Notation

As before, let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra, with $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. We fix a nondegenerate invariant symmetric form $(-,-)$ on $\mathfrak{h}^*$.

Let T be a local deformation ring with residue field $\mathbb{K}$.

Let $S = S(\mathfrak{h})$ and let $\tau : S \to T \in \mathfrak{h}^*_T = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, T)$ be the structure morphism. Extend $(-,-)$ to a bilinear form $(-,-)_T : \mathfrak{h}^*_T \times \mathfrak{h}^*_T \to T$.

We'll be especially concerned with the following special cases.

- Let $R = S_{(\mathfrak{h})}$, $Q = \mathrm{Frac}(R)$ and $\mathbb{K}$ the residue field. The structure morphism $\tau : S \to R$ is just inclusion.
- For any $\alpha \in \Delta$, let $h_\alpha = (\tau, \alpha)_R \in \mathfrak{h} \subset R$. Then define R_α to be the localisation of R at Rh_α and $\mathbb{K}_\alpha$ to be its residue field.

Decomposition of $\mathcal{O}_T$ into blocks

Define an equivalence relation on $\mathfrak{h}^*$ generated by:

$$\lambda \sim_T \mu \text{ if and only if } (P_T(\lambda) : M_T(\mu)) = [M_{\mathbb{K}}(\lambda) : L_{\mathbb{K}}(\mu)] \neq 0.$$

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Let $\theta \subset \mathfrak{h}^*$ be a union of equivalence classes under $\sim_T$. Then set $\mathcal{O}_{T,\theta}$ to be the full subcategory of all modules M such that every highest weight of a subquotient of M lies in θ.

Theorem

The functor

$$\bigoplus_{\Lambda \in \mathfrak{h}^*/\sim_T} \mathcal{O}_{T,\Lambda} \to \mathcal{O}_T; \{M_\Lambda\} \mapsto \oplus M_\Lambda,$$

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Remark. If we have a morphism of local deformation rings $T \to T'$, then $\lambda \sim_{T'} \mu \Rightarrow \lambda \sim_T \mu$. Thus the block decomposition for T' refines that for T.

Blocks and Weyl group orbits

Theorem (Bernstein-Gelfand-Gelfand)

An equivalence class Λ *is equal to the dot orbit* $\mathcal{W}_T(\Lambda) \cdot \lambda$*, where*

$$\mathcal{W}_T(\Lambda) =< s_\beta : 2(\lambda + \rho + \tau, \beta)_{\mathbb{K}} \in \mathbb{Z}(\beta, \beta)_{\mathbb{K}} > .$$

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Let's calculate $\mathcal{W}_T(\Lambda)\cdot\lambda$ in two special cases.

- Let $T = R$: Since $(\tau,\beta)_R \in \mathfrak{h}$ we get $(\tau,\beta)_{\mathbb{K}} = 0$. Thus the orbits $\mathcal{W}_T(\Lambda)\cdot\lambda$ are just the usual Weyl group orbits $\mathcal{W}\cdot\lambda$, and in particular contain a unique antidominant weight.
- Let $T = R_\alpha$: Since $(\tau,\beta)_{R_\alpha} = \tau(h_\beta) = h_\beta$, the condition

  $$2(\lambda+\rho+\tau,\beta)_{\mathbb{K}_\alpha} \in \mathbb{Z}(\beta,\beta)$$

  is equivalent to

  $$2(\lambda+\rho,\beta) \in \mathbb{Z}(\beta,\beta) \text{ and } h_\beta \in Rh_\alpha. \tag{1}$$

  Therefore, only $\beta = \pm\alpha$ can satisfy the conditions of (1), and the possible equivalence classes are $\Lambda = \{\lambda\}$ or $\{\lambda, s_\alpha\cdot\lambda\}$.

$\mathcal{O}_{T,\Lambda}$ in the subgeneric cases

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Theorem

Suppose that $\lambda > \mu = s_\alpha \cdot \lambda$. Then $P_{R_\alpha}(\lambda) \cong M_{R_\alpha}(\lambda)$ and there is a short exact sequence

$$0 \to M_{R_\alpha}(\lambda) \to P_{R_\alpha}(\mu) \to M_{R_\alpha}(\mu) \to 0.$$

The block $\mathcal{O}_{R_\alpha,\Lambda}$ is equivalent to to the category of representations (over R_α) of the quiver

$$\lambda \underset{j}{\overset{i}{\rightleftarrows}} \mu\ ,$$

with relation $ji = h_\alpha$.

Proof of subgeneric case

Proof.

(Sketch) The isomorphism $P_{R_\alpha}(\lambda) \cong M_{R_\alpha}(\lambda)$ follows from BGG reciprocity. By using the Jantzen sum formula, one sees that

$$[M_{\mathbb{K}_\alpha}(\lambda) : L_{\mathbb{K}_\alpha}(\mu)] = [P_{R_\alpha}(\mu) : M_{R_\alpha}(\lambda)] = 1.$$

This yields the short exact sequence, and the quiver description follows by calculating $\mathrm{End}(P_{R_\alpha}(\lambda) \oplus P_{R_\alpha}(\mu))$. □

The centre of $\mathcal{O}_{R_\alpha,\Lambda}$

Define $Z_{T,\Lambda}$ to be centre of $\mathcal{O}_{T,\Lambda}$, that is, the ring of endotransformations of the identity functor. One can view the centre as a subring of $\bigoplus \mathrm{End} P_i$ satisfying certain commutation conditions. Here the P_i are projective modules such that $\bigoplus P_i$ is a projective generator in $\mathcal{O}_T$.

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Corollary

Suppose that we are in the subgeneric situation. The centre $Z_{R_\alpha,\{\lambda,s_\alpha\cdot\lambda\}}$ is isomorphic to $\{(t_\lambda,t_\mu)\in R_\alpha\oplus R_\alpha : t_\lambda = t_\mu \mod h_\alpha\}$.

This follows by evaluating the action of the centre on $M_{R_\alpha}(\lambda)\oplus M_{R_\alpha}(\mu)$ using the quiver description above.

The calculation of $Z_{R,\Lambda}$

We can now use the subgeneric case to calculate $Z_{R,\Lambda}$.

Theorem

Evaluation on Verma modules induces an isomorphism of algebras

$$Z_{R,\Lambda} \cong \{(t_\nu) \in \prod_{\nu\in\Lambda} R : t_\nu = t_{s_\alpha\cdot\nu} \bmod h_\alpha \text{ for all } \alpha \text{ such that } s_\alpha \in \mathcal{W}\}.$$

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Proof

View $Z_{R,\Lambda}$ *as a subring of* $\bigoplus \mathrm{End} P_i$, *where* $\bigoplus P_i$ *is a projective generator in* $\mathcal{O}_{R,\Lambda}$. *By basechange, there is an injective map* $\bigoplus \mathrm{End} P_i \hookrightarrow \bigoplus \mathrm{End}(P_i \otimes_R Q)$, *which in turn induces an injective map* $Z_{R,\Lambda} \hookrightarrow Z_{Q,\Lambda}$. *Furthermore,*

$$Z_{R,\Lambda} = \bigcap_{\mathfrak{p} \subset R, \mathrm{ht}\mathfrak{p} = 1} Z_{R_\mathfrak{p},\Lambda} \subset Z_{Q,\Lambda}.$$

The calculation of $Z_{R,\Lambda}$ (continued)

Proof continued.

Fix a height one prime $\mathfrak{p} \subset R$. There are two cases:

- If $h_\alpha \notin \mathfrak{p}$ for all α, then Λ decomposes under $\sim_{R_\mathfrak{p}}$ into trivial equivalence classes. Thus

$$Z_{R_\mathfrak{p},\Lambda} = \prod_{\nu\in\Lambda} R_\mathfrak{p}.$$

- If $h_\alpha \in \mathfrak{p}$ for some α, then Λ splits under $\sim_{R_\alpha}$ into classes of the form $\{\lambda, s_\alpha \cdot \lambda\}$. The subgeneric case implies

$$Z_{R_\mathfrak{p},\Lambda} \cong \{(t_\nu \in \prod_{\nu\in\Lambda} R_\alpha : t_\nu = t_{s_\alpha\cdot\nu} \mod h_\alpha\}.$$

Taking the intersection of all of these yields the result. □

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Connection to moment graphs: The algebra $Z_{R,\Lambda}$ is the structure algebra of the Bruhat graph associated to the Weyl group $\mathcal{W}$.

The endomorphism algebra of an antidominant projective

Proposition

Let $\lambda \in \mathfrak{h}^*$ be antidominant, with $\lambda \in \Lambda$. Then the natural map

$$Z_{R,\Lambda} \to \operatorname{End} P_R(\lambda)$$

is an isomorphism.

Proof.

There is a commutative diagram

$$\begin{array}{ccc} \operatorname{End} P_R(\lambda) & = & \bigcap_{\mathfrak{p} \subset R, \mathrm{ht}\mathfrak{p}=1} \operatorname{End}(P_R(\lambda) \otimes_R R_{\mathfrak{p}}) \\ \uparrow & & \| \\ Z_{R,\Lambda} & = & \bigcap_{\mathfrak{p} \subset R, \mathrm{ht}\mathfrak{p}=1} Z_{R_{\mathfrak{p}},\Lambda}. \end{array}$$

□

The functor $\mathbb{V}$

Let $\lambda \in \mathfrak{h}^*$ be antidominant and let Λ be the equivalence class under $\sim_R$ containing λ.

Definition

The functor $\mathbb{V} = \mathbb{V}_{R,\Lambda}$ is the exact functor

$$\mathcal{O}_{R,\Lambda} \to Z_{R,\Lambda} - \mathrm{mod}; M \mapsto \mathrm{Hom}(P_R(\mu), M).$$

This definition can be made for any deformation ring T.

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Remark. As noted before, $Z_{R,\Lambda}$ is isomorphic to the structure sheaf $\mathcal{Z}$ of the Bruhat graph associated to $\mathcal{W}$. So we can think of $\mathbb{V}$ as a functor $\mathcal{O}_{R,\Lambda} \to \mathcal{Z} - \mathrm{mod}$. By applying the localisation functor $\mathcal{L}$, one obtains a functor from $\mathcal{O}_{R,\Lambda}$ to sheaves on this moment graph.

Properties of $\mathbb{V}$:

Theorem

Let T be a deformation ring. Then we have:

(1) *For any $w \in \mathcal{W}$ there is an isomorphism $\mathbb{V}M_T(w \cdot \lambda) \cong Z_{T,\Lambda}/m_w$, where $m_w \subset Z_{T,\Lambda}$ is the ideal generated by all elements acting trivially on $M_T(w \cdot \lambda)$. In particular, $\mathbb{V}M_T(\lambda)$ is free of rank one over T.*

(2) *If $M \in \mathcal{O}_{T,\Lambda}$ has a Verma flag, then $\mathbb{V}M$ is free of finite rank over T.*

(3) *$\mathbb{V}$ commutes with basechange $T \to T'$, that is,*

$$\mathbb{V}_{T,\Lambda}(\cdot) \otimes_T T' \cong \prod_i \mathbb{V}_{T',\Lambda_i}(\cdot \otimes_T T'),$$

where $\Lambda = \bigcup_i \Lambda_i$ is the splitting of Λ under $\sim_{T'}$.

(4) *$\mathbb{V} = \mathbb{V}_{R,\Lambda}$ is fully faithful on modules with a Verma flag, that is, if $M, M' \in \mathcal{O}_{R,\Lambda}$ have a Verma flag, then*

$$\mathrm{Hom}(M, M') \cong \mathrm{Hom}(\mathbb{V}M, \mathbb{V}M').$$

Sketch of proof

Proof.

(Sketch)

(1) By basechange,

$$\mathbb{V}M_T(w\cdot\lambda)\otimes_T\mathbb{K} = \mathrm{Hom}(P_{\mathbb{K}}(\lambda), M_{\mathbb{K}}(w\cdot\lambda)).$$

By BGG reciprocity, this has dimension 1. Thus $\mathbb{V}M_T(w\cdot\lambda)$ is a cyclic T-module (Nakayama's Lemma). The result follows since $Z_{T,\Lambda}/m_w \cong T$.

(2) Follows from (1) and exactness of $\mathbb{V}$.

(3) Follows from basechange.

(4) This is by proved by using (2) and (3) to reduce to the subgeneric case, where it is clear.

□