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A SURGERY PROOF OF BING'S THEOREM CHARACTERIZING THE 3-SPHERE

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ABSTRACT

A classical theorem of R.H. Bing states that a closed connected 3-manifold M is homeomorphic to the 3-sphere if and only if every knot in M is contained in a 3-ball. We give a simple proof of this characterization based on the surgery presentation of 3-manifolds.

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The 3-dimensional Poincaré conjecture states that every simply connected closed 3-manifold is homeomorphic to the 3-sphere. As a possible approach, R.H. Bing [1] proved the following knot-theoretical characterization:

Theorem 1 (Bing, 1958). A closed connected 3-manifold M is homeomorphic to the 3-sphere if and only if every knot in M is contained in a 3-ball.

Beside the original proof given by Bing [1], alternative proofs can be found in the textbooks by Hempel [2, Theorem 14.3] and Rolfsen [3, §9E]. The theorem also follows from the existence of an open book decomposition [4]. In addition, Bing's theorem has been generalized in various ways, most notably to a characterization of $\mathbb{R}^3$ among all contractible open 3-manifolds [5]. The 3-dimensional Poincaré conjecture, however, remains unsolved to the present day.

The work of Bing foreshadowed the development of surgery on 3-manifolds, as documented by Bing's question at the end of his article [1] and Lickorish's answer in [6]. The purpose of this note is to give a simple proof of Bing's theorem based on the surgery presentation of 3-manifolds and the Alexander-Schönflies Theorem.

A knot or link in M will be called *local* if it is contained in an open 3-ball.

Lemma 1. A closed connected 3-manifold M is homeomorphic to the 3-sphere if and only if every link in M is local.

Proof. If $M \cong \mathbb{S}^3$, then obviously every link in M is local. Conversely assume that M is a closed connected 3-manifold such that every link in M is local. In particular, M is simply connected and hence orientable. Let M_L denote the 3-manifold obtained from M by doing surgery along the link L, that is, remove a tubular neighbourhood of L and sew it back in according to a framing of L. The surgery theorem of Lickorish [6] and Wallace [7] ensures that there exists L in M such that $M_L \cong \mathbb{S}^3$. On the other hand, surgery of M along the local link L produces a connected sum $M_L \cong M \sharp M'$. From $\mathbb{S}^3 \cong M \sharp M'$ we conclude that $M \cong M' \cong \mathbb{S}^3$ by appealing to the Alexander-Schönflies Theorem [8, 9, 10]. □

The preceding proof uses the seemingly stronger hypothesis that every link in M is local. It thus remains to establish the transition from knots to links:

Lemma 2. If each knot in M is local then so is every link.

Proof. The following argument is parallel to the one given by Rolfsen [3, §9E], where he shows that every 4-valent graph in M is local. We will prove the lemma by induction on the number n of components. For $n = 1$ we are dealing with knots, so there is nothing to prove. Let L be a link with $n \geq 2$ components $K_1, K_2, \dots, K_n$, and suppose that all links with less than n components are local. Let $B \subset M$ be a closed 3-ball such that $B \cap L$ is a trivial 2-string tangle as in Figure 1a. We can tie the components K_1 and K_2 together by replacing the trivial tangle T by the tangle U shown in Figure 1b.

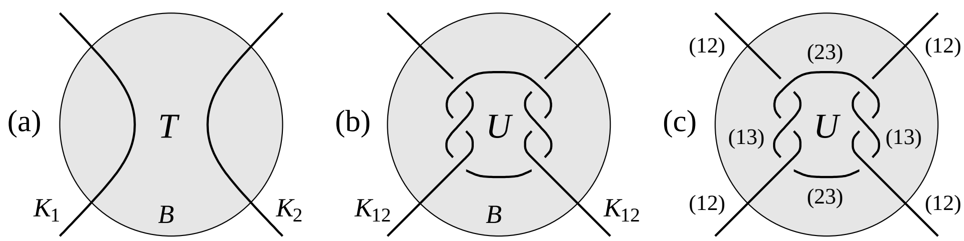

Figure 1: (a), (b) Tying two components together in order to form a knot.
(c) A 3-colouring showing that the tangle U is unsplittable.

The tangle U has been so chosen as to be *unsplittable*, which means that the two strings cannot be separated by a properly embedded disk $D \subset B$. To see this, first note that each string is unknotted. If the two could be separated by a disk, then the pair (B, U) would be homeomorphic to (B, T). But the complements $X = B \smallsetminus T$ and $Y = B \smallsetminus U$ are non-homeomorphic: the 3-colouring displayed in Figure 1c defines a surjective homomorphism of $\pi_1(Y)$ onto the symmetric group S_3 while $\pi_1(\partial Y)$ is mapped onto S_2. This is clearly impossible for X. We conclude that the tangle U is unsplittable, and every properly embedded disk $D \subset Y$ is parallel to the boundary ∂Y.

Having replaced T by U, the resulting link $L^* = K_{12} \cup K_3 \cup \ldots \cup K_n$ has one component less than L, and by induction L^* is contained in the interior of a closed 3-ball $B^* \subset M$. We can assume that the boundary ∂B^* is transverse to ∂B and the number of intersection curves is minimal. Since U is unsplittable, we must have $\partial B^* \cap \partial B = \emptyset$, whence $B \subset B^*$. (This argument is detailed in Rolfsen [3, §9E].) Finally L^* and B both lie in B^*, so we can untie K_{12} to reconstruct K_1 and K_2 within B^*. We conclude that L, too, lies in B^*, which completes the proof. $\square$

Remark. There are many ways to show that the tangle U is unsplittable. The following geometric argument was communicated to me by W.B.R. Lickorish, and I would like to include it for its elegance: If (B, U) were homeomorphic to the trivial tangle (B, T), then gluing them together along their boundaries could only produce 2-bridge knots. The obvious gluing, however, produces a connected sum of two trefoils, which is a 3-bridge knot.

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References

[1] R.H. Bing, *Necessary and sufficient conditions that a* 3*-manifold be* S^3. Ann. of Math. (2) 68 (1958) 17–37.

[2] J. Hempel, 3*-Manifolds.* Princeton University Press, Princeton, N. J., 1976. Ann. of Math. Studies, No. 86.

[3] D. Rolfsen, *Knots and links.* Publish or Perish Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7.

[4] R. Myers, *Open book decompositions of* 3*-manifolds.* Proc. Amer. Math. Soc. 72 (1978) 397–402.

[5] O.L. Costich, P.H. Doyle, and D.E. Galewski, *A characterisation of punctured open* 3*-cells.* Proc. Amer. Math. Soc. 28 (1971) 295–298.

[6] W.B.R. Lickorish, *A representation of orientable combinatorial* 3*-manifolds.* Ann. of Math. (2) 76 (1962) 531–540.

[7] A.H. Wallace, *Modifications and cobounding manifolds.* Canad. J. Math. 12 (1960) 503–528.

[8] J.W. Alexander, *On the subdivision of* 3*-space by a polyhedron.* Proc. Nat. Acad. Sci. U.S.A. 10 (1924) 6–8.

[9] E.E. Moise, *Geometric topology in dimensions* 2 *and* 3. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, Vol. 47.

[10] W.B.R. Lickorish, *The irreducibility of the* 3*-sphere.* Michigan Math. J. 36 (1989) 345–349.