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HOMOLOGICAL CHARACTERIZATION OF THE UNKNOT

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Abstract. Given a knot K in the 3-sphere, let Q_K be its fundamental quandle as introduced by D. Joyce. Its first homology group is easily seen to be $H_1(Q_K) \cong \mathbb{Z}$. We prove that $H_2(Q_K) = 0$ if and only if K is trivial, and $H_2(Q_K) \cong \mathbb{Z}$ whenever K is non-trivial. An analogous result holds for links, thus characterizing trivial components.

More detailed information can be derived from the conjugation quandle: let Q_K^π be the conjugacy class of a meridian in the knot group $\pi_1(\mathbb{S}^3 \smallsetminus K)$. We show that $H_2(Q_K^\pi) \cong \mathbb{Z}^p$, where p is the number of prime summands in a connected sum decomposition of K.

Introduction and statement of results

The fundamental group of a knot. For a knot K in the 3-sphere $\mathbb{S}^3$ let $\pi_K := \pi_1(\mathbb{S}^3 \smallsetminus K)$ be the fundamental group of the knot complement. All higher homotopy groups vanish [27], which means that $\mathbb{S}^3 \smallsetminus K$ is an Eilenberg-MacLane space. By Poincaré duality, its integral homology is given by $H_0 \cong H_1 \cong \mathbb{Z}$ and $H_n = 0$ for all $n \geq 2$. This means that among these classical invariants of algebraic topology, only the group π_K contains information about the knot K.

The knot group is indeed a very strong invariant: it classifies unoriented prime knots [15, 33]. To capture the complete information, one can consider a meridian-longitude pair $m_K, l_K \in \pi_K$ (see §1). It follows from the work of F. Waldhausen [31] that the group system (π_K, m_K, l_K) classifies knots.

The fundamental quandle of a knot. A quandle, as introduced by D. Joyce [19], is a set Q with a binary operation whose axioms model conjugation in a group, or equivalently, the Reidemeister moves of knot diagrams. Quandles have been intensively studied by different authors and under various names: as "distributive groupoids" by S.V. Matveev [22], as "crossed G-sets" by P.J. Freyd and D.N. Yetter [14], as "crystals" by L.H. Kauffman [20], and — slightly generalized — as "automorphic sets" by E. Brieskorn [1], and as "racks" by R. Fenn and C. Rourke [12]. We review the relevant definitions in §2.

The Wirtinger presentation of the knot group π_K involves only conjugation and thus may be re-interpreted as defining a quandle. The quandle Q_K so presented is called the fundamental quandle of the knot K (see §2). Using Waldhausen's results, Joyce [19] showed that the knot quandle is a classifying invariant: if Q_K and $Q_{K'}$ are isomorphic, then the knots K and K' are equivalent up to inversion.

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Homological characterization of the unknot. Like the knot group, the knot quandle is in general very difficult to analyze. It is therefore natural to ask: how can we extract partial information?

R. Fenn, C. Rourke, and B. Sanderson [13] have developed a homology theory for racks, which has been adapted to quandles by J.S. Carter et al. [5, 6, 7]. The relevant definitions are recalled in §5.1 and §6.1 below. Our main theorem determines the second homology group of the knot quandle Q_K with integer coefficients:

Theorem 1. *Let K be a knot and let Q_K be its fundamental quandle. If K is trivial, then $H_2(Q_K) = 0$. If K is non-trivial, however, then $H_2(Q_K) \cong \mathbb{Z}$.*

Unlike the knot group π_K, the knot quandle Q_K thus has interesting homology. Indeed, $H_2(Q_K)$ seems ideally suited to tackle the *unknotting problem*: given a knot, how can we decide whether or not it is trivial? We will discuss algorithmic questions in Section 10 at the end of this paper.

Remark 2. It is immediate from the definitions that $H_1(Q_K) = H^1(Q_K) = \mathbb{Z}$. By the Universal Coefficient Theorem we have $H_2(Q_K, \Lambda) = H_2(Q_K) \otimes \Lambda$ and $H^2(Q_K, \Lambda) = \operatorname{Hom}(H_2(Q_K), \Lambda)$ for every abelian group Λ. Hence the preceding theorem completely determines the second (co)homology groups of knot quandles.

Central extensions. In order to prove Theorem 1, it will be useful to compare each closed knot K to the corresponding long knot L (see §1.2). We define the fundamental quandle Q_L just as we defined Q_K, with the sole exception that the first and the last arc of L are not identified. They correspond to distinguished elements q_L and q_L^* in Q_L, and the quandle Q_K is obtained from Q_L by adjoining the additional relation $q_L = q_L^*$. For the knot groups π_L and π_K this relation is redundant, but for quandles the situation differs remarkably:

Theorem 3 (§4.2)**.** *Let L be a long knot and let K be the corresponding closed knot. The natural projection $Q_L \to Q_K$ is the universal covering of Q_K. If K is the trivial knot, then both Q_L and Q_K are trivial quandles. If K is non-trivial, however, then $Q_L \to Q_K$ is a central extension with covering group $\Lambda = \langle l_K \rangle \cong \mathbb{Z}$.*

This result is the key to Theorem 1. The notions of quandle covering and central extension are introduced and discussed in §3. In order to translate central extensions to cohomology classes, we establish the following classification theorem:

Theorem 4 (§5.2)**.** *Suppose that Q is a quandle and Λ is an abelian group. Let $\mathscr{E}(Q, \Lambda)$ be the set of equivalence classes of central extensions of Q by Λ. Then there is a natural bijection $\mathscr{E}(Q, \Lambda) \cong H^2(Q, \Lambda)$.*

This is a direct analogue of a classical result in group cohomology: central extensions of a group G with kernel Λ are classified by cohomology classes in $H^2(G, \Lambda)$.

Liftings and obstructions. The following unique lifting property will serve to show that $H_2(Q_L)$ vanishes and, in a second step, to calculate $H_2(Q_K)$.

Lemma 5 (§3.2)**.** *Suppose that L is a long knot and $f : Q_L, q_L \to Q, q$ is a quandle morphism. If $p : \tilde{Q}, \tilde{q} \to Q, q$ is a covering, then there exists a unique quandle morphism $\tilde{f} : Q_L, q_L \to \tilde{Q}, \tilde{q}$ with $f = p\tilde{f}$.*

For closed knots, we deduce the following lifting criterion:

Lemma 6 (§6.3)**.** *Every closed knot K can be equipped with an orientation class $[K] \in H_2(Q_K)$. Suppose that $f : Q_K, q_K \to Q, q$ is a quandle morphism and $p : \tilde{Q}, \tilde{q} \to Q, q$ is a central extension with associated cohomology class $[\lambda] \in H^2(Q, \Lambda)$. Then there exists a lifting $\tilde{f} : Q_K, q_K \to \tilde{Q}, \tilde{q}$ if and only if $\langle [\lambda] \mid f \mid [K] \rangle$ vanishes.*

To explain the notation, we remark that every quandle morphism $f : Q_K \to Q$ induces a map on homology, $f_* : H_*(Q_K) \to H_*(Q)$, and a map on cohomology, $f^* : H^*(Q, \Lambda) \to H^*(Q_K, \Lambda)$. The evaluation

$$\langle [\lambda] \mid f \mid [K] \rangle = \langle [\lambda] \mid f_*[K] \rangle = \langle f^*[\lambda] \mid [K] \rangle$$

is thus an element in the coefficient group Λ.

State-sum invariants. The orientation class has been used implicitly by Carter et al. [5] to define a state-sum invariant of knots. Their definition can now be reformulated as follows: for every finite quandle Q and every cocycle $\lambda \in Z^2(Q, \Lambda)$, the associated state-sum invariant is given by

$$S_Q^\lambda(K) = \sum_f \langle [\lambda] \mid f \mid [K] \rangle,$$

where f ranges over all quandle morphisms $f : Q_K \to Q$. Here Λ is written multiplicatively, so that the above sum is an element of the group ring $\mathbb{Z}\Lambda$. (As the referee pointed out, this interpretation of S_Q^λ has independently been developed by J.S. Carter, S. Kamada, and M. Saito in [8].)

Classifying oriented knots. As mentioned above, the knot quandle Q_K characterizes the knot K only up to inversion, that is, simultaneously changing the orientations of K and $\mathbb{S}^3$. The orientation class $[K]$, as its name suggests, removes the remaining ambiguity:

Theorem 7 (§6.4)**.** *Each oriented knot K is characterized by the pair $(Q_K, [K])$.*

More explicitly, two oriented knots K and K' are isotopic if and only if there exists a quandle isomorphism $\phi : Q_K \to Q_{K'}$ with $\phi_*[K] = [K']$.

Characteristic classes. As we have seen above, every knot K comes equipped with two characteristic classes: the central extension $Q_L \to Q_K$ defines a cohomology class $[L] \in H^2(Q_K)$, and dually the orientation of K defines $[K] \in H_2(Q_K)$. We can now state the following more detailed version of Theorem 1:

Theorem 8 (§7)**.** *If K is a non-trivial knot, then $H_2(Q_K) \cong \mathbb{Z}$, and the orientation of K defines a canonical generator $[K] \in H_2(Q_K)$. Dually, we have $H^2(Q_K) \cong \mathbb{Z}$, and the central extension $Q_L \to Q_K$ defines a canonical generator $[L] \in H^2(Q_K)$ whose evaluation yields $\langle [L] \mid [K] \rangle = 1$.*

This result answers a fundamental question about quandle homology, raised by J.S. Carter, S. Kamada, and M. Saito in [7], Question 7.3: the orientation class $[K]$ vanishes if and only if the knot K is trivial.

The conjugation quandle. By construction, the fundamental quandle Q_K allows a natural representation $Q_K \to \pi_K$ on the knot group π_K. Its image Q_K^π is the conjugacy class of the meridian m_K and is called the conjugation quandle of K. It is easy to see that $H_1(Q_K^\pi) \cong H^1(Q_K^\pi) \cong \mathbb{Z}$. The rank of the second homology group, however, depends on the number of prime summands:

Theorem 9 (§8)**.** *If K is the connected sum of prime knots $K^1, \dots, K^p$, then $H_2(Q^\pi_K) \cong H^2(Q^\pi_K) \cong \mathbb{Z}^p$. Moreover, the orientation classes $[K^1], \dots, [K^p]$ map to a basis of $H_2(Q^\pi_K)$, and their dual classes $[L^1], \dots, [L^p]$ map to a basis of $H^2(Q^\pi_K)$.*

In particular, the preceding theorem characterizes prime knots: a knot K is trivial if and only if $H_2(Q^\pi_K) = 0$, and it is prime if and only if $H_2(Q^\pi_K) \cong \mathbb{Z}$.

Generalization to links. Let $K \subset \mathbb{S}^3$ be a link with components $K^1, \dots, K^n$. In this case we find $H_1(Q_K) \cong H^1(Q_K) \cong \mathbb{Z}^n$. As before we can define characteristic classes $[K^1], \dots, [K^n] \in H_2(Q_K)$, one for each component of K, and dually $[L^1], \dots, [L^n] \in H^2(Q_K)$. A component K^i of K is called *trivial* if there exists an embedded disk $D \subset \mathbb{S}^3$ with $K^i = K \cap D = \partial D$.

Theorem 10 (§9)**.** *Let K be a link with non-trivial components $K^1, \dots, K^m$ and trivial components $K^{m+1}, \dots, K^n$. Then the second homology group $H_2(Q_K)$ is freely generated by $[K^1], \dots, [K^m]$, and the classes $[K^{m+1}], \dots, [K^n]$ vanish. Dually, the second cohomology group $H^2(Q_K)$ is freely generated by $[L^1], \dots, [L^m]$, and the classes $[L^{m+1}], \dots, [L^n]$ vanish. For all $i, j \in \{1, \dots, m\}$, evaluation yields $\big\langle [L^i] \bigm| [K^j] \big\rangle = \delta_{ij}$.*

In particular, the theorem characterizes trivial components: given a link K, the component K^i is trivial if and only if its orientation class $[K^i] \in H_2(Q_K)$ vanishes.

How this paper is organized. The paper roughly follows the outline given in this introduction. In order to make the presentation as self-contained as possible, Section 1 recalls some facts about knot groups, while Section 2 collects the basic definitions concerning knot quandles.

Section 3 introduces the notions of quandle covering and quandle extension. Section 4 shows that Q_L is the universal central extension of Q_K and determines its structure in terms of the group system (π_K, m_K, l_K). Central extensions are translated into quandle cohomology in Section 5. The dual notion of quandle homology allows to define the orientation class of a knot, as explained in Section 6.

These tools are applied in Section 7 to determine the second (co)homology group of knot quandles, thus proving our main result. The arguments are extended to conjugation quandles in Section 8. The generalization to link quandles is sketched in Section 9. We conclude this article with some remarks on algorithms and decidability questions in Section 10.

1. Knot groups

This first section recalls some facts about the knot group system and its Wirtinger presentation. It serves primarily to fix our notation.

1.1. **Peripheral system.** Let $\mathbb{D}^2$ be the closed unit disk in the complex plane, its boundary $\partial\mathbb{D}^2 = \mathbb{S}^1$ being the unit circle. A *knot* is a smooth embedding $k \colon \mathbb{S}^1 \hookrightarrow \mathbb{S}^3$, considered up to isotopy. This is the same as considering the oriented image $K = k(\mathbb{S}^1)$ in $\mathbb{S}^3$, again up to isotopy. A *standard framing* of K is an embedding $f \colon \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{S}^3$ such that:

- The central axis $f|_{\mathbb{S}^1 \times 0}$ parametrizes the oriented knot K.
- The meridian curve $f|_{1 \times \mathbb{S}^1}$ has linking number $+1$ with K.
- The longitude curve $f|_{\mathbb{S}^1 \times 1}$ has linking number 0 with K.

Every knot K has a standard framing, and any two standard framings of K are isotopic. As basepoint of the space $\mathbb{S}^3 \smallsetminus K$ we choose $p = f(1,1)$. In the fundamental group $\pi_K := \pi_1(\mathbb{S}^3 \smallsetminus K, p)$, the homotopy class $m_K = [f|_{1\times\mathbb{S}^1}]$ is called the *meridian* of the knot K, and the homotopy class $l_K = [f|_{\mathbb{S}^1\times 1}]$ is called the *longitude* of K.

Up to isomorphism, the triple (π_K, m_K, l_K) is an invariant of the knot K: if K and K' are isotopic, then there is an isomorphism $\phi : \pi_K \to \pi_{K'}$ with $\phi(m_K) = m_{K'}$ and $\phi(l_K) = l_{K'}$. Remarkably, the converse also holds:

Theorem 11 ([31])**.** *Two knots K and K' are isotopic if and only if there is an isomorphism $\phi : \pi_K \to \pi_{K'}$ with $\phi(m_K) = m_{K'}$ and $\phi(l_K) = l_{K'}$.* □

This is a special case of Waldhausen's theorem on sufficiently large 3-manifolds. See [31], Corollary 6.5, as well as [3], §3C, for its application to knots.

A knot K is called *trivial* if there exists an embedded disk $D \subset \mathbb{S}^3$ with $\partial D = K$. Up to isotopy, there is exactly one trivial knot. Dehn's lemma [27] reformulates the geometric condition in terms of the fundamental group:

Theorem 12 ([27])**.** *K is trivial if and only if its longitude $l_K \in \pi_K$ vanishes.* □

1.2. **Long knots versus closed knots.** Besides closed knots $k : \mathbb{S}^1 \hookrightarrow \mathbb{S}^3$ it will be useful to consider long knots $\ell : \mathbb{R} \hookrightarrow \mathbb{R}^3$, i.e. smooth embeddings such that $\ell(t) = (t, 0, 0)$ for all parameters t outside some compact interval. We regard long knots only up to isotopy with compact support. This is the same as considering the image $L = \ell(\mathbb{R})$ in $\mathbb{R}^3$ up to isotopy with compact support.

The closure of a long knot is a closed knot, defined in the obvious way. The closure map is well-defined on isotopy classes and establishes a bijection between long knots and closed knots. Conversely, the passage from a closed knot $K \subset \mathbb{S}^3$ to a long knot $L \subset \mathbb{R}^3$ is essentially the choice of a basepoint $P \in K$, from which we obtain a diffeomorphism $(\mathbb{S}^3 \smallsetminus \{P\}, K \smallsetminus \{P\}) \cong (\mathbb{R}^3, L)$. In particular, we have a homeomorphism of the knot complements $\mathbb{S}^3 \smallsetminus K \cong \mathbb{R}^3 \smallsetminus L$, and the knot groups π_K and π_L are isomorphic.

As far as the knot group is concerned, there is thus no difference between a closed knot K and its corresponding long knot L, and we can freely choose the point of view that is most convenient. One nice feature about long knots, for example, is that the group π_L has a canonical meridian-longitude pair as shown in Figure 1.

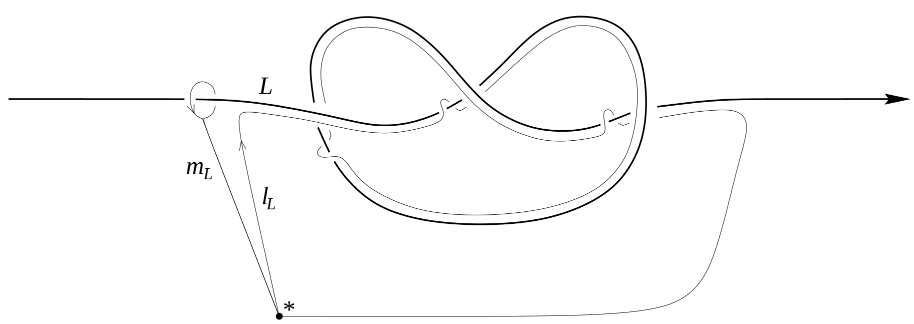

FIGURE 1. Meridian and longitude of a long knot

1.3. **Wirtinger presentation.** The following well-known method produces a presentation of the knot group by generators and relations, called the *Wirtinger presentation*. Given a long knot L, we represent it by a long diagram as in Figure 1. Travelling along the knot from $-\infty$ on the left to $+\infty$ on the right, we number the arcs consecutively from 0 to n, where n is the number of crossings. At the end of arc number $i-1$, we undercross arc number $\kappa(i)$ and continue on arc number i. Let $\varepsilon(i) = \pm 1$ be the sign of this crossing, as depicted in Figure 2. The maps $\kappa : \{1, \dots, n\} \to \{0, \dots, n\}$ and $\varepsilon : \{1, \dots, n\} \to \{\pm 1\}$ are the *Wirtinger code* of the diagram. From this we derive the following presentation:

Theorem 13. *Suppose that L is represented as a long knot diagram with Wirtinger code (κ, ε) as above. Then the knot group allows the presentation*

$$\pi_L \cong \langle x_0, x_1, \dots, x_n \,|\, r_1, \dots, r_n \rangle \quad \textit{with relations} \quad r_i \;:\; x_i = x_{\kappa i}^{-\varepsilon i}\, x_{i-1}\, x_{\kappa i}^{\varepsilon i}.$$

Moreover, as peripheral system we can choose the meridian $m_L = x_0$ and the longitude $l_L = \prod_{i=1}^{i=n} x_{i-1}^{-\varepsilon i}\, x_{\kappa i}^{\varepsilon i}$. □

Since the Wirtinger presentation will be used throughout this article, it seems worthwhile to make the isomorphism explicit: Suppose that the knot L lies in the diagram plane and coincides with the diagram D except for the crossings, where the undercrossing strand traces a small half-circle below the plane. As basepoint of the complement $\mathbb{R}^3 \smallsetminus L$ we choose some point p above the diagram plane. Let γ_i be the loop that starts at p, runs to arc number i in a straight line, encircles it once in a right-handed loop (in order to achieve linking number $+1$), and returns to p in a straight line. We define $\phi : \langle x_0, x_1, \dots, x_n \,|\, r_1, \dots, r_n \rangle \to \pi_L$ by $\phi(x_i) = \gamma_i$. It is easy to see that the relations $r_1, \dots, r_n$ are satisfied, so ϕ is a well-defined homomorphism of groups. The theorem of Seifert and van Kampen shows that ϕ is indeed an isomorphism. (For details see for example Crowell-Fox [10], §VI.3, or Burde-Zieschang [3], §3B.) It now follows from our definitions that (m_L, l_L) is a meridian-longitude pair of the knot L.

Remark 14. The Wirtinger presentation works just as well for a closed knot diagram. Since arcs 0 and n are then identified, this amounts to adding the relation $x_0 = x_n$ to the above presentation. The group is, of course, the same.

1.4. **Group homology.** In order to illustrate our calculation of quandle homology, we emphasize that the Wirtinger presentation allows to determine the low-dimensional homology groups of π_L by elementary methods: To start with, it is easy to read off the abelianization, which yields $H_1(\pi_L) \cong \mathbb{Z}$. Moreover, π_L is given by $n+1$ generators and n relations, which implies $H_2(\pi_L) = 0$ (see for example Brown [2], §II.5, Exercise 5).

These observations illustrate the main theme of this article. For knot quandles we will see $H_1(Q_L) \cong H_1(Q_K) \cong \mathbb{Z}$ and $H_2(Q_L) = 0$. So far this is completely analogous to the homology of knot groups. Our proof that $H_2(Q_K) \cong \mathbb{Z}$, however, is based on the fact that passing from a long knot quandle Q_L to a closed knot quandle Q_K adds a non-trivial relation (provided that the knot K is non-trivial).

2. Knot quandles

This section collects the basic definitions concerning quandles and in particular fundamental quandles of knots.

2.1. **Knot colourings.** The Wirtinger presentation allows to interpret knot group homomorphisms $\pi_K \to G$ as colourings of knot diagrams. More precisely: let D be a long diagram, its arcs being numbered by $0, \ldots, n$. A G-*colouring* of D is a map $f : \{0, \ldots, n\} \to G$ such that at each coloured crossing as in Figure 2 the colours a and c are conjugated via $a^b = c$. Such a colouring is denoted by $f : D \to G$.

FIGURE 2. Wirtinger rules for colouring a knot diagram

Remark 15. The Wirtinger theorem says that the knot group π_K is the universal colouring group for K in the following sense: each diagram D representing K comes with a canonical colouring $D \to \pi_K$, and every group colouring $D \to G$ factors as a composition of $D \to \pi_K$ and a unique group homomorphism $\pi_K \to G$. Thus colourings $D \to G$ are in bijection with group homomorphisms $\pi_K \to G$.

2.2. **Quandles and automorphic sets.** The Wirtinger presentation of π_K involves only conjugation but not the group multiplication itself. The underlying algebraic structure can be described as follows:

Definition 16. A *quandle* is a set Q with two binary operations $*, \overline{*} : Q \times Q \to Q$ satisfying the following axioms for all $a, b, c \in Q$:

(Q1) $a * a = a$ (idempotency)
(Q2) $(a * b) \overline{*} b = a = (a \overline{*} b) * b$ (right invertibility)
(Q3) $(a * b) * c = (a * c) * (b * c)$ (self-distributivity)

The name "quandle" was introduced by D. Joyce [19]. The same notion was studied by S.V. Matveev [22] under the name "distributive groupoid", and by P.J. Freyd and D.N. Yetter [14] under the name "crossed G-set".

Definition 17. A *homomorphism* of quandles is a map $\phi : Q \to Q'$ that satisfies $\phi(a * b) = \phi(a) * \phi(b)$, and hence $\phi(a \overline{*} b) = \phi(a) \overline{*} \phi(b)$, for all $a, b \in Q$. The automorphism group $\mathrm{Aut}(Q)$ consists of all bijective homomorphisms $\phi : Q \to Q$. We adopt the convention that automorphisms of Q act on the right, written a^ϕ, which means that their composition $\phi\psi$ is defined by $a^{(\phi\psi)} = (a^\phi)^\psi$ for all $a \in Q$.

Axioms (Q2) and (Q3) are equivalent to saying that for every $a \in Q$ the right translation $\varrho_a : x \mapsto x * a$ is an automorphism of Q. Such structures were studied by E. Brieskorn [1] under the name "automorphic sets" and by C. Rourke and R. Fenn [12] under the name "rack".

Definition 18. The group $\mathrm{Inn}(Q)$ of *inner automorphisms* is the subgroup of $\mathrm{Aut}(Q)$ generated by all right translations ϱ_a with $a \in Q$. The quandle Q is called *connected* if the action of $\mathrm{Inn}(Q)$ on Q is transitive.

In view of the map $\varrho : Q \to \mathrm{Inn}(Q)$, we also write a^b for the operation $a * b$ in a quandle. Conversely, it will sometimes be convenient to write $a * b$ for the conjugation $b^{-1}ab$ in a group. In neither case will there be any danger of confusion.

Definition 19. A *representation* of a quandle Q on a group G is a map $\phi : Q \to G$ such that $\phi(a * b) = \phi(a) * \phi(b)$ for all $a, b \in Q$. We call $\varrho : Q \to \mathrm{Inn}(Q)$ the *natural* representation of Q. An *augmentation* consists of a representation $\phi : Q \to G$ together with a group homomorphism $\alpha : G \to \mathrm{Inn}(Q)$ such that $\alpha\phi = \varrho$.

In general we will simplify matters by assuming that G is generated by the image $\phi(Q)$. In this case the action of G on Q is uniquely determined by the representation ϕ, and we simply say that $\phi : Q \to G$ is an augmentation. For example, every quandle Q comes equipped with the natural augmentation $\varrho : Q \to \mathrm{Inn}(Q)$.

2.3. **Fundamental quandles.** As before, let D be a long knot diagram, its arcs being numbered by $0, \dots, n$. A *Q-colouring* is a map $f : \{0, \dots, n\} \to Q$ such that at each crossing as in Figure 2 the three colours a, b, c satisfy the relation $a * b = c$. Such a colouring is denoted by $f : D \to Q$. The quandle axioms ensure that each Reidemeister move $D \leftrightharpoons D'$ induces a bijection between the Q-colourings of D and the Q-colourings of D', see Joyce [19], §15.

The Wirtinger presentation says that $D \to \pi_L$ is universal for group colourings. The analogue for the category of quandles can be defined as follows:

Definition 20. Given a diagram D representing the long knot L, let Q_L be the quandle generated by $q_0, \dots, q_n$ subject to the Wirtinger relations: $q_i = q_{i-1} * q_{\kappa i}$ for each positive crossing and $q_i = q_{i-1} \mathbin{\overline{*}} q_{\kappa i}$ for each negative crossing, respectively. We call Q_L the *knot quandle* or *fundamental quandle* of L.

The quandle axioms guarantee that Q_L (up to isomorphism) is indeed an invariant of the knot L. For details of this construction, and for an alternative topological definition, we refer to the article by D. Joyce [19], §14–15.

Remark 21. The generators $q_0, \dots, q_n$ of the knot quandle Q_L are connected via their mutual action, hence Q_L is connected. Moreover, Q_L has two special elements, q_L and q_L^*, corresponding to the first and the last arc respectively.

Remark 22. The preceding definition serves equally well to define the fundamental quandle Q_K of a closed knot K. The only difference is the additional relation $q_L = q_L^*$, because the first and the last arc are now identified.

Remark 23. Let K be a (long or closed) knot. By definition, Q_K is the universal colouring quandle for K: each diagram D representing K comes with a canonical colouring $D \to Q_K$, and every quandle colouring $D \to Q$ factors as a composition of $D \to Q_K$ and a unique quandle homomorphism $Q_K \to Q$. Thus colourings $D \to Q$ are in bijection with quandle homomorphisms $Q_K \to Q$.

Remark 24. Let K be a (long or closed) knot. The universal property of Q_K induces a canonical representation $Q_K \to \pi_K$. The universal property of π_K, in turn, induces a canonical group homomorphism $\pi_K \to \mathrm{Inn}(Q_K)$. Both fit together to form an augmentation $Q_K \to \pi_K$.

3. Quandle extensions

This section introduces the notions of quandle covering and quandle extension. Given a long knot L with closure K, Theorem 30 shows that Q_L is the universal covering quandle of Q_K.

3.1. **Meridians and partial longitudes.** We begin by explaining how quandle colourings can be used to encode longitudinal information. The Wirtinger presentation produces at each crossing a *local* relation between three meridians. The longitude, however, is *global* in the sense that it involves a certain product over all meridians. We can decompose this product into a sequence of local calculations as follows: Consider a long knot diagram D with Wirtinger code (κ, ε). We colour each arc not only with its meridian x_i but also with its partial longitude $l_i := \prod_{j=1}^{j=i} x_{j-1}^{-\varepsilon j} x_{\kappa j}^{\varepsilon j}$. In particular, the first arc is coloured with $(m_L, 1)$ and the last arc is coloured with (m_L, l_L), the meridian-longitude pair of the knot L. At each crossing we find the situation shown in Figure 3.

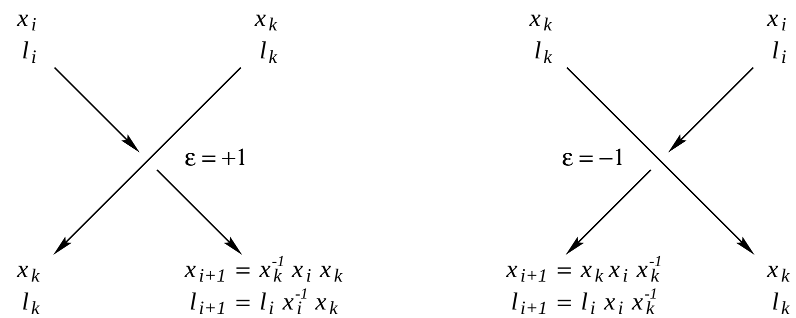

FIGURE 3. Meridians and partial longitudes

This crossing relation can be encoded in a quandle as follows:

Lemma 25. *Let G be a group that is generated by a conjugacy class $Q = x^G$. Then Q is a connected quandle with respect to conjugation $a * b = b^{-1}ab$ and its inverse $a \mathbin{\overline{*}} b = bab^{-1}$. Let G' be the commutator subgroup and define*

$$\tilde{Q} = \tilde{Q}(G, x) := \{ \ (a, g) \in G \times G' \mid a = x^g \ \}.$$

The set $\tilde{Q}$ becomes a connected quandle when equipped with the operations

$$(a, g) * (b, h) = (a * b, ga^{-1}b) \quad \text{and} \quad (a, g) \mathbin{\overline{*}} (b, h) = (a \mathbin{\overline{*}} b, gab^{-1}).$$

Moreover, the projection $p : \tilde{Q} \to Q$ given by $p(a, g) = a$ is a surjective quandle homomorphism. It becomes an equivariant map when we let G' act on Q by conjugation and on $\tilde{Q}$ by $(a, g)^b = (a^b, gb)$. In both cases G' acts transitively and as a group of inner automorphisms.

Proof. Obviously, the operations $*$ and $\overline{*}$ turn Q into a quandle. Since $G = \langle Q \rangle$, the quandle Q is connected. Moreover, the abelianized group G/G' is generated by the image of x, thus $G = \langle x \rangle G'$ and $x^G = x^{G'}$. In particular, G' acts transitively on Q and the map $p : \tilde{Q} \to Q$ is surjective.

It is easily verified that the operations $*$ and $\overline{*}$ on $\tilde{Q}$ are well-defined and satisfy the quandle axioms. Obviously, G' acts transitively on $\tilde{Q}$ and turns p into an equivariant map. It remains to show that G' acts by inner automorphisms. To see this, first notice that the quandle operations of $\tilde{Q}$ can be reformulated as

$$(a, g) * (b, h) = (a * b, x^{-1}gb) \quad \text{and} \quad (a, g) \mathbin{\overline{*}} (b, h) = (a \mathbin{\overline{*}} b, xgb^{-1}).$$

Every $b \in G'$ can be written as $b = b_1^{\varepsilon_1} \cdots b_n^{\varepsilon_n}$ with $b_i \in Q$ and $\varepsilon_i \in \{\pm 1\}$ such that the exponent sum $\varepsilon = \sum \varepsilon_i$ vanishes. Since p is surjective, each $b_i \in Q$ can be lifted to some $(b_i, h_i) \in \tilde{Q}$. Let $\beta_i \in \operatorname{Inn}(\tilde{Q})$ be the corresponding right translation, and set $\beta = \beta_1^{\varepsilon_1} \cdots \beta_n^{\varepsilon_n}$. We obtain $(a, g)^\beta = (a^b, x^\varepsilon g b) = (a, g)^b$ as desired. □

Remark 26. The action of G' on $\tilde{Q}$ can be extended to an action of the whole group G provided that there exists a homomorphism $\varepsilon : G \to \mathbb{Z}$ with $\varepsilon(Q) = 1$. In this case every $b \in G$ acts on $\tilde{Q}$ by

$$(a, g)^b := (a^b, g a^{-\varepsilon b} b) = (a^b, x^{-\varepsilon b} g b).$$

Since $(a, g) * (b, h) = (a, g)^b$ for every $(b, h) \in \tilde{Q}$, we see that G acts by inner automorphisms and $p : \tilde{Q} \to Q \subset G$ is an augmentation.

Remark 27. Since all the information of $(a, g) \in \tilde{Q}$ is contained in g, we can just as well consider the set G' equipped with quandle operations

$$g * h = x^{-1} g h^{-1} x h \quad \text{and} \quad g \mathbin{\overline{*}} h = x g h^{-1} x^{-1} h.$$

These operations already appear in the work of D. Joyce [19], §7, on the representation theory of quandles. The notation proposed in the preceding lemma emphasizes the meridian-longitude interpretation.

3.2. **Quandle coverings.** The quandle $\tilde{Q} = \tilde{Q}(G, x)$ constructed above is tailor-made to capture longitude information. Considered purely algebraically, it is a covering in the following sense:

Definition 28. A surjective quandle homomorphism $p : \tilde{Q} \to Q$ is called a *covering* if $p(\tilde{x}) = p(\tilde{y})$ implies $\tilde{a} * \tilde{x} = \tilde{a} * \tilde{y}$ for all $\tilde{a}, \tilde{x}, \tilde{y} \in \tilde{Q}$. In other words, the natural representation $\tilde{Q} \to \operatorname{Inn}(\tilde{Q})$ factors through p.

This property allows to define an action of Q on $\tilde{Q}$ by setting $\tilde{a} * x := \tilde{a} * \tilde{x}$ with $\tilde{x} \in p^{-1}(x)$. Note that every augmentation $\phi : \tilde{Q} \to G$ defines a covering $\tilde{Q} \to Q$ when restricted to its image $Q = \phi(\tilde{Q})$.

Example 29. The natural representation $Q_L \to \operatorname{Inn}(Q_L)$ factors through the quotients Q_K and Q_K^π, hence $Q_L \to Q_K$ and $Q_L \to Q_K^\pi$ are coverings.

The term "covering" is motivated by formal similarities with covering maps in the category of topological spaces. The unique path lifting property, for example, corresponds to the unique lifting property for long knots:

Theorem 30. *Suppose that L is a long knot and $f : Q_L, q_L \to Q, q$ is a quandle morphism. If $p : \tilde{Q}, \tilde{q} \to Q, q$ is a covering, then f lifts to a unique quandle morphism $\tilde{f} : Q_L, q_L \to \tilde{Q}, \tilde{q}$ with $f = p\tilde{f}$.*

In particular, the natural projection $Q_L, q_L \to Q_K, q_K$ is the universal covering of Q_K, and $Q_L, q_L \to Q_K^\pi, m_K$ is the universal covering of Q_K^π.

Proof. Let D be a long knot diagram representing L. As usual we number the arcs by $0, \ldots, n$ and denote by (κ, ε) the Wirtinger code of D. Let $q_0, \ldots, q_n$ be the corresponding generators of Q_L. Given a homomorphism $f : Q_L, q_L \to Q, q$, we inductively define $\tilde{f} : \{q_0, \ldots, q_n\} \to \tilde{Q}$ as follows: Since $q_0 = q_L$, we have $\tilde{f}(q_0) = \tilde{q}$ to start with. At each positive resp. negative crossing we set $\tilde{f}(q_i) = \tilde{f}(q_{i-1}) * f(q_{\kappa i})$ resp. $\tilde{f}(q_i) = \tilde{f}(q_{i-1}) \mathbin{\overline{*}} f(q_{\kappa i})$. By induction we have $p\tilde{f}(q_i) = f(q_i)$ for all i. This implies that $\tilde{f}$ satisfies the Wirtinger relations. It thus uniquely extends to a homomorphism $\tilde{f} : Q_L, q_L \to \tilde{Q}, \tilde{q}$ with $f = p\tilde{f}$, as claimed. □

Remark 31. The preceding proof is based on the fact that each generator $q_1, \dots, q_n$ is connected to q_0 in a unique way via the Wirtinger relations. The argument holds more generally for every quandle that is given by such a tree-like presentation.

On the other hand, the lifting property does not hold for closed knots, because the Wirtinger relations form a cycle. Lemma 49 below determines the lifting obstruction in terms of quandle homology.

3.3. **Quandle extensions.** We will be mostly interested in coverings that are *galois* in the sense to be defined below. To illustrate this notion, let us consider a group $G = \langle x^G \rangle$, from which we construct quandles $Q = x^G$ and $\tilde{Q} = \tilde{Q}(G, x)$ as in Lemma 25. As we have seen above, the projection $p : \tilde{Q} \to Q$ is a covering map. Moreover, covering transformations are given by the left action of $\Lambda = C(x) \cap G'$ defined by $\lambda \cdot (a, g) = (a, \lambda g)$. This action satisfies the following axioms:

(E1) $(\lambda\tilde{x}) * \tilde{y} = \lambda(\tilde{x} * \tilde{y})$ and $\tilde{x} * (\lambda\tilde{y}) = \tilde{x} * \tilde{y}$ for all $\tilde{x}, \tilde{y} \in \tilde{Q}$ and $\lambda \in \Lambda$.
(E2) Λ acts freely and transitively on each fibre $p^{-1}(x)$.

Axiom (E1) is equivalent to saying that Λ acts by automorphisms and the left action of Λ commutes with the right action of $\operatorname{Inn}(\tilde{Q})$. We denote such an action by $\Lambda \curvearrowright \tilde{Q}$. In this situation the quotient $Q := \Lambda \backslash \tilde{Q}$ carries a unique quandle structure that turns the projection $p : \tilde{Q} \to Q$ into a quandle covering.

Definition 32. A *galois covering* or *extension* $E : \Lambda \curvearrowright \tilde{Q} \to Q$ consists of a surjective quandle homomorphism $\tilde{Q} \to Q$ and a group action $\Lambda \curvearrowright \tilde{Q}$ satisfying axioms (E1) and (E2). We call E a *central extension* if Λ is abelian.

Quandle extensions are an analogue of group extensions, and central quandle extensions come as close as possible to imitating central group extensions. This will become even more evident in §5 where we classify central quandle extensions via the second cohomology group $H^2(Q, \Lambda)$.

4. The structure of knot quandles

As before, we consider a long knot L and the corresponding closed knot K. Theorems 33 and 35 explicitly determine the structure of the covering $Q_L \to Q_K$ in terms of the group system (π_K, m_K, l_K). Corollary 39 extends this result to the covering $Q_L \to Q_K^\pi$, where Q_K^π is the conjugation quandle of K.

4.1. **The fundamental quandle of a long knot.** As a first step, we provide a concrete presentation of Q_L in terms of the fundamental group π_L.

Theorem 33. *For every long knot L there exists a unique quandle isomorphism $Q_L \cong \tilde{Q}(\pi_L, m_L)$ sending q_L to $(m_L, 1)$ and respecting both augmentation maps $Q_L \to \pi_L$ and $\tilde{Q}(\pi_L, m_L) \to \pi_L$.*

Proof. First recall that the universal properties of Q_L and π_L induce the augmentation $\phi : Q_L \to \pi_L$. It respects basepoints in the sense that $\phi(q_L) = m_L$, and its image Q_L^π is the conjugacy class of m_L in π_L. On the other hand we consider the covering $p : \tilde{Q} \to Q_L^\pi$ with $\tilde{Q} = \tilde{Q}(\pi_L, m_L)$ as defined in Lemma 25. According to Remark 26, $p : \tilde{Q} \to Q_L^\pi \subset \pi_L$ is an augmentation as well.

Theorem 30 tells us that ϕ lifts to a unique quandle morphism $\Phi : Q_L \to \tilde{Q}$ with $\Phi(q_L) = (m_L, 1)$. This map is equivariant with respect to the action of π_L: it suffices to prove this for the action of $a = \phi(q)$ with $q \in Q_L$, thus

$$\Phi(q_0^a) = \Phi(q_0 * q) = \Phi(q_0) * \Phi(q) = \Phi(q_0) * (a, g) = \Phi(q_0)^a.$$

We define the inverse map $\Psi : \tilde{Q} \to Q_L$ by $\Psi(a,g) = q_L^g$. This map, too, is equivariant with respect to the action of π_L: for every $x \in \pi_L$ we have

$$\Psi\big((a,g)^x\big) = \Psi(a^x, m_L^{-\varepsilon x} gx) = q_L^{gx} = \Psi(a,g)^x.$$

The composition $\Psi\Phi : Q_L \to Q_L$ fixes q_L, and $\Phi\Psi : \tilde{Q} \to \tilde{Q}$ fixes $(m_L, 1)$. Both quandles are connected, which means that π_L acts transitively. Equivariance thus implies $\Psi\Phi = \mathrm{id}$ and $\Phi\Psi = \mathrm{id}$. □

Corollary 34. *For a long knot L, the natural representation $\phi : Q_L \to \pi_L$ is an embedding if and only if L is trivial. Stated differently, Q_L can be embedded into a group if and only if L is the trivial knot.*

Proof. We continue to use the notation of the previous proof. Comparing the first and the last arc of L, we find that $\Phi(q_L) = (m_L, 1)$ while $\Phi(q_L^*) = (m_L, l_L)$, hence $q_L \neq q_L^*$ for every non-trivial long knot. Since $\phi(q_L) = \phi(q_L^*) = m_L$, the natural representation $\phi : Q_L \to \pi_L$ is an embedding if and only if L is trivial. Moreover, any other representation $Q_L \to G$ factors through ϕ. Hence, if L is non-trivial, then Q_L cannot be embedded into any group. □

This non-embedding result should be contrasted with the situation for a closed knot K: H. Ryder [28] proved that the natural representation $Q_K \to \pi_K$ is an embedding if and only if K is trivial or prime (see Corollary 40 below).

4.2. **The fundamental quandle of a closed knot.** Besides the long knot L we now consider its closure K. Recall that Q_K is obtained from Q_L by adjoining one extra relation identifying the elements q_L and q_L^*, corresponding respectively to the first arc and the last arc of L.

Theorem 35. *The natural projection $Q_L \to Q_K$ is the universal covering of Q_K. If K is trivial, then both Q_L and Q_K are trivial quandles. If K is non-trivial, however, then $Q_L \to Q_K$ is a central extension with covering group $\Lambda = \langle l_K \rangle \cong \mathbb{Z}$.*

Proof. We have seen in §3.2 that $Q_L \to Q_K$ is the universal covering of Q_K. Theorem 33 allows to identify the fundamental quandle Q_L with $\tilde{Q}(\pi_K, m_K)$. The group $\Lambda = \langle l_K \rangle$ acts on $\tilde{Q}(\pi_K, m_K)$ on the left by $\lambda \cdot (a,g) = (a, \lambda g)$. This action satisfies axiom (E1) as defined in §3.3. We can thus pass to the quotient quandle $\bar{Q}$ and obtain a central extension $\Lambda \curvearrowright Q_L \to \bar{Q}$.

Since this quotient identifies $q_L = (m_K, 1)$ and $q_L^* = (m_K, l_K)$, it factors through $Q_L \to Q_K$ and induces a quandle homomorphism $\Phi : Q_K \to \bar{Q}$. We define the inverse map $\Psi : \bar{Q} \to Q_K$ by $\Psi([a,g]) = q_K^g$. This is well-defined because l_K acts trivially on q_K. By construction we have $\Phi(q_K) = [m_K, 1]$ and $\Psi([m_K, 1]) = q_K$. The group π_K acts transitively on Q_K and on $\bar{Q}$, and both quandle homomorphisms Φ and Ψ are equivariant. We conclude that $\Psi\Phi = \mathrm{id}$ and $\Phi\Psi = \mathrm{id}$. This means that the action of $\Lambda = \langle l_K \rangle$ defines a central quandle extension $\Lambda \curvearrowright Q_L \to Q_K$. □

Remark 36. As a corollary, we can present Q_K as the quotient $\langle l_K \rangle \backslash \tilde{Q}(\pi_K, m_K)$. An equivalent presentation was obtained by Joyce [19], Corollary 16.2. Conversely, Q_K determines $(\pi_K, m_K, \langle l_K \rangle)$. As a consequence, the knot quandle classifies knots up to inversion, cf. Joyce [19], Corollary 16.3. The remaining ambiguity can be removed by the orientation class $[K] \in H_2(Q_K)$, as explained in §6.

4.3. **The conjugation quandle of a knot.** Having examined the fundamental quandles Q_L and Q_K, we can now compare them to their common conjugation quandle. Recall that Q_L allows a natural representation $Q_L \to \pi_L$ on the knot group π_L. Its image Q_L^π is the conjugacy class of the meridian m_L and is called the conjugation quandle of L. Since the knot groups π_L and π_K are identical, so are the conjugation quandles Q_L^π and Q_K^π.

In order to understand Q_K^π, we have to determine the centralizer of m_K in π_K. Notice that $\pi_K = \langle m_K \rangle \ltimes \pi_K'$, hence $C(m_K) = \langle m_K \rangle \times \Lambda$ with $\Lambda = C(m_K) \cap \pi_K'$. The geometric significance of Λ is highlighted by the following theorem:

Theorem 37. *If K is the connected sum of prime knots $K^1, \dots, K^p$, then their longitudes $l_K^1, \dots, l_K^p \in \pi_K$ freely generate $\Lambda = C(m_K) \cap \pi_K'$.*

Proof. The connected sum $K = K^1 \sharp \dots \sharp K^p$ allows to define a family of group homomorphisms $\pi_{K^i} \to \pi_K$, mapping every meridian m_{K^i} to m_K and every longitude l_{K^i} to some element l_K^i such that $l_K = l_K^1 \cdots l_K^p$. The theorem of Seifert and van Kampen shows that π_K is the amalgamated product $\pi_{K^1} * \dots * \pi_{K^p}$, where the subgroups $\langle m_{K^1} \rangle, \dots, \langle m_{K^p} \rangle$ are identified via $m_{K^1} = \dots = m_{K^p}$. In particular, $\pi_K' = \pi_{K^1}' * \dots * \pi_{K^p}'$ is a free product, hence $\Lambda = \Lambda^1 * \dots * \Lambda^p$. According to Lemma 38 (proved below), we have $\Lambda^i = \langle l_{K^i} \rangle$ for every prime knot K^i. □

In the case of fibered knots, the preceding theorem was proven by D. Noga [26], §4, and by W. Whitten [32], §2. The following lemma settles the general case. It is a straightforward application of the annulus theorem, for which we refer to the article by J.W. Cannon and C.D. Feustel [4] or the book by W. Jaco [18], Theorem VIII.13. Similar applications have been worked out by J. Simon [29] and H. Ryder [28].

Lemma 38. *If K is a prime knot, then $C(m_K) = \langle m_K, l_K \rangle$.*

Proof. Given a closed knot $K \subset \mathbb{S}^3$, we choose a standard framing $f : \mathbb{S}^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{S}^3$ as in §1.1. The exterior $M = \mathbb{S}^3 \smallsetminus f(\mathbb{S}^1 \times \operatorname{int} \mathbb{D}^2)$ is a compact oriented 3-manifold with boundary $\partial M = f(\mathbb{S}^1 \times \mathbb{S}^1)$. As before we choose $p = f(1,1)$ as basepoint and represent the meridian $m_K \in \pi_1(M)$ by the curve $\mu = f|_{1 \times \mathbb{S}^1}$. Given another element $c \in \pi_1(M)$, we represent c by a loop $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = p$.

Let $A = [0,1] \times \mathbb{S}^1$ be the standard annulus. If c and m_K commute, then there exists a continuous map $g : A \to M$ such that $g|_{0 \times \mathbb{S}^1} = g|_{1 \times \mathbb{S}^1} = \mu$ and $g|_{[0,1] \times 1} = \gamma$. If moreover $c \notin \pi_1(\partial M)$, then g is essential in the sense of [4, 18]. By a slight deformation we can then obtain an essential map $\bar{g} : A \to M$ such that $\bar{g}|_{\partial A}$ is an embedding. More explicitly, we can arrange that $\bar{g}|_{1 \times \mathbb{S}^1} = \bar{\mu}$ is a different meridian, $\bar{\mu} = f|_{\vartheta \times \mathbb{S}^1}$ say, while $\bar{g}|_{0 \times \mathbb{S}^1} = \mu$ remains unchanged. The annulus theorem [4, 18] then guarantees the existence of an essential embedding $h : A \hookrightarrow M$ with $h|_{\partial A} = \bar{g}|_{\partial A}$.

Considering again $M \subset \mathbb{S}^3$, we can cap off the embedded annulus $h(A)$ by two meridian disks $f(1 \times \mathbb{D}^2)$ and $f(\vartheta \times \mathbb{D}^2)$. This produces an embedded sphere $S \subset \mathbb{S}^3$ that meets K transversely in exactly two points. The annulus $h(A)$ being essential in M means that S splits K into two non-trivial summands (cf. [3], §15C). We conclude that the existence of $c \in C(m_K) \smallsetminus \pi_1(\partial M)$ implies that K is a non-trivial connected sum. In other words, if K is prime, then $C(m_K) = \langle m_K, l_K \rangle$. □

Corollary 39. *For every long knot L, the natural projection $Q_L \to Q_L^\pi$ is the universal quandle covering of Q_L^π. It is an extension whose covering group Λ is* *freely generated by the partial longitudes $l_L^1, \dots, l_L^p \in \pi_L$ coming from the prime summands of a connected sum decomposition $L = L^1 \sharp \cdots \sharp L^p$.*

Proof. We have already seen in §3.2 that $Q_L \to Q_L^\pi$ is the universal covering of Q_L^π. Theorem 33 allows to identify $Q_L \to Q_L^\pi$ with the canonical projection $\tilde{Q}(\pi_L, m_L) \to Q_L^\pi$. By construction, the latter is an extension with covering group $\Lambda = C(m_L) \cap \pi_L'$. □

Corollary 40 ([28])**.** *For a closed knot K, the natural projection $Q_K \to Q_K^\pi$ is an isomorphism if and only if K is trivial or prime. Stated differently, Q_K can be embedded into a group if and only if K is trivial or prime.*

Proof. Theorem 35 allows to identify Q_K with the quotient $\langle l_L \rangle \backslash Q_L$, where L is the long knot corresponding to K. On the other hand, Corollary 39 identifies Q_K^π with $\langle l_L^1, \dots, l_L^p \rangle \backslash Q_L$, where $l_L^1, \dots, l_L^p$ are the longitudes of the prime summands of L. Hence the projection $Q_K \to Q_K^\pi$ is injective if and only if $p \leq 1$. □

We remark that for $p \geq 2$ the projection $Q_K \to Q_K^\pi$ is a covering but not galois. The universal covering $Q_L \to Q_K^\pi$, however, is always galois. This is another reason why it it easier to work with Q_L instead of Q_K.

5. Quandle cohomology

We begin this section by recalling the definition of quandle cohomology [5]. Theorem 44 shows that central extensions of a quandle Q by an abelian group Λ are classified by elements of the second cohomology group $H^2(Q, \Lambda)$.

5.1. **Quandle cohomology.** Let Q be a quandle and let Λ be an abelian group. An n-cochain is a map $\lambda : Q^n \to \Lambda$ satisfying $\lambda(a_1, \dots, a_n) = 0$ whenever $a_i = a_{i+1}$ for some index i. The set $C^n = C^n(Q, \Lambda)$ of all n-cochains forms a $\mathbb{Z}$-module. The coboundary operator $\delta^n : C^n \to C^{n+1}$ is defined by

$$(\delta^n \lambda)(a_0, \dots, a_n) = \sum_{i=1}^{n} (-1)^i \Big[\lambda(a_0^{a_i}, \dots, a_{i-1}^{a_i}, a_{i+1}, \dots, a_n) - \lambda(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \Big].$$

This defines a cochain complex (C^*, δ^*). As usual, the kernel $Z^n = \ker(\delta^n)$ is called the submodule of n-cocycles, and the image $B^n = \operatorname{im}(\delta^{n-1})$ is called the submodule of n-coboundaries. Their quotient $H^n(Q, \Lambda) = Z^n / B^n$ is the n-th cohomology group of the quandle Q with coefficients in Λ.

Example 41. A 1-cochain is simply a map $\mu : Q \to \Lambda$. It is a cocycle if and only if $\mu(a * b) - \mu(a) = 0$ for all $a, b \in Q$. In other words, 1-cocycles are exactly the class functions, that is, constant on orbits under the action of $\operatorname{Inn}(Q)$. In particular, if Q is connected, then $H^1(Q, \Lambda) = \Lambda$.

5.2. **Classification of central extensions.** It is a classical result of group cohomology that central extensions of a group G with kernel Λ are classified by the second cohomology group $H^2(G, \Lambda)$, see for example Brown [2], §IV.3, or MacLane [21], §IV.4. We will now prove that an analogous theorem holds for quandles. As the referee pointed out, this result has independently been developed in [8, 9].

Lemma 42. *Let $E : \Lambda \curvearrowright \tilde{Q} \to Q$ be a central extension. Each section $s : Q \to \tilde{Q}$ defines a map $\lambda : Q \times Q \to \Lambda$ such that $s(a) * s(b) = \lambda(a, b) \cdot s(a * b)$. This map λ is a 2-cocycle. Furthermore, if $s' : Q \to \tilde{Q}$ is another section, then the associated 2-cocycle*

λ' differs from λ by a 2-coboundary. Thus each central extension E determines a cohomology class $\Phi(E) := [\lambda] \in H^2(Q,\Lambda)$.

Proof. Since the action of Λ is free and transitive on each fibre, the above equation uniquely defines the map λ. Idempotency of $\tilde{Q}$ implies $\lambda(a,a) = 0$ for each $a \in Q$, so λ is a cochain. Self-distributivity implies the cocycle condition $\delta\lambda = 0$.

If s' is another section, then there exists $\mu : Q \to \Lambda$ with $s'(a) = \mu(a) \cdot s(a)$, and we find $\lambda - \lambda' = \delta\mu$. This shows that the cohomology class $[\lambda]$ is independent of the chosen section, and hence characteristic of the extension E. □

Conversely, we will associate with each $[\lambda] \in H^2(Q,\Lambda)$ a central extension of Q by Λ. There is essentially only one possibility to do this. More precisely:

Definition 43. Let Q be a quandle and Λ an abelian group. An *equivalence* between two central extensions $\Lambda \curvearrowright Q_1 \to Q$ and $\Lambda \curvearrowright Q_2 \to Q$ is a quandle isomorphism $\phi : Q_1 \to Q_2$ that respects projections, $p_1 = p_2\phi$, and is equivariant, i.e. $\phi t = t\phi$ for all $t \in \Lambda$. We define $\mathscr{E}(Q,\Lambda)$ to be the set of equivalence classes of central extensions of Q by Λ.

Theorem 44. *Let Q be a quandle and let Λ be an abelian group. For each central extension $E : \Lambda \curvearrowright \tilde{Q} \to Q$ let $\Phi(E)$ be the associated cohomology class in $H^2(Q,\Lambda)$. This map induces a bijection $\Phi : \mathscr{E}(Q,\Lambda) \cong H^2(Q,\Lambda)$.*

Proof. First note that Φ is well-defined on equivalence classes of extensions. To prove the theorem, we will construct an inverse map $\Psi : H^2(Q,\Lambda) \to \mathscr{E}(Q,\Lambda)$ as follows. Given a 2-cocycle $\lambda \in Z^2(Q,\Lambda)$, we define the quandle $\Lambda \times_\lambda Q$ as the set $\Lambda \times Q$ equipped with the binary operation

$$(u,a) * (v,b) = (u + \lambda(a,b), a * b).$$

Indeed, this defines a quandle: idempotency is guaranteed by $\lambda(a,a) = 0$, the inverse operation is given by

$$(u,a) \mathbin{\overline{*}} (v,b) = (u - \lambda(a \mathbin{\overline{*}} b, b), a \mathbin{\overline{*}} b),$$

and self-distributivity follows from the cocycle condition $\delta\lambda = 0$. The action of Λ is defined by $t \cdot (u,a) = (t+u,a)$. We obtain a central extension $\Lambda \curvearrowright \Lambda \times_\lambda Q \to Q$.

If $\lambda' = \lambda + \delta\mu$, then the corresponding extensions are equivalent via the isomorphism $\phi : \Lambda \times_\lambda Q \to \Lambda \times_{\lambda'} Q$ defined by $\phi(u,a) = (u + \mu(a), a)$. Hence we have indeed constructed a map $\Psi : H^2(Q,\Lambda) \to \mathscr{E}(Q,\Lambda)$.

To see that $\Phi\Psi = \mathrm{id}$, let $\lambda \in Z^2(Q,\Lambda)$ and consider the section $s : Q \to \Lambda \times_\lambda Q$ with $s(a) = (0,a)$. The corresponding cocycle is λ, hence $\Phi\Psi = \mathrm{id}$.

It remains to show that $\Psi\Phi = \mathrm{id}$. Given an extension $E : \Lambda \curvearrowright \tilde{Q} \to Q$, we choose a section $s : Q \to \tilde{Q}$ and consider the corresponding 2-cocycle $\lambda \in Z^2(Q,\Lambda)$. The map $\phi : \Lambda \times_\lambda Q \to \tilde{Q}$ given by $\phi(u,a) = u \cdot s(a)$ is then an equivalence of extensions, which proves $\Psi\Phi = \mathrm{id}$. □

6. Orientation classes

This section recalls the definition of quandle homology [6] and defines the orientation class $[K] \in H_2(Q_K)$ of a knot K. Theorem 51 shows that the oriented knot K is characterized by the pair $(Q_K, [K])$.

6.1. **Quandle homology.** Given a quandle Q, let C'_n be the free abelian group generated by n-tuples $(a_1, \dots, a_n) \in Q^n$. We define $\partial_n : C'_n \to C'_{n-1}$ by

$$\partial_n(a_1, \dots, a_n) = \sum_{i=2}^{n} (-1)^i \Big[(a_1^{a_i}, \dots, a_{i-1}^{a_i}, a_{i+1}, \dots, a_n) \\ - (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \Big].$$

It is easily verified that (C'_*, ∂_*) is a chain complex. Let C''_n be the submodule generated by all n-tuples $a \in Q^n$ with $a_i = a_{i+1}$ for some index i. Quandle axiom (Q1) ensures that $\partial_n(C''_n) \subset C''_{n-1}$, hence (C''_*, ∂_*) is a sub-complex of (C'_*, ∂_*).

We define the chain complex for our quandle Q as the quotient $C_* := C'_*/C''_*$. The induced boundary operator is again denoted by ∂_*. As usual, the kernel $Z_n = \ker(\partial_n)$ is called the submodule of n-cycles, and the image $B_n = \operatorname{im}(\partial_{n+1})$ is called the submodule of n-boundaries. Their quotient $H_n(Q) = Z_n/B_n$ is the n-th homology group of the quandle Q with integer coefficients.

Example 45. Every 1-cycle is just a formal sum $\sum_{i=1}^{i=n} \alpha_i a_i$ of elements $a_i \in Q$ with coefficients $\alpha_i \in \mathbb{Z}$. The submodule of 1-boundaries is generated by differences $a * b - a$ with $a, b \in Q$. This means that the equivalence classes of homologous elements in Q coincide with orbits under the $\operatorname{Inn}(Q)$-action, and these classes freely generate $H_1(Q)$. In particular, we have $H_1(Q) = \mathbb{Z}$ if and only if Q is connected.

Given an abelian group Λ, we define $C_n(Q, \Lambda) = C_n(Q) \otimes \Lambda$. Notice that $C^n(Q, \Lambda) = \operatorname{Hom}(C_n(Q), \Lambda)$ is the cochain complex defined in §5.1. This allows to define quandle homology $H_n(Q, \Lambda) = H_n(C_*(Q, \Lambda))$ and quandle cohomology $H^n(Q, \Lambda) = H^n(C^*(Q, \Lambda))$ in the usual way, and both notions are dual to each other. Whenever the coefficient group is not specified, we tacitly assume $\Lambda = \mathbb{Z}$.

Lemma 46. *For every quandle Q and every abelian group Λ there are natural isomorphisms $H_2(Q, \Lambda) \cong H_2(Q) \otimes \Lambda$ and $H^2(Q, \Lambda) \cong \operatorname{Hom}(H_2(Q), \Lambda)$.*

Proof. This follows from the Universal Coefficient Theorem and the fact that $H_1(Q)$ is free. See for example MacLane [21], Theorems III.4.1 and V.11.1. □

6.2. **The orientation class of a closed knot.** Let K be a closed knot and let D be a diagram representing K. Let each arc of D be coloured with the corresponding generator of Q_K. For each coloured crossing p with sign $\varepsilon = \pm 1$ as in Figure 2, we define its weight to be $(p) := \varepsilon \cdot (a, b)$, considered as an element in $C_2(Q_K)$. Let $(D) \in C_2(Q_K)$ be the sum of the weights of all crossings.

Lemma 47. *For every closed diagram, $(D) \in C_2(Q_K)$ is a cycle. Its homology class $[D] \in H_2(Q_K)$ is invariant under Reidemeister moves. Every closed knot K can thus be equipped with a characteristic class $[K] := [D] \in H_2(Q_K)$.*

The orientation class has been used implicitly by Carter et al. [5] to define a state-sum invariant of knots. The explicit definition given here has independently been studied by J.S. Carter, S. Kamada, and M. Saito in [7], building on the Ph.D. thesis of M.T. Greene [17].

Proof of the lemma. Consider a positive crossing as in Figure 2a. Its weight is $+(a, b)$, which maps to $\partial(a, b) = c - a$. We interpret this as saying that the arc coloured with a contributes the weight $-a$ at its end, and the arc coloured with c contributes $+c$ at its beginning. The same holds for a negative crossing as in

Figure 2b. Its weight is $-(a, b)$, which maps to $-\partial(a, b) = a - c$. Again we interpret this as saying that the arc coloured with a contributes the weight $+a$ at its beginning, and the arc coloured with c contributes $-c$ at its end. In total, every arc, coloured with some $a \in Q$, contributes $+a$ at its beginning and $-a$ at its end. For a closed knot diagram all contributions cancel each other, and we have $\partial(D) = 0$.

It is a routine exercise to check that $[D]$ does not change under Reidemeister moves. For R1-moves notice that we have quotiented out degeneracies, so that $(a, a) = 0$ in $C_2(Q_K)$. An R2-move introduces two extra crossings, which cancel each other because of our sign convention. An R3-move, finally, adds a 2-boundary. The homology class $[D]$ is thus an invariant of the knot, as claimed. □

Remark 48. We call $[K]$ the *orientation class* of K. As its name suggests, it encodes the orientation of K (see §6.4 below). Let K^* be the inverse knot, obtained from K by inverting the orientation of K and of the sphere $\mathbb{S}^3$. This operation produces a canonical isomorphism $\phi : Q_K \to Q_{K^*}$ satisfying $\phi_*[K] = -[K^*]$.

6.3. **The lifting lemma for closed knots.** The orientation class $[K]$ also encodes the obstruction to the following lifting problem. Suppose that $p : \tilde{Q}, \tilde{q} \to Q, q$ is a central extension with covering group Λ and cohomology class $[\lambda] \in H^2(Q, \Lambda)$.

Lemma 49. *Let K be a closed knot. A quandle morphism $f : Q_K, q_K \to Q, q$ can be lifted to a morphism $\tilde{f} : Q_K, q_K \to \tilde{Q}, \tilde{q}$ if and only if $\langle [\lambda] \mid f \mid [K] \rangle$ vanishes.*

This is a consequence of the following monodromy calculation:

Lemma 50. *Let L be the long knot corresponding to K. Every homomorphism $f : Q_K, q_K \to Q, q$ can be lifted to a unique homomorphism $h : Q_L, q_L \to \tilde{Q}, \tilde{q}$. For all $a \in Q_L$ we have $h(l_K \cdot a) = \ell \cdot h(a)$ with $\ell = \langle [\lambda] \mid f \mid [K] \rangle$.*

Proof. The quandle homomorphism $f : Q_K, q_K \to Q, q$ can be composed with $Q_L, q_L \to Q_K, q_K$ to define a homomorphism $g : Q_L, q_L \to Q, q$. By Theorem 30, there exists a unique lifting $h : Q_L, q_L \to \tilde{Q}, \tilde{q}$. We first prove that $h(q_L^*) = \ell \cdot \tilde{q}$.

Let $s : Q \to \tilde{Q}$ be a section with $s(q) = \tilde{q}$ and $s(a) * s(b) = \lambda(a, b) \cdot s(a * b)$. As usual, we represent L by a long diagram D, its arcs being numbered by $0, \ldots, n$. Let $q_0, \ldots, q_n$ be the corresponding generators of Q_L. Define $\ell_i \in \Lambda$ by the condition $h(q_i) = \ell_i \cdot sg(q_i)$. We have $\ell_0 = 0$ to start with. For $i = 1, \ldots, n$ we find $\ell_i = \ell_{i-1} + \langle \lambda \mid f \mid (p_i) \rangle$, where (p_i) is the weight of the crossing p_i. In total we get $\ell = \ell_n = \langle [\lambda] \mid f \mid [K] \rangle$, as claimed.

The equality $h(l_K \cdot a) = \ell \cdot h(a)$ thus holds for the basepoint $a = q_L$. If it holds for $a \in Q_L$, then it also holds for $a * b$ and $a \mathbin{\overline{*}} b$ with $b \in Q_L$:

$$h(l_K \cdot a * b) = h(l_K \cdot a) * h(b) = \ell \cdot h(a) * h(b) = \ell \cdot h(a * b).$$

Since Q_L is connected, we conclude that $h(l_K \cdot a) = \ell \cdot h(a)$ for all $a \in Q_L$. □

6.4. **Classifying oriented knots.** Joyce [19], building on the work of Waldhausen [31], showed that the fundamental quandle Q_K characterizes the knot K up to inversion. The remaining ambiguity is removed by the orientation class $[K] \in H_2(Q_K)$:

Theorem 51. *An oriented knot K is characterized by the pair $(Q_K, [K])$.*

Proof. With the fundamental quandle Q_K we associate its universal group representation $\phi : Q_K \to \pi$. Choosing a basepoint $q \in Q_K$ defines a basepoint $m = \phi(q)$ in the group π. Furthermore, the group π acts on Q_K with stabilizer $\Lambda := \mathrm{Stab}(q)$.

The abstract data (π, m, Λ) can easily be interpreted in terms of the fundamental group. Since the natural representation $Q_K \to \pi_K$ satisfies the same universal property as $\phi : Q_K \to \pi$, there is a canonical isomorphism $\pi \cong \pi_K$, which is in fact an isomorphism between (π, m, Λ) and $(\pi_K, m_K, \langle l_K \rangle)$. Choosing a generator $l \in \Lambda$ thus yields $(\pi, m, l) \cong (\pi_K, m_K, l_K^{\pm 1})$. According to Theorem 11, this means that (π, m, Λ) characterizes the knot K up to inversion (see [3], §3C).

If Λ is trivial, then the knot K is trivial and there is nothing to prove. Otherwise, let $\tilde{Q}, \tilde{q} \to Q_K, q$ be the universal covering of Q_K. We know from Theorem 35 that it is a central extension with covering group Λ. The choice of a generator $l \in \Lambda$ defines an action of $\mathbb{Z}$, where $z \in \mathbb{Z}$ acts by $l^z : \tilde{Q} \to \tilde{Q}$. This corresponds to a cohomology class $[l] \in H^2(Q_K)$ with $\langle [l] \mid [K] \rangle = \pm 1$. We choose the generator $l \in \Lambda$ such that $\langle [l] \mid [K] \rangle = 1$. By Lemma 50, the extensions $\mathbb{Z} \curvearrowright \tilde{Q} \to Q_K$ and $\mathbb{Z} \curvearrowright Q_L \to Q_K$ are then equivalent. It follows that $(\pi, m, l) \cong (\pi_K, m_K, l_K)$.

We can thus translate $(Q_K, [K])$ to the knot group system (π_K, m_K, l_K). According to Theorem 11, this data characterizes the knot K. □

7. Homology of knot quandles

As we have seen in the preceding sections, every knot K comes equipped with two characteristic classes: the central extension $Q_L \to Q_K$ defines a cohomology class $[L] \in H^2(Q_K)$, and the orientation of K defines a homology class $[K] \in H_2(Q_K)$.

We will now prove our main result. Recall that the quandles Q_L and Q_K are connected, which implies $H_1(Q_L) = H_1(Q_K) = \mathbb{Z}$ and $H^1(Q_L) = H^1(Q_K) = \mathbb{Z}$. The interesting point is the second (co)homology group:

Theorem 52. *Let L be a long knot and K the corresponding closed knot. We have $H_2(Q_L) = H^2(Q_L) = 0$, and in the case of the trivial knot $H_2(Q_K) = H^2(Q_K) = 0$ as well. If K is non-trivial, however, then $H_2(Q_K) \cong \mathbb{Z}$, and the orientation of K defines a canonical generator $[K] \in H_2(Q_K)$. Dually, we have $H^2(Q_K) \cong \mathbb{Z}$, and the central extension $Q_L \to Q_K$ defines a canonical generator $[L] \in H^2(Q_K)$ whose evaluation yields $\langle [L] \mid [K] \rangle = 1$.*

Proof. We first show $H^2(Q_L, \Lambda) = 0$. Consider $[\lambda] \in H^2(Q_L, \Lambda)$ and the associated central extension $p : \tilde{Q} \to Q_L$. Theorem 30 says that there exists a quandle morphism $s : Q_L \to \tilde{Q}$ with $ps = \mathrm{id}$, hence $[\lambda] = 0$ as claimed. Specializing to $\Lambda = H_2(Q_L)$, the isomorphism $H^2(Q_L, \Lambda) \cong \mathrm{Hom}(H_2(Q_L), \Lambda)$ implies $H_2(Q_L) = 0$.

If the knot K is trivial, then Q_K consists of a single element, and $H_n(Q_K) = 0$ for all $n \geq 2$. In the sequel we can thus assume that K is non-trivial. In this case, Theorem 35 says that $Q_L \to Q_K$ is a central extension with covering group $\Lambda \cong \mathbb{Z}$ generated by the longitude $l_K \in \pi_K$. By Lemma 50, the associated cohomology class $[L] \in H^2(Q_K)$ satisfies $\langle [L] \mid [K] \rangle = 1$. This shows that $[K]$ generates an infinite cyclic subgroup in $H_2(Q_K)$.

It remains to prove $H_2(Q_K) = \langle [K] \rangle$. Consider an abelian group Λ and a linear map $\lambda : H_2(Q_K) \to \Lambda$. By the Universal Coefficient Theorem, λ corresponds to a cohomology class in $H^2(Q_K, \Lambda)$. By Theorem 44, this class is realized by a central extension $\Lambda \curvearrowright \tilde{Q} \to Q_K$. If $\lambda([K]) = 0$, then Lemma 49 implies that the extension splits and λ vanishes. In particular this is true for the quotient map $\lambda : H_2(Q_K) \to H_2(Q_K)/\langle[K]\rangle$, which proves that $H_2(Q_K) = \langle[K]\rangle$. □

8. Homology of conjugation quandles

In this section we determine the homology of the conjugation quandle Q_K^π, that is, the conjugacy class of the meridian m_K in π_K. Since the quandle Q_K^π is connected, we have $H_1(Q_K^\pi) = H^1(Q_K^\pi) = \mathbb{Z}$. The rank of the second homology group, however, depends on the number of prime summands:

Theorem 53. *If K is the connected sum of prime knots $K^1, \dots, K^p$, then we have $H_2(Q_K^\pi) \cong H^2(Q_K^\pi) \cong \mathbb{Z}^p$. Moreover, the orientation classes $[K^1], \dots, [K^p]$ map to a basis of $H_2(Q_K^\pi)$, and their dual classes $[L^1], \dots, [L^p]$ map to a basis of $H^2(Q_K^\pi)$.*

Proof. The connected sum $K = K^1 \sharp \dots \sharp K^p$ allows to define quandle monomorphisms $\alpha^i : Q_{K^i}^\pi \to Q_K^\pi$ and epimorphisms $\beta_i : Q_K^\pi \to Q_{K^i}^\pi$ such that $\beta_i \alpha^i$ is the identity for all i, whereas $\beta_i \alpha^j$ is the constant map to m_{K^i} whenever $i \neq j$.

Since each summand K^i is prime, Corollary 40 tells us that the fundamental quandle Q_{K^i} and the conjugation quandle $Q_{K^i}^\pi$ are canonically isomorphic. We can thus identify the orientation class $[K^i] \in H_2(Q_{K^i}^\pi)$ with its image under $\alpha_*^i : H_2(Q_{K^i}^\pi) \to H_2(Q_K^\pi)$, and the dual class $[L^i] \in H^2(Q_{K^i}^\pi)$ with its image under $\beta_i^* : H^2(Q_{K^i}^\pi) \to H^2(Q_K^\pi)$. It follows from this construction that $\langle [L^i] \mid [K^j] \rangle = \delta_{ij}$. This shows that $[K^1], \dots, [K^p] \in H_2(Q_K^\pi)$ freely generate an abelian subgroup of rank p, and the same holds for $[L^1], \dots, [L^p] \in H^2(Q_K^\pi)$.

In order to show that $H_2(Q_K^\pi)$ is generated by $[K^1], \dots, [K^p]$, we consider an abelian group A and a linear map $\lambda : H_2(Q_K^\pi) \to A$. By the Universal Coefficient Theorem, λ corresponds to a cohomology class in $H^2(Q_K^\pi, A)$, and by Theorem 44, this class is realized by a central extension. If $\lambda([K^1]) = \dots = \lambda([K^p]) = 0$, then Lemma 54 (proved below) implies that the extension splits and λ vanishes. In particular this is true for the quotient map $\lambda : H_2(Q_K) \to H_2(Q_K)/\langle[K^1], \dots, [K^p]\rangle$, which proves that $H_2(Q_K) = \langle[K^1], \dots, [K^p]\rangle$. □

Lemma 54. *Let $\eta : \tilde{Q}, \tilde{q} \to Q, q$ be a central extension with covering group A and cohomology class $[\lambda] \in H^2(Q, A)$. A quandle morphism $f : Q_K^\pi, m_K \to Q, q$ can be lifted to a morphism $\tilde{f} : Q_K^\pi, m_K \to \tilde{Q}, \tilde{q}$ if and only if $\langle [\lambda] \mid f \mid [K^i] \rangle$ vanishes for all $i = 1, \dots, p$.*

This is a consequence of the following monodromy calculation. Recall from Corollary 39 that the universal covering of Q_K^π is given by $\Lambda \curvearrowright Q_L \to Q_K^\pi$, where Q_L is the fundamental quandle of the corresponding long knot L, and the covering group Λ is freely generated by the partial longitudes $l_K^1, \dots, l_K^p \in \pi_K$ coming from the prime summands $K^1, \dots, K^p$.

Lemma 55. *Let $\eta : \tilde{Q}, \tilde{q} \to Q, q$ be a central extension with covering group A and cohomology class $[\lambda] \in H^2(Q, A)$. Every homomorphism $f : Q_K^\pi, m_K \to Q, q$ can be lifted to a unique homomorphism $h : Q_L, q_L \to \tilde{Q}, \tilde{q}$. For all $a \in Q_L$ we have $h(l_K^i \cdot a) = \ell^i \cdot h(a)$ with $\ell^i = \langle [\lambda] \mid f \mid [K^i] \rangle$.*

Proof. The quandle homomorphism $f : Q_K^\pi, m_K \to Q, q$ can be composed with the natural projection $Q_L, q_L \to Q_K^\pi, m_K$ to define a homomorphism $g : Q_L, q_L \to Q, q$. By Theorem 30, there exists a unique lifting $h : Q_L, q_L \to \tilde{Q}, \tilde{q}$ with $g = \eta h$. Since $g(l_K^i \cdot a) = g(a)$, there exists $\ell^i \in A$ such that $h(l_K^i \cdot a) = \ell^i \cdot h(a)$. It follows that

$h(l_K^i \cdot a^\phi) = \ell^i \cdot h(a^\phi)$ for all $\phi \in \mathrm{Inn}(Q_L)$. Since Q_L is connected, the factor ℓ^i is the same for all $a \in Q_L$.

We represent the knot $K = K^1 \sharp \dots \sharp K^p$ as a planar diagram $D = D^1 \sharp \dots \sharp D^p$ realizing the connected sum decomposition. The monodromy ℓ^i can be read off the diagram D as in the proof of Lemma 50: it suffices to travel along the summand D^i. Each crossing $p_1^i, \dots, p_n^i$ contributes $\langle \lambda \mid f \mid (p_j^i) \rangle$, and in total we obtain $\ell^i = \langle \lambda \mid f \mid (D^i) \rangle$. Since the 2-cocycle (D^i) represents the cohomology class $[K^i]$, we conclude that $\ell^i = \langle [\lambda] \mid f \mid [K^i] \rangle$. □

9. Homology of link quandles

Our calculation of homology groups can be generalized from knot quandles to link quandles. All proofs parallel those given for knots — they are slightly more technical but introduce no new ideas. We will therefore only sketch the main ingredients.

In this section, let $K \subset \mathbb{S}^3$ be a link with n components. A component K^i of K is called *trivial* if there exists an embedded disk $D \subset \mathbb{S}^3$ with $K^i = K \cap D = \partial D$. As in the case of knots, Dehn's lemma [27] allows to formulate this geometric condition in terms of the fundamental group: the component K^i is trivial if and only if its longitude $l_K^i \in \pi_K$ is trivial.

Remark 56. Just as K has n components, its fundamental quandle Q_K decomposes into n orbits under the action of $\mathrm{Inn}(Q_K)$. More precisely, if we choose meridians $q_K^1, \dots, q_K^n \in Q_K$, one for each component $K^1, \dots, K^n \subset K$, then their orbits $Q_K^1, \dots, Q_K^n$ form the desired decomposition of Q_K. This implies $H_1(Q_K) = H^1(Q_K) = \mathbb{Z}^n$. See Examples 41 and 45, or [6], Proposition 3.8.

Orientation classes $[K^1], \dots, [K^n] \in H_2(Q_K)$ can be defined as before, summing over each component separately. The lifting lemma now takes the following form:

Lemma 57. *Suppose that K is a closed link and $f : Q_K \to Q$ is a quandle morphism with $f(q_K^i) = q^i$ for all i. Let $p : \tilde{Q} \to Q$ be a central extension with $p(\tilde{q}^i) = q^i$ for all i, let Λ be its covering group, and let $[\lambda] \in H^2(Q, \Lambda)$ be the associated cohomology class. Then there exists a lifting $\tilde{f} : Q_K \to \tilde{Q}$ with $f = p\tilde{f}$ and $\tilde{f}(q_K^i) = \tilde{q}^i$ for all i if and only if all evaluations $\langle [\lambda] \mid f \mid [K^i] \rangle$ vanish.* □

For each i consider the long link L^i obtained from K by opening the component K^i while leaving all other components closed. Up to isotopy there is only one way of doing this, so we can speak of L^i as the i-th long link associated with K. As before the quandle Q_K can be obtained from Q_{L^i} by adjoining one extra relation identifying both ends of L^i.

Lemma 58. *The natural projection of fundamental quandles $p_i : Q_{L^i} \to Q_K$ is a quandle covering. For each closed component $j \neq i$, it induces an isomorphism $p_i^j : Q_{L^i}^j \cong Q_K^j$. For the open component i, the restriction $p_i^i : Q_{L^i}^i \to Q_K^i$ is a central extension with covering group $\Lambda_i = \langle l_{K^i} \rangle$.* □

Notice that the covering $p_i : Q_{L^i} \to Q_K$ is not a central extension in the sense of §3.3, because the action Λ_i is not free on each fibre. Nevertheless, it is possible to associate cohomology classes $[L^1], \dots, [L^n] \in H^2(Q_K)$ with the maps $p_1, \dots, p_n$.

These prerequisites being in place, it is now an easy matter to determine the second (co)homology group of the link quandle Q_K, thus proving Theorem 10 stated in the introduction. The proof simply reformulates the above proof of Theorem 52, so we will omit it.

A similar analysis can be carried out for the conjugation quandle Q_K^π.

10. Algorithmic questions

We conclude this article with a few remarks on algorithmic questions.

10.1. **The unknotting problem.** The discussion of knot groups and knot quandles touches upon the notoriously difficult *classification problem*: given two knots, how can we decide whether or not they are equivalent? W. Haken, G. Hemion, and others proved that this problem is algorithmically solvable [23].

In this article we have restricted our attention to the less ambitious but still very difficult *unknotting problem*: given a knot, how can we decide whether or not it is trivial? Dehn's lemma [27] translates this into a group-theoretical criterion: a knot K is trivial if and only if its longitude $l_K \in \pi_K$ vanishes.

Remark 59. Dehn's lemma can be turned into an algorithm, using a result of W. Thurston (see [30], Theorem 3.3): knot groups are residually finite, i.e. for every non-trivial element $x \in \pi_K$ there exists a finite group G and a homomorphism $\phi : \pi_K \to G$ such that $\phi(x) \neq 1$.

Restricted to the class of residually finite groups, the word problem can be solved in a uniform way (see [25] or [24], Theorem 5.3): there exists an algorithm that, given a residually finite group $G = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$ and a word $w \in \langle x_1, \ldots, x_m \rangle$, decides whether or not w vanishes in G. Applied to a Wirtinger presentation of (π_K, l_K), this algorithm thus solves the unknotting problem. Despite its theoretical importance, however, this algorithm is far from being practical.

10.2. **Computing quandle homology.** Theorem 1 says that a knot K is trivial if and only if $H_2(Q_K) = 0$. At first glance this seems to be a step towards a practical algorithm. Alas, the main difficulty resides in the following question:

Question 60. Given a knot quandle Q_K, is there an algorithm for computing $H_2(Q_K)$? If so, what is the complexity of this problem?

In order to estimate the difficulty of this question, let us consider a group-theoretical analogue. For a finitely presented group G, it is easy to compute $H_1(G) = G/[G, G]$. For the second homology group, we quote the well-known theorem of H. Hopf (cf. Brown [2], §II.5): given $G = F/R$, where F is a free group, there is an isomorphism

$$H_2(G) \cong \frac{R \cap [F, F]}{[F, R]}.$$

In particular, if G has a finite presentation P, then $H_2(G)$ is finitely generated. Does this mean that we can effectively compute $H_2(G)$ from P? Quite surprisingly, this is not the case! C. Gordon [16] has shown that there is no algorithm for deciding, given a finite presentation of a group G, whether or not $H_2(G) = 0$.

Returning to our initial Question 60, it is worth emphasizing that knot quandles are a special class of quandles, and their Wirtinger presentations form a special class of presentations. It is in this restricted setting that we are looking for an algorithm to compute $H_2(Q_K)$. As far as I know, the question remains open. A detailed investigation would certainly be desirable.

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