An Introduction to Derived and Triangulated Categories

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The Category of Complexes

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Definition

Let $\mathscr{A}$ be an Abelian category. The category $\mathbf{Kom}(\mathscr{A})$ *has*

- *objects: chain complexes of objects of $\mathscr{A}$*
- *morphisms: chain complex maps.*

Quasi-Isomorphisms

Definition

Let $f^\bullet$ be a morphism in $\mathbf{Kom}(\mathscr{A})$ between two complexes $A^\bullet$ and $B^\bullet$. Then $f^\bullet$ is a **quasi-isomorphism** *or* **quis** *if $H^*(f^\bullet)$ is an isomorphism.*

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$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d"] \arrow[dd, "f^{n-1}"] & A^n \arrow[r] \arrow[dd, "f^n"] & \cdots & & \cdots & H^{n-1}(A^\bullet) \arrow[dd, "H^{n-1}(f^\bullet)"] & H^n(A^\bullet) \arrow[dd, "H^{n-1}(f^\bullet)"] & \cdots \\
 & & & {} \arrow[r, rightsquigarrow, "H^*"] & {} & & & & \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "d"] & B^n \arrow[r] & \cdots & & \cdots & H^{n-1}(B^\bullet) & H^n(B^\bullet) & \cdots
\end{tikzcd}$$

The derived category: a universal property definition

The **derived catergory of $\mathscr{A}$** is a category $\mathcal{D}(\mathscr{A})$ such that there exists a functor

$$Q\colon \mathbf{Kom}(\mathscr{A}) \longrightarrow \mathcal{D}(\mathscr{A}),$$

such that

- Q sends quasi-isomorphisms to isomorphisms
- Q satisfies a universal property with respect to this.

Homotopy Equivalence of chain complex maps

Definition

Let $f^\bullet$ and $g^\bullet$ be two chain maps between complexes $A^\bullet$ and $B^\bullet$. Then there is an equivalence relation $\sim$ called **chain homotopy** *defined by*

$$f^\bullet \sim g^\bullet \Longleftrightarrow \begin{array}{c} \textit{there exist a collection of morphisms} \\ \textit{(in } \mathscr{A}\textit{)}\ s^i : A^i \longrightarrow B^{i-1} \textit{ such that} \\ f - g = sd + ds. \end{array}$$

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$$\begin{tikzcd}
\cdots \arrow[r, "d"] & A^{n-1} \arrow[r, "d"] \arrow[dl, "s^{n-1}"'] \arrow[d, bend right, "f^{n-1}"'] \arrow[d, bend left, "g^{n-1}"] & A^n \arrow[r, "d"] \arrow[dl, "s^n"'] \arrow[d, bend right, "f^n"'] \arrow[d, bend left, "g^n"] & \cdots \arrow[dl, "s^{n+1}"'] \\
\cdots \arrow[r, "d"] & B^{n-1} \arrow[r, "d"] & B^n \arrow[r, "d"] & \cdots
\end{tikzcd}$$

A chain complex map $f^\bullet$ is a **null-homotopy** if $f^\bullet \sim 0$.

An overview of the construction

We construct the derived category in *two* steps:

$$\mathbf{Kom}(\mathscr{A}) \xrightarrow{\quad Q \quad} \mathcal{D}(\mathscr{A})$$

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- Step 1- Factor out null-homotopies to form the category $\mathcal{K}(\mathscr{A})$

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& \mathcal{K}(\mathscr{A}) &
\end{tikzcd}$$

factor out null-homotopies

An overview of the construction

We construct the derived category in *two* steps:

- Step 1- Factor out null-homotopies to form the category $\mathcal{K}(\mathscr{A})$
- Step 2- Localise at the collection of quasi-isomorphisms.

$$\begin{tikzcd}
\mathbf{Kom}(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, "\pi"] & & \mathcal{D}(\mathscr{A}) \\
& \mathcal{K}(\mathscr{A}) \arrow[ur, "\mathrm{loc}"] &
\end{tikzcd}$$

factor out null-homotopies

localise at quis's

Additive and Abelian Categories

Definition

A category $\mathscr{A}$ is **additive** *if you can 'add and subtract morphisms nicely', that is,*

- *for any two objects A and B in $\mathscr{A}$,* $\mathbf{Hom}_{\mathscr{A}}(A, B)$ *is an Abelian group*
- *the addition in each such group distributes over the composition of morphisms*
- *finite products exist.*

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A functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ between two additive categories is **additive** *if $F(f) + F(g) = F(f + g)$ for all morphisms f, g in $\mathscr{A}$ wherever $f + g$ is defined.*

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An additive category is **Abelian** *if it has kernels and cokernels and every mono is some kernel and every epi is some cokernel.*

The homotopy category

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Let $\mathscr{A}$ be an Abelian category. The category $\mathcal{K}(\mathscr{A})$ has

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$\mathcal{K}(\mathscr{A})$ is an additive category and the natural functor which identifies homotopic maps is additive.

$\mathcal{K}(\mathscr{A})$ has some extra structure which we will abstract to define what we call a *triangulated category*.

Translation and Cones

Definition

In $\mathbf{Kom}(\mathscr{A})$ *and* $\mathcal{K}(\mathscr{A})$ *we have a functor that takes a complex* $A^\bullet$ *and shifts the complex* k *times by degree.*

For example, $A[1]$ *denotes the shifted 'to the left' complex* $A^\bullet$ *with* n^{th} *degree* A^{n+1}*. The image of a morphism* f *is written* $f[1]$*.*

Given any chain map $f^\bullet \colon A^\bullet \longrightarrow B^\bullet$ *we define* $\mathbf{Cone}(f)$ *the* **cone of** f *to be the complex with* n^{th} *degree* $A^{n+1} \oplus B^n$ *and differential* $\tilde{d}$*, defined by*

$$\begin{aligned} A^{n+1} \oplus B^n &\xrightarrow{\quad\tilde{d}\quad} A^{n+2} \oplus B^{n+1} \\ (a, b) &\longmapsto (d(a), d(b) - f(a)) \end{aligned}$$

Triangles

Given any chain map $f^\bullet \colon A^\bullet \longrightarrow B^\bullet$ we get a short exact sequence of chain complexes defined by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^\bullet & \xhookrightarrow{\iota} & \mathbf{Cone}(f) & \xrightarrow{\delta} & A[1] & \longrightarrow & 0 \\
 & & b & \longmapsto & (0, b) & & & & \\
 & & & & (a, b) & \longmapsto & -a & &
\end{array}$$

This is our model for an abstract *triangle*.

Triangles

Given any chain map $f^\bullet \colon A^\bullet \longrightarrow B^\bullet$ we get a short exact sequence of chain complexes defined by

$$
\begin{tikzcd}
0 \arrow[r] & B^\bullet \arrow[r, hook, "\iota"] & \mathbf{Cone}(f) \arrow[r, two heads, "\delta"] & A[1] \arrow[r] \arrow[ll, dotted, bend right, "f^\bullet"'] & 0 \\
& b \arrow[r, mapsto] & (0, b) & & \\
& & (a, b) \arrow[r, mapsto] & -a &
\end{tikzcd}
$$

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The dotted line above is not strictly a map of chain complexes because of the shift of degree.

Abstract triangles

Definition

A **standard triangle** *is a triple* (f, ι, δ) *of morphisms in* $\mathcal{K}(\mathscr{A})$*, where* ι *and* δ *are defined as above and depend on* f*.*

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Definition

A **standard triangle** *is a triple* (f, ι, δ) *of morphisms in* $\mathcal{K}(\mathscr{A})$, *where* ι *and* δ *are defined as above and depend on* f.

An **distinguished triangle** *is a triple* $(A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1])$ *of morphisms in* $\mathcal{K}(\mathscr{A})$ *such that there exists a morphism* f *and isomorphisms a,b,c such that the following diagram commutes in* $\mathcal{K}(\mathscr{A})$ *(that is, up to chain homotopy)*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "a", "\cong"'] & B \arrow[r, "v"] \arrow[d, "b", "\cong"'] & C \arrow[r, "w"] \arrow[d, "c", "\cong"'] & A[1] \arrow[d, "a[1]", "\cong"'] \\
A' \arrow[r, "f"] & B' \arrow[r, "\iota"] & \mathbf{Cone}(f) \arrow[r, "\delta"] & A'[1]
\end{tikzcd}$$

In the above case we say that the triangle (u, v, w) is **isomorphic** to the triangle (f, ι, δ).

Triangulated categories

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Definition

An additive category $\mathscr{K}$ is called a **triangulated category** *if it has an automorphism $T : \mathscr{K} \longrightarrow \mathscr{K}$ called the* **translation functor** *and a distinguished family of triangles $\left\{ (A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA) \right\}$ called* **distinguished triangles** *satifying the following four axioms:*

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$\left\{(A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA)\right\}$ *called* **distinguished triangles** *satifying the following four axioms:*

TR1 Every morphism belongs to some distinguished triangle:
Given a morphism $A \xrightarrow{f} B$, there exists a distinguished triangle $\left(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA\right)$ and distinguishedness of triangles is preserved by isomorphisms of triangles. Also $(\mathrm{id}, 0, 0)$ is a distinguished triangle.

The rotation axiom

TR2 Distinguished triangles can be '*rotated*', that is, if $(A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA)$ is a distinguished triangle then

$$(B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{-Tu} TB)$$

and

$$(\ T^{-1}C \xrightarrow{-T^{-1}w} A \xrightarrow{u} B \xrightarrow{v} C\ )$$

are distinguished.

The morphisms axiom

TR3 Given any two distinguished triangles
$(A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA)$ and $(A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} TA')$ and morphisms f and g such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'
\end{tikzcd}$$

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$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & TA \arrow[d, "Tf"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA', \end{tikzcd}$$

then there exists a morphism h which makes the diagram commute. Note that this is **not unique**.

The octahedral axiom

TR4 The octahedral axiom.

$$\begin{tikzcd}
 & & & C \arrow[dl, "x"] & \\
 & & A' \arrow[dll, "i"] & & \\
B \arrow[urrr, "v"] \arrow[ddrr, "j"] & & & & B' \\
 & & & A \arrow[ulll, "u"] & \\
 & & C' \arrow[ur, "\epsilon"] & &
\end{tikzcd}$$

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$$
\begin{tikzcd}
 & & C \arrow[dl, "x"] \arrow[ddr, "y"] & \\
 & A' \arrow[dl, "i"] & & \\
B \arrow[uurr, "v"] \arrow[ddr, "j"'] & & & B' \arrow[dl, "\delta"] \\
 & & A \arrow[ull, "u"] \arrow[uuu, "v\circ u"'] & \\
 & C' \arrow[ur, "\epsilon"'] & &
\end{tikzcd}
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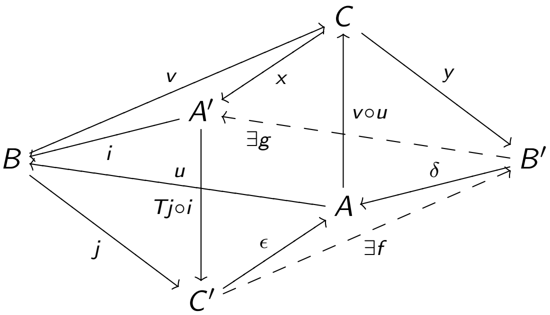

What's so good about triangulated categories?

In general, passing from $\mathbf{Kom}(\mathscr{A})$ to $\mathcal{K}(\mathscr{A})$ removes the Abelian structure of the category.

Exact triangles are good substitutes for the short exact sequences that we have lost.

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$(A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA)$ in a triangulated category $\mathscr{K}$ induces a long exact sequence on the level of cohomology

$$\cdots \xrightarrow{w^*} H^i(A) \xrightarrow{u^*} H^i(B) \xrightarrow{v^*} H^i(C) \xrightarrow{w^*} H^{i+1}(A) \xrightarrow{u^*} \cdots$$

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Localisation

Definition

Let S be a collection of morphisms in a category $\mathscr{C}$. We define the **localisation of $\mathscr{C}$ with respect to** *S to be a category $S^{-1}\mathscr{C}$ with a funtor*

$$q \colon \mathscr{C} \longrightarrow S^{-1}\mathscr{C}$$

such that

- *For all $s \in S$, $q(s)$ is an isomorphism*
- *q satisfies a universal property with respect to this.*

Examples of Localisations

$$\begin{array}{c}\textbf{Kom}(\mathscr{A}) \longrightarrow \mathcal{K}(\mathscr{A}) \\ \text{universally makes} \\ \text{homotopy equivalences} \\ \text{into isomorphisms}\end{array} \Longleftrightarrow \mathcal{K}(\mathscr{A}) = \left\{\begin{array}{c}\text{homotopy} \\ \text{equivalences}\end{array}\right\}^{-1} \textbf{Kom}(\mathscr{A})$$

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$$\begin{array}{c}\textbf{Kom}(\mathscr{A}) \longrightarrow \mathcal{D}(\mathscr{A}) \\ \text{universally makes} \\ \text{quasi-isomorphisms} \\ \text{into isomorphisms}\end{array} \Longleftrightarrow \mathcal{D}(\mathscr{A}) = \{\text{quis's}\}^{-1} \textbf{Kom}(\mathscr{A})$$

However, the following definition is better.

The derived category

Definition

Let Σ be the set of all quasi-isomorphisms in $\mathcal{K}(\mathscr{A})$ (not **Kom**$(\mathscr{A})$*) then we define the* **derived category of** $\mathscr{A}$ *to be the localisation of $\mathcal{K}(\mathscr{A})$ with respect to Σ,*

$$\mathscr{D}(\mathscr{A}) := \Sigma^{-1}\mathcal{K}(\mathscr{A}).$$

We still haven't shown that the derived category even exists!

Multiplicative systems

Definition

A collection S of morphisms in a category $\mathscr{C}$ is called a **multiplicative system** *if:*

MS1 S is closed under composition and contains all identity morphisms.

MS2 If $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$ are morphisms in $\mathscr{C}$ then

$$sf = sg \text{ for some s} \iff ft = gt \text{ for some t}$$

The Øre condition

MS3 If $Z \xrightarrow{t} Y$ is in S, then for every $X \xrightarrow{g} Y$ in $\mathscr{C}$ there exists maps making the diagram below commute

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "s"'] & Z \arrow[d, "t"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

In practise, we are 'rewriting' $t^{-1}g$ as fs^{-1}.

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If S is a multiplicative system, then a fraction $s^{-1}f$ written

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$$\begin{tikzcd} & Z_1 \arrow[dl, "s_1"'] \arrow[drrr, "f"' near end] & & Z_2 \arrow[dlll, "s_2" near end] \arrow[dr, "g"] & \\ X & & & & Y \end{tikzcd}$$

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Two fractions are equivalent if there exists a third which maps into both:

$$\begin{tikzcd}
& & Z_3 \arrow[ddll, bend right, "s_3"'] \arrow[ddrr, bend left, "h"] & & \\
& Z_1 \arrow[dl, "s_1"'] \arrow[drrr, "f" near end] & & Z_2 \arrow[dlll, "s_2"' near end] \arrow[dr, "g"] & \\
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Two fractions are equivalent if there exists a third which maps into both:

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\end{tikzcd}$$

Conclusion

The set of quasi-isomorphisms in $\mathcal{K}(\mathscr{A})$ is a multiplicative system.

Thus, by passing to $\mathcal{K}(\mathscr{A})$ we not only get that $\mathcal{D}(\mathscr{A})$ exists, but are able to manipulate morphisms more concretely using the Øre condition.

Derived functors - motivation

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$$\mathrm{Hom}, \otimes, \Gamma,$$
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- More explicitly, let $f_*\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories (recall that f_* being left exact means that if $0 \to X \to Y \to Z$ is exact in $\mathscr{A}$, then $0 \to f_*X \to f_*Y \to f_*Z$ is exact in $\mathscr{B}$).

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- The functor $\mathbf{R}f_*$ will be exact in the sense that it will map distinguished triangles to distinguished triangles.

Derived functors - the formal definition

Let $f_* \colon \mathscr{A} \to \mathscr{B}$ be an additive functor. The derived functor is the data of a triangulated functor $F \colon D(\mathscr{A}) \to D(\mathscr{B})$ and a natural transformation

$$s \colon \mathrm{loc} \circ f_* \to F \circ \mathrm{loc}$$

such that the induced morphism

$$\mathrm{Hom}(F, G) \to \mathrm{Hom}(\mathrm{loc} \circ f_*, G \circ \mathrm{loc})$$

is an isomorphism for any triangulated functor $G \colon D(\mathscr{A}) \to D(\mathscr{B})$. Here loc denotes the localization functor.

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- for every object $X \in \mathscr{A}$ there is a monomorphism $X \hookrightarrow I$ with $I \in \mathscr{I}$;
- if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$, and if X, Y are in $\mathscr{I}$, then Z is also in $\mathscr{I}$;

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A full additive subcategory $\mathscr{I} \subset \mathscr{A}$ is *f_*-injective* or an *adapted class* for f_* if:

- for every object $X \in \mathscr{A}$ there is a monomorphism $X \hookrightarrow I$ with $I \in \mathscr{I}$;
- if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$, and if X, Y are in $\mathscr{I}$, then Z is also in $\mathscr{I}$;
- if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X, Y, Z \in \mathscr{I}$, then $0 \to f_*X \to f_*Y \to f_*Z \to 0$ is exact in $\mathscr{B}$.

What's the point?

- The first axiom allows us to replace an object $X \in \mathscr{A}$ with a 'resolution' of X by f_*-injective objects: embed X into I, take the cokernel and embed this into another f_*-injective object, wash, rinse, repeat.

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More precisely:

The formal construction

Let $f_*\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor. Let $\mathscr{I} \subseteq \mathscr{A}$ be a f_*-injective category. Let S (resp. S') be quasi-isomorphisms in $\mathbf{Kom}(\mathscr{A})$ (resp. $\mathbf{Kom}(\mathscr{I})$).

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The *right derived functor* $\mathbf{R}f_*\colon D^+(\mathscr{A}) \to D^+(\mathscr{B})$ is given by the composition

$$D^+(\mathscr{A}) \xrightarrow{\mathbf{i}} S'^{-1}\mathcal{K}^+(\mathscr{I}) \xrightarrow{f_*} S'^{-1}\mathcal{K}^+(\mathscr{B}) \to D^+(\mathscr{B}).$$

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- If $\mathscr{A}$ has enough injectives, then the injectives form an adapted class for $\mathrm{Hom}_{\mathscr{A}}(X, -)$.

A universal adapted class (contd.)

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- **Key point:** If $I^\bullet$ is a complex of injectives and $X^\bullet$ is a complex of objects in $\mathscr{A}$ such that the cohomology of $X^\bullet$ is zero in every degree, then every chain map $X^\bullet \to I^\bullet$ is homotopic to zero.

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- In fact, if $\mathscr{A}$ has enough injectives, then the injectives are f_*-injective for *any* left exact functor f_*.
- **Key point:** If $I^\bullet$ is a complex of injectives and $X^\bullet$ is a complex of objects in $\mathscr{A}$ such that the cohomology of $X^\bullet$ is zero in every degree, then every chain map $X^\bullet \to I^\bullet$ is homotopic to zero.
- **Sidenote:** The above also implies that the evident functor

$$\mathcal{K}^+(\mathscr{I}) \to D^+(\mathscr{A})$$

is an equivalence (here $\mathscr{I} \subseteq \mathscr{A}$ is the full subcategory consisting of injectives and we are assuming that $\mathscr{A}$ has enough injectives).

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- There may not be enough injectives (or rather, as in real life, there may not be enough projectives in the dual setting of left derived functors).

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Or, if you are so inclined, systematically work with opposite categories.

Toy example

Let A be the algebra of 2×2 matrices over $\mathbb{C}$.

$$A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}.$$

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As an A-module, A decomposes as

$$A \simeq V_1 \oplus P$$

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