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VASSILIEV INVARIANTS AND THE POINCARÉ CONJECTURE

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ABSTRACT. This article examines the relationship between 3-manifold topology and knot invariants of finite type. We prove that in every Whitehead manifold there exist knots that cannot be distinguished by Vassiliev invariants. If, on the other hand, Vassiliev invariants distinguish knots in each homotopy sphere, then the Poincaré conjecture is true (i.e. every homotopy 3-sphere is homeomorphic to the standard 3-sphere).

INTRODUCTION AND STATEMENT OF RESULTS

Knot invariants of finite type, also called Vassiliev invariants, were initially conceived to study knots in euclidean 3-space [30, 9]. Many important knot invariants are of finite type, most notably the coefficients of the Alexander-Conway and the Jones polynomial, after a suitable change of variables [9, 4]. The same holds for all quantum invariants of knots [2].

Since Vassiliev invariants exist in abundance, it soon became a central question whether or not they distinguish all knots. For the classical case of knots in $\mathbb{R}^3$ this remains unsolved to the present day. The aim of this article is to show that the question is intimately related to the topology of the ambient 3-manifold.

The combinatorial definition of finite type invariants, as given by Gusarov [9] and later by Birman and Lin [4], immediately extends to knots in an arbitrary 3-manifold. We recall the relevant facts below. Lin [17] studied the case of simply connected manifolds and proved that the algebra of Vassiliev invariants so obtained is canonically isomorphic to the algebra obtained for knots in $\mathbb{R}^3$ (possibly modulo 2-torsion). Lin's result does not suffice, however, to provide knots that are indistinguishable by Vassiliev invariants. The present article develops several techniques to produce such examples.

Knots in Whitehead manifolds. A *Whitehead manifold* is a contractible open 3-manifold that is not homeomorphic to $\mathbb{R}^3$ but embeddable therein. The first example of such a manifold was discovered by J.H.C. Whitehead [31]. There exists an uncountable infinity of such manifolds, no two of which are homeomorphic [18]. In §3 we present an elementary proof of Lin's result for Whitehead manifolds and establish a considerably stronger conclusion:

Theorem 1 (proved in §3.3)**.** *In every Whitehead manifold there exist distinct knots that cannot be distinguished by Vassiliev invariants.*

This is the first known example where Vassiliev invariants fail to distinguish knots. The construction given in §3 is very simple, and concrete examples are provided in §4. The price for this simplicity is, of course, the exotic nature of the ambient manifold.

Knots in homotopy spheres. One may ask whether such a pathological situation can arise in a closed manifold as well. Lin [17] already mentioned that there possibly exists a connection between this knot-theoretic question and the Poincaré conjecture. As our main result, we prove that this is indeed the case.

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Recall that a *homotopy* 3*-sphere* is a closed 3-manifold that is simply connected. Such a manifold is homotopy equivalent to the standard 3-sphere, whence the name. The Poincaré conjecture [24] states that every homotopy 3-sphere is in fact homeomorphic to the standard 3-sphere. We prove:

Theorem 2. *If Vassiliev invariants distinguish all knots in each homotopy sphere, then the Poincaré conjecture is true.*

To prove this theorem, we proceed as follows. Suppose that V is a *fake* 3*-sphere*, that is, a homotopy 3-sphere that is not homeomorphic to $\mathbb{S}^3$. By Bing's characterization of the 3-sphere [3], V contains a knot K that is not contained in any open ball. The connected sum $M = V \sharp V$ is again a homotopy sphere, and it contains two copies K_1 and K_2 of the knot K. We first establish that these two knots cannot be distinguished by any Vassiliev invariant:

Theorem 3 (proved in §3.2)**.** *Let M be a simply connected* 3*-manifold and $h\colon M \hookrightarrow M$ be an orientation preserving embedding. Then Vassiliev invariants cannot distinguish between a knot K_1 and its image $K_2 = hK_1$.*

In our construction it is true, but far from obvious, that K_1 and K_2 are actually distinct. This is established, at the end of this article, by the following result:

Theorem 4 (proved in §5.4)**.** *Let $M = V_1 \sharp V_2$ be a connected sum of two* 3*-manifolds. If two knots $K_1 \subset V_1$ and $K_2 \subset V_2$ are isotopic in M, then each is contained in an open ball.*

This concludes the case of homotopy 3-spheres and proves Theorem 2. Somewhat more generally, a *fake* 3*-ball* is a compact contractible 3-manifold C, with boundary ∂C homeomorphic to the 2-sphere $\mathbb{S}^2$, such that C is not homeomorphic to the standard 3-ball. For an arbitrary 3-manifold M we obtain the following conclusion: If Vassiliev invariants distinguish knots in $M \sharp M$, then M does not contain any fake 3-balls. It seems plausible to strengthen this conclusion by replacing $M \sharp M$ with M, but this problem will require a different approach.

How this paper is organized. Section 1 collects some background material about knots in 3-manifolds and Bing's characterization of the 3-sphere. Section 2 establishes the equivalence between homotopy of singular knots and discrete homotopy via crossing changes. These preliminaries being in place, Section 3 constructs knots that cannot be distinguished by any Vassiliev invariant. Section 4 provides concrete examples of such knots in Whitehead manifolds. Finally, Section 5 establishes an isotopy version of the Alexander-Schönflies theorem, which then serves to distinguish certain knots in homotopy spheres.

1. Knots in 3-manifolds

This first section collects some basic facts about singular knots in 3-manifolds. Throughout this article we will work in the category of smooth manifolds and smooth maps. For standard notions in differential topology we refer to Hirsch [13]. Unless otherwise stated, every 3-manifold will be assumed to be connected, oriented, and without boundary. Such a manifold can be compact (hence closed) or non-compact (hence open). In $\mathbb{R}^n$ we let $\mathbb{D}^n$ denote the closed unit ball, $\mathbb{B}^n = \operatorname{int} \mathbb{D}^n$ the open unit ball, and $\mathbb{S}^{n-1} = \partial \mathbb{D}^n$ the standard $n-1$ dimensional sphere. The unit interval is denoted by $\mathbb{I} = [0, 1]$.

1.1. **Knots and singular knots.** A *knot* in a 3-manifold M is an embedding $\kappa\colon \mathbb{S}^1 \hookrightarrow M$. More generally, a *singular knot* is an immersion $\kappa\colon \mathbb{S}^1 \looparrowright M$ such that every multiple point is a non-degenerate double point according to the local model $\times$. More formally this means that for every double point $p = \kappa(s) = \kappa(\bar{s})$ with $s \neq \bar{s}$ the tangent vectors $\kappa'(s)$ and $\kappa'(\bar{s})$ are linearly independent in T_pM.

In particular, non-degeneracy implies that κ can only have a finite number of double points, and we will assume that they are numbered by $1, \dots, n$. A double point is also called a *singularity*, and a knot with n double points is called *n-singular*. This includes the special case of (0-singular) knots.

Two n-singular knots $\kappa, \kappa' \colon \mathbb{S}^1 \looparrowright M$ are *equivalent* if they differ only by diffeotopies of the circle $\mathbb{S}^1$ and the ambient manifold M. This is the same as considering the oriented image $K = \kappa(\mathbb{S}^1)$ modulo diffeotopies of M. The equivalence class is denoted by $[\kappa]$ or $[K]$, respectively. We let $\mathcal{K}_n$ denote the set of equivalence classes of n-singular knots. Again this definition includes the set $\mathcal{K} = \mathcal{K}_0$ of equivalence classes of non-singular knots.

It is customary not to insist on the distinction between an n-singular knot $\kappa \colon \mathbb{S}^1 \looparrowright M$, its image $\kappa(\mathbb{S}^1) \subset M$, and its equivalence class $[\kappa] \in \mathcal{K}_n$. We will adopt this slight abuse of notation whenever there is no danger of confusion

It is essential for the sequel that the equivalence of knots is preserved under embeddings of 3-manifolds. To this end we will employ Thom's isotopy extension theorem [29], in a generalized form given by Hirsch [13, Theorem 8.1.4]:

Theorem 5 (Isotopy extension [29, 13]). *Let $K \subset M$ be a compact subset and let $\phi \colon \mathbb{I} \times M \to N$ be an isotopy. Then there exists a diffeotopy $\Phi \colon \mathbb{I} \times N \to N$ having compact support such that $\Phi_0 = \mathrm{id}_N$ and $\Phi_t \phi_0(x) = \phi_t(x)$ for all x in some neighbourhood of K.* □

Corollary 6. *Every embedding $\phi \colon M \hookrightarrow N$ of 3-manifolds induces a natural map $\mathcal{K}_* \phi \colon \mathcal{K}_* M \to \mathcal{K}_* N$, which depends only on the isotopy class of ϕ.* □

1.2. **Knot theoretic characterization of R^3 and S^3.** We will make frequent use of the following fact, which is a special case of the n-dimensional version established independently by Milnor [20, Theorem 2.2], Palais [23, Theorem B], and Cerf [5, Proposition II.5.7]. See also Hirsch [13, Theorem 8.3.1].

Theorem 7 (Disk embedding [20, 23, 5]). *Any two orientation-preserving embeddings $\phi_0, \phi_1 \colon \mathbb{B}^3 \hookrightarrow M$ are isotopic. Any two orientation-preserving embeddings $\phi_0, \phi_1 \colon \mathbb{D}^3 \hookrightarrow M$ are ambient isotopic, that is, there exists a diffeotopy $\Phi \colon \mathbb{I} \times M \to M$ such that $\Phi_0 = \mathrm{id}_M$ and $\Phi_1 \phi_0 = \phi_1$.* □

An n-singular knot $\kappa \colon \mathbb{S}^1 \looparrowright M$ is called *local* if it is contained in an open ball in M. By the preceding theorem, the subset of local knots in $\mathcal{K}_* M$ is exactly the image of the map $\mathcal{K}_* \phi \colon \mathcal{K}_* \mathbb{R}^3 \to \mathcal{K}_* M$ induced by any given embedding $\phi \colon \mathbb{R}^3 \hookrightarrow M$.

Corollary 8. *Given an orientation-preserving embedding $h \colon M \hookrightarrow M$, every local knot K is equivalent to its image hK.* □

Obviously, if every knot in M is local, then M is necessarily simply connected. The converse, however, is false: Whitehead manifolds are simply connected but they contain non-local knots (see §4). More generally we have the following characterization of euclidean 3-space, due to Costich, Doyle, and Galewski [6]:

Theorem 9 ([6]). *Let M be a contractible open 3-manifold. If every knot in M is local, then M is homeomorphic to euclidean space $\mathbb{R}^3$.* □

This is based on an earlier result of Bing [3] characterizing the 3-sphere:

Theorem 10 ([3]). *Let M be a closed connected 3-manifold. If every knot in M is local, then M is homeomorphic to the standard sphere $\mathbb{S}^3$.* □

Beside the original proof given by Bing [3], alternative proofs can be found in the textbooks by Hempel [12, Theorem 14.3] and Rolfsen [25, §9E]. The theorem also follows from the existence of an open book decomposition [22], and from the surgery presentation of 3-manifolds [8].

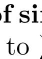
2. Homotopy of knots in 3-manifolds

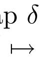
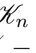
In order to understand Vassiliev invariants of knots in a simply connected 3-manifold, we obviously have to make use of its homotopy properties. On the other hand, we will work with the combinatorial definition of Vassiliev invariants given by Gusarov [9] and by Birman and Lin [4]. This section recalls the essential constructions and establishes the equivalence between homotopy and discrete homotopy (Lemma 14) to be exploited in the sequel.

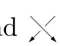
2.1. **Homotopy of knots and singular knots.** Two knots $\kappa_0, \kappa_1 \colon \mathbb{S}^1 \hookrightarrow M$ are *homotopic* in M if there exists a smooth map $h\colon \mathbb{I} \times \mathbb{S}^1 \to M$ with $h_0 = \kappa_0$ and $h_1 = \kappa_1$. In the case where κ_0 and κ_1 are n-singular knots, however, we have to take some extra precautions in order to ensure that h preserves singularities. Given a singular knot $\kappa\colon \mathbb{S}^1 \looparrowright M$, its *singular set* Σ_κ is the preimage of its singularities. It is a finite subset of $\mathbb{S}^1$ equipped with a fixed-point free involution τ that associates to each singular parameter s the unique parameter $\bar{s} \neq s$ with $\kappa(\bar{s}) = \kappa(s)$. We extend τ by $\tau r = r$ for all regular parameters $r \in \mathbb{S}^1 \smallsetminus \Sigma_\kappa$. The involution τ contains all the information about the configuration of singularities: by construction κ induces a homeomorphism $\bar{\kappa}\colon \mathbb{S}^1/\tau \xrightarrow{\sim} \kappa(\mathbb{S}^1)$. Graphically, τ is encoded as a *chord diagram* on the circle $\mathbb{S}^1$: the points of Σ_κ are marked as vertices and each singular pair $\{s, \tau s\}$ is connected by a *chord*, i.e. an edge between s and τs.

Definition 11. Two n-singular knots $\kappa_0, \kappa_1\colon \mathbb{S}^1 \looparrowright M$ are *combinatorially equivalent* if their chord diagrams τ_0 and τ_1 differ by an orientation-preserving diffeomorphism of the circle. After reparametrization we can thus assume that κ_0 and κ_1 have the same chord diagram τ. We then say that κ_0 and κ_1 are *homotopic* in M if there exists a smooth homotopy $h\colon \mathbb{I} \times \mathbb{S}^1 \to M$ such that $h_0 = \kappa_0$ and $h_1 = \kappa_1$ and $h_t(s) = h_t(\tau s)$ for all $t \in [0,1]$ and $s \in \mathbb{S}^1$. This is the same as saying that h descends to a homotopy $\bar{h}$ of the quotient $\mathbb{S}^1/\tau$ such that $\bar{h}_0 = \bar{\kappa}_0$ and $\bar{h}_1 = \bar{\kappa}_1$.

Remark 12. Homotopies of singular knots preserve the combinatorial data and the numbering of singularities. In a simply connected manifold, homotopy of singular knots thus coincides with combinatorial equivalence of chord diagrams. This can also be achieved for a non-simply connected 3-manifold M if we label chord diagrams with elements of $\pi_1(M)$, as explained in Lieberum [16].

2.2. **Resolution of singularities and discrete homotopy.** If we regard a generic homotopy from  to  then at some intermediate time we will encounter a singular knot that looks locally like . Conversely, we can define the notion of *discrete homotopy* as follows:

Definition 13. Let $\mathscr{K}_n = \mathbb{Z}\mathcal{K}_n$ be the $\mathbb{Z}$-module with basis $\mathcal{K}_n$. We define the linear map $\delta\colon \mathscr{K}_n \to \mathscr{K}_{n-1}$ by resolving the n-th singularity according to the local model  $\mapsto$  $-$ .

In this definition the figures  and  and  represent knots $\kappa_\bullet$ and κ_+ and κ_-, respectively, that differ as shown in a small ball B around the singularity and are identical outside of B. Based on our definition of equivalence, this map is indeed well-defined: the classes $[\kappa_+]$ and $[\kappa_-]$ are uniquely defined by the local model and the orientation of M, and they depend only on the class $[\kappa_\bullet]$, not on $\kappa_\bullet$ itself.

Lemma 14 (Homotopy discretization)**.** *Two n-singular knots κ_0, κ_1 in M are homotopic if and only if their equivalence classes satisfy $[\kappa_0] \equiv [\kappa_1]$ modulo $\delta\mathscr{K}_{n+1}$.*

The lemma can be thus rephrased: every homotopy between n-singular knots can be discretized into a finite sequence of crossing changes. A proof in the piecewise linear setting has been sketched by Lin [17, Lemma 6.4] and Kalfagianni [14, Lemma 4.3]. Since a detailed account seems not to be available in the literature, we provide a full proof in the smooth setting:

Proof. The two singular knots κ_+ and κ_- appearing in the resolution of an $(n+1)$-singular knot $\kappa_\bullet$ are homotopic. It follows that for every formal sum $A \in \mathscr{K}_{n+1}$ the resolution δA has coefficient sum 0 with respect to each homotopy class. Thus if two n-singular knots κ_0 and κ_1 satisfy $[\kappa_0] - [\kappa_1] = \delta A$, then they are homotopic.

To prove the converse, we start with a homotopy $h\colon \mathbb{I} \times \mathbb{S}^1 \to M$. The idea is, of course, to put the homotopy in general position. For simplicity's sake we begin with the case where κ_0 and κ_1 are non-singular knots.

Putting a homotopy in general position. We define $H\colon \mathbb{I} \times \mathbb{S}^1 \to \mathbb{I} \times M$ by $H(t,s) = (t, h_t(s))$. Since H maps a 2-manifold to a 4-manifold, we can apply Whitney's immersion theorem, see Hirsch [13, Theorem 2.2.12]: the map H can be approximated by an immersion $\bar{H}$, still satisfying $\bar{H}(0,s) = (0, \kappa_0(s))$ and $\bar{H}(1,s) = (1, \kappa_1(s))$. Furthermore we can assume that every multiple point of $\bar{H}$ is a transverse double point. Notice, however, that the projection $\pi\colon \mathbb{I} \times M \to \mathbb{I}$, given by $\pi(t,m) = t$, does not necessarily yield $\pi\bar{H}(t,s) = t$. But $\phi\colon \mathbb{I} \times \mathbb{S}^1 \to \mathbb{I} \times \mathbb{S}^1$ given by $\phi(t,s) = (\pi\bar{H}(t,s), s)$ is close to the identity. Hence ϕ is a diffeomorphism (see Munkres [21, Theorem 3.10]), and $K = \bar{H}\phi^{-1}$ satisfies $\pi K(t,s) = t$ as desired.

Since every multiple point of K is a transverse double point, there can only be a finite number of them. In particular we can arrange that they all appear at different values of t. We finally obtain a homotopy $\kappa\colon \mathbb{I} \times \mathbb{S}^1 \to M$ with $K(t,s) = (t, \kappa_t(s))$. In view of its above properties we say that the homotopy κ is *generic* or in *general position.*

Discretizing a generic homotopy. For all but a finite number of values $t \in \mathbb{I}$ the map κ_t defines a non-singular knot $\mathbb{S}^1 \hookrightarrow M$. Each singular time t is isolated in $\mathbb{I}$ and we have $\kappa_t(s) = \kappa_t(\bar{s})$ for exactly one pair of parameters $s \neq \bar{s}$ in $\mathbb{S}^1$. It remains to identify this situation with the local model of crossing changes given earlier.

We set $\dot{\kappa} = \frac{\partial}{\partial t}\kappa$ and $\kappa' = \frac{\partial}{\partial s}\kappa$. Every double point $p = \kappa_t(s) = \kappa_t(\bar{s})$ corresponds to a double point $P = K(t,s) = K(t,\bar{s})$. Transversality in $N = \mathbb{I} \times M$ means that the four tangent vectors

$$(1, \dot{\kappa}(t,s)), \quad (0, \kappa'(t,s)), \quad (1, \dot{\kappa}(t,\bar{s})), \quad (0, \kappa'(t,\bar{s}))$$

form a basis of $T_P N$. This is equivalent to saying that the three vectors

$$\kappa'(t,s), \quad \kappa'(t,\bar{s}), \quad \dot{\kappa}(t,s) - \dot{\kappa}(t,\bar{s})$$

form a basis of $T_p M$. Define $\sigma(t) = \pm 1$ according to whether this basis is positive or negative with respect to the orientation of M. Note that this sign is well-defined: it is invariant if we exchange s and $\bar{s}$.

It is now easy to understand the passage of the critical time t from $\kappa_{t-\varepsilon}$ via κ_t to $\kappa_{t+\varepsilon}$ in terms of our local model: for $\sigma(t) = +1$ we pass from via to , and for $\sigma(t) = -1$ we pass from via to . To conclude, let $t_1, t_2, \ldots, t_r$ be the critical times of κ and define $A = \sum_{i=1}^{i=r} \sigma(t_i)[\kappa_{t_i}]$. By construction we find $\delta A = [\kappa_1] - [\kappa_0]$ as desired.

Homotopies of singular knots. In the general case, where κ_0 and κ_1 are n-singular, one proceeds as follows. Let $\Sigma = \{s_1, \bar{s}_1, \ldots, s_n, \bar{s}_n\} \subset \mathbb{S}^1$ be the set of singular parameters of κ_0, which by hypothesis are the same for κ_1. First of all, we can assume that during the homotopy h different double points $h(t, s_i) = h(t, \bar{s}_i)$ and $h(t, s_j) = h(t, \bar{s}_j)$ never collide. Up to a diffeotopy of M we can then assume that double points are not moved at all. We can even arrange that a neighbourhood of the singular points is fixed, that is, $h(t,s) = \kappa_0(s)$ for all $t \in \mathbb{I}$ and $s \in U$, where U is a closed neighbourhood of Σ. After these preparations the existing n singularities do not play any further rôle: we can put h in general position without disturbing it on the set $A = (\{0,1\} \times \mathbb{S}^1) \cup (\mathbb{I} \times U)$. The above discretization applies exactly as before. This completes the proof of Lemma 14. $\square$

3. Vassiliev theory in simply connected 3-manifolds

This section shows how to construct knots in a simply connected 3-manifold that cannot be distinguished by Vassiliev invariants (Theorem 17). Our approach exploits the fact that Vassiliev theory is functorial with respect to embeddings. This point of view is ideally suited for Whitehead manifolds, our main example. With some more effort we then extend our result to homotopy spheres.

3.1. **Vassiliev theory is functorial.** The *Vassiliev filtration* of $\mathscr{K} = \mathscr{K}_0$ is defined by $\mathscr{F}_n = \operatorname{im}(\delta^n \colon \mathscr{K}_n \to \mathscr{K})$. The quotients $\mathscr{K}/\mathscr{F}_n$ form a projective system, and its limit is the $\mathbb{Z}$-module denoted by $\hat{\mathscr{K}}$. The canonical map $\alpha \colon \mathscr{K} \to \hat{\mathscr{K}}$ has kernel $\mathscr{F}_\omega = \bigcap_n \mathscr{F}_n$. According to the discrete homotopy lemma, the limit $\hat{\mathscr{K}}$ can be considered as the homotopy completion of the module of knots in M.

Dually, a knot invariant $v \colon \mathcal{K} \to A$ with values in some abelian group A is called *invariant of finite type* or *Vassiliev invariant* of degree n if $v(\mathscr{F}_{n+1}) = 0$.

Corollary 6 and the local nature of the resolution map δ allow us to interpret Vassiliev theory as a functor:

Lemma 15. *Vassiliev theory behaves functorially: To every 3-manifold M we associate a sequence of $\mathbb{Z}$-modules $(\mathscr{K}_* M, \delta)$, and every orientation-preserving embedding $\phi \colon M \hookrightarrow N$ induces a natural family of linear maps $\mathscr{K}_*\phi \colon \mathscr{K}_* M \to \mathscr{K}_* N$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathscr{K} M \arrow[d, "\mathscr{K}\phi"'] & \mathscr{K}_1 M \arrow[l, "\delta"'] \arrow[d, "\mathscr{K}_1\phi"'] & \mathscr{K}_2 M \arrow[l, "\delta"'] \arrow[d, "\mathscr{K}_2\phi"'] & \mathscr{K}_3 M \arrow[l, "\delta"'] \arrow[d, "\mathscr{K}_3\phi"'] & \dots \arrow[l, "\delta"'] \\
\mathscr{K} N & \mathscr{K}_1 N \arrow[l, "\delta"'] & \mathscr{K}_2 N \arrow[l, "\delta"'] & \mathscr{K}_3 N \arrow[l, "\delta"'] & \dots \arrow[l, "\delta"']
\end{tikzcd}$$

All of the above constructions are thus functorial. In particular the induced map $\mathscr{K}\phi$ respects the Vassiliev filtration in the sense that $\phi(\mathscr{F}_n M) \subset \mathscr{F}_n N$.

Example 16. As a rather trivial but frequent example consider a 3-manifold N and the punctured manifold $M = N \smallsetminus \{p_1, \dots, p_n\}$. The inclusion $M \hookrightarrow N$ induces isomorphisms $\mathscr{K}_* M \xrightarrow{\sim} \mathscr{K}_* N$. Knot theory, and in particular Vassiliev theory, is insensitive to adding punctures.

3.2. **The self-embedding trick.** In order to produce non-trivial examples, we turn to Whitehead manifolds and homotopy spheres in the sequel. All of our constructions are based on the following key observation:

Theorem 17. *Let M be a simply connected manifold and let $h \colon M \hookrightarrow M$ be an orientation-preserving embedding. Then Vassiliev invariants cannot distinguish between a knot κ and its image $h\kappa$.*

Proof. Every local knot κ^* is equivalent to its image $h\kappa^*$, cf. Corollary 8. Since M is simply connected, every n-singular knot κ is homotopic to some local knot κ^*, cf. Remark 12. According to Lemma 14, this homotopy can be discretized: there exists $A \in \mathscr{K}_{n+1}$ such that $\delta A = [\kappa] - [\kappa^*]$. By functoriality we obtain $\delta hA = [h\kappa] - [h\kappa^*]$ hence $\delta(A - hA) = [\kappa] - [h\kappa]$. We can extend this construction by linearity: for every $A_n \in \mathscr{K}_n$ there exists $A_{n+1} \in \mathscr{K}_{n+1}$ such that $\delta(A_{n+1} - hA_{n+1}) = A_n - hA_n$. This argument can now be iterated. For a knot κ this means that $[\kappa] - [h\kappa] \in \mathscr{F}_1 \cap \mathscr{F}_2 \cap \dots = \mathscr{F}_\omega$. We conclude that Vassiliev invariants cannot distinguish κ and $h\kappa$. □

3.3. **Application to Whitehead manifolds.** According to Kister and McMillan [18, 15] there exist uncountably many contractible open 3-manifolds, no two of which are homeomorphic. They can be divided into two uncountable families depending on whether they embed into $\mathbb{R}^3$ or not. A contractible open 3-manifold $W \ncong \mathbb{R}^3$

that embeds into $\mathbb{R}^3$ is called a *Whitehead manifold* [31, 18]. We immediately derive the following isomorphism theorem:

Corollary 18. *Let W be a Whitehead manifold. Then any two orientation preserving embeddings $f\colon \mathbb{R}^3 \hookrightarrow W$ and $g\colon W \hookrightarrow \mathbb{R}^3$ induce mutually inverse isomorphisms $\mathscr{K}\mathbb{R}^3/\mathscr{F}_n\mathbb{R}^3 \cong \mathscr{K}W/\mathscr{F}_nW$ for all n. In particular, the algebra of Vassiliev invariants is the same for $\mathbb{R}^3$ and for W, up to a canonical isomorphism.*

Proof. On the one hand, the composition $gf\colon \mathbb{R}^3 \hookrightarrow W \hookrightarrow \mathbb{R}^3$ induces the identity on $\mathscr{K}_*\mathbb{R}^3$, hence on each quotient $\mathscr{K}\mathbb{R}^3/\mathscr{F}_n\mathbb{R}^3$. On the other hand, $fg\colon W \hookrightarrow \mathbb{R}^3 \hookrightarrow W$ induces the identity not on $\mathscr{K}_*W$ but on each quotient $\mathscr{K}W/\mathscr{F}_nW$, as shown by the previous theorem. □

Using a more general approach, Lin [17] obtained a similar result: for every contractible open 3-manifold W every embedding $\mathbb{R}^3 \hookrightarrow W$ induces isomorphisms $\mathscr{K}\mathbb{R}^3/\mathscr{F}_n \cong \mathscr{K}W/\mathscr{F}_n$, possibly modulo 2-torsion. In the case of a Whitehead manifold, not only is the proof considerably simplified by the above argument, but our techniques also allow a much stronger conclusion:

Corollary 19. *In every Whitehead manifold W there exist knots that are distinct but cannot be distinguished by any Vassiliev invariant.*

Proof. Let $h\colon W \hookrightarrow \mathbb{R}^3 \hookrightarrow W$ be an orientation-preserving embedding. Theorem 9 guarantees the existence of a non-local knot K in W. Its image hK is local, hence K and hK are not isotopic in W. According to the previous theorem we have $K \equiv hK$ modulo $\mathscr{F}_\omega$. □

3.4. **Application to homotopy spheres.** Conjecturally the preceding corollary holds for every contractible open 3-manifold, even if it does not embed into $\mathbb{R}^3$. We will consider such examples in §4.2 below. For the time being, we content ourselves with the following weaker version:

Theorem 20. *Let M be a simply connected 3-manifold that contains a non-local knot K. Then the two copies of K in $M \sharp M$ are distinct but cannot be distinguished by any Vassiliev invariant.*

Proof. We consider two copies $M_\pm$ of M and form the connected sum $N = M_+ \sharp M_-$. This manifold is again simply connected. Moreover, it allows a diffeomorphism h of period 2 that preserves orientation and exchanges M_+ and M_-. By hypothesis, M contains a non-local knot K, thus N contains two copies $K_\pm$ of K. By construction h exchanges these two knots, and Theorem 17 implies that they cannot be distinguished by any Vassiliev invariant.

The only difficulty is to show that $K_\pm$ are actually distinct. This is achieved by an isotopy version of the Alexander-Schönflies theorem, see Theorem 41 and Corollary 45 below. □

The previous theorem applies, for example, to every contractible open 3-manifold that does not embed into $\mathbb{R}^3$. Via the theorem of Bing [3] we arrive at the following conclusion:

Corollary 21. *Suppose that M is a homotopy 3-sphere that is not homeomorphic to $\mathbb{S}^3$. Then the connected sum $M \sharp M$ contains distinct knots that cannot be distinguished by any Vassiliev invariant.* □

Note that $M \sharp M$ is again a homotopy sphere. Hence, if Vassiliev invariants distinguish knots in each homotopy sphere, then the Poincaré conjecture is true. For an arbitrary 3-manifold M we conclude: if Vassiliev invariants distinguish all knots in $M \sharp M$, then M does not contain any fake 3-balls.

4. Knots in Whitehead manifolds

In order to provide some concrete examples of knots in contractible open 3-manifolds, this section is devoted to Whitehead manifolds [18, 15].

We begin with some preliminary notation. Suppose that K is a knot in the interior of a solid torus $T \cong \mathbb{S}^1 \times \mathbb{D}^2$. Following Schubert [26] we define the *wrapping number* $[K:T]$ to be the minimal number of points in $K \cap D$, where D varies over all meridional disks of T. Obviously, the wrapping number is invariant under isotopies of K in T. Moreover, $[K:T] = 0$ if and only if the knot K is local in T, cf. [26, §9, Satz 1]. Analogously, if T_1 contains a solid torus T_0 with central axis K_0, then we define its wrapping number to be $[T_0:T_1] := [K_0:T_1]$.

Example 22. In the example of Figure 1 we have $[T_0:T_1] = 2$. To see this, first notice that the wrapping number is at most 2. Moreover it must be even, because the homology class of K_0 in T_1 vanishes. Finally it cannot be zero, because K_0 is not local in T_1: its pre-images are linked in the universal cover $\mathbb{R} \times \mathbb{D}^2$ of T_1.

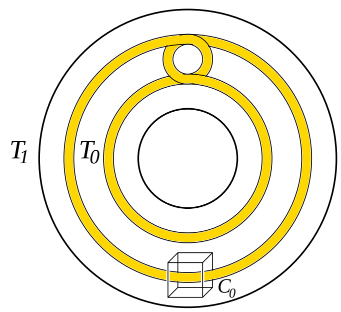


Figure 1. Construction of a Whitehead manifold

The crucial point is that the wrapping number is multiplicative, as proved by Schubert [26, §9, Satz 3]:

Theorem 23 ([26])**.** *Given three solid tori $T_0 \subset T_1 \subset T_2$, the wrapping number satisfies $[T_0:T_2] = [T_0:T_1] \cdot [T_1:T_2]$.* □

4.1. **Whitehead manifolds.** A *Whitehead sequence* is a family $\{T_n \mid n \in \mathbb{N}\}$ of solid tori $T_n \subset \operatorname{int} T_{n+1}$ such that each inclusion $T_n \hookrightarrow T_{n+1}$ is null-homotopic and has wrapping number $[T_n:T_{n+1}] \geq 2$. The central axis of the torus T_n is denoted by K_n.

Theorem 24. *Given a Whitehead sequence $T_0 \subset T_1 \subset T_2 \subset \dots$ the union $W = \bigcup_n T_n$ is a contractible open 3-manifold. The knots $K_0, K_1, K_2, \dots$ are all distinct and non-local in W. In particular W is not homeomorphic to $\mathbb{R}^3$.*

Proof. Obviously $W = \bigcup_n \operatorname{int} T_n$ is an open 3-manifold. Every compact subset in W lies in some T_n and is thus contractible in T_{n+1}. This implies $\pi_i W = 0$ for all $i \geq 0$. Since W is a CW-complex, it follows that W is contractible.

By hypothesis each wrapping number $[T_n:T_{n+1}]$ is at least 2, which implies $[T_i:T_n] \geq 2^{n-i}$ for all $n > i$. If K_i were local in W, then it would be local in some T_n with $n > i$, contradicting $[K_i:T_n] \neq 0$. Similarly, if K_i and K_j with $i < j$ were isotopic in W, then they would be isotopic in some T_n with $n > j$. We have $[T_i:T_n] = [T_i:T_j] \cdot [T_j:T_n]$ with $[T_i:T_j] \geq 2$, which shows $[K_i:T_n] \neq [K_j:T_n]$. We conclude that K_i and K_j cannot be isotopic in W. □

Example 25. Let T_0 be embedded in $T_1 \subset \mathbb{R}^3$ as shown in Figure 1. As we have seen above, $[T_0:T_1] = 2$. Although T_0 is knotted in T_1, it can be untwisted in $\mathbb{R}^3$:

there is a diffeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ that maps T_0 to T_1. We define $T_n = h^n(T_0)$ to obtain a Whitehead sequence $T_0 \subset T_1 \subset T_2 \subset \cdots \subset \mathbb{R}^3$. The union $W = \bigcup_n T_n$ is a contractible open 3-manifold embedded in $\mathbb{R}^3$ but not homeomorphic to it.

Remark 26. By varying Whitehead's construction, McMillan [18] obtained an uncountable family of contractible open 3-manifolds in $\mathbb{R}^3$, no two of which are homeomorphic. The application of Schubert's results on wrapping numbers to the analysis of Whitehead manifolds goes back to McMillan's article.

Corollary 27. *Let W be the Whitehead manifold of the preceding example. Then there exists an infinite family $\{K_n \mid n \in \mathbb{N}\}$ in W of distinct non-local knots, none of which can be distinguished from the trivial knot by Vassiliev invariants.*

Proof. Let $T_0 \subset T_1 \subset T_2 \subset \cdots \subset \mathbb{R}^3$ be a nested family of tori as in Example 25 and let K_n be the axis of the torus T_n. The inclusion $g\colon W \hookrightarrow \mathbb{R}^3$ maps K_n to the trivial knot in $\mathbb{R}^3$. The embedding $h\colon W \hookrightarrow \mathbb{R}^3 \hookrightarrow W$ thus maps K_n to the trivial knot in W. By Theorem 17, Vassiliev invariants cannot distinguish the knots $K_0, K_1, K_2, \dots$ from the trivial knot in W. □

Remark 28. The corollary can also be derived from Gusarov's n-equivalence of knots [10]. In our example, the knot K_m is n-trivial in T_{m+n+1}, hence Vassiliev invariants cannot distinguish K_m from the trivial knot in W.

4.2. **Generalized Whitehead manifolds.** To complete the picture, we will also consider contractible open 3-manifolds that cannot be embedded in $\mathbb{R}^3$. The interest in this question stems from the following observation: If M is a homotopy sphere not homeomorphic to $\mathbb{S}^3$, then $W = M \smallsetminus \{\text{point}\}$ is a contractible open 3-manifold that cannot be embedded into $\mathbb{R}^3$. While the existence of such a fake 3-sphere M remains unsolved, the following example constructs W not embeddable in $\mathbb{R}^3$.

Following Milnor [19] and Schubert [27], we will make use of the bridge number, defined as follows: given a knot $K \subset \mathbb{R}^3$, we define its bridge number $|K|$ to be the minimal number of local maxima with respect to a fixed linear projection $p\colon \mathbb{R}^3 \to \mathbb{R}$, where the minimum is taken over all knots isotopic to K.

Suppose that K_0 is a knot contained in a solid torus $T_1 \subset \mathbb{R}^3$ with axis K_1. If $[K_0 : T_1] \geq 1$ and K_1 is knotted in $\mathbb{R}^3$, then T_1 is called a *companion torus* and its axis K_1 is called a *companion knot* of K_0. Conversely, K_0 is called a *satellite* of K_1. According to Schubert [27, Satz 3], reproved in [28], we have:

Theorem 29 ([27])**.** *The bridge number of a satellite knot $K_0 \subset T_1 \subset \mathbb{R}^3$ with companion K_1 satisfies the inequality $|K_0| \geq [K_0 : T_1] \cdot |K_1|$.* □

As a consequence we obtain a geometric obstruction for embeddings in $\mathbb{R}^3$:

Corollary 30. *Suppose that each torus T_n is embedded in T_{n+1} as in Figure 1 except that a non-trivial knot has been tied into T_n inside the cube C_n. Then the union $W = \bigcup_n T_n$ is a contractible open 3-manifold that cannot be smoothly embedded into $\mathbb{R}^3$.*

Proof. Theorem 24 shows that W is a contractible open 3-manifold. It remains to show that W cannot be embedded into $\mathbb{R}^3$. To arrive at a contradiction, suppose that there exists a smooth embedding $f\colon W \hookrightarrow \mathbb{R}^3$. Each knot K_n in W maps to a knot $K_n' = fK_n$ in $\mathbb{R}^3$. We can thus consider its bridge number $|K_n'|$. To begin with, we have $|K_n'| \geq 2$ for all $n \in \mathbb{N}$, because K_n' has a non-trivial summand: the knot in the cube C_n. Moreover, K_{n+1}' is a companion of K_n' with wrapping number $[T_n : T_{n+1}] \geq 2$. Theorem 29 implies that $|K_n'| \geq \left|K_{n+1}'\right| \cdot [T_n : T_{n+1}] \geq 4$ for all $n \in N$. Reiterating this argument we see that $|K_n'| \geq 2^k$ for all k, which is impossible for a smooth knot in $\mathbb{R}^3$. □

Remark 31. By varying this technique, Kister and McMillan [15] constructed an uncountable family of contractible open 3-manifolds, no two of which are isomorphic, and none of which can be embedded in $\mathbb{R}^3$. They gave essentially the above argument, but without any reference to the bridge number.

Remark 32. For simplicity we have restricted attention to Whitehead manifolds of genus 1. The approach can be generalized to construct contractible open 3-manifolds $W = \bigcup_n T_n$ from a family of handlebodies $T_0 \subset T_1 \subset T_2 \subset \dots$ where each inclusion $T_n \hookrightarrow T_{n+1}$ is null-homotopic, see McMillan [18].

4.3. **Periodic generalized Whitehead manifolds.** We will finally investigate a family of generalized Whitehead manifolds W for which we can explicitely construct indistinguishable knots in W, even if W is not embeddable into $\mathbb{R}^3$. A generalized Whitehead manifold $W = \bigcup_n T_n$ defined by a family of tori $T_0 \subset T_1 \subset T_2 \subset \dots$ is called *periodic* if there is a diffeomorphism $h\colon W \to W$ with $hT_n = T_{n+1}$ for all n.

Example 33. The Whitehead manifold of Example 25 is periodic by construction. It is not necessary to realize this construction in $\mathbb{R}^3$: let $T = \mathbb{S}^1 \times \mathbb{D}^2$ be the standard solid torus and let $\phi\colon T \hookrightarrow \operatorname{int} T$ be a self-embedding that is null-homotopic and has wrapping number ≥ 2. For example, we could realize a pair $\phi T \subset T$ as in Figure 1 with some non-trivial knot in C. By telescoping we then form the family $T_n = T \times \{n\}$ with embeddings $\phi_n\colon T_n \hookrightarrow T_{n+1}$ given by $\phi_n(x,n) = (\phi(x), n{+}1)$. By construction, the direct limit $W = \lim T_n$ is periodic: the diffeomorphism $h\colon W \xrightarrow{\sim} W$ is induced by the shift $h\colon T_n \to T_{n+1}$ with $h(x,n) = (x, n+1)$.

Corollary 34. *In every periodic generalized Whitehead manifold, the knots $K_0, K_1, K_2, \dots$ are distinct but cannot be distinguished by any Vassiliev invariant.*

Proof. According to Theorem 24, the knots $K_0, K_1, K_2, \dots$ are non-isotopic in W. By hypothesis $h\colon W \to W$ satisfies $hT_n = T_{n+1}$ and thus $hK_n = K_{n+1}$. Theorem 17 implies that these knots cannot be distinguished by Vassiliev invariants. $\square$

These results motivate the following plausible generalization:

Question 35. Suppose that $W \not\cong \mathbb{R}^3$ is a contractible open 3-manifold. Is it true that W contains distinct knots that cannot be distinguished by Vassiliev invariants?

Corollary 19 answers this question for Whitehead manifolds, using the embedding $W \hookrightarrow \mathbb{R}^3$. The preceding Corollary 34 settles the question for generalized Whitehead manifolds granting the existence of a period $W \hookrightarrow W$. In general we know that the conclusion is true at least for $W \sharp W$, cf. Theorem 20.

5. Embedded surgery

We will show in this section that a non-local knot in a 3-manifold M cannot traverse a 2-sphere, which completes our proof of Theorem 20. The main result, Theorem 41, is interesting in its own right: it establishes an isotopy version of the Alexander-Schönflies theorem. In order to state and prove the theorem, we first recall some standard cut-and-paste techniques.

5.1. **Decomposition along a sphere system.** Let N be an oriented 3-manifold whose boundary is a collection of 2-spheres. If N is connected, we define its *closure* $\langle N \rangle$ to be the manifold obtained from glueing a 3-ball to each 2-sphere in ∂N. If N has several connected components $N_1, \dots, N_k$, we define its *connected closure* by $\langle N \rangle := \langle N_1 \rangle \sharp \dots \sharp \langle N_k \rangle$. It will be convenient to include the exceptional case of the empty manifold, in which case we set $\langle \emptyset \rangle := \mathbb{S}^3$.

Conversely, let M be an oriented connected 3-manifold without boundary. A *sphere system* $S \subset M$ is a non-empty collection of disjoint 2-spheres in M, each separating M into two connected components. The *decomposition* of M along S is the non-connected 3-manifold $M|S := M \smallsetminus \operatorname{int} T$, where T is a tubular neighbourhood of S. The boundary of $M|S$ thus consists of two parallel copies of S.

Remark 36. For every sphere system $S \subset M$ we have $M \cong \langle M|S \rangle$.

A coorientation of S induces a coorientation of the boundary of $M|S$. A connected component of $M|S$ is called *positive* or *negative* if the coorientation of its boundary points to the interior or the exterior, respectively. We will assume that S is *coherently cooriented* in the sense that each component of $M|S$ is either positive or negative. Since M is connected and every sphere in S separates M, there are exactly two coherent coorientations of S.

Definition 37. Let $S \subset M$ be a coherently cooriented sphere system. We let $M|S^+$ and $M|S^-$ denote the union of the positive and negative components, respectively, of the decomposition $M|S$. We thus obtain a presentation of M as the connected sum of two manifolds $M \cong \langle M|S^+ \rangle \sharp \langle M|S^- \rangle$.

5.2. **Embedded surgery on a surface.** As before let $S \subset M$ be a collection of 2-spheres. Given a surface F transverse to S, we want to replace F by a modified surface F_* disjoint from S. This is realized by *surgery* on F along S as follows.

The intersection $C := S \cap F$ is a finite collection of circles. Let T be a tubular neighbourhood of S parametrized by $\tau\colon [-1,+1] \times S \xrightarrow{\sim} T$ such that $\tau_0\colon S \to S$ is the identity and $T \cap F = \tau([-1,+1] \times C)$. We choose an innermost circle $C_0 \subset C$, i.e. a circle bounding a disk $D_0 \subset S$ with $\partial D_0 = D_0 \cap C = C_0$. We can then replace the cylinder $\tau([-\frac{1}{2},+\frac{1}{2}] \times C)$ by two disks $\tau(\{-\frac{1}{2},+\frac{1}{2}\} \times D)$. The result is a new surface which has one fewer intersection circle with S. Reiterating this process we finally obtain a surface F_* disjoint from S. Two steps of this iteration are sketched in Figure 2.

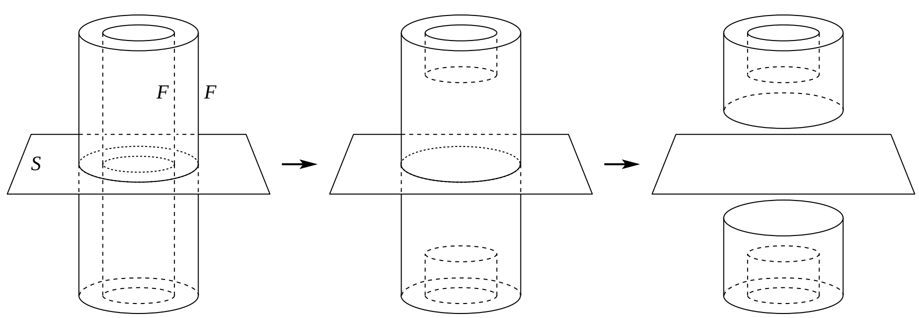


FIGURE 2. Surgery on a surface F along a sphere S

Remark 38. If F is compact or oriented or cooriented, then so is F_*. Each surgery step increases the Euler characteristic by 2, and the number of connected components by 0 or 1. In particular, if F is a collection of spheres, then so is F_* and every intermediate surface produced by surgery.

Definition 39. Suppose that S and F are transverse coherently cooriented sphere systems. We can then cut M along S or along F, and the preceding construction even allows us to combine both: surgery on F along S produces a coherently cooriented sphere system F_* disjoint from S. We can thus define the 3-manifold $M|S^+|F^+ := (M|S^+) \cap (M|F^+_*)$.

Proposition 40. *The choices involved in the construction of F_* change $M|S^+|F^+$ only in cutting out 3-balls or pasting them back to boundary 2-spheres. In particular the connected closure $\langle M|S^+|F^+ \rangle$ is well-defined. We thus obtain a presentation of M as the connected sum of four manifolds:*

$$M \cong \langle M|S^+|F^+ \rangle \sharp \langle M|S^+|F^- \rangle \sharp \langle M|S^-|F^+ \rangle \sharp \langle M|S^-|F^- \rangle.$$

5.3. **Invariance under isotopies.** The following theorem says that the decomposition of M along S and F is invariant under isotopies of S and F. This result and its proof can be seen as an isotopy version of the Alexander-Schönflies theorem, which is displayed as a special case in §5.4 below.

Theorem 41. *Let M be a connected 3-manifold and suppose that S, F_0, F_1 are sphere systems, each coherently cooriented. We further assume that F_0 and F_1 intersect S transversely, so the manifolds $\langle M|S^+|F_0^+\rangle$ and $\langle M|S^+|F_1^+\rangle$ are well-defined. If F_0 and F_1 are isotopic, then $\langle M|S^+|F_0^+\rangle \cong \langle M|S^+|F_1^+\rangle$.*

Remark 42. The preceding theorem allows us to define $\langle M|S^+|F^+\rangle$ even in the case where F is not transverse to S: by an arbitrarily small isotopy of F we can obtain a sphere system F_0 that is transverse to S. We then set $\langle M|S^+|F^+\rangle := \langle M|S^+|F_0^+\rangle$. According to the theorem, any other isotopy will lead to a diffeomorphic manifold $\langle M|S^+|F_1^+\rangle$.

Proof of Theorem 41. The proof developed in the sequel is a straightforward generalization of the classical proof of Alexander [1]. The presentation that follows has been inspired by Allen Hatcher's notes on 3-manifold topology [11].

By hypothesis there exists an isotopy $\phi\colon \mathbb{I} \times F \to M$ from $F_0 = \phi_0(F)$ to $F_1 = \phi_1(F)$. After a small perturbation of ϕ fixing ϕ_0 and ϕ_1 we can assume that every surface $F_t := \phi_t(F)$ is transverse to S, except for a finite number of critical times. Moreover, we can assume that every critical surface F_t is tangent to S in a single non-degenerate point.

For every regular parameter $t \in \mathbb{I}$, we can consider the manifold $M_t := M|S^\pm|F_t^\pm$. Clearly M_a and M_b are diffeomorphic if the interval $[a, b]$ does not contain a critical parameter. (Here we tacitly assume that the choices involved in the surgery on F_t along S are made in a uniform way.)

For every critical parameter t, we have to distinguish several cases according to the type of the tangency and the coorientations of S and F_t. We claim that only three transformations are possible, together with their inverses:

- Addition or deletion of a component diffeomorphic to a 3-ball.
- Cutting out the interior of a properly embedded 3-ball,
  or gluing a 3-ball to a boundary 2-sphere.
- Splitting one component along a properly embedded separating disk, or merging two components by gluing them together along boundary disks.

A detailed discussion of the model cases is given below. In each case we obtain that the connected closures $\langle M_{t-\varepsilon}\rangle$ and $\langle M_{t+\varepsilon}\rangle$ are diffeomorphic, which proves the theorem.

First case. We first consider the case where the critical point is a minimum or maximum. Since these play symmetric rôles, we need only consider a maximum as depicted in Figure 3.

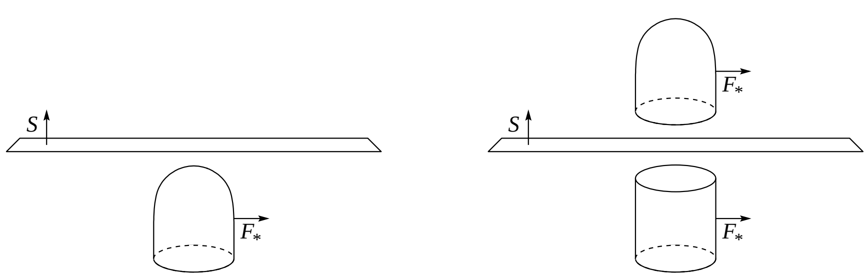


FIGURE 3. First case: a maximum (or minimum)

The critical point crosses the sphere S at time t, say. The picture on the left shows S and the surgered surface F_* at time $t - \varepsilon$, whereas the picture on the right shows S and the surgered surface F_* at time $t + \varepsilon$. Depending on the four possible coorientations, exactly one of the following transformations takes place when passing from $M_{t-\varepsilon}$ to $M_{t+\varepsilon}$:

- $M|S^+|F_*^+$: cutting out the interior of a properly embedded 3-ball.
- $M|S^+|F_*^-$: addition of a new component diffeomorphic to a 3-ball.
- $M|S^-|F_*^+$: remains unchanged.
- $M|S^-|F_*^-$: remains unchanged.

Second case. If the critical point is neither maximum nor minimum, then it is necessarily a saddle point. Let us consider the situation depicted in Figure 4, which is the standard pair of pants picture.

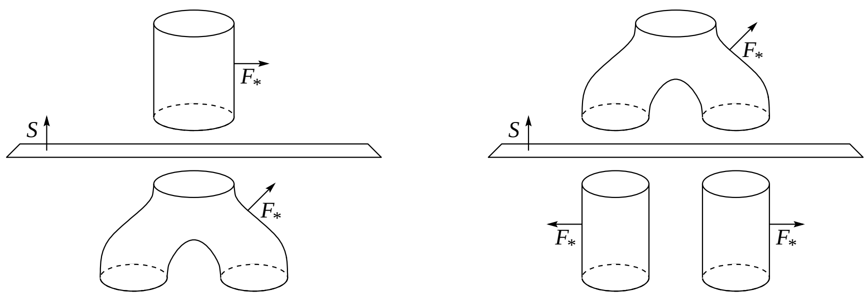


FIGURE 4. Second case: a saddle point yielding a pair of pants

Note that every component of the surface F_* is a 2-sphere, cf. Remark 38. As a consequence, only the following transformations occur from $M_{t-\varepsilon}$ to $M_{t+\varepsilon}$:

- $M|S^+|F_*^+$: remains unchanged.
- $M|S^+|F_*^-$: remains unchanged.
- $M|S^-|F_*^+$: cutting out the interior of a properly embedded 3-ball.
- $M|S^-|F_*^-$: splitting along a properly embedded separating disk

Third case. In the third and last case, the critical point is again a saddle point, but embedded surgery produces a pair of pants with one leg turned outside-in.

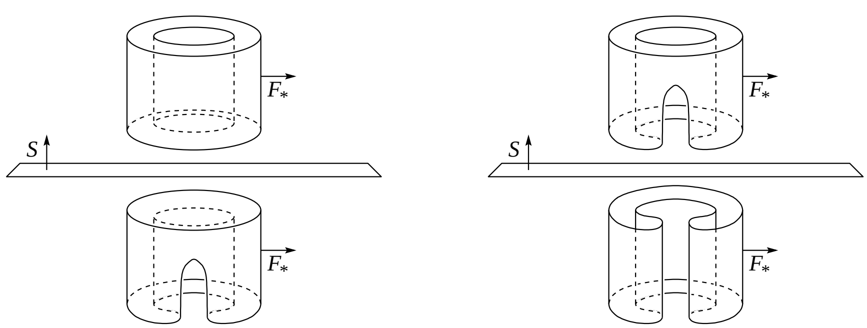


FIGURE 5. Third case: a pair of pants with one leg outside-in

In the third case only the following transformations occur from $M_{t-\varepsilon}$ to $M_{t+\varepsilon}$:

- $M|S^+|F_*^+$: gluing two components together along boundary disks.
- $M|S^+|F_*^-$: gluing a 3-ball to a boundary 2-sphere.
- $M|S^-|F_*^+$: remains unchanged.
- $M|S^-|F_*^-$: remains unchanged.

Conclusion. For every passage of a critical parameter t the transformation from $M_{t-\varepsilon}$ to $M_{t+\varepsilon}$ is given by one the the three models above, read from left to right or from right to left. In each case the connected closures $\langle M_{t-\varepsilon}\rangle$ and $\langle M_{t+\varepsilon}\rangle$ are diffeomorphic. We conclude that $\langle M_0\rangle = \langle M|S^+|F_0^+\rangle$ and $\langle M_1\rangle = \langle M|S^+|F_1^+\rangle$ are diffeomorphic, which completes the proof. $\square$

5.4. **Applications of isotopy invariance.** In order to illustrate Theorem 41, we first explain how the Alexander-Schönflies theorem appears as a special case.

Corollary 43 (Alexander-Schönflies)**.** *Let $F \subset \mathbb{S}^3$ be a smoothly embedded 2-sphere. Then $\mathbb{S}^3|F$ has two components, each of which is diffeomorphic to the 3-ball.*

Proof. As usual, one first establishes that $\mathbb{S}^3|F$ has two connected components by Poincaré-Alexander duality. In particular, F can be coherently cooriented. We will prove that $\langle \mathbb{S}^3|F^+\rangle \cong \mathbb{S}^3$, which implies that $\mathbb{S}^3|F^+$ is a 3-ball, cf. Theorem 7.

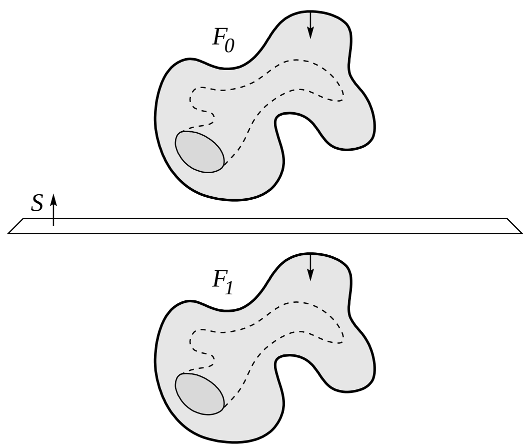


FIGURE 6. Proof of the Alexander-Schönflies theorem

Let S be the equator of $\mathbb{S}^3$. By a diffeotopy of $\mathbb{S}^3$ we can move F to F_0 in the positive hemisphere $\mathbb{S}^3|S^+$, with coorientations as shown in Figure 6. Analogously we can move F to F_1 in the negative hemisphere $\mathbb{S}^3|S^-$. We obtain $\mathbb{S}^3|S^+|F_0^+ \cong \mathbb{S}^3|F^+$ and $\mathbb{S}^3|S^+|F_1^+ = \emptyset$. By the preceding theorem we conclude that $\langle \mathbb{S}^3|F^+\rangle \cong \langle\emptyset\rangle = \mathbb{S}^3$. $\square$

Remark 44. The Alexander-Schönflies theorem implies that $M \sharp N \cong \mathbb{S}^3$ if and only if $M \cong N \cong \mathbb{S}^3$. As an immediate consequence we obtain that $\langle M\rangle \cong \mathbb{S}^3$ if and only if M is a collection of holed 3-spheres.

As promised, we finally deduce that a non-local knot cannot traverse a 2-sphere, which finishes the proof of Theorem 20.

Corollary 45. *Let $S \subset M$ be a cooriented separating 2-sphere. If a knot K_0 in $M|S^+$ is equivalent to a knot K_1 in $M|S^-$, then both knots are local.*

Proof. Let $S_0 \subset M|S^+$ be a parallel copy of S situated on the positive side and equipped with the same coorientation. This implies that $M|S^-|S_0^+ = \emptyset$ and $M|S^+|S_0^+ = M|S_0^+$ contains K_0. By hypothesis there exists a diffeotopy $\Phi\colon \mathbb{I}\times M \to M$ with $\Phi_0 = \mathrm{id}_M$ and $\Phi_1 K_0 = K_1$. We can assume that the 2-sphere $S_1 := \Phi_1 S_0$ is transverse to S. According to the preceding theorem we have $\langle M|S^-|S_1^+\rangle \cong \langle M|S^-|S_0^+\rangle \cong \mathbb{S}^3$, hence $M|S^-|S_1^+$ is a collection of holed 3-spheres. By hypothesis K_1 is contained in $M|S^-$ as well as $M|S_1^+$. This implies that K_1 is contained in $M|S^-|S_1^+$. Being contained in a holed 3-sphere, K_1 is local in $M|S^-$, hence in M. Symmetrically, K_0 is local in $M|S^+$, hence in M. $\square$

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References

1. J. W. Alexander, *On the subdivision of* 3*-space by a polyhedron*, Proc. Nat. Acad. Sci. U.S.A. **10** (1924), 6–8.
2. D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), no. 2, 423–472. MR **97d:**57004
3. R. H. Bing, *Necessary and sufficient conditions that a* 3*-manifold be* S^3, Ann. of Math. (2) **68** (1958), 17–37. MR 20 #1973
4. J. S. Birman and X.-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), no. 2, 225–270. MR **94d:**57010
5. J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961), 227–380. MR 25 #3543
6. O. L. Costich, P. H. Doyle, and D. E. Galewski, *A characterisation of punctured open* 3*-cells*, Proc. Amer. Math. Soc. **28** (1971), 295–298. MR 42 #6800
7. M. Eisermann, *Invariants de Vassiliev et conjecture de Poincaré*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 11, 1005–1010. MR **2003e:**57017
8. ———, *A surgery proof of Bing's theorem characterizing the* 3*-sphere*, J. Knot Theory Ramifications **13** (2004), no. 2, 307–309. MR **MR2048115 (2005b:**57029**)**
9. M. N. Gusarov, *A new form of the Conway-Jones polynomial of oriented links*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **193** (1991), 4–9, also in: *Topology of manifolds and varieties* (O. Viro, ed.), Adv. Soviet Math., vol. 18, Amer. Math. Soc., Providence, RI, 1994, pp. 167–172. MR **93b:**57007
10. ———, *On n-equivalence of knots and invariants of finite degree*, Topology of manifolds and varieties (O. Viro, ed.), Adv. Soviet Math., vol. 18, Amer. Math. Soc., Providence, RI, 1994, pp. 173–192. MR **96i:**57005
11. A. Hatcher, *Notes on basic 3-manifold topology*, electronic document available at `http://www.math.cornell.edu/~hatcher/`.
12. J. Hempel, 3*-Manifolds*, Ann. of Math. Studies, vol. 86, Princeton University Press, Princeton, N. J., 1976. MR 54 #3702
13. M. W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1976. MR 56 #6669
14. E. Kalfagianni, *Finite type invariants for knots in* 3*-manifolds*, Topology **37** (1998), no. 3, 673–707. MR **2000a:**57009
15. J. M. Kister and D. R. McMillan, Jr., *Locally euclidean factors of* E^4 *which cannot be imbedded in* E^3, Ann. of Math. (2) **76** (1962), 541–546. MR 26 #1868
16. J. Lieberum, *Invariants de Vassiliev pour les entrelacs dans* S^3 *et dans les variétés de dimension trois*, Prépublication de l'Institut de Recherche Mathématique Avancée, vol. 1998/30, Université Louis Pasteur, Strasbourg, 1998. MR **2000g:**57025
17. X.-S. Lin, *Finite type link invariants of* 3*-manifolds*, Topology **33** (1994), no. 1, 45–71. MR **94m:**57020
18. D. R. McMillan, Jr., *Some contractible open* 3*-manifolds*, Trans. Amer. Math. Soc. **102** (1962), 373–382. MR 25 #561
19. J. W. Milnor, *On the total curvature of knots*, Ann. of Math. (2) **52** (1950), 248–257. MR 12,273c
20. ———, *Differentiable structures*, Mimeographed notes, Princeton, 1961.
21. J. R. Munkres, *Elementary differential topology*, Ann. of Math. Studies, vol. 54, Princeton University Press, Princeton, N.J., 1963. MR 29 #623
22. R. Myers, *Open book decompositions of* 3*-manifolds*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 397–402. MR **80a:**57004
23. R. S. Palais, *Extending diffeomorphisms*, Proc. Amer. Math. Soc. **11** (1960), 274–277. MR 22 #8515
24. H. Poincaré, *Cinquième complément à l'analysis situs*, Rend. Circ. Mat. Palermo **18** (1904), 45–110, also in: *Œuvres de Henri Poincaré*, tome VI, Gauthier-Villars, Paris, 1953, pp. 435–498.
25. D. Rolfsen, *Knots and links*, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Berkeley, Calif., 1976. MR 58 #24236
26. H. Schubert, *Knoten und Vollringe*, Acta Math. **90** (1953), 131–286. MR 17,291d

27. ———, *Über eine numerische Knoteninvariante*, Math. Z. **61** (1954), 245–288. MR 17,292a
28. J. Schultens, *Additivity of bridge numbers of knots*, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 3, 539–544. MR **MR2018265 (2004i:**57010)
29. R. Thom, *La classification des immersions*, Séminaire Bourbaki, 10ème année: 1957/1958, Exposé 157.
30. V. A. Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its applications (V. I. Arnold, ed.), Adv. Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990, pp. 23–69. MR **92a:**57016
31. J. H. C. Whitehead, *A certain open manifold whose group is unity*, Quart. J. Math. **6** (1935), 268–279.

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