# TORSION-FREE CONNECTIONS ON $G$-STRUCTURES

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ABSTRACT. We prove that for a Lie group $\mathrm{SO}_n(\mathrm{R}) \subset G \subset \mathrm{GL}_n(\mathrm{R})$, any $G$-structure on a smooth manifold can be endowed with a torsion free connection which is locally the Levi-Civita connection of a Riemannian metric in a given conformal class. In this process, we classify the admissible groups.

## 1. Introduction

Let $M$ be a smooth manifold of dimension $n$ and $G$ a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. A $G$-structure on $M$ is a reduction of the frame bundle of $M$ to $G$ [2, Chapter 4].

We recall the following well-known result (see for example [3]):

**Proposition 1.** *Let $G$ be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$ and let $P$ be a $G$-structure on $M$. Then, there exists a torsion-free connection on $P$.*

Proposition 1 was stated as an exercise in [2, Section 17.4, exercise (1)]. However, the author of this exercise believed that a more precise result occurs if we consider the following strategy of proof: we take a reduction of $P$ to $\mathrm{O}_n(\mathbb{R})$ or $\mathrm{SO}_n(\mathbb{R})$, which defines a Riemannian metric $g$, and then the Levi-Civita connection of $g$ is the desired torsion-free connection. This implies the stronger result that the connection on $P$ can be taken to be the Levi-Civita connection of a metric on $M$. However, such a reduction fails to exist in general, as shown by the following example:

**Example 2.** *We consider the circle $S^1 \subset \mathbb{C}$, parametrized by the map $\psi : [0, 2\pi) \ni \theta \to e^{i\theta}$. Its tangent bundle is given by $TS^1 \simeq S^1 \times \mathbb{R}$, and its frame bundle is $\mathrm{Fr}(S^1) \simeq S^1 \times \mathbb{R}^*$. Let $G$ be the closed subgroup of $\mathbb{R}^*$ generated by 2, and let $P$ be the $G$-structure of $S^1$ given by $P_{\psi(\theta)} = \{\psi(\theta)\} \times 2^{\frac{\theta}{2\pi}} G$ for any $\theta \in [0, 2\pi)$. There is no reduction of $P$ to $G \cap \mathrm{O}_1(\mathbb{R}) = \{1\}$ because $P$ is a non-trivial principal bundle. Note that the bundle $P$ is an embedding of the universal cover of $S^1$ into its frame bundle.*

Nevertheless, we can prove that the torsion-free connection in the setting of Proposition 1 is locally induced by a Riemannian metric, and this is the object of this note. This can be described more precisely using conformal geometry. A discussion about the following definitions can be found in [1, Section 2.2.2].

We recall that the conformal group $\mathrm{CO}_n(\mathbb{R})$ is the group of all matrices $\lambda S$ for $(\lambda, S) \in \mathbb{R}^* \times \mathrm{O}_n(\mathbb{R})$. Given a Riemannian metric $g$ on $M$, the conformal class $[g]$ of $g$ is the set of all the metrics $g'$ such that there exists a function $f : M \to \mathbb{R}$ satisfying $e^{2f} g' = g$. There is a one-to-one correspondence between $\mathrm{CO}_n(\mathbb{R})$-structures and conformal classes $[g]$, since any restriction of a $\mathrm{CO}_n(\mathbb{R})$-structure to $\mathrm{O}_n(\mathbb{R})$ defines a Riemannian metric $g$. We can then define the Weyl structures,

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which are the analogue in conformal geometry of the Levi-Civita connection in Riemannian geometry:

**Definition 3.** *A Weyl connection (or structure) on a conformal manifold* $(M,[g])$ *is a torsion-free connection* $\nabla$ *such that there exists a* 1*-form* $\theta$*, called the Lee form of* $\nabla$ *with respect to* $g$*, satisfying* $\nabla g = -2\theta\otimes g$*.*

*Equivalently, if* $P$ *is the* $\mathrm{CO}_n(\mathbb{R})$*-structure associated to the conformal class* $[g]$*, a Weyl structure is a torsion-free connection on* $P$*.*

In Definition 3, when the Lee form $\theta$ is closed, -which does not depend on the choice of the metric in the conformal class,- then the Weyl structure $\nabla$ is called *closed*. This is equivalent to the fact that around any point in $M$, $\nabla$ is the Levi-Civita connection of a metric in $[g]$.

With this definition, we state the result we will prove in this note:

**Theorem 4.** *Let* $M$ *be a smooth* $n$*-dimensional manifold. Let* $G$ *be a closed subgroup of* $\mathrm{GL}_n(\mathbb{R})$ *containing* $\mathrm{SO}_n(\mathbb{R})$ *and let* $P$ *be a* $G$*-structure on* $M$*. Then, there is a reduction* $Q$ *of* $P$ *to* $G\cap\mathrm{CO}_n(\mathbb{R})$ *and a torsion-free connection on* $Q$ *such that the connection induced on the* $\mathrm{CO}_n(\mathbb{R})$*-structure given by the extension of* $Q$ *to* $\mathrm{CO}_n(\mathbb{R})$ *is a closed Weyl structure.*

In order to illustrate this theorem and to give an idea of its proof, we come back to Example 2. Here, even if there is no reduction of the structure group to $\mathrm{SO}_1(\mathbb{R}) = \{1\}$, we can observe that $G$ seats inside $\mathrm{CO}_1(\mathbb{R})$ and is a discrete group. Hence, the pull-back of the bundle $P$ to the universal cover $\mathbb{R}$ of $S^1$ is a trivial bundle, whose total space is equal to $\{(x, 2^{\frac{x}{2\pi}+k}) \mid x\in\mathbb{R},\ k\in\mathbb{Z}\}$, and the projection onto the basis is the first projection. The metric on $\mathbb{R}$ given by $2^{-\frac{x}{\pi}}\langle\cdot,\cdot\rangle$ (where $\langle\cdot,\cdot\rangle$ stands for the standard metric) then induces a covariant derivative $\tilde{\nabla}$ on $\mathbb{R}$. Seeing the smooth sections of $T\mathbb{R}$ as smooth maps from $\mathbb{R}$ to $\mathbb{R}$, one has

$$\tilde{\nabla}_X Y = X\cdot\frac{d}{dx}Y - \frac{\ln 2}{2\pi}X\cdot Y$$

for any vector fields $X$, $Y$ on $\mathbb{R}$. This connection descends to a connection $\nabla$ on $S^1$ because the translation $x\mapsto x+2\pi$ is an affine map. Moreover, $\nabla$ is torsion-free and compatible with $P$, so it is the connection given by Theorem 4 (which is here unique since $G$ is discrete).

We quickly outline the proof of Proposition 1, using the analysis of [3, Chapter 4]. Denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $\mathrm{ad}P$ the adjoint bundle of $P$ (which is a vector subbundle of the bundle of endomorphisms of $TM$). The set of connections on $TM$ compatible with $P$ is an affine space modeled on $\Omega^1(M,\mathrm{ad}P)$. For any $\xi\in\Omega^1(M,\mathrm{ad}P)$, we define $(\partial\xi)(X,Y) := \xi(X)(Y)-\xi(Y)(X)$ where $X,Y\in TM$ and we consider the set

$$\mathcal{T}_P := \frac{\Omega^2(M,TM)}{\partial(\Omega^1(M,\mathrm{ad}P))}. \tag{1}$$

The *intrinsic torsion* $T_P^{\mathrm{int}}$ of $P$ is the equivalence class $[T_\nabla]\in\mathcal{T}_P$ where $T_\nabla$ is the torsion of any connection $\nabla$ compatible with $P$. This is well-defined because if $\nabla'$ is another connection, there is $\xi\in\Omega^1(M,\mathrm{ad}P)$ such that $\nabla' = \nabla+\xi$, and an easy computation leads to $T_{\nabla'} = T_\nabla + \partial(\xi)$. Then, there exists a torsion-free connection on $P$ if and only if $T_P^{\mathrm{int}} = 0$.

For any $x\in M$, fix a frame $u\in P_x$ (which identifies $\mathbb{R}^n$ with $T_xM$). For any $\phi\in\Lambda^2(\mathbb{R}^n)^*\otimes\mathbb{R}^n$, let $\xi\in(\mathbb{R}^n)^*\otimes\mathrm{End}(\mathbb{R}^n)$ be given by

$$2\xi(X)(Y) := \phi(X,Y) - \phi(X,\cdot)^*(Y) - \phi(Y,\cdot)^*(X) \qquad X,Y\in\mathbb{R}^n, \tag{2}$$

where "$*$" denotes the adjoint with respect to the standard metric on $\mathbb{R}^n$. By construction, one has $\partial\xi = \phi$ and $\xi(X)$ is skew-symmetric for every $X \in \mathbb{R}^n$. Since $\mathfrak{o}_n(\mathbb{R}) \subset \mathfrak{g}$, we have $\xi \in (\mathbb{R}^n)^* \otimes \mathfrak{g}$. We deduce that $\partial(\Omega^1(M, \mathrm{ad}P)) = \Omega^2(M, TM)$, implying $\mathfrak{T}_P = 0$, thus $T_P^{\mathrm{int}} = 0$, which gives the result.

## 2. Proof of Theorem 4

The proof of Theorem 4 relies on the classification of the subgroups of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$.

In all this text, we will denote by $\mathrm{Diag}(a_1, \dots, a_n)$ the diagonal matrix with diagonal coefficients $a_1, \dots, a_n$ (we will also use this notation for any block diagonal matrix) and we will denote by $sgn : \mathbb{R} \to \{-1, 0, 1\}$ the sign function. We first show the maximality of $\mathrm{SO}_n(\mathbb{R})$ in $\mathrm{SL}_n(\mathbb{R})$, which is a known result, but we recall a proof for the reader's convenance following partly an answer given by Yves Cornulier on the forum MathStackExchange.

**Lemma 5.** *Let $G$ be a subgroup of $\mathrm{SL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$. Then, $G = \mathrm{SL}_n(\mathbb{R})$ or $G = \mathrm{SO}_n(\mathbb{R})$.*

**Proof**. For $n = 1$ there is nothing to prove. For $n = 2$, suppose that there exists $A \in G \setminus \mathrm{SO}_2(\mathbb{R})$. Using the singular value decomposition, one can assume that $A = \mathrm{Diag}(a, \frac{1}{a})$ with $a > 1$. For $\theta \in \mathbb{R}$ let $R_\theta$ be the rotation of angle $\theta$. Let $\psi$ be the map which associates to an element of $\mathrm{SL}_n(\mathbb{R})$ the largest eigenvalue of the symmetric part of its polar decomposition. This map is continuous and one has $\psi(AR_0A) = a^2$ and $\psi(AR_{\pi/2}A) = 1$. Thus, by the intermediate value theorem, for any $x \in [1, a^2]$, the matrix $\mathrm{Diag}(x, \frac{1}{x})$ is in $G$, and this is true for any $k \in \mathbb{N}$ and $x \in [1, a^k]$ by induction, so it is true for any $x > 1$, which gives the result.

It remains to treat the case where $n \geq 3$ using the case $n = 2$. Assume that there is $A \in G \setminus \mathrm{SO}_n(\mathbb{R})$. We can assume that $A$ is diagonal with positive coefficients using the singular value decomposition, thus $A = \mathrm{Diag}(a_1, a_2, \dots, a_n)$, and conjugating by a suitable matrix in $\mathrm{SO}_n(\mathbb{R})$ we can assume that $a_1 \neq a_2$. Let $B \in \mathrm{SL}_n(\mathbb{R})$. We want to prove that $B \in G$. By another use of the singular value decomposition, we can assume that $B = \mathrm{Diag}(b_1, b_2, \dots, b_n)$ where the coefficients are positive. Moreover, one has

$$\begin{aligned}\mathrm{Diag}(b_1, b_2, \dots, b_n) = \mathrm{Diag}(b_1, \frac{1}{b_1}, 1, \dots, 1) \cdot \mathrm{Diag}(1, b_1b_2, \frac{1}{b_1b_2}, 1, \dots, 1)\\ \dots \mathrm{Diag}(1, \dots, 1, (b_1 \dots b_{n-1}), \frac{1}{b_1 \dots b_{n-1}} = b_n),\end{aligned}$$

thus it is sufficient to prove that each of the factors appearing in the right-hand side are in $G$. By conjugating by a suitable element of $\mathrm{SO}_n(\mathbb{R})$, it is sufficient to prove that for any $x > 0$ the matrix $\mathrm{Diag}(x, x^{-1}, 1, \dots, 1)$ is in $G$. We now consider the matrix $R = \mathrm{Diag}(R_{\pi/2}, I_{n-2}) \in \mathrm{SO}_n(\mathbb{R})$, and we remark that

$$A^{-1}RAR^{-1} = \mathrm{Diag}(\frac{a_2}{a_1}, \frac{a_1}{a_2}, 1, \dots, 1).$$

Since $\frac{a_2}{a_1} \neq 1$, the case $n = 2$ implies that $G$ contains all the matrices $\mathrm{Diag}(x, x^{-1}, 1, \dots, 1)$ for $x > 0$, which gives that $B \in G$ and concludes the proof. □

The following lemma will also be important in the description of the groups containing $\mathrm{SO}_n(\mathbb{R})$.

**Lemma 6.** *Let $G$ be a subgroup of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$, and let $x \in \det(G)$. Then, $|x|^{\frac{1}{n}} \mathrm{Diag}(sgn(x), 1, \dots, 1) \in G$.*

**Proof**. Let $x \in \det(G)$. There is a matrix $A \in G$ such that $\det(A) = x$, and using the singular value decomposition of $A$, there is a diagonal matrix $D \in G$ with $\det(D) = x$. If $D$ is of the form $|x|^{\frac{1}{n}}\mathrm{Diag}(\pm 1, \ldots, \pm 1)$ we have the conclusion of the lemma after multiplying by an element of $\mathrm{SO}_n(\mathbb{R})$ of the form $\mathrm{Diag}(\pm 1, \ldots, \pm 1)$, so we assume that $D^2 \notin \mathrm{Span}(I_n)$. There is a matrix $S \in \mathrm{SO}_n(\mathbb{R})$ with $SD^2 \neq D^2 S$. Let $B := D^{-1}S^T DS \in \mathrm{SL}_n(\mathbb{R})$. One has

$$BB^T = D^{-1}S^T DSS^T DSD^{-1} = D^{-1}S^T D^2 SD^{-1} = (D^{-1}SD)^{-1}(DSD^{-1}),$$

then

$$BB^T = I_n \Leftrightarrow D^{-1}SD = DSD^{-1} \Leftrightarrow D^2 S = SD^2,$$

and this last assertion is false, thus $BB^T \neq I_n$ and $B \notin \mathrm{SO}_n(\mathbb{R})$. By Lemma 5, we conclude that $G \cap \mathrm{SL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})$, and in particular $|x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \ldots, 1)D^{-1} \in G$, so $|x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \ldots, 1) \in G$ after multiplication by $D$ on the right. $\square$

One writes $\mathrm{GL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R}) \rtimes \mathbb{R}^*$ with the identification $\{\mathrm{Id}\} \rtimes \mathbb{R}^* \to \mathrm{GL}_n(\mathbb{R}), x \mapsto |x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \ldots, 1)$. We finally give the classification result:

**Proposition 7.** *Let $G$ be a subgroup of $\mathrm{GL}_n(\mathbb{R})$ containing $\mathrm{SO}_n(\mathbb{R})$. There exists a subgroup $H$ of $(\mathbb{R}^*, \times)$ such that $G$ is equal to either $\mathrm{SO}_n(\mathbb{R}) \rtimes H$ or $\mathrm{SL}_n(\mathbb{R}) \rtimes H$. Moreover, if $G$ is closed, so is $H$.*

**Proof**. One has the following short exact sequence:

$$0 \to \mathrm{SL}_n(\mathbb{R}) \cap G \to G \overset{\det}{\to} \det(G) \to 1. \tag{3}$$

Now, let $\phi : H := \det(G) \to G$ given by $\phi(x) = |x|^{\frac{1}{n}}\mathrm{Diag}(sgn(x), 1, \ldots, 1)$, which is well-defined by Proposition 6. It is clear that $\phi$ is a morphism and $\det \circ \phi = id_H$, thus one has $G = (\mathrm{SL}_n(\mathbb{R}) \cap G) \rtimes H$. Moreover, by Lemma 5 one has $\mathrm{SL}_n(\mathbb{R}) \cap G = \mathrm{SL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{R}) \cap G = \mathrm{SO}_n(\mathbb{R})$ because $G$ contains $\mathrm{SO}_n(\mathbb{R})$.

It remains to show that $H$ is closed when $G$ is closed. But if $H$ is non-discrete, $H \cap \mathbb{R}^*_+$ has to be dense in $\mathbb{R}^*_+$, so, $G$ being closed, it contains all the matrices of the form $|x|^{\frac{1}{n}} I_n$, $x \in \mathbb{R}$, and then $H = \det G = \mathbb{R}^*_+$ or $\mathbb{R}^*$. $\square$

**Remark 8.** Note that in Proposition 7, the semi-direct product is actually direct when $H \subset \mathbb{R}^*_+$ or when $n$ is odd.

Finally, we give the proof of the main theorem, for which we recall the following definition:

**Definition 9.** *Let $(N_1, g_1)$, $(N_2, g_2)$ be two Riemannian manifolds. A similarity between $N_1$ and $N_2$ is a diffeomorphism $\phi : N_1 \to N_2$ such that there exists $\lambda \in \mathbb{R}^+_*$ with $\lambda^2 g_1 = \phi^* g_2$. In this case, $\lambda$ is called the ratio of the similarity.*

*Proof of Theorem 4.* According to Lemma 7, there is a closed subgroup $H$ of $\mathbb{R}^*$ such that $G \simeq \mathrm{SO}_n(\mathbb{R}) \rtimes H$ or $\mathrm{SL}_n(\mathbb{R}) \rtimes H$. From the classification of the subgroups of $\mathbb{R}^*$, $H$ is either $\mathbb{R}^*$, $\mathbb{R}^*_+$ or discrete.

**First case: $H = \mathbb{R}^*$ or $H = \mathbb{R}^*_+$.** In this case, $G$ is either $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{CO}_n(\mathbb{R})$ or $\mathrm{GL}^+_n(\mathbb{R})$ or $\mathrm{CO}^+_n(\mathbb{R})$. In all these cases, there is a metric $g$ compatible with the $G$-structure, i.e. a reduction $P'$ of $P$ to $G \cap \mathrm{O}_n(\mathbb{R})$. Then, the Levi-Civita connection of $g$ is torsion-free, so it induces a torsion-free connection on $P'$, and thus a torsion-free connection on the extension $Q$ of $P'$ to $G \cap \mathrm{CO}_n(\mathbb{R})$. The resulting connection on the extension of $Q$ to $\mathrm{CO}_n(\mathbb{R})$ is a closed (actually exact) Weyl structure because it is induced by the Levi-Civita connection of a metric on $M$.

**Second case: $H$ is discrete.** Let $\widetilde{M}$ be the universal cover of $M$ and let $\widetilde{P}$ be the pull-back of $P$ to $\widetilde{M}$.

We first study the case $G = \mathrm{SO}_n(\mathbb{R}) \rtimes H$. Then, the $H$-principal bundle $\widetilde{P}/\mathrm{SO}_n(\mathbb{R})$ is a covering of $\widetilde{M}$ so it is trivial. Every element $a \in H$ thus defines an $\mathrm{SO}_n(\mathbb{R})$-structure on $\widetilde{M}$ i.e. a metric $\widetilde{g}$. Since $\pi_1(M)$ acts on $P/\mathrm{SO}_n(\mathbb{R})$ by multiplication by an element of $H$, we deduce that $\pi_1(M)$ acts by similarities on $(\widetilde{M}, \widetilde{g})$. Consequently, the Levi-Civita connection of $\widetilde{g}$ induces a torsion-free connection on $\widetilde{P}$ which descends to a torsion-free connection on $P$. We can take $Q := P$ in the statement of the theorem since $G \subset \mathrm{CO}_n(\mathbb{R})$. Finally, the resulting connection on the extension of $P$ to $\mathrm{CO}_n(\mathbb{R})$ is a closed Weyl structure because it is locally given by the Levi-Civita covariant derivative of a Riemannian metric defined by a local reduction of $P$ to $G \cap \mathrm{O}_n(\mathbb{R})$.

We consider now the case $G = \mathrm{SL}_n(\mathbb{R}) \rtimes H$. Just as before, the $H$-principal bundle $\widetilde{P}/\mathrm{SL}_n(\mathbb{R})$ is trivial. Choosing an element $a \in H$ defines a $\mathrm{SL}_n(\mathbb{R})$-structure $\widetilde{Q}$ on $\widetilde{M}$ i.e. a volume form $\widetilde{v}$, and in particular an orientation on $\widetilde{M}$. Let $h$ be a Riemannian metric on $M$, and let $\widetilde{h}$ be its pull-back to $\widetilde{M}$. Let $v_h$ be the volume with respect to $\widetilde{v}$ of a $\widetilde{h}$-orthonormal frame of $T\widetilde{M}$ (note that $v_h^2$ does not depend on the choice of the frame). We define $\widetilde{g} := (v_h^2)^{\frac{1}{n}}\widetilde{h}$. Then, any oriented $\widetilde{g}$-orthonormal frame has volume 1 with respect to $\widetilde{v}$. This implies that $\widetilde{g}$ defines a reduction of $\widetilde{Q}$ to $\mathrm{SO}_n(\mathbb{R})$. As in the previous case, $\pi_1(M)$ acts on $P/\mathrm{SL}_n(\mathbb{R})$ by multiplication by an element of $H$, so for $\gamma \in \pi_1(M)$, $\gamma^*\widetilde{v}$ is a multiple of $\widetilde{v}$. Since, $\pi_1(M)$ acts by isometries on $(\widetilde{M}, \widetilde{h})$, it acts by similarities on $(\widetilde{M}, \widetilde{g})$. We finally conclude in the same way as for the case $G = \mathrm{SO}_n(\mathbb{R}) \rtimes H$. □

From the proof we see that the principal bundle $Q$ defined in Theorem 4 has $\mathrm{SO}_n(\mathbb{R}) \rtimes H'$ as structure group, where $H'$ is a discrete subgroup of $\mathbb{R}_+^*$ (just take $H' := \{1\}$ when $H = \mathbb{R}^*$ or $\mathbb{R}_+^*$, and $H' := H$ otherwise).

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