Geometry of Schubert Varieties

RepNet Workshop

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Flag Varieties

Throughout, let k be a fixed algebraically closed field and V a finite-dimensional k-vector space.

A *flag* in V is a nested sequence of strictly increasing subvector spaces:

$$F_\bullet := (0 = F_0 < F_1 < \cdots < F_r = V)$$

If $r = \dim V$ we speak of a *complete flag*, otherwise we call $F_\bullet$ a *partial flag*.

For $\underline{\mathrm{d}} \in \mathbb{N}^r$ we set

$$\mathrm{Fl}(V, \underline{\mathrm{d}}) := \{F_\bullet \mid \dim(F_i/F_{i-1}) = d_i, i = 1 \ldots r\}$$

the set of $\underline{\mathrm{d}}$-flags.

When is $\mathsf{Fl}(V, \underline{\mathrm{d}})$ a variety? Some examples:

- $\underline{\mathrm{d}} = (1, \dim V - 1)$ we obtain $\mathsf{Fl}(V, \underline{\mathrm{d}}) = \mathbb{P}(V)$
- $\underline{\mathrm{d}} = (l, \dim V - l)$ we have $\mathsf{Fl}(V, \underline{\mathrm{d}}) = \mathsf{Gr}(V, l)$, the Grassmannian of l-dimensional subspaces of V

Promising! We could proceed by embedding $\mathsf{Fl}(V, \underline{\mathrm{d}})$ in a product of Grassmannians

$$\mathsf{Fl}(V, \underline{\mathrm{d}}) \hookrightarrow \mathsf{Gr}(V, \dim F_1) \times \mathsf{Gr}(V, \dim F_2) \ldots \times \mathsf{Gr}(V, \dim F_r).$$
$$(F_0 < F_1 < \ldots < F_r) \mapsto (F_1, F_2, \ldots, F_r)$$

But let us give a description that is easier to generalize (of course the same up to isomorphism):

We clearly have a natural action of $\mathsf{GL}(V)$ on $\mathsf{Fl}(V,\underline{\mathrm{d}})$ by

$$g \cdot F_\bullet = (0 = g(F_0) < g(F_1) < \cdots < g(F_{r-1}) < g(F_r) = V)$$

and this action is transitive. So for any fixed flag $F_\bullet \in \mathsf{Fl}(V,\underline{\mathrm{d}})$ we have

$$\mathsf{Fl}(V,\underline{\mathrm{d}}) \cong \mathsf{GL}(V)/\mathsf{Stab}(F_\bullet)$$

and we define this to be an isomorphism of varieties (well-defined because two stabilizers are conjugate).

EXAMPLE

For $V = k^n$ and the *standard flag*

$$E_\bullet = (0 < \langle e_1 \rangle < \langle e_1, e_2 \rangle < \ldots < \langle e_1, e_2, \ldots, e_n \rangle = k^n)$$

we have

$$\begin{aligned} \mathrm{Stab}(E_\bullet) = B &:= \{\text{upper triangular matrices}\} \\ &:= \textit{standard Borel subgroup} \end{aligned}$$

and so

$$\mathrm{Fl}(k^n, (1, 1, \ldots, 1)) = \mathrm{GL}_n(k)/B$$

Let (W, S) be a Coxeter system with W finite, i.e. W is a finite group with presentation $\langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$, where $m_{ij} \in \mathbb{N} \cup \{\infty\}$ with $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ for $i \neq j$. Here, $(s_i s_j)^{\infty} = 1$ denotes the empty relation.

Definition

- $\ell : W \to \mathbb{N}$, $\ell(w) := \min \#\{l \mid w = s_1 \cdots s_l \text{ with } s_i \in S\}$ is the *length function*.
- Any expression $w = s_1 \cdots s_{\ell(w)}$ with $s_i \in S$ is called a *reduced expression* of w.
- $R := \{wsw^{-1} \mid w \in W, s \in S\}$ are the *reflections* of (W, S).
- If $u^{-1}w \in R$ and $\ell(u) < \ell(w)$, one writes $u \to w$ and says u *covers* w. (Note: $u^{-1}w \in R$ if and only if $wu^{-1} \in R$)
- If there exists a sequence $u = u_0 \to u_1 \to \cdots \to u_r = w$, one writes $u \leq w$.

Theorem

(i) $\leq$ is a partial order on W (*Bruhat order*).

(ii) If $w = s_1 \cdots s_l$ is a reduced expression of w, then $u \leq w$ if and only if a subexpression of $s_1 \cdots s_l$ is equal to u.

(iii) If $u \leq w$, then there exists a sequence $u = u_0 \to u_1 \to \cdots \to u_r = w$ with $\ell(u_{i+1}) = \ell(u_i) + 1$.

(iv) $1 \leq w$ for all $w \in W$.

(v) $\leq$ is directed, i.e. if $u, v \in W$, then $u \leq w$, $v \leq w$ for some $w \in W$.

(vi) There exists a unique element $w_0 \in W$ of maximal length. Moreover, $w \leq w_0$ for all $w \in W$.

Our primary example of a Coxeter system:

Theorem

(S_n, S) with $S := \{s_i \mid 1 \leq i \leq n-1\}$ and $s_i := (i, i+1)$ is a Coxeter system.

Some properties:

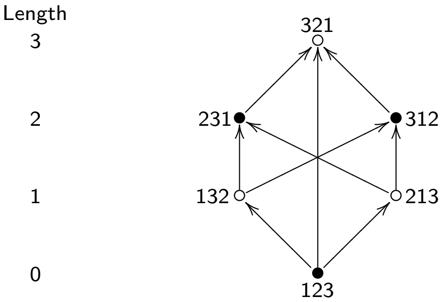
(i) $\ell(w) = \#\{(i,j) \mid i < j, w(i) > w(j)\}$ for all $w \in \mathrm{S}_n$.

(ii) $w_0 = (n \quad n-1 \quad \cdots \quad 1)$.

Definition

The *Bruhat graph* of (W, S) is the directed graph with vertices W and arrows given by the covering relation $\to$.

The Bruhat graph of S_3 is:



Throughout, we fix $G := \mathrm{GL}_n(k)$. We denote by B the group of upper triangular matrices in G and by T the group of diagonal matrices in G.

DEFINITION

(i) $\mathrm{N}_G(T)$ is the group of monomial matrices. Hence, $W := \mathrm{N}_G(T)/T \cong \mathrm{S}_n$ canonically. The permutation matrices in G form a complete set of representatives $\{\dot{w}\}$ of W.
(ii) $S := \{s_i \mid 1 \leq i < n\}$, where $s_i := (i, i+1)$.
(iii) $\Phi := \{\chi_{ij} \mid 1 \leq i, j \leq n, i \neq j\}$, where $\chi_{ij} : T \to k^\times$, $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i t_j^{-1}$.
(iv) $\Phi^+ := \{\chi_{ij} \mid 1 \leq i < j \leq n\}$.
(v) $s_{\chi_{ij}} := s_{ij} := (i, j)$.
(vi) W acts on $\mathrm{Hom}(T, k^\times)$ by $(w.\chi)(t) := \chi(t^w)$.

Definition

(i) $U :=$ the group of upper unitriangular matrices.

(ii) $B^- :=$ the group of lower triangular matrices.

(iii) $U^- :=$ the group of lower unitriangular matrices.

(iv) $U'_w := U \cap wU^-w^{-1}$ for $w \in W$.

(v) $\Phi^-_w := \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in -\Phi^+\}$
$= \{\chi_{ij} \mid i < j, w^{-1}(i) > w^{-1}(j)\}$.

(vi) $u_{\chi_{ij}} := u_{ij} : k \to G$, $c \mapsto 1 + cE_{ij}$.

Theorem

(i) $G = \coprod_{w \in W} BwB$ (*Bruhat decomposition*).

(ii) The map $U'_w \times B \to BwB$ given by $(u, b) \mapsto u\dot{w}b$ is an isomorphism.

(iii) $G/B = \coprod_{w \in W} BwB/B$. We call $C_w := BwB/B$ a *Schubert cell* in G/B.

(iv) The map $U'_w \to C_w$ given by $u \mapsto u\dot{w}B$ is an isomorphism.

(v) Let $\alpha_1, \ldots, \alpha_l$ be the roots of Φ_w^- (any ordering). Then the map $\mathbb{A}^{\ell(w)} \to U'_w$, $(a_1, \ldots, a_l) \mapsto u_{\alpha_1}(a_1) \cdots u_{\alpha_l}(a_l)$, is an isomorphism of varieties. Hence, $\mathbb{A}^{\ell(w)} \cong C_w$.

Definition

For each $w \in W$ we set $X_w := \overline{BwB/B} \subseteq G/B$ and refer to this as the *Schubert variety* of w.

Theorem

The following holds:

(i) X_w is a projective variety and $\dim X_w = \ell(w)$.

(ii) $X_w = \coprod_{u \leq w} C_u$, where $\leq$ denotes the Bruhat order.

(iii) $X_w = \bigcup_{u \leq w} X_u$.

(iv) $X_u \subseteq X_w$ if and only if $u \leq w$.

(v) $X_{w_0} = G/B$ and $X_{id} = pt$.

Let $n = 4$. For $w = (2413)$ we have

$$U'_w = \left\{ \begin{pmatrix} 1 & \star & 0 & \star \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathbb{A}^3 = \mathbb{A}^{\ell(w)} \Rightarrow C_w = \left\{ \begin{pmatrix} \star & \star & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \star & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} B \right\}.$$

We have $\Phi_w^- = \{\chi_{12}, \chi_{14}, \chi_{34}\}$ and indeed the map $\mathbb{A}^3 \to U'_w$ defined above as

$$\begin{aligned}(a_1, a_2, a_3) \mapsto\ & u_{12}(a_1)u_{14}(a_2)u_{34}(a_3) \\ &= 1 + a_3 E_{34} + a_2 E_{14} + a_1 E_{12} \\ &= \begin{pmatrix} 1 & a_1 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

is an isomorphism.

Let us consider an example now:

$$\begin{pmatrix} 6 & 9 & 2 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 6 & 2 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix} \xrightarrow[-1\cdot\mathrm{I}+\mathrm{IV}]{-1\cdot\mathrm{I}+\mathrm{II}} \begin{pmatrix} 6 & 3 & 2 & -7 \\ 1 & 0 & 0 & 0 \\ 0 & 6 & 2 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix} \xrightarrow{\frac{1}{3}\cdot\mathrm{II}} \begin{pmatrix} 6 & 1 & 2 & -7 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{-1\cdot\mathrm{II}+\mathrm{III}} \begin{pmatrix} 6 & 1 & 1 & -7 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{7\cdot\mathrm{III}+\mathrm{IV}} \begin{pmatrix} 6 & 1 & \textcircled{1} & 0 \\ \textcircled{1} & 0 & 0 & 0 \\ 0 & 2 & 0 & \textcircled{1} \\ 0 & \textcircled{1} & 0 & 0 \end{pmatrix}$$

So this is in C_{2413} with coordinates $(6, 1, 2)$.

We can also determine the decomposition $M = u\dot{w}b$ with $u \in U'_w$ and $b \in B$:
The column operations give b^{-1}, $\dot{w}$ can be read off from M and u computed from $M, b, \dot{w}$:

$$b = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \dot{w} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\ u = \begin{pmatrix} 1 & 6 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The maximal torus T acts canonically on G/B. The root morphisms u_α and their property $tu_\alpha(c)t^{-1} = u_\alpha(\alpha(t)c)$ are the central ingredients in the proof of the following theorem:

THEOREM

(i) The set of T-fixed points is $\{wB \mid w \in W\}$, i.e. $(G/B)^T \cong W$.

(ii) The one-dimensional T-orbits are precisely the

$$\mathcal{O}_{w,\alpha} := \{u_\alpha(x)wB \mid x \in k^\times\} = Tu_\alpha(1)wB$$

for $w \in W$ and $\alpha \in \Phi_w^-$.

(iii) $\mathcal{O}_{w,\alpha} \subseteq C_w$ for all $\alpha \in \Phi_w^-$.

(iv) If $\ell(w) = 1$, then $\overline{\mathcal{O}}_{w,\alpha} = X_w$ for the unique $\alpha \in \Phi_w^-$. In particular, all Schubert curves in G/B are among the closures of the one-dimensional T-orbits.

The moment graph of G/B

There is a T-equivariant embedding $G/B \to \mathbb{P}(V)$ for some V, where the T-action on $\mathbb{P}(V)$ is induced by a linear action of T on V.

Hence, if $\mathcal{O}$ is a one-dimensional T-orbit, then there exists a T-equivariant isomorphism $\varphi : \overline{\mathcal{O}} \to \mathbb{P}^1$, where $\overline{\mathcal{O}}$ is the closure of $\mathcal{O}$ and the T-action on $\mathbb{P}^1$ is induced by a linear action on k^2.

In particular, $\overline{\mathcal{O}}$ is obtained from $\mathcal{O}$ by adding the two T-fixed points $\varphi^{-1}(0)$ and $\varphi^{-1}(\infty)$.

Theorem

The two T-fixed points of the closure of $\mathcal{O}_{w,\alpha}$ are wB and $s_\alpha wB$.

Combining the results above with the following lemma:

LEMMA

Let (W, S) be a Coxeter system with W finite. Let $w \in W$ and $\alpha \in \Phi^+$. Then $\ell(ws_\alpha) < \ell(w)$ if and only if $w(\alpha) \in -\Phi^+$.

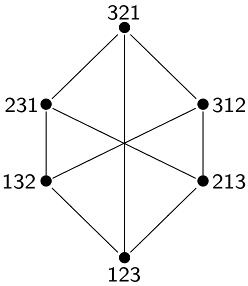
we arrive at the following amazing result:

THEOREM

The moment graph of G/B with respect to the action of T is precisely the undirected Bruhat graph of (W, S).

The moment graph of G/B

Let $n = 3$. Then $\Phi^+ = \{\chi_{12}, \chi_{13}, \chi_{23}\}$, $\Phi^-_{123} = \emptyset$, $\Phi^-_{132} = \{\chi_{23}\}$, $\Phi^-_{213} = \{\chi_{12}\}$, $\Phi^-_{231} = \{\chi_{12}, \chi_{13}\}$, $\Phi^-_{312} = \{\chi_{13}, \chi_{23}\}$, $\Phi^-_{321} = \{\chi_{12}, \chi_{13}, \chi_{23}\}$. We have $s_{23} \cdot (132) = (123)$, $s_{12} \cdot (213) = (123)$, $s_{12} \cdot (231) = (132)$, $s_{13} \cdot (231) = (213)$, $s_{13} \cdot (312) = (132)$, $s_{23} \cdot (312) = (213)$, $s_{12} \cdot (321) = (312)$, $s_{13} \cdot (321) = (123)$, $s_{23} \cdot (321) = (231)$.
Hence, the moment graph of G/B is



which is just the (undirected) Bruhat graph of S_3.

Bott-Samelson resolution

X_w need not be smooth, although C_w always is.

Theorem

For $G = \mathrm{GL}_n(k)$ and $y \in S_n$ the dimension of the tangent space $T_{yB}X_w$ is

$$\dim T_{yB}X_w = \#\{(i,j) \in \mathrm{S}_n \mid y(i,j) \leq w\}$$

So for $\mathrm{GL}_4(k)$ we see that $X_{(1,4)}$ is singular: It has dimension $\ell((1,4)) = 5$ but $T_B X_{(1,4)}$ has dimension 6 corresponding to the permutations $(1,2), (2,3), (3,4), (1,4), (2,4), (1,3)$.

It is an important problem in algebraic geometry to find resolutions of singular spaces. For Schubert varieties this can be done via the Bott-Samelson varieties.

Definition

A *parabolic subgroup* P of G is any closed subgroup containing B. If X is a space with a left B action, the *induced P-space* is the quotient

$$P \overset{B}{\times} X := (P \times X)/B := P \times X/((g,x) \sim (gb^{-1}, bx) \; \forall b \in B).$$

$P \overset{B}{\times} X$ has a canonical B-action via left multiplication and thus for any $P_1, P_2, \ldots, P_l$ of parabolics this construction can be iterated:

$$P_1 \overset{B}{\times} P_2 \overset{B}{\times} \ldots \overset{B}{\times} P_l \overset{B}{\times} X := P_1 \overset{B}{\times} (P_2 \overset{B}{\times} (\ldots \overset{B}{\times} (P_l \overset{B}{\times} X) \ldots))$$

Definition

P_s is the subgroup of G generated by B and $s \in S$. This will be referred to as a *minimal parabolic*. We have $P_s = B \cup BsB$.

Consider $w \in W$ with reduced expression $w = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_l}$. We then define the *Bott-Samelson variety* Z_w as

$$Z_w := P_{s_{\alpha_1}} \overset{B}{\times} P_{s_{\alpha_2}} \overset{B}{\times} \ldots \overset{B}{\times} (P_{s_{\alpha_l}}/B).$$

Theorem

Z_w is a smooth projective variety of dimension $\ell(w)$.

The proof uses the inductive construction of the Z_w as $Z_w = P_s \overset{B}{\times} Z_{w'}$ for some $w' < w$ as well as the following lemma:

Lemma

(i) The induced P-space gives a fiber bundle over P/B with fiber X and total space $P \overset{B}{\times} X$:

$$X \to P \overset{B}{\times} X \to P/B$$

(ii) For any minimal parabolic P_s the space P_s/B is isomorphic to $\mathbb{P}^1$ as a variety.

The last line gives us the first step in the induction. Note that P_{s_i}/B corresponds to the flags

$$(0 = E_0 < E_1 < \dots E_{i-1} < L < E_{i+1} < \dots E_n = k^n)$$

and L is uniquely determined by the choice of a line in E_{i+1}/E_{i-1} giving the isomorphism with $\mathbb{P}^1 = \mathbb{P}(E_{i+1}/E_{i-1})$.

As the name suggests, the Z_w will give us the Bott-Samelson resolution. We can organize the key results into one theorem:

Theorem

Consider a Schubert variety $X_w \subseteq G/B$. Then the following holds:

(i) The morphism

$$\begin{array}{rcl} Z_w & \to & X_w \\ (p_1, \ldots, p_{\ell(w)})B & \mapsto & p_1 p_2 \ldots p_{\ell(w)} B \end{array}$$

is a resolution of singularities.

(ii) X_w is normal.

(iii) X_w has at most rational singularities.

Corollary

The singular locus of a Schubert variety has codimension ≥ 2.