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YANG–BAXTER DEFORMATIONS AND RACK COHOMOLOGY

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Abstract. In his study of quantum groups, Drinfeld suggested to consider set-theoretic solutions of the Yang–Baxter equation as a discrete analogon. As a typical example, every conjugacy class in a group, or more generally every rack Q provides such a Yang–Baxter operator $c_Q\colon x \otimes y \mapsto y \otimes x^y$. In this article we study deformations of c_Q within the space of Yang–Baxter operators. Infinitesimally these deformations are classified by Yang–Baxter cohomology. We show that the Yang–Baxter cochain complex of c_Q homotopy-retracts to a much smaller subcomplex, called quasi-diagonal. This greatly simplifies the deformation theory of c_Q, including the modular case which had previously been left in suspense, by establishing that every deformation of c_Q is gauge equivalent to a quasi-diagonal one. In a quasi-diagonal deformation only behaviourally equivalent elements of Q interact; if all elements of Q are behaviourally distinct, then the Yang–Baxter cohomology of c_Q collapses to its diagonal part, which we identify with rack cohomology. This establishes a strong relationship between the classical deformation theory following Gerstenhaber and the more recent cohomology theory of racks, both of which have numerous applications in knot theory.

1. Introduction and statement of results

1.1. **Motivation and background.** Yang–Baxter operators (recalled in §2) first appeared in theoretical physics: in a 1967 paper by Yang [44] on the many-body problem in one dimension, during the 1970s in work by Baxter [3, 4] on exactly solvable models in statistical mechanics, and later in quantum field theory (Faddeev [19]). They correspond to tensor representations of braid groups and have thus played a prominent rôle in knot theory and low-dimensional topology ever since the discovery of the Jones polynomial [28] in 1984. Attempts to systematically construct solutions of the Yang–Baxter equation have led to the theory of *quantum groups*, see Drinfeld [11] and Turaev, Kassel, Rosso [40, 41, 31, 32].

Yang–Baxter operators resulting from the quantum approach are deformations of the transposition operator $\tau\colon x \otimes y \mapsto y \otimes x$. As a consequence, the associated knot invariants are of finite type in the sense of Vassiliev [42] and Gusarov [27], see also Birmann–Lin [6] and Bar-Natan [2]. These invariants continue to have a profound impact on low-dimensional topology; their interpretation in terms of classical algebraic topology, however, remains difficult and most often mysterious.

As a discrete analogon, Drinfeld [12] pointed out set-theoretic solutions, which have been studied by Etingof–Schedler–Soloviev [18] and Lu–Yan–Zhu [36], among others. An important class of such solutions is provided by *racks* or *automorphic*

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sets $(Q,*)$, which have been studied by Brieskorn [7] in the context of braid group actions. Here the Yang–Baxter operator takes the form $c_Q\colon x\otimes y\mapsto y\otimes x^y$, where $x^y = x*y$ denotes the action of the rack Q on itself. The transposition τ corresponds to the trivial action, whereas conjugation $x^y = y^{-1}xy$ in a group provides many non-trivial examples. Applications to knot theory were independently developed by Joyce [30] and Matveev [38]. Freyd and Yetter [24] observed that the knot invariants obtained from c_Q are the well-known colouring numbers of classical knot theory. These invariants, in contrast, are not of finite type [13].

Freyd and Yetter [24, 45] also initiated the question of deforming set-theoretic solutions within the space of Yang–Baxter operators over a ring $\mathbb{A}$, following Gerstenhaber's paradigm of algebraic deformation theory [25], and illustrated their general approach by the simplified ansatz of diagonal deformations [24, §4]. The latter are encoded by rack cohomology, which was independently developed by Fenn and Rourke [21] from a homotopy-theoretic viewpoint via classifying spaces. Carter *et al.* [10] have applied rack and quandle cohomology to knots by constructing state-sum invariants. These, in turn, can be interpreted in terms of classical algebraic topology as colouring polynomials associated to knot group representations [16].

1.2. **Yang–Baxter deformations.** In this article we continue the study of Yang–Baxter deformations of racks linearized over a ring $\mathbb{A}$. Detailed definitions will be given in §2, in particular we will review Yang–Baxter operators (§2.1), set-theoretic solutions coming from racks (§2.2) and their deformation theory (§2.3). In this introduction we merely recall the basic definitions in order to state our main result.

Notation (modules)**.** Throughout this article $\mathbb{A}$ denotes a commutative ring with unit. All modules will be $\mathbb{A}$-modules, all maps between modules will be $\mathbb{A}$-linear, and all tensor products will be formed over $\mathbb{A}$. For every $\mathbb{A}$-module V we denote by $V^{\otimes n}$ the tensor product $V\otimes\cdots\otimes V$ of n copies of V. Given a set Q we denote by $\mathbb{A}Q$ the free $\mathbb{A}$-module with basis Q. We identify the n-fold tensor product $(\mathbb{A}Q)^{\otimes n}$ with $\mathbb{A}Q^n$. This choice of bases allows us to identify $\mathbb{A}$-linear maps $f\colon \mathbb{A}Q^n\to\mathbb{A}Q^n$ with matrices $f\colon Q^n\times Q^n\to\mathbb{A}$, whose coefficients are denoted by $f\left[\begin{smallmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{smallmatrix}\right]$.

For the purposes of deformation theory we equip $\mathbb{A}$ with a fixed ideal $\mathfrak{m}\subset\mathbb{A}$. Most often we require that $\mathbb{A}$ be complete with respect to $\mathfrak{m}$, that is, we assume that the natural map $\mathbb{A}\to\varprojlim \mathbb{A}/\mathfrak{m}^n$ is an isomorphism (§6). A typical setting is the power series ring $\mathbb{K}[\![h]\!]$ over a field $\mathbb{K}$, equipped with its maximal ideal $\mathfrak{m}=(h)$, or the ring of p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/_{p^n}$ with its maximal ideal (p).

Notation (racks)**.** A *rack* or *automorphic set* $(Q,*)$ is a set Q equipped with an operation $*\colon Q\times Q\to Q$ such that every right translation $x\mapsto x*y$ is an automorphism of $(Q,*)$. This is equivalent to saying that the $\mathbb{A}$-linear map $c_Q\colon \mathbb{A}Q\otimes\mathbb{A}Q\to\mathbb{A}Q\otimes\mathbb{A}Q$ defined by $c_Q\colon x\otimes y\mapsto y\otimes(x*y)$ for all $x,y\in Q$ is a Yang–Baxter operator over the ring $\mathbb{A}$ (see §2.1).

Two rack elements $y,z\in Q$ are called *behaviourally equivalent*, denoted $y\equiv z$, if they satisfy $x*y=x*z$ for all $x\in Q$. This is equivalent to saying that y,z have the same image under the inner representation $\rho\colon Q\to\operatorname{Inn}(Q)$. As usual, a matrix $f\colon Q^n\times Q^n\to\mathbb{A}$ is called *diagonal* if $f\left[\begin{smallmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{smallmatrix}\right]$ vanishes whenever $x_i\neq y_i$. It is called *quasi-diagonal* if $f\left[\begin{smallmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{smallmatrix}\right]$ vanishes whenever $x_i\not\equiv y_i$.

Quasi-diagonal maps play a crucial rôle in the classification of deformations:

Theorem 1.1. *Suppose that the ring $\mathbb{A}$ is complete with respect to the ideal $\mathfrak{m}$. Then every Yang–Baxter deformation c of c_Q over $\mathbb{A}$ is equivalent to a quasi-diagonal deformation. More explicitly this means that c is conjugated to a deformation of the form $c_Q \circ (\mathrm{id}^{\otimes 2} + f)$ where the deformation term $f\colon \mathbb{A}Q^2 \to \mathfrak{m}Q^2$ is quasi-diagonal.*

There are thus two extreme cases in the deformation theory of racks:

(1) In the one extreme the rack Q is trivial, whence $\rho\colon Q \to \mathrm{Inn}(Q)$ is trivial and all elements of Q are behaviourally equivalent. This is the initial setting in the theory of quantum invariants and we cannot add anything new here.
(2) In the other extreme, where $\rho\colon Q \to \mathrm{Inn}(Q)$ is injective, all elements of Q are behaviourally distinct, and every deformation of c_Q is equivalent to a diagonal deformation. This is the setting of rack cohomology.

In other words, the more inner symmetries Q has, the less deformations c_Q admits. Our theorem makes the transition between the two extremes precise and quantifies the degree of deformability of set-theoretic Yang–Baxter operators.

Example 1.2. Consider a group $(G, \cdot)$ that is generated by one of its conjugacy classes $Q \subset G$. Then $(Q, *)$ is a rack with respect to conjugation $x*y = y^{-1}\cdot x\cdot y$, and we have a natural isomorphism $\mathrm{Inn}(Q, *) \cong \mathrm{Inn}(G, \cdot)$. If the centre of G is trivial, then the injectivity of $\rho\colon Q \to \mathrm{Inn}(Q)$ implies that every Yang–Baxter deformation of c_Q is equivalent to a diagonal deformation.

As pointed out above, diagonal deformations have received much attention over the last 20 years [24, 21, 45, 10]. It is reassuring that Theorem 1.1 justifies this short-cut in the case where $\rho\colon Q \to \mathrm{Inn}(Q)$ is injective. In general, however, the simplified ansatz of diagonal deformations may miss some interesting Yang–Baxter deformations, namely those that are quasi-diagonal but not diagonal. For more detailed examples and applications see §7.

1.3. **Yang–Baxter cohomology.** Our approach to proving Theorem 1.1 follows the classical paradigm of studying algebraic deformation theory via cohomology, as expounded by Gerstenhaber [25]. Since it may be of independent interest, we state here our main cohomological result, which in degree 2 proves the infinitesimal version of Theorem 1.1.

The previous article [15] introduced Yang–Baxter cohomology $H^*_{\mathrm{YB}}(c_Q; \mathfrak{m})$ to encode infinitesimal deformations of c_Q over a ring $\mathbb{A}$ with respect to the ideal $\mathfrak{m}$ (§2.3). There the second cohomology $H^2_{\mathrm{YB}}(c_Q; \mathfrak{m})$ was calculated under the hypothesis that the order of the inner automorphism group $\mathrm{Inn}(Q)$ is finite and invertible in the ring $\mathbb{A}$. The main application was to Yang–Baxter operators c_Q derived from a finite rack Q and deformed over the ring $\mathbb{A} = \mathbb{Q}[\![h]\!]$. In many cases the results of [15] imply that c_Q is rigid over $\mathbb{Q}[\![h]\!]$.

In the present article we calculate Yang–Baxter cohomology $H^*_{\mathrm{YB}}(c_Q; \mathfrak{m})$ in general, including the modular case that had previously been left in suspense [15, Question 39]. As our main result we establish the following classification; for detailed definitions and proofs we refer to §5.

Theorem 1.3. *The quasi-diagonal subcomplex $C^*_\Delta(c_Q; \mathfrak{m}) \subset C^*_{\mathrm{YB}}(c_Q; \mathfrak{m})$ is a homotopy retract, whence the induced map $H^*_\Delta(c_Q; \mathfrak{m}) \to H^*_{\mathrm{YB}}(c_Q; \mathfrak{m})$ is an isomorphism.*

Contrary to [15] we no longer require the rack Q to be finite, nor do we impose any restrictions on the characteristic of the ring $\mathbb{A}$. This opens up the way to study the modular case, for example p-adic deformations of c_Q where p divides $|\mathrm{Inn}(Q)|$.

Remark 1.4. Yang–Baxter cohomology includes rack cohomology $H^*_{\mathrm{R}}(Q;\Lambda)$ as its diagonal part, as explained in §3, where Λ is a module over some ring $\mathbb{K}$. If $|\mathrm{Inn}(Q)|$ is invertible in $\mathbb{K}$, then $H^*_{\mathrm{R}}(Q;\Lambda)$ is trivial in a certain sense, see Etingof–Graña [17]. The modular case, however, leads to topologically interesting rack deformations (§7.6). Since the full Yang–Baxter cohomology of racks vastly extends rack cohomology, the modular case stood out as a difficult yet promising problem.

Theorem 1.3 solves this problem: it shows that the right object to study is the quasi-diagonal subcomplex C^*_Δ, situated between the strictly diagonal complex C^*_{Diag} and the full Yang–Baxter complex C^*_{YB}, i.e., we have $C^*_{\mathrm{Diag}} \subset C^*_\Delta \subset C^*_{\mathrm{YB}}$. We will see that the inclusion $C^*_{\mathrm{Diag}} \subset C^*_{\mathrm{YB}}$ allows for a retraction $C^*_{\mathrm{YB}} \twoheadrightarrow C^*_{\mathrm{Diag}}$, which entails that H^*_{Diag} is a direct summand of H^*_{YB}. In general, however, this is not a homotopy retraction and $H^*_{\mathrm{Diag}} \subsetneqq H^*_{\mathrm{YB}}$. The inclusion $\iota\colon C^*_\Delta \subset C^*_{\mathrm{YB}}$ also allows for a retraction $\pi\colon C^*_{\mathrm{YB}} \twoheadrightarrow C^*_\Delta$, such that $\pi\circ\iota = \mathrm{id}_\Delta$, and our main result is the construction of a homotopy $\iota\circ\pi \simeq \mathrm{id}_{\mathrm{YB}}$.

Remark 1.5. Again we have two extreme cases that are particularly clear-cut:

(1) In the one extreme, if Q is trivial, then all elements of Q are behaviourally equivalent. In this case we trivially have $C^*_\Delta = C^*_{\mathrm{YB}}$.
(2) If $\rho\colon Q \to \mathrm{Inn}(Q)$ is injective, then all elements of Q are behaviourally distinct. In this case quasi-diagonal means diagonal, whence $C^*_\Delta = C^*_{\mathrm{Diag}}$.

In general C^*_Δ lies strictly between C^*_{Diag} and C^*_{YB}, and retracting the full Yang–Baxter complex C^*_{YB} to its quasi-diagonal subcomplex C^*_Δ greatly simplifies the problem. It often reduces the complexity from $|Q|^4$ unknowns to the order of $|Q|^2$ unknowns, which in many cases makes it amenable to computer calculations (§7).

1.4. **How this article is organized.** Section 2 recollects the relevant definitions concerning Yang–Baxter operators, their deformations and cohomology. It also gives explicit formulae in the case of racks, which is our main focus here. Section 3 identifies diagonal deformations with rack cohomology, and Section 4 introduces quasi-diagonal deformations. Section 5 proves our main result in the infinitesimal case, by constructing a homotopy retraction from the full Yang–Baxter complex to its quasi-diagonal subcomplex. Section 6 extends the infinitesimal result to complete deformations, and Section 7 provides examples and applications. We conclude with some open questions in Section 8.

2. Yang–Baxter operators, deformations, and cohomology

This section provides the necessary background of Yang–Baxter operators (§2.1) and racks (§2.2) and fixes notation. The notion of Yang–Baxter deformation and cohomology (§2.3) is recalled from [15]. We add here the dual notion of Yang–Baxter homology (§2.4) and the observation of non-functoriality (§2.5).

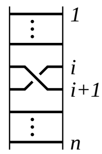
2.1. Yang–Baxter operators.

Definition 2.1. Let V be a module over the ring $\mathbb{A}$. A *Yang–Baxter operator* on V is an automorphism $c\colon V\otimes V \to V\otimes V$ that satisfies the *Yang–Baxter equation*

$$(2.1)\quad (c\otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c\otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c\otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c) \quad\text{in}\quad \mathrm{Aut}_{\mathbb{A}}(V^{\otimes 3}).$$

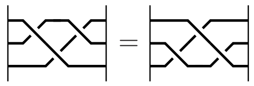
This equation first appeared in theoretical physics (Yang [44], Baxter [3, 4], Faddeev [19]). It also has a natural interpretation in terms of Artin's braid group B_n [1, 5] and its tensor representations:

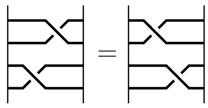
The automorphisms $c_1, \dots, c_{n-1} \colon V^{\otimes n} \to V^{\otimes n}$ defined by

$$c_i = \mathrm{id}_V^{\otimes (i-1)} \otimes c \otimes \mathrm{id}_V^{\otimes (n-i-1)}, \qquad \text{or in graphical notation } c_i =$$ 

,

satisfy the well-known braid relations

(2.2) $$c_i c_j c_i = c_j c_i c_j \quad \text{if } |i-j| = 1,$$ 
,

(2.3) $$c_i c_j = c_j c_i \quad \text{if } |i-j| \geq 2,$$ 
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Equation (2.2) is a reformulation of the Yang–Baxter equation (2.1), while the commutativity relation (2.3) follows automatically from the tensor construction.

Remark 2.2. A graphical notation for tensor calculus was first used by Penrose [39]; for a brief discussion of its history see Joyal–Street [29]. This notation has the obvious advantage to appeal to our geometric vision. More importantly, it incorporates a profound relationship with knot theory and leads to invariants of knots and links in $\mathbb{R}^3$:

$$\begin{array}{ccc} B_n & \xrightarrow[\text{representation}]{\text{Yang–Baxter}} & \mathrm{Aut}(V^{\otimes n}) \\ {\scriptstyle\text{closure}}\big\downarrow & & \big\downarrow{\scriptstyle\text{trace}} \\ \{\text{links}\} & \xrightarrow[\text{invariant}]{} & \mathbb{A} \end{array}$$

Each link L can be presented as the closure of some braid. This braid acts on $V^{\otimes n}$ as defined above, and a suitably deformed trace maps it to the ring $\mathbb{A}$. In favourable cases the result does not depend on the choice of braid, and thus defines an invariant of the link L, see Turaev [41, chap. I] or Kassel [31, chap. X].

2.2. **Quandles and racks.** In every group $(G, \cdot)$ the conjugation $a * b = b^{-1} \cdot a \cdot b$ enjoys the following properties:

- (Q1) For every element a we have $a * a = a$. (idempotency)
- (Q2) Every right translation $\varrho(b) \colon a \mapsto a * b$ is a bijection. (right invertibility)
- (Q3) For all a, b, c we have $(a * b) * c = (a * c) * (b * c)$. (self-distributivity)

Taking these properties as axioms, Joyce [30] defined a *quandle* to be a set Q equipped with a binary operation $* \colon Q \times Q \to Q$ satisfying (Q1–Q3). Independently, Matveev [38] studied the equivalent notion of *distributive groupoid*. Following Brieskorn [7], an *automorphic set* is only required to satisfy (Q2–Q3): these two axioms are equivalent to saying that every right translation is an automorphism of $(Q, *)$. The shorter term *rack* was suggested by Fenn and Rourke [21], going back to the terminology *wrack* used by J.H. Conway in correspondence with G.C. Wraith in 1959. Such structures appear naturally in the study of braid actions [7] and provide set-theoretic solutions of the Yang–Baxter equation [12]:

Proposition 2.3. *Given a set Q with binary operation $*\colon Q \times Q \to Q$, we can consider the free module $V = \mathbb{A}Q$ with basis Q over the ring $\mathbb{A}$ and define the $\mathbb{A}$-linear operator*

$$c_Q\colon \mathbb{A}Q \otimes \mathbb{A}Q \to \mathbb{A}Q \otimes \mathbb{A}Q \quad \textit{by} \quad a \otimes b \mapsto b \otimes (a * b) \quad \textit{for all} \quad a, b \in Q.$$

Then c_Q is a Yang–Baxter operator if and only if Q is a rack. □

A *homomorphism* between two racks $(Q, *)$ and $(Q', *')$ is a map $\phi\colon Q \to Q'$ satisfying $\phi(a * b) = \phi(a) *' \phi(b)$ for all $a, b \in Q$. Racks and their homomorphisms form a category in the usual way. The *automorphism group* $\operatorname{Aut}(Q)$ of the rack Q consists of all bijective homomorphisms $\phi\colon Q \to Q$. We adopt the convention that automorphisms of Q act on the right, written a^ϕ, which means that their composition $\phi\psi$ is defined by $a^{(\phi\psi)} = (a^\phi)^\psi$ for all $a \in Q$.

Each $a \in Q$ defines an automorphism $\rho(a) \in \operatorname{Aut}(Q)$ by $x \mapsto x * a$. For every $\phi \in \operatorname{Aut}(Q)$ we have $\rho(a^\phi) = \rho(a)^\phi$. The group $\operatorname{Inn}(Q)$ of *inner automorphisms* is the normal subgroup of $\operatorname{Aut}(Q)$ generated by all right translations $\rho(a)$, where $a \in Q$. The *inner representation* $\rho\colon Q \to \operatorname{Inn}(Q)$ satisfies $\rho(a * b) = \rho(a) * \rho(b)$, that is, it maps the operation of the rack Q to conjugation in the group $\operatorname{Inn}(Q)$.

Notation. In view of the representation $\rho\colon Q \to \operatorname{Inn}(Q)$, we often write a^b for the operation $a^{\rho(b)} = a * b$. Conversely, it will sometimes be convenient to write $a * b$ for the conjugation $a^b = b^{-1}ab$ in a group.

Definition 2.4. Two elements $x, y \in Q$ are *behaviourally equivalent* if $a * x = a * y$ for all $a \in Q$. This means that $\rho(x) = \rho(y)$, and will be denoted by $x \equiv y$ for short.

2.3. **Deformations and cohomology.** We are interested here in set-theoretic solutions of the Yang–Baxter equation and their deformations within the space of Yang–Baxter operators over some ring.

Definition 2.5. We fix an ideal $\mathfrak{m}$ in the ring $\mathbb{A}$. Consider an $\mathbb{A}$-module V and a Yang–Baxter operator $c\colon V \otimes V \to V \otimes V$.

A map $\tilde{c}\colon V \otimes V \to V \otimes V$ is called a *Yang–Baxter deformation* of c with respect to $\mathfrak{m}$ if $\tilde{c}$ is itself a Yang–Baxter operator and satisfies $\tilde{c} \equiv c$ modulo $\mathfrak{m}$.

An *equivalence transformation*, or *gauge equivalence* with respect to $\mathfrak{m}$, is an automorphism $\alpha\colon V \to V$ satisfying $\alpha \equiv \mathrm{id}_V$ modulo $\mathfrak{m}$.

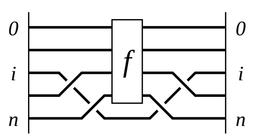
Two Yang–Baxter operators c and $\tilde{c}$ are called *equivalent* if there exists an equivalence transformation $\alpha\colon V \to V$ such that $\tilde{c} = (\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha)$.

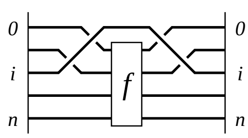
In order to study deformations it is useful to linearize the problem by considering infinitesimal deformations, where $\mathfrak{m}^2 = 0$. To this end we recall the definition of Yang–Baxter cohomology $H^*_{\mathrm{YB}}(c; \mathfrak{m})$ that encodes infinitesimal deformations.

Definition 2.6. The Yang–Baxter cochain complex $C^*_{\mathrm{YB}}(c; \mathfrak{m})$ consists of the $\mathbb{A}$-modules $C^n = \operatorname{Hom}(V^{\otimes n}, \mathfrak{m}V^{\otimes n})$. For each $f \in C^n$ we define the partial coboundary $d_i^n f \in C^{n+1}$ by

$$d_i^n f = (c_n \cdots c_{i+1})^{-1} \, (f \otimes \mathrm{id}_V) \, (c_n \cdots c_{i+1}) - (c_1 \cdots c_i)^{-1} \, (\mathrm{id}_V \otimes f) \, (c_1 \cdots c_i) \,. \tag{2.4}$$

This formula becomes more suggestive in graphical notation:

(2.5) $d_i^n f = \; +$ 

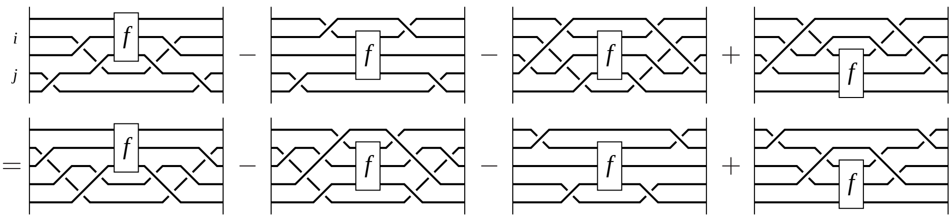
$-$ 

.

The coboundary is defined as the alternating sum $d^n = \sum_{i=0}^{n} (-1)^i d_i^n$.

Proposition 2.7. *We have $d_j^{n+1} d_i^n = d_i^{n+1} d_{j-1}^n$ for $i < j$, whence $d^{n+1} d^n = 0$.*

Proof. Our hypothesis that c is a Yang–Baxter operator ensures that $d_j^{n+1} d_i^n = d_i^{n+1} d_{j-1}^n$ for $i < j$. This can be proven by a straightforward computation, and is most easily verified using the graphical tensor calculus suggested above:



It follows that all terms cancel each other in pairs to yield $d^{n+1} \circ d^n = 0$. □

Definition 2.8. The cochain complex $C^*_{\mathrm{YB}}(c; \mathfrak{m}) = (C^*, d^*)$ is called the *Yang–Baxter cochain complex*, and its cohomology $H^*_{\mathrm{YB}}(c; \mathfrak{m})$ is called the *Yang–Baxter cohomology* of the operator c with respect to the ideal $\mathfrak{m}$.

Proposition 2.9. *The second cohomology $H^2_{\mathrm{YB}}(c; \mathfrak{m})$ classifies infinitesimal Yang–Baxter deformations: assuming $\mathfrak{m}^2 = 0$, the deformation $\tilde{c} = c \circ (\mathrm{id}_V^{\otimes 2} + f)$ satisfies*

$$\begin{aligned}(\mathrm{id}_V \otimes \tilde{c})^{-1} (\tilde{c} \otimes \mathrm{id}_V)^{-1} (\mathrm{id}_V \otimes \tilde{c})^{-1} (\tilde{c} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \tilde{c})(\tilde{c} \otimes \mathrm{id}_V) \\ = (\mathrm{id}_V \otimes c)^{-1} (c \otimes \mathrm{id}_V)^{-1} (\mathrm{id}_V \otimes c)^{-1} (c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) + d^2 f.\end{aligned}$$

This means that $\tilde{c}$ is a Yang–Baxter operator if and only if $d^2 f = 0$. Likewise, c and $\tilde{c}$ are equivalent via conjugation by $\alpha = (\mathrm{id}_V + g)$ if and only if $f = d^1 g$, because

$$(\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha) = c \circ (\mathrm{id}_V^{\otimes 2} + d^1 g).$$

Remark 2.10. In the quantum case, where $c = \tau$, we obtain $df = 0$ for all $f \in C^*_{\mathrm{YB}}$. In particular there are no infinitesimal obstructions to deforming τ: *every* deformation of τ satisfies the Yang–Baxter equation modulo $\mathfrak{m}^2$, and only higher-order obstructions are of interest. This explains why Yang–Baxter cohomology is absent in the quantum case.

Infinitesimal obstructions are important, however, if $c \neq \tau$, for example for an operator c_Q coming from a non-trivial rack Q, the main object of interest to us here. In extreme cases they even allow us to conclude that c_Q is rigid.

Example 2.11. Yang–Baxter cohomology can in particular be applied to study the deformations of the Yang–Baxter operator c_Q associated with a rack Q. The canonical basis Q of $V = \mathbb{A}Q$ allows us to identify each $\mathbb{A}$-linear map $f \colon \mathbb{A}Q^n \to \mathbb{A}Q^n$ with its matrix $f \colon Q^n \times Q^n \to \mathbb{A}$, related by the definition

$$f \colon (x_1 \otimes \cdots \otimes x_n) \mapsto \sum_{y_1, \ldots, y_n} f \begin{bmatrix} x_1, \ldots, x_n \\ y_1, \ldots, y_n \end{bmatrix} \cdot (y_1 \otimes \cdots \otimes y_n).$$

For example, the identity $\mathrm{id} \colon \mathbb{A}Q \to \mathbb{A}Q$ will be identified with the following matrix $Q \times Q \to \mathbb{A}$, which is usually called the Kronecker delta function:

$$\mathrm{id} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

In this notation the coboundary can be rewritten more explicitly as follows:

$$(d_i^n f)\begin{bmatrix} x_0,\dots,x_n \\ y_0,\dots,y_n \end{bmatrix} = +f\begin{bmatrix} x_0 \ ,\dots,x_{i-1},x_{i+1},\dots,x_n \\ y_0 \ ,\dots,y_{i-1},y_{i+1},\dots,y_n \end{bmatrix}\cdot \mathrm{id}\begin{bmatrix} x_i^{x_{i+1}\cdots x_n} \\ y_i^{y_{i+1}\cdots y_n} \end{bmatrix}$$
$$-f\begin{bmatrix} x_0^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n \\ y_0^{y_i},\dots,y_{i-1}^{y_i},y_{i+1},\dots,y_n \end{bmatrix}\cdot \mathrm{id}\begin{bmatrix} x_i \\ y_i \end{bmatrix}. \tag{2.6}$$

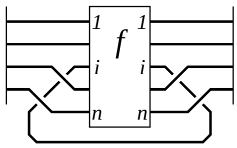
Remark 2.12. Instead of an ideal $\mathfrak{m}$ in a ring $\mathbb{A}$ one can also define the Yang–Baxter cochain complex $C^*_{\mathrm{YB}}(c;\mathfrak{m})$ and its cohomology $H^*_{\mathrm{YB}}(c;\mathfrak{m})$ for any module $\mathfrak{m}$ over a ring $\mathbb{K}$. Both points of view become equivalent in the infinitesimal setting: If $\mathfrak{m}^2 = 0$ in $\mathbb{A}$, then $\mathfrak{m}$ is a module over the quotient ring $\mathbb{K} = \mathbb{A}/\mathfrak{m}$. Conversely, every $\mathbb{K}$-module $\mathfrak{m}$ defines a $\mathbb{K}$-algebra $\mathbb{A} = \mathbb{K} \oplus \mathfrak{m}$ with $\mathfrak{m}^2 = 0$.

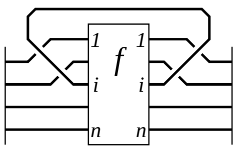
2.4. **Yang–Baxter homology.** As could be expected, there is a homology theory dual to Yang–Baxter cohomology. Let $\mathbb{A}$ be a ring and let $c\colon V \otimes V \to V \otimes V$ be a Yang–Baxter operator. We will assume that the $\mathbb{A}$-module V is free of finite rank, so that we can define a trace $\mathrm{tr}\colon \operatorname{End}(V) \to \mathbb{A}$, see Lang [35, §XVI.5]. Slightly more general, it suffices to assume V projective and finitely generated over $\mathbb{A}$, see Turaev [41, chap. 1]. Even though this hypothesis may seem restrictive, it is precisely the setting of quantum knot invariants, where a trace is indispensable. Notice further that then $\operatorname{End}(V^{\otimes n}) = \operatorname{End}(V)^{\otimes n}$, and for each index $i = 1,\dots,n$ we have a partial trace $\mathrm{tr}_i\colon \operatorname{End}(V)^{\otimes n} \to \operatorname{End}(V)^{\otimes(n-1)}$ defined by contracting the ith tensor factor.

Definition 2.13. Given a Yang–Baxter operator $c\colon V \otimes V \to V \otimes V$, the Yang–Baxter chain complex $C_*^{\mathrm{YB}}(c)$ consists of the $\mathbb{A}$-modules $C_n = \operatorname{End}(V^{\otimes n})$. We define the partial boundary $\partial_n^i\colon C_n \to C_{n-1}$ by

$$\partial_n^i f = \mathrm{tr}_n\left[(c_{n-1}\cdots c_i)\, f\, (c_{n-1}\cdots c_i)^{-1}\right] - \mathrm{tr}_1\left[(c_1\cdots c_{i-1})\, f\, (c_1\cdots c_{i-1})^{-1}\right]. \tag{2.7}$$

Again this formula becomes more suggestive in graphical notation:

(2.8) $\qquad \partial_n^i f = \quad +$ 

$-$ 

.

As Equation (2.5) above, this is reminiscent of rope skipping. The boundary $\partial_n\colon C_n \to C_{n-1}$ is defined as the alternating sum $\partial_n = \sum_{i=1}^n (-1)^{i-1}\partial_n^i$.

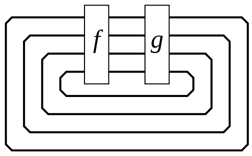
Proposition 2.14. *We have* $\partial_{n-1}^j \partial_n^i = \partial_{n-1}^i \partial_n^{j+1}$ *for* $i \le j$*, whence* $\partial_{n-1}\partial_n = 0$.

Proof. Our hypothesis that c is a Yang–Baxter operator ensures that $\partial_{n-1}^j \circ \partial_n^i = \partial_{n-1}^i \circ \partial_n^{j+1}$ for $i \le j$. This can be proven by a straightforward computation, and is most easily verified using the graphical tensor calculus suggested above. It follows, as usual, that all terms cancel each other in pairs to yield $\partial_{n-1} \circ \partial_n = 0$. □

Definition 2.15. The chain complex $C_*^{\mathrm{YB}}(c) = (C_*, \partial_*)$ is called the *Yang–Baxter chain complex*, and its homology $H_*^{\mathrm{YB}}(c)$ is called the *Yang–Baxter homology* of c.

Proposition 2.16. *The dual complex* $\operatorname{Hom}(C_*^{\mathrm{YB}}, \mathfrak{m})$ *is naturally isomorphic to the Yang–Baxter cochain complex* $C^*_{\mathrm{YB}}(c;\mathfrak{m})$ *defined above.*

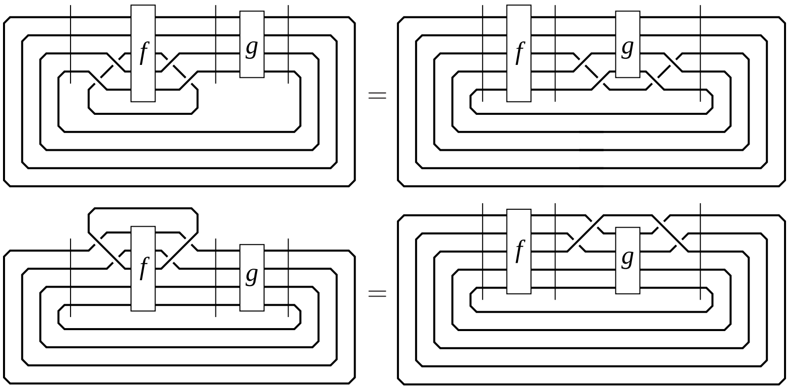
Proof. The duality is induced by the duality pairing $\operatorname{End}(V^{\otimes n}) \otimes \operatorname{End}(V^{\otimes n}) \to \mathbb{A}$ defined by $\langle f \mid g \rangle = \operatorname{tr}(fg)$. In graphical notation this reads as

$\langle f \mid g \rangle =$  .

The advantage of this notation is that all calculations become self-evident. In particular, we see that the coboundary operator d^* of Equation (2.5) is the dual of the boundary operator ∂_* of Equation (2.8): for $f \in C^{\mathrm{YB}}_{n+1}$ and $g \in C^n_{\mathrm{YB}}$ and all $i = 1, \dots, n+1$ we have

$$\langle \partial^i_{n+1} f \mid g \rangle = \langle f \mid d^n_{i-1} g \rangle.$$

In graphical notation this can be seen as follows:



We conclude that $\langle \partial_{n+1} f \mid g \rangle = \langle f \mid d^n g \rangle$ as claimed. □

Remark 2.17. In the case of a finite rack Q and its associated Yang–Baxter operator c_Q, the chain complex C^{YB}_* can be described as follows. Starting from the canonical basis Q of $V = \mathbb{A}Q$, we obtain the basis Q^n of $V^{\otimes n}$ and then a basis $Q^n \times Q^n$ of $\operatorname{End}(V^{\otimes n})$. In analogy with our previous notation we denote by $\left(\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right)$ the endomorphism that maps $x_1 \otimes \cdots \otimes x_n$ to $y_1 \otimes \cdots \otimes y_n$, while mapping all other elements of the basis Q^n to zero. The boundary operator can then be rewritten more explicitly as follows:

(2.9)
$$\partial_n \begin{pmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{pmatrix} = \sum_{i=1}^{n} (-1)^{i-1} \Bigg[\begin{pmatrix} x_1 \;, \dots, x_{i-1}, x_{i+1}, \dots, x_n \\ y_1 \;, \dots, y_{i-1}, y_{i+1}, \dots, y_n \end{pmatrix} \cdot \operatorname{id} \begin{pmatrix} x_i^{x_{i+1} \cdots x_n} \\ y_i^{y_{i+1} \cdots y_n} \end{pmatrix}$$
$$- \begin{pmatrix} x_1^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n \\ y_1^{y_i}, \dots, y_{i-1}^{y_i}, y_{i+1}, \dots, y_n \end{pmatrix} \cdot \operatorname{id} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \Bigg].$$

We see that the boundary formula (2.9) is dual to the coboundary formula (2.6), which nicely illustrates the preceding proposition.

Remark 2.18. The duality exhibited above is graphically appealing and it is reassuring to have the standard homology-cohomology pairing. We have to restrict to free modules of finite rank, however, or finitely generated projective modules.

Yang–Baxter cohomology alone can be defined over arbitrary Yang–Baxter modules (V, c), not necessarily projective or finitely generated. From this viewpoint Yang–Baxter cohomology seems more natural than homology.

2.5. **Non-Functoriality.** Yang–Baxter (co)homology suffers from a curious defect: it is not functorial with respect to homomorphisms of Yang–Baxter operators.

Definition 2.19. A *homomorphism* between Yang–Baxter operators $c\colon V \otimes V \to V \otimes V$ and $\bar{c}\colon \bar{V} \otimes \bar{V} \to \bar{V} \otimes \bar{V}$ is an $\mathbb{A}$-linear map $\phi\colon V \to \bar{V}$ such that $\bar{c} \circ (\phi \otimes \phi) = (\phi \otimes \phi) \circ c$, making the following diagram commute:

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\;c\;} & V \otimes V \\ {\scriptstyle \phi\otimes\phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi\otimes\phi} \\ \bar{V} \otimes \bar{V} & \xrightarrow{\;\bar{c}\;} & \bar{V} \otimes \bar{V} \end{array}$$

This ensures that ϕ induces for each n a homomorphism $\phi^{\otimes n}\colon V^{\otimes n} \to \bar{V}^{\otimes n}$ that is equivariant with respect to the natural action of the braid group B_n.

Example 2.20. A map $\phi\colon Q \to \bar{Q}$ is a homomorphism between two racks Q and $\bar{Q}$ if and only if only if its $\mathbb{A}$-linear extension $\phi\colon \mathbb{A}Q \to \mathbb{A}\bar{Q}$ is a homomorphism between the associated Yang–Baxter operators c_Q and $c_{\bar{Q}}$.

Given a homomorphism ϕ between Yang–Baxter operators c and $\bar{c}$, we would expect a cochain homomorphism $\phi^*\colon C^*_{\mathrm{YB}}(\bar{c}; \mathfrak{m}) \to C^*_{\mathrm{YB}}(c; \mathfrak{m})$ and a chain homomorphism $\phi_*\colon C_*^{\mathrm{YB}}(c) \to C_*^{\mathrm{YB}}(\bar{c})$. The definitions of $C_n^{\mathrm{YB}}(c) = \operatorname{End}(V^{\otimes n})$ and $C^n_{\mathrm{YB}}(c; \mathfrak{m}) = \operatorname{Hom}(V^{\otimes n}, \mathfrak{m}V^{\otimes n})$, however, do not lend themselves to any obvious construction. The problem already occurs in degree 2 shown in the above diagram: in a deformation of $c\colon V \otimes V \to V \otimes V$ both factors may interact, and this does not respect the product structure of $\phi \otimes \phi$.

Example 2.21. Consider a homomorphism $\phi\colon Q \to \bar{Q}$ between racks. We can define a map $\phi^*\colon C^*_{\mathrm{YB}}(c_{\bar{Q}}; \mathfrak{m}) \to C^*_{\mathrm{YB}}(c_Q; \mathfrak{m})$ by setting

$$(\phi^* f)\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} = f\begin{bmatrix} \phi(x_1), \dots, \phi(x_n) \\ \phi(y_1), \dots, \phi(y_n) \end{bmatrix}. \tag{2.10}$$

Even though this is the natural candidate, it does in general not define a cochain map, that is, we usually have $\phi^* \circ d^*_{\bar{Q}} \neq d^*_Q \circ \phi^*$. In order to illustrate this, we shall construct an explicit example.

The inner automorphism group $\operatorname{Inn}(Q)$ acts naturally on Q. The set of orbits $\bar{Q} = Q/\operatorname{Inn}(Q)$ can be regarded as a trivial rack, in which case the quotient map $\phi\colon Q \to \bar{Q}$ becomes a rack homomorphism.

Consider a cochain $f \in C^n_{\mathrm{YB}}(c_{\bar{Q}}; \mathfrak{m})$. The coboundary $d^*_{\bar{Q}}$ vanishes, so that $\phi^* d^*_{\bar{Q}} f = 0$. In general, however, we have $d^*_Q \phi^* f \neq 0$. To see this consider $y, z \in Q$ satisfying $y \not\equiv z$, which means that there exists $x \in Q$ such that $x^y \neq x^z$. We find

$$\begin{aligned} \Big(d^1_Q(\phi^* f)\Big)\begin{bmatrix} x, y \\ x, z \end{bmatrix} &= (\phi^* f)\begin{bmatrix} y \\ z \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x^y \\ x^z \end{bmatrix} - (\phi^* f)\begin{bmatrix} y \\ z \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x \\ x \end{bmatrix} \\ &\quad - (\phi^* f)\begin{bmatrix} x \\ x \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} y \\ z \end{bmatrix} + (\phi^* f)\begin{bmatrix} x^y \\ x^z \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} y \\ z \end{bmatrix} = -f\begin{bmatrix} \phi(y) \\ \phi(z) \end{bmatrix}. \end{aligned}$$

This is in general not zero, whence $d^*_Q \phi^* f \neq \phi^* d^*_{\bar{Q}} f$. We conclude that the natural candidate $\phi^*\colon C^*_{\mathrm{YB}}(c_{\bar{Q}}; \mathfrak{m}) \to C^*_{\mathrm{YB}}(c_Q; \mathfrak{m})$ is not a cochain map.

3. Diagonal deformations

In §2 we have considered general Yang–Baxter deformations. For c_Q coming from a rack Q the theory becomes much easier if we concentrate on deformations of the form $\tilde{c}(a \otimes b) = \lambda(a,b) \cdot c_Q(a \otimes b)$ where $\lambda \colon Q \times Q \to \Lambda$ is a map to some abelian group Λ. Such deformations are classified by rack cohomology:

Definition 3.1. Let Q be a rack and let Λ be an abelian group (written additively). We consider the cochain complex $C^n_{\mathrm{R}} = C^n_{\mathrm{R}}(Q;\Lambda)$ formed by all maps $\lambda \colon Q^n \to \Lambda$. The coboundary $\delta^n \colon C^n_{\mathrm{R}} \to C^{n+1}_{\mathrm{R}}$ is defined by

$$
\begin{aligned}
(\delta^n \lambda)(a_0, \dots, a_n) = \sum_{i=1}^{n} (-1)^i \Big[\; & \lambda(a_0 \;, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \\
& - \lambda(a_0^{a_i}, \dots, a_{i-1}^{a_i}, a_{i+1}, \dots, a_n) \; \Big].
\end{aligned}
\tag{3.1}
$$

This defines a cochain complex $(C^*_{\mathrm{R}}, \delta^*)$, whose cohomology $H^n_{\mathrm{R}}(Q;\Lambda)$ is called the *rack cohomology* of Q with coefficients in Λ.

Remark 3.2. It is easily seen that $\tilde{c}$ is a Yang–Baxter operator if and only if λ is a rack cocycle, see Graña [26]. Likewise, $\tilde{c}$ and c_Q are equivalent if and only if λ is a coboundary. As in group theory, the second cohomology group $H^2_{\mathrm{R}}(Q;\Lambda)$ corresponds to equivalence classes of central extension $\Lambda \curvearrowright \tilde{Q} \to Q$ [14, 15].

Let us make explicit how rack cohomology fits into the more general framework of Yang–Baxter cohomology. Suppose that Λ is a module over some ring $\mathbb{K}$. We can form the $\mathbb{K}$-algebra $\mathbb{A} = \mathbb{K} \oplus \Lambda$ by setting $uv = 0$ for all $u, v \in \Lambda$, that is, we equip $\mathbb{A}$ with the product $(a,u) \cdot (b,v) = (ab, av + bu)$. For $\mathbb{K} = \Lambda$, for example, we obtain $\mathbb{A} = \mathbb{K}[h]/(h^2)$. We have an augmentation homomorphism $\varepsilon \colon \mathbb{A} \to \mathbb{K}$ defined by $\varepsilon(1) = 1$ and $\varepsilon(u) = 0$ for all $u \in \Lambda$. The augmentation ideal $\mathfrak{m} = \ker(\varepsilon)$ thus coincides with Λ. Notice also that the additive group Λ is isomorphic to the multiplicative subgroup $1 + \mathfrak{m}$ of the ring $\mathbb{A}$.

If we consider diagonal deformations

$$
\tilde{c}(a \otimes b) = \big(1 + \lambda(a,b)\big) \cdot c_Q(a \otimes b) \quad \text{with } \lambda(a,b) \in \mathfrak{m},
$$

then we see that rack cohomology naturally embeds into Yang–Baxter cohomology:

Proposition 3.3. *The rack cochain complex $C^*_{\mathrm{R}}(Q;\Lambda)$ is naturally isomorphic to the diagonal subcomplex C^*_{Diag} of the Yang–Baxter cohomology $C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$.* □

Remark 3.4. Unlike the full Yang–Baxter complex $(C^*_{\mathrm{YB}}(c_Q,\mathfrak{m}), d^*)$, the diagonal subcomplex $C^*_{\mathrm{Diag}}(c_Q,\mathfrak{m})$ and rack cohomology (3.1) *are* functorial in Q.

Proposition 3.5. *There exists a retraction $r \colon C^*_{\mathrm{YB}} \to C^*_{\mathrm{Diag}}$ of cochain complexes, whence rack cohomology $H^*_{\mathrm{R}}(Q;\Lambda)$ is a direct summand of Yang–Baxter cohomology $H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$.*

Proof. The obvious idea turns out to work. We define $r^n \colon C^n_{\mathrm{YB}} \to C^n_{\mathrm{Diag}}$ by

$$
(r^n f)\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] := \begin{cases} f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] & \text{if } x_i = y_i \text{ for all } i = 1,\dots,n, \\ 0 & \text{otherwise.} \end{cases}
$$

The coboundary formula (2.6) shows that $d^n_i \circ r^n = r^{n+1} \circ d^n_i$, whence $d \circ r = r \circ d$. By construction we have $r(C^*_{\mathrm{YB}}) = C^*_{\mathrm{Diag}}$ and $r|C^*_{\mathrm{Diag}} = \mathrm{id}$. □

Remark 3.6. The example of a trivial rack Q shows that Yang–Baxter $H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ is in general much bigger than rack cohomology $H^*_{\mathrm{R}}(Q;\Lambda)$, so we cannot capture all information by diagonal deformations alone. In order to do so, we have to consider the more general notion of *quasi-diagonal* deformations, which we explain next.

4. Quasi-diagonal deformations

As before we consider a rack Q and the associated Yang–Baxter operator c_Q over some ring $\mathbb{A}$. Within the Yang–Baxter complex we can now define the quasi-diagonal subcomplex. Recall from §1.2 that a matrix $f\colon Q^n \times Q^n \to \mathbb{A}$ is called *quasi-diagonal* if $f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \not\equiv y_i$ for some index $i = 1,\dots,n$.

Proposition 4.1. *The quasi-diagonal cochains of the Yang–Baxter complex form a subcomplex, denoted (C^*_Δ, d^*).* □

Remark 4.2. Restricted to the subcomplex C^*_Δ of quasi-diagonal cochains, the coboundary $d^n\colon C^n_\Delta \to C^{n+1}_\Delta$ takes the form $d^n f = \sum_{i=1}^n (-1)^i d^n_i f$ with

$$(d^n_i f)\left[\begin{smallmatrix} x_0,\dots,x_n \\ y_0,\dots,y_n \end{smallmatrix}\right] = \left(f\left[\begin{smallmatrix} x_0 & ,\dots,x_{i-1},x_{i+1},\dots,x_n \\ y_0 & ,\dots,y_{i-1},y_{i+1},\dots,y_n \end{smallmatrix}\right] - f\left[\begin{smallmatrix} x_0^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n \\ y_0^{y_i},\dots,y_{i-1}^{y_i},y_{i+1},\dots,y_n \end{smallmatrix}\right] \right) \cdot \mathrm{id}\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right].$$

This illustrates, in explicit terms, that the quasi-diagonal subcomplex is half-way between Yang–Baxter cohomology (2.6) and rack cohomology (3.1). As pointed out in the introduction, the quasi-diagonal subcomplex C^*_Δ coincides with the Yang–Baxter complex C^*_{YB} if the rack Q is trivial. On the other hand, C^*_Δ coincides with the rack complex C^*_{R} if the inner representation $\rho\colon Q \to \mathrm{Inn}(Q)$ is injective: in this case $x \equiv y$ means $x = y$, and quasi-diagonal means diagonal.

The main goal of this article is to show that $C^*_\Delta \subset C^*_{\mathrm{YB}}$ is a homotopy retract. We point out that a much weaker statement follows easily from the definition:

Proposition 4.3. *There exists a retraction $r\colon C^*_{\mathrm{YB}} \to C^*_\Delta$.*

Proof. Again the obvious idea turns out to work. We define $r^n\colon C^n_{\mathrm{YB}} \to C^n_\Delta$ by

$$(r^n f)\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] := \begin{cases} f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] & \text{if } x_i \equiv y_i \text{ for all } i = 1,\dots,n, \\ 0 & \text{otherwise.} \end{cases}$$

The coboundary formula (2.6) shows that $d^n_i \circ r^n = r^{n+1} \circ d^n_i$, whence $d \circ r = r \circ d$. By construction we have $r(C^*_{\mathrm{YB}}) = C^*_\Delta$ and $r|C^*_\Delta = \mathrm{id}$, so r is a retraction. □

Remark 4.4. Like the full Yang–Baxter complex $C^*_{\mathrm{YB}}(c_Q,\mathfrak{m})$, the quasi-diagonal complex $C^*_\Delta(c_Q,\mathfrak{m})$ is *not* functorial in the rack Q. Every rack homomorphism $\phi\colon Q \to \bar{Q}$ induces a map $\phi^*_\Delta\colon C_\Delta(c_{\bar{Q}},\mathfrak{m}) \to C_\Delta(c_Q,\mathfrak{m})$ defined by

$$(\phi^*_\Delta f)\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} = f\begin{bmatrix} \phi(x_1),\dots,\phi(x_n) \\ \phi(y_1),\dots,\phi(y_n) \end{bmatrix} \tag{4.1}$$

for all $x_1 \equiv y_1,\dots,x_n \equiv y_n$ in Q. This natural map, however, is in general not a cochain map. A concrete example can be constructed as follows.

Example 4.5. Consider a non-trivial rack $\bar{Q}$ and choose $\bar{x},\bar{y} \in \bar{Q}$ such that $\bar{x}^{\bar{y}} \neq \bar{x}$. Assume that $\phi\colon Q \to \bar{Q}$ is a rack homomorphism, $\phi(x) = \bar{x}$, $\phi(y) = \bar{y}$, $\phi(z) = \bar{z}$, with $y \neq z$ but $\bar{y} = \bar{z}$. The easiest example is the trivial extension $Q = \bar{Q} \times \{1,2\}$, where $(a,i)*(b,j) = (a*b,i)$, which also ensures that $y = (\bar{y},1)$ and $z = (\bar{y},2)$ are behaviourally equivalent. For each cochain $f \in C^1(c_{\bar{Q}},\mathfrak{m})$ we find

$$(4.2)\quad (d^1\phi^* f)\begin{bmatrix} x, y \\ x, z \end{bmatrix} = \left((\phi^* f)\begin{bmatrix} x^y \\ x^z \end{bmatrix} - (\phi^* f)\begin{bmatrix} x \\ x \end{bmatrix} \right) \cdot \mathrm{id}\begin{bmatrix} y \\ z \end{bmatrix} = 0 \qquad \text{as opposed to}$$

$$(4.3)\quad (\phi^* d^1 f)\begin{bmatrix} x, y \\ x, z \end{bmatrix} = (d^1 f)\begin{bmatrix} \bar{x}, \bar{y} \\ \bar{x}, \bar{z} \end{bmatrix} = \left(f\begin{bmatrix} \bar{x}^{\bar{y}} \\ \bar{x}^{\bar{y}} \end{bmatrix} - f\begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix} \right) \mathrm{id}\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} = f\begin{bmatrix} \bar{x}^{\bar{y}} \\ \bar{x}^{\bar{y}} \end{bmatrix} - f\begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix}.$$

Since $\bar{x}^{\bar{y}} \neq \bar{x}$, the cochain f can be so chosen that the last difference is non-zero.

The difference between (4.2) and (4.3) disappears for equivariant cochains:

Definition 4.6. A cochain $f \in C^n(c_Q,\mathfrak{m})$ is *fully equivariant* if it satisfies

$$f\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} = f\begin{bmatrix} x_1^{g_1}, \dots, x_n^{g_n} \\ y_1^{g_1}, \dots, y_n^{g_n} \end{bmatrix}$$

for all $x_1,\dots,x_n,y_1,\dots,y_n \in Q$ and $g_1,\dots,g_n \in \mathrm{Inn}(Q)$. It is called *entropic* if it is fully equivariant and quasi-diagonal. Such cochains are characterized by the condition $d_0^n f = \dots = d_n^n f = 0$, in other words, all partial coboundaries vanish [15, Lemma 30]. In particular, entropic cochains are cocycles; the submodule of entropic cocycles is denoted by $E^*(c_Q,\mathfrak{m}) \subset Z^*_{\mathrm{YB}}(c_Q,\mathfrak{m})$.

Remark 4.7. For every rack Q we have $C^*_{\mathrm{YB}}(c_Q,\mathfrak{m}) \supset C^*_\Delta(c_Q,\mathfrak{m}) \supset E^*(c_Q,\mathfrak{m})$ by definition. Every rack homomorphism $\phi\colon Q \to \bar{Q}$ induces maps

$$\begin{array}{ccccc}
C^*_{\mathrm{YB}}(c_Q,\mathfrak{m}) & \xleftarrow{\ \supset\ } & C^*_\Delta(c_Q,\mathfrak{m}) & \xleftarrow{\ \supset\ } & E^*(c_Q,\mathfrak{m}) \\
{\scriptstyle \phi^*}\big\uparrow & & \big\uparrow{\scriptstyle \phi^*_\Delta} & & \big\uparrow{\scriptstyle \phi^*_E} \\
C^*_{\mathrm{YB}}(c_{\bar{Q}},\mathfrak{m}) & \xleftarrow{\ \supset\ } & C^*_\Delta(c_{\bar{Q}},\mathfrak{m}) & \xleftarrow{\ \supset\ } & E^*(c_{\bar{Q}},\mathfrak{m}).
\end{array}$$

Here ϕ^* is defined by (2.10), whereas ϕ^*_Δ is defined by (4.1), and the map ϕ^*_E is obtained from ϕ^*_Δ by restriction. In general ϕ^* and ϕ^*_Δ are *not* cochain maps, as pointed out above. Only the third map ϕ^*_E is always a cochain map because $C^*_E \subset Z^*_{\mathrm{YB}}$ is a trivial subcomplex.

Entropic deformations are symmetric unter $\mathrm{Inn}(Q)$. If the order of $\mathrm{Inn}(Q)$ is finite and invertible in $\mathbb{A}$, then symmetrization can be used to show that every Yang–Baxter deformation of c_Q is equivalent to an entropic deformation [15]. In the present article we will not impose such restrictions and thus not use symmetrization.

5. Constructing a homotopy retraction

Having set the scene in the preceding sections, we can now study the subcomplex $C^*_\Delta \subset C^*_{\mathrm{YB}}$ of quasi-diagonal cochains. It is easy to see that it is a retract but it is more delicate to prove that it is a homotopy retract. The construction of Proposition 4.3 is nice and simple, but unfortunately the retraction $r\colon C^*_{\mathrm{YB}} \to C^*_\Delta$ does not seem to be homotopic to the identity of C^n_{YB}.

To resolve this difficulty we introduce an auxiliary filtration

$$C^*_{\mathrm{YB}} = C^*_0 \supset C^*_1 \supset C^*_2 \supset \dots \supset C^*_\infty = C^*_\Delta.$$

of subcomplexes. We then prove that each complex homotopy-retracts to its successor (§5.1). The advantage is that the partial retractions $p^*_m\colon C^*_m \twoheadrightarrow C^*_{m+1}$ are much easier to understand. Composing these homotopies we obtain the desired homotopy retraction $C^*_{\text{YB}} \twoheadrightarrow C^*_\Delta$ (§5.2). Figuratively speaking, we thus construct the deformation from C^*_{YB} to C^*_Δ by a piecewise linear path.

Definition 5.1. For each $m \in \mathbb{N}$ we define $C^*_m \subset C^*_{\text{YB}}$ to be the subcomplex of cochains that are quasi-diagonal in the last m variables. More explicitly:

$$C^n_m := \left\{ f \in C^n_{\text{YB}} \mid f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] = 0 \text{ if } x_i \not\equiv y_i \text{ for some index } i \text{ with } n-m < i \leq n \right\}.$$

In each degree n we thus obtain a filtration $C^n_{\text{YB}} = C^n_0 \supset C^n_1 \supset \cdots \supset C^n_n$ that stabilizes at C^n_n: obviously $C^n_m = C^n_n$ for all $m > n$.

Lemma 5.2. *The coboundary $d^n \colon C^n_{\text{YB}} \to C^{n+1}_{\text{YB}}$ satisfies $d^n(C^n_m) \subset C^{n+1}_m$ for each $m \in \mathbb{N}$. In other words, $(C^*_m, d^*|_{C^*_m})$ is a subcomplex of (C^*_{YB}, d^*).*

Proof. Suppose that $f \in C^n_m$. Formula (2.6) for the partial coboundary shows that $d^n_i f$ is in C^{n+1}_m. The same thus holds for the coboundary $d^n f = \sum_{i=0}^n (-1)^i d^n_i f$. □

Notation. We will suppress the explicit mention of the coboundary map and denote the complex (C^*_{YB}, d^*) simply by C^*_{YB}. Likewise we write C^*_m for $(C^*_m, d^*|_{C^*_m})$.

5.1. **Cochain homotopies.** We wish to show that the inclusion $\iota^*_{m+1} \colon C^*_{m+1} \hookrightarrow C^*_m$ is a *homotopy retract*. To this end we shall construct a cochain map $p^*_m \colon C^*_m \twoheadrightarrow C^*_{m+1}$ such that $p^*_m \circ \iota^*_{m+1} = \mathrm{id}^*_{m+1}$ and a cochain homotopy $\iota^*_{m+1} \circ p^*_m \simeq \mathrm{id}^*_m \colon C^*_m \to C^*_m$. Such a projection p^*_m is called a *homotopy retraction* see Mac Lane [37, §II.2]. Recall that a *cochain homotopy* is a map $s^n_m \colon C^n_m \to C^{n-1}_m$ such that $p^n_m - \mathrm{id}^n_m = d^{n-1} \circ s^n_m + s^{n+1}_m \circ d^n$. In the sequel we will prefer the sign convention $d^{n-1} \circ s^n_m - s^{n+1}_m \circ d^n$, which is logically equivalent.

Remark 5.3. We call the set $\Delta = \{(x,y) \in Q^2 \mid x \equiv y\}$ the *quasi-diagonal*. On its complement $\Delta^c = \{(x,y) \in Q^2 \mid x \not\equiv y\}$ we choose a map $\psi \colon \Delta^c \to Q^2$, $(x,y) \mapsto (u,v)$ such that $u \neq v$ but $u^x = v^y$. It is easy to see that such a map exists: the inequivalence $x \not\equiv y$ means that the inner automorphisms $z \mapsto z * x$ and $z \mapsto z * y$ are different. This is equivalent to saying that their inverses $z \mapsto z \mathbin{\overline{*}} x$ and $z \mapsto z \mathbin{\overline{*}} y$ are different: there exists $z \in Q$ such that $u = z \mathbin{\overline{*}} x$ differs from $v = z \mathbin{\overline{*}} y$. In other words we have $u \neq v$ but $u^x = v^y$.

Definition 5.4. Fix $n, m \in \mathbb{N}$. For $m \geq n$ we define $s^n_m \colon C^n_m \to C^{n-1}_m$ to be the zero map. For $0 \leq m \leq n-1$ we set $k := n - m$ and define $s^n_m \colon C^n_m \to C^{n-1}_m$ by

$$(s^n_m f)\left[\begin{smallmatrix} x_2,\dots,x_n \\ y_2,\dots,y_n \end{smallmatrix}\right] := \begin{cases} f\left[\begin{smallmatrix} x_2,\dots,x_{k-1},u,x_k,\dots,x_n \\ y_2,\dots,y_{k-1},v,y_k,\dots,y_n \end{smallmatrix}\right] & \text{if } x_k \not\equiv y_k, \text{ with } (u,v) = \psi(x_k,y_k), \\ 0 & \text{if } x_k \equiv y_k. \end{cases}$$

This induces a map $t^n_m := d^{n-1} \circ s^n_m - s^{n+1}_m \circ d^n \colon C^n_m \to C^n_m$.

Theorem 5.5. *The cochain map $p^n_m := \mathrm{id}^n_m - (-1)^{n-m} t^n_m \colon C^n_m \to C^n_m$ sends C^n_m to the subcomplex C^n_{m+1} and restricts to the identity on C^n_{m+1}. By construction, the maps id^*_m and p^*_m are homotopy equivalent, and thus $C^*_{m+1} \hookrightarrow C^*_m$ is a homotopy retract and thus induces an isomorphism $H^*(C^*_{m+1}) \xrightarrow{\sim} H^*(C^*_m)$ on cohomology.*

Proof. The fact that p is a cochain map follows at once from its definition:

$$\begin{aligned} d^n \circ p^n_m &= d^n - (-1)^{n-m}\left[d^n d^{n-1} s^n_m - d^n s^{n+1}_m d^n\right], \\ p^{n+1}_m \circ d^n &= d^n + (-1)^{n-m}\left[d^n s^{n+1}_m d^n - s^{n+2}_m d^{n+1} d^n\right]. \end{aligned}$$

The two properties $p^n_m(C^n_m) \subset C^n_{m+1}$ and $p^n_m|C^n_{m+1} = \mathrm{id}^n_{m+1}$ will be established in the following two lemmas. The remaining statements are standard consequences of cochain homotopy, see Mac Lane [37, §II.2]. □

Lemma 5.6. *We have* $(t^n_m f)\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right] = (-1)^k f\left[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}\right]$ *whenever* $x_k \not\equiv y_k$.

Proof. As before we set $k := n - m$. We will calculate $t^n_m \colon C^n_m \to C^n_m$ by making $d^{n-1}_i \circ s^n_m$ and $s^{n+1}_m \circ d^n_i$ explicit for $i = 0, \dots, n$. Let $f \in C^n_m$ and assume $x_k \not\equiv y_k$. We shall distinguish the three cases $i \le k-2$ and $i = k-1$ and $i \ge k$.

First case. For $i = 0, \dots, k-2$ we find:

$$\begin{aligned}
&(d^{n-1}_i s^n_m f)\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} \\
&\qquad = +(s^n_m f)\begin{bmatrix} x_1 & ,\dots,x_i & ,x_{i+2},\dots,x_k,\dots,x_n \\ y_1 & ,\dots,y_i & ,y_{i+2},\dots,y_k,\dots,y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\
&\qquad\quad -(s^n_m f)\begin{bmatrix} x_1^{x_{i+1}},\dots,x_i^{x_{i+1}},x_{i+2},\dots,x_k,\dots,x_n \\ y_1^{y_{i+1}},\dots,y_i^{y_{i+1}},y_{i+2},\dots,y_k,\dots,y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} \\
&\qquad = +f\begin{bmatrix} x_1 & ,\dots,x_i & ,x_{i+2},\dots,u,x_k,\dots,x_n \\ y_1 & ,\dots,y_i & ,y_{i+2},\dots,v,y_k,\dots,y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\
&\qquad\quad -f\begin{bmatrix} x_1^{x_{i+1}},\dots,x_i^{x_{i+1}},x_{i+2},\dots,u,x_k,\dots,x_n \\ y_1^{y_{i+1}},\dots,y_i^{y_{i+1}},y_{i+2},\dots,v,y_k,\dots,y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} \\
&\qquad = \quad (d^n_i f)\begin{bmatrix} x_1,\dots,x_{k-1},u,x_k,\dots,x_n \\ y_1,\dots,y_{k-1},v,y_k,\dots,y_n \end{bmatrix} \\
&\qquad = \quad (s^{n+1}_m d^n_i f)\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix}.
\end{aligned}$$

The third of these four equalities needs justification. We have to verify that

$$x_{i+1}^{x_{i+2}\cdots x_{k-1}x_k\cdots x_n} = y_{i+1}^{y_{i+2}\cdots y_{-1}y_k\cdots y_n}$$

is equivalent to

$$x_{i+1}^{x_{i+2}\cdots x_{k-1}ux_k\cdots x_n} = y_{i+1}^{y_{i+2}\cdots y_{k-1}vy_k\cdots y_n}.$$

We can assume that $x_j \equiv y_j$ for all $k < j \le n$, otherwise the factors involving f vanish by our hypothesis $f \in C^n_m$. So we only have to show that

$$x_{i+1}^{x_{i+2}\cdots x_{k-1}x_k} = y_{i+1}^{y_{i+2}\cdots y_{-1}y_k}$$

is equivalent to

$$x_{i+1}^{x_{i+2}\cdots x_{k-1}ux_k} = y_{i+1}^{y_{i+2}\cdots y_{k-1}vy_k}.$$

This follows from $(a*u)*x_k = (a*x_k)*(u*x_k)$ and $(b*v)*y_k = (b*y_k)*(v*y_k)$, and our construction $(u,v) = \psi(x_k,y_k)$ ensures that $u*x_k = v*y_k$.

Second case. For $i = k-1$ we find:

$$\begin{aligned}
(d^{n-1}_{k-1} s^n_m f)\begin{bmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{bmatrix} &= +(s^n_m f)\begin{bmatrix} x_1 & ,\dots,x_{k-1},x_{k+1},\dots,x_n \\ y_1 & ,\dots,y_{k-1},y_{k+1},\dots,y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_k^{x_{k+1}\cdots x_n} \\ y_k^{y_{k+1}\cdots y_n} \end{bmatrix} \\
&\quad -(s^n_m f)\begin{bmatrix} x_1^{x_k},\dots,x_{k-1}^{x_k},x_{k+1},\dots,x_n \\ y_1^{y_k},\dots,y_{k-1}^{y_k},y_{k+1},\dots,y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_k \\ y_k \end{bmatrix} \\
&= \quad 0.
\end{aligned}$$

The first factors vanish whenever $x_j \not\equiv y_j$ for some j with $k < j \le n$; otherwise the second factors vanish because of our hypothesis $x_k \ne y_k$. On the other hand

$$\begin{aligned}(s_m^{n+1} d_{k-1}^n f)\begin{bmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{bmatrix} &= +(d_{k-1}^n f)\begin{bmatrix}x_1,\dots,x_{k-1},u,x_k,\dots,x_n\\ y_1,\dots,y_{k-1},v,y_k,\dots,y_n\end{bmatrix}\\ &= +f\begin{bmatrix}x_1,\dots,x_{k-1},x_k,\dots,x_n\\ y_1,\dots,y_{k-1},y_k,\dots,y_n\end{bmatrix}\cdot \mathrm{id}\begin{bmatrix}u^{x_k\cdots x_n}\\ v^{y_k\cdots y_n}\end{bmatrix}\\ &\quad -f\begin{bmatrix}x_1^u,\dots,x_{k-1}^u,x_k,\dots,x_n\\ y_1^v,\dots,y_{k-1}^v,y_k,\dots,y_n\end{bmatrix}\cdot \mathrm{id}\begin{bmatrix}u\\ v\end{bmatrix}\\ &= \quad f\begin{bmatrix}x_1,\dots,x_{k-1},x_k,\dots,x_n\\ y_1,\dots,y_{k-1},y_k,\dots,y_n\end{bmatrix}.\end{aligned}$$

The first factors vanish whenever $x_j \not\equiv y_j$ for some j with $k < j \le n$; otherwise we have $u \ne v$ with $u^{x_k} = v^{y_k}$, whence $u^{x_k\cdots x_n} = v^{y_k\cdots y_n}$.

Third case. For $i \ge k$ we find:

$$\begin{aligned}&(d_i^{n-1} s_m^n f)\begin{bmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{bmatrix}\\ &\quad = +(s_m^n f)\begin{bmatrix}x_1 & ,\dots,x_k & ,\dots,x_i & ,x_{i+2},\dots,x_n\\ y_1 & ,\dots,y_k & ,\dots,y_i & ,y_{i+2},\dots,y_n\end{bmatrix}\cdot \mathrm{id}\begin{bmatrix}x_{i+1}^{x_{i+2}\cdots x_n}\\ y_{i+1}^{y_{i+2}\cdots y_n}\end{bmatrix}\\ &\qquad -(s_m^n f)\begin{bmatrix}x_1^{x_{i+1}},\dots,x_k^{x_{i+1}},\dots,x_i^{x_{i+1}},x_{i+2},\dots,x_n\\ y_1^{y_{i+1}},\dots,y_k^{y_{i+1}},\dots,y_i^{y_{i+1}},y_{i+2},\dots,y_n\end{bmatrix}\cdot \mathrm{id}\begin{bmatrix}x_{i+1}\\ y_{i+1}\end{bmatrix}\\ &\quad = \quad 0.\end{aligned}$$

The first summand vanishes because $x_k \not\equiv y_k$; the second summand vanishes because $x_{i+1} \ne y_{i+1}$ or $x_k^{x_{i+1}} \not\equiv y_k^{y_{i+1}}$. Analogously:

$$\begin{aligned}(s_m^{n+1} d_k^n f)\begin{bmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{bmatrix} &= +(d_k^n f)\begin{bmatrix}x_1,\dots,x_{k-1},u,x_k,\dots,x_n\\ y_1,\dots,y_{k-1},v,y_k,\dots,y_n\end{bmatrix}\\ &= +f\begin{bmatrix}x_1 & ,\dots,x_{k-1},u & ,x_{k+1},\dots,x_n\\ y_1 & ,\dots,y_{k-1},v & ,y_{k+1},\dots,y_n\end{bmatrix}\cdot \mathrm{id}\begin{bmatrix}x_k^{x_{k+1}\cdots x_n}\\ y_k^{y_{k+1}\cdots y_n}\end{bmatrix}\\ &\quad -f\begin{bmatrix}x_1^{x_k},\dots,x_{k-1}^{x_k},u^{x_k},x_{k+1},\dots,x_n\\ y_1^{y_k},\dots,y_{k-1}^{y_k},v^{y_k},y_{k+1},\dots,y_n\end{bmatrix}\cdot \mathrm{id}\begin{bmatrix}x_k\\ y_k\end{bmatrix}\\ &= \quad 0.\end{aligned}$$

The first factors vanish whenever $x_j \not\equiv y_j$ for some j with $k < j \le n$; otherwise the second factors vanish because of our hypothesis $x_k \ne y_k$.

The same conclusion holds for $i > k$:

$$\begin{aligned}&(s_m^{n+1} d_i^n f)\begin{bmatrix}x_1,\dots,x_n\\ y_1,\dots,y_n\end{bmatrix}\\ &\quad = +(d_i^n f)\begin{bmatrix}x_1,\dots,x_{k-1},u,x_k,\dots,x_n\\ y_1,\dots,y_{k-1},v,y_k,\dots,y_n\end{bmatrix}\\ &\quad = +f\begin{bmatrix}x_1 & ,\dots,x_{k-1},u & ,x_k & ,\dots,x_{i-1},x_{i+1},\dots,x_n\\ y_1 & ,\dots,y_{k-1},v & ,y_k & ,\dots,y_{i-1},y_{i+1},\dots,y_n\end{bmatrix}\cdot \mathrm{id}\begin{bmatrix}x_i^{x_{i+1}\cdots x_n}\\ y_i^{y_{i+1}\cdots y_n}\end{bmatrix}\\ &\qquad -f\begin{bmatrix}x_1^{x_i},\dots,x_{k-1}^{x_i},u^{x_i},x_k^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n\\ y_1^{y_i},\dots,y_{k-1}^{y_i},v^{y_i},y_k^{y_i},\dots,y_{i-1}^{y_i},y_{i+1},\dots,y_n\end{bmatrix}\cdot \mathrm{id}\begin{bmatrix}x_i\\ y_i\end{bmatrix}\\ &\quad = \quad 0.\end{aligned}$$

The first summand vanishes because $x_k \not\equiv y_k$; the second summand vanishes because $x_i \neq y_i$ or $x_k^{x_i} \not\equiv y_k^{y_i}$. $\square$

Lemma 5.7. *The map t_m^n satisfies $t_m^n f = 0$ whenever $f \in C_{m+1}^n$.*

Proof. We show $(t_m^n f)[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}] = 0$ for $f \in C_{m+1}^n$ and $x_1, \dots, x_n, y_1, \dots, y_n \in Q$. The previous lemma resolves the case $x_k \not\equiv y_k$, so it suffices to consider the remaining case where $x_k \equiv y_k$. By definition of s_m^{n+1} we have $(s_m^{n+1} d_i^n f)[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}] = 0$ because $x_k \equiv y_k$. Likewise, for $i \leq k-2$ we find:

$$\begin{aligned}
&(d_i^{n-1} s_m^n f)\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} \\
&\qquad = +(s_m^n f)\begin{bmatrix} x_1 & , \dots, x_i & , x_{i+2}, \dots, x_k, \dots, x_n \\ y_1 & , \dots, y_i & , y_{i+2}, \dots, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\
&\qquad\quad -(s_m^n f)\begin{bmatrix} x_1^{x_{i+1}}, \dots, x_i^{x_{i+1}}, x_{i+2}, \dots, x_k, \dots, x_n \\ y_1^{y_{i+1}}, \dots, y_i^{y_{i+1}}, y_{i+2}, \dots, y_k, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} \\
&\qquad = \quad 0.
\end{aligned}$$

For $i \geq k-1$, however, we find:

$$\begin{aligned}
&(d_i^{n-1} s_m^n f)\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} \\
&\qquad = +(s_m^n f)\begin{bmatrix} x_1 & , \dots, x_{k-1}, \dots, x_i & , x_{i+2}, \dots, x_n \\ y_1 & , \dots, y_{k-1}, \dots, y_i & , y_{i+2}, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1}^{x_{i+2}\cdots x_n} \\ y_{i+1}^{y_{i+2}\cdots y_n} \end{bmatrix} \\
&\qquad\quad -(s_m^n f)\begin{bmatrix} x_1^{x_{i+1}}, \dots, x_{k-1}^{x_{i+1}}, \dots, x_i^{x_{i+1}}, x_{i+2}, \dots, x_n \\ y_1^{y_{i+1}}, \dots, y_{k-1}^{y_{i+1}}, \dots, y_i^{y_{i+1}}, y_{i+2}, \dots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix}.
\end{aligned}$$

The summands are non-zero only if $x_{i+1} = y_{i+1}$ and $x_j \equiv y_j$ for all j with $k \leq j \leq n$: in this case their difference measures the defect of $s_m^n f$ to being equivariant (jointly in the first i variables).

Both summands vanish if $x_{k-1} \equiv y_{k-1}$, so let us assume $x_{k-1} \not\equiv y_{k-1}$:

$$\begin{aligned}
(d_i^{n-1} s_m^n f)\begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} &= +f\begin{bmatrix} x_1 & , \dots, u', x_{k-1}, \dots, x_i & , x_{i+2}, \dots, x_n \\ y_1 & , \dots, v', y_{k-1}, \dots, y_i & , y_{i+2}, \dots, y_n \end{bmatrix} \\
&\quad -f\begin{bmatrix} x_1^{x_{i+1}}, \dots, u'', x_{k-1}^{x_{i+1}}, \dots, x_i^{x_{i+1}}, x_{i+2}, \dots, x_n \\ y_1^{y_{i+1}}, \dots, v'', y_{k-1}^{y_{i+1}}, \dots, y_i^{y_{i+1}}, y_{i+2}, \dots, y_n \end{bmatrix}
\end{aligned} \tag{5.1}$$

Here $(u', v') = \psi(x_{k-1}, y_{k-1})$ and $(u'', v'') = \psi(x_{k-1}^{x_{i+1}}, y_{k-1}^{y_{i+1}})$. The contributions do in general not cancel for $f \in C_m^n$, but both summands vanish if $f \in C_{m+1}^n$. $\square$

Remark 5.8. Equation (5.1) shows that $(t_m^n f)[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}]$ can be non-zero for $f \in C_m^n$, if $x_k \equiv y_k$ but $x_{k-1} \not\equiv y_{k-1}$. This equation measures the defect of the cochain f, and our auxiliary map $\psi\colon (x_{k-1}, y_{k-1}) \to (u, v)$, to be equivariant under the action of $|\mathrm{Inn}(Q)|$. In the equivariant setting of [15] this defect disappears, and the projection p_m^n becomes

$$(p_m^n f)[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}] := \begin{cases} 0 & \text{if } x_j \not\equiv y_j \text{ for some } j \text{ with } n-m \leq j \leq n, \\ f[\begin{smallmatrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{smallmatrix}] & \text{otherwise.} \end{cases}$$

This simplified formula has been used in [15], where symmetrization was applied throughout to simplify calculations. In our present setting we cannot apply symmetrization and thus cannot assume equivariance. It is remarkable, therefore, that the above calculations carry through. The price to pay is that the projection p^n_m has a more complicated form.

5.2. **Composition of homotopy retractions.** Having constructed homotopy retractions $C^*_0 \twoheadrightarrow C^*_1 \twoheadrightarrow \dots \twoheadrightarrow C^*_{m-1} \twoheadrightarrow C^*_m$ it now suffices to put the pieces together:

Corollary 5.9. *The subcomplex C^*_Δ of quasi-diagonal cochains is a homotopy retract of the full Yang–Baxter cochain complex C^*_{YB}. As a consequence the inclusion $C^*_\Delta \hookrightarrow C^*_{\text{YB}}$ induces an isomorphism on cohomology, $H^*(C^*_\Delta) \xrightarrow{\sim} H^*(C^*_{\text{YB}})$.*

Proof. The composition of homotopic cochain maps yields again homotopic cochain maps. As a consequence, the composition of our partial homotopy retractions yields again a homotopy retraction

$$P^*_m := p^*_{m-1} \circ p^*_{m-2} \circ \dots \circ p^*_1 \circ p^*_0 \colon C^*_0 \to C^*_m.$$

This shows that the inclusion $C^*_m \hookrightarrow C^*_{\text{YB}}$ is a homotopy retract. We wish to pass to the limit $C^*_\Delta = \bigcap_m C^*_m$. In each degree n we have $p^n_m = \mathrm{id}^n_n$ for all $m \geq n$, and thus $P^n_m = P^n_n$. We can thus define $P^*_\infty = \lim_{m\to\infty} P^*_m$ as the degree-wise limit $P^n_\infty = P^n_n$. We conclude that $C^*_\Delta \hookrightarrow C^*_{\text{YB}}$ is a homotopy retract. □

6. From infinitesimal to complete deformations

In this section we will pass from infinitesimal to complete deformations. In order to do so, we will assume that the ring $\mathbb{A}$ is complete with respect to the ideal $\mathfrak{m}$, that is, we assume that the natural map $\mathbb{A} \to \varprojlim \mathbb{A}/\mathfrak{m}^n$ is an isomorphism.

Example 6.1. A polynomial ring $\mathbb{K}[h]$ is not complete with respect to the ideal (h). Its completion is the power series ring $\mathbb{K}[\![h]\!] = \varprojlim \mathbb{K}[h]/(h^n)$. The latter is complete with respect to its ideal $\mathfrak{m} = (h)$. If $\mathbb{K}$ is a field, then $\mathbb{K}[\![h]\!]$ is a complete local ring, i.e., complete with respect to its unique maximal ideal $\mathfrak{m}$.

Example 6.2. The ring of integers $\mathbb{Z}$ is not complete with respect to the ideal (p), where p will be assumed to be prime. Its completion is the ring of p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/_{p^n}$, which is complete with respect to its unique maximal ideal $\mathfrak{m} = (p)$.

Completions lend themselves to induction techniques: we solve the problem first for $\mathfrak{m} = 0$, and then inductively for $\mathfrak{m}^n = 0$ where $n = 2, 3, \dots$. One can always force this condition by passing to the quotient $\mathbb{A}/\mathfrak{m}^n$, and finally to the limit $\varprojlim \mathbb{A}/\mathfrak{m}^n$.

Lemma 6.3. *Let $\mathbb{A}$ be a ring with ideal $\mathfrak{m}$ such that $\mathfrak{m}^{n+1} = 0$. If $c \colon \mathbb{A}Q^2 \to \mathbb{A}Q^2$ ie a Yang–Baxter operator that satisfies $c \equiv c_Q$ modulo $\mathfrak{m}$ and is quasi-diagonal modulo $\mathfrak{m}^n$, then there exists $\alpha \colon \mathbb{A}Q \to \mathbb{A}Q$ with $\alpha \equiv \mathrm{id}_V$ modulo $\mathfrak{m}^n$, such that $(\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha)$ is a quasi-diagonal deformation of c_Q.*

Proof. We have $c = c_Q \circ F$ with $F \equiv \mathrm{id}_V^{\otimes 2}$ modulo $\mathfrak{m}$. We write F in matrix notation as a map $F \colon Q^2 \times Q^2 \to \mathbb{A}$. Its non-quasi-diagonal part $f \colon Q^2 \times Q^2 \to \mathbb{A}$ is defined by

$$f\left[\begin{smallmatrix} x_1, x_2 \\ y_1, y_2 \end{smallmatrix}\right] := \begin{cases} 0 & \text{if } x_1 \equiv y_1 \text{ and } x_2 \equiv y_2, \\ F\left[\begin{smallmatrix} x_1, x_2 \\ y_1, y_2 \end{smallmatrix}\right] & \text{otherwise.} \end{cases}$$

By hypothesis f takes values in $\mathfrak{m}^n \subset \mathbb{A}$, and can thus be considered as a cochain $C^2_{\text{YB}}(c_Q; \mathfrak{m}^n)$. The map $\bar{c} = c_Q \circ (F - f) = c \circ (\mathrm{id}_V^{\otimes 2} - f)$ is quasi-diagonal, by construction. We claim that $\bar{c}$ is actually a Yang–Baxter operator. We know that c satisfies the Yang–Baxter equation; its deformation $\bar{c}$ thus satisfies

$$\mathrm{id}_V^{\otimes 3} - (\mathrm{id}_V \otimes \bar{c})^{-1}(\bar{c} \otimes \mathrm{id}_V)^{-1}(\mathrm{id}_V \otimes \bar{c})^{-1}(\bar{c} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \bar{c})(\bar{c} \otimes \mathrm{id}_V) = d^2 f. \tag{6.1}$$

It is easy to check that the left-hand side is a quasi-diagonal map, whereas the right-hand side is zero on the quasi-diagonal. We conclude that *both* sides must vanish. This means that $\bar{c}$ satisfies the Yang–Baxter equation, and that $f \in C^2_{\mathrm{YB}}(c_Q; \mathfrak{m}^n)$ is a cocycle.

By Theorem 1.3, the inclusion $C^*_\Delta(c_Q; \mathfrak{m}^n) \subset C^*_{\mathrm{YB}}(c_Q; \mathfrak{m}^n)$ induces an isomorphism on cohomology. The class $[f] \in C^2_{\mathrm{YB}}(c_Q; \mathfrak{m}^n)$ can thus be presented by a quasi-diagonal cocycle $\tilde{f} \in C^2_\Delta(c_Q; \mathfrak{m}^n)$. This means that there exists a cochain $g \in C^1_{\mathrm{YB}}(c_Q; \mathfrak{m}^n)$ such that $\tilde{f} = f + d^1 g$. We conclude that $\alpha = \mathrm{id}_V + g$ conjugates c to a quasi-diagonal Yang–Baxter operator $\tilde{c} = (\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha)$, as desired. □

Remark 6.4. In the preceding proof the construction and analysis of $\bar{c}$ serve to show that f is a 2-cocycle. The separation trick for Equation (6.1) is taken from [15, §4]. I seize the opportunity to point out that there the difference is misprinted and lacks the term $\mathrm{id}_V^{\otimes 3}$; with this obvious correction the argument applies as intended.

Remark 6.5. In the proof of Lemma 6.3 we do not claim that c is conjugate to $\bar{c}$. This is true in the equivariant setting of [15], but without equivariance it is false in general: the coboundary $d^1 g$ kills the non-quasi-diagonal part but usually also changes the quasi-diagonal part (see Remark 5.8).

To conclude the passage from infinitesimal to complete deformations, it only remains to put the ingredients together:

Theorem 6.6. *Suppose that the ring $\mathbb{A}$ is complete with respect to the ideal $\mathfrak{m}$. Then every Yang–Baxter deformation c of c_Q over $\mathbb{A}$ is equivalent to a quasi-diagonal deformation $c_Q f$ where $f\colon \mathbb{A}Q^2 \to \mathbb{A}Q^2$ is quasi-diagonal and $f \equiv \mathrm{id}$ modulo $\mathfrak{m}$.*

Proof. Starting the induction with $c_1 := c = c_Q f_1$, suppose that $c_n = c_Q f_n$ has a deformation term f_n that is quasi-diagonal modulo $\mathfrak{m}^n$. By Lemma 6.3, there exists $\alpha_n\colon \mathbb{A}Q \to \mathbb{A}Q$ with $\alpha_n \equiv \mathrm{id}_V$ modulo $\mathfrak{m}^n$, such that $c_{n+1} := (\alpha_n \otimes \alpha_n)^{-1} c_n (\alpha_n \otimes \alpha_n)$ is given by $c_{n+1} = c_Q f_{n+1}$ with f_{n+1} quasi-diagonal modulo $\mathfrak{m}^{n+1}$. The lemma ensures that such a map $\bar{\alpha}_n$ exists modulo $\mathfrak{m}^{n+1}$; this can be lifted to a map $\alpha_n\colon \mathbb{A}Q \to \mathbb{A}Q$, which is invertible because $\mathbb{A}$ is complete. Completeness of $\mathbb{A}$ also ensures that we can pass to the limit and define the infinite product $\alpha = \alpha_1\alpha_2\alpha_3\cdots$: for each $n \in \mathbb{N}$ this product is finite modulo $\mathfrak{m}^n$. By construction, $(\alpha\otimes\alpha)^{-1}\, c\, (\alpha\otimes\alpha)$ is quasi-diagonal and equivalent to c, as desired. □

Corollary 6.7. *If $H^2_{\mathrm{YB}}(c_Q; \mathfrak{m}/\mathfrak{m}^2) = \mathfrak{m}/\mathfrak{m}^2$, then c_Q is rigid over $(\mathbb{A}, \mathfrak{m})$.*

Proof. For every unit $u \in 1 + \mathfrak{m}$ we obtain a trivially deformed Yang–Baxter operator $\tilde{c} = u \cdot c_Q$. On the cochain level this corresponds to a constant multiple of the identity, which induces an injection $\mathfrak{m}/\mathfrak{m}^2 \hookrightarrow H^2_{\mathrm{YB}}(c_Q; \mathfrak{m}/\mathfrak{m}^2)$. If these trivial classes exhaust all cohomology classes, then degree-wise elimination as in the preceding proof conjugates any given deformation of c_Q to one of the form $u \cdot c_Q$. □

7. Examples and applications

7.1. **Trivial quandles.** Consider first a trivial quandle Q, with $x * y = x$ for all x, y, so that $c_Q = \tau$ is simply the transposition operator. Here our results cannot add anything new, because the Yang–Baxter complex C^*_{YB} is trivial, i.e., $df = 0$ for all $f \in C^*_{\mathrm{YB}}$. In particular there are no infinitesimal obstructions: *every* deformation of τ satisfies the Yang–Baxter equation modulo $\mathfrak{m}^2$. There are, however, higher-order obstructions: these form a subject of their own and belong to the much deeper theory of quantum invariants [11, 40, 31, 32].

7.2. **Faithful quandles.** Next we consider the other extreme, where Theorem 1.1 applies most efficiently. Let G be a centreless group, so that conjugation induces an isomorphism $G \xrightarrow{\sim} \mathrm{Inn}(G)$. Suppose that $Q \subset G$ is a conjugacy class, or a collection of conjugacy classes, that generates G. Then we have $\mathrm{Inn}(Q) \cong \mathrm{Inn}(G) \cong G$, and the inner representation $\rho \colon Q \to \mathrm{Inn}(Q)$ is injective. In this case every Yang–Baxter deformation of c_Q over a complete ring $\mathbb{A}$ is equivalent to a diagonal deformation. If the order $|G|$ is finite and invertible in $\mathbb{A}$, then c_Q is rigid [15].

7.3. **The dihedral quandle of order 3.** The smallest non-trivial example of a rigid operator c_Q is given by the quandle $Q = \{(12), (13), (23)\}$, formed by transpositions in the symmetric group S_3. The associated link invariant is the number of 3-colourings, as defined by Fox [22, 23]. The operator c_Q does not admit any non-trivial deformation over $\mathbb{Q}[\![h]\!]$. In this sense it is an isolated solution of the Yang–Baxter equation. We can now prove more:

Proposition 7.1. *For the quandle $Q = \{(12), (13), (23)\} \subset S_3$ the associated Yang–Baxter operator c_Q is rigid over every complete ring.*

Proof. According to [15], the operator c_Q is rigid over every ring $\mathbb{A}$ in which the order $|S_3| = 6$ is invertible. Potentially there could exist non-trivial deformations in characteristic 2 or 3. Theorem 1.3 ensures that infinitesimal deformations are quasi-diagonal, which means diagonal in the present example because $\rho \colon Q \to \mathrm{Inn}(Q)$ is injective (see §7.2). According to Proposition 3.3, diagonal deformations correspond to rack cohomology. A direct calculation shows that $H^2_{\mathrm{R}}(Q; \mathbb{Z}/_2) \cong \mathbb{Z}/_2$ and $H^2_{\mathrm{R}}(Q; \mathbb{Z}/_3) \cong \mathbb{Z}/_3$, whence Corollary 6.7 implies rigidity. $\square$

7.4. **The other quandle of order 3.** The smallest quandle that is non-trivial yet deformable is $Q = \{a, b, c\}$ with operation given by the table below. Ordering the basis $Q \times Q$ lexicographically, we obtain the matrix of c_Q as indicated. We consider $\mathbb{A} = \mathbb{K}[h]/(h^2)$ with $\mathfrak{m} = h\mathbb{K}$. The group $\mathrm{Inn}(Q)$ is of order 2: if 2 is invertible in $\mathbb{K}$, then $H^2_{\mathrm{YB}}(c_Q; \mathbb{K})$ is free of rank 9 and can easily be made explicit using the results of [15]. We state it here in form of a 9-parameter deformation $c = c_Q \circ (\mathrm{id}_V^{\otimes 2} + f)$, where $f \in C^2_{\mathrm{YB}}(c_Q, \mathfrak{m})$ is quasi-diagonal and equivariant under $\mathrm{Inn}(Q) \times \mathrm{Inn}(Q)$:

$$Q = \begin{array}{|c|ccc|} \hline * & a & b & c \\ \hline a & a & a & b \\ b & b & b & a \\ c & c & c & c \\ \hline \end{array}, \quad c_Q = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad f = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdot & \lambda_3 & \lambda_4 & \cdot & \cdot & \cdot & \cdot \\ \lambda_2 & \lambda_1 & \cdot & \lambda_4 & \lambda_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \lambda_5 & \cdot & \cdot & \lambda_6 & \cdot & \cdot & \cdot \\ \lambda_3 & \lambda_4 & \cdot & \lambda_1 & \lambda_2 & \cdot & \cdot & \cdot & \cdot \\ \lambda_4 & \lambda_3 & \cdot & \lambda_2 & \lambda_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \lambda_6 & \cdot & \cdot & \lambda_5 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_7 & \lambda_8 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_8 & \lambda_7 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_9 \end{pmatrix}.$$

For every choice of parameters $\lambda_1, \ldots, \lambda_9 \in \mathfrak{m}$ the deformed operator satisfies the Yang–Baxter equation (to all orders) and thus deforms c_Q over $(\mathbb{A}, \mathfrak{m})$.

A priori there could exist more deformations over $\mathbb{Z}/_2$, but a computer calculation shows that $\dim H^2_{\mathrm{YB}}(c_Q; \mathbb{Z}/_2) = 9$. So for this quandle there are no additional deformations in the modular case.

7.5. **The dihedral quandle of order** 4. There are quandles for which the modular case offers more deformations than the rational case. We wish to illustrate this by an example where the additional deformations are not diagonal but quasi-diagonal. The smallest example is given by the set of reflections of a square,

$$Q = \{\ (13)\,,\ (24)\,,\ (12)(34)\,,\ (14)(23)\ \}.$$

This set is closed under conjugation in the symmetric group S_4, hence a quandle. With respect to the lexicographical basis, c_Q is represented by the following permutation matrix:

$$c_Q = \begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}.$$

By construction, this matrix is a solution of the Yang–Baxter equation. According to [15] it admits a 16-parameter deformation $c(\lambda) = c_Q \circ (\mathrm{id}_V^{\otimes 2} + f)$ given by the following matrix, which is quasi-diagonal and equivariant under $\mathrm{Inn}(Q) \times \mathrm{Inn}(Q)$:

$$f = \begin{pmatrix}
\lambda_1&\lambda_2&\cdot&\cdot&\lambda_3&\lambda_4&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\lambda_2&\lambda_1&\cdot&\cdot&\lambda_4&\lambda_3&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\lambda_5&\lambda_6&\cdot&\cdot&\lambda_7&\lambda_8&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\lambda_6&\lambda_5&\cdot&\cdot&\lambda_8&\lambda_7&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\lambda_3&\lambda_4&\cdot&\cdot&\lambda_1&\lambda_2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\lambda_4&\lambda_3&\cdot&\cdot&\lambda_2&\lambda_1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\lambda_7&\lambda_8&\cdot&\cdot&\lambda_5&\lambda_6&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\lambda_8&\lambda_7&\cdot&\cdot&\lambda_6&\lambda_5&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\lambda_9&\lambda_{10}&\cdot&\cdot&\lambda_{11}&\lambda_{12}&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\lambda_{10}&\lambda_9&\cdot&\cdot&\lambda_{12}&\lambda_{11}&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\lambda_{13}&\lambda_{14}&\cdot&\cdot&\lambda_{15}&\lambda_{16}\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\lambda_{14}&\lambda_{13}&\cdot&\cdot&\lambda_{16}&\lambda_{15}\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\lambda_{11}&\lambda_{12}&\cdot&\cdot&\lambda_9&\lambda_{10}&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\lambda_{12}&\lambda_{11}&\cdot&\cdot&\lambda_{10}&\lambda_9&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\lambda_{15}&\lambda_{16}&\cdot&\cdot&\lambda_{13}&\lambda_{14}\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\lambda_{16}&\lambda_{15}&\cdot&\cdot&\lambda_{14}&\lambda_{13}
\end{pmatrix}.$$

For every choice of parameters $\lambda_1, \dots, \lambda_{16} \in \mathfrak{m}$ the matrix $c(\lambda)$ satisfies the Yang–Baxter equation (to all orders) and thus deforms $c(0) = c_Q$ over $(\mathbb{A}, \mathfrak{m})$.

The quandle Q has the inner automorphism group $\mathrm{Inn}(Q) \cong \mathbb{Z}/_2 \times \mathbb{Z}/_2$, of order 4. If 2 is invertible in $\mathbb{K}$, then $H^*_{\mathrm{YB}}(c_Q; \mathbb{K})$ can be calculated using the results of [15] and is easily seen to be free of rank 16 such that f is the most general deformation. In particular we have $\dim H^*_{\mathrm{YB}}(c_Q; \mathbb{K}) = 16$ for every field $\mathbb{K}$ of characteristic $\neq 2$.

Over $\mathbb{K} = \mathbb{Z}/_2$, however, a computer calculation shows that $\dim H^2_{\mathrm{YB}}(c_Q; \mathbb{Z}/_2) = 20$, which means that there exists a 20-parameter deformation, at least infinitesimally. We state the result in the form $c = c_Q(\mathrm{id}_V^{\otimes 2} + f + g)$ as follows.

First we have the 16-parameter family that appears in every characteristic:

$$f = \begin{pmatrix}
\lambda_1 & \lambda_2 & \cdot & \cdot & \lambda_3 & \lambda_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_2 & \lambda_1 & \cdot & \cdot & \lambda_4 & \lambda_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_5' & \lambda_6 & \cdot & \cdot & \lambda_7' & \lambda_8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_6 & \lambda_5' & \cdot & \cdot & \lambda_8 & \lambda_7' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_3 & \lambda_4 & \cdot & \cdot & \lambda_1 & \lambda_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\lambda_4 & \lambda_3 & \cdot & \cdot & \lambda_2 & \lambda_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_7'' & \lambda_8 & \cdot & \cdot & \lambda_5'' & \lambda_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda_8 & \lambda_7'' & \cdot & \cdot & \lambda_6 & \lambda_5'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_9' & \lambda_{10} & \cdot & \cdot & \lambda_{11}' & \lambda_{12} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{10} & \lambda_9' & \cdot & \cdot & \lambda_{12} & \lambda_{11}' & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{13} & \lambda_{14} & \cdot & \cdot & \lambda_{15} & \lambda_{16} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{14} & \lambda_{13} & \cdot & \cdot & \lambda_{16} & \lambda_{15} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{11}'' & \lambda_{12} & \cdot & \cdot & \lambda_9'' & \lambda_{10} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{12} & \lambda_{11}'' & \cdot & \cdot & \lambda_{10} & \lambda_9'' & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{15} & \lambda_{16} & \cdot & \cdot & \lambda_{13} & \lambda_{14} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{16} & \lambda_{15} & \cdot & \cdot & \lambda_{14} & \lambda_{13}
\end{pmatrix}.$$

For every choice of parameters in $\mathfrak{m}$ the matrix $c(\lambda) = c_Q \circ (\mathrm{id}_V^{\otimes 2} + f)$ satisfies the Yang–Baxter equation modulo $\mathfrak{m}^2$. It even satisfies the Yang–Baxter equation to any order provided that $\lambda_5' = \lambda_5''$, $\lambda_7' = \lambda_7''$, $\lambda_9' = \lambda_9''$, $\lambda_{11}' = \lambda_{11}''$.

We set $\lambda_5 = \lambda_5' + \lambda_5''$, $\lambda_7 = \lambda_7' + \lambda_7''$, $\lambda_9 = \lambda_9' + \lambda_9''$, $\lambda_{11} = \lambda_{11}' + \lambda_{11}''$. Two deformations $c(\lambda)$ and $c(\tilde{\lambda})$ are gauge equivalent if and only if $\lambda_k = \tilde{\lambda}_k$ for all $k = 1, \dots, 16$. We have chosen the redundant formulation above in order to highlight the symmetry resp. the symmetry breaking. If 2 were invertible, we would simply set $\lambda_5' = \lambda_5'' = \frac{1}{2}\lambda_5$ etc. In characteristic 2, however, we can realize $\lambda_5 = 1$ either by $\lambda_5' = 1$ and $\lambda_5'' = 0$, or by $\lambda_5' = 0$ and $\lambda_5'' = 1$: both deformations are gauge equivalent, but no symmetric form is possible.

Next we have a 4-parameter deformation that appears only in characteristic 2:

$$g = \begin{pmatrix}
\alpha' & \cdot & \cdot & \cdot & \beta' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \alpha'' & \cdot & \cdot & \cdot & \beta'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \alpha' & \cdot & \cdot & \cdot & \beta' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \alpha'' & \cdot & \cdot & \cdot & \beta'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\beta'' & \cdot & \cdot & \cdot & \alpha'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \beta' & \cdot & \cdot & \cdot & \alpha' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \beta'' & \cdot & \cdot & \cdot & \alpha'' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \beta' & \cdot & \cdot & \cdot & \alpha' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma' & \cdot & \cdot & \cdot & \delta' & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma'' & \cdot & \cdot & \cdot & \delta'' & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma' & \cdot & \cdot & \cdot & \delta' & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma'' & \cdot & \cdot & \cdot & \delta'' \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta'' & \cdot & \cdot & \cdot & \gamma'' & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta' & \cdot & \cdot & \cdot & \gamma' & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta'' & \cdot & \cdot & \cdot & \gamma'' & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta' & \cdot & \cdot & \cdot & \gamma'
\end{pmatrix}.$$

For every choice of parameters in $\mathfrak{m}$ the matrix $c_Q \circ (\mathrm{id}_V^{\otimes 2} + g)$ satisfies the Yang–Baxter equation modulo $\mathfrak{m}^2$. Two such deformations are gauge equivalent if and only if they share the same values $\alpha = \alpha' + \alpha''$, $\beta = \beta' + \beta''$, $\gamma = \gamma' + \gamma''$, $\delta = \delta' + \delta''$. They satisfy the Yang–Baxter equation modulo $\mathfrak{m}^3$ if and only if $\alpha' = \alpha''$, $\beta' = \beta''$, $\gamma' = \gamma''$, $\delta' = \delta''$, whence $\alpha = \beta = \gamma = \delta = 0$. The implications of such modular deformations still have to be worked out (§8.2).

7.6. **Colouring polynomials.** We conclude with a class of examples where the modular case provides non-trivial diagonal deformations. Interesting knot invariants, called colouring polynomials, arise already at the infinitesimal level.

For concreteness' sake, consider the alternating group $G = A_5$ and the conjugacy class $Q = (12345)^G$ of order 12. The knot invariant associated to c_Q counts for each knot $K \subset \mathbb{R}^3$ the number of knot group representations $\pi_1(\mathbb{R}^3 \smallsetminus K) \to G$ sending meridians of K to elements of Q. According to §7.2 the operator c_Q has only trivial deformations over $\mathbb{Q}[\![h]\!]$ or any ring $\mathbb{A}$ in which $|\mathrm{Inn}(Q)| = 60$ is invertible.

The modular case is more interesting: if we consider $\mathbb{A} = \mathbb{Z}/_5[h]/(h^2)$, then c_Q allows non-trivial diagonal deformations that are topologically interesting [16, Exm. 1.3]. The associated knot invariants can be identified as *colouring polynomials*, counting knot group representations $\pi_1(\mathbb{R}^3 \smallsetminus K) \to G$ while keeping track of longitudinal information [14, 16]. According to Theorem 1.3, *all* infinitesimal deformations of c_Q are encoded in this way by rack cohomology, which has been intensely studied in recent years and is fairly well understood.

8. Open questions

8.1. **Topological interpretation.** Our calculations of Yang–Baxter cohomology have been entirely algebraic. Unfortunately we do not have any topological model to guide our intuition or to translate Yang–Baxter cohomology to a geometric situation that would be easier to understand. By way of contrast, for group cohomology we have the topological notion of classifying space, see Brown [8, §I.4]. For racks an analogous concept was developed by Fenn and Rourke [21]. It would be interesting to set up a topological model for Yang–Baxter cohomology. Is this possible? Does the non-functoriality of §2.5 obstruct such a construction?

8.2. **From infinitesimal to complete deformations.** As explained in §3, rack cohomology $H^2_{\mathrm{R}}(Q;\mathbb{K})$ encodes infinitesimal deformations of c_Q, i.e., deformations over $\mathbb{A} = \mathbb{K}[h]/(h^2)$. Even at the infinitesimal level this approach leads to interesting knot invariants, as illustrated by colouring polynomials (§7.6). In the framework of Yang–Baxter deformations, the following generalization appears natural:

Question 8.1. What is the classification of complete deformations, that is, deformations of c_Q over the power series ring $\mathbb{K}[\![h]\!]$ or the p-adic integers $\mathbb{Z}_p$?

For deformations of finite racks over $\mathbb{Q}[\![h]\!]$ this question has been solved in [15]. The modular case is still open and potentially more interesting.

Question 8.2. Given a deformation of c_Q, which topological information is contained in the associated knot invariant?

For knot invariants coming from rack or quandle cohomology, this question was answered in [16]. For non-diagonal deformations the question is still open. Notice that the problem gets more complicated and more intriguing as we approach the quantum case: the closer Q is to the trivial quandle, the more deformations will appear. Their topological interpretation, however, becomes more difficult, and for the time being remains mysterious.

8.3. **From racks to biracks.** Given a set Q and a bijective map $c\colon Q\times Q \to Q\times Q$, we can formulate the set-theoretic Yang–Baxter equation [12] as

$$(\mathrm{id} \times c)(c \times \mathrm{id})(\mathrm{id} \times c) = (c \times \mathrm{id})(\mathrm{id} \times c)(c \times \mathrm{id}).$$

In general c will have the form $c(x, y) = (x \triangleright y, x \triangleleft y)$ with two binary operations $\triangleright, \triangleleft\colon Q \times Q \to Q$, see [18, 36] for details. Recently, Kauffman's theory of virtual knots [33] has rekindled interest in such set-theoretic solutions $(Q, \triangleright, \triangleleft)$ called *biracks* or *biquandles* [20, 34]. Racks correspond to the case where the operation

$x \triangleright y = y$ is trivial whereas $x \triangleleft y = x^y$ is the rack operation. Our notion of Yang–Baxter cohomology [15] has been conceived for arbitrary Yang–Baxter operators, and in particular it covers set-theoretic solutions such as biracks and biquandles.

Question 8.3. Can our results on quasi-diagonal deformations be extended to set-theoretic solutions of the Yang–Baxter equation that do not come from racks?

The restricted setting of diagonal deformations has been studied by Carter *et al.* [9]. More general deformations still need to be examined.

8.4. **Higher dimensional knots.** It is worth noting that quandle 3-cocycles have been used to construct state-sum invariants of knotted surfaces $K\colon M^2 \hookrightarrow \mathbb{R}^4$, see Carter *et al.* [10]. These invariants count finite representations of the fundamental quandle Q_K with fundamental class $[K] \in H_3(Q_K)$. On the other hand, deformation theory interprets 3-cocycles as higher-order obstructions to algebraic deformations. What is the relationship between these viewpoints? For classical knots see [14, 16]. Does this relationship generalize to (framed) n-knots $M^n \hookrightarrow \mathbb{R}^{n+2}$?

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