Two applications of parity sheaves

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In this talk I will discuss two applications of parity sheaves, one topological, and one representation theoretic.

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One similarly defines $\mathbb{Q}$- and $\mathbb{Z}$-smooth.

If X is p-smooth then one has Poincaré duality with $\mathbb{F}_p$-coefficients:

$$H^\bullet(X;\mathbb{F}_p) \cong H_c^{2n-\bullet}(X;\mathbb{F}_p)^*.$$

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In general one has inclusions:

smooth locus	$\subset$	$\mathbb{Z}$-smooth locus	$\subset$	p-smooth locus	$\subset$	rationally smooth locus

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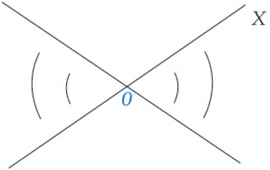

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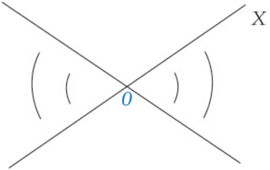

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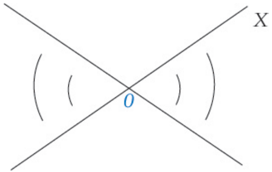

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Why?

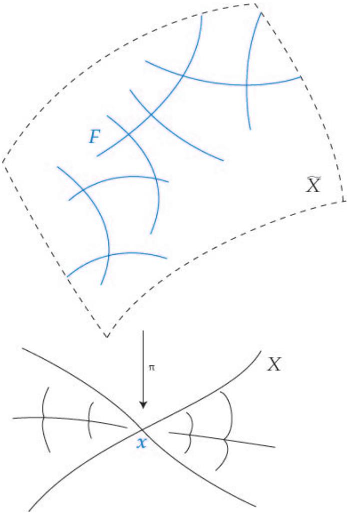
$$\begin{aligned} \text{link of } X \text{ at } 0 &:= X \cap \text{small sphere around } 0 \\ &\cong S^3/\pm 1 \\ &\cong \mathbb{RP}^3 \end{aligned}$$

which has 2-torsion in its cohomology.

An (exceptional) surface singularity

Let X be a surface singularity of type E_8.

That is, $X = \mathbb{C}^2/H$, where $H \subset SL_2(\mathbb{C})$ is the binary icosahedral group.



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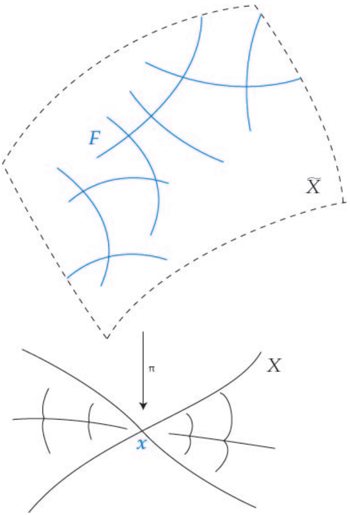
Then X is $\mathbb{Z}$-smooth, and hence p-smooth for all p.



Schubert varieties

Let G be a reductive algebraic group with Borel subgroup B and maximal torus T and Weyl group W.

We can consider the flag variety G/B. We will abuse notation and identify points of W with their images in G/B.

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A natural question is:

Question

What is the p-smooth locus of Schubert varieties?

The question of smoothness and rational smoothness have been addressed by many authors.

For smoothness these include answers in classical types usually by some form of "pattern avoidance".

For rational smoothness one has:

$x \leqslant w$ belongs to the rationally smooth locus	$\Leftrightarrow P_{x,w} = 1 \Leftrightarrow$	for all $x \leqslant y \leqslant w$, there are $\ell(w)$ closed T-invariant curves in X_w containing y in their closure

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It is known that in simply laced type the rationally smooth locus and smooth locus agree.

In the exercises we have seen an example (in SP_4/B) where the 2-smooth locus is not equal to the rationally smooth locus. (In fact we saw that we found an A_1 singularity.)

Hence one cannot expect equality of the rationally smooth locus and p-smooth locus in general.

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Theorem

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This answers a (stronger version of a) question of Soergel.

Similar results hold for Kac-Moody Schubert varieties X if the moment graph satisfies the Goresky-Kottwitz-MacPherson (GKM) condition: for any T-fixed point $x \in X$ the characters of T corresponding to any two one-dimensional orbits L and L' having x in their closure do not become linear dependent modulo p.

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Using the self-duality of $\mathcal{E}(w, \mathbb{F}_p)$ one may show that the p-smooth locus of X_w is equal to the locus U over which $\mathcal{E}(w, \mathbb{F}_p)$ is isomorphic to (a suitable shift of) the constant sheaf.

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In Olaf's talk we saw that that we can calculate the stalks of the the indecomposable parity sheaves using the moment graph of the flag variety.

Hence the determination of the p-smooth locus is reduced to a problem about the Braden-MacPherson sheaf on the moment graph.

To finish off, we use a result of Fiebig:

If the moment graph of X_w is Goresky-Kottwitz-MacPherson for k, then the stalk of $B(w,k)$ at x is of rank one if and only if there are only $\dim X_w$ edges at x.

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2. Is there a more geometrical explanation of these results?
3. What about torsion in general? (Compare results of Braden.)

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Recall that G denotes a reductive algebraic group over $\mathbb{C}$ with fixed maximal torus and Borel subgroup $T \subset B$.

To G one may associate its affine Grassmannian:

$$\mathcal{G}r_G := G((t))/G[[t]]$$

$\mathcal{G}r_G$ is a projective ind-variety. That is, it is the direct limit of finite dimensional projective algebraic varieties under closed inclusions.

Basic geometry of the affine Grassmannian

We have a decomposition of $\mathcal{G}r_G$ into $G[[t]]$-orbits:

$$\mathcal{G}r_G = \bigsqcup_{\lambda \in X_*(T)^+} \mathcal{G}r_\lambda$$

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(Note that $X_*(T)$ also parametrises the simple $G^\vee$-modules, where $G^\vee$ denotes the Langlands dual group of G.)

Let $\mathbf{P}_{G[[t]]}(\mathcal{G}r, k)$ denote the category of $G[[t]]$-equivariant perverse sheaves of k-vector spaces on $\mathcal{G}r$. Then $\mathbf{P}_{G[[t]]}(\mathcal{G}r, k)$ has a convolution product $*$.

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Theorem (Mirkovic-Vilonen)

There is an equivalence of tensor categories

$$(\mathbf{P}_{G[[t]]}(\mathcal{G}r, k), *) \xrightarrow{\sim} (\mathrm{Rep}(G_k^\vee), \otimes)$$

with fiber functor given by the hypercohomology functor.

Here $G_k^\vee$ denotes the Langlands dual group scheme over k.

Parity sheaves on the afffine Grassmannian

Because the affine Grassmannian is an example of a Kac-Moody flag variety, similar arguments to the finite dimensional case guarantee that, given $\lambda \in X^+(\lambda)$ there is a unique (up to isomorphism) indecomposable parity sheaf $\mathcal{E}(\lambda, k)$ on $\mathcal{G}r_G$ with support $\overline{\mathcal{G}r_\lambda}$.

It is interesting to ask if $\mathcal{E}(\lambda, k)$ is perverse and, if so, what is corresponds to under the geometric Satake correspondence.

Assume from now on that k is the algebraic closure of a finite field $\mathbb{F}_q$.

Recall that $G_k^\vee$ denotes a reductive algebraic group over k. Let $T^\vee$ denote a maximal torus dual to $T \subset G$.

The simple $G^\vee$-modules are classified by their highest weight $\lambda \in X^*(T^\vee)^+$. Given $\lambda \in X^*(T^\vee)^+$ let $L(\lambda)$ denote the corresponding simple module.

Given $\lambda \in X^*(T^\vee)$ one also has standard and costandard modules $\Delta(\lambda)$ and $\nabla(\lambda)$.

Tiltings modules

Definition

A tilting module is a $G_k^\vee$-module which has both a Δ and a ∇ filtration.

General arguments show indecomposable tilting modules are also classified by highest weight. Given any $\lambda \in X^*(T^\vee)$ we denote by $T(\lambda)$ the corresponding indecomposable tilting module.

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It is an open problem to determine the characters of indecomposable tilting modules, and a solution to this problem would have many consequences in representation theory (e.g. formulas for dimensions of simple kS_n-modules).

Recall the geometric Satake isomorphism:

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and let h denote the Coxeter number of G:

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It follows that the Braden-MacPherson algorithm may be used to calculate the characters of tilting modules.

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Under the equivalence of Mirkovic and Vilonen, the 0^{th} perverse cohomology of an indecomposable parity sheaf corresponds to an indecomposable tilting module:

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Because parity sheaves are preserved under convolution, and convolution is p-exact, this would give a geometric explanation of the algebraic fact (proved by Donkin and Matthieu) that a tensor product of two tilting modules is tilting.

Let $\mathcal{H}$ denote the spherical Hecke algebra of G:

$$\mathcal{H} = \bigoplus_{\lambda \in X_*(T)^+} \mathbb{Z}[v^{\pm 1}]\widetilde{H}_\lambda$$

Specialisation $v \mapsto 1$ yields a ring homomorphism

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q-characters of spherical perverse sheaves

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Under this homomorphism the Kazhdan-Lusztig basis C_λ maps to the character of the standard module $\Delta(\lambda)$.

One can define a character map

$$\mathrm{ch} : \mathbf{P}_{G[[t]]}(\mathcal{G}r_G) \to \mathcal{H}$$

by

$$\mathrm{ch}(\mathcal{F}) = \sum_{\lambda \in X^+(T)^+} v^{-\dim \mathcal{G}r_\lambda} \dim_v H^\bullet(\mathcal{F}_\lambda) \widetilde{H}_\lambda$$

where $\dim_v H = \sum \dim H^i v^i \in \mathbb{Z}[v^{\pm 1}]$.

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For example, in characteristic zero Lusztig showed that

$$\mathrm{ch}(\mathbf{IC}(\lambda)) = C_\lambda.$$

If we fix an $\mathbb{F}_p$-rational structure on our algebraic group $G^\vee$ we get a Frobenius morphism

$$F : G_k^\vee \to G_k^\vee$$

(For example, if G is embedded in some GL_n with the standard rational structure, then F is just elevation of each matrix entry to the p^{th} power.)

… and now the twist

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Precomposing by F yields a functor of "Frobenius twist":

$$F^* : \mathrm{Rep}\, G_k^\vee \to \mathrm{Rep}\, G_k^\vee$$

On characters F^* induces the "dilation by p" map.

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We can however conjecture what it should do to the q-characters of parity sheaves.

Tilting tensor product formula

To motivate this we recall the tilting tensor product formula. Suppose that we can express λ as a sum

$$\lambda = \mu + p\xi$$

such that $p - 1 \leqslant \langle \mu, \alpha^\vee \rangle < 2p - 1$ for all simple roots α.

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A result of Donkin says that if $p > 2h - 1$ then we have an isomorphism

$$T(\lambda) \cong T(\mu) \otimes T(\xi)^F$$

where $(-)^F$ denotes Frobenius twist. (It is conjectured by Donkin that the restrictions on p can be dropped.)

Consider the "p-stretch" map:

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We conjecture that the tilting tensor product formula lifts to q-characters as follows:

Conjecture (Tilting, stretching and twisting)

$$\operatorname{ch} \mathcal{E}(\lambda) = \operatorname{ch} \mathcal{E}(\mu) * F \operatorname{ch} \mathcal{E}(\xi).$$

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Hence whatever $(-)^F$ is, it is certainly a strange one!