Universität Stuttgart

Neuigkeiten über Lie-Algebren

Kolloquium Tübingen

17. Juli 2017

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The greatest mathematical paper of all time

A. J. Coleman, The Mathematical Intelligencer **11** (1989), 29–38

Euclid's *Elements*, Newton's *Principia*, . . . and in the past 200 years:

Wilhelm Killing (1847–1923), *Die Zusammensetzung der stetigen, endlichen Transformationsgruppen II*, Mathematische Annalen **33** (1888), 1–48.

MathSciNet Review MR1007036 by Jean Dieudonné:

"Many mathematicians will disagree, [...]. One may however observe that:

(1) nobody had tackled the problem before Killing;

(2) he solved it by methods he invented;

(3) nobody collaborated with him for that solution;

(4) the result became a most important milestone in modern mathematics.

I think it is not so easy to find papers exhibiting all those features"

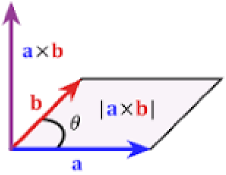
Lie algebra = infinitesimal version of a transformation group

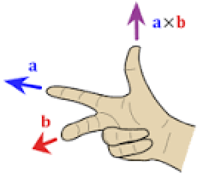
A vector space L equipped with a product $x \cdot y$ such that

- $x \cdot x = 0$,
- $x \cdot (y \cdot z) + y \cdot (z \cdot x) + z \cdot (x \cdot y) = 0$ (Jacobi identity).

Usually write $x \cdot y$ as $[x, y]$ (Lie bracket).

Example. Vector product in $L = \mathbb{R}^3$:





Example. Matrices: $L = M_n(\mathbb{C})$ with bracket $[A, B] = A \cdot B - B \cdot A$; denote $\mathfrak{gl}_n(\mathbb{C})$.
Subalgebra $\mathfrak{sl}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \mathrm{Trace}(A) = 0\}$.

An infinite-dimensional example

Start with $R = \mathbb{C}[X, X^{-1}]$ (Laurent polynomials).

A linear map $D\colon R \to R$ is called a derivation if

$$D(f \cdot g) = f \cdot D(g) + D(f) \cdot g \qquad \text{for all } f, g \in R.$$

Example: $D =$ usual (formal) derivative with respect to X.

$$D(X^n) = nX^{n-1}, \qquad D(X^{-1}) = -X^{-2}, \qquad \text{etc.}$$

Let $L =$ vector space of all derivations of R.

Exercise: If $D, D' \in L$, then $[D, D'] = D \circ D' - D' \circ D \in L$. So L is a Lie algebra.

For $m \in \mathbb{Z}$, we set $L_m(f) = -X^{m+1} \cdot D(f)$ for $f \in R$. Then:

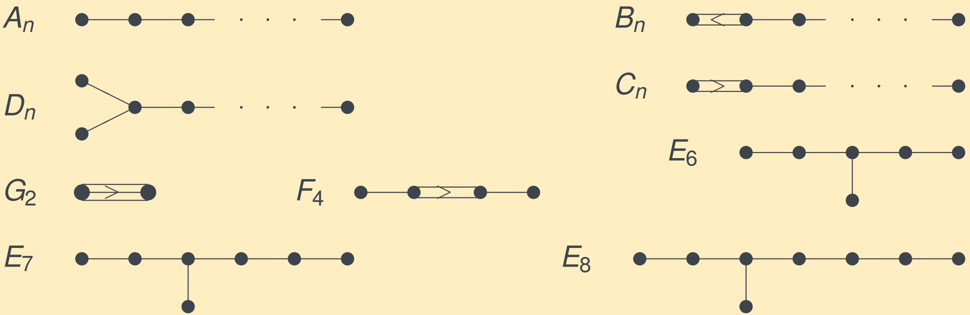
- $\{L_m \mid m \in \mathbb{Z}\}$ is a vector space basis of L.
- $[L_m, L_n] = (m - n)L_{m+n}$ for all $m, n \in \mathbb{Z}$.

L is called “Witt algebra” $\rightsquigarrow$ important in mathematical physics.

A subspace $U \subseteq L$ is called an ideal if $[u, x] \in U$ and $[x, u] \in U$ for all $x \in L$, $u \in U$. In this case, $\bar{L} = L/U$ also is a Lie algebra. So L “built up” from U and $\bar{L}$.

Definition. L is called simple if $L \neq \{0\}$, the bracket is not identically zero and there is no proper ideal in L.

Cartan–Killing (∼1890): The finite-dimensional simple Lie algebras over $\mathbb{C}$ are classified by “Dynkin diagrams”.



Infinite families: Lie algebras of matrices

$$A_n \leftrightarrow \mathfrak{sl}_{n+1}(\mathbb{C}), \quad B_n \leftrightarrow \mathfrak{so}_{2n+1}(\mathbb{C}), \quad C_n \leftrightarrow \mathfrak{sp}_{2n}(\mathbb{C}), \quad D_n \leftrightarrow \mathfrak{so}_{2n}(\mathbb{C}).$$

Exceptional algebras:

$$\dim \mathfrak{g}_2 = 14, \quad \dim \mathfrak{f}_4 = 52, \quad \dim \mathfrak{e}_6 = 78, \quad \dim \mathfrak{e}_7 = 133, \quad \dim \mathfrak{e}_8 = 248.$$

S. GARIBALDI, E_8, the most exceptional group. Bull. AMS **53** (2016), 643–671.

“The Lie algebra $\mathfrak{e}_8$ or Lie group E_8 was first sighted by a human being sometime in summer or early fall of 1887, by Wilhelm Killing as part of his program to classify the semisimple finite-dimensional Lie algebras over the complex numbers. [...]

Since then, it has been a source of fascination for mathematicians and others in its role as the largest of the exceptional Lie algebras. [...]

If we know some statement for all groups except E_8, then we do not really know it.”

How does a Dynkin diagram determine a simple Lie algebra ?

If diagram has nodes $1, \ldots, n$, then form "Cartan matrix" $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ where:

- $a_{ii} = 2$ for all i; $a_{ij} = 0$ if $i \neq j$ are not joined by an edge;
- if $i \neq j$ are joined by a single edge, set $a_{ij} = a_{ji} = -1$;
- if $i \neq j$ are joined by $m \geqslant 2$ edges (arrow towards j), set $a_{ij} = -1$, $a_{ji} = -m$.

B_3 

$\rightsquigarrow$

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Chevalley generators ("épinglage") of corresponding Lie algebra L:

- $L = \langle e_i, f_i, h_i \mid 1 \leqslant i \leqslant n \rangle_{\text{Lie}}$,
- $[e_i, f_i] = h_i$, $[e_i, f_j] = 0$ (if $i \neq j$), $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ji} f_j$.
- $H := \langle h_1, \ldots, h_n \rangle_{\mathbb{C}}$ "Cartan" subalgebra of L.

This determines L up to isomorphism.

Vector space basis of L

Consider $\mathbb{R}^n$ with standard basis $\{\alpha_1, \ldots, \alpha_n\}$. Define

$$s_i \colon \mathbb{R}^n \to \mathbb{R}^n, \qquad \alpha_j \mapsto \alpha_j - a_{ij}\alpha_i.$$

Then $s_i^2 = \mathrm{id}_n$ and $W = \langle s_1, \ldots, s_n \rangle \subseteq \mathrm{GL}_n(\mathbb{R})$ is a finite reflection group.

$\Phi := \{w(\alpha_i) \mid w \in W, 1 \leqslant i \leqslant n\}$ is the abstract "root system" associated with A.
Corresponding Lie algebra L has basis $\{h_1, \ldots, h_n\} \cup \{e_\alpha \mid \alpha \in \Phi\}$.

"Root strings" (Killing): Let $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$.
Let $p = p_{\alpha,\beta} \geqslant 0$ and $q = q_{\alpha,\beta} \geqslant 0$ be maximal such that

$$\beta - q\alpha, \quad \ldots, \quad \beta - \alpha, \quad \beta, \quad \beta + \alpha, \quad \ldots, \quad \beta + p\alpha \in \Phi.$$

C. Chevalley (1955): e_α can be chosen such that $e_{\alpha_i} = \pm e_i$, $e_{-\alpha_i} = \pm f_i$ and

$$[e_\alpha, e_\beta] = \pm(q_{\alpha,\beta} + 1)e_{\alpha+\beta} \quad \text{whenever } \alpha, \beta, \alpha + \beta \in \Phi$$

(and with this normalization, the e_α are unique up to sign).

An aside: root systems and lattices

Let $\Gamma :=$ all $\mathbb{Z}$-linear combinations of roots $\alpha \in \Phi$. Then Γ is a "lattice" in $\mathbb{R}^n$.
Conversely, let $\Gamma \subseteq \mathbb{R}^n$ be a lattice.

- Γ is integral if $(x, y) \in \mathbb{Z}$ for $x, y \in \Gamma$; then Γ is even if $(x, x) \in 2\mathbb{Z}$ for $x \in \Gamma$.
- Dual lattice $\Gamma^* := \{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in \Gamma\}$; if Γ integral, then $\Gamma \subseteq \Gamma^*$.
- Γ is unimodular if Γ is integral and $\Gamma = \Gamma^*$.

Up to isomorphism, there is only one even unimodular lattice in $\mathbb{R}^8$.
(History goes back to around 1870, before Lie and Killing.) This is given by

$$\Gamma_8 := \Big\{(x_i) \in \mathbb{R}^8 \;\Big|\; 2x_i \in \mathbb{Z},\ \ x_i - x_j \in \mathbb{Z},\ \ \textstyle\sum_i x_i \in 2\mathbb{Z}\Big\}.$$

The set $R := \{x \in \Gamma_8 \mid (x, x) = 2\}$ contains precisely 240 vectors;
these form the "root system" associated with the Dynkin diagram E_8.

(A further famous example in dimension 24: the Leech lattice $\rightsquigarrow$ Conway's finite simple groups.
See, e.g., J-P. Serre, A Course in Arithmetic, or W. Ebeling, Lattices and Codes, Springer-Verlag.)

C. Chevalley ($\sim$1955/1960): Algebraic analogues of Lie groups.

$$\text{Simple Lie algebra } L \quad + \quad \text{field } k \quad \rightsquigarrow \quad \text{group } G_L(k).$$

(Mimic exponential "$\exp(e_\alpha)$" over arbitrary field k; then $G_L(k) = \langle \text{“}\exp(te_\alpha)\text{”} \mid \alpha \in \Phi, t \in k \rangle$.)

- k algebraically closed: $G_L(k)$ simple algebraic group over k,

  $\mathrm{SL}_n(k), \quad \mathrm{Sp}_{2n}(k), \quad \mathrm{SO}_n(k), \quad \mathrm{E}_8(k), \quad \ldots$
- k finite $\rightsquigarrow$ new families (at the time) of finite simple groups,

  e.g., $|E_8(\mathbb{F}_q)| = q^{248} +$ lower powers of q (where q is a prime power).

Textbook references:

- Bourbaki, Groupes et algèbres de Lie, 1968/1975;
- Steinberg, Lectures on Chevalley groups, 1967/68 (now available from AMS);
- Humphreys, Introduction to Lie algebras and representation theory, 1972;
- Carter, Simple groups of Lie type, 1972;
- Erdmann–Wildon, Introduction to Lie algebras, 2006.

So what are the news about Lie algebras ?

GEORGE LUSZTIG,
arXiv:1309.1382 (3 pages):

“The Lie group of type E_8 can be obtained from the graph E_8 by a method of Chevalley (1955),

simplified using theory of canonical basis (1990).”

$\rightsquigarrow$ 3 more papers:
one by Lusztig, two by myself.

Why look for a simplification ?

- Construction of L is an issue, especially for exceptional types.
- Chevalley's construction of $G_L(k)$ relies on choice of e_α's.

J. Tits 1966: Start with vector space M of correct dimension, fix basis $\{h_1, \ldots, h_n\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ and try to define Lie bracket for basis elements. Relies on systematic study of sign choices in Chevalley's normalisaztion of e_α.

J-P. Serre 1966: Start with free Lie algebra generators $\{e_i, f_i \mid 1 \leqslant i \leqslant n\}$ and add defining relations (nowadays called "Serre relations") to obtain a presentation. Very elegant, does not resolve issue of making choices for the e_α's.

C.-M. Ringel 1990: Fix orientation of Dynkin diagram and use the representation theory of quivers and Hall polynomials. Then e_α's correspond to well-defined indecomposable objects in certain category of modules.
Pre-cursor of "canonical bases", still relies on 2^{n-1} choices of orientations.

The simplified construction

Recall "root strings": Let $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$. Then

$$p_{\alpha,\beta} := \max\{i \geqslant 0 \mid \beta + i\alpha \in \Phi\} \quad \text{and} \quad q_{\alpha,\beta} := \max\{j \geqslant 0 \mid \beta - j\alpha \in \Phi\}.$$

Let M be a vector space over $\mathbb{C}$ of the "correct" dimension, with a given basis $\{u_1, \ldots, u_n\} \cup \{v_\alpha \mid \alpha \in \Phi\}$. Define linear maps $e_i, f_i \colon M \to M$ by

$$e_i(u_j) := |a_{ji}| v_{\alpha_i}, \qquad e_i(v_\alpha) := \begin{cases} (q_{\alpha_i,\alpha} + 1) v_{\alpha+\alpha_i} & \text{if } \alpha + \alpha_i \in \Phi, \\ u_i & \text{if } \alpha = -\alpha_i, \\ 0 & \text{otherwise;} \end{cases}$$

$$f_i(u_j) := |a_{ji}| v_{-\alpha_i}, \qquad f_i(v_\alpha) := \begin{cases} (p_{\alpha_i,\alpha} + 1) v_{\alpha-\alpha_i} & \text{if } \alpha - \alpha_i \in \Phi, \\ u_i & \text{if } \alpha = \alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

(Note: Matrices of e_i, f_i have entries in $\mathbb{Z}_{\geqslant 0}$ — in fact, in $\{0, 1, 2, 3\}$.)

Theorem. Lusztig (1990) $+$ G. (Proc. Amer. Soc., 2017)

Consider the Lie algebra $\mathfrak{gl}(M)$ and let $L := \langle e_i, f_i \mid 1 \leqslant i \leqslant n \rangle_{\text{Lie}} \subseteq \mathfrak{gl}(M)$. Then L is a simple Lie algebra corresponding to the given Dynkin diagram.

Example: Type G_2 with Cartan matrix $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

- $\mathbb{R}^2$ with standard basis $\{\alpha_1, \alpha_2\}$. Matrices $s_1, s_2 \in \mathrm{GL}_2(\mathbb{R})$ given by

$$s_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}.$$

- Applying s_1, s_2 repeatedly to $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$, we obtain

$$\Phi = \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(1,2), \pm(1,3), \pm(2,3)\}.$$

- Simple Lie algebra $L = \langle e_1, e_2, f_1, f_2 \rangle_{\mathrm{Lie}} \subseteq \mathfrak{gl}_{14}(\mathbb{C})$. For example:

$$e_1 = \begin{pmatrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

e_1, e_2	upper triangular
f_1, f_2	lower triangular

(A dot stands for 0)

Resolving sign issue in Chevalley's basis

We can transport the basis $\{u_1, \ldots, u_n\} \cup \{v_\alpha \mid \alpha \in \Phi\}$ from M to L and obtain:

Corollary.

- There is a basis $\{\tilde{h}_1, \ldots, \tilde{h}_n\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ of L such that:

$$[e_i, \tilde{h}_j] = |a_{ji}| e_{\alpha_i}, \qquad [e_i, e_\alpha] = \begin{cases} (q_{\alpha_i,\alpha} + 1) e_{\alpha+\alpha_i} & \text{if } \alpha + \alpha_i \in \Phi, \\ \tilde{h}_i & \text{if } \alpha = -\alpha_i, \\ 0 & \text{otherwise;} \end{cases}$$

$$[f_i, \tilde{h}_j] = |a_{ji}| e_{-\alpha_i}, \qquad [f_i, e_\alpha] = \begin{cases} (p_{\alpha_i,\alpha} + 1) e_{\alpha-\alpha_i} & \text{if } \alpha - \alpha_i \in \Phi, \\ \tilde{h}_i & \text{if } \alpha = \alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

- This basis of L is unique up to a global constant.
- In particular, the structure constants in the equations $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$ (for $\alpha, \beta, \alpha + \beta \in \Phi$) are uniquely determined up to a global constant.

Where do the formulae for the action of e_i, f_i come from?

- $\approx$ 1985: Introduction of quantised enveloping algebras of Lie algebras (Drinfeld, Jimbo), as Hopf algebra deformations depending on a parameter v. Origins in mathematical physics, quantum integrable systems, ...
- Early 1990's: Discovery of "canonical bases" (Lusztig) and "crystal bases" (Kashiwara). Specialisation $v \mapsto 1$ gives rise to canonical bases in irreducible representations of ordinary simple Lie algebras.

LUSZTIG (1990 + J. Comb. Algebra 2017). Consider $L \hookrightarrow \mathfrak{gl}(M)$, as above. This is an irreducible representation and $\{u_1, \ldots, u_n\} \cup \{v_\alpha \mid \alpha \in \Phi\}$ is the "canonical basis" of M in the above sense. ($\rightsquigarrow$ Explains why formulae are "natural".)

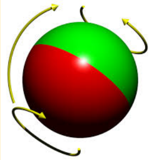
Explicit/simple construction of L and, hence, also of $G_L(k)$, useful for:

- algorithmic problems: nilpotent orbits, matrix group recognition project, ...;
- teaching courses on Lie algebras ("existence theorem").

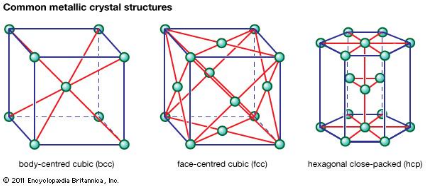
My motivation for studying Lie algebras and Chevalley groups

Group theory $=$ study of symmetries

Continuous $\rightsquigarrow$ Lie groups



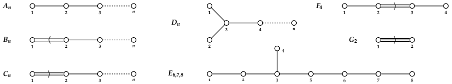
Discrete $\rightsquigarrow$ finite groups



- "Atoms" of symmetry: finite **simple** groups.
- A highlight of 20th century mathematics: **Classification.**
  (first announced 1981 • completed 2004: Aschbacher, Smith • 12000 pages of proof)
- 2nd generation proof: D. Gorenstein, R. Lyons, R. Solomon, Math. Survey and Monographs, Amer. Math. Soc., 1994 –?? (currently 6 volumes).

The Periodic Table Of Finite Simple Groups

Dynkin Diagrams of Simple Lie Algebras



$0, C_1, \mathbb{Z}_1$ 1 1																	C_2 2
$A_1(4), A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											$^2A_3(4)$ $B_2(3)$ 25 920	$C_3(3)$ 4 585 351 680	$D_4(2)$ 174 182 400	$^2D_4(2^2)$ 197 406 720	$G_2(2)'$ $^2A_2(9)$ 6 048	C_3 3
$A_1(9), B_2(2)'$ A_6 360	$^2G_2(3)'$ $A_1(8)$ 504											$B_2(4)$ 979 200	$C_3(5)$ 228 501 000 000 000	$D_4(3)$ 4 952 179 814 400	$^2D_4(3^2)$ 10 151 968 619 520	$^2A_2(16)$ 62 400	C_5 5
A_7 2 520	$A_1(11)$ 660	$E_6(2)$ 214 841 575 522 005 575 270 400	$E_7(2)$ 7 997 476 042 075 799 759 100 487 262 680 802 918 400	$E_8(2)$ 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	$F_4(2)$ 3 311 126 603 366 400	$G_2(3)$ 4 245 696	$^3D_4(2^3)$ 211 341 312	$^2E_6(2^2)$ 76 532 479 683 774 853 939 200	$^2B_2(2^3)$ 29 120	Tits* $^2F_4(2)'$ 17 971 200	$^2G_2(3^3)$ 10 073 444 472	$B_3(2)$ 1 451 520	$C_4(3)$ 65 784 756 654 489 600	$D_5(2)$ 23 499 295 948 800	$^2D_5(2^2)$ 25 015 379 558 400	$^2A_2(25)$ 126 000	C_7 7
$A_3(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ 7 257 703 347 541 463 210 028 258 395 214 643 200	$E_7(3)$ 1 271 375 236 818 136 742 240 479 751 139 021 644 554 379 203 770 766 254 617 395 200	$E_8(3)$ [illegible]	$F_4(3)$ 5 734 420 792 816 671 844 761 600	$G_2(4)$ 251 596 800	$^3D_4(3^3)$ 20 560 831 566 912	$^2E_6(3^2)$ 14 636 855 916 969 695 633 965 120 680 532 377 600	$^2B_2(2^5)$ 32 537 600	$^2F_4(2^3)$ 264 905 352 699 586 176 614 400	$^2G_2(3^5)$ 49 825 657 439 340 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 604 953 600	$D_4(5)$ 8 911 539 000 000 000 000	$^2D_4(4^2)$ 67 536 471 195 648 000	$^2A_3(9)$ 3 265 920	C_{11} 11
A_9 181 440	$A_1(17)$ 2 448	$E_6(4)$ 85 528 710 781 342 640 103 833 619 055 142 765 466 746 880 000	$E_7(4)$ 111 131 458 114 940 385 379 597 233 477 884 941 280 664 199 527 155 056 307 251 745 263 504 588 800 000 000	$E_8(4)$ [illegible]	$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000	$G_2(5)$ 5 859 000 000	$^3D_4(4^3)$ 67 802 350 642 790 400	$^2E_6(4^2)$ 85 696 576 147 617 709 485 896 772 387 584 983 695 360 000 000	$^2B_2(2^7)$ 34 093 383 680	$^2F_4(2^5)$ 1 318 633 155 799 591 447 702 161 609 782 722 560 000	$^2G_2(3^7)$ 239 189 910 264 352 349 332 632	$B_2(7)$ 138 297 600	$C_3(9)$ 54 025 731 402 499 584 000	$D_5(3)$ 1 289 512 799 941 305 139 200	$^2D_4(5^2)$ 17 880 203 250 000 000 000	$^2A_2(64)$ 5 515 776	C_{13} 13
A_n $\frac{n!}{2}$	$PSL_{n+1}(q), L_{n+1}(q)$ $A_n(q)$ $\frac{q^{n(n+1)/2}}{(n+1,q-1)}\prod_{i=1}^{n}(q^{i+1}-1)$	$E_6(q)$ $\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	$E_7(q)$ $\frac{q^{63}}{(2,q-1)}\prod_{i=1, i\neq 3,5}^{9}(q^{2i}-1)$	$E_8(q)$ $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	$F_4(q)$ $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	$G_2(q)$ $q^6(q^6-1)(q^2-1)$	$^3D_4(q^3)$ $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	$^2E_6(q^2)$ $\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	$^2B_2(2^{2n+1})$ $q^2(q^2+1)(q-1)$	$^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q), \Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O^+_{2n}(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O^-_{2n}(q)$ $^2D_n(q^2)$ $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$PSU_{n+1}(q)$ $^2A_n(q^2)$ $\frac{q^{n(n+1)/2}}{(n+1,q+1)}\prod_{i=2}^{n+1}(q^i-(-1)^i)$	$\mathbb{Z}_p$ C_p p

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Ree Groups and Tits Group*
- Sporadic Groups
- Cyclic Groups

Alternates†
Symbol
Order‡

M_{11} 7 920	M_{12} 95 040	M_{22} 443 520	M_{23} 10 200 960	M_{24} 244 823 040	$J(1), J(11)$ J_1 175 560	HJ J_2 604 800	HJM J_3 50 232 960	J_4 86 775 571 046 077 562 880	HS 44 352 000	McL 898 128 000	F_7, HHM, HTH He 4 030 387 200	Ru 145 926 144 000
Sz Suz 448 345 497 600	$O'NS, O–S$ $O'N$ 460 815 505 920	$\cdot 3$ Co_3 495 766 656 000	$\cdot 2$ Co_2 42 305 421 312 000	$\cdot 1$ Co_1 4 157 776 806 543 360 000	F_5, D HN 273 030 912 000 000	LyS Ly 51 765 179 004 000 000	F_3, E Th 90 745 943 887 872 000	$M(22)$ Fi_{22} 64 561 751 654 400	$M(23)$ Fi_{23} 4 089 470 473 293 004 800	$F_{3+}, M(24)'$ Fi'_{24} 1 255 205 709 190 661 721 292 800	F_2 B 4 154 781 481 226 426 191 177 580 544 000 000	F_1, M_1 M 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

*The Tits group $^2F_4(2)'$ is not a group of Lie type, but is the (index 2) commutator subgroup of $^2F_4(2)$. It is usually given honorary Lie type status.

The groups starting on the second row are the classical groups. The sporadic suzuki group is unrelated to the families of Suzuki groups.

†For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate isomorphims. All such isomorphisms appear on the table except the family $B_n(2^m) \cong C_n(2^m)$.

‡Finite simple groups are determined by their order with the following exceptions:
$B_n(q)$ and $C_n(q)$ for q odd, $n > 2$;
$A_8 \cong A_3(2)$ and $A_2(4)$ of order 20160.

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