Deformed category $\mathcal{O}$

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Notation

$\mathfrak{g}$ = finite dimensional semisimple Lie algebra over $\mathbb{C}$.

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ = Borel subalgebra.

$\mathfrak{h}^* \supseteq \Delta$ = root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Δ_+ = positive roots determined by $\mathfrak{b}$.

$S = S(\mathfrak{h}) = U(\mathfrak{h})$

$(\cdot\,,\cdot)$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$; inducing a form on $\mathfrak{h}^*$.

Deformation rings

A *deformation ring* T is a commutative noetherian S-algebra, which is an integral domain.

T is a *local deformation ring* if it is local as an ring.

Let $\tau : S \to T$ be the structure map.

Let $\lambda \in \mathfrak{h}^*$. Then $\mathbb{C}$ is a local deformation ring with $\tau = \lambda$.

The localization $S_{(h)}$ of S at $0 \in \mathfrak{h}^*$ is a local deformation ring, with τ the inclusion.

The field of fractions Q of $S_{(h)}$ in a local deformation ring.

Deformation rings 2

Given a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, we define

$$\mathfrak{a}_T = \mathfrak{a} \otimes_{\mathbb{C}} T,$$

and we have

$$U(\mathfrak{a}_T) \cong U(\mathfrak{a}) \otimes_{\mathbb{C}} T.$$

Deformed Verma modules

Let $\lambda \in \mathfrak{h}^*$. Define T_λ to be the $\mathfrak{b}_T$-module, such that

1. it is equal to T as a T-module;
2. $\mathfrak{t}$ acts by multiplication by $\lambda(t) + \tau(t)$;
3. $\mathfrak{n}$ acts as zero.

Then

$$M_T(\lambda) = U(\mathfrak{g}_T) \otimes_{U(\mathfrak{b}_T)} T_\lambda$$

is the *deformed Verma module* associated to λ.

$M_T(\lambda)$ is free as a T-module.

Weight spaces

Let M be a $U(\mathfrak{g}_T)$-module and $\lambda \in \mathfrak{h}^*$. Then

$$M_\lambda = \{m \in M \mid hm = (\lambda + \tau)(h)m \text{ for all } h \in \mathfrak{h}\}$$

is the λ-*weight space* of M.

M is a *weight module* if

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda.$$

A deformed Verma module $M_T(\lambda)$ is a weight module with set of weights $\lambda - \mathbb{Z}_{\geq 0}\Delta_+$.

Deformed category $\mathcal{O}$

The *deformed category* $\mathcal{O}$ is the category $\mathcal{O}_T$ of finitely generated $U(\mathfrak{g}_T)$-modules such that

1. M is a weight module
2. $U(\mathfrak{n}_T)m$ is finitely generated as a T-module for all $m \in M$.

It is an abelian category and contains all deformed Verma modules.

Given a homomorphism $T \to T'$ of deformation rings there is a *base change functor*

$$? \otimes_T T' : \mathcal{O}_T \to \mathcal{O}_{T'}.$$

Clearly $M_T(\lambda) \otimes_T T' = M_{T'}(\lambda)$.

Verma flags

$M \in \mathcal{O}_T$ is said to have a *Verma flag* if there is a finite filtration

$$M = M_1 \supseteq \ldots \supseteq M_{r+1} = 0$$

such that $M_i/M_{i+1} \cong M_T(\lambda_i)$ for some $\lambda_i \in \mathfrak{h}^*$ for $i = 1, \ldots, r$.

We define $\mathcal{M}_T$ to be the full subcategory of $\mathcal{O}_T$ consisting of all modules with a Verma flag.

Given $M \in \mathcal{M}_T$ and $\lambda \in \mathfrak{h}^*$, we write $(M : M_T(\lambda))$ for the number of times $M_T(\lambda)$ occurs in a Verma flag for M. This is independent of the choice of Verma flag.

Any $M \in \mathcal{M}_T$ is free as a T-module.

Deformed projectives

From now on T is a local deformation ring, and we write $\mathbb{K}$ for the residue field of T.

We may consider the category $\mathcal{O}_{\mathbb{K}}$ of $\mathfrak{g}_{\mathbb{K}} = \mathfrak{g}_T \otimes_T \mathbb{K}$-modules. This contains projective modules $P_{\mathbb{K}}(\lambda)$ and simple modules $L_{\mathbb{K}}(\lambda)$ for each $\lambda \in \mathfrak{h}^*$.

The category $\mathcal{O}_T$ "contains enough projectives". More precisely:

Theorem

For each $\lambda \in \mathfrak{h}^*$, there is a projective module $P_T(\lambda)$ such that $P_T(\lambda) \otimes_T \mathbb{K} \cong P_{\mathbb{K}}(\lambda)$.

More on deformed projectives

Let $\lambda, \mu \in \mathfrak{h}*$. Then:

1. $P_T(\lambda)$ is indecomposable;
2. each projective in $\mathcal{O}_T$ is a finite direct sums of $P_T(\lambda)$s;
3. $P_T(\lambda) \in \mathcal{M}_T$; and
4. $(P_T(\lambda) : M_T(\mu)) = [M_{\mathbb{K}}(\lambda) : L_{\mathbb{K}}(\mu)]$.

Let T' be another local deformation ring with a homomorphism $T \to T'$, and let $P \in \mathcal{O}_T$ be projective. Then there is a isomorphism of functors

$$\mathrm{Hom}_{\mathfrak{g}_T}(P, ?) \otimes_T T' \cong \mathrm{Hom}_{\mathfrak{g}_{T'}}(P_{T'}, ?) : \mathcal{O}_T \to T'$$

Deformed simples

The simples in $\mathcal{O}_T$ are indexed by $\mathfrak{h}^*$, more precisely:

Theorem

The functor $? \otimes_T \mathbb{K}$ *gives rise to a bijection between isomorphism classes of simple objects in* $\mathcal{O}_T$ *and those in* $\mathcal{O}_{\mathbb{K}}$.

$P_T(\lambda)$ is the projective cover of $L_T(\lambda)$.