A localization of modules with a Verma flag

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Basic setting

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Want to associate a moment graph to the data given by (R, R^+, W):

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- We define the order by

$$w \le w' \Leftrightarrow w \cdot \lambda \le w' \cdot \lambda$$

for all $w, w' \in \mathsf{W}$.

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- $\mathsf{Z} = \mathsf{Z}(\mathcal{G})$ is the structure algebra over A.
- $\mathsf{Z}\text{-mod}^f$ is the category of Z-modules which are finitely generated as A-modules and torsion free over A.

The structure functor

We already know:

$$\begin{aligned}\mathrm{End}_{\mathfrak{g}_A}(P_A(\lambda)) &\cong \{(t_x) \in \prod_{x \in W} A \mid t_x \equiv t_{s_\alpha x} \bmod \alpha^\vee \forall x \in W, \alpha \in R^+\} \\ &\cong Z(\mathcal{G})\end{aligned}$$

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$$\begin{aligned}\mathbb{V} : \mathcal{O}^{\mathrm{VF}}_{\mathrm{A},\lambda} &\to \mathrm{Z}\text{-mod}^f \\ M &\mapsto \mathrm{Hom}_{\mathfrak{g}_\mathrm{A}}(\mathrm{P}_\mathrm{A}(\lambda), M)\end{aligned}$$

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for every edge $E : x \overset{\alpha^\vee}{- - -} y$. We get $\mathcal{L}(M)^E$ via the pushout

$$\begin{array}{ccc} M(E) & \xrightarrow{\pi_x} & \mathcal{L}(M)^x \\ \downarrow{\scriptstyle \pi_y} & & \downarrow{\scriptstyle \rho_{x,E}} \\ \mathcal{L}(M)^y & \xrightarrow{\rho_{y,E}} & \mathcal{L}(M)^E \end{array}$$

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$$\mathcal{L}(M) \xrightarrow{\sim} \mathcal{L}(\Gamma(\mathcal{L}(M)))$$

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$$\Gamma(\mathcal{SH}_A(\mathcal{G})) \xrightarrow{\mathcal{L}} \mathcal{L}(Z\text{-mod}^f)$$

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- Each $\mathcal{P}^x$ is a projective A-module
- For each $E : y \stackrel{\alpha^\vee}{\rightarrow} x$ the map
$$\rho_{y,E} : \mathcal{P}^y \rightarrow \mathcal{P}^E$$
induces an isomorphism
$$\mathcal{P}^y / \alpha^\vee \mathcal{P}^y \stackrel{\sim}{\longrightarrow} \mathcal{P}^E$$

F-projective objects

One can show that the sheafs in the image of

$$\mathcal{L} : \mathsf{Z}\text{-mod}^f \quad \to \quad \mathcal{SH}_{\mathsf{A}}(\mathcal{G})$$

fulfill the second property in the preceding definition.

The Braden-MacPherson sheaf

Let $\mathcal{B}(z)$ be the Braden-MacPherson sheaf associated to a vertex $z \in \mathcal{V}$. We denote by $\mathcal{B}_{\mathsf{A}}(z) := \mathcal{B}(z) \otimes_{\mathsf{S}} \mathsf{A}$ the sheaf

$$(\{\mathcal{B}(z)^x \otimes_{\mathsf{S}} \mathsf{A}\}, \{\mathcal{B}(z)^E \otimes_{\mathsf{S}} \mathsf{A}\}, \{\rho_{x,E} \otimes_{\mathsf{S}} \mathrm{id}_{\mathsf{A}}\})$$

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Theorem

The sheaf $\mathcal{B}_{\mathrm{A}}(z)$ is F-projective and indecomposable.

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Proposition

Let $\mathcal{P}$ be a F-projective sheaf. Then there exists an isomorphism of sheafs

$$\mathcal{P} \xrightarrow{\sim} \mathcal{B}_{\mathsf{A}}(z_1) \oplus \ldots \oplus \mathcal{B}_{\mathsf{A}}(z_n)$$

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In particular every indecomposable F-projective object is isomorphic to a BMP.

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$$0 \to L[D] \to M[D] \to N[D] \to 0$$

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- Induction on the length of Verma flag yields the claim.

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$$M[\mathcal{H}] := M[D_{\mathcal{H}}]$$

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Proposition

Let $M \in \mathcal{O}_{A,\lambda}^{VF}$*. Then* $\mathcal{L}(\mathbb{V}(M))$ *is flabby and for each open subgraph* $\mathcal{H} \subset \mathcal{G}$ *we get an isomorphism*

$$\Gamma(\mathcal{H}, \mathcal{L}(\mathbb{V}(M))) \quad \tilde{\rightarrow} \quad \mathbb{V}(M[\mathcal{H}])$$

A functor into sheaves

Lemma

Let $P \in \mathcal{O}_{A,\lambda}^{VF}$ *be projective and* $w \in W$*. Then*

$$\mathcal{L}(\mathbb{V}P)^w$$

is a free A*-module of rank* $(P : \Delta_A(w \cdot \lambda))$.

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Left to show: For each $x \in \mathrm{W}$ and $\alpha \in \mathrm{R}^+$ with $x \cdot \lambda \leq s_\alpha x \cdot \lambda$ and $E : x - -s_\alpha x$ that

$$\mathcal{L}(\mathbb{V}P)^x / \alpha^\vee \mathcal{L}(\mathbb{V}P)^x \;\; \tilde{\to} \;\; \mathcal{L}(\mathbb{V}P)^E$$

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We use

- $\rho_{x,E} : \mathcal{L}(\mathbb{V}P)^x \to \mathcal{L}(\mathbb{V}P)^E$ is surjective

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- and similarly for $\alpha^\vee \mathcal{L}(\mathbb{V}P)^x$.
- It suffices to show

$$\ker \rho_{x,E} \subset \alpha^\vee(\mathcal{L}(\mathbb{V}P)^x \otimes_{\mathsf{A}} \mathsf{A}_{\mathfrak{p}}) \qquad \text{for all } \mathfrak{p} \subset \mathsf{A} \text{ with ht } \mathfrak{p} = 1$$

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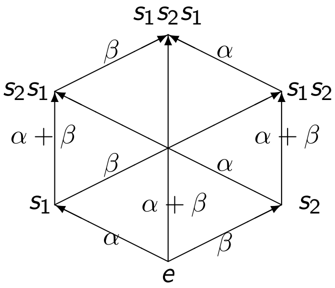

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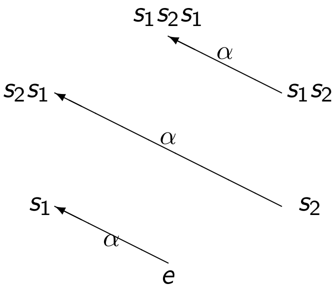

A functor into sheaves

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$$\mathcal{L}(\mathbb{V}\,\mathsf{P}_{\mathsf{A}}(w \cdot \lambda)) \cong \mathcal{B}_{\mathsf{A}}(w) \qquad \textit{for all } w \in \mathsf{W}$$

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- Hence $\mathcal{L}(\mathbb{V}\,\mathrm{P}_\mathrm{A}(w\cdot\lambda))$ is indecomposable

A functor into sheaves

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$$0 \neq (\mathrm{P}_{\mathrm{A}}(w \cdot \lambda) : \Delta_{\mathrm{A}}(x \cdot \lambda)) = [\Delta_{\mathrm{A}}(x \cdot \lambda) : \mathrm{L}_{\mathrm{A}}(w \cdot \lambda)]$$

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- Hence $z = w$ $\square$

A functor into sheaves

Corollary

$$[\Delta_A(x \cdot \lambda) : L_A(w \cdot \lambda)] = \mathrm{rk}_S \, \mathcal{B}_A(w)^x \qquad \textit{for all } w, x \in W$$

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- Applying the Theorem yields the claim. □

Relation to the Kazdhan-Lusztig conjecture

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Conjecture

(A combinatorial version of the Kazdhan-Lusztig conjecture)

$$\mathrm{rk}_S\, \mathcal{B}(w)^x = h_{x,w}(1) \qquad \textit{for all } x, w \in \mathcal{W}$$

Bibliography

[Fie03] Peter Fiebig, *Centers and translation functors for category $\mathcal{O}$ over Kac-Moody algebras*, Math. Z. **243** (2003), no. 4, 689–717.

[Fie06] ______, *The combinatorics of category $\mathcal{O}$ over symmetrizable Kac-Moody algebras*, Transform. Groups **11** (2006), no. 1, 29–49.

[Fie08] ______, *Sheaves on moment graphs and a localization of Verma flags*, Adv. Math. **217** (2008), 683–712.

[Jan08] Jens Carsten Jantzen, *Moment graphs and representations*, Summer school on geometric methods in representation theory, Grenoble, France (2008).