors being a complete Euclidean space [4], thanks to a classification of flat similarity structures by Fried [5].

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One can give a different approach of LCP manifolds using conformal geometry, which is equivalent to the previous exposition. Indeed, a non-Riemannian similarity structure can be defined by endowing the compact manifold M with a conformal structure c and a connection D on M preserving the conformal class, such that D is always locally but not globally the Levi-Civita connection of a metric in c. The lifted connection $\tilde{D}$ on $\tilde{M}$ is then globally the Levi-Civita of a metric h in the lifted conformal class $\tilde{c}$, uniquely defined up to a multiplication by a constant. This point of view enlighten some similarities with the intensively studied Locally Conformally Kähler (LCK) manifolds. For this reason, one of the first known examples of LCP manifolds was a subfamily of the OT-manifolds [13], where the algebraic number field used for the construction has exactly 2 complex embeddings. In this case, one can define a Kähler potential on the universal cover of the manifold inducing a non-Riemannian similarity structure with reducible non-flat holonomy. This example was further studied in [4] where it was shown that all the OT-manifolds admit LCP structures.

The OT-manifolds are moreover the only known examples of LCP manifolds admitting a compatible LCK structure. However, they do not admit a Kähler metric belonging to the induced conformal class. It has been proved in [2] that the conformal class of an LCP manifold does not contain a Kähler or an Einstein metric, illustrating again the strong rigidity of these structures.

One can then observe that for a compact manifold to admit a non-Riemannian similarity structure already imposes important constraints, but the LCP manifolds are even more restrictive, and only few examples were given until a previous work of the author [4]. It seems that, despite the variety of examples that one can construct, we always need the same basic ingredients, giving hope for the possibility of finding a general construction pattern providing a good understanding of these structures. In particular, all examples come from solvmanifolds, a track which was followed by Andrada, del Braco and Moroianu [1] in order to describe the LCP manifolds arising from solvable unimodular Lie algebras up to dimension 5. Our goal in this paper is to continue investigating, in order to obtain a general construction. For this, we will start from objects introduced in previous works and characterizing LCP structures.

In order to prove that the number of possibilities for the holonomy of $(\tilde{M}, h)$ is limited, Kourganoff [8] used an important tool on LCP manifolds, which is the restriction P of the fundamental group $\pi_1(M)$ to the non-flat part N. The closure $\bar{P}$ of this restriction is a subgroup of the group of similarities of (N, g_N), and its identity component $\bar{P}^0$ is an abelian Lie group containing only isometries. The group $\mathbb{R}^q \times \bar{P}^0$, where $\mathbb{R}^q$ has to be understood as the translations of the Euclidean space, is called the *characteristic group*. It consists of isometric transformations of $(\tilde{M}, h)$ and encodes important information on the LCP structure. In particular it helps understanding the action of the fundamental group. For instance, using the action of $\pi_1(M)$ by conjugation on the characteristic group, it was shown in [4] that the similarity ratios of elements of $\pi_1(M)$ are units of an algebraic number field.

The characteristic group being a connected abelian Lie group, it is a product between an Euclidean space and a torus. However, all examples provided so far have simply connected characteristic group, suggesting the significance of understanding this particular case. In this article, we then focus on giving a description of LCP manifolds with simply connected characteristic groups, using the remarkable fact that this group then acts freely and properly on the manifold $\tilde{M}$, thus implying the existence of a new decomposition of $\tilde{M}$ as a product

$\mathbb{R}^p \times C$ with $\mathbb{R}^q \subset \mathbb{R}^p$, no longer adapted to the metric, but allowing a nice understanding of the action of $\pi_1(M)$.

A glance at the known examples lets us expect that the deck-transformations admit the following description: there exists a discrete group H acting freely, properly and co-compactly on the factor C and a subgroup Ω of $\mathrm{GL}_p(\mathbb{Z}) \times H$ such that

$$\pi_1(M) = \mathbb{Z}^p \rtimes \Omega, \tag{1}$$

where $\mathbb{Z}^p$ acts on $\mathbb{R}^p$ by translations. However, the situation appears to be less ideal, and the group Ω can take values in the automorphisms of the trivial $(S^1)^p$-principal bundle over C, lifted to $\mathbb{R}^p \times C$. The group $\pi_1(M)$ is then not always a semi-direct product. Moreover, the action of the group Ω on C is not necessarily free, turning C/Ω into a good compact orbifold rather than a manifold (see Theorem 4.3). It is then natural to ask if, given a good compact orbifold together with a suitable lift of its fundamental group to the automorphisms of a trivial torus bundle over its universal cover, one can in turn construct an LCP manifold. The answer is positive, as we show in Theorem 4.9. In this process, we also describe all the possible LCP structures by providing a way to construct the metrics for which the deck-transformations act by similarities. In the last section, we discuss new examples and open questions, showing why some results we could conjecture are actually false. We also give a necessary and sufficient condition for the construction of an LCP structure when C/Ω is a manifold (see Proposition 5.7).

2. Preliminaries

Let M be a manifold endowed with a conformal structure, i.e. a set c of Riemannian metrics such that for all $g, g' \in c$, there exists $f : M \to \mathbb{R}$ satisfying $g = e^{2f} g'$.

A Weyl connection on the conformal manifold (M, c) is a a torsion-free connection D such that D preserves c in the sense that for any $g \in c$, there exists a 1-form $\theta_g \in \Omega^1(M)$ with

$$Dg = -2\theta_g \otimes g. \tag{2}$$

In this case, θ_g is called the Lee form of D with respect to g.

If there is $g \in c$ such that θ_g is closed, then $\theta_{g'}$ is closed for any metric $g' \in c$ and this is equivalent to D being locally the Levi-Civita connection of a metric in c, which means that for any $x \in M$ there is an open set $U \subset M$ with $x \in U$ and a metric $g \in c$ such that the Levi-Civita connection of g coincides with D on U. The Weyl connection D is then said to be *closed*. This statement holds globally if and only if there is a metric $g \in c$ such that θ_g is exact and in this case, D is said to be *exact*.

We recall that a similarity (also called a homothety) between two Riemannian manifolds (M_1, g_1), (M_2, g_2) is a diffeomorphism $\varphi : M_1 \to M_2$ satisfying

$$\varphi^* g_2 = \lambda^2 g_1 \tag{3}$$

for some $\lambda > 0$ called *the similarity ratio of* φ. A similarity structure on M, as defined in [8], is a metric h on the universal cover $\tilde{M}$ such that $\pi_1(M)$ acts by similarities on (M, h). When $\pi_1(M)$ acts only by isometries, this similarity structure is called Riemannian. If the Weyl connection D is closed, its lift $\tilde{D}$ to the universal cover $\tilde{M}$ of M together with the lifted conformal class $\tilde{c}$ is an exact Weyl connection, and there exists a metric $h_D \in \tilde{c}$, unique up to multiplication by a constant, such that $\tilde{D}$ is the Levi-Civita connection of h_D. The fundamental group $\pi_1(M)$ acts on $(\tilde{M}, h_D)$ by similarities, defining a similarity structure on M. As it was discussed in [4], there is a one-to-one correspondence between similarity structures up to constant multiplication and closed Weyl structure on M. Through this identification, Riemannian similarity structures correspond to exact Weyl structures.

From now on, we consider a compact conformal manifold (M, c) endowed with a closed, non-exact Weyl connection D, which is non-flat and has reducible holonomy. The structure (M, c, D) is called a Locally Conformally Product structure (or LCP for short) [4]. A theorem due to Kourganoff [8, Theorem 1.5] allows one to understand the structure of these LCP manifolds by looking at the Riemannian metric h_D induced on $\tilde{M}$ by D.

Theorem 2.1 (Kourganoff). *Let D be a closed, non-exact Weyl structure on a compact conformal manifold (M, c). Assume moreover that D is non-flat and has reducible holonomy. Then, there exists $q \geq 1$ and an irreducible incomplete Riemannian manifold (N, g_N) such that the universal cover $\tilde{M}$ of M endowed with the metric h_D induced by D is isometric to the Riemannian product $\mathbb{R}^q \times (N, g_N)$.*

It was proved in [4] that (M, c, D) is an LCP structure if and only if $(\tilde{M}, D)$ is reducible and has a Riemannian factor which is an Euclidean space. This fact will be often used in order to show that the examples we will give are indeed LCP manifolds.

According to Theorem 2.1, one has $(\tilde{M}, h_D) \simeq \mathbb{R}^q \times (N, g_N)$, where $\mathbb{R}^q$ and (N, g_N) will respectively be called *the flat part* and the *non-flat part* of $\tilde{M}$. This can be reformulated by saying that $(\tilde{M}, h_D)$ has a de Rham decomposition with 2 factors, and since $\pi_1(M)$ preserves the connection $\tilde{D}$, it must preserves the de Rham decomposition up to the order of the factors, but it must also preserves the flat factor $\mathbb{R}^q$. Consequently, $\pi_1(M)$ preserves the de Rham decomposition, and any $\gamma \in \pi_1(M)$ can be written as $\gamma =: (\gamma_E, \gamma_N)$ where γ_E and γ_N are similarities of $\mathbb{R}^q$ and N respectively. We then define $P := \{\gamma_N \mid \gamma \in \pi_1(M)\}$, and we denote by $\bar{P}$ the closure of P in $\mathrm{Sim}(N, g_N)$ with respect to the compact-open topology.

The connected component of the identity in $\bar{P}$ is denoted by $\bar{P}^0$. It has been shown in [8, Lemma 4.1] that $\bar{P}^0$ is an abelian Lie group, satisfying $\bar{P}^0 \subset \mathrm{Iso}(N, g_N)$. In particular, it is the product of a real vector space and a torus.

The decomposition $\tilde{M} = \mathbb{R}^q \times N$ induces a natural foliation $\tilde{\mathfrak{F}}$ on $\tilde{M}$ via the submersion $\tilde{M} \to N$, whose leaves are the sets $\tilde{\mathfrak{F}}_x := \mathbb{R}^q \times \{x\}$ for $x \in N$. In turn, the foliation $\tilde{\mathfrak{F}}$ induces a foliation $\mathfrak{F}$ on the compact manifold M. The closures of the leaves of $\mathfrak{F}$ define a singular Riemannian foliation $\bar{\mathfrak{F}}$ on M [8, Theorem 1.9], and if one denotes by $\pi_M : \tilde{M} \to M$ the canonical projection, the leaves of $\bar{\mathfrak{F}}$ are exactly the sets $\pi_M(\mathbb{R}^q \times \bar{P}^0 x)$ for $x \in N$ [8, Lemma 4.11]. In view of this last property we define for $x \in N$ the set $\tilde{\mathfrak{CF}}_x := \mathbb{R}^q \times \bar{P}^0 x$, so that $\tilde{\mathfrak{CF}} = \{\tilde{\mathfrak{CF}}_x \mid x \in N\}$ is a singular foliation of $\tilde{M}$. Since $\bar{P}^0$ is an abelian Lie group which acts by isometries on (N, g_N), the Riemannian manifolds $\mathbb{R}^q \times \bar{P}^0 x$, $x \in N$, with the metric induced by g_N are products of an Euclidean space with a flat torus, as it was noticed in [8] and [4], however their dimensions depend on x a priori.

We call a *lattice* of an abelian Lie group G any discrete subgroup H of G. If G/H is compact, then H is called a *full lattice*. It was shown in [8, Lemma 4.18] that the group

$$\Gamma_0 := \pi_1(M) \cap (\mathrm{Sim}(\mathbb{R}^q) \times \bar{P}^0) \tag{4}$$

is a full lattice of $\mathbb{R}^q \times \bar{P}^0$. In particular, it is an abelian subgroup of $\mathrm{Isom}(\tilde{M}, h_D)$. In addition, it was shown in [4, Lemma 2.10], by adapting the incorrect proof of [8, Lemma 4.17], that P

is isomorphic to $\pi_1(M)$. Since $\bar{P}^0$ is a normal subgroup of $\bar{P}$, being its identity component, Γ_0 is a normal subgroup of $\pi_1(M)$ by definition.

We will study in this article the case where $\bar{P}^0 \simeq \mathbb{R}^{p-q}$ for some $p \geq q$, in order to give a precise description of these manifolds.

3. General results

3.1. **Properties of some group actions on LCP manifolds.** Let (M, c, D) be an LCP manifold. We keep the notations of the preliminary section. The group Γ_0 is a finitely generated abelian group since it is a lattice of $\mathbb{R}^q \times \bar{P}^0$, thus $\Gamma_0 = \Gamma_0^{tor} \oplus \mathbb{Z}^p$ where Γ_0^{tor} is the torsion subgroup of Γ_0 and $p \geq q$. Our first goal in this section is to give a special decomposition of the universal cover $\tilde{M}$ of M. For this purpose, we consider a representative Γ_0^F of $\Gamma_0/\Gamma_0^{tor} \simeq \mathbb{Z}^p$ in Γ_0 with basis $(\gamma_1, \ldots, \gamma_p)$. We denote by "exp" the exponential map of the Lie group $\mathbb{R}^q \times \bar{P}^0$, and we consider $(X_1, \ldots, X_p) \in (T_e(\mathbb{R}^q \times \bar{P}^0))^p$, where e is the neutral element, such that $\gamma_i = \exp(X_i)$ for any $1 \leq i \leq p$. The subgroup $\exp^{-1}(\Gamma_0)$ is a full lattice of $T_e(\mathbb{R}^q \times \bar{P}^0)$ and one easily sees that $\exp^{-1}(\Gamma_0) = \langle X_1, \ldots, X_p \rangle \oplus \exp^{-1}(\Gamma_0^{tor})$, thus $T_e(\mathbb{R}^q \times \bar{P}^0) = \mathrm{Span}(X_1, \ldots, X_p) \oplus \mathrm{Span}(\exp^{-1}(\Gamma_0^{tor}))$.

We define the subgroup F of $\mathbb{R}^q \times \bar{P}^0$ by

$$F := \exp(\mathrm{Span}(X_1, \ldots, X_p)). \tag{5}$$

We claim that $\exp : \mathrm{Span}(X_1, \ldots, X_p) \to F$ is an isomorphism. Indeed, it is sufficient to prove that the map is into. But for any $X \in \mathrm{Span}(X_1, \ldots, X_p)$ such that $\exp(X) = 0$, one has $X \in \exp^{-1}(\Gamma_0^{tor}) \cap \mathrm{Span}(X_1, \ldots, X_n) = \{0\}$, so $X = 0$ and we deduce the injectivity.

In addition, F is a closed subgroup. Indeed, if we write $\mathbb{R}^q \times \bar{P}^0$ as the product $\mathbb{R}^p \times (S^1)^m$ of an Euclidean space and a torus, the projection of the basis $(\gamma_1, \ldots, \gamma_p)$ onto $\mathbb{R}^p$ is a basis of $\mathbb{R}^p$ because otherwise Γ_0 would not be a full lattice of $\mathbb{R}^q \times \bar{P}^0$. From the point of view of the Lie algebra, it implies that the projection of $(X_1, \ldots, X_p)$ onto the Lie algebra of $\mathbb{R}^p$ is a basis. We then easily define a continuous bijection between $\mathbb{R}^p$ and $\mathrm{Span}(X_1, \ldots, X_p) \simeq F$, proving that $\mathbb{R}^p \simeq F$ and that F is closed.

The group F thus represents the non-compact part of the group $\mathbb{R}^q \times \bar{P}^0$. In order to show that F acts freely on $\tilde{M}$, we first prove two technical lemmata:

Lemma 3.1. *Let $\Gamma \subset \Gamma'$ be two full lattices of $\mathbb{R}^p$. Then, for any $\gamma \in \Gamma'$ there exists $r \geq 1$ such that $\gamma^r \in \Gamma$.*

Proof. The space $\mathbb{R}^p/\Gamma$ is a covering of $\mathbb{R}^p/\Gamma'$ with fiber Γ'/Γ. This is a finite covering because both spaces are compact, thus Γ'/Γ is finite and each one of its elements has finite order, proving the lemma. □

Lemma 3.2. *Let $\mathcal{M}$ be a smooth manifold on which acts the group $\mathbb{R}^p$. Assume that a full lattice of $\mathbb{R}^p$ acts freely and properly on $\mathcal{M}$. Then, $\mathbb{R}^p$ acts freely on $\mathcal{M}$.*

Proof. We denote by Γ the full lattice of the statement. Let $x \in \mathcal{M}$ and consider the set $\mathbb{R}^p \cdot x$. Let $S(x) = \{a \in \mathbb{R}^p \mid a \cdot x = x\}$ be the stabilizer of x in $\mathbb{R}^p$. Then one has $\mathbb{R}^p \cdot x \simeq \mathbb{R}^p/S(x)$. We want to prove that $S(x)$ only contains the identity.

By definition, the free abelian group Γ is a full lattice of $\mathbb{R}^p$ and Γ acts freely and properly on $\mathcal{M}$. Since Γ stabilizes $\mathbb{R}^p \cdot x$, one has that Γ acts freely and properly on $\mathbb{R}^p \cdot x$, so it is a full lattice of the abelian group $\mathbb{R}^p/S(x)$. A lattice of an abelian lie group has a rank lower than the dimension of the group, so $p \leq \dim(\mathbb{R}^p/S(x))$, and moreover $\dim(\mathbb{R}^p/S(x)) \leq p$ because $\dim(\mathbb{R}^p) = p$, thus $\dim(\mathbb{R}^p/S(x)) = p$ and $S(x)$ is a discrete subgroup of $\mathbb{R}^p$.

One has $(\mathbb{R}^p \cdot x)/\Gamma \simeq \mathbb{R}^p/\langle S(x), \Gamma\rangle$, where $\langle S(x), \Gamma\rangle$ is the group generated by $S(x)$ and Γ. In particular $\langle S(x), \Gamma\rangle$ is a full lattice of $\mathbb{R}^p$.

We now pick $a \in S(x)$. By Lemma 3.1 applied to the full lattices $\Gamma \subset \langle S(x), \Gamma\rangle$, there exists $r \geq 1$ such that $a^r \in \Gamma$. Since $a^r \cdot x = x$ and Γ acts freely, one has that $a^r = \mathrm{id}$. In addition, $S(x)$ has no torsion because it is a subgroup of $\mathbb{R}^p$ thus $a = \mathrm{id}$ and $S(x) = \{\mathrm{id}\}$. □

Corollary 3.3. *The group F acts freely on $\tilde{M}$.*

Proof. We apply Lemma 3.2 to the action of $F \simeq \mathbb{R}^p$ on $\tilde{M}$, knowing that the full lattice Γ_0^F of F acts freely and properly on $\tilde{M}$ because $\Gamma_0^F \subset \Gamma_0 \subset \pi_1(M)$. □

Corollary 3.4. *If $\bar{P}^0$ is simply connected, then $\mathbb{R}^q \times \bar{P}^0$ acts freely on $\tilde{M}$. In particular, $\bar{P}^0$ acts freely on N.*

Proof. If $\bar{P}^0$ is simply connected, the group F defined in equation 5 is equal to the whole group $\mathbb{R}^q \times \bar{P}^0$, and by Corollary 3.3 it acts freely on $\tilde{M}$. □

We now discuss a problem tackled in [8, Lemma 4.16], whose proof was shown to be incorrect. We would like to understand the action of the group P on N. As we explained before, it is known that P is isomorphic to $\pi_1(M)$. However, we don't know if the action of P is free. We give here a first result which states that this is true if and only if the restricted action of Γ_0 is free.

Proposition 3.5. *The group P acts freely on N if and only if the restriction of Γ_0 to N, which coincides with $P \cap \bar{P}^0$, acts freely.*

Proof. If P acts freely on N, then it is easily seen that the restriction of Γ_0 to N acts freely. It remains to prove the converse.

Assume that there is $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ and $x \in N$ such that $\gamma_N(x) = x$ (we recall that γ_N is the part of γ acting on N). The transformation γ stabilizes the closed leaf $\tilde{\mathcal{CF}}_x$. Since $\pi_1(M)$ acts freely and properly discontinuously on $\tilde{M}$, the group $\pi_1(M)/\Gamma_0$ acts freely and properly discontinuously on $\tilde{M}/\Gamma_0$ (see [4, Proposition 4.1] for example). We denote by $\bar{\gamma}$ the equivalence class of γ in $\pi_1(M)/\Gamma_0$. As $\bar{\gamma}$ stabilizes the compact set $\tilde{\mathcal{CF}}_x/\Gamma_0$ (which is compact because Γ_0 is a full lattice of $\mathbb{R}^q \times \bar{P}^0$ and $\tilde{\mathcal{CF}}_x = \mathbb{R}^q \times \bar{P}^0 x$), the set $\{\bar{\gamma}^m(0, x) \mid m \in \mathbb{N}\}$ is finite because $\pi_1(M)/\Gamma_0$ acts properly on $\tilde{M}/\Gamma_0$. Thus there exists $m \in \mathbb{N} \setminus \{0\}$ such that $\bar{\gamma}^m(0, x) = (0, x)$, but $\pi_1(M)/\Gamma_0$ acts freely, so $\bar{\gamma}^m = \mathrm{id}$ and $\gamma^m \in \Gamma_0$.

The restriction γ_E of γ to the flat part $\mathbb{R}^q$ is of the form $\mathbb{R}^q \ni a \mapsto Aa+b$ where $A \in \mathrm{GL}_q(\mathbb{R})$ and $b \in \mathbb{R}^q$. Since γ has no fixed point and $\gamma_N(x) = x$, γ_E has no fixed point. One has that γ_E^m is a translation because $\gamma^m \in \Gamma_0$, so $A^m = I_q$, and considering the polynomial $X^m - 1 =: (X-1)R(X)$ one has $\mathbb{R}^q = \ker(A - I_q) \oplus \ker R(A) =: V_1 \oplus V_2$. According to this decomposition, $b =: b_1 + b_2$, and $(I_q - A)|_{V_2}$ being invertible there exists a unique $v_2 \in V_2$ such that $(I_q - A)v_2 = b_2$. One then has $\gamma_E(v_2) = v_2 + b_1$, implying that $b_1 \neq 0$. Hence $\gamma^m(v_2, x) = (v_2 + mb_1, x)$ giving that γ^m is a non-trivial element of Γ_0. Since $\pi_1(M)$ is isomorphic to P, $(\gamma^m)_N$ is a non-trivial element of P, which is the restriction of an element of Γ_0 to N and which has x as a fixed point. □

Corollary 3.6. *If $\bar{P}^0$ is simply connected, then P acts freely on N.*

Proof. Assume that $\bar{P}^0$ is simply connected. By Corollary 3.4, $\bar{P}^0$ acts freely on N. In particular, the restriction of Γ_0 to N, which is contained in $\bar{P}^0$, acts freely and Proposition 3.5 implies that P acts freely on N. □

The results of Corollaries 3.4 and 3.6 motivate the following definition:

Definition 3.7. *The group $\mathbb{R}^q \times \bar{P}^0$ will be called the* characteristic group *of the LCP manifold. The LCP structure (M, c, D) is said to be* simple *if its characteristic group is simply connected.*

Remark 3.8. The action of the group $\pi_1(M)$ on $\tilde{M}$ descends to an action of $\pi_1(M)/\Gamma_0$ on $N/\bar{P}^0 \simeq \tilde{M}/(\mathbb{R}^q \times \bar{P}^0)$ because $\mathbb{R}^q \times \bar{P}$ is stable under conjugation by elements of $\pi_1(M)$. By the proof of Proposition 3.5, we know that this action is proper. Indeed, if K is a compact subset of $N/\bar{P}^0$, the set $E := \{\bar{\gamma} \in \pi_1(M)/\Gamma_0 \mid (\bar{\gamma}K) \cap K \neq \emptyset\}$ is equal to $\{\bar{\gamma} \in \pi_1(M)/\Gamma_0 \mid (\bar{\gamma}K') \cap K' \neq \emptyset\}$ where K' is the inverse image of K by the projection $\tilde{M}/\Gamma_0 \to N/\bar{P}^0$, and K' is compact because $(\mathbb{R}^q \times \bar{P}^0)/\Gamma_0$ is compact, thus the set E is finite.

We finally prove a useful property, which can be used to identify the characteristic group in some special situations.

Proposition 3.9. *The image of $\mathbb{R}^q$ in the torus $(\mathbb{R}^q \times \bar{P}^0)/\Gamma_0$ is dense.*

Proof. Let $(a, p) \in \mathbb{R}^q \times \bar{P}^0$ and let U, V be neighborhoods of a and p respectively. Since $P \cap \bar{P}^0 = \Gamma_0|_N$ is dense in $\bar{P}^0$, there exists $\gamma_0 \in \Gamma_0$ such that $\gamma_0 \in \mathbb{R}^q \times V$, and then one can find $a' \in \mathbb{R}^q$ such that $a' + \gamma_0 \in U \times V$. This implies that $\langle \mathbb{R}^q, \Gamma_0 \rangle$ is dense in $\mathbb{R}^q \times \bar{P}^0$, and thus the image of $\mathbb{R}^q$ in $(\mathbb{R}^q \times \bar{P}^0)/\Gamma_0$ is dense. □

3.2. **Finite coverings of LCP manifolds.** In [8, Theorem 1.10], it was shown using Selberg's lemma that there exists a finite covering M' of M such that the closures of the leaves of the foliation $\mathcal{F}'$ induced by $\mathcal{F}$ on M' are flat tori. In particular, M' is still an LCP manifold. We recall here the key point of the proof for the convenience of the reader, and also because we will use it later.

Proposition 3.10. *Let (M, c, D) be an LCP structure. Then, up to a finite covering of M, the action of $\pi_1(M)$ on the characteristic group by conjugation has no torsion.*

Proof. The action of $\pi_1(M)$ on the characteristic group by conjugation is well-defined, because $\bar{P}^0$ is a normal subgroup of $\bar{P}$, and the restriction of $\pi_1(M)$ to $\mathbb{R}^q$ contains only similarities which are in particular affine maps of the form $\mathcal{A} := \mathbb{R}^q \ni a \mapsto Ra + t$ with $(R, t) \in \mathrm{GL}_q(\mathbb{R}) \times \mathbb{R}^q$. For any $b \in \mathbb{R}^q$, if we define the translation $\tau_b : \mathbb{R}^q \ni a \mapsto a + b$, one has $\mathcal{A}\tau_b\mathcal{A}^{-1} = (\mathbb{R}^q \ni a \mapsto a + Rb) =: \tau_{Rb}$.

We then define $J : \pi_1(M) \to \mathrm{Aut}(\mathbb{R}^q \times \bar{P}^0)$, $\gamma \mapsto (\gamma_0 \mapsto \gamma\gamma_0\gamma^{-1})$. For all $\gamma \in \pi_1(M)$, $J(\gamma)$ preserves Γ_0, which is a full-lattice of $\mathbb{R}^q \times \bar{P}^0$. This implies that $J(\gamma)$ descends to an automorphism of the torus $(\mathbb{R}^q \times \bar{P}^0)/\Gamma_0$, and $J(\gamma)$ defines a unique linear transformation $\tilde{J}(\gamma)$ on the universal cover $\mathbb{R}^p$ of $\mathbb{R}^q \times \bar{P}^0$, which preserves the full-lattice given by the lift of Γ_0. Up to a linear transformation, one can assume that this lattice is the canonical lattice $\mathbb{Z}^p$ of $\mathbb{R}^p$. It follows that $\tilde{J}(\gamma) \in \mathrm{GL}_p(\mathbb{R})$. The map $\tilde{J} : \gamma \mapsto \tilde{J}(\gamma)$ is then a group homomorphism, and by Selberg's lemma there exists a subgroup G of $\tilde{J}(\pi_1(M))$ with finite index and without torsion element. Thus $\tilde{J}^{-1}(G)$ is a subgroup of $\pi_1(M)$ of finite index, such that $J(\tilde{J}^{-1}(G))$ has no torsion element. $\square$

The question of wether this finite cover M' can always be taken to be M was raised, since this result is true when $\mathcal{F}$ is of dimension 1. Here we answer negatively by providing a counter-example.

Example 3.11. Let $\tilde{M} := \mathbb{R}^4 \times \mathbb{R}_+^*$. We consider the matrices

$$A := \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{pmatrix}, \qquad B := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The matrices A and B commutes so they can be diagonalized in a common basis (X_1, Y_1, X_2, Y_2) where $(X_1, Y_1) \in (\mathbb{R}^2 \times \{0\}^2)^2$ and $(X_2, Y_2) \in (\{0\}^2 \times \mathbb{R}^2)^2$. In this basis, A is written as $\mathrm{Diag}(\lambda, -\lambda^{-1}, \lambda, -\lambda^{-1})$ with $\lambda = 2 + \sqrt{5} > 1$ and B remains the same. We define a group G of transformations of $\tilde{M}$ by

$$G := \langle \mathbb{Z}^4, T_A : (a,t) \mapsto (Aa, \lambda t), T_B : (a,t) \mapsto (Ba + (0, 1/2, 0, 0)^T, t) \rangle, \tag{6}$$

where $\mathbb{Z}^4$ is the standard lattice acting on $\mathbb{R}^4$. Simple computations provide

$$T_A \circ T_B(a,t) = T_B \circ T_A(a,t) + ((1,1,0,0)^T, 0), \qquad T_B^2(a,t) = (a + (0,1,0,0)^T, t). \tag{7}$$

Let now $(a,t) \in \tilde{M}$ and $g \in G$ such that $g(a,t) = (a,t)$. According to the relations (7) one has that there exist $\delta \in \{0,1\}$, $n \in \mathbb{Z}$ and $Z \in \mathbb{Z}^4$ such that

$$g = Z \circ T_B^\delta \circ T_A^n, \tag{8}$$

implying

$$(a,t) = g(a,t) = (B^\delta A^n a + (0, \delta/2, 0, 0)^T + Z, \lambda^n t). \tag{9}$$

From the identity $t = \lambda^n t$ it follows that $n = 0$, and $B^\delta a + (0, \delta/2, 0, 0)^T + Z = a$ gives $Z = 0$ and $\delta = 0$, so we conclude that $g = \mathrm{id}$ and that G acts freely. In order to prove that G acts properly discontinuously, it is sufficient to notice that $\tilde{M}/\mathbb{Z}^4 \simeq (S^1)^4 \times \mathbb{R}_+^*$ and $G/\mathbb{Z}^4$ acts properly on $(S^1)^4 \times \mathbb{R}_+^*$ since the class of T_A acts by multiplication by λ on the coordinate t and T_B has order 2. Consequently, $M := \tilde{M}/G$ is a manifold, and it is compact since $G([0,1]^4 \times [1, \lambda]) = \tilde{M}$.

We consider the Riemannian metric h on $\tilde{M}$ given by

$$h := dx_1^2 + dx_2^2 + t^4(dy_1^2 + dy_2^2) + dt^2 \tag{10}$$

where (x_1, y_1, x_2, y_2) are the coordinates in the basis (X_1, Y_1, X_2, Y_2) of $\mathbb{R}^4$ and t is the parameter of the factor $\mathbb{R}_+^*$. For any $g \in G$, if we use the decomposition $g = Z \circ T_B^\delta \circ T_A^n$ given by (8), we have $g^* h := \lambda^{2n} h$, meaning that G acts on $\tilde{M}$ by similarities which are not all isometries. The Levi-Civita connection of h thus descends to a connection D on M while h induces a conformal class c on M. The triple (M, c, D) is an LCP structure. From the point of view of the universal cover $\tilde{M}$, the flat part of this LCP structure is identified to the subspace of $\mathbb{R}^4$ given by $\mathrm{Span}(X_1, X_2)$, and its non-flat part is identified to the manifold $\mathrm{Span}(Y_1, Y_2) \times \mathbb{R}_+^*$.

It is easily seen that the group $\bar{P}^0$ in this case consists of all the translations lying in $\mathrm{Span}(Y_1, Y_2)$, and thus for any (y_1, y_2, t) in the non-flat part, one has $\tilde{\mathcal{CF}}_{(y_1, y_2, t)} = \mathbb{R}^4 \times \{t\}$.

We deduce that the closed leaf $\pi_M(\tilde{\mathcal{CF}}_{(y_1,y_2,t)})$ is diffeomorphic to $\mathbb{R}^4 \times \{t\}/S$ (see [4, Lemma 3.5] for additional details), with $S := \{g \in G \mid \forall (x_1, x_2) \in \mathbb{R}^2,\ g(x_1, x_2, y_1, y_2, t) \in \mathbb{R}^4 \times \{t\}\} = \langle \mathbb{Z}^4, T_B\rangle$, where the last equality comes from the decomposition (8). Since S acts freely and properly discontinuously on $\mathbb{R}^4 \times \{t\} \simeq \mathbb{R}^4$, one has $\pi_1(\pi_M(\tilde{\mathcal{CF}}_{(y_1,y_2,t)})) \simeq S$, but S is not abelian, so $\pi_M(\tilde{\mathcal{CF}}_{(y_1,y_2,t)})$ is not a torus.

We now come back to the finite cover M' of M introduced before. From the proof of [8, Theorem 1.10] already cited above, we have the following property: denoting by $\bar{P}'^0$ the equivalent of the group $\bar{P}^0$ for M' (since M' is also an LCP manifold), one has $\bar{P}'^0 = \bar{P}^0$ and the action of $\pi_1(M') \subset \pi_1(M)$ on $\mathbb{R}^q \times \bar{P}'^0$ by conjugation has no torsion. This motivates the following definition:

Definition 3.12. *We say that the LCP structure (M, c, D) is* torsion-free *if the action of $\pi_1(M)$ by conjugation on the characteristic group has no torsion.*

4. Simple LCP manifolds

In this section we give a description of simple LCP manifolds (i.e. with simply connected characteristic group). In a first part, we study the structure of such manifolds to derive the existence of necessary conditions, and we prove in a second part that any manifold satisfying these conditions can be endowed with an LCP structure.

4.1. **Analysis of the LCP structure.** Let (M, c, D) be an LCP manifold. We keep the notations of the preliminary section, and we assume that $\bar{P}^0 \simeq \mathbb{R}^{p-q}$ for some $p \geq q$, i.e. that the LCP structure is simple and its characteristic group is isomorphic to $\mathbb{R}^p$. By Corollary 3.4 this group acts freely on $\tilde{M}$. Moreover, we obtain the following result:

Proposition 4.1. *If (M, c, D) is simple, then for any $x \in N$, the Riemannian manifold $\tilde{\mathcal{CF}}_x = \mathbb{R}^q \times \bar{P}^0 x$ is isometric to the Euclidean space $\mathbb{R}^p$.*

Proof. Let $x \in N$. By Corollary 3.4, the group $\bar{P}^0$ acts freely on N, so $\tilde{\mathcal{CF}}_x = \mathbb{R}^q \times \bar{P}^0 x \simeq \mathbb{R}^q \times \bar{P}^0 \simeq \mathbb{R}^p$. Moreover, we recall that $\bar{P}^0$ is a subgroup of the isometries of (N, g_N), hence the metric h_D restricted to $\tilde{\mathcal{F}}_x$ is invariant by the group $\mathbb{R}^q \times \bar{P}^0 \simeq \mathbb{R}^p$, which implies that $\tilde{\mathcal{CF}}_x$ is an Euclidean space isometric to $\mathbb{R}^p$. □

Corollary 3.4 gives that the group $\bar{P}^0$ acts freely on N. It has been shown in [8, Lemma 4.9] that $\bar{P}$ acts properly on N, and so does $\bar{P}^0$ as a closed subgroup of $\bar{P}$. Consequently, by [9, Theorem 21.10] the quotient $C := N/\bar{P}^0$ is a smooth manifold. The canonical submersion $\pi_N : N \to C$ is a Riemannian submersion because $\bar{P}^0$ acts by isometries on $(\tilde{M}, h_D)$, and it defines a $\mathbb{R}^{p-q}$-principal bundle. Here, we do not need to specify whether the action of $\mathbb{R}^{p-q}$ is on the right or the left because the group $\mathbb{R}^{p-q}$ is abelian. In particular, the fibers of the principal bundle $\pi : \tilde{M} \to C$ are contractible and by [14, Corollary 29.3] this bundle admits a section, so it is trivial because it is a principal bundle. Thus we can write $N \simeq \mathbb{R}^{p-q} \times C$.

The metric g_N is invariant under the action of the group $\bar{P}^0 \simeq \mathbb{R}^{p-q}$ (which consists of isometries of g_N), thus it is a $\mathbb{R}^{p-q}$-invariant metric of the principal bundle $\mathbb{R}^{p-q} \times C \to C$.

We now study the action of $\pi_1(M)$. One has $\tilde{M} \simeq \mathbb{R}^q \times N \simeq \mathbb{R}^p \times C$ by the previous analysis, and since Γ_0 is a full-lattice of $\mathbb{R}^q \times \bar{P}^0 \simeq \mathbb{R}^p$ it induces a full-lattice on each fiber of the $\mathbb{R}^p$-principal bundle $\tilde{M} \to C = N/\bar{P}^0$. Let us fix a basis of Γ_0. Using this basis, we can now identify $\mathbb{R}^p$ with $\mathbb{R}^q \times \bar{P}^0$ in such a way that Γ_0 is the canonical lattice $\mathbb{Z}^p$ of $\mathbb{R}^p$. Under this identification, $\mathbb{R}^q$ and $\bar{P}^0$ are vector subspaces E^q and E^{p-q} of $\mathbb{R}^p$ satisfying $\mathbb{R}^p = E^q \oplus E^{p-q}$. We will from now on identify Γ_0 with the canonical full-lattice $\mathbb{Z}^p$ of $\mathbb{R}^p$ and $\tilde{M}$ with $\mathbb{R}^p \times C$.

For any $a \in \mathbb{R}^p$, we denote by τ_a the action of a on $\tilde{M}$. We recall that γ acts by conjugation on $\mathbb{R}^p$. Indeed, the action of γ_N by conjugation on $E^{p-q} \simeq \bar{P}^0$ is well-defined, because $\bar{P}^0$ is a normal subgroup of $\bar{P}$, so it is stable by the action of P by conjugation. This transformation is an automorphism of the group E^{p-q}, and in particular it is a linear invertible map of E^{p-q} viewed as a vector space. Moreover, γ_E acts on E^q as an affine transformation $E^{p-q} \ni v \mapsto R_\gamma v + t_\gamma$, where $(R_\gamma, t_\gamma) \in \mathrm{GL}(E^q) \times E^q$. Consequently, if for any $u \in E^q$ we denote by $t(u)$ the translation by u in E^q, one has $\gamma_E t(v) \gamma_E^{-1} = t(R_\gamma v)$. Altogether, there exists a matrix $A_\gamma \in \mathrm{GL}_p(\mathbb{R})$ such that for any $a \in \mathbb{R}^p$ one has $\gamma \tau_a \gamma^{-1} = \tau_{A_\gamma a}$, and A_γ preserves the decomposition $\mathbb{R}^p = E^q \oplus E^{p-q}$.

The matrix A_γ stabilizes $\Gamma_0 \subset \mathbb{R}^p$ because it is a normal subgroup of $\pi_1(M)$, and so does $(A_\gamma)^{-1} = A_{\gamma^{-1}}$, thus it is an element of $\mathrm{GL}_p(\mathbb{Z})$. In addition, the transformation γ on $\tilde{M}$ descends to a transformation $\bar{\gamma}$ on $N/\bar{P}^0$, because for any $(a, y) \in \mathbb{R}^p \times N$ one has $\gamma\tau_a(0, y) = (\gamma\tau_a\gamma^{-1})\gamma(0, y)$, thus $\gamma\tau_a(0, y)$ and $\gamma(0, y)$ are in the same coset modulo $\bar{P}^0$ since $\gamma\tau_a\gamma^{-1} = \tau_{A_\gamma a} \in \mathbb{R}^p$. We deduce that for any $(a, x) \in \mathbb{R}^p \times C$, one has

$$\gamma(a, x) = \gamma\tau_a\gamma^{-1}\gamma(0, x) = \tau_{A_\gamma a}\gamma(0, x) = (A_\gamma a + f_\gamma(x), \bar{\gamma}(x)), \tag{11}$$

where $f_\gamma : C \to \mathbb{R}^p$ is a function whose projection on $\mathbb{R}^q$ is constant, because γ preserves the product structure $\tilde{M} \simeq \mathbb{R}^q \times N$. Consequently, γ is an automorphism of the trivial $\mathbb{R}^p$-principal bundle $\mathbb{R}^p \times C \to C$ (see [7, Section 5] for the definition). Moreover, since $A \in \mathrm{GL}_p(\mathbb{Z})$, the map γ descends to an automorphism of the trivial $(S^1)^p$-principal bundle $(\mathbb{R}^p/\mathbb{Z}^p) \times C \simeq (S^1)^p \times C \to C$. The only elements of $\pi_1(M)$ which descend to the identity this way are the elements of $\mathbb{Z}^p \simeq \Gamma_0 \subset \pi_1(M)$, thus $\pi_1(M)/\Gamma_0$ is a subgroup of the automorphisms of $(S^1)^p \times C \to C$.

We introduce the following definitions:

Definition 4.2. *For any principal bundle $P \to B$, we denote by $\mathrm{Aut}(P \to B)$ the set of its automorphisms.*

An automorphism of the trivial $\mathbb{R}^p$-principal over C can be written as a map $\mathbb{R}^p \times C \ni (a, x) \mapsto (Aa + f(x), \varphi(x))$ where $A \in \mathrm{GL}_p(\mathbb{R})$, $f \in C^\infty(C, \mathbb{R}^p)$ and $\varphi \in \mathrm{Diff}(C)$. We call A the linear part *of the automorphism and f its* translation part. *We define $\mathrm{Aut}^{\mathbb{Z}}_{E^q}(\mathbb{R}^p \times C \to C)$ as the set of automorphisms of $\mathbb{R}^p \times C \to C$ such that $A \in \mathrm{GL}_p(\mathbb{Z})$ and f composed with the projection onto E^q parallel to E^{p-q} is constant.*

We identify the automorphisms of $(S^1)^p$ with the matrices of $\mathrm{GL}_p(\mathbb{Z})$. An automorphism of the trivial $(S^1)^p$-principal bundle over C can be written as $(S^1)^p \times C \ni (\bar{a}, x) \mapsto (A\bar{a} + \bar{f}(x), \varphi(x))$ where $A \in \mathrm{GL}_p(\mathbb{Z})$, $\bar{f} \in C^\infty(C, (S^1)^p)$ and $\varphi \in \mathrm{Diff}(C)$. Since the manifold C is simply-connected, any element of $\mathrm{Aut}((S^1)^p \times C \to C)$ can be lifted to an element of $\mathrm{Aut}(\mathbb{R}^p \times C \to C)$, uniquely defined up to a choice of base-points (we will omit this choice since it does not impact our results). We define $\mathrm{Aut}_{E^q}((S^1)^p \times C \to C)$ as the set of automorphisms of the trivial $(S^1)^p$-principal bundle over C whose lifts lie in $\mathrm{Aut}^{\mathbb{Z}}_{E^q}(\mathbb{R}^p \times C \to C)$.

With these definitions, we deduce from the previous discussion that $\pi_1(M)$ is a subgroup of $\mathrm{Aut}^{\mathbb{Z}}_{E^q}(\mathbb{R}^p \times C \to C)$ and $\pi_1(M)/\Gamma_0$ is a subgroup of $\mathrm{Aut}_{E^q}((S^1)^p \times C \to C)$. This implies that, if we consider the projection

$$\mathcal{P} : \mathrm{Aut}^{\mathbb{Z}}_{E^q}(\mathbb{R}^p \times C \to C) \to \mathrm{Aut}_{E^q}((S^1)^p \times C \to C),$$

one has $\pi_1(M) = \mathcal{P}^{-1}(\pi_1(M)/\Gamma_0)$ because $\mathbb{Z}^p \simeq \Gamma_0 \subset \pi_1(M)$.

We know from Remark 3.8 that $\pi_1(M)/\Gamma_0$ is a discrete group acting properly on C, so $N/\bar{P} \simeq C/(\pi_1(M)/\Gamma_0)$ has an orbifold structure. Since C is a simply connected manifold, this orbifold is *good*, and it is compact because, M being compact, the projection onto C of any compact fundamental domain for the co-compact action of $\pi_1(M)$ on $\tilde{M}$ is a compact fundamental domain for the action of $\pi_1(M)/\Gamma_0$ on C.

We assume from now on that $\pi_1(M)$ acts on $\bar{P}^0$ without torsion element, which is always possible up to a finite covering as explained in Section 3.2. If we pick $\gamma \in \pi_1(M)$ such that $\bar{\gamma} \in \pi_1(M)/\Gamma_0$ acts as the identity on C, we know thanks to Remark 3.8 that γ has finite order, thus $A_\gamma = \mathrm{I}_p$ because A_γ cannot be a torsion element, and the translation part f_γ has to be constant because it exists $k \geq 1$ such that $kf_\gamma \in \Gamma_0 \simeq \mathbb{Z}^p$. Hence f_γ is an element of $\bar{P}^0$ and $\gamma \in \Gamma_0$ by definition. We conclude that the only elements of $\pi_1(M)$ whose induced action on C is the identity are the elements of Γ_0. Consequently, the group $\pi_1(M)/\Gamma_0$ acts effectively on C, and it can be identified to the fundamental group of the good orbifold $N/\bar{P}$. Thus there is a short exact sequence

$$0 \to \mathbb{Z}^p \to \pi_1(M) \to \pi_1(N/\bar{P}) \to 0. \tag{12}$$

Altogether, we have proved the following theorem:

Theorem 4.3. *Let (M, c, D) be a simple LCP manifold. Then, there exists a simply connected manifold C and an integer $p \geq 2$ such that $\tilde{M} \simeq \mathbb{R}^p \times C$. There is a discrete subgroup Ω of $\mathrm{Aut}_{E^q}((S^1)^p \times C \to C)$ such that, if we consider the canonical projection*

$$\mathcal{P} : \mathrm{Aut}^{\mathbb{Z}}_{E^q}(\mathbb{R}^p \times C \to C) \to \mathrm{Aut}_{E^q}((S^1)^p \times C \to C),$$

one has $\pi_1(M) = \mathcal{P}^{-1}(\Omega)$, and the action of Ω restricted to C is proper and co-compact. Moreover, there is a decomposition $\mathbb{R}^p = E^q \oplus E^{p-q}$, such that $\dim E^q = q$, E^q is the flat part of the LCP manifold, and the linear part (see Definition 4.2) of Ω preserves this decomposition.

If in addition the LCP manifold is torsion-free, then the group Ω acts effectively on C, so it can be identified to the fundamental group of the compact good orbifold C/Ω, and there is a short exact sequence $0 \to \mathbb{Z}^p \to \pi_1(M) \to \Omega \to 0$.

Remark 4.4. From the discussion of Section 3.2, for any LCP structure (M, c, D), up to considering a finite cover of M one can assume that $\pi_1(M)$ acts by conjugation on $\mathbb{R}^q \times \bar{P}^0 \simeq \mathbb{R}^p$ without torsion. This means exactly that the linear part of Ω is a group with no torsion element and that the LCP structure is torsion-free. Then, up to a finite covering, Ω is isomorphic to $\pi_1(C/\Omega)$.

4.2. **Construction of an LCP structure from admisible data.** We now investigate the converse of the above statement, i.e. we consider a simply connected manifold C, an integer $p \geq 2$ and a discrete subgroup Ω of the automorphisms of $(S^1)^p \times C \to C$ satisfying the hypotheses of Theorem 4.3 and we will construct an LCP structure on $M := (\mathbb{R}^p \times C)/\Omega$. More precisely, we assume that there exists a decomposition $\mathbb{R}^p = E^q \oplus E^{p-q}$ with E^q a

q-dimensional subspace of $\mathbb{R}^p$, Ω is a subgroup of $\mathrm{Aut}_{E^q}((S^1)^p \times C \to C)$, the linear part of Ω preserves the decomposition $E^q \oplus E^{p-q}$ and its restriction to E^q contains only similarities with respect to a given scalar product g_{E^q}, not all being isometries. We also assume that Ω restricted to C acts properly and co-complactly (but not necessarily effectively), and for any element $\omega \in \Omega \setminus \{\mathrm{id}\}$ with a fix point $x \in C$, ω acts on $\mathbb{R}^p \times \{x\}$ (which is preserved by the transformation ω) with no fixed point. This last condition is implicit in the conclusion of Theorem 4.3 since $\pi_1(M)$ acts freely on $\tilde{M}$.

Remark 4.5. In view of Remark 4.4, we could assume that the linear part of Ω is a group with no torsion element. In this case, for any $x \in C$ and $\omega \in \Omega \setminus \{\mathrm{id}\}$ in the isotropy group of x, this subgroup of Ω being finite because the action is proper, ω has finite order. Consequently, the linear part of ω is the identity, and the last condition listed above just means that the translation part does not vanish at x.

The first step is to find a candidate for the fundamental group of the LCP manifold. For this, we introduce a group G of transformations of $\tilde{M} := \mathbb{R}^p \times C$. If

$$\mathcal{P} : \mathrm{Aut}^{\mathbb{Z}}_{E^q}(\mathbb{R}^p \times C \to C) \to \mathrm{Aut}_{E^q}((S^1)^p \times C \to C)$$

denotes the canonical projection, we define the subgroup G of $\mathrm{Aut}^{\mathbb{Z}}_{E^q}(\mathbb{R}^p \times C \to C)$ by

$$G := \mathcal{P}^{-1}(\Omega), \tag{13}$$

which clearly contains $\mathbb{Z}^p$.

Lemma 4.6. *The group G acts freely, properly discontinuously and co-compactly on $\mathbb{R}^p \times C$.*

Proof. According to [4, Proposition 4.1], since $\mathbb{Z}^p \trianglelefteq G$ by construction, G acts freely and properly discontinuously on $\mathbb{R}^p \times C$ if and only if $\mathbb{Z}^p$ acts freely and properly discontinuously on $\mathbb{R}^p \times C$ and $G/\mathbb{Z}^p$ acts freely and properly discontinuously on $(\mathbb{R}^p \times C)/\mathbb{Z}^p \simeq (S^1)^p \times C$. The first claim is obvious, so we are left to check the second one.

Let $\omega \in \Omega$ which has a fix point $(\bar{a}, x) \in T^p \times C$. Then, the restriction of ω to C satisfies $\omega(x) = x$, so ω is in the isotropy group of x, and by assumption $\omega|_{\mathbb{R}^p \times \{x\}}$ would have no fixed point if $\omega \neq \mathrm{id}$, thus $\omega = \mathrm{id}$ since ω has a fix point.

The projection map $(S^1)^p \times C \to C$ is proper, so the action of $G/\mathbb{Z}^p$ on $T^p \times C \to C$ is proper since Ω acts properly on C.

The co-compactness of the action is easily checked by choosing a compact elementary domain D of C for the action of Ω and considering $D' := [0, 1]^p \times D$. Then one has $G(D') = \mathbb{R}^p \times C$. □

From Lemma 4.6, we know that $M := (\mathbb{R}^p \times C)/G$ is a compact manifold. In order to define an LCP structure on M, it remains to construct a Riemannian metric h on its universal cover $\tilde{M} := \mathbb{R}^p \times C$, which is G-equivariant (in the sense that G acts by similarities on $(\tilde{M}, h)$) and with reducible holonomy. This second point is the easiest: using the hypothesis made at the beginning of this section concerning the restriction of Ω to E^q and writing $\tilde{M} = E^q \times (E^{p-q} \times C)$, the metric should be of the form $g_{E^q} + g_N$ where g_{E^q} was introduced before, and g_N is a metric on $N := E^{p-q} \times C$. The action of G preserves the product structure, and its restriction to E^q consists only on similarities, not all being isometries. Hence we can define the group homomorphism $\tilde{\rho} : G \to \mathbb{R}^*_+$ which gives the similarity ratio of any element of this restriction.

We will now describe all possible Riemannian metrics for g_N such that the group E^{p-q} acts by isometries. For such a metric, the group $\mathbb{Z}^p$ acts by isometries on $\tilde{M}$ and $\tilde{\rho}$ descends to a group homomorphism $\rho : \Omega \to \mathbb{R}_+^*$. These metrics are given in a basis adapted to the decomposition $E^{p-q} \times C$ by fields of matrices over C of the form

$$\begin{pmatrix} Q & b_{FB} \\ b_{BF} & g_B \end{pmatrix} \tag{14}$$

with g_B being a Riemannian metric on C, Q being a field of positive definite quadratic forms on E^{p-q}, and $b_{FB} : E^{p-q} \times TC \to \mathbb{R}$ and $b_{BF} : TC \times E^{p-q} \to \mathbb{R}$ are two bilinear forms related by the symmetry of the metric, i.e. b_{FB} is determined by b_{BF}.

Let $\omega \in \Omega$ and we denote by $A \in \mathrm{GL}_p(\mathbb{Z})$ its linear part and by $f \in C^\infty(C, E^{p-q})$ the E^{p-q}-component of its translation part. The representatives of ω in G (i.e. the elements of $\mathcal{P}^{-1}(\{\omega\})$) all have the same differential, since they differ only by a translation element of $\mathbb{Z}^p$, thus the group Ω acts on TN by push-forward. The restriction of ω_* to E^{p-q} is a constant matrix corresponding to the linear part A of ω restricted to E^{p-q} and for any $X \in TC$, one has $\omega_* X = d\omega(X) + X(f)$. In particular, the transformation ω_* is E^{p-q}-invariant because f is, and so is the action of Ω by push-forward.

The admssible metrics on $\tilde{M}$ should be G-equivariant, which is equivalent to the metric g_N under the form (14) being Ω-equivariant, i.e. the admissible metrics correspond to the positive definite matrices satisfying:

$$\omega^* \begin{pmatrix} Q & b_{FB} \\ b_{BF} & g_B \end{pmatrix} = \rho(\omega)^2 \begin{pmatrix} Q & b_{FB} \\ b_{BF} & g_B \end{pmatrix}, \qquad \forall \omega \in \Omega, \tag{15}$$

where the pull-back is well-defined by the previous discussion on the action of Ω by push-forward. We thus need to construct such an invariant metric. Using the fact that Ω acts co-compactly on C, there exists a compact $K \subset C$ such that $C = \Omega \cdot K$. By compactness, there is a finite cover $(U_i)_{i \in I}$ of K by open sets whose closures are contained in charts of C and are closed ball of the Euclidean space in these charts, so in particular each U_i is relatively compact. Defining $U = \cup_{i\in I} U_i$, it is easily seen that U is a relatively compact open set such that $K \subset U \subset \bar{U}$. On each U_i one can construct a Riemannian metric g_i of the form (14) (by taking b_{BF} and b_{FB} to be zero for example), and one can find a function $\chi_i : \tilde{M} \to \mathbb{R}$ with support lying in $\bar{U}_i$ such that $\chi_i > 0$ on U_i. The metric $g := \sum_{i \in I} \chi_i g_i$ is then of the form (14) and is a Riemannian metric on U. We now define g_N by the formula:

$$g_N := \sum_{\omega \in \Omega} \rho(\omega)^{-2} \omega^* g. \tag{16}$$

Lemma 4.7. *The metric g_N given by Equation* (16) *is well-defined and is a Ω-invariant Riemannian metric on $E^{p-q} \times C$. Moreover, any Ω-invariant metric arises from this construction.*

Proof. First, we prove that g_N is well-defined. It is sufficient to prove that the sum has only a finite number of non-zero terms on any sufficiently small neighborhood of a point in C.

Let $x \in C$. Since $C = \Omega \cdot \bar{K} \subset \Omega \cdot U$, there exists a small open subset V of U and $\omega_0 \in \Omega$ such that $x \in \omega_0 \cdot V$. For any $\omega \in \Omega$, the term $\omega^* g$ is not identically vanishing on $\omega_0 \cdot V$ only if $\omega \cdot \omega_0 \cdot V \cap \bar{U} \neq \emptyset$. The group Ω acts properly on C, so the set of elements $\omega' \in \Omega$ such that $\omega' \cdot \bar{V} \cap \bar{U} \neq \emptyset$ is finite because $\bar{U}$ is compact, thus the set of elements $\omega \in \Omega$ such that

$\omega \cdot \omega_0 \cdot V \cap \bar{U} \neq \emptyset$ is finite. This implies that g_N is a sum of a finite number of terms on $\omega_0 \cdot V$, so it is well-defined and smooth on this neighborhood of x. This analysis holds for any point, hence g_N is well-defined and smooth.

As a sum of positive-definite or null terms, g_N is positive definite or null at any point of C. We need to prove that it is non-zero everywhere. But for any $x \in C$ there exists $\omega \in \Omega$ and $y \in U$ such that $\omega(y) = x$ and $\omega^* g(x)$ is non-zero. Thus g_N is a Riemannian metric on $E^{p-q} \times C$.

We now check the equivariance property (15). Let $\omega_0 \in \Omega$. One has:

$$\omega_0^* g_N = \omega_0^* \sum_{\omega \in \Omega} \rho(\omega)^{-2} \omega^* g = \rho(\omega_0)^2 \sum_{\omega \in \Omega} \rho(\omega \cdot \omega_0)^{-2} \omega_0^* \omega^* g = \rho(\omega_0)^2 \sum_{\omega \in \Omega} \rho(\omega)^{-2} \omega^* g = \rho(\omega_0)^2 g_N.$$

To prove that any equivariant metric is obtained by this construction, we first consider a non-negative smooth function χ with support in $\bar{U}$ and such that $\chi > 0$ on U. Such a function exists due to the way we constructed U. For any $\omega \in \Omega$, let $\chi_\omega := \omega^* \chi$. With the same arguments we used for the metric g_N, one can prove that $\chi_T := \sum_{\omega \in \Omega} \chi_\omega$ is well-defined, positive, smooth and Ω-invariant. Now, let g be any Ω-equivariant metric, and we define

$$\forall \omega \in \Omega, \qquad g_\omega := \frac{\chi_\omega}{\chi_T} g.$$

Then for any $\omega \in \Omega$ one has $g_\omega := \omega^* g_{\mathrm{id}}$. This yields:

$$g = \sum_{\omega \in \Omega} \frac{\chi_\omega}{\chi_T} g = \sum_{\omega \in \Omega} \omega^* g_{\mathrm{id}}$$

and g is constructed in the same way as g_N. □

The previous discussion together with Lemma 4.7 allows us to define an LCP structure on M induced by the Riemannian structure $(\tilde{M}, g_{E^q} + g_N)$. The flat part of this LCP manifold contains E^q and we can prove the following:

Proposition 4.8. *If E^q is exactly the flat part of the LCP structure, then the characteristic group of the LCP manifold is the smallest vector subspace F of $\mathbb{R}^p$ containing E^q and generated by a subfamily of $\mathbb{Z}^p$. In particular, the projection of E^q onto $\mathbb{R}^p/\mathbb{Z}^p$ is dense in the projection of F.*

Proof. Since we are working on an LCP manifold, we will use the notations of Section 2 in this proof. A reasoning similar as the one of Section 3.2 allow us to consider, up to a finite covering, that the linear part of Ω has no torsion, without changing the characteristic group and the lattice Γ_0.

We first prove that the vector subspace F introduced in the statement of this proposition is unique and well-defined. In order to do so, it is sufficient to prove that the intersection of two subspaces F_1, F_2 of $\mathbb{R}^p$ containing E^q and generated by subfamilies $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathbb{Z}^p$ still has this property. The fact that $E^q \subset F_1 \cap F_2$ is obvious, so it remains to prove that $F_1 \cap F_2$ is generated by a subfamily of $\mathbb{Z}^p$. Denote by F_1' and F_2' the $\mathbb{Q}$-subspaces of $\mathbb{Q}^p$ generated by $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, so that $F_1 = F_1' \otimes_{\mathbb{Z}} \mathbb{R}$ and $F_2 = F_2' \otimes_{\mathbb{Z}} \mathbb{R}$. Then one has

$$F_1 \cap F_2 = (F_1' \otimes_{\mathbb{Z}} \mathbb{R}) \cap (F_2' \otimes_{\mathbb{Z}} \mathbb{R}) = (F_1' \cap F_2') \otimes_{\mathbb{Z}} \mathbb{R},$$

and any $\mathbb{Q}$-basis of $F_1' \cap F_2'$ gives a basis with vectors in $\mathbb{Z}^p$ after multiplying each vector by a suitable integer. This induces a basis of $F_1 \cap F_2$ with elements in $\mathbb{Z}^p$.

The non-flat part of the LCP manifold is $N := E^{p-q} \times C$ since E^q is exactly the flat part and N is orthogonal to E^q. By definition, $\bar{P}^0$ is the connected component of the identity in $\bar{P}$, the closure of the restriction of G to N.

Let $\omega \in \Omega$ such that $\omega|_N \in \bar{P}^0$. There exists a continuous path $\sigma : [0,1] \to \bar{P}^0$ such that $\sigma(0) = \mathrm{id}$ and $\sigma(1) = \omega|_N$. Let $x \in C$. The set $\sigma([0,1])|_C(\{x\})$ is a path-connected subset of C. Since the compact-open topology on metric spaces is characterized by the uniform convergence on compacts and the elements of $\bar{P}^0$ are in the closure of $G|_N$, the closure of $G|_N(\{(0,x)\})$ (where $0 \in E^{p-q}$) must contain $\sigma([0,1])(\{(0,x)\})$, so the closure of $G|_C(\{x\})$ must contain $\sigma([0,1])|_C(\{x\})$. Yet, $G|_C(\{x\}) = \Omega|_C(\{x\})$ is a discrete subset of C because Ω acts properly discontinuously on C. Thus, $\sigma([0,1])|_C(\{x\})$ is reduced to a single point, yielding $x = \sigma(0)|_C(x) \in \sigma([0,1])|_C(\{x\}) = \{x\}$, and we deduce that $\sigma(1)|_C(x) = x$. It follows that $\sigma(1)|_C = \mathrm{id}$ and $\sigma(1) \in E^{p-q}$ because the linear part of Ω is has no torsion. Consequently, the elements of $G \cap (E^q \times \bar{P}^0) = \Gamma_0$ are translations of $\mathbb{R}^p$ and there exists $m \in \mathbb{N}$ such that $\Gamma_0 \subset \frac{1}{m}\mathbb{Z}^p$ because G acts properly discontinuously on $\mathbb{R}^p \times C$.

We can now prove that $F = E^q \times \bar{P}^0$. Indeed, the vector space $E^q \times \bar{P}^0$ admits $\Gamma_0 \subset \frac{1}{m}\mathbb{Z}^p$ as a full lattice, so it is generated by a subfamily of $\mathbb{Z}^p$, thus it contains F. On the other hand, E^q/Γ_0 has to be dense in $(E^q \times \bar{P}^0)/\Gamma_0$ by Property 3.9, so it is dense in $(E^q \times \bar{P}^0)/(m\Gamma_0)$. In particular, if F' is a subspace of $\mathbb{R}^p$ generated by a subfamily of $\mathbb{Z}^p$ with $F \subset F'$, then $F/\mathbb{Z}^p$ and $F'/\mathbb{Z}^p$ are two sub-tori of $\mathbb{R}^p/\mathbb{Z}^p$ with $F/\mathbb{Z}^p \subsetneq F'/\mathbb{Z}^p$, but the image of E^q is contained in $F/\mathbb{Z}^p$, so it is not dense in $F'/\mathbb{Z}^p$. We conclude that $E^q \times \bar{P}^0 = F$ and the image of E^q in $\mathbb{R}^p/\mathbb{Z}^p$ is dense in the image of F because it is dense in $(E^q \times \bar{P}^0)/\Gamma^0$. ☐

The whole discussion of this section is summarized in the following theorem:

Theorem 4.9. *Let C be simply connected manifold. Let $p \geq 2$ be an integer and Ω be a discrete subgroup of $\mathrm{Aut}_{E^q}((S^1)^p \times C \to C)$ whose restriction to C acts properly and co-compactly (i.e. C/Ω is a compact good orbifold). Assume that there exists a decomposition $\mathbb{R}^p = E^q \oplus E^{p-q}$ with $\dim E^q = q$ preserved by the linear part of Ω, and such that the restriction of this linear part to E^q contains only similarities with respect to a given scalar product g_{E^q}, not all being isometries. We also assume that for any element $\omega \in \Omega \setminus \{id\}$ with a fix point $x \in C$, $\omega|_{\mathbb{R}^q \times \{x\}}$ has no fixed point. Then, considering the canonical projection (see Definition 4.2)*

$$\mathcal{P} : \mathrm{Aut}^{\mathbb{Z}}_{E^q}(\mathbb{R}^p \times C \to C) \to \mathrm{Aut}_{E^q}((S^1)^p \times C \to C),$$

the group $G := \mathcal{P}^{-1}(\Omega)$ acts freely, properly and co-compactly on $\tilde{M} := E^q \times N$ where $N := E^{p-q} \times C$, and there exists a Riemannian metric g_N on N such that G acts by similarities, not all being isometries on $(\tilde{M}, h := g_{E^q} + g_N)$. All such metrics g_N are constructed as in Equation (16)*. This induces an LCP structure on $M := \tilde{M}/G$.*

5. Discussion on the hypotheses and further examples

Theorem 4.3 together with Theorem 4.9 give a description of simple LCP manifolds. However, there are still open questions concerning the hypotheses and possible simplifications that one could make in these statements. Indeed, the existence of the group Ω in Theorem 4.9 is the only obstruction for the construction of an LCP manifold starting from a simply connected manifold C. In this section we discuss cases where one can construct such a group and we provide examples proving that some hypotheses cannot be removed. We will use the notations of these two theorems in the following section.

5.1. **The orbifold hypothesis and the structure of the fundamental group.** We begin this section by providing an example satisfying the hypotheses of Theorem 4.9 where C/Ω is a compact orbifold which is not a manifold, i.e. Ω acts effectively on C with fixed points.

Example 5.1. Consider $C := S^2 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$ and the group $\Omega \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ of automorphisms of $(S^1)^2 \times C \to C$ acting for any $(\bar{a}, (x, y, z), s) \in (S^1)^2 \times S^2 \times \mathbb{R}$, and for all $m \in \mathbb{Z}$ as:

$$(\bar{1}, 0) \cdot (\bar{a}, (x, y, z), s) = (\bar{a} + (0, 1/2)^T, (-y, -x, z), s),$$
$$(\bar{0}, m) \cdot (\bar{a}, (x, y, z), s) = \left(\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^m \bar{a}, (x, y, z), s + m \right),$$

i.e. $(\bar{1}, 0)$ is the rotation of axis z and angle π on S^2. One easily check that this satisfies all the necessary hypotheses, since the matrix is diagonalizable with real eigenvalues different from ± 1, and the elements $(\bar{1}, m)$ for $m \in \mathbb{Z}$ have non-zero translation part. The orbifold C/Ω is not a manifold since the point $((0, 0, 1), 0)$ has a non-trivial isotropy group.

Example 5.1 proves that we cannot drop the case where C/Ω is an orbifold. However, in this example there is a finite covering (or equivalently a subgroup Ω' of Ω with finite index) of the LCP manifold constructed via Theorem 4.9 such that C/Ω' is a manifold, since C/Ω is actually *very good*, being finitely covered by $S^2 \times S^1$. One can then ask whether for any LCP manifold it is possible to find a finite covering such that C/Ω is always a manifold in Theorem 4.3. The answer is positive when C/Ω is the product of a manifold with a 2-dimensional orbifold for example, because any good compact 2-orbifold is very good [12, Theorem 2.5]. We do not know if there exists a counter-example to this statement, so this question is still open.

Another natural question is the following: is the group G defined in Theorem 4.9 always a semi-direct product $\mathbb{Z}^p \rtimes \Omega$, as it is in all the examples we gave so far? The answer to this question is no, as shown by Example 5.2 below.

Example 5.2. Consider $C := \mathbb{R}^2$ and $\Omega \simeq \mathbb{Z}^2$ the group of automorphisms of $(S^1)^2 \times C \to C$ given by:

$$(m_1, m_2) \cdot (\bar{a}, (x, y)) := (A^{m_1} \bar{a} + m_2 \tau, (m_1 x, m_2 y)), \qquad \forall (\bar{a}, (x, y)) \in (S^1)^2 \times \mathbb{R}^2, \tag{17}$$

where

$$A := \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \qquad \tau := \frac{1}{5} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

This group is well-defined and satisfies the hypotheses of Theorem 4.9 since the matrix A is diagonalizable with real eigenvalues different from ± 1. Now, for the group G defined by (13) to be a semi-direct product, there should exist a subgroup H of G such that $H \cap \mathbb{Z}^2 = \{\mathrm{id}\}$ and H is isomorphic to $\Omega \simeq \mathbb{Z}^2$. However, for such a subgroup to exist, one should find representatives of $(1, 0)$ and $(0, 1)$ whose commutator is zero since Ω is Abelian. This is equivalent to finding $\tau_1, \tau_2 \in \mathbb{Z}^p$ so that the affine transformations of $\mathbb{R}^2$ given by $(A, \tau_1), (I_2, \tau + \tau_2) \in \mathrm{GL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ commute. We have

$$(A, \tau_1)^{-1} \cdot (I_2, \tau + \tau_2)^{-1} \cdot (A, \tau_1) \cdot (I_2, \tau + \tau_2) = (I_2, (A - I_2)(\tau + \tau_2)) = (1, 0)^T + (A - I_2)\tau_2.$$

For the commutator to be zero, we should thus have $(1, 0)^T = (I_2 - A)\tau_2$, and writing $\tau_2 =: (a, b)^T$ one obtains the system

$$\begin{cases} -2a + b = 1 \\ a - 3b = 0 \end{cases},$$

which has no solution for $a, b \in \mathbb{Z}$. Thus, the group G is not a semi-direct product of the form $\mathbb{Z}^p \rtimes \Omega$.

5.2. **The linear part of Ω.** In order to understand which manifold can lead to the construction of an LCP manifold, it is important to know what are the possible group Ω, and in particular what are their linear part. We study here the subgroups of $\mathrm{GL}_p(\mathbb{Z})$ appearing as the linear part of Ω. These groups should be finitely generated because the LCP manifold is compact, they should preserve a decomposition $E^q \oplus E^{p-q}$ of $\mathbb{R}^p$ with $\dim(E^q) = q$, their restriction to E^q consisting only of similarities for a given scalar product on E^q, but not all being isometries. Moreover, in regard of Proposition 4.8 together with Theorem 4.3 we can assume that the image of E^q is dense in $\mathbb{R}^p/\mathbb{Z}^p$ since we want to describe exactly the characteristic groups of LCP manifolds. Let U be such a group.

The elements of U have the following property, which is a consequence of the Jordan-Chevalley decomposition:

Proposition 5.3. *All the elements of U are semi-simple.*

Proof. Let $A \in U$. Since $\mathbb{Q}$ is a perfect field, A admits a Jordan-Chevalley decomposition $A =: D + N$ where $D \in M_p(\mathbb{Q})$ is semi-simple, $N \in M_p(\mathbb{Q})$ is nilpotent and $[D, N] = 0$. There exists an integer $m \in \mathbb{N}$ such that $mD \in M_p(\mathbb{Z})$ and $mN \in M_p(\mathbb{Z})$. The linear transformations $mA = mD + mN$ and mD descend to two group endomorphisms of the p-torus $\mathbb{R}^p/\mathbb{Z}^p$, because they are matrices of $M_p(\mathbb{Z})$.

Since $A|_{E^q}$ is a similarity, there exists $(\lambda, O, P) \in \mathbb{R}^*_+ \times O(q) \times \mathrm{GL}_q(\mathbb{R})$ such that

$$A|_{E^q} = \lambda P^{-1} O P \tag{18}$$

and in particular, $A|_{E^q}$ is diagonalizable in $\mathbb{C}$, then semi-simple. The Jordan-Chevalley decomposition of $A|_{E^q}$ is given by $D|_{E^q} + N|_{E^q}$, so we have $N|_{E^q} = 0$ because $A|_{E^q}$ is semi-simple, implying $A|_{E^q} = D|_{E^q}$. Consequently, the endomorphisms of $\mathbb{R}^p/\mathbb{Z}^p$ induced by mA and mD coincide on the image of E^q in $\mathbb{R}^p/\mathbb{Z}^p$, which is dense, so they are equal by continuity. We conclude that $mA = mD$ and $A = D$, so A is semi-simple. □

Corollary 5.4. *There exists a basis of $\mathbb{R}^p$ containing only vectors of $\mathbb{Z}^p$ such that in this basis the matrix A is diagonal by blocks, where the diagonal blocks have irreducible characteristic polynomial in $\mathbb{Z}$.*

Proof. We will work in the $\mathbb{Q}$-vector space $\mathbb{Q}^p$, in which A is a well-defined linear transformation, and we will extend the result to $\mathbb{R}^p$.

The characteristic polynomial χ_A of A can be decomposed into monic irreducible factors over $\mathbb{Z}$ as

$$\chi_A = \prod_{k=1}^{m} P_k^{\alpha_k}, \tag{19}$$

where the polynomials P_k are pairwise prime. This gives a decomposition of $\mathbb{Q}^p$ into invariant subspaces

$$\mathbb{Q}^p = \bigoplus_{k=1}^{m} \ker\, P_k^{\alpha_k} := \bigoplus_{k=1}^{m} \ker E_k, \tag{20}$$

and the projections onto the $P_k^{\alpha_k}$ are polynomials in A with coefficients in $\mathbb{Q}$, so they are matrices with coefficients in $\mathbb{Q}$. Let $1 \leq k \leq m$. The vector space E_k admits as a basis any basis $\mathcal{B}_k$ of the full-lattice given by $\mathbb{Z}^p \cap E_k$. For a vector $v \in \mathcal{B}_k$, $Av \in \mathbb{Z}^p \cap E_k$ and then Av is a linear combination of elements of $\mathcal{B}_k$ with coefficients in $\mathbb{Z}$. This means that the matrix of $A|_{E_k}$ written in $\mathcal{B}_k$ has coefficients in $\mathbb{Z}$. If we define the basis $\mathcal{B}$ of $\mathbb{Q}^p$ as the concatenation of the bases $\mathcal{B}_k$, then A is diagonal by blocks with coefficients in $\mathbb{Z}$.

It remains to look at what happens for the restriction to each E_k, so we can assume that $\chi_A = P^\alpha$ where P is an irreducible polynomial in $\mathbb{Z}$. Since A is semi-simple by Proposition 5.3, P is the minimal polynomial of A. The Frobenius decomposition gives the existence of a decomposition of the ambient vecto-space $\bigoplus_{k=1}^{\ell} F_k$ where the F_k are cyclic, stable by A and the characteristic polynomial of $A|_{F_k}$ is P. The same argument as in the first part of the proof then gives us a basis of vectors with coefficients in $\mathbb{Z}$ adapted to the decomposition, in which $A|_{F_k}$ has coefficients in $\mathbb{Z}$ for each k. $\square$

Let $A \in U$, and assume that $A|_{E^q}$ is not an isometry for the scalar product given on E^q, i.e. $\lambda \neq 1$ in equation (18). Using Corollary 5.4, we can write A under the form $\mathrm{Diag}(A1, \dots A_m)$, where the blocks A_k have irreducible characteristic polynomials. The subspace E^q is spanned by real and complex part of complex eigenvectors of A (since E^q and E^{p-q} are stable by A), i.e. it is a subspace of the sum of the eigenspaces of $A_1, \dots, A_m$ whose associated eigenvalues have absolute value λ, which is stable by A. Moreover, the elements of U all commute with the projector $\mathbf{P}$ on E^q parallel to E^{p-q}.

Conversely, starting from a matrix $A \in \mathrm{GL}_p(\mathbb{Z})$ of the form of Corollary 5.4, we can give a theoretical way of constructing an admissible group U. First, there should exist $\lambda \neq 1$ such that each block of A has at least one eigenvalue of absolute value λ. Choose a q-dimensional subspace E^q of $\mathbb{R}^p$ stable by A, such that $A|_{E^q}$ has only eigenvalues with absolute value λ and the image of E^q in $\mathbb{R}^p/\mathbb{Z}^p$ is dense (it is always possible, since we can just take the space spanned by the real and complex parts of eigenvectors with eigenvalues of absolute value λ). Since A is semi-simple, one can choose a stable space E^{p-q} supplementary to E^q, inducing a projector $\mathbf{P}$ on E^q parallel to E^{p-q}. Let $\mathrm{Com}(\mathbf{P}) := \{M \in \mathrm{GL}_p(\mathbb{Z}) \mid M\mathbf{P} = \mathbf{P}M\}$. Let $\Theta : \mathrm{Com}(\mathbf{P}) \to \mathrm{GL}_q(\mathbb{R})$, $M \mapsto |\det M|_{E^q}|^{-1} M|_{E^q}$ and $S := \Theta(\mathrm{Com}(\mathbf{P}))$. Remark that $|\det A|_{E^q}|^{-1} A|_{E^q}$ has only eigenvalues with absolute value 1, so it is contained into a maximal compact subgroup H of $\mathrm{GL}_q(\mathbb{R})$. This subgroup H is conjugated to $O(q)$ and then defines a scalar product on E^q. Taking $U := \Theta^{-1}(H)$ defines an admissible group, and any admissible group can be obtained by this construction, up to a choice of $\mathbf{P}$, H and taking a subgroup of U.

5.3. **The non-constant translations.** A visible difference between the group Ω introduced in Theorem 4.9 and all the examples we gave so far is the presence of a possibly non-constant translation part in an element of Ω. Nevertheless, one can easily obtain non-constant translations from any example with constant translation part introduced before. For that, it is sufficient to consider the trivial bundle $\mathbb{R}^p \times C$ in Theorem 4.9 and the diffeomorphism

$$\varphi : \mathbb{R}^p \times C \to \mathbb{R}^p \times C, (a, x) \mapsto (a + s(x), x) \tag{21}$$

where $s : C \to E^{p-q}$ is any smooth function. Now, if one takes an element $\omega \in \Omega$ which has $\tilde{\omega} = (\mathbb{R}^p \times C \ni (a, x) \mapsto (Aa + b, \omega|_C(x)))$ as a lift, it follows:

$$\varphi^{-1}\tilde{\omega}\varphi(a, x) = (Aa + b + As(x) - s(\omega|_C(x)), \omega|_C(x)), \qquad \forall (a, x) \in \mathbb{R}^p \times C,$$

and taking a function $s : C \to E^{p-q}$ such that s is not equivariant will lead to non-constant translations. The new LCP structure is then isomorphic to the modified one, and one can then ask wether all the examples with non-constant translations arise from this construction. In other words, is it always possible to find a function $s : C \to E^{p-q}$ such that for any $\omega \in \Omega$ with $\tilde{\omega} = (\mathbb{R}^p \times C \ni (a, x) \mapsto (Aa + f(x), \omega|_C(x))$ as a lift, there exists a constant $c \in \mathbb{R}^p$ such that

$$\tilde{\omega}(s(x), x) = (s(\omega|_C(x)) + c, \omega|_C(x)), \qquad \forall x \in C. \tag{22}$$

Indeed, using the diffeomorphism φ defined in (21) one has in this case:

$$\varphi^{-1}\tilde{\omega}\varphi(a, x) = (Aa + c, \omega|_C(x)), \qquad \forall (a, x) \in \mathbb{R}^p \times C,$$

and all the translations are constant. Conversely, if there exists a diffeomorphism φ and a section s as in (21) such that through this transformation the translation part of Ω only contains constants, then s has to satisfy the condition given by Equation (22). Such a function s does not always exists, as shown by the following example:

Example 5.5. We define the matrix

$$A_0 := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \tag{23}$$

This matrix is diagonalizable with eigenvalues $\lambda := \frac{3+\sqrt{5}}{2}$ and λ^{-1}. The matrix of eigenvectors is

$$P := \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}.$$

We consider the affine transformations of $\mathbb{R}^4$ depending on a parameter $z \in \mathbb{R}$ given by:

$$\alpha : X \mapsto \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix} X, \quad \beta_1(z) \quad : X \mapsto X + (0, 0, z, 0), \quad \beta_2(z) : X \mapsto X + (0, 0, 0, z). \tag{24}$$

These maps canonically descend to $(S^1)^4$. One easily checks that the commutators of these transformations are:

$$[\alpha, \beta_1(z)] = \beta_1(z)^{-1}\beta_2(z), \quad [\alpha, \beta_2(z)] = \beta_1(z), \quad [\beta_1(z), \beta_2(z)] = \mathrm{id}. \tag{25}$$

We consider the manifold $C_0 := \mathbb{R}^2 \times \mathbb{R}^*_+$ on which acts co-compactly, freely and properly the group

$$H := \langle T : (y, t) \mapsto (A_0 y, \lambda t), T_1 : (y, t) \mapsto (y + (1, 0)^T, t), T_2 : (y, t) \mapsto (y + (0, 1)^T, t)\rangle,$$

where $(y, t) \in \mathbb{R}^2 \times \mathbb{R}$. Note that C_0/H can actually be given the structure of an LCP manifold. The universal cover of the compact manifold $C_0/H \times S^1$ is $C := C_0 \times \mathbb{R}$, and its fundamental group is $H \times \mathbb{Z}$. We have the relations

$$[T, T_1] = T_1^{-1}T_2, \quad [T, T_2] = T_1, \quad [T_1, T_2] = \mathrm{id}, \tag{26}$$

so for any $z \in \mathbb{R}$ the groups H and $\langle \alpha, \beta_1(z), \beta_2(z)\rangle$ are isomorphic via the isomorphism ι_z determined by $\iota_z(T) = \alpha$, $\iota_z(T_j) = \beta_j(z)$. We consider the subgroup $\Omega \simeq H \times \mathbb{Z}$ of $\mathrm{Aut}((S^1)^4 \times C \to C)$ given by

$$(h, n) \cdot (\bar{a}, (y, t)) = (\iota_z(h)\bar{a}, h(y, t), z), \quad \forall (h, n) \in H \times \mathbb{Z}, \forall (\bar{a}, (y, t), z) \in (S^1)^4 \times C_0 \times \mathbb{R}.$$

In order to prove that this group is well-defined, it is sufficient to show that the commutators of the generators $(T,0),(T_1,0),(T_2,0),(\mathrm{id},1)$ of Ω satisfy the suitable relations. It is easily seen that

$$[(T,0)\cdot,(T_1,0)\cdot]=(T_1^{-1}T_2,0)\cdot,\quad [(T,0)\cdot,(T_2,0)\cdot]=(T_1,0)\cdot,\quad [(T_1,0)\cdot,(T_2,0)\cdot]=(\mathrm{id},0)\cdot$$

because these commutators can be computed at a fixed $z\in\mathbb{R}$ and then correspond to the relations (26). We compute the remaining commutators. For any $(\bar a,(y,t),z)\in(S^1)^4\times C_0\times\mathbb{R}$ one has

$$\begin{aligned}
&[(T,0)\cdot,(\mathrm{id},1)\cdot](\bar a,(y,t),z)=(\bar a,(y,t),z)\\
&[(T_1,0)\cdot,(\mathrm{id},1)\cdot](\bar a,(y,t),z)=(\bar a+(0,0,1,0)^T,(y,t),z)=(\bar a,(y,t),z)\\
&[(T_2,0)\cdot,(\mathrm{id},1)\cdot](\bar a,(y,t),z)=(\bar a+(0,0,0,1)^T,(y,t),z)=(\bar a,(y,t),z),
\end{aligned}$$

then all these commutators are equal to the identity, proving that Ω is well-defined. Now, the linear part of α is a matrix A, which is diagonalizable under the form $\mathrm{Diag}(\lambda,\lambda^{-1},\lambda,\lambda^{-1})$. We choose a new coordinate system on $\mathbb{R}^4\times C=\mathbb{R}^4\times\mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{R}$ in the following way: if $(x_1,x_2,y_1,y_2,r_1,r_2,t,z)$ is the canonical system of coordinates we set

$$\begin{pmatrix}u_1\\u_2\\v_1\\v_2\\w_1\\w_2\\t\\z\end{pmatrix}=\begin{pmatrix}P^{-1}&&&&\\&P^{-1}&&&\\&&P^{-1}&&\\&&&1&\\&&&&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\y_1\\y_2\\r_1\\r_2\\t\\z\end{pmatrix}$$

In these new coordinates, A is diagonal and we define $\frac{\partial}{\partial u}:=\pi\frac{\partial}{\partial u_1}+\frac{\partial}{\partial v_1}$, so that the image of $\mathrm{Span}(u)$ under the canonical projection is dense in the torus $\mathbb{R}^4/\mathbb{Z}^4$ (written in the old coordinates). The matrix A preserves the decomposition $\mathbb{R}^4=\mathrm{Span}(\frac{\partial}{\partial u})\oplus\mathrm{Span}(\frac{\partial}{\partial u_2},\frac{\partial}{\partial v_1},\frac{\partial}{\partial v_2})$. This construction satisfies all the hypotheses of Theorem 4.9, so we obtain a group G such that $(\mathbb{R}^4\times C)/G$ can be given an LCP structure with flat part containing $\mathrm{Span}(\frac{\partial}{\partial u})$. In particular, $\mathbb{R}^4$ is contained in the characteristic group by Proposition 4.8.

We consider the lift of $(T_1,0)\in\Omega$ and $(\mathrm{id},1)\in\Omega$ respectively given by

$$\tilde\omega_1=(\mathbb{R}^4\times C_0\times\mathbb{R}\ni(a,(y,t),z)\mapsto(\beta_1(z)(a),T_1(y,t),z))\in G$$

and

$$\tilde\omega=(\mathbb{R}^4\times C_0\times\mathbb{R}\ni(a,(y,t),z)\mapsto(a,(y,t),z+1))\in G.$$

If there existed a section s satisfying (22), then using the transformation φ given by (21), one would get that

$$\begin{aligned}
\varphi\tilde\omega_1\varphi^{-1}&=(\mathbb{R}^4\times C_0\times\mathbb{R}\ni(a,(y,t),z)\mapsto(a+c_1,T_1(y,t),z))\\
\varphi\tilde\omega\varphi^{-1}&=(\mathbb{R}^4\times C_0\times\mathbb{R}\ni(a,(y,t),z)\mapsto(a+c_2,(y,t),z+1))
\end{aligned}$$

for $c_1,c_2\in\mathbb{R}$, so these two elements would commute. But one has

$$[\varphi\tilde\omega_1\varphi^{-1},\varphi\tilde\omega\varphi^{-1}]=\varphi[\tilde\omega_1,\tilde\omega]\varphi^{-1}=(\mathbb{R}^4\times C_0\times\mathbb{R}\ni(a,(y,t),z)\mapsto(a+(0,0,1,0)^T,(y,t),z)),$$

which is a contradiction. We can give an explicit metric on $\mathbb{R}^4 \times C$ which defines an LCP structure: this metric is written in the coordinate system $(u_1, u_2, v_1, v_2, w_1, w_2, t, z)$ as

$$\begin{pmatrix} \frac{2}{\pi^2} & 0 & -\frac{1}{\pi} & 0 & 0 & 0 & 0 & \frac{w_1}{\pi} \\ 0 & t^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\pi} & 0 & 1 & 0 & 0 & 0 & 0 & -w_1 \\ 0 & 0 & 0 & t^4 & 0 & 0 & 0 & -t^4 w_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{w_1}{\pi} & 0 & -w_1 & -t^4 w_2 & 0 & 0 & 0 & t^2 + w_1^2 + t^4 w_2^2 \end{pmatrix}. \tag{27}$$

An orthonormal basis for this metric is given by:

$$\pi\frac{\partial}{\partial u_1} + \frac{\partial}{\partial v_1}, t^{-2}\frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_1}, t^{-2}\frac{\partial}{\partial v_2}, \frac{\partial}{\partial w_1}, t^{-2}\frac{\partial}{\partial w_2}, \frac{\partial}{\partial t}, t^{-1}\left(w_1\frac{\partial}{\partial v_1} + w_2\frac{\partial}{\partial v_2} + \frac{\partial}{\partial z}\right)$$

and the dual frame is

$$\frac{1}{\pi}du_1, t^2 du_2, dv_1 - \frac{1}{\pi}du_1, t^2(dv_2 - w_2 dz), dw_1, t^2 dw_2, dt, tdz.$$

Straightforward computations show that the flat part of this LCP structure is exactly the integral manifold of the distribution spanned by $\frac{\partial}{\partial u}$. In addition, one has

$$\left[\frac{\partial}{\partial w_1}, w_1\frac{\partial}{\partial v_1} + w_2\frac{\partial}{\partial v_2} + \frac{\partial}{\partial z}\right] = \frac{\partial}{\partial v_1},$$

showing that the distribution orthogonal to the fibers of $\mathbb{R}^4 \times C \to C$ is not integrable, and in particular it is not possible to find a metric of the form (14) with $b_{FB} = 0$.

However, it is always possible to remove the non-constant translation part when G is a semi-direct product:

Proposition 5.6. *Assume that in Theorem 4.9 $G = \mathbb{Z}^p \rtimes \Omega$ and Ω acts freely on C. Then the translation part Ω can be assumed to contain only constants belonging to E^q up to a diffeomorphism.*

Proof. The assumption $G = \mathbb{Z}^p \rtimes \Omega$ allows us to define a subgroup $\tilde{\Omega}$ of G inducing a splitting of the short exact sequence $0 \to \mathbb{Z}^p \to G \to \Omega \to 0$. In particular, there is an isomorphism $\iota : \Omega \to \tilde{\Omega}$.

From the previous discussion, it is sufficient to find a function $s : C \to E^{p-q}$ which is Ω-invariant, i.e. which satisfies Equation 22 with $c \in E^q$ for any $\omega \in \Omega$. Since the linear part of Ω preserves the decomposition $E^q \oplus E^{p-q}$, it is sufficient to find s such that

$$\iota(\omega)|_{E^{p-q}\times C}(s(x), x) = (s(\omega|_C(x)), \omega|_C(x)), \qquad \forall \omega \in \Omega. \tag{28}$$

Consider the associated bundle

$$\mathcal{B} = C \times_\Omega E^{p-q},$$

where Ω acts on $C \times E^{p-q}$ by its natural action. This is a bundle over C/Ω, which is a compact manifold since Ω acts freely on C. Its typical fiber E^{p-q} is contractible, so it has a global smooth section [14, Corollary 29.3]. By a standard result, the sections of $\mathcal{B}$ are in one-to-one correspondence with the equivariant maps satisfying (28), which implies the existence of the map s we were searching for. $\square$

5.4. **Existence of the group Ω.** One question remains: when can one construct a group Ω as in Theorem 4.9? We know that C/Ω is a good compact orbifold since Ω acts properly and co-compactly on C. Thus, a way to answer this question is, starting from a good compact orbifold $\bar{C} = C/\pi_1(\bar{C})$ (where C is the universal cover of $\bar{C}$), to check if we can lift $\pi_1(\bar{C})$ to a subgroup Ω of the automorphisms of a trivial principal torus bundle over C. We give here a necessary condition for the existence of Ω, which turns out to be sufficient when $\bar{C}$ is manifold.

Proposition 5.7. *Let C be a simply-connected manifold and Ω be a group as given in the statement of Theorem 4.9. then Ω is isomorphic to a semi-direct product $\Omega' \rtimes \mathbb{Z}$ where Ω' is a subgroup of Ω.*

Conversely, if $\bar{C}$ is a compact manifold with universal cover C such that $\pi_1(\bar{C}) \simeq H \rtimes \mathbb{Z}$ for a subgroup H of $\pi_1(\bar{C})$, then there exists an integer $p \geq 2$ and a group $\Omega \subset \mathrm{Aut}((S^1)^p \times C \to C)$ as in Theorem 4.9.

Proof. Let $\rho : \Omega \to \mathbb{R}_+^*$ associating to $\omega \in \Omega$ the similarity ratio of its linear part restricted to E^q. The group $\rho(\Omega)$ is a subgroup of $\mathbb{R}_+^*$ with finite rank, generated by a finite number of independent elements. Let λ be one of these elements and let $\pi_1 : \rho(\Omega) \to \langle \lambda \rangle$ be the canonical projection. One has a short exact sequence

$$0 \to \ker(\pi_1 \circ \rho) \to \Omega \to \langle \lambda \rangle \simeq \mathbb{Z} \to 0,$$

and this sequence splits because one can find a section consisting of the map sending λ to an element $\omega \in \Omega$ such that $\rho(\omega) = \lambda$. Thus $\Omega \simeq \ker(\pi_1 \circ \rho) \rtimes \mathbb{Z}$.

We now prove the converse part. We can take $p = 2$ and define $\Omega \simeq H \rtimes \mathbb{Z}$ by $\omega = \mathrm{id}$ for $\omega \in H$ and $\omega = ((S^1)^p \times C \ni (\bar{a}, x) \mapsto (A\bar{a}, \omega(x)))$ for $\omega \in \mathbb{Z}$, where A is any suitable matrix (take for example the matrix defined in (23)). $\square$

Remark 5.8. if C/Ω is a manifold in Theorem 4.3, we can actually say more. Indeed, since the fibration $\mathbb{R}^p \times C \to C$ is a Riemannian fibration, the metric h on $\mathbb{R}^p \times C$ induces a metric g_B on C (we already emphasize this in this text). The fundamental group of the compact manifold C/Ω acts by similarities, not all being isometries on (C, g_B), so g_B is a non-Riemannian similarity structure on C/Ω. An application of Theorem 2.1 gives that (C, g_B) is either flat (a case which is classified [5]), irreducible, or an LCP manifold. The two main conjectures remaining are then the following: are all LCP manifolds simple? and given an LCP manifold, can we always say, up to a finite cover, that C/Ω is a manifold? This last problem was tackled in section 5.1. If the answers to these two questions are positive, then one could again decompose (C, g_B) using Theorem 4.3, and continue this process. This would end in a finite number of iterations, because the flat part is of positive dimension, leaving us with a flat or an irreducible manifold at the end.

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