Introduction to Algebraic Topology

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Motivation (I)

- Simplicial homology groups “measure” n-dimensional holes in topological spaces.
- It provides “global” information.
- It is a topological tool, mainly used in topology, but also in geometry and algebra.
- There are other definitions for (co)homology (Morse, singular, deRham, quantum, sheaf...) and (often) isomorphisms among them.

Motivation (II)

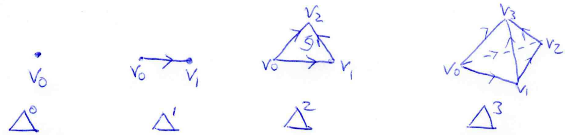
- Simplicial homology is easy to compute (for easy examples).
- 'Abelianising' the fundamental group of a space $\pi_1(X)$ gives the first homology group of X.
- An orientation-free approach requires more computations (triangulations [Mun84]).

Δ-complexes

Definition

Let $\{a_0, \ldots, a_n\}$ be a geometrically independent set in $\mathbb{R}^n$. The n-simplex Δ^n spanned by $a_0, \ldots, a_n$ is:

$$\Delta^n = \{x = \sum_{i=0}^{n} t_i a_i \mid \sum_{i=0}^{n} t_i = 1\}.$$



The subsimplices obtained by removing one single vertex are called faces (there are $n+1$ of them). The union of all faces is $\partial\Delta^n$. Δ^n has a topology induced by the usual topology of $\mathbb{R}^n$. The open simplex of Δ^n is $\overset{\circ}{\Delta}{}^n := \Delta^n \setminus \partial\Delta$.

Definition

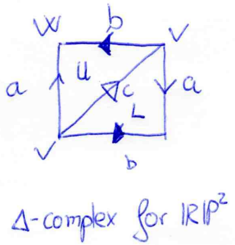
A Δ-complex structure on a topological space X is a collection of maps $\Delta_X = \{\sigma_\alpha : \Delta^n \longrightarrow X,\ n \text{ depending on } \alpha\}$, such that:

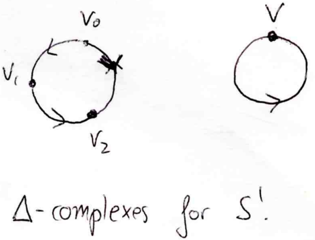
(i) $\sigma_\alpha|_{\overset{\circ}{\Delta}{}^n}$ *is injective* $\forall\alpha$, *and all $x \in X$ is in the image of one and only one such restriction.*

(ii) *Each restriction of σ_α to a face of Δ^n is a map* $\sigma_\beta : \Delta^{n-1} \longrightarrow X, \quad \sigma_\beta \in \Delta_X$.

(iii) *A set $A \subseteq X$ is open iff $\sigma_\alpha^{-1}(A)$ is open* $\forall\sigma_\alpha$.

Examples of Δ-complexes: $\mathbb{RP}^2$ and S^1





Simplicial chain complex

Definition

Let $C_n(X)$ be the free abelian group with basis the open n-simplices e^n_α of X. Its elements $\sigma_\alpha = \sum_\alpha n_\alpha e^n_\alpha$, are called n-chains.
The boundary homomorphism $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ is defined on basis elements as:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.$$

Lemma ([Hat02]: lemma 2.1)

The composition $C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$ is zero.

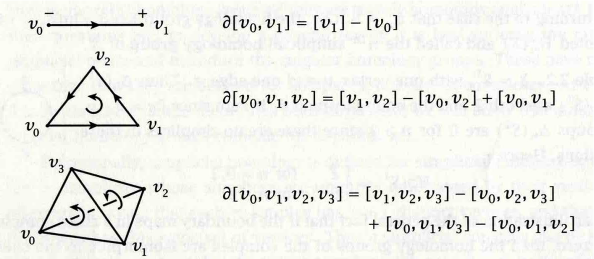


Figure: Boundaries of the standard simplices. [Hat02]

Definition

If for $\sigma \in C_n(X)$, $\partial_n(\sigma) = 0$, then σ is a cycle. If $\sigma = \partial_{n+1}(\hat{\sigma})$, then σ is a boundary.

Definition

The n-th simplicial homology group of X is:

$$H_n(X) = \frac{\text{Ker } \partial_n}{\text{Im } \partial_{n+1}}.$$

The n-th betti number of X is the rank of H_n as a $\mathbb{Z}$-module:

$$b_n(X) = \text{rk}_{\mathbb{Z}}(H_n(X)).$$

Lemma

The definition does not depend on the Δ-complex structure chosen for X.

Cohomology

Definition

Consider the dual, cochain group $C^n(X) := \mathrm{Hom}(C_n(X), \mathbb{Z})$. *We also have a coboundary homomorphism* $\delta^n : C^{n-1} \longrightarrow C^n$ *defined, for* $\gamma : C_n(X) \longrightarrow \mathbb{Z}$, *by its action:*

$$\delta(\gamma)(\sigma) := \gamma(\partial(\sigma)).$$

Now, we have cocycles ($\delta(\sigma) = 0$) *and coboundaries* ($\sigma = \delta\hat{\sigma}$) *and, since* $\delta^2 = 0$, *the cohomology groups:*

$$H^n(X) = \frac{\mathrm{Ker}\ \delta_n}{\mathrm{Im}\ \delta_{n+1}}.$$

Affine space

If $f : X \longrightarrow Y$ is continous, we get a $f_\sharp : C_n(X) \longrightarrow C_n(Y)$ and $\partial \circ f = f \circ \partial$. Therefore we have $f_* : H_n(X) \longrightarrow H_n(Y)$.

Theorem

If $f, g : X \longrightarrow Y$ are homotopic then $f_* = g_*$ and therefore if X, Y are homotopic, $H_*(X) = H_*(Y)$.

Hence, since $\Delta^0 \cong \mathbb{R}^k$, $H_*(\mathbb{R}^k) = H_*(\Delta^0)$. Its chain complex is:

$$0 \to 0 \to \cdots \to C_0 = \mathbb{Z} \to 0.$$

And:

$$H_n(\mathbb{R}^k) = \begin{cases} \mathbb{Z} & n = 0; \\ 0 & n \neq 0. \end{cases}$$

Borel–Moore Homology

Simplicial Homology With Closed Support is defined similarly to homology with compact support, but allowing the chains to be infinite combinations of simplices.

There are both more cycles and more boundaries.

In the case where X has a finite triangulation, $H_k^{BM}(X) = H_k(X)$.

Affine space

Suppose that $\sigma = \sum a_n[n]$ is a cycle in $\mathbb{R}$.

Define $\tau = \sum b_n[n, n+1]$ by setting $b_0 = 0$, and setting b_n so that the coefficient of $[n-1]$ in $\partial\tau$ is 0. We get that σ is a boundary.

$$\Rightarrow H_0^{BM}(\mathbb{R}) = 0$$

Affine space

The chain $\sigma = \sum [n, n+1]$ is a cycle. Conversely, any 1-cycle is a multiple of σ.

$$H_1^{BM}(\mathbb{R}) = K$$

Similarly, one shows

$$H_k^{BM}(\mathbb{R}^n) = \begin{cases} 0 & k \neq n \\ K & k = n \end{cases}$$

Poincaré Duality

If X is a smooth, n-dimensional, orientable manifold, then any fine-enough triangulation has a dual

Poincaré Duality

- The vertices of the dual correspond to the top simplices of the original triangulation.
- The edges of the dual correspond to the facets of the original triangulation.

  $\vdots$
- The top cells of the dual correspond to the vertices of the original triangulation.

$$\Rightarrow C_i^{BM}(X) = C_{n-i}(X)^\vee$$

In fact, $H_i^{BM}(X) = H_{n-i}(X)^\vee$.

LES for a closed set and its complement

Suppose that X is a triangulated space, and let $F \subset X$ be a sub-complex. Let $U = X \setminus F$, and triangulate it by dividing each open simplex into (infinitly many) sub-simplices.

LES for a closed set and its complement

If σ is a k-cycle in F, then it is a k-cycle in X. We get a map $BM_k(F) \to BM_k(X)$.

If τ is a k-cycle in X, then its restriction to U can be presented as a cycle. We get a map $BM_k(X) \to BM_k(U)$.

If ξ is a k-cycle in U, then its **boundary in** X is a cycle in F. We get a map $BM_k(U) \to BM_{k-1}(F)$

LES for a closed set and its complement

We get a sequence

$$\cdots \to H_k^{BM}(F) \to H_k^{BM}(X) \to H_k^{BM}(U) \to H_{k-1}^{BM}(F) \to H_{k-1}^{BM}(X) \to \cdots$$

which is exact.

Computation: Complex Projective Space

$\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{P}^{n-1}$. Looking at the LES of open/closed subsets, we get

$$\to H^{BM}_{k+1}(\mathbb{C}^n) \to H^{BM}_k(\mathbb{P}^{n-1}) \to H^{BM}_k(\mathbb{P}^n) \to H^{BM}_k(\mathbb{C}^n) \to H^{BM}_{k-1}(\mathbb{P}^{n-1}) \to$$

If $k \neq 2n, 2n-1$ then $BM_k(\mathbb{P}^n) = BM_k(\mathbb{P}^{n-1})$. By induction,

$BM_{2n}(\mathbb{P}^{n-1}) = BM_{2n-1}(\mathbb{P}^{n-1}) = 0$. So if $k = 2n$, then $BM_{2n}(\mathbb{P}^n) = BM_{2n}(\mathbb{C}^n) = K$, Whereas if $k = 2n-1$, then $BM_{2n-1}(\mathbb{P}^n) = 0$.

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